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DISSERTATION

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By

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## CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGMENTS</td>
</tr>
<tr>
<td>VITA</td>
</tr>
<tr>
<td>ILLUSTRATIONS</td>
</tr>
</tbody>
</table>

### PART ONE

**CONVENTIONAL SHELL ANALYSIS**

1. Formulation of Equations | 1 |
2. Conventional Methods of Solution | 19 |
3. Deficiencies of Conventional Methods | 27 |

### PART TWO

**DISPLACEMENT FUNCTION SOLUTIONS**

4. Displacement Equations | 31 |
5. Continuous Displacement Functions | 47 |
6. Matrix Equations | 67 |

### PART THREE

**METHODS OF NUMERICAL SOLUTION**

7. Conventional Methods | 71 |
8. Multiple Regression | 80 |
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>PART FOUR</td>
<td></td>
</tr>
<tr>
<td>RESULTS AND CONCLUSIONS</td>
<td></td>
</tr>
<tr>
<td>9. Numerical Solution</td>
<td>88</td>
</tr>
<tr>
<td>10. Conclusions</td>
<td>130</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>135</td>
</tr>
<tr>
<td>APPENDIX</td>
<td>138</td>
</tr>
<tr>
<td>Figure</td>
<td>Illustration Description</td>
</tr>
<tr>
<td>--------</td>
<td>---------------------------------------------------------------</td>
</tr>
<tr>
<td>1</td>
<td>Geometry of Shell Element</td>
</tr>
<tr>
<td>2</td>
<td>Direct Forces on Shell Element</td>
</tr>
<tr>
<td>3</td>
<td>Bending Moments on Shell Element</td>
</tr>
<tr>
<td>4</td>
<td>Geometry of Spherical Shell</td>
</tr>
<tr>
<td>5</td>
<td>Geometry of Cylindrical Shell</td>
</tr>
<tr>
<td>6</td>
<td>Geometry of Conical Shell</td>
</tr>
<tr>
<td>7</td>
<td>Complete Sphere - Tangential Pressure Comparison</td>
</tr>
<tr>
<td>8</td>
<td>Complete Sphere - Normal Pressure Comparison</td>
</tr>
<tr>
<td>9</td>
<td>Complete Sphere - Meridional Deflection</td>
</tr>
<tr>
<td>10</td>
<td>Complete Sphere - Normal Deflection</td>
</tr>
<tr>
<td>11</td>
<td>Complete Sphere Meridional Direct Force</td>
</tr>
<tr>
<td>12</td>
<td>Complete Sphere - Circumferential Direct Force</td>
</tr>
<tr>
<td>13</td>
<td>Complete Sphere - Meridional Bending Moment</td>
</tr>
<tr>
<td>14</td>
<td>Complete Sphere - Circumferential Bending Moment</td>
</tr>
<tr>
<td>15</td>
<td>Completed Sphere - Meridional Transverse Shear</td>
</tr>
<tr>
<td>16</td>
<td>Truncated Sphere - Constant Edge Load Tangential Pressure Comparison</td>
</tr>
<tr>
<td>17</td>
<td>Truncated Sphere - Constant Edge Load Normal Pressure Comparison</td>
</tr>
<tr>
<td>18</td>
<td>Truncated Sphere - Constant Edge Load Meridional Deflection</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------------------------------------------------</td>
</tr>
<tr>
<td>19</td>
<td>Truncated Sphere - Constant Edge Load Normal Deflection</td>
</tr>
<tr>
<td>20</td>
<td>Truncated Sphere - Constant Edge Load Meridianal Direct Force</td>
</tr>
<tr>
<td>21</td>
<td>Truncated Sphere - Constant Edge Load Circumferential Direct Force</td>
</tr>
<tr>
<td>22</td>
<td>Truncated Sphere - Constant Edge Load Meridianal Bending Moment</td>
</tr>
<tr>
<td>23</td>
<td>Truncated Sphere - Constant Edge Load Circumferential Bending Moment</td>
</tr>
<tr>
<td>24</td>
<td>Truncated Sphere - Constant Edge Load Meridianal Transverse Shear</td>
</tr>
<tr>
<td>25</td>
<td>Truncated Sphere - Statically Null Edge Load Meridianal Pressure Comparison</td>
</tr>
<tr>
<td>26</td>
<td>Truncated Sphere - Statically Null Edge Load Circumferential Pressure Comparison</td>
</tr>
<tr>
<td>27</td>
<td>Truncated Sphere - Statically Null Edge Load Normal Pressure Comparison</td>
</tr>
<tr>
<td>28</td>
<td>Truncated Sphere - Statically Null Edge Load Meridianal Deflection</td>
</tr>
<tr>
<td>29</td>
<td>Truncated Sphere - Statically Null Edge Load Circumferential Deflection</td>
</tr>
<tr>
<td>30</td>
<td>Truncated Sphere - Statically Null Edge Load Normal Deflection</td>
</tr>
<tr>
<td>31</td>
<td>Truncated Sphere - Statically Null Edge Load Meridianal Direct Force</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
</tbody>
</table>
| 32     | Truncated Sphere - Statically Null Edge Load  
|        | Circumferential Direct Force | 122  |
| 33     | Truncated Sphere - Statically Null Edge Load  
|        | Meridinal Bending Moment | 123  |
| 34     | Truncated Sphere - Statically Null Edge Load  
|        | Circumferential Bending Moment | 124  |
| 35     | Truncated Sphere - Statically Null Edge Load  
|        | In-Surface Shear | 125  |
| 36     | Truncated Sphere - Statically Null Edge Load  
|        | Twisting Moment | 126  |
| 37     | Truncated Sphere - Statically Null Edge Load  
|        | Meridinal Transverse Shear | 127  |
| 38     | Truncated Sphere - Statically Null Edge Load  
|        | Circumferential Transverse Shear | 128  |
PART 1
CONVENTIONAL SHELL ANALYSIS

Part 1 is a brief review of the present status of thin shell theory. It is primarily designed to demonstrate the need of some useful method of solution, such as that set forth in Part 2.

Chapter 1
Formulation of Equations

From the beginning of serious investigation into the behavior of elastic thin shells there has been much disagreement as to the proper formulation of the equations to be solved. The first noteworthy attempt to set forth a useful shell theory was made by the great English elastician, A.E.H. Love (1)*, whose work remains to this day the basis for the main stream of work in this field.

Love's theory was based on the Kirchhoff hypothesis of thin plates, which states that all elements normal to the middle surface of the shell remain normal, straight, and undeformed. A by-product of this hypothesis is the assumption that direct stresses

*(1) indicates Reference (1)
normal to the surface and transverse shear strains may be neglected in comparison with other stresses. These conditions are only approximately true but the inaccuracies that result are, as in strength of materials theory, rarely significant.

The conventional method of formulating the necessary equations of thin shells consists of writing three sets of relations - six equilibrium equations, seven relations between generalized forces and strains, and six strain-curvature-displacement equations. There are nineteen unknown quantities - four forces, four moments, two transverse shears, three strains, three curvature changes, and three displacements - and nineteen equations.

Prior to writing any of the equations mentioned above one must establish a reference system. Under the assumptions made the theory of shells is essentially the study of the behavior of a surface. Hence a two-dimensional curvilinear reference system suffices to describe a shell. It is shown by Struik (2) that the principal directions of a surface are always orthogonal. The principal directions are simply the directions of tangents to the arcs through the point having the maximum and minimum curvature. For
convenience we select an orthogonal curvilinear coordinate system whose axes coincide with the principal directions at every point.

An example is shown in

\[
\begin{align*}
\left| B + \frac{\partial B}{\partial \alpha} \, d\alpha \right| d\beta & \quad \left| A + \frac{\partial A}{\partial \beta} \, d\beta \right| d\alpha
\end{align*}
\]

**Figure 1**

Geometry of Shell Element

Figure 1, which depicts an infinitesimal element of a surface. A and B are Lamé' parameters which relate the change in coordinates \( \alpha \) and \( \beta \), respectively, to distance along the arc. First order terms only are kept in the expansions for A and B on the positive faces of the element.
With a coordinate system established the equations of equilibrium for an element of shell can be written. The forces and moments are presented in Figures 2 and 3 respectively. A double headed vector indicates the axis of rotation using the right hand rule. Equilibrium equations are

\[ \begin{align*}
T_1 & = T_2 \\
N_1 & = N_2 \\
T_{12} & = T_{21}
\end{align*} \]

\[ \begin{align*}
M_1 & = M_{12} \\
M_2 & = M_{21}
\end{align*} \]

Figure 2

Direct Forces on Shell Element

Figure 3

Bending Moments on Shell Element
as given by Novozhilov (3):

\[
\frac{1}{AB} \left[ \frac{\partial (BT)}{\partial \alpha} + \frac{\partial (AT)}{\partial \beta} \right] + \frac{N_1}{R_1} = 0,
\]

\[
\frac{1}{AB} \left[ \frac{\partial (BT_{12})}{\partial \alpha} + \frac{\partial (AT_{12})}{\partial \beta} \right] + \frac{N_2}{R_2} + q_2 = 0,
\]

\[
\frac{1}{AB} \left[ \frac{\partial (BN)}{\partial \alpha} + \frac{\partial (AN)}{\partial \beta} \right] - \frac{T_1}{R_1} - \frac{T_2}{R_2} + q_n = 0,
\]

\[
\frac{1}{AB} \left[ \frac{\partial (BN_{12})}{\partial \alpha} + \frac{\partial (AN_{12})}{\partial \beta} \right] - N_1 = 0,
\]

\[
\frac{1}{AB} \left[ \frac{\partial (BM_{12})}{\partial \alpha} + \frac{\partial (AM_{12})}{\partial \beta} \right] - \frac{M_{12}}{R_1} - \frac{M_{21}}{R_2} = 0.
\]

The three components of displacement at a point are \( u, v, \) and \( w, \) acting as shown in Figure 1. According to what Love called the first approximation, terms of the order \( z/R \) are very small and can be neglected in comparison with 1. In this case \( z \) is the normal distance of an element from the neutral surface and \( R \) is any radius of curvature of the shell. This first approximation theory is a
restating of the Kirchhoff hypothesis. Based on it the following relations between strain and displacement ensue:

\[
\begin{align*}
\epsilon_1 &= \frac{1}{A} \frac{\partial u}{\partial \alpha} + \frac{1}{AB} \frac{\partial A}{\partial \beta} + \frac{w}{R_1}, \\
\epsilon_2 &= \frac{1}{B} \frac{\partial v}{\partial \beta} + \frac{1}{AB} \frac{\partial B}{\partial \alpha} + \frac{w}{R_2}, \\
\omega &= \frac{B}{A} \frac{\partial}{\partial \alpha} \left( \frac{v}{B} \right) + \frac{A}{B} \frac{\partial}{\partial \beta} \left( \frac{u}{A} \right), \\
K_1 &= -\frac{1}{A} \frac{\partial}{\partial \alpha} \left[ \frac{1}{A} \frac{\partial w}{\partial \alpha} - \frac{u}{R_1} \right] - \frac{1}{AB} \frac{\partial A}{\partial \beta} \left( \frac{1}{B} \frac{\partial w}{\partial \beta} - \frac{v}{R_2} \right), \\
K_2 &= -\frac{1}{B} \frac{\partial}{\partial \beta} \left[ \frac{1}{B} \frac{\partial w}{\partial \beta} - \frac{v}{R_2} \right] - \frac{1}{AB} \frac{\partial B}{\partial \alpha} \left( \frac{1}{A} \frac{\partial w}{\partial \alpha} - \frac{u}{R_1} \right), \\
\tau &= -\frac{1}{AB} \left( \frac{\partial^2 w}{\partial \alpha \partial \beta} - \frac{1}{A} \frac{\partial A}{\partial \beta} \frac{\partial w}{\partial \alpha} - \frac{1}{B} \frac{\partial B}{\partial \alpha} \frac{\partial w}{\partial \beta} \right) \\
&\quad + \frac{1}{R_1} \left( \frac{1}{B} \frac{\partial u}{\partial \beta} - \frac{1}{AB} \frac{\partial A}{\partial \beta} u \right) + \frac{1}{R_2} \left( \frac{1}{A} \frac{\partial}{\partial \alpha} - \frac{1}{AB} \frac{\partial B}{\partial \alpha} \right). 
\end{align*}
\]  

\(\epsilon_1, \epsilon_2, \omega\) are defined as the relative changes in length and the shear of the middle surface respectively. \(K_1, K_2,\) and \(\tau\) characterize the changes in curvature in the directions \(\alpha\) and \(\beta\), and the twist of the middle surface. The first three are often denoted as membrane strains, the second three as bending strains.

Finally the strains and internal forces must be related. First consider the last of equations (1.1). It can be readily shown by assuming the symmetry of the stress tensor that this equation is identically zero. Hence we introduce the notation
\[ S = T_{12} - \frac{M_{12}}{R_2} = T_{21} - \frac{M_{12}}{R_2}, \quad H = \frac{M_{12} + M_{21}}{2} \quad (1.3) \]

Now the force-strain relations are, for an isotropic, homogeneous shell,

\[
T_1 = \frac{E t}{1 - \nu^2} (\varepsilon_1 + \nu \varepsilon_2), \quad T_2 = \frac{E t}{1 - \nu^2} (\varepsilon_2 + \nu \varepsilon_1),
\]

\[
M_1 = \frac{E t^3}{12(1 - \nu^2)} (K_1 + \nu K_2), \quad M_2 = \frac{E t^3}{12(1 - \nu^2)} (K_1 + \nu K_2),
\]

\[
S = \frac{E t}{2(1 + \nu)} \omega, \quad H = \frac{F t^3}{12(1 + \nu)} T, \quad (1.4)
\]

where \( E \) = modulus of elasticity,

\( t \) = shell thickness,

\( \nu \) = Poisson's ratio.

There have been numerous variants on the above force-strain equations but none appear to be any better for our purposes. Love (1) and Timoshenko (2) propose somewhat simpler equations but they do not completely satisfy the equilibrium equations. Lurye (5) and others developed more complex relations which have the alleged virtue of higher accuracy. However the fact is that the corrective terms are smaller than the accuracy originally assumed
in the first approximation theory. Hence the advantage of simplicity would be sacrificed for no proven increased accuracy.

It will be noted from Figures 2 and 3 that five forces exist on any edge of a shell segment. This would indicate that five boundary conditions are necessary to define the situation at each edge. It was, however, shown by Kirchhoff that four boundary conditions suffice to give complete definition of the stresses existing on the edge of a plate. This was accomplished by replacing the twisting moment by corresponding distributed transverse forces.

The same principle applies in the case of a shell, where in addition the twisting moment contributes distributed tangential (shear) forces.

On a boundary line $\phi = \text{constant}$ the four internal loads are

$$T_1, \ T_{12} + \frac{M_{12}}{R_2}, \ N_1 - \frac{1}{B} \frac{\partial M_{12}}{\partial \beta}, \ M_1.$$

On an edge $\beta = \text{constant}$ they are

$$T_2, \ T_{21} + \frac{M_{21}}{R_1}, \ N_2 + \frac{1}{B} \frac{\partial M_{21}}{\partial \phi}, \ M_2.$$

Boundary conditions may also be given by prescribing displacements (translations or rotations) at an edge. In any event four values of load or deflection or four relations between load and deflection (so-called elastic conditions) are adequate to describe the conditions at a boundary.
In thin shell theory as in all problems in elasticity it is necessary that the strains satisfy the conditions of compatibility. Unlike plane elasticity which requires only one such relation or three dimensional elasticity, which has six, the theory of surfaces has three compatibility equations. They are (3)

\[
B \left( \frac{\partial K_2}{\partial \alpha} + \frac{\partial B}{\partial \alpha} |K_2 - K_1| - \frac{\partial A T}{\partial \beta} - 2 \frac{\partial A T}{\partial \beta} \right)
+ \frac{1}{R_2} \frac{\partial A \omega}{\partial \beta} + \frac{1}{R_1} \left[ \frac{A \partial \omega}{\partial \beta} + \frac{\partial A \omega}{\partial \beta} - B \frac{\partial \epsilon_2}{\partial \alpha} \right]
- \frac{\partial B}{\partial \alpha} (\epsilon_2 - \epsilon_1) = 0.
\]

\[
A \left( \frac{\partial K_1}{\partial \beta} + \frac{\partial A}{\partial \beta} (K_1 - K_2) - \frac{B \partial T}{\partial \alpha} - 2 \frac{\partial B T}{\partial \alpha} \right)
+ \frac{1}{R_1} \frac{\partial B \omega}{\partial \alpha} + \frac{1}{R_2} \left[ \frac{B \partial \omega}{\partial \alpha} + \frac{\partial B \omega}{\partial \alpha} - A \frac{\partial \epsilon_1}{\partial \beta} \right]
- \frac{\partial A}{\partial \beta} (\epsilon_1 - \epsilon_2) = 0.
\]

\[
\frac{K_1}{R_1} + \frac{K_2}{R_2} + \frac{1}{AB} \left\{ \frac{\partial}{\partial \alpha} \left[ \frac{1}{A} \left[ B \frac{\partial \epsilon_2}{\partial \alpha} + \frac{\partial B}{\partial \alpha} (\epsilon_2 - \epsilon_1) \right] \right]
- \frac{A}{2} \frac{\partial \omega}{\partial \beta} - \frac{\partial A \omega}{\partial \beta} \right\} + \frac{1}{B} \left[ \frac{A \partial \epsilon_1}{\partial \beta} \right]
+ \frac{\partial A}{\partial \beta} (\epsilon_1 - \epsilon_2) - \frac{B}{2} \frac{\partial \omega}{\partial \alpha} - \frac{\partial B \omega}{\partial \alpha} \right\} = 0.
\]
Now the six equations of equilibrium, (1.1) and (1.2), can be reduced to three equations by substituting values of \( N_1 \) and \( N_2 \) from (1.2) into (1.1). Also using (1.3) we have

\[
\frac{\partial(BT_1)}{\partial \alpha} + \frac{\partial(AM_1)}{\partial \alpha} + \frac{\partial A}{\partial \beta} \left[ S - \frac{\partial B}{\partial \alpha} T_2 + \frac{1}{R_1} \left[ \frac{\partial(BM_1)}{\partial \alpha} \right. \right. \\
- \frac{\partial B}{\partial \alpha} M_2 \frac{2\partial(AH)}{\partial B} + 2 \frac{R_1}{R_2} \frac{\partial A}{\partial \beta} H \left. \right] + AB q_1 = 0, \\
\frac{\partial(BS)}{\partial \alpha} + \frac{\partial(AT_2)}{\partial \beta} + \frac{\partial B}{\partial \alpha} \left[ S - \frac{\partial A}{\partial \beta} T_1 + \frac{1}{R_2} \left[ \frac{\partial(AM_2)}{\partial \beta} \right. \right. \\
- \frac{\partial A}{\partial \beta} M_1 \frac{2\partial(BH)}{\partial \alpha} + 2 \frac{R_2}{R_1} \frac{\partial B}{\partial \alpha} H \left. \right] + AB q_2 = 0, \\
\frac{T_1}{R_1} + \frac{T_2}{R_2} - \frac{1}{AB} \left\{ \frac{\partial}{\partial \alpha} \frac{1}{A} \left[ \frac{\partial(BM_1)}{\partial \alpha} + \frac{\partial(AH)}{\partial \beta} + \frac{\partial A}{\partial \beta} H \right. \right. \\
- \frac{\partial B}{\partial \alpha} M_2 \left[ \right. \frac{1}{B} \left( \frac{\partial(BH)}{\partial \alpha} + \frac{\partial(AM_2)}{\partial \beta} + \frac{\partial B}{\partial \alpha} H \right. \right. \\
- \frac{\partial A}{\partial \beta} M_1 \frac{1}{A} \left. \right] \right\} - q_n = 0.
\]

It is obvious from equations (1.4) that each strain can be uniquely expressed in terms of internal loads. Thus

\[
\varepsilon_1 = \frac{1}{E_t} (T_1 - v T_2), \quad \varepsilon_2 = \frac{1}{E_t} (T_2 - v T_1), \\
K_1 = \frac{12}{E_t^3} (M_1 - v M_2), \quad K_2 = \frac{12}{E_t^3} (M_2 - v M_1), \\
\omega = \frac{2(1 + v)}{E_t} S, \quad \tau = \frac{12(1 + v)}{E_t^3} H.
\]
Substituting from (1.7) into the compatibility equations (1.5) produces three more equations for the six internal forces, or a total of six in all.

This system of six equations is of eighth order, i.e., solution for any one quantity necessitates the solution of an eighth order partial differential equation. One expedient frequently used to reduce the complexity of the problem, albeit at the expense of accuracy, is to neglect all bending effects. In this case the so-called membrane equations ensue. They are obtained by reducing equations (1.6):

\[
\begin{align*}
\frac{\partial(BT_1)}{\partial \alpha} &+ \frac{\partial(AS)}{\partial \beta} + \frac{\partial A}{\partial \beta} S - \frac{\partial B}{\partial \alpha} T_2 + AB q_1 = 0, \\
\frac{\partial(BS)}{\partial \alpha} &+ \frac{\partial(AT_2)}{\partial \beta} + \frac{\partial B}{\partial \alpha} S - \frac{\partial A}{\partial \beta} T_1 + AB q_2 = 0,
\end{align*}
\]

\[\frac{T_1}{R_1} + \frac{T_2}{R_2} = q_n.\]

This system is of fourth order.

An alternate formulation of the thin shell equations is obtained by converting the first three equations (1.1) into relations between displacements and external load. Substitute the force
strain equations (1.4) into (1.6), then replace strains by displacements from (1.2). The resulting equations are expressed in matrix form as shown by Goldenweiser (6).

\[
\begin{bmatrix}
L_{11} & L_{12} & L_{13} \\
L_{21} & L_{22} & L_{23} \\
L_{31} & L_{32} & L_{33}
\end{bmatrix}
\begin{bmatrix}
u \\ v \\ w
\end{bmatrix}
+ \frac{t^2}{12}
\begin{bmatrix}
N_{11} & N_{12} & N_{13} \\
N_{21} & N_{22} & N_{23} \\
N_{31} & N_{32} & N_{33}
\end{bmatrix}
\begin{bmatrix}
u \\ v \\ w
\end{bmatrix}
= \frac{-t(1-\nu^2)}{Et}
\begin{bmatrix}
q_1 \\
q_2 \\
q_n
\end{bmatrix}
\tag{1.9}
\]

\([L]\) and \([N]\) are the matrices of differential operators for membrane and bending effects respectively. Expanded they are

\[
L_{11} = \frac{1}{A} \frac{\partial}{\partial \alpha} - \frac{1}{AB} \frac{\partial}{\partial \beta} + \frac{1}{B} \frac{\partial}{\partial \beta} + \frac{(1-\nu)}{R_1 R_2}
\]

\[
L_{12} = \frac{1}{A} \frac{\partial}{\partial \alpha} \frac{1}{AB} \frac{\partial}{\partial \beta} + \frac{1}{B} \frac{\partial}{\partial \beta} A, \tag{1.10}
\]

\[
L_{13} = \frac{1}{A} \frac{\partial}{\partial \alpha} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) - \frac{1}{AB} \frac{\partial}{\partial \alpha} + \frac{(1-\nu)}{A} \frac{\partial}{\partial \alpha} A,
\]

\[
L_{21} = \frac{1}{B} \frac{\partial}{\partial \beta} - \frac{1}{AB} \frac{\partial}{\partial \alpha} B - \frac{1}{B} \frac{\partial}{\partial \beta} \frac{1}{A} \frac{\partial}{\partial \alpha} + \frac{(1-\nu)}{2} \frac{\partial}{\partial \alpha} A,
\]

\[
L_{22} = \frac{1}{B} \frac{\partial}{\partial \beta} + \frac{1}{AB} \frac{\partial}{\partial \alpha} + \frac{(1-\nu)}{2} \frac{\partial}{\partial \alpha} A, \tag{1.11}
\]

\[
L_{23} = \frac{1}{B} \frac{\partial}{\partial \beta} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) - \frac{1}{B} \frac{\partial}{\partial \beta} + \frac{(1-\nu)}{R_1} \frac{\partial}{\partial \beta} A,
\]

\[
L_{31} = - \frac{1}{R_1} \frac{\partial}{\partial \alpha} - \frac{1}{R_2} \frac{\partial}{\partial \alpha} B + \frac{1}{R_2} \frac{\partial}{\partial \alpha} B + \frac{1}{R_2} \frac{\partial}{\partial \alpha} B,
\]

\[
L_{33} = - \frac{1}{R_1} \frac{\partial}{\partial \alpha} - \frac{1}{R_2} \frac{\partial}{\partial \alpha} B + \frac{1}{R_2} \frac{\partial}{\partial \alpha} B + \frac{1}{R_2} \frac{\partial}{\partial \alpha} B,
\]
\[
L^{32} = - \left[ \frac{1}{R_1} + \frac{1}{R_2} \right] \frac{1}{AB} \frac{\partial}{\partial \beta} A + \frac{(1 - \nu)}{\partial \beta} \frac{1}{R_1} A,
\]
\[
L^{33} = - \frac{1}{R_1^2} - \frac{2 \nu}{R_1 R_2} - \frac{1}{R_2^2},
\]
\[
N^{11} = \frac{1}{R_1} \frac{1}{A} \frac{\partial}{\partial \alpha} \frac{1}{AB} \frac{\partial}{\partial \alpha} B + \frac{(1 - \nu)}{2} \frac{1}{R_1^2} \frac{1}{B} \frac{\partial}{\partial \beta} \frac{1}{AB} \frac{\partial}{\partial \beta},
\]
\[
N^{12} = \frac{1}{R_1} \frac{1}{A} \frac{\partial}{\partial \alpha} \frac{1}{AB} \frac{\partial}{\partial \beta} \frac{1}{R_2} A - \frac{(1 - \nu)}{2} \frac{1}{R_1^2} \frac{1}{B} \frac{\partial}{\partial \beta} \frac{1}{AB} \frac{\partial}{\partial \alpha}
\]
\[
N^{13} = - \frac{1}{R_1} \frac{1}{A} \frac{\partial}{\partial \alpha} \nabla^2 \left( \frac{1}{R_1 R_2} \frac{1}{AR_1} \frac{\partial}{\partial \alpha} \right),
\]
\[
N^{21} = \frac{1}{BR_2} \frac{\partial}{\partial \beta} \frac{1}{AB} \frac{\partial}{\partial \alpha} B - \frac{(1 - \nu)}{2} \frac{1}{R_2} \frac{1}{AR_2} \frac{\partial}{\partial \alpha} \frac{1}{AB} \frac{\partial}{\partial \beta}
\]
\[
N^{22} = \frac{1}{BR_2} \frac{\partial}{\partial \beta} \frac{1}{AB} \frac{\partial}{\partial \beta} A + \frac{(1 - \nu)}{2} \frac{1}{R_2} \frac{1}{AR_2} \frac{\partial}{\partial \alpha} \frac{1}{AB} \frac{\partial}{\partial \alpha}
\]
\[
+ \frac{(1 - \nu)}{R_1 R_2} \frac{1}{R_2^2}
\]
\[
(1.10) \quad \text{Cont'd}
\]
\[
N^{23} = - \frac{1}{BR_2} \frac{\partial}{\partial \beta} \nabla^2 \left( \frac{1}{R_1 R_2} \frac{1}{BR_2} \frac{\partial}{\partial \beta} \right),
\]
\[
N^{31} = \nabla^2 \frac{1}{AB} \frac{\partial}{\partial \alpha} B + \frac{(1 - \nu)}{2} \frac{1}{AB} \left[ \frac{\partial}{\partial \alpha} \frac{1}{R_1} \frac{\partial}{\partial \beta} + \frac{\partial}{\partial \beta} \frac{\partial}{\partial \alpha} \right]
\]
\[
\quad - \frac{\partial}{\partial \beta} \frac{1}{R_1} \frac{\partial}{\partial \alpha}, \frac{1}{AB} \frac{\partial}{\partial \beta} A + \frac{(1 - \nu)}{2} \frac{1}{AB} \left[ \frac{\partial}{\partial \beta} \frac{1}{R_2} \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \alpha} \frac{\partial}{\partial \beta} \right]
\]
\[
N^{32} = \nabla^2 \frac{1}{AB} \frac{\partial}{\partial \beta} A + \frac{(1 - \nu)}{2} \frac{1}{AB} \left[ \frac{\partial}{\partial \beta} \frac{1}{R_2} \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \alpha} \frac{\partial}{\partial \beta} \right]
\]
\[
\quad - \frac{\partial}{\partial \alpha} \frac{1}{R_1} \frac{\partial}{\partial \beta} B + \frac{(1 - \nu)}{2} \frac{1}{AB} \left[ \frac{\partial}{\partial \alpha} \frac{1}{R_1} \frac{\partial}{\partial \beta} A + \frac{(1 - \nu)}{R_1 R_2} \frac{1}{R_1 R_2} \right],
\]
\[
N^{33} = - \nabla^2 \nabla^2 \left( \frac{1}{AB} \left[ \frac{\partial}{\partial \alpha} \frac{1}{R_1 R_2} \frac{1}{A} \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta} \frac{A}{B} \frac{\partial}{\partial \beta} \right] \right),
\]
where
\[
\nabla^2 = \frac{1}{AB} \left[ \frac{\partial}{\partial \alpha} \frac{B}{A} \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta} \frac{A}{B} \frac{\partial}{\partial \beta} \right].
\]
As previously stated, \( [N] \) is the matrix of operators obtained from considering bending effects. Hence the membrane equations are obtained by neglecting \( [N] \). They are

\[
\begin{bmatrix}
L^{11} & L^{12} & L^{13} \\
L^{21} & L^{22} & L^{23} \\
L^{31} & L^{32} & L^{33}
\end{bmatrix}
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix}
= -\frac{(1 - \nu^2)}{Et}
\begin{bmatrix}
q_1 \\
q_2 \\
q_n
\end{bmatrix}. \tag{1.11}
\]

A quite different approach has been taken to the derivation of the thin shell elastic equations by Vlasov (7), who starts with the generalized three-dimensional elasticity equations and simplifies to the case of a shell. Green and Zerna (8) also adopted this approach.

We start with the six strain-displacement equations, written in three dimensions for an arbitrary curvilinear coordinate system, \( \alpha, \beta, \gamma \). In this case there are three Lame coefficients, \( A, B, C \), functions of \( \alpha, \beta, \gamma \). It is assumed that \( \gamma \) corresponds to the shell normal and that \( C = 1 \). This yields the following:

\[
e^{\alpha\alpha} = \frac{1}{A} \frac{\partial u}{\partial \alpha} + \frac{1}{AB} \frac{\partial A}{\partial \beta} v + \frac{1}{A} \frac{\partial A}{\partial \gamma} w,
\]

\[
e^{\beta\beta} = \frac{1}{B} \frac{\partial v}{\partial \beta} + \frac{1}{B} \frac{\partial B}{\partial \gamma} w + \frac{1}{AB} \frac{\partial B}{\partial \alpha} u, \tag{1.12}
\]

\[
e^{\gamma\gamma} = \frac{\partial w}{\partial \gamma}.
\]
Further simplifications are made by setting

\[ \epsilon_{\alpha\gamma} = \epsilon_{\beta\gamma} = \epsilon_{\gamma\gamma} = 0, \]

which is the Kirchhoff approximation. It is also assumed that \( A \) and \( B \) vary linearly with distance from the middle surface and that

\[ \epsilon_{\alpha\alpha} = \epsilon_1 + \kappa_1 \gamma, \]
\[ \epsilon_{\beta\beta} = \epsilon_2 + \kappa_2 \gamma, \]
\[ \epsilon_{\alpha\beta} = \omega + \tau \gamma. \]

Combining all of these assumptions with equations (1.12) finally produces middle surface strains,

\[ \epsilon_1 = \frac{1}{A} \frac{\partial u}{\partial \alpha} + \frac{1}{AB} \frac{\partial A}{\partial \beta} v + k_1 w, \]
\[ \epsilon_2 = \frac{1}{AB} \frac{\partial B}{\partial \alpha} u + \frac{1}{B} \frac{\partial v}{\partial \beta} + k_2 w, \]
\[ \omega = \frac{A}{B} \frac{\partial}{\partial \beta} \left( \frac{u}{A} \right) + \frac{B}{A} \frac{\partial A}{\partial \alpha} \left( \frac{v}{B} \right), \]

\[ \kappa_1 = \frac{\partial k_1}{\partial \alpha} \frac{u}{A} + \frac{\partial k_1}{\partial \beta} \frac{v}{B} - k_1^2 w - \frac{1}{A} \frac{\partial}{\partial \alpha} \left( \frac{1}{A} \frac{\partial w}{\partial \alpha} \right) - \frac{1}{AB^2} \frac{\partial A}{\partial \beta} \frac{\partial w}{\partial \beta}. \]
\[ K_2 = \frac{\partial k_2}{\partial \alpha} \frac{u}{A} + \frac{\partial k_2}{\partial \beta} \frac{v}{B} - k_2^2 w - \frac{1}{B} \frac{\partial}{\partial \beta} \left( \frac{1}{B} \frac{\partial w}{\partial \beta} \right) \]

\[ - \frac{1}{A^2 B} \frac{\partial B}{\partial \alpha} \frac{\partial w}{\partial \alpha} \] (1.15 Cont'd)

\[ T' = (k_1 - k_2) \left[ \frac{A}{B} \frac{\partial}{\partial \beta} \left( \frac{u}{A} \right) - \frac{B}{A} \frac{\partial}{\partial \alpha} \left( \frac{v}{B} \right) \right] \]

\[ - \frac{2}{AB} \left[ \frac{\partial^2 w}{\partial \alpha \partial \beta} - \frac{1}{A} \frac{\partial A}{\partial \beta} \frac{\partial w}{\partial \alpha} - \frac{1}{B} \frac{\partial B}{\partial \alpha} \frac{\partial w}{\partial \alpha} \right] \]

where \( k_1 = \frac{1}{R_1}, \quad k_2 = \frac{1}{R_2} \)

It will be seen that these equations differ substantially from (1.2) in the expressions for curvature. If the process is followed that led to equations (1.9) the resultant matrices \( [L] \) and \( [N] \) now prove to be symmetric.

Closer inspection of some of the strain equations derived by Vlasov indicates more realistic expressions than those obtained from the Love - Timoshenko equations. Additionally the symmetric matrices \( [L] \) and \( [N] \) are to be expected as a result of the Maxwell - Betti Theorem. Consequently a choice between the two approaches should lead one to select the second.

The discrepancies noted prove to be more apparent than real. Even Vlasov (9) states that there is scant difference between the results obtained by the use of the two theories. Since most of the literature is based on the work of Love et al., it was decided to utilize these equations as the basis for this investigation.
In many cases a shell can be considered to fall in the category known as shallow shells. This means essentially that the slope of the undistorted shell is very small with respect to 1.

A theory for the analysis of shallow shells was first put forward by Marguerre (10). Subsequently Reissner (11, 12) investigated its application to shallow spherical shells and set forth a simplified form of the solutions pertaining to this kind of shell. Vlasov (9) has presented a comprehensive extension to all types of shallow shells.

The basic assumption in this theory is that products of tangential displacements, u and v, with the curvatures or their derivatives may be neglected. By this means the changes of curvature may be simplified from (1.2):

\[ K_1 = -\frac{1}{A} \frac{\partial}{\partial \alpha} \frac{1}{A} \frac{\partial w}{\partial \alpha} - \frac{1}{AB^2} \frac{\partial A}{\partial \beta} \frac{\partial w}{\partial \beta} \]

\[ K_2 = -\frac{1}{B} \frac{\partial}{\partial \beta} \frac{1}{B} \frac{\partial w}{\partial \beta} - \frac{1}{A^2 B} \frac{\partial B}{\partial \alpha} \frac{\partial w}{\partial \alpha} \]

\[ ? = -\frac{1}{AB} \left( \frac{\partial^2 w}{\partial \alpha \partial \beta} - \frac{1}{A} \frac{\partial A}{\partial \beta} \frac{\partial w}{\partial \alpha} \right) \]

\[ - \frac{1}{B} \frac{\partial B}{\partial \alpha} \frac{\partial w}{\partial \beta} \]  

(1.16)
The membrane stresses are represented in terms of a stress function $F(x, \beta)$:

$$
\bar{T}_1 = \frac{1}{B} \frac{\partial}{\partial \beta} \frac{\partial F}{\partial \beta} + \frac{1}{A^2 B} \frac{\partial}{\partial \alpha} \frac{\partial F}{\partial \alpha},
$$

$$
\bar{T}_2 = \frac{1}{A} \frac{\partial}{\partial \alpha} \frac{\partial F}{\partial \alpha} + \frac{1}{AB^2} \frac{\partial}{\partial \beta} \frac{\partial F}{\partial \beta},
$$

$$
\bar{S} = -\frac{1}{AB} \left( \frac{\partial^2 F}{\partial \alpha \partial \beta} - \frac{1}{B} \frac{\partial B}{\partial \alpha} \frac{\partial F}{\partial \beta} - \frac{1}{A} \frac{\partial A}{\partial \beta} \frac{\partial F}{\partial \alpha} \right). \tag{1.17}
$$

If the same simplifying assumption is made that the product of curvature and force is negligible and if no tangential loading is assumed the first two of equations (1.1) become

$$
\frac{\partial}{\partial \alpha} (B \bar{T}_1) - \bar{T}_2 \frac{\partial B}{\partial \alpha} + \frac{\partial (AS)}{\partial \beta} + \bar{S} \frac{\partial A}{\partial \beta} = 0,
$$

$$
\frac{\partial (AT_2)}{\partial \beta} - \bar{T}_1 \frac{\partial A}{\partial \beta} + \frac{\partial (BS)}{\partial \alpha} + \bar{S} \frac{\partial B}{\partial \alpha} = 0. \tag{1.18}
$$

Equations 1.17), when substituted into equations (1.18), identically satisfy these equations of equilibrium.

By means of various substitutions two equations eventually emerge:

$$
\frac{1}{Eh} \nabla^4 F - \nabla_k^2 w = 0, \tag{1.19}
$$

$$
\nabla_k^2 F + \frac{Eh^3}{12(1-v^2)} \nabla^4 w - q_n = 0,
$$

where \( \nabla_k^2 = \frac{1}{AB} \left[ \frac{\partial}{\partial \alpha} \left( \frac{B}{A} k_2 \frac{\partial}{\partial \alpha} \right) + \frac{\partial}{\partial \beta} \left( \frac{A}{B} k_1 \frac{\partial}{\partial \beta} \right) \right] \).

These are the classical shallow shell equations.
Chapter 2

Conventional Methods of Solution

It is obvious that a consideration of the generalized equations is exceedingly difficult. As has been stated an eighth order system of partial differential equations must be solved. The most effective device to reduce the complexity has been to utilize what is known as membrane theory. If all moments and transverse shearing forces are neglected in comparison to in-surface forces, one has from (1.3)

\[ T_{1*} = T_{2*} = S*, \]
\[ T_1 = T_1*, \]
\[ T_2 = T_2*. \] (2.1)

Now the first three equations of (1.1) become

\[ \frac{\partial (BT_1*)}{\partial \alpha} + \frac{\partial (AS*)}{\partial \beta} + S* \frac{\partial A}{\partial \beta} - T_2* \frac{\partial B}{\partial \alpha} + AB q_1 = 0, \]
\[ \frac{\partial (BS*)}{\partial \alpha} + \frac{\partial (AT_2*)}{\partial \beta} + S* \frac{\partial B}{\partial \alpha} - T_1* \frac{\partial A}{\partial \beta} + AB q_2 = 0, \]
\[ \frac{T_1*}{R_1} + \frac{T_2*}{R_2} = q_n. \] (2.2)
This system of three equations in three unknowns is equivalent to a fourth order equation. It may be solved directly for the three membrane force components $T_1^*$, $T_2^*$, and $S^*$. These may be substituted in turn into equations (1.7) to obtain membrane strains $\varepsilon_1$, $\varepsilon_2$, and $\omega$. Finally the first three of equations (1.2) can be solved for displacements $u$, $v$, and $w$. Two boundary conditions are specified at each edge from among $T_1^*$, $T_2^*$, $S^*$ and $u$, $v$, $w$.

A detailed examination of equations (2.2) will show that discontinuities in the stress resultants occur whenever discontinuities appear in the applied loadings or geometrical characteristics. Additionally certain boundary conditions may be inconsistent with the membrane stress condition. Satisfaction of compatibility requirements necessitates a consideration of bending effects.

The loci of geometric and loading discontinuities, in addition to the edges, will be called lines of distortion. These are in effect lines at which the basic membrane state of stress is distorted by bending effects. It has been demonstrated that the effects of applying shears or moments on non-asymptotic lines of distortion are relatively localized. Outside of the zones affected the membrane theory in numerous instances is found to be essentially
correct. Hence spacing lines of distortion far enough apart that the zones of disturbed state of stress do not overlap enables one to investigate each such region separately.

The usual approach to correcting the membrane solution is to superimpose another solution assuming no external loading on the shell. This solution uses equilibrium equations which are homogeneous. Manipulation of the constants of integration enables one to satisfy all boundary conditions and to eliminate internal discontinuities. Again it can be shown that ignoring the external loads in the moment equations is permissible, primarily due to the narrowness of the affected zone.

A powerful tool for considering all effects simultaneously is the method of asymptotic integration. It is presented in detail by Goldenweiser (6) and in a modified form by Novozhilov (3).

Consider the homogeneous displacement-equilibrium equations from (1.9)

\[
\begin{bmatrix} L \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + h^2 \begin{bmatrix} N \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = 0, \quad (2.3)
\]

where \( h^2 = t^2 / 12\eta^2 \) and \( \eta \) is some characteristic linear dimension.
Assume solutions in the form
\[
\begin{align*}
  u &= k^\lambda U (\alpha, \beta; k) \, e^{kf(\alpha, \beta)}, \\
  v &= k^\mu V (\alpha, \beta; k) \, e^{kf(\alpha, \beta)}, \\
  w &= k^\nu W (\alpha, \beta; k) \, e^{kf(\alpha, \beta)},
\end{align*}
\]
where \( k = h^{-s} \) (\( s > 0 \)) and \( 2/s \) may be an integer. The functions \( U, V, W \) may be represented by the asymptotic expansions
\[
\begin{align*}
  U &= u_0 (\alpha, \beta) + u_1 \left( \frac{\alpha \beta}{k} \right) + \frac{u_2(\alpha, \beta)}{k^2} + \ldots, \\
  V &= v_0 (\alpha, \beta) + v_1 \left( \frac{\alpha \beta}{k} \right) + \frac{v_2(\alpha, \beta)}{k^2} + \ldots, \\
  W &= w_0 (\alpha, \beta) + w_1 \left( \frac{\alpha \beta}{k} \right) + \frac{w_2(\alpha, \beta)}{k^2} + \ldots,
\end{align*}
\]
where we consider that
\[
u_0 \neq 0; \quad v_0 \neq 0; \quad w_0 \neq 0.
\]
As the shell becomes increasingly thin \( h \) approaches 0 and \( k \) becomes very large. In this case \( U, V, W \) approach \( u_0, v_0, w_0 \) asymptotically. On the other hand the series expressions of (2.5) always approach \( U, V, W \) asymptotically with increasing number of terms.

The operators \( L^{ij} \) and \( N^{ij} \) may be expanded into asymptotic series with respect to decreasing powers of \( k \) as follows:
\[
\begin{align*}
  L^{i1} (u) &= k^{\lambda} \left\{ k^{n_{i1}} L_0^{i1} u_0 + k^{n_{i1}-1} \left[ L_1^{i1} (u_0) + L_0^{i1} u_1 \right] + \ldots \right\} e^{kf}, \\
  L^{i2} (v) &= k^{\mu} \left\{ k^{n_{i2}} L_0^{i2} v_0 + k^{n_{i2}-1} \left[ L_1^{i2} (v_0) + L_0^{i2} v_1 \right] + \ldots \right\} e^{kf},
\end{align*}
\]
\[ L_{ij}^{13}(w) = k \left\{ k_{mi1}^{n_j1} L_0^{i13} w_0^1 k^{(n_j3-1)} \left[ L_1^{i13}(w_0) + L_0^{i13} w_1 \right]^+ \right\} e^{\lambda t} \]

\[ N_{ij}^{11}(u) = k \left\{ k_{mi1}^{n_j1} N_0^{i11} u_0^1 k^{(m_j1-1)} \left[ N_1^{i11}(u_0) + N_0^{i11} u_1 \right]^+ \right\} e^{\lambda t}, \] (2.6 Cont'd)

\[ N_{ij}^{12}(v) = k \left\{ k_{mi2}^{n_j2} N_0^{i12} v_0^1 k^{(m_j2-1)} \left[ N_1^{i12}(v_0) + N_0^{i12} v_1 \right]^+ \right\} e^{\lambda t}, \]

\[ N_{ij}^{13}(w) = k \left\{ k_{mi3}^{n_j3} N_0^{i13} w_0^1 k^{(m_j3-1)} \left[ N_1^{i13}(w_0) + N_0^{i13} w_1 \right]^+ \right\} e^{\lambda t}, \]

where \( n_{ij}, m_{ij} \) indicate the order of \( L_{ij}, N_{ij} \), respectively. Some typical values for \( L_{ij}^{11}, N_{ij}^{11} \) are

\[ L_{0}^{11} = \frac{1}{A^2} \left( \frac{\partial f}{\partial \alpha} \right)^2 + \frac{(1 - l_i^1)}{2B^2} \left( \frac{\partial f}{\partial \beta} \right)^2, \]

\[ L_{0}^{12} = L_{0}^{21} = \frac{1 + \nu}{2} \frac{1}{AB} \frac{\partial f}{\partial \alpha} \frac{\partial f}{\partial \beta}, \] \hspace{1cm} (2.7)

\[ N_{0}^{11} = \frac{1}{R_1^2} \frac{1}{A^2} \left( \frac{\partial f}{\partial \alpha} \right)^2 + \frac{1 - \nu}{2} \frac{1}{R_1^2} \frac{1}{B^2} \left( \frac{\partial f}{\partial \beta} \right)^2, \]

\[ N_{0}^{13} = \frac{1}{AR_1} \frac{\partial f}{\partial \alpha} \left[ \frac{1}{A^2} \left( \frac{\partial f}{\partial \alpha} \right)^2 + \frac{1}{B^2} \left( \frac{\partial f}{\partial \beta} \right)^2 \right]. \]

It is seen that \( L_{ij}^{11}, N_{ij}^{11} \) are not differential operators.
The first, or principal, system of equations is obtained by equating to zero the coefficients of the highest powers of $k$. In this case we replace $h^2$ by $k^{-2/5}$ and insert the values of $n_{ij}$ and $m_{ij}$. This produces

$$k \lambda + 2 \left( L_1^{11} + k^{-2} \frac{2}{5} N_0^{11} \right) u_0 + k \mu + 2 \left( L_2^{12} + k^{-2} \frac{2}{5} N_0^{12} \right) v_0$$

$$+ k \lambda + 1 \left( L_3^{13} + k^{2} \frac{2}{5} N_0^{13} \right) w_0 = 0,$$

$$k \lambda + 2 \left( L_2^{21} + k^{-2} \frac{2}{5} N_0^{21} \right) u_0 + k \mu + 2 \left( L_3^{22} + k^{-2} \frac{2}{5} N_0^{22} \right) v_0$$

$$+ k \lambda + 1 \left( L_3^{23} + k^{2} \frac{2}{5} N_0^{23} \right) w_0 = 0,$$

$$k \lambda + 2 \left( L_3^{31} + k^{2} \frac{2}{5} N_0^{31} \right) u_0 + k \mu + 2 \left( L_3^{32} + k^{-2} \frac{2}{5} N_0^{32} \right) v_0$$

$$+ k \lambda + 1 \left( L_3^{33} + k^{4} \frac{2}{5} N_0^{33} \right) w_0 = 0.$$

Superfluous terms exist in these equations since only terms multiplied by the highest power of $k$ are required. If $s$ is specified it develops that five cases may exist at $s = 1/2$, $s = 1$, or $s$ in the three regions otherwise possible. For example, at $s = 1/2$,

$$k \lambda + 2 L_1^{11} u_0 + k \mu + 2 L_2^{12} v_0 + k \lambda + 1 L_3^{13} w_0 = 0,$$

$$k \lambda + 2 L_2^{21} u_0 + k \mu + 2 L_3^{22} v_0 + k \lambda + 1 L_3^{23} w_0 = 0,$$

$$k \lambda + 2 L_3^{31} u_0 + k \mu + 2 L_3^{32} v_0 + k \lambda + 1 L_3^{33} w_0 = 0.$$
In order to keep either \( u_0 \), \( v_0 \), or \( w_0 \) from vanishing we set
\[
\lambda = \lambda = \lambda' = 1.
\]
This reduces (2.9) to
\[
L_0^{\text{11}} u_0 + L_0^{\text{12}} v_0 + L_0^{\text{13}} w_0 = 0,
\]
\[
L_0^{\text{21}} u_0 + L_0^{\text{22}} v_0 + L_0^{\text{23}} w_0 = 0 \tag{2.10}
\]
\[
L_0^{\text{31}} u_0 + L_0^{\text{32}} v_0 + (L_0^{\text{33}} + N_0^{\text{33}}) w_0 = 0.
\]
For a non-trivial solution we set the determinant of the displacement coefficients to zero. This provides an equation for \( f \), as seen in equations (2.7).

A further step, to the second system of equations, is required to solve for \( u_0 \), \( v_0 \), \( w_0 \). In this case terms of the type \( L_1^{\text{11}} (u_0) + L_0^{\text{11}} u_1 \) appear in a more complex form of equations (2.8). This produces a first approximation for \( U \), \( V \), \( W \) in equations (2.4). Further steps produce an asymptotically decreasing error in the displacement functions.

The actual case to which equations (2.8) reduces is dependent upon \( s \), the index of variation. This value is determined by a process of expanding the applied load, or boundary value, into a series analogous to the expressions (2.4).

With the increasing availability and economy of large high-speed electronic digital computers, numerical methods of solution of thin shell problems have become much more effective, especially
for shells of revolution. Grafton and Strome (13) set forth a method of analysis of axisymmetric shells based on the direct stiffness method first popularized by Turner et al (14). A shell of revolution is considered to comprise a series of truncated cones which approximate the actual configuration. For each cone a stiffness matrix is developed to relate translations and rotations at the edges to applied edge forces. These matrices are added to obtain an overall stiffness matrix, which is solved in the conventional manner.

A finite difference method for solution of a shell with axisymmetric loading is presented by Sepetoski et al (15). This method was extended to the case of shells of revolution under arbitrary loading by Budiansky and Radkowski (16).
Chapter 3

Deficiencies of Conventional Methods

The methods of solution discussed in Chapter 2 have several serious deficiencies. In many cases the membrane theory gives results which are completely unrealistic. In others no solution at all can be attained. The primary reason for these phenomena is that no compatibility requirements are imposed in membrane theory.

An outstanding example of incompatibility in a membrane solution can be demonstrated in the case of a shell of non-positive Gaussian curvature loaded on its edges. Any load along an asymptotic line, or generator, propagates undiminished along that line. Hence a concentrated load would provide one element in compression with adjacent elements unloaded, and thus unstrained. This is obviously incompatible. Other fallacies frequently occur in such shells as shown by Novozhilov (3) and Vlasov (9).

Examples also exist in which membrane theory is invalid for shells of positive Gaussian curvature. Some such cases are shown by Bouma (17) and Goldenweiser (6). An example will be
shown later in which a spherical shell membrane solution is in violation of the St. Venant Theory which states (among other things) that a statically null set of forces should produce diminishing stresses with increased distance from the load.

It naturally follows that a completely invalid membrane solution is of no further value. One does not add corrections, such as edge effects, to a false preliminary answer. In such a case the complete set of equations including bending effects must be solved directly.

Now it is obvious that the generalized thin shell equations presented in Chapter 1 are not easily amenable of solution. In fact no general solution is known. Only in very limited cases have methods for solution been derived. These invariably involve the evaluation of unusual series, frequently with complex algebra.

If we restrict our attention to axisymmetric shells (shells of revolution) we find a solution derived by both Flügge (18) and Timoshenko (4) in terms of the hypergeometric series. It is significant that this is valid only for the homogeneous equations. Even then the series have been evaluated and solutions produced for axisymmetric loading only. The conclusion is that this approach
is of prohibitive complexity in the solution of meaningful problems. And as previously stated it applies only to the homogeneous bending equations and consequently requires a concurrent membrane solution.

The method of asymptotic integration is basically a rational way of simplifying equation (1.9) by discarding terms not necessary in a particular problem. The essential difficulties of solution just detailed still remain. It would appear that greater accuracy could be obtained by continuing the asymptotic process beyond the first step. However it develops that the complexity of solutions increases drastically, even in comparison to those already discussed. Additionally it is found that corrective terms are of the order of magnitude of those originally ignored when formulating the basic theory. Therefore their inclusion would not be consistent.

The stiffness method and finite difference methods both yield approximate solutions to axisymmetric shell problems. In general the error decreases asymptotically with increasing number of sections, or points. All numerical methods are limited, however, by the precision available on the particular computer involved. In any case round-off errors eventually place a ceiling on the number of points which can be considered. The stiffness method has been
derived for axisymmetric loads only. In view of the complexity of the assumed displacement functions for each section consideration of unsymmetric loads would be extremely difficult. The finite difference method has not yet been developed to solve the problem of singularities occurring at the closed end of a shell.

The foregoing discussion demonstrates that useful solutions to thin shell problems are few and fraught with pitfalls. A method to provide a valid solution for an axisymmetric shell of any configuration, under an arbitrary loading, would be extremely useful. Such a method was first suggested by Leissa (18, 19) and demonstrated for the special case of the spherical shell. The next Part generalizes upon this approach.
PART 2

DISPLACEMENT FUNCTION SOLUTIONS

Chapter 4

Displacement Equations

The generalized equations of equilibrium in terms of displacements are

\[
\begin{bmatrix}
L_{11} & L_{12} & L_{13} \\
L_{21} & L_{22} & L_{23} \\
L_{31} & L_{32} & L_{33}
\end{bmatrix}
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix}
+ \frac{t^2}{12}
\begin{bmatrix}
N_{11} & N_{12} & N_{13} \\
N_{21} & N_{22} & N_{23} \\
N_{31} & N_{32} & N_{33}
\end{bmatrix}
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix}
= \frac{1}{E} \frac{1}{R_1}{\begin{bmatrix} q_1 \\ q_2 \\ q_n \end{bmatrix}}. 
\]

Expressions for \([L]\) and \([N]\) are set forth in equations (1.10).

They are expanded and listed below for the case of an axisymmetric shell. The geometric restriction is

\[
\frac{\partial A}{\partial \beta} = \frac{\partial B}{\partial \beta} = \frac{\partial R_1}{\partial \beta} = \frac{\partial R_2}{\partial \beta} = 0. 
\]

The \([L]\) and \([N]\) operators become

\[
L_{11} = \frac{1}{A^2} \frac{\partial^2}{\partial \alpha^2} + \left[ \frac{B'}{A^2 B} - \frac{A'}{B^3} \right] \frac{\partial}{\partial \alpha} + \frac{1 - \nu}{2B^2} \frac{\partial^2}{\partial \beta^2} \\
L_{12} = (1 + \nu) \frac{1}{AB} \frac{\partial^2}{\partial \alpha \partial \beta} - \frac{3 - \nu}{2} \frac{B'}{AB^2} \frac{\partial}{\partial \beta} \\
L_{13} = \frac{1}{A} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \frac{\partial}{\partial \alpha} - \frac{1}{A} \left( \frac{R_1'}{R_1^2} + \frac{R_2'}{R_2^2} \right) 
\]

31
\[ L_{21} = \frac{1 + V}{2} \frac{1}{AB} \frac{\partial^2}{\partial \alpha \partial \beta} + \frac{3 - V}{2} \frac{B'}{AB^2} \frac{\partial}{\partial \beta} \]

\[ L_{22} = \frac{(1 - V)}{2} \frac{1}{A^2} \left[ \frac{\partial^2}{\partial \alpha^2} + \left( \frac{B'}{B} - \frac{A'}{A} \right) \frac{\partial}{\partial \alpha} + \frac{1}{B} \left( B'' - \frac{A'B'}{A} \right) - \frac{B'^2}{B} \right] + \frac{1}{B^2} \frac{\partial^2}{\partial \beta^2} + \frac{(1 - V)}{R_1 R_2} \]

\[ L_{23} = \frac{1}{B} \left[ \frac{V}{R_1} + \frac{1}{R_2} \right] \frac{\partial}{\partial \beta} \]

\[ L_{31} = -\frac{1}{A} \left[ \frac{1 + V}{R_1} + \frac{1}{R_2} \right] \frac{\partial}{\partial \alpha} + \frac{B'}{AB} \left[ \frac{1}{R_1} + \frac{V}{R_2} \right] \frac{1}{1 - \cdot \cdot \cdot \cdot R_2^2} \]

\[ L_{32} = \frac{1}{B} \left[ \frac{V}{R_1} + \frac{1}{R_2} \right] \frac{\partial}{\partial \beta} \]

\[ L_{33} = -\left[ \frac{1}{R_1^2} + \frac{2 V}{R_1 R_2} + \frac{1}{R_2^2} \right] \]

\[ N_{11} = -\frac{1}{A^2 R_1} \left[ \frac{\partial^2}{\partial \alpha^2} + \left( \frac{B'}{B} - \frac{A'}{A} - 2 R_1 \right) \frac{\partial}{\partial \alpha} + \frac{B''}{B} \frac{A''}{A} - \frac{B'}{B} \frac{A'}{A} - \frac{B'^2}{B^2} - \frac{R_1''}{R_1} \right. \]

\[ + \frac{2 R_1 \frac{B'}{B}}{R_1^2} + \frac{R_1}{R_1} \left( \frac{A'}{A} - \frac{B'}{B} \right) + \frac{1 - V}{R_1} \frac{1}{R_1^2} \left[ \frac{1}{2 B^2} \frac{\partial^2}{\partial \beta^2} + \frac{1}{R_1 R_2} \right] \]

\[ N_{12} = \frac{1}{A^2 B R_1} \left[ \frac{1}{R_2} - \frac{V}{2 R_1} \right] \frac{\partial^2}{\partial \alpha \partial \beta} - \frac{1}{A B R_1} \left[ \frac{1 - V}{2 R_1} \frac{B'}{B} + \frac{1}{R_1} \frac{B'}{B} + \frac{R_2'}{R_2} \right] \frac{\partial}{\partial \beta} \]

\[ N_{13} = -\frac{1}{A^3 R_1} \left[ \frac{\partial^3}{\partial \alpha^3} + \left( \frac{B'}{B} - \frac{3 A'}{A} \right) \frac{\partial^2}{\partial \alpha^2} + \left( \frac{B''}{B} - \frac{B'}{B} + \frac{2 A'}{A} \right) - \frac{A''}{A} \right. \]

\[ + \frac{3 A'^2}{A} \frac{B'}{R_1 R_2} \frac{1 - V}{A} \frac{R_1 R_2}{B^2} \frac{\partial}{\partial \alpha} + \frac{A''}{B^2} - \frac{3 A''}{B^2} - \frac{2 A'' B'}{B^2} \frac{\partial^2}{\partial \beta^2} \]

\[ N_{21} = \frac{1}{A B R_1} \left[ \left[ \frac{1}{R_2} - \frac{1 - V}{2 R_1} \right] \frac{\partial^2}{\partial \alpha \partial \beta} + \frac{1}{R_2} \left( \frac{B'}{B} - \frac{R_1'}{R_1} \right) + \frac{1 - V}{2 R_1} \frac{B'}{B} \frac{\partial}{\partial \beta} \right] \]

\[ N_{22} = \frac{1}{R_2^2} \left[ \frac{1 - V}{2 A^2} \left[ \frac{\partial^2}{\partial \alpha^2} + \left( \frac{B'}{B} - \frac{A'}{A} \right) \frac{\partial}{\partial \alpha} + \frac{B''}{B} - \frac{B'}{B} \right. \right. \]

\[ + \frac{1}{B^2} \frac{\partial^2}{\partial \beta^2} + \frac{1}{R_1 R_2} \left( 1 - \frac{B'}{B} \right) \frac{\partial}{\partial \beta} \]

\[ N_{23} = -\frac{1}{A B R_2} \left[ \frac{\partial^3}{\partial \alpha^2 \partial \beta} + \left( \frac{B'}{B} - \frac{A'}{A} \right) \frac{\partial^2}{\partial \alpha^2} + \frac{A''}{B^2} \frac{\partial^3}{\partial \beta^3} \right. \]

\[ - \frac{1 - V}{B R_2} \frac{1 - V}{R_2^2} \frac{\partial}{\partial \beta} \]

\[ N_{31} = \frac{1}{A^2 R_1} \left[ \frac{\partial^3}{\partial \alpha^3} + \frac{1}{A^2} \left[ \frac{2 B'}{B} - \frac{3 A'}{A} \frac{R_1'}{R_1} \frac{\partial^2}{\partial \alpha^2} + \frac{1}{A^2} \left( \frac{B''}{B} - \frac{A''}{A} - \frac{B'}{B} \right) \right. \right. \]
The typical term on the left-hand side of the matrix equation is $L_{ij}^t + \frac{t^2}{12} N_{ij}$. Since by definition in thin shell theory the ratio $t/R$ is small we can say

$$1 + \frac{t^2}{12R_1^2} = 1 + \frac{t^2}{12R_2^2} = 1 + \frac{t^2}{12R_1R_2} = 1 \quad (4.4)$$

In light of equation (4.4) each term of the matrix $[L] + \frac{t^2}{12} [N]$ becomes

$\frac{L_{11}^t + \frac{t^2}{12} N_{11}}{A^2} = \frac{1}{A^2} \left[ \frac{\partial^2}{\partial \alpha^2} + \left( \frac{B'}{B} - \frac{A'}{A} - \frac{t^2}{12R_1^2} R_1^2 \right) \frac{\partial}{\partial \alpha} \right] + \frac{2R_1^2}{R_1^2} \left[ R_1^2 \frac{\partial^2}{\partial \beta^2} \right] + \frac{R_1}{R_1^2} \left[ \left( B' - A' R_1 \right)^2 + \frac{1 - \nu}{2} \frac{A^2}{B^2} \frac{\partial^2}{\partial \beta^2} \right]$

$\frac{L_{12}^t + \frac{t^2}{12} N_{12}}{B} = \frac{1}{AB} \left[ \frac{\partial^2}{\partial \alpha^2} + \left( \frac{B'}{B} - \frac{A'}{A} - \frac{t^2}{12R_1^2} R_1^2 \right) \frac{\partial}{\partial \alpha} \right] + \frac{3 - \nu}{2} \frac{B'}{B} + \frac{3 - \nu}{2} \frac{R_1^2}{R_2 R_1^2} \frac{\partial}{\partial \beta}$

$\frac{L_{21}^t + \frac{t^2}{12} N_{21}}{A} = \frac{1}{AB} \left[ \frac{\partial^2}{\partial \alpha^2} + \left( \frac{B'}{B} - \frac{A'}{A} - \frac{t^2}{12R_1^2} R_1^2 \right) \frac{\partial}{\partial \alpha} \right] + \frac{3 - \nu}{2} \frac{B'}{B} + \frac{3 - \nu}{2} \frac{R_1^2}{R_2 R_1^2} \frac{\partial}{\partial \beta}$

$\frac{L_{22}^t + \frac{t^2}{12} N_{22}}{12} = \frac{1}{B} \left[ \frac{\partial}{\partial \alpha} + \frac{1}{12A^2} \frac{1}{BR_2} \left[ \frac{\partial^3}{\partial \alpha^2 \partial \beta} \right] + \frac{B'}{B} - \frac{A'}{A} \right] + \frac{3 - \nu}{2} \frac{B'}{B} + \frac{3 - \nu}{2} \frac{R_1^2}{R_2 R_1^2} \frac{\partial^3}{\partial \beta^2}$

$\frac{L_{31}^t + \frac{t^2}{12} N_{31}}{A} = -\frac{1}{A} \left[ \left( \frac{1}{R_1} + \frac{\nu}{R_2} \right) \frac{\partial}{\partial \alpha} + \frac{B'}{B} + \left( 1 - \nu \right) \frac{R_2^2}{R_1^2} \right] + \frac{t^2}{12A^2} \frac{3 - \nu}{2} \frac{B'}{B} + \frac{3 - \nu}{2} \frac{R_1^2}{R_2 R_1^2} \frac{\partial^2}{\partial \beta^2} + \frac{B''}{B} - \frac{B'}{A} - \frac{3A' R''}{AB} - \frac{5A' R_1^2}{BR_1} - \frac{2R_1'}{R_1}$
The terms not listed are unaffected by the reduction given in equation (4.4).

Consider the case of several restricted types of shells of revolution. The first example is a sphere. In this instance we have

\[ R_1 = R_2 = R, \]
\[ A = R, \quad B = R \sin \theta, \]
\[ \alpha = \theta, \quad \beta = \phi, \]
\[ B' = R \cos \theta, \quad B'' = -R \sin \theta, \quad B''' = -R \cos \theta, \quad (4.6) \]
as shown in Figure 4.
Figure 4

Geometry of Spherical Shell

Substituting,

\[ \begin{align*}
L_{11} + \frac{t^2}{12} N_{11} &= \frac{1}{R^2} \left[ \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{1 - \nu}{2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} 
- \left( \nu + \cot^2 \theta \right) \right], \\
L_{12} + \frac{t^2}{12} N_{12} &= \frac{1}{R^2 \sin \theta} \left[ \frac{1 + \nu}{2} \frac{\partial^2}{\partial \theta \partial \phi} - \frac{3 - \nu \cot \theta}{2} \frac{\partial}{\partial \phi} \right], \\
L_{13} + \frac{t^2}{12} N_{13} &= \frac{K}{R^2} \left[ - \frac{\partial^3}{\partial \theta^3} - \cot \theta \frac{\partial^2}{\partial \theta^2} + \left[ \cot^2 \theta + \frac{1 + \nu}{K} \right] \frac{\partial}{\partial \phi} 
- \frac{1}{\sin^2 \theta} \frac{\partial}{\partial \theta} \left( \frac{\partial^2}{\partial \phi^2} + 2 \cot \theta \frac{\partial^2}{\partial \phi^2} \right) \right], \\
L_{21} + \frac{t^2}{12} N_{21} &= \frac{1}{R^2 \sin \phi} \left[ \frac{1 + \nu}{2} \frac{\partial^2}{\partial \phi \partial \phi} + \frac{3 - \nu}{2} \cot \theta \frac{\partial}{\partial \phi} \right], \\
L_{22} + \frac{t^2}{12} N_{22} &= \frac{1}{2R^2} \left[ \frac{\partial^2}{\partial \phi^2} + \cot \phi \frac{\partial}{\partial \phi} + 1 - \cot^2 \theta \right].
\end{align*} \]
\[ L_{23} + t^2 \frac{N_{23}}{12} = -\frac{K}{R^2 \sin \epsilon} \left[ \frac{\partial^3}{\partial \theta^3} + 2 \cot \theta \frac{\partial^2}{\partial \epsilon^2} - \left( \cot^2 \theta + \frac{1 + \nu}{K} \right) \frac{\partial}{\partial \epsilon} \right] \]
\[ + \frac{1}{\sin^2 \epsilon} \frac{\partial^2}{\partial \phi^2} \]

\[ L_{31} + t^2 \frac{N_{31}}{12} = \frac{K}{R^2 \sin \epsilon} \left\{ \frac{\partial^3}{\partial \theta^3} + \frac{2}{\sin^2 \epsilon} \frac{\partial^3}{\partial \epsilon^2 \partial \phi^2} + \frac{3 + \nu}{2} \frac{\cot \theta}{\sin^2 \epsilon} \frac{\partial^2}{\partial \theta \partial \phi^2} \right\} \]
\[ - \cot \theta \left( 1 + \nu \right) + \frac{\cot \theta}{\sin^2 \epsilon} \right\}, \]

\[ L_{32} + t^2 \frac{N_{32}}{12} = \frac{K}{R^2 \sin \epsilon} \left[ \frac{\partial^3}{\partial \theta^3} - \cot \theta \frac{\partial^2}{\partial \epsilon^2} \phi \right] \]

\[ L_{33} + t^2 \frac{N_{33}}{12} = -\frac{K}{R^2} \left[ \frac{\partial^4}{\partial \theta^4} + \frac{2}{\sin \theta} \frac{\cot \theta}{\partial \theta} \frac{\partial^4}{\partial \phi^2} \frac{\partial^2}{\partial \phi^2} \right] \]
\[ + \left( \nu + \cot^2 \theta \right) \cot \theta \frac{\partial}{\partial \theta} \frac{\partial^4}{\partial \phi^2} \frac{\partial^2}{\partial \phi^2} \]
\[ - 2 \frac{\cot \theta}{\sin \theta} \frac{\partial^3}{\partial \theta \partial \phi^2} + \frac{1}{\sin^2 \theta} \left[ 1 + \nu + 4 \cot \theta \right] \frac{\partial^2}{\partial \phi^2} \]
\[ + \frac{1}{\sin^2 \theta} \left[ \frac{\partial^2}{\partial \phi^2} + \frac{1}{\sin^4 \theta} \right] \frac{\partial^4}{\partial \phi^4} \right] - \frac{2(1 + \nu)}{R_2} \]
where \( K = \frac{t^2}{12R^2} \).

For a cylinder we have

\[ \alpha = x, \quad \beta = \phi, \]
\[ A = 1, \quad B = R, \]  \hspace{1cm} (4.9)
\[ R_1 = \infty, \quad R_2 = R, \]

as shown in Figure 5.

![Figure 5](image)

**Figure 5**

**Geometry of Cylindrical Shell**

Now the matrix of operators becomes

\[
L^{11} + \frac{t^2}{12} N^{11} = \frac{\partial^2}{\partial x^2} + \frac{1-\nu}{2R^2} \frac{\partial^2}{\partial \phi^2},
\]

\[
L^{12} + \frac{t^2}{12} N^{12} = \frac{1}{R} \left( \frac{1+\nu}{2} \frac{\partial^2}{\partial x \partial \phi} \right),
\]

\[
L^{13} + \frac{t^2}{12} N^{13} = \frac{\nu}{R} \frac{\partial}{\partial x},
\]

\[
L^{21} + \frac{t^2}{12} N^{21} = \frac{1}{R} \left( \frac{1+\nu}{2} \frac{\partial^2}{\partial x^2} \right). \hspace{1cm} (4.10)
\]
\[ L_{22} + \frac{t^2}{12} N_{22} = \frac{1 - \nu}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{R^2} \frac{\partial^2 \theta}{\partial \phi^2}, \quad (4.10 \text{ Cont'd}) \]

\[ L_{23} + \frac{t^2}{12} N_{23} = -K \left[ \frac{\partial^3}{\partial x^2 \partial \phi} + \frac{1}{R^2} \frac{\partial^3 \gamma}{\partial \phi^3} + \frac{1}{R^2} \frac{\partial}{\partial \phi} \right] \]

\[ L_{31} + \frac{t^2}{12} N_{31} = -\frac{\nu}{R} \frac{\partial}{\partial x} - \frac{K}{R} (3 - \nu) \frac{\partial^3 \gamma}{\partial x^2 \partial \phi} \]

\[ L_{32} + \frac{t^2}{12} N_{32} = K \left[ \frac{3 - \nu}{2} \frac{\partial^3}{\partial x^2 \partial \phi} + \frac{1}{R^2} \frac{\partial^3 \gamma}{\partial \phi^3} \right] - \frac{1}{R^2} \frac{\partial}{\partial \phi} \]

\[ L_{33} + \frac{t^2}{12} N_{33} = -\frac{1}{R^2} - K \left[ R^2 \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial \phi^2} + \frac{1}{R^2} \frac{\partial^2}{\partial \phi^2} + \frac{1}{R^2} \frac{\gamma^4}{\partial \phi^4} \right] \]

where \( K = \frac{t^2}{12R^2} \).

Finally consider a right circular cone. From Figure 6 we have

\[ \text{Figure 6} \]

Geometry of Conical Shell
\[ \alpha = x, \]
\[ A = \frac{1}{\cos \gamma}, \quad B = x \tan \gamma, \quad B' = \tan \gamma, \quad (4.11) \]
\[ R_1 = \infty, \quad R_2 = x \sin \gamma, \quad R_2' = \sin \gamma. \]

Using these expressions the operators become

\[ L_{11}^{11} + \frac{t^2}{12} N_{11}^{11} = \cos^2 \gamma \left[ \frac{\partial^2}{\partial x^2} + \frac{1}{x} \frac{\partial}{\partial x} - \frac{1}{x^2} \right] \]
\[ + \frac{1 - \nu}{2x^2 \tan^2 \gamma} \frac{\partial^2}{\partial \phi^2}, \]

\[ L_{12}^{12} + \frac{t^2}{12} N_{12}^{12} = \frac{\cos \gamma}{x \tan \gamma} \left[ \frac{1 + \nu}{2} \frac{\partial^2}{\partial x \partial \phi} - \frac{3 - \nu}{2} \frac{1}{x} \frac{\partial}{\partial \phi} \right], \]

\[ L_{13}^{13} + \frac{t^2}{12} N_{13}^{13} = \frac{1}{x \tan \gamma} \left[ \nu \frac{\partial}{\partial x} - 1 \right], \]

\[ L_{21}^{21} + \frac{t^2}{12} N_{21}^{21} = \frac{\cos \gamma}{x \tan \gamma} \left[ \frac{1 + \nu}{2} \frac{\partial^2}{\partial x \partial \phi} + \frac{3 - \nu}{2} \frac{1}{x} \frac{\partial}{\partial \phi} \right], \quad (4.12) \]

\[ L_{22}^{22} + \frac{t^2}{12} N_{22}^{22} = \frac{1 - \nu}{2} \cos^2 \gamma \left[ \frac{\partial^2}{\partial x^2} + \frac{1}{x} \frac{\partial}{\partial x} - \frac{1}{x^2} \right] \]
\[ + \frac{1}{x^2 \tan^2 \gamma} \frac{\partial^2}{\partial \phi^2}, \]

\[ L_{23}^{23} + \frac{t^2}{12} N_{23}^{23} = \frac{x^2 \sin \gamma \tan \gamma}{\sin \gamma} \frac{\partial}{\partial \phi} - K \cos \gamma \left[ \frac{\partial^3}{x^2 \partial \phi} \right] \]
\[ + \frac{1}{x} \frac{\partial^2}{\partial x \partial \phi} + \frac{1}{x^2 \sin^2 \gamma} \frac{\partial^3}{\partial \phi^3}. \]
\[
\begin{align*}
L^{31} + \frac{t^2}{12} N^{31} &= -\frac{1}{x \tan \gamma} \left[ \frac{\partial}{\partial x} + \frac{1}{x} \right] \\
& \quad + \frac{K}{x^3 \tan \gamma} \left[ \frac{(3 - \nu)}{2} \frac{\partial^3}{\partial x^3 \partial \phi^2} + \frac{1}{x} \frac{\partial^2}{\partial \gamma^2} \right], \\
L^{32} + \frac{t^2}{12} N^{32} &= \frac{1}{x^2 \sin \gamma \tan \gamma} \frac{\partial}{\partial \phi} - \frac{K \cos \gamma}{x^2} \left[ (2 - \nu) \frac{\partial^3}{\partial \gamma^2 \partial \phi} \\
& \quad - \frac{5 + \nu}{2} \frac{1}{x} \frac{\partial^2}{\partial x^2 \partial \phi} + \frac{7 + \nu}{2} \frac{1}{x^2} \frac{\partial}{\partial \phi} \\
& \quad + \frac{1}{x^2 \sin^2 \gamma} \frac{\partial^3}{\partial \phi^3} \right], \\
L^{33} + \frac{t^2}{12} N^{33} &= -\frac{1}{x^2 \sin^2 \gamma} - \frac{K \sin^2 \gamma \cos \gamma}{\partial x^4} + \frac{2}{x} \frac{\partial^3}{\partial x^3 \partial \phi^2} \\
& \quad - \frac{1}{x^2} \frac{\partial^2}{\partial x^2} + \frac{1}{x^3} \frac{\partial}{\partial x} + \frac{x^2 \sin^2 \gamma}{\partial x^2} \frac{\partial^2}{\partial \phi^2} \\
& \quad - \frac{2}{x^3 \sin^2 \gamma} \frac{\partial^3}{\partial x \partial \phi^2} + \frac{1}{x^4 \sin^2 \gamma} \left[ 4 \frac{\partial^4}{\partial \phi^4} \\
& \quad + \frac{1}{\sin^2 \gamma} \left( \frac{\partial^2}{\partial \phi^2} + \frac{1}{x^4 \sin^4 \gamma} \frac{\partial^4}{\partial \phi^4} \right) \right].
\end{align*}
\]

where \( K = \frac{t^2}{12 \tan^2 \gamma} \).

In any real problem the boundary conditions must be satisfied as well as the equilibrium equations. The discussion will be restricted further to the case of rotationally closed axisymmetric shells.
with boundaries limited to lines $\alpha =$ constant. In general, four boundary conditions must be specified on each edge.

The state of stress on an edge is determined by four quantities as previously stated in Chapter 1. They are membrane direct force, $T$; membrane shear, $Q = S + 2H$; transverse shear, $V = N_1 + \frac{1}{B} \frac{\partial M_2}{\partial \beta}$; and direct bending moment, $M_1$. Alternatively, geometric boundary conditions - $u$, $v$, $w$, - may be specified. $\Omega$ is the rotation, in a plane through the polar axis, of a line normal to the shell.

A condition may occur in which both stress and geometric boundary conditions are specified. Frequently, so-called elastic boundary conditions occur in which stress and geometric values are related by a spring constant. An example is

$$T_1 + k_1 u = 0.$$ 

The generalized boundary equations are

$$\rho_1 T_1 + k_1 u = B_1,$$
$$\rho_2 Q_1 + k_2 v = B_2,$$
$$\rho_3 V_1 + k_3 w = B_3,$$
$$\rho_4 M_1 + k_4 \Omega = B_4,$$ 

(4.15)
where $\rho_1, \ldots, \rho_4 = 0$ or 1 and $B_1, \ldots, B_4$ are externally applied
generalized line loads acting along the boundary. If the $k$ values
(or stiffnesses) are zero the equations reduce to stress type
boundary conditions. Geometric boundary conditions are indicated
by $\rho = 0$.

All of these values to be specified at a point can be
expressed in terms of displacements. The stresses can be re-
written from (1.2) and (1.4),

$$
T_1 = \frac{E t}{1 - \nu^2} \left[ \frac{1}{A} \frac{\partial u}{\partial \alpha} + \frac{w}{R_1} + \nu \left( \frac{1}{B} \frac{\partial v}{\partial \beta} + \frac{1}{A} \frac{B'}{B} u + \frac{w}{R_2} \right) \right],
$$

$$
Q_1 = \frac{E t}{2(1+\nu)} \left\{ \frac{1}{A} \frac{\partial u}{\partial \beta} + \frac{1}{A} \frac{\partial v}{\partial \alpha} - \frac{B'}{AB} v + \frac{t^2}{3R^2} \left[ \frac{1}{BR_1} \frac{\partial u}{\partial \beta} 
+ \frac{1}{R_2} \left( \frac{1}{A} \frac{\partial v}{\partial \alpha} - \frac{B'}{AB} v \right) \right] - \frac{1}{AB} \left( \frac{\partial^2 w}{\partial \alpha \partial \beta} - \frac{B'}{B} \frac{\partial w}{\partial \beta} \right) \right\},
$$

$$
V_1 = \frac{E t^3}{12(1 - \nu^2)} \left\{ \frac{1}{A^2 R_1} \frac{\partial^2 u}{\partial \alpha^2} + \frac{1}{A^2 R_1} \left[ \frac{B'}{B} - \frac{A'}{A} - \frac{2R_1'}{R_1} \right] \frac{\partial u}{\partial \alpha} 
+ \frac{1}{A^2 R_1} \left[ -\frac{R_1''}{R_1} + \frac{2R_1'^2}{R_1^2} + \frac{A'''R_1}{AR_1} + \frac{V B''}{B} - \frac{V A'B'}{AB} - \frac{B''}{B^2} 
- \frac{B'R'}{BR_1} \right] u + \frac{2(1 - \nu)}{B^2 R_1} \frac{\partial^2 u}{\partial \beta^2} + \frac{2 - \nu}{ABR_2} \frac{\partial^2 v}{\partial \alpha \partial \beta} 
- \frac{1}{ABR_2} \left[ (3 - 2\nu) \frac{B'}{B} + \frac{V R_2'}{R_2} \right] \frac{\partial v}{\partial \beta} - \frac{1}{A^3} \frac{\partial^3 w}{\partial \alpha^3} 
+ \frac{1}{A^3} \left[ \frac{3A'}{A} - \frac{B'}{B} \right] \frac{\partial^2 w}{\partial \alpha^2} + \frac{1}{A^3} \left[ \frac{A'''}{A} - \frac{3A'^2}{A^2} 
- \frac{VB''}{B} + (1 + \nu) \frac{A'B'}{AB} + \frac{B''^2}{B^2} \right] \frac{\partial w}{\partial \alpha} - \frac{2 - \nu}{AB^2} \frac{\partial^3 w}{\partial \alpha \partial \beta^2} \right\}.
$$
\[ M_1 = \frac{Et^3}{12(1-\nu^2)} \left[ \frac{1}{AR_1} \frac{\partial u}{\partial \alpha} - \frac{1}{AR_1} \left( \frac{R_1}{R_1} - \frac{\nu B'}{B} \right) u + \frac{\nu}{BR_2} \frac{\partial v}{\partial \beta} - \frac{1}{A^2} \frac{\partial^2 w}{\partial \alpha^2} + \frac{1}{A^2} \left( \frac{A'}{A} - \frac{\nu B'}{B} \right) \frac{\partial w}{\partial \alpha} \right. \\
- \frac{1}{B^2} \left. \left( \frac{\partial^2 w}{\partial \beta^2} + (3 - \nu) B' \frac{\partial^2 w}{\partial \beta^2} \right) \right] . \]

The rotation is
\[ \Omega = - \frac{1}{A} \frac{\partial w}{\partial \alpha} + \frac{u}{R_1} . \]
\[ M_1 = \frac{EtK}{1 - \nu^2} \left[ \frac{\partial u}{\partial \xi} \cot \phi + \cot \phi \frac{\partial u}{\partial \alpha} + \frac{1}{\sin \theta} \frac{\partial v}{\partial \theta} - \frac{\partial^2 w}{\partial \theta^2} \right. \]
\[ \quad - \nu \cot \theta \frac{\partial w}{\partial \theta} - \frac{1}{\sin^2 \theta} \frac{\partial^2 w}{\partial \phi^2} \]  
\[ \Omega = \frac{1}{R} \left[ u - \frac{\partial w}{\partial \xi} \right] \]  

For a cylinder we have,
\[ T_1 = \frac{Et}{1 - \nu^2} \left[ \frac{\partial u}{\partial \xi} + \frac{\nu}{R} \left( \frac{\partial v}{\partial \phi} + \frac{w}{\nu} \right) \right], \quad (4.19) \]
\[ Q_1 = \frac{EtK}{2(1+\nu)} \left[ \frac{\partial u}{\partial \phi} + \frac{\nu}{\xi} \frac{\partial v}{\partial \xi} + 4K \left( \frac{\partial v}{\partial \xi} - \frac{\partial^2 w}{\partial \xi \partial \phi} \right) \right], \]
\[ V_1 = \frac{EtK}{1 - \nu^2} \left[ (2 - \nu) \frac{\partial^2 v}{\partial \xi \partial \phi} - \frac{R^2}{\partial x} \frac{\partial^3 w}{\partial x^3} - (2 - \nu) \frac{\partial^3 w}{\partial x \partial \phi^2} \right], \]
\[ M_1 = \frac{EtK}{1 - \nu^2} \left[ \nu \frac{\partial v}{\partial \phi} - \frac{R^2}{\partial x} \left( \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial \phi^2} \right) \right], \]
\[ \Omega = -\frac{\partial w}{\partial \xi} \]  

And these quantities become, for a cone
\[ T_1 = \frac{Et}{1 - \nu^2} \left[ \cos \gamma \frac{\partial u}{\partial \xi} + \nu \left( \frac{1}{\tan \gamma} \frac{\partial v}{\partial \phi} + \frac{1}{\xi} \frac{\partial v}{\partial \xi} - w \right) \right], \]
\[ Q_1 = \frac{Et}{2(1+\nu)} \left\{ \cos \gamma \frac{\partial u}{\partial \phi} + \cos \gamma \frac{\partial v}{\partial \xi} - \frac{v}{\xi} \right\} + \frac{4K}{\xi^2} \left[ \frac{1}{\cos \gamma} \left( \frac{\partial v}{\partial \xi} - \frac{v}{\xi} \right) - \frac{\partial^2 w}{\partial \xi \partial \phi} + \frac{1}{\xi} \frac{\partial w}{\partial \phi} \right], \quad (4.20) \]
\[ V_1 = \frac{EtK}{1 - \nu^2} \left[ \frac{2 - \nu}{\xi^2} \frac{\partial^2 v}{\partial \xi \partial \phi} - \frac{3 - \nu}{\xi^3} \frac{\partial v}{\partial \phi} - \sin^2 \gamma \cos \gamma \frac{\partial^3 w}{\partial x \partial \phi} \right] \]
+ \frac{1}{x} - \frac{\partial^2 w}{\partial x^2} = \frac{1}{x^2} \frac{\partial w}{\partial x} - (2 - \nu) \frac{\cos \gamma}{x^2} \frac{\partial^3 w}{\partial x \partial \gamma^2} \\
+ (3 - \nu) \frac{\cos \gamma}{x^3} \frac{\partial^2 w}{\partial \gamma^2}, \tag{4.20}

M_1 = \frac{EtK}{1 - \nu^2} \left[ \frac{\nu}{x \cos \gamma} \frac{\partial v}{\partial \gamma} - \sin^2 \gamma \left( \frac{\partial^2 w}{\partial x^2} + \frac{\nu}{x} \frac{\partial w}{\partial x} \right) \right] \\
- \frac{1}{x^2} \frac{\partial^2 w}{\partial \gamma^2},

\Omega = - \cos \gamma \frac{\partial w}{\partial x}.

Frequently shells of revolution are closed at one end, or even both ends. In such a case boundary conditions cannot be imposed in the conventional sense. However the physical reality of the structure requires that all generalized forces and displacements be finite and continuous at a pole, as well as elsewhere. Certain relations, called regularity conditions, must be satisfied by the displacement expressions in order for this to be true. These regularity conditions will be considered together with, or in place of, the boundary conditions. They will be examined in more detail subsequently for the case of a spherical shell.
Chapter 5
Continuous Displacement Functions

Let it be assumed that the middle surface displacements can be expressed as

\[ u = \sum_{m} u_m(\alpha) \cos m \phi, \]
\[ v = \sum_{m} v_m(\alpha) \sin m \phi, \]  \hspace{1cm} (5.1)
\[ w = \sum_{m} w_m(\alpha) \cos m \phi. \]

In order to allow separation of variables in the equilibrium equations consider similarly

\[ q_1 = q_1(\alpha) \cos m \phi, \]
\[ q_2 = q_2(\alpha) \sin m \phi, \]  \hspace{1cm} (5.2)
\[ q_n = q_n(\alpha) \cos m \phi. \]

It is obvious that these equations assume that the loading has a plane of symmetry through the zero meridian \((\phi=0)\). However, any asymmetrical loading can be constructed of symmetric and anti-symmetric components about the zero meridian. Preferably several reference meridians would be used, with a symmetric loading pattern corresponding to each. In this manner any arbitrary loading can be considered to reduce to systems as given in equations (5.2). A complete solution is then obtained by simple superposition.
For each value of \( m \), or Fourier mode, a separate problem can be considered to exist. Again, the final solution consists of a linear combination of the solutions for each Fourier mode. Future consideration will be restricted to the case of a single loading component, or mode.

The displacement functions (e.g. \( u_m(\alpha) \)) are expressed as continuous linear series with unknown coefficients. The most obvious such series is a power series in the variable \( \alpha \), as used by Leissa (18):

\[
\begin{align*}
    u &= \sum_1^i \sum_{-m}^{m} a_{im} \alpha^i \cos m\phi, \\
    v &= \sum_1^i \sum_{-m}^{m} b_{jm} \alpha^j \sin m\phi, \\
    w &= \sum_{k} \sum_{-m}^{m} c_{km} \alpha^k \cos m\phi.
\end{align*}
\]

The power series expressions for displacements result in serious difficulties, primarily the numerical problem associated with digital computer solutions. Let the boundaries be determined by \( \alpha_1, \alpha_2 \), where \( \alpha_2 \) exceeds \( \alpha_1 \). In the case of a closed shell the value of \( \alpha_1 \) is zero. If \( \alpha_2 \) approaches \( \pi \) then \( \alpha_2^i \) becomes very large when a large number of coefficients is used. For example if \( \alpha_2 > 3 \) (radians) and \( i > 20 \) then \( \alpha_2^i > 10^{10} \).

At the same time \( \alpha^i \) can be very small (\(< 10^{-20}\)) as \( \alpha \) approaches \( \alpha_1 \) and \( i \) is large. A complete solution requires a simultaneous consideration of all such values.
Leissa (19) was able to use a solution in the form of (5.3) on a spherical shell having boundaries $\alpha_1 = \theta_1 = 15^\circ$, $\alpha_2 = \theta_2 = 90^\circ$. Larger $\alpha_2/\alpha_1$ ratios gave excessive computational errors. The writer has demonstrated repeatedly that these large variations in order of magnitude of terms cause computer solutions, based on single precision arithmetic, to break down.

A sample problem consists of a clamped hemisphere with $R/t = 1000$ under a constant unit normal pressure. Repeated attempts were made to solve this problem with varying combinations of number of displacement coefficients and pressure points. In no case however were acceptable answers obtained. The forces at the edge $\alpha_2 = 90^\circ$ were computed consistently to be wrong by orders of magnitude. Average values are tabulated in Table 1 for a 32 parameter solution (12 a's, 20 c's).

<table>
<thead>
<tr>
<th></th>
<th>$T_1$</th>
<th>$T_2$</th>
<th>$M_1$</th>
<th>$V_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Computed</td>
<td>15800</td>
<td>14700</td>
<td>-13</td>
<td>-400</td>
</tr>
<tr>
<td>Actual</td>
<td>594</td>
<td>178</td>
<td>-.16</td>
<td>-12.4</td>
</tr>
</tbody>
</table>

Table 1

Edge Values for Clamped Hemisphere under Unit Normal Load
Normalizing the system \( (\alpha_1 = 0, \alpha_2 = 1) \) tends to alleviate the problem but difficulties remain. Severe sensitivity to minute variations in coefficients still exists, especially when computing values of moment and transverse shear. The problem appears to be insurmountable so this method was discarded.

It was determined that a trigonometric series would remedy the aforementioned problem.

Consider

\[
\begin{align*}
    u &= \sum_i \sum_m (A_{im} \sin i \alpha + B_{im} \cos i \alpha) \cos m \phi, \\
    v &= \sum_j \sum_m (C_{jm} \sin j \alpha + D_{jm} \cos j \alpha) \sin m \phi, \quad (5.4) \\
    w &= \sum_k \sum_m (E_{km} \sin k \alpha + F_{km} \cos k \alpha) \cos m \phi.
\end{align*}
\]

Substituting these expressions into equations (4.1) with the assistance of (4.3) and (4.5),

\[
\frac{1}{A^2} \left\{ \left[ \frac{B''}{B} - \frac{A'B'}{AB} - \frac{B'^2}{B^2} + \frac{(1 - \nu)}{R_1 R_2} - \frac{t^2}{12 R_1^2} \right] \frac{R_1''}{R_1} \\
    - \frac{2R_1^2}{R_1'^2} + \frac{R_1'}{R_1} \left\{ \frac{B'}{B} - \frac{A'}{A} \right\} \right. \\
    - \frac{(1 - \nu)}{2} \frac{A^2}{B^2} m^2 - i^2 \left\{ (A_{im} \sin i \alpha + B_{im} \cos i \alpha) \\
    + i \left\{ \frac{B'}{B} - \frac{A'}{A} - \frac{t^2}{12 R_1^2} \frac{2R_1'}{R_1} \right\} (A_{im} \cos i \alpha - B_{im} \sin i \alpha) \right\} \\
\left. + \sum_j \frac{1}{AB} \left\{ -m \left[ \frac{(3 - \nu)}{2} \frac{B'}{B} + \frac{t^2}{12 R_1 R_2} \frac{R_2'}{R_2} \right] (C_{jm} \sin j \alpha + D_{jm} \cos j \alpha) \right. \\
    + \frac{(1 + \nu)}{2} j m (C_{jm} \cos j \alpha - D_{jm} \sin j \alpha) \right\}
\]
\[ + \sum_{k} \frac{1}{k} \left\{ \left[ - \left( \frac{R_1}{R_{12}} + \frac{R_2}{R_{22}} \right) + \frac{t^2}{12A^2R_1} \right] \left( k^2 \left\{ \frac{B'}{B} - \frac{3A'}{A} \right\} \right) \right. \\
- \frac{m^2}{B^2} \left( \frac{2A^2B'}{B^3} \right) \right\} (E_{km} \sin k \alpha + F_{km} \cos k \alpha) \]

\[ + k \left[ \frac{1}{R_1} + \frac{U}{R_2} + \frac{t^2}{12A^2R_1} \right] \left( k^2 \left\{ \frac{B''}{B} - \frac{B'}{B} \left( \frac{B'}{B} + \frac{2A'}{A} \right) \right\} \right. \\
- \frac{A''}{A} - \frac{3A'^2}{A^2} - (1 - U) \frac{A^2}{R_1R_2} \left( \frac{m^2}{B^2} \left\{ \frac{A^2}{B^2} \right\} \right) \right] \\
(E_{km} \cos k \alpha - F_{km} \sin k \alpha) = - \frac{(1 - U^2)}{Et} q_{lm} \quad (5.5-a) \]

\[ - \frac{m}{AB} \sum_{i} \left\{ \left( \frac{3}{2} - \frac{U}{2} \right) \frac{B'}{B} - \frac{t^2}{12R_1R_2} \frac{R_1'}{R_1} \right\} (A_{im} \sin i \alpha \right. \\
+ B_{im} \cos i \alpha \left. + i \frac{(1+U)}{2} \right\} (A_{im} \cos i \alpha - B_{im} \sin i \alpha) \]

\[ + \frac{1}{A^2} \left( \frac{1 - U}{2} \right) \sum_{j} \left\{ \left[ \frac{1}{B} - \left( \frac{B''}{B} - \frac{A'B'}{A} - \frac{B'^2}{B} \right) + \frac{2A^2}{R_1R_2} \right. \\
- \frac{m^2}{B^2} \frac{A^2}{B^2} - j^2 \right\} (C_{jm} \sin j \alpha + D_{jm} \cos j \alpha) \]

\[ + j \left\{ \frac{B'}{B} - \frac{A'}{A} \right\} (C_{jm} \cos j \alpha - D_{jm} \sin j \alpha) \]

\[ - \frac{m}{B} \sum_{k} \left\{ \left( \frac{U}{B} + \frac{1}{R_2} - \frac{t^2}{12A^2} \frac{1}{R_2} \left( \frac{m^2}{B^2} \frac{A^2}{B^2} + k^2 \right) \right. \right. \\
(E_{km} \sin k \alpha + F_{km} \cos k \alpha) + \frac{t^2}{12A^2} \frac{k}{R_2} \left[ \frac{B'}{B} - \frac{A'}{A} \right] \right. \\
(E_{km} \cos k \alpha - F_{km} \sin k \alpha) \right\} = - \frac{(1 - U^2)}{Et} q_{lm} \quad (5.5-b) \]
\[
\frac{1}{A} \sum_i \left\{ \left[ - \frac{B'}{B} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) - \left( 1 - \nu \right) \frac{R_2'}{R_2^2} + \frac{t^2}{12A^2R_1} \right] \right. \\
\left. - \ i^2 \left\{ \left[ 2B' - \frac{3A'}{A} - \frac{3R_1'}{R_1} \right] \right. \\
\left. - \ m^2 \frac{A^2}{B^2} \left[ \frac{1}{2} + \frac{R_1}{R_2} \right] \right. \\
\left. \left. \frac{B'}{B} - \frac{(3 - \nu)}{2} \frac{R_1'}{R_1} \right) \right) \\
+ \frac{B''}{B} - \frac{4B'}{B^2} - \frac{A''}{AB} - \frac{3A'}{AB} \\
- \frac{A'}{AB} \left( \frac{2A'}{A} + \frac{B'}{B^3} - \frac{R_1''}{R_1} + \frac{6R_1'R_1''}{R_1^2} - \frac{6R_1'}{R_3^3} \right) \\
+ \frac{A'' R_1'}{AR_1} + \frac{3A'R_1'}{AR_1} \left[ \frac{2R_1'}{R_1} + \frac{A'}{A} \right] - \frac{2}{BR_1} \left[ \frac{B''R_1'}{B R_1} + \frac{B'}{BR_1} \right] \\
+ \frac{B'R_1'}{BR_1} \left[ \frac{B'}{B} + \frac{2R_1'}{R_1} \right] + \frac{4A'B'R_1'}{AB R_1} + \frac{A'^2(1 - \nu)}{R_1 R_2} \frac{2R_1'}{R_1} \right) \\
\left[ A \sin i(\alpha) + B \sin \nu \right] \\
\left. + \frac{i}{\left[ \frac{1}{R_1} + \frac{\nu}{R_2} \right] + \frac{t^2}{12A^2R_1} \right] \right) \left. \left[ \frac{B''}{B} - \frac{B'^2}{B^2} - \frac{A''}{A} - \frac{3A^2}{A^2} - \frac{3A'B'}{AB} \right. \right] \\
\left. + \frac{5}{\left[ \frac{A' R_1'}{AR_1} - \frac{B'R_1'}{BR_1} - \frac{2R_1''}{R_1} + \frac{4R_1'^2}{R_1^2} - \left( 1 - \nu \right) \frac{A'^2}{R_1 R_2} \right] \right) \left. \frac{(A \cos i(\alpha - B \sin i(\alpha)) \right]} \\
\left. + \frac{m}{B} \sum_j \left\{ \left[ \frac{\nu}{R_1} + \frac{1}{R_2} + \frac{t^2}{12A^2} \right] \left[ j^2 \left\{ \frac{2(1 - \nu)}{R_2} \frac{(1 - \nu)}{2R_1} \right] \right. \right) \\
\left. - \left\{ \frac{1 + \nu}{2R_2} + \frac{1 - \nu}{2R_1} \right\} \left[ \frac{B''}{B} - \frac{B'^2}{B^2} + \frac{A'B'}{AB} \right] \right. \right) + \frac{1}{R_2} \frac{B'R_2'}{BR_2} \\
- \frac{1}{R_2} \frac{R_2''}{R_2} + \frac{2}{R_2} \frac{R_2'^2}{R_2^2} - \frac{1}{R_2} \frac{A'R_2'}{AR_2} + \frac{(1 - \nu)}{2R_1} \frac{B'R_1'}{BR_1} \right\}
\]
For the sphere the equilibrium equations become

\[
\frac{1}{R^2} \left\{ -\sum_i \left[ i^2 + \frac{(1 - \nu)m^2}{2 \sin^2 \theta} + \nu + \cot^2 \theta \right] (A_{im} \sin i \theta - B_{im} \cos i \theta) \right\} = -\frac{(1 - \nu^2) \Omega_n}{E_t}
\]
\[
\begin{align*}
- \frac{m}{\sin \ell} \sum_j \left[ \frac{(3 - \ell^2)}{2} \cot \theta \left( C_{jm} \sin j \theta + D_{jm} \cos j \theta \right) \right. \\
- \left. \frac{(1+\nu)}{2} \sum_j \left( C_{jm} \cos j \theta - D_{jm} \sin j \theta \right) \right] \\
+ K \sum_k \left[ k^2 \cot \theta - 2 m^2 \frac{\cot \theta}{\sin^2 \theta} \right] \left( E_{km} \sin k \theta + F_{km} \cos k \theta \right) \\
+ \sum_k \left[ (1+\nu) + K \left( k^2 + 1 + \cot^2 \theta + \frac{1 - \nu}{\sin \theta} + \frac{m^2}{\sin^2 \theta} \right) \right] \\
\left( E_{km} \cos k \theta - F_{km} \sin k \theta \right) \right] \\
= - \frac{(1 - \nu^2)}{E \ell m} \quad (5.6-a)
\end{align*}
\]

\[
\frac{1}{R^2} \left\{ - \frac{m}{\sin \theta} \sum_i \left[ \frac{(3 - \nu)}{2} \cot \theta \left( A_{im} \sin i \theta + B_{im} \cos i \theta \right) \right. \\
+ i \frac{(1+\nu)}{2} \left( A_{im} \cos i \theta - B_{im} \sin i \theta \right) \right] \\
+ \sum_j \left[ \frac{(1 - \nu)}{2} \left( 1 - \cot^2 \theta - j^2 \right) - \frac{m^2}{\sin^2 \theta} \right] \left( C_{jm} \sin j \theta \\
+ D_{jm} \cos j \theta \right) + \frac{(1 - \nu)}{2} \cot \theta \\
+ \sum_j \left( C_{jm} \cos j \theta - D_{jm} \sin j \theta \right) \right] \\
- \frac{m}{\sin \theta} \sum_k \left[ K \left( k^2 + \frac{m^2}{\sin^2 \theta} \right) + (1+\nu) \right] \left( E_{km} \sin k \theta \\
+ F_{km} \cos k \theta \right) \right]
\]
\[ + m \cot \frac{\theta}{\sin \theta} K \sum_k k \left( E_{km} \cos k \theta - F_{km} \sin k \theta \right) \]

\[ = - \frac{(1 - \nu^2)}{Et} q_{2m} \quad \ldots \quad (5.6-b) \]

\[ \frac{1}{R^2} \left\{ - \sum_i \left[ (1+\nu) \cot \theta + K \left( 2 \cot \theta \frac{i^2 + (3+\nu) \cot \theta}{2 \sin^2 \theta} m^2 \right. \right. \right. \]
\[ \left. \left. - \cot \frac{\theta}{\sin^2 \theta} \right) \right] (A_{im} \sin i \theta + B_{im} \cos i \theta) \]
\[ - \sum_i \left[ (1+\nu) + K \left( i^2 + \cot^2 \theta + \frac{3 - \nu}{2 \sin^2 \theta} m^2 \right) \right] \]
\[ (A_{im} \cos i \theta - B_{im} \sin i \theta) \]
\[ - \frac{m}{\sin \theta} \sum_j \left[ (1+\nu) + K \left( j^2 - \frac{1}{\sin^2 \theta} + \frac{m^2}{\sin^2 \theta} \right) \right] \]
\[ (C_{jm} \sin j \theta + D_{jm} \cos j \theta) \]
\[ - \frac{m K}{\sin \theta} \cot \theta \sum_j \left( C_{jm} \cos j \theta - D_{jm} \sin j \theta \right) \]
\[ \sum_k \left[ 2 (1+\nu) + K \left( k^4 + k^2 \left\{ 2 - \nu - \frac{1}{\sin^2 \theta} \right\} + 2 \frac{k^2 m^2}{\sin^2 \theta} \right. \right. \right. \]
\[ \left. \left. - m^2 \left\{ 1 + \nu + 4 \frac{\cot^2 \theta + 1}{\sin^2 \theta} + \frac{m^4}{\sin^4 \theta} \right\} \right) \right] \]
\[ (E_{km} \sin k \theta + F_{km} \cos k \theta) \]
\[ + K \sum_k \left[ 2 \cot \theta k^2 - (\nu \cot \theta + \cot^3 \theta) + m^2 \frac{2 \cot \theta}{\sin^2 \theta} \right] \]
\[ (E_{km} \cos k \theta - F_{km} \sin k \theta) = - \frac{(1 - \nu^2)}{Et} q_{nm} \quad \ldots \quad (5.6-c) \]
The equilibrium equations for a cylinder become

\[- \sum_{i} \left[ i^2 + \frac{(1 - V) m^2}{2R^2} \right] (A_{im} \sin i x + B_{im} \cos i x)\]

\[+ \frac{(1 + V)}{2R} m \sum_{j} j (C_{jm} \cos j x - D_{jm} \sin j x)\]

\[+ \frac{V}{R} \sum_{k} k (E_{km} \cos k x - F_{km} \sin k x)\]

\[= - \frac{(1 - V^2)}{\nu} \Omega_{im} \quad (5.7-a)\]

\[+ \frac{(1 + V)}{2R} m \sum_{i} i (A_{im} \cos i x - B_{im} \sin i x)\]

\[+ \sum_{j} \left[ \frac{(1 - V)}{2} j^2 + \frac{m^2}{R^2} \right] (C_{jm} \sin j x + D_{jm} \cos j x)\]

\[+ m \sum_{k} k \left[ \frac{1}{R^2} + K \left( \frac{k^2 + m^2}{R^2} \right) \right] (E_{km} \sin k x + F_{km} \cos k x)\]

\[= \frac{(1 - V^2)}{\nu} \Omega_{2m} \quad (5.7-b)\]

\[\frac{1}{R} \sum_{i} \left[ \frac{(3 - V)}{2} \right] (A_{im} \sin i x + B_{im} \cos i x)\]

\[+ \nu i (A_{im} \cos i x - B_{im} \sin i x)\]

\[+ \sum_{j} \left[ \frac{1}{R^2} + K \left( \frac{(3 - V) j^2 + m^2}{2 R^2} \right) \right] (C_{jm} \sin j x + D_{jm} \cos j x)\]

\[+ \sum_{k} k \left[ \frac{1}{R^2} + K \left( R^2 k^4 + 2 k^2 m^2 - \frac{m^2}{R^2} + \frac{m^4}{R^2} \right) \right] \quad (5.7-c)\]

\[(E_{km} \sin k x + F_{km} \cos k x) = \frac{(1 - V^2)}{\nu} \Omega_{km} .\]
Finally the equations for a cone are

\[
\begin{align*}
\cos^2 \gamma \sum_i &\left\{ - \left[ \frac{1}{x^2} + \frac{(1-U)}{2} \frac{m^2}{x^2 \sin^2 \gamma} \right] + i^2 j (A_{im} \sin i x + B_{im} \cos i x) \right. \\
+ &\frac{i}{x} (A_{im} \cos i x - B_{im} \sin i x) \left. \right\} \\
+ &\frac{m \cos \gamma}{x \tan \gamma} \sum_j \left\{ - \frac{(1+U)}{2} \frac{1}{x} (C_{jm} \sin j x + D_{jm} \cos j x) \\
+ &\frac{(1+U)}{2} j (C_{jm} \cos j x - D_{jm} \sin j x) \right\} \\
+ &\cos \gamma \sum_k \left\{ - \frac{1}{x^2} (E_{km} \sin k x + F_{hm} \cos k x) \\
+ &k \frac{U}{x \sin \gamma} (E_{km} \cos k x - F_{km} \sin k x) \right\} \\
= &- \frac{(1-U^2)}{\Et} \phi \Im \tag{5.8-a}
\end{align*}
\]
\[- \frac{m}{x^2 \tan^2 \gamma} \sum_k \left\{ \frac{1}{\cos} + \mathcal{U} - K \left[ \frac{m^2}{x \sin \gamma} + k^2 x \sin \gamma \right] \right\} \]

\((E_{km} \sin k x + F_{km} \cos k x)\)

\[+ K k \sin \gamma \left( E_{km} \cos k x - F_{km} \sin k x \right) \\]

\[= - \frac{(1 - U^2)}{\mathcal{E} \mathcal{T}} q_{2m}, \quad (5.8\text{-b})\]

\[- \frac{1}{x \tan \gamma} \sum_i \left[ \frac{1}{x} \right. \]

\[\left( A_{im} \sin i x + B_{im} \cos i x \right) \]

\[+ i \mathcal{U} \left( A_{im} \cos i x - B_{im} \sin i x \right) \]

\[+ \frac{m}{x^2} \sum_j \left\{ \left[ \frac{1}{\tan \gamma \sin \gamma} + K \cos \gamma \left( j^2 (2 - \mathcal{U}) + \left( \frac{7 + \mathcal{U}}{2} \right) \frac{m^2}{\sin^2 \gamma} \right) \right] \right\} \]

\((C_{jm} \sin j x + D_{jm} \cos j x)\)

\[- j K \sin \gamma \left( \frac{5 + \mathcal{U}}{2x} \right) \left( C_{jm} \cos j x - D_{jm} \sin j x \right) \]

\[- \sum_k \left\{ \left[ \frac{1}{x^2 \tan^2 \gamma} + K \left( \sin^2 \gamma \cos^2 \gamma \left( k^4 + \frac{k^2}{x^2} \right) + 2 k^2 m^2 \cos^2 \gamma \right) \right] \right\} \]

\[- m^2 \left\{ \frac{4 \cos^2 \gamma}{x^4} + \frac{1}{x^2 \tan^2 \gamma} \right\} \left( E_{km} \sin k x \right. \]

\[+ F_{km} \cos k x) \]

\[+ k K \left[ \sin^2 \gamma \cos^2 \gamma \left( \frac{2 k^2}{x} - \frac{1}{x^3} \right) - \frac{\cos^2 \gamma}{x} - 2m^2 \right] \quad (5.8\text{-c})\]

\[(E_{km} \cos k x + F_{km} \sin k x) \quad = - \frac{(1 - U^2)}{\mathcal{E} \mathcal{T}} q_{nm} \]
Similar expressions for other shells of revolution can be derived readily from equations (5.5). Henceforth, the discussion will be restricted to the case of a spherical shell. Substituting (5.1) into (1.2) and then into (1.4) produces expressions for internal loads as follows:

\[
T_{1m} = \frac{Et}{R(1 - \nu^2)} \left\{ \sum_i \left[ \nu \cot \theta \left( A_{im} \sin \nu \theta + B_{im} \cos \nu \theta \right) \right. \right. \\
+ i \left( A_{im} \cos \nu \theta - B_{im} \sin \nu \theta \right) \right. \\
+ \frac{m \nu}{\sin \theta} \sum_j \left( C_{jm} \sin \nu \theta + D_{jm} \cos \nu \theta \right) \\
+ \left(1 + \nu \right) \sum_k \left( E_{km} \sin \nu \theta + F_{km} \cos \nu \theta \right) \left. \right\}, \quad (5.9-a)
\]

\[
T_{2m} = \frac{Et}{R(1 - \nu^2)} \left\{ \sum_i \left[ \nu \cot \theta \left( A_{im} \sin \nu \theta + B_{im} \cos \nu \theta \right) \right. \right. \\
+ i \nu \left( A_{im} \cos \nu \theta - B_{im} \sin \nu \theta \right) \right. \\
+ \frac{m}{\sin \theta} \sum_j \left( C_{jm} \sin \nu \theta + D_{jm} \cos \nu \theta \right) \\
+ \left(1 + \nu \right) \sum_k \left( E_{km} \sin \nu \theta + F_{km} \cos \nu \theta \right) \left. \right\}, \quad (5.9-b)
\]

\[
S_{m} = \frac{Et}{2R(1 + \nu)} \left\{ - \frac{m}{\sin \theta} \sum_i \left( A_{im} \sin \nu \theta + B_{im} \cos \nu \theta \right) \right. \\
- \sum_j \left[ \cot \theta \left( C_{jm} \sin \nu \theta + D_{jm} \cos \nu \theta \right) \right. \\
\left. \left. + i \left( C_{jm} \cos \nu \theta - D_{jm} \sin \nu \theta \right) \right] \right\}, \quad (5.9-c)
\]
\[ M_{1m} = - \frac{E \mu K}{(1 - \nu^2)} \left\{ - \sum_i \left[ \psi \cot \theta \ (A_{im} \sin i \ell + B_{im} \cos i \ell) \right] + i \left( A_{im} \cos i \ell - B_{im} \sin i \ell \right) \right\} \]

\[ - \sum_j \left( C_{jm} \sin j \ell + D_{jm} \cos j \ell \right) \]

\[ - \sum_k \left[ \left( \frac{k^2}{\sin^2 \ell} + \frac{\nu \mu m^2}{\sin^2 \ell} \right) \ (E_{km} \sin k \ell + F_{km} \cos k \ell) \right] \]

\[ \cot \ell \ (E_{km} \cos k \ell - F_{km} \sin k \ell) \] (5.9-d)

\[ M_{2m} = - \frac{E \mu K}{(1 - \nu^2)} \left\{ - \sum_i \left[ \cot \theta \ (A_{im} \sin i \ell + B_{im} \cos i \ell) \right] + i \sqrt{\nu} \left( A_{im} \cos i \ell - B_{im} \sin i \ell \right) \right\} \]

\[ - \sum_j \left( C_{jm} \sin j \ell + D_{jm} \cos j \ell \right) \]

\[ - \sum_k \left[ \left( \frac{\nu \mu k^2}{\sin^2 \ell} + \frac{m^2}{\sin^2 \ell} \right) \ (E_{km} \sin k \ell + F_{km} \cos k \ell) \right] \]

\[ - k \cot \ell \ (E_{km} \cos k \ell - F_{km} \sin k \ell) \] (5.9-e)

\[ H_m = \frac{E \mu K}{(1 + \nu)} \frac{1}{\sin \theta} \left\{ - m \sum_i \left( A_{im} \sin i \theta + B_{im} \cos i \theta \right) \right\} \]

\[ - \sum_j \left[ \cos \theta \ (C_{jm} \sin j \theta + D_{jm} \cos j \theta) \right] - j \sin \theta \left( C_{jm} \cos j \theta - D_{jm} \sin j \theta \right) \]

\[ - m \sum_k \left[ \cot \theta \ (E_{km} \sin k \theta + F_{km} \cos k \theta) \right] - k \left( E_{km} \cos k \theta - F_{km} \sin k \theta \right) \] (5.9-f)

\[ Q_m = S_m + \frac{2H_m}{R} = \frac{E \mu K}{2R(1 + \nu)} \left\{ - m \sum_i \left( A_{im} \sin i \theta + B_{im} \cos i \theta \right) \right\} \]

\[ - \sum_j \left[ \cot \theta \ (C_{jm} \sin j \theta + D_{jm} \cos j \theta) \right] \]

\[ - j \left( C_{jm} \cos j \theta - D_{jm} \sin j \theta \right) \]

\[ - \frac{4 K m}{\sin \theta} \sum_k \left[ \cot \theta \ (E_{km} \sin k \theta + F_{km} \cos k \theta) \right] \]

\[ - k \left( E_{km} \cos k \theta - F_{km} \sin k \theta \right) \] (5.9-g)
\[ V_{1m} = -\frac{EtK}{R(1 - U^2)} \left\{ \sum_i \left[ \left( i^2 + \mathcal{U} + \cot^2 \theta + \frac{2(1 - \mathcal{U}) m^2}{\sin^2 \theta} \right) \right] \\ (A_{im} \sin i \theta + B_{im} \cos i \theta) \\
- i \cot \theta (A_{im} \cos i \theta - B_{im} \sin i \theta) \right\} \\
+ \frac{m}{\sin \theta} \sum_j \left[ (3 - 2\mathcal{U}) (C_{jm} \sin j \ell + D_{jm} \cos j \ell) \\
- j (2 - \mathcal{U}) (C_{jm} \cos j \ell - D_{jm} \sin j \ell) \right] \\
- \sum_k \left[ \cot \theta \left( k^2 \left( 3 - \mathcal{U} \right) m^2 \right) (E_{km} \sin k \ell + F_{km} \cos k \ell) \\
+ k (k^2 + \mathcal{U} + \cot^2 \theta + \frac{2(1 - \mathcal{U}) m^2}{\sin^2 \theta} (E_{km} \cos k \ell \\
- F_{km} \sin k \ell) \right) \right], \quad (5.9-h) \]

\[ V_{2m} = -\frac{EtK}{R(1 - U^2)} \left\{ \frac{m}{\sin \ell} \sum_i \left[ \cot \theta (A_{im} \sin i \theta + B_{im} \cos i \theta) \\
+ i (2 - \mathcal{U}) (A_{im} \cos i \theta - B_{im} \sin i \theta) \right] \\
+ \sum_j \left[ 2(1 - \mathcal{U}) (j^2 - 1) + \frac{m^2}{\sin^2 \theta} \right] (C_{jm} \sin j \theta + D_{jm} \cos j \theta) \\
+ \frac{m}{\sin \theta} \sum_k \left[ k^2 \left( 2 - \mathcal{U} \right) + n^2 - 2(1 - \mathcal{U}) (E_{km} \sin k \ell + F_{km} \cos k \ell) \\
+ k (1 - 2 \mathcal{U}) \cot \theta (E_{km} \cos k \ell - F_{km} \sin k \ell) \right] \right), \quad (5.9-i) \]

The rotation becomes

\[ \Omega_m = \frac{1}{R} \left\{ \sum_i (A_{im} \sin i \theta + B_{im} \cos i \theta) \\
- \sum_k (E_{km} \cos k \ell - F_{km} \sin k \ell) \right\}, \quad (5.10) \]
Expressions for loads and displacements as given in (5.4), (5.9), and (5.10) can be substituted into the boundary equations (4.16) to obtain boundary conditions in terms of the undetermined constants \( A_{im}, B_{im}, C_{jm}, \) etc. When the shell does not have two boundaries in the conventional sense but has one or both ends closed then regularity conditions at \( \theta = 0 \) or \( \theta = \pi \) must be satisfied.

Consider the equilibrium equations (5.6) for \( \theta = 0 \). Under the assumption that the external loading varies continuously over the surface of the shell it can be stated that the computed pressures determined from equations (5.6) will have finite values at all times. Hence certain requirements must be imposed on the displacement coefficients. Simple substitution will verify that these conditions will also maintain the finiteness of computed internal loads at the apex.

The details of computation of the regularity conditions and the evaluation of loads at \( \theta = 0 \) or \( \theta = \pi \) are given in the Appendix. The regularity equations are

\[
\sum_{i} A_{im} \cos i \theta = 0, \quad m > 0,
\]

\[
\sum_{i} B_{im} \cos i \theta \cos \theta + \sum_{j} D_{jm} \cos j \epsilon = 0, \quad m = 1,
\]
\[ \sum_i B_{im} \cos i \theta = 0, \quad m \neq 1, \]
\[ \sum_i B_{im} i^2 \cos i \theta = 0, \]
\[ \sum_j C_{jm} j \cos j \theta = 0, \]
\[ \sum_j D_{jm} \cos j \theta = 0, \quad m \neq 1, \quad (5.11) \]
\[ \sum_j D_{jm} j^2 \cos j \theta = 0, \]
\[ \sum_k E_{km} k \cos k \theta = 0, \]
\[ \sum_k E_{km} k^3 \cos k \theta = 0, \]
\[ \sum_k F_{km} k \cos k \theta = 0, \quad m > 0, \]
\[ \sum_k F_{km} k^2 \cos k \theta = 0. \]

At \( \theta = 0, \cos i \theta = 1 \). Hence, for example,
\[ \sum_i A_{im} i = 0, \quad m > 0, \]
\[ \sum_i B_{im} + \sum_j D_{jm} = 0, \quad m = 1. \quad (5.12) \]
Likewise at \( \theta = \pi \), \( \cos \theta = (-1)^i \). In this case we have

\[
\sum_i A_{im} (-1)^i = 0, \quad m > 0,
\]

(5.13)

\[
\sum_i B_{im} (-1)^i - \sum_j D_{jm} (-1)^j = 0.
\]

These regularity conditions have physical interpretations.

For \( m = 0 \), the case of axisymmetric loading, we can say that there is no tangential displacement at the apex. This is the same as the statement

\[
\sum_i B_{i0} = 0.
\]

For higher modes there is no normal displacement at the apex.

This is expressed

\[
\sum_k F_{km} = 0, \quad m > 0.
\]

For the first mode the apex tangential displacement at the zero meridian is equal, but opposite in sign, to the apex circumferential displacement at \( \phi = \frac{\pi}{2} \). This is simply

\[
\sum_i B_{i1} + \sum_j D_{j1} = 0.
\]

For modes higher than the first there is no displacement at \( \theta = 0 \).

This is in agreement with the third, sixth, and ninth of equations (5.11).
When the regularity conditions of equations (5.11) are imposed we have the following computed values at \( \Theta = 0 \) or \( \Theta = \pi \):

\[
\sum_j D_{jm} j^2 \cos j \Theta \frac{(1 + \nu)}{2} \cos \Theta = (1 - \nu^2) \frac{R^2 q_l m}{E_t}, \quad m = 1,
\]

\[
\sum_j D_{jm} j^2 \cos j (1 - \nu) = (1 - \nu^2) \frac{R^2}{E_t} q_2 m, \quad m = 1, \quad (5.14)
\]

\[
\sum_i A_{im} 3 i^3 \cos i \Theta - \sum_k F_{km} k^4 \cos k \Theta = R^2 \frac{(1 - \nu^2)}{E_t K} q_{no}, \quad m = 0.
\]

The internal loads become:

\[
T_{1m} = \frac{E_t}{R(1 - \nu)} \left[ \sum_i A_{i0} i \cos i \Theta + \sum_k F_{ko} \cos k \Theta \right], \quad m = 0,
\]

\[
T_{1m} = 0 \quad , \quad m > 0 \quad ,
\]

\[
T_{2m} = T_{1m} \quad ,
\]

\[
S_m = 0 \quad ,
\]

\[
M_{1m} = \frac{E_t K}{1 - \nu} \sum_i A_{im} i \cos i \Theta, \quad m = 0 \quad ,
\]

\[
M_{2m} = M_{1m} = 0 \quad , \quad m > \quad ,
\]

\[
H_m = 0 \quad ,
\]

\[
V_{1m} = 0 \quad ,
\]

\[
V_{2m} = 0 \quad .
\]
The applied normal load is non-zero at the apex only for the axisymmetric case as the last of equations (5.14) indicates. Tangential loads appear at the apex only in the first mode (wind type load as the first two of (5.14) show.

Internal loads at $\Theta = 0$ or $\Theta = \pi$ appear only in the axisymmetric case. This excludes all shearing and twisting effects.
Chapter 6
Matrix Equations

With the assumption of the trigonometric displacement functions of Chapter 5 the primary problem is to solve for the \( r_m \) unknown coefficients \( A_{im}, \ldots, F_{km} \). For example equation (5.6-a) can be rewritten as follows,

\[
\sum_i \left\{ \left[ i \cot \theta \cos i \theta - \left( i^2 + \frac{(1-\nu)m^2}{2 \sin^2 \theta} \right) \sin i \theta \right] A_{im} - \left[ i^2 + \frac{(1-\nu)m^2}{2 \sin^2 \theta} \right] \cos i \theta + i \cot \theta \sin i \theta \right\} B_{im} + \frac{m}{\sin \theta} \sum_j \left\{ \left[ \frac{(1+\nu)j \cos j \theta - (3-\nu) \cot \theta \sin j \theta}{2} \right] C_{jm} - \left[ \frac{(3-\nu) \cot \theta \cos j \theta - (1+\nu)j \sin j \theta}{2} \right] D_{jm} \right\} + \sum_k \left\{ k \left[ 1 + \nu + K \left\{ k^2 \cot \theta + \frac{m^2}{\sin^2 \theta} \right\} \right] \cos k \theta + K (k^2 - 2m^2) \cot \theta \sin k \theta \right\} F_{km} + \left[ K (k^2 - 2m^2) \cot \theta \cos k \theta + k (1+\nu) + K \left\{ k^2 + \cot \theta \right\} \sin k \theta \right] \right\} F_{km} = - R^2 \frac{(1-\nu^2)}{Et} q_{im}.
\]

The subscript \( m \) will be discarded in the remainder of the discussion. It is assumed that we will be dealing with just one case at a time. The actual Fourier mode \( (m) \) will be implied but not noted.
The three equations for equilibrium at a point can be written as

\[
\begin{bmatrix}
\psi
\end{bmatrix}
\begin{bmatrix}
X
\end{bmatrix} = \begin{bmatrix}
Q
\end{bmatrix},
\]

(6.2)

where \([\psi]\) is the matrix of coefficients of the form shown in equation (6.1), \(\{X\}\) is comprised of sub-matrices as follows

\[
\begin{bmatrix}
A \\
B \\
C \\
D \\
E \\
F
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
q_1 \\
q_2 \\
q_n
\end{bmatrix}
\]

For example,

\[
\psi_{11} = \cot \theta \cos \theta - \left[ 1 + \nu + \cot^2 \theta + \frac{1 - \nu}{2 \sin^2 \theta} m^2 \right] \sin \theta.
\]

(6.3)

Writing the equations for \(p\) points along the meridian of a shell produces a total of \(3p\) equations (\(2p\) for axisymmetric loading).

Boundary conditions (or regularity conditions) must also be satisfied by the displacement coefficients. They vary in number from a minimum of six for a doubly truncated sphere with axisymmetric loading to a maximum of eighteen for a doubly closed sphere with a higher mode (\(m > 1\)) type loading. These too can be revised into a form similar to that of equation (6.1) and then written

\[
[A] \begin{bmatrix}
X
\end{bmatrix} = \begin{bmatrix}
B
\end{bmatrix}.
\]

(6.4)
Let the number of boundary and regularity conditions be \( q \).

Then the total number of unknown coefficients \( r \) can be divided into two groups - \( q \) coefficients \( X_1 \), and \( r-q \) coefficients \( X_2 \). Equation (6.4) can be rewritten

\[
\begin{bmatrix}
A_1 & A_2
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix} = \begin{bmatrix}
B
\end{bmatrix},
\]

or

\[
\begin{bmatrix}
A_1
\end{bmatrix}\{X_1\} + \begin{bmatrix}
A_2
\end{bmatrix}\{X_2\} = \begin{bmatrix}
B
\end{bmatrix}. \tag{6.5}
\]

Since \( [A_1] \) is square, of order \( qxq \), we can solve for \( \{X_1\} \) as follows:

\[
\{X_1\} = [A_1]^{-1} \left( \begin{bmatrix} B \end{bmatrix} - [A_2] \{X_2\} \right). \tag{6.6}
\]

The \( q \) coefficients \( X_1 \) are determined by the boundary and regularity conditions. Hence it follows that only \( r-q \), or \( s \), coefficients can be obtained from the equilibrium equations. Rewrite equation (6.2)

\[
\begin{bmatrix}
\psi_1 & \psi_2
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix} = \begin{bmatrix}
Q
\end{bmatrix},
\]

or

\[
\begin{bmatrix}
\psi_1
\end{bmatrix}\{X_1\} + \begin{bmatrix}
\psi_2
\end{bmatrix}\{X_2\} = \begin{bmatrix}
Q
\end{bmatrix}. \tag{6.7}
\]

Substituting for \( X_1 \) from equation (6.6) yields

\[
\begin{bmatrix}
\psi_2 \\
\psi_1
\end{bmatrix} - \begin{bmatrix}
\psi_1
\end{bmatrix} [A_1]^{-1} [A_2] \{X_2\} = \begin{bmatrix}
Q
\end{bmatrix} - [\psi_1] [A_1]^{-1} \{B\}. \tag{6.8}
\]

This can be rewritten as

\[
\begin{bmatrix}
\overline{\psi}
\end{bmatrix}\{\overline{X}\} = \{\overline{Q}\}. \tag{6.9}
\]
Now a unique solution for s coefficients requires the writing of \(3p = s\) equilibrium equations. Such a solution will exist when a problem is properly defined. It will satisfy all boundary conditions and will agree precisely with the equilibrium equations at all points under consideration. However it frequently occurs that such solutions correspond to values of internal or external loads which vary wildly between the \(p\) points selected. Such uniqueness is obviously not a virtue.

The general method of attack is to write \(t\) equations, where \(t > s\). Recall that in the general case \(t = 3p\). Now an overdetermined set of equations exists. There is generally no unique solution to such a system. Instead one tries to obtain coefficients such that the error is minimal.

Having solved for the displacement coefficients one can calculate the corresponding applied loads. The difference between the actual and computed loads is a measure of the error in the solution. The solutions can be denoted as exact solutions for the corresponding computed loadings.
The problem of evaluating a spherical shell under arbitrary loading conditions finally reduces to finding solutions to a set of systems of the type

$$\begin{bmatrix} \psi \end{bmatrix} \{x\} = \{Q\} \quad (7.1)$$

where the number of equations $t$ in general exceeds the number of unknowns $s$. As shown by Tricomi (26), a unique solution exists if and only if the rank of the matrix $\begin{bmatrix} \psi \end{bmatrix}$ is $s$ and the remaining $t-s$ equations are linear combinations of the $s$ independent equations. In general the second condition is not satisfied and a unique solution does not exist. Rather the effort is to minimize the total error.

Let the error in any equation be defined as

$$\epsilon_i = Q_i - \sum_j \psi_{ij} x_j \quad (7.2)$$

The method of least squares minimizes the total square error

$$\sum \epsilon_i^2. \text{ This can be accomplished by simply pre-multiplying both}$$

$$j$$
sides of the matrix equation (7.1) by the transpose of
\[
[\psi']^T[\psi']\{x\} = [\psi]^T\{Q\}.
\] (7.3)

This can be rewritten
\[
[E] \{x\} = \{F\}.
\] (7.4)

If matrix \([E]\) is non-singular we can solve
\[
\{x\} = [E]^{-1} \{F\}.
\] (7.5)

This solution for displacement coefficients is the least squares solution; i.e. the squared error is minimized.

A computer program was written in FORTRAN IV language for the IBM 7090 computer. It generates the system of equations (7.1) preparatory to solution for the unknown coefficients. This system was subjected to the operations shown in equations (7.3) and (7.5) in an effort to obtain least squares solutions. In theory this is a simple process if a unique solution exists. The reality however is that the inversion of matrix \([E]\) frequently becomes difficult, if not impossible, to obtain by numerical methods.

For any real physical system the matrix \([E]\) is positive definite and symmetric. The positive definiteness simply means that all of the principal minors of the matrix are always positive. In many cases however such a matrix can have a determinant of very small value. This frequently implies near dependence (or ill
conditioning) of the equations on which it is based. Ill conditioned equations are defective in that small changes in coefficients can produce wide variations in the values of the solution. Accurate solutions usually require much more precision than is available on the average digital computer. It is evident that increasing the number of points at which equations are to be written tends to make those equations more nearly dependent, since the distance between points is thereby decreased.

Now an increase in the number of points, and thereby equations, would tend to decrease the least square error. On the other hand the effect of this decreased error is more than offset by the decrease in magnitude of the determinant of the matrix $[E]$. When Gaussian elimination, the typical method of matrix inversion on a digital computer, is used, round-off errors tend to increase to the point where completely useless values are computed for the matrix inverse.

This phenomenon of round-off error is illustrated by a sample problem. Consider a spherical cap with closed apex and lower boundary at $\theta = 15^\circ$. The radius/thickness ratio is 1000. The shell is simply supported at the outer edge and subjected to a continuous unit normal load $q_n$. It was found that at most four to
six parameters, corresponding to \( \{X_2\} \) in Chapter 6, could be computed. With the use of double precision arithmetic a digital computer solution was obtained for ten parameters. This simply means that 16 significant figures are retained in all operations, whereas single precision arithmetic utilizes eight figures only. It is evident that a direct attack on the least squares problem is futile.

Each column of matrix \( \{\psi_i\} \) in equation (7.1) can be considered to be a vector \( \{\psi_i\} \) corresponding to the unknown displacement coefficient \( X_i \). The correlation between vectors \( \{\psi_i\} \) and \( \{\psi_j\} \) can be expressed as

\[
\rho_{ij} = \frac{\langle \psi_i, \psi_j \rangle}{\|\psi_i\| \|\psi_j\|},
\]

(7.6)

where \( \langle \psi_i, \psi_j \rangle \) is the inner (or scalar) product of the two vectors and \( \|\psi_i\| \) is the norm of vector \( \{\psi_i\} = (\psi_i, \psi_i)^{1/2} \). It is obvious that \( \rho_{ii} = 1.0 \). Linear dependence exists if one or more cross-correlation coefficients equals 1.0.

If matrix \( \left[ E \right] \) of equation (7.4) is normalized then a matrix of the coefficients \( \rho_{ij} \) appears. The determinant of this matrix is the Gram determinant

\[
G = \begin{vmatrix}
1 & \rho_{12} & \ldots & \rho_{1s} \\
\rho_{21} & 1 & \ldots & \rho_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{n1} & \rho_{n2} & \ldots & 1
\end{vmatrix}
\]

(7.7)
If all of the vectors \( \{ \psi_1 \} \) are mutually orthogonal, then no off-diagonal terms appear. In this case \( G = 1 \). Any deviation from orthogonality tends to decrease the magnitude of the Gram determinant. A linearly dependent set of vectors yields a value of \( G = 0 \). In general a very small non-zero value of the Gram determinant indicates that the matrix is ill-conditioned.

Attempts to invert an ill-conditioned matrix by digital computer methods, which use arithmetic of limited precision, are rarely successful. Experience indicates that least squares matrices are usually ill-conditioned. In most cases the matrix of correlation coefficients contains numerous off-diagonal terms with values exceeding .99. For this reason direct attack on a least squares matrix must be set aside.

It has been demonstrated that any set of linearly independent vectors can be transformed into a set of mutually orthogonal vectors. Perhaps the most popular transformation is the Gram-Schmidt orthogonalization process, which is described by Friedman (19).

Consider any set of linearly independent vectors \( \psi_1, \psi_2 \).

Set

\[
\phi_1 = \psi_1 \\
\phi_2 = \psi_2 - (\phi_1, \psi_2)\phi_1. 
\]  

(7.8)
It is evident that

\[(\phi_2, \phi_1) = 0.\]  \hspace{1cm} (7.9)

This process is continued for each vector so that

\[\phi_n = \psi_n - \sum_{i=1}^{n-1} \left( \frac{\phi_i}{\psi_i} \frac{\psi_n}{\phi_i} \right) \phi_i.\]  \hspace{1cm} (7.10)

We still have, for \(i \neq j,\)

\[(\phi_i, \phi_j) = 0.\]  \hspace{1cm} (7.11)

Now the representation of an arbitrary function \(Q\) as a linear sum of orthogonal functions is a simple matter. We set

\[Q = y_1 \phi_1 + y_2 \phi_2 + \ldots.\]  \hspace{1cm} (7.12)

As shown by Churchill (22) the unknown coefficients \(y_i\) can be obtained by the expression

\[y_i = \frac{(Q, \phi_i)}{(\phi_i, \phi_i)}.\]  \hspace{1cm} (7.13)

It remains to replace the coefficients \(y_i\) by the desired coefficients \(x_i\). Consider a system of two functions \(\psi_1, \psi_2,\) which is orthogonalized to \(\phi_1, \phi_2.\) By equation (7.8) we have

\[\phi_1 = \psi_1\]
\[\phi_2 = \psi_2 - \left( \frac{\phi_1}{\psi_1} \frac{\psi_2}{\phi_1} \right) \psi_1.\]  \hspace{1cm} (7.14)
From (7.12),

\[ Q = y_1 \phi_1 + y_2 \phi_2 \quad (7.15) \]

\[ = y_1 \psi_1 + y_2 \left[ \psi_2 - \left( \frac{\psi_1}{\phi_1, \phi_1} \right) \psi_1 \right] \]

\[ = y_2 \psi_2 + \left[ y_1 - y_2 \left( \frac{\phi_1, \psi_2}{(\phi_1, \phi_1)} \right) \psi_1 \right]. \]

Since, from (7.1)

\[ Q = x_1 \psi_1 + x_2 \psi_2, \quad (7.16) \]

we have

\[ x_2 = y_2, \]

\[ x_1 = y_1 - \left( \frac{\phi_1, \psi_2}{(\phi_1, \phi_1)} \right) y_2. \quad (7.17) \]

Likewise for a system of three functions,

\[ x_3 = y_3. \]

\[ x_2 = y_2 - \left( \frac{\phi_2, \psi_3}{(\phi_2, \phi_2)} \right) y_3 \]

\[ x_1 = y_1 - \left( \frac{\phi_1, \psi_2}{(\phi_1, \phi_1)} \right) y_2 - \left[ \left( \frac{(\phi_1, \psi_3)}{(\phi_1, \phi_1)} \right) - \left( \frac{(\phi_2, y_2)}{(\phi_2, \phi_2)} \right) \right] y_3. \]

In general, for a system of \( n \) functions

\[ x_n = y_n. \]

\[ x_{n-1} = y_{n-1} - \left( \frac{\phi_{n-1}, \psi_n}{(\phi_{n-1}, \phi_{n-1})} \right) y_n. \]
This technique was programmed for the IBM 7090 digital computer and applied to the aforementioned problem with disappointing results. The Gram-Schmidt process permitted solution for up to twelve parameters. With double precision it was possible to solve for sixteen coefficients. Beyond this point round-off error again destroyed the value of the solutions.

The reason for the continued numerical difficulty is now obvious. In the generation of the orthogonal vector system each vector is determined from a consideration of all previous orthogonalized vectors. Thus small errors in \( \Phi_2 \) produce larger errors in \( \Phi_3 \), still larger errors in \( \Phi_4 \), and so on until the whole system degenerates into a set of meaningless numbers.

Davis and Rabinowitz (21) confirmed the limitations of the Gram-Schmidt method and proposed a superior alternative. Assume

\[
x_i = y_i - \left( \frac{\phi_i}{\phi_i, \phi_i} \right) y_{i-1} - \left[ \frac{(\phi_i, \psi_{i+2})}{(\phi_i, \phi_i)} \right] y_{i+2} - \left[ \frac{(\phi_i, \psi_{i+3})}{(\phi_i, \phi_i)} \right] y_{i+3} - \cdots
\]

(7.19)
that the original system of vectors is $1, x, x^2, \ldots, x^{M-1}$. This system may be orthogonalized into polynomials $P_n(x)$ while $P_n^*(x)$ will be normal. Let

$$P_{-1} = 0, \quad P_0 = 1$$ (7.20)

$$P_{n+1} = xP_n^* - (xP_n^*, P_n^*)P_n^* - (P_n, P_n)^{1/2}P_n^{*\ast}$$

where $P_n^* = \frac{P_n}{(P_n, P_n)^{1/2}}$.

This three term recurrence relationship demonstrates substantial improvement over the Gram-Schmidt process. In a sample problem undertaken by Davis and Rabinowitz the single precision recurrence method was still good at $n = 20$, while the Gram-Schmidt method with double precision broke down at $n = 16$.

Useful numerical solutions to many shell problems would require a large number of parameters, 40 or more. None of the methods discussed show promise of satisfying the need. In the next chapter an iterative regression routine is set forth which appears to be the solution required.
Chapter 8

Multiple Regression

It occurs that some of the deficiencies in the orthogonality processes described in Chapter 7 are caused by the order of appearance of the vectors $\psi_i$. If, for example, one tries to orthogonalize two vectors which correlate very highly with each other, the resultant second vector is of very small magnitude. Obviously subsequent steps in the process will be of dubious accuracy. A logical improvement would be to reorder the vectors in order to minimize this effect.

The numerical difficulties previously detailed make a practical computer solution to the least squares problem difficult, if not impossible, of achievement. An alternative method of computing the required displacement coefficients is the multiple regression analysis developed by Efroymson (24). This does not really invert the matrix $[E]$ of equation (7.4) but provides a pseudo-inverse containing essentially all the practical information to be extracted from the matrix.
This process consists of solving the problem, of fitting an equation to a number of points, in stepwise fashion. In this case for example the desired equation is

\[ Q = \psi_1 x_1 + \psi_2 x_2 + \psi_3 x_3 + \ldots + \psi_n x_n, \quad (8.1) \]

where \( x_1, \ldots, x_n \) are the coefficients to be determined. Now equation (8.1) can be developed a step at a time. Thus

\[ Q = \psi_1 x_1, \]

\[ Q = \psi_1 x_1' + \psi_2 x_2', \quad (8.2) \]

\[ Q = \psi_1 x_1'' + \psi_2 x_2''' + \psi_3 x_3''', \]

\[ \vdots \]

It is assumed here that \( \psi_1, \psi_2, \ldots, \psi_n \) are ordered according to the improvement in fit obtained by equations (8.2). Obviously this is not normally the case. The method selects each new vector to be added to the equation as well as computing the best values of the coefficients at each step.

The method commences by normalizing the matrix \( \begin{bmatrix} E_j \end{bmatrix} \) to obtain the matrix of correlation coefficients \( r_{ij} \). The load vector \( Q \) is also normalized and cross-correlations computed with the displacement function vectors. This vector is added at the right and
on the bottom to produce a square matrix of order \( s + 1 \). There are \( s \) coefficients so that a typical correlation with load is \( r_{i, s + 1} \).

First the largest value of \( r_{i, s + 1} \) is determined. This identifies the vector \( \psi_k \) having the highest correlation with the load \( Q \). Or, in other words, of all the vectors \( \psi_i, \psi_k x_k \) most nearly defines \( Q \). In this case

\[
x_k = r_k, s + 1 \frac{\sigma_{s+1}}{\sigma_k},
\]

where \( \sigma_i = (\psi_i, \psi_i)^{1/2} \),

\[
\sigma_{s+1} = (Q, Q)^{1/2}.
\]

Now the matrix \( [r] \) is modified by an exchange step as defined by Stiefel (25). Consider a set of equations

\[
y_1 = ax_1 + bx_2
\]
\[
y_2 = cx_1 + dx_2.
\]

Assuming \( a \neq 0 \) we can rewrite this set as follows:

\[
x_1 = \frac{1}{a} y_1 - \frac{b}{a} x_2
\]
\[
y_2 = \frac{c}{a} y_1 + \left( d - \frac{bc}{a} \right) x_2.
\]

This step merely exchanges the roles of the previously independent variable \( x_1 \) and dependent variable \( y_1 \). The equations may be stated in matrix notation:

\[
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix} =
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix},
\]
\[
\begin{bmatrix}
x_1 \\
y_2
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{a} & -\frac{b}{a} \\
c & d - \frac{bc}{a}
\end{bmatrix}
\begin{bmatrix}
y_1 \\
x_2
\end{bmatrix}.
\]

\[
\begin{bmatrix}
\frac{1}{a} & -\frac{b}{a} \\
c & d - \frac{bc}{a}
\end{bmatrix}
\begin{bmatrix}
y_1 \\
x_2
\end{bmatrix}.
\]

\[
(8.7)
\]
It is obvious that $x_2$ can be exchanged with $y_1$ by another such step, thus providing the inverse to the original matrix of coefficients.

For a more general case assume a system of four equations in four unknowns:

$${}\begin{align*}
\{ y_1 \} &= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix} \begin{bmatrix} x_1 \end{bmatrix}
\{ y_2 \} &= \begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \begin{bmatrix} x_2 \end{bmatrix}
\{ y_3 \} &= \begin{bmatrix} a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \begin{bmatrix} x_3 \end{bmatrix}
\{ y_4 \} &= \begin{bmatrix} a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_4 \end{bmatrix}
\end{align*}$$

An exchange of $y_3$ and $x_3$ results in the following matrix of coefficients:

$$\begin{bmatrix}
\alpha_{11} & \alpha_{12} & \frac{a_{13}}{a_{33}} & \alpha_{14} \\
\alpha_{21} & \alpha_{22} & \frac{a_{23}}{a_{33}} & \alpha_{24} \\
\frac{-a_{31}}{a_{33}} & \frac{-a_{32}}{a_{33}} & \frac{1}{a_{33}} & \frac{-a_{34}}{a_{33}} \\
\alpha_{41} & \alpha_{42} & \frac{a_{43}}{a_{33}} & \alpha_{44}
\end{bmatrix}$$

where, for example,

$$\alpha_{11} = a_{11} - \frac{a_{31}}{a_{33}} a_{13},$$

$$\alpha_{12} = a_{12} - \frac{a_{32}}{a_{33}} a_{13},$$

$$\alpha_{44} = a_{44} - \frac{a_{34}}{a_{33}} a_{43}.$$
The rules are fourfold:

1. Replace the pivot element by its reciprocal value.
2. Divide other elements of the pivot column by the pivot element.
3. Divide other elements of the pivot row by the pivot element and change sign.
4. Each remaining element is transformed by considering the rectangle of four matrix elements which contains the element itself in one corner and the pivot element in the corner diagonal to that. The elements forming the co-diagonal of the rectangle are multiplied, the product divided by the pivot element, and the result subtracted from the element in question.

This process can be continued until a complete inverse is obtained.

After one exchange step the matrix $[r]$ is transformed to a matrix $[r']$ in accordance with the rules stated above. It is desired to identify the next most significant variable. The vector of load correlations $r_{i, s+1}$ is normalized as follows:

$$V_i = \frac{r_{i, s+1} - r_{i, s+1, s+1}}{r_{i, i} - r_{s+1, s+1}}$$  (8.10)
Those variables not yet in the equation will have positive values of $V_i$. Rule (3) above causes $V_i$ to be negative for variables that have been considered. The desired variable $k$ corresponds to the largest positive value $V_k$.

Each step consists of selecting the variable to be added to the equation and then transforming the matrix $[r]$ by means of an exchange step pivoting on the diagonal term corresponding to that variable. At each step an equation can be written in which the coefficients are computed by means of equation (8.3). For example at the end of $n$ steps there are $n$ variables in the equation. They are identified as those variables for which $V_i$ is negative. The displacement coefficients $x_i$ are

$$x_i = r_{i,s+1} \frac{\sigma_{s+1}}{\sigma_i} \tag{8.11}$$

The equation (7.1) has at this point the form

$$\sum_i \psi_i x_i = Q, \tag{8.12}$$

where the summation contains $n$ terms.

As the process continues the matrix $[E]$ is progressively modified to approach the matrix $[E]^{-1}$. Given a computer of sufficient precision the inverse could actually be attained. In general, however, the usual numerical difficulties intrude and prevent a solution. It can be shown, for example, that the diagonal terms in
the normalized matrix can never be negative. Actual numerical solutions frequently find some of these terms going negative.

A certain method to check the validity of the regression process is to compare actual and computed values at each step. The main disadvantage is the considerable increase in elapsed computer time caused thereby. A reasonable compromise is to commence this process at the first step after a diagonal term is computed to be negative. The standard deviation at each step is the square root of the sum of the errors squared at each point. When this value ceases to decrease it can be assumed that the process is, for all practical purposes, complete. In general, subsequent steps show the standard deviation increasing rapidly.

It has been shown that a solution obtained by terminating at the first relative minimum is not always optimum. For example a hemisphere with clamped boundaries was subjected to an anti-symmetric \((m = 1)\) normal load of value \(\sin \theta\) (loosely called a wind load). In a solution assuming 48 displacement terms a relative minimum standard deviation of .0350 was computed after 32 steps. However, when the process was carried to 39 steps the standard deviation was computed to be .0308.
It appears that termination at the first relative minimum would not be too inaccurate. Increased familiarity with the method should enable one to anticipate better the behavior of the solution. Perhaps ultimately the number of steps allowed can be set in advance and the costly practice of making repeated comparisons will be discarded.

Obviously the result is not a pure least squares solution. Those terms which have not been added to the equation do not figure further in solution for displacements and forces. Their values remain zero. The standard deviation has been essentially minimized as it is determined on the particular computer in use. For this reason it is stated that a pseudo-least squares solution has been obtained.

At this point the coefficients $x_2$ of Chapter 6 have been computed. They are substituted back into equation (6.6) to yield the remaining coefficients $x_1$. Now all displacements and forces may be computed by the equations of Chapter 5.
PART 4
RESULTS AND CONCLUSIONS

Chapter 9
Numerical Solutions

Consider first the case of a complete sphere of 750 foot diameter. This is a sandwich structure with a total depth of 6.28 inches and face sheet thickness 0.1 inch. The modulus of elasticity \((E)\) is \(29 \times 10^6\) psi and Poisson's ratio 0.3. With a uniform normal load of 1.0 pounds per square inch (psi.) we have the sphere behaving as a pure membrane.

A solution having 60 coefficients (10 A, 10 B, 20 E, 20 F) was assumed for this problem. From equations (5.11) we have five regularity equations at each pole, causing the number of independent parameters to be reduced to 50. The regression analysis optimized the system in just nine steps. Five additional coefficients were computed from the regularity coefficients (the other five were zero). The total of 14 coefficients is distributed as follows - 3B, 6E, 5F. The agreement of computed and actual values is almost perfect, as shown by Figures 7 and 8.
Figure 7

Complete Sphere - Tangential Pressure Comparison
Figure 8

Complete Sphere - Normal Pressure Comparison
The theoretical value for direct force per unit length for this problem is

\[ \frac{pR}{2} = 2250 \text{ lbs.} \quad (9.1) \]

Maximum deviation of \( T_1 \) or \( T_2 \) from this number is 0.19 lbs. for \( T_2 \) at \( \theta = 90^\circ \). The normal deflection is

\[ w = (1 - \nu) \frac{pR^2}{2 Et} = 1.222 \text{ inch.} \quad (9.2) \]

This is the computed value for \( w \) at all points. As would be expected the computed values for tangential displacement, moment, and shear are of such small magnitude as to be trivial. All displacements and internal forces are shown in Figures 9 through 15.

Next we examine the behavior of a hemisphere with its edge clamped against displacement or rotation. This is comparable to a cantilever support. Its radius is 200 inches and the cross section is one inch solid. Other values are - \( E \) of \( 30 \times 10^6 \) psi and \( \nu \) of 0.3.

This structure was analyzed for the case of a constant unit normal pressure (\( q_n = 1.0 \text{ psi.} \)). Based on membrane theory the direct force is 500 lbs. per inch and the normal displacement is 0.1167 inch. The method of computation for the edge effect corrections is set forth quite clearly by Flügge (20).
Figure 9

Complete Sphere - Meridional Deflection
Figure 10

Complete Sphere - Normal Deflection
Figure 11

Complete Sphere Meridional Direct Force
Figure 12

Complete Sphere - Circumferential Direct Force
Figure 13

Complete Sphere - Meridional Bending Moment
Figure 14

Complete Sphere - Circumferential Bending Moment
Figure 15

Completed Sphere - Meridional Transverse Shear
The best solution obtained assumed 32 displacement coefficients - 12 A and 20 F - and evaluated for pressures at 37 points - 2.5° intervals. Standard deviation of the computed loading from the applied load is .030 psi. in this case. Tables 1 and 2 contain a comparison of "exact" and computed values for internal load and displacement. It is probable that the inclusion of B and E coefficients would produce a substantial decrease in the error of computed quantities.

<table>
<thead>
<tr>
<th>w (inch)</th>
<th>T₁ (lb./inch)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>.01167</td>
</tr>
<tr>
<td>Computed</td>
<td>.0120</td>
</tr>
</tbody>
</table>

Table 1

Apex Values
Clamped Hemisphere with Unit pressure

<table>
<thead>
<tr>
<th>T₁ (lb./inch)</th>
<th>T₂ (lb./inch)</th>
<th>M₁ (in.-lb./inch)</th>
<th>V₁ (lb./inch)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>508</td>
<td>150</td>
<td>105</td>
</tr>
<tr>
<td>Computed</td>
<td>510</td>
<td>150</td>
<td>47.7</td>
</tr>
</tbody>
</table>

Table 2

Edge Values
Clamped Hemisphere with Unit pressure
Now apply an anti-symmetrical load \((m = 1)\) to the same structure as follows:

\[ q_n = \sin \theta \cos \phi. \]  

This loading is loosely denoted a wind load, to which it corresponds roughly in practice. A solution was obtained in which 48 deflection parameters were assumed - 8A, 10B, 8C, 10D, and 12E - and 37 points were considered. All three deflection components and three applied loads appear in this case with a standard deviation of 0.0308 psi. The membrane solutions were computed and a comparison is shown in Table 3. Computation of edge effects by the method of hypergeometric series was abandoned because of the numerical complexity involved.

It is seen that the results agree quite closely with membrane theory for \(\phi\) up to 60°. As the edge is approached of course the membrane solutions alone are inadequate. \(T_1\) and \(T_2\) are evaluated at \(\phi = 0\), \(S\) at \(\phi = 90°\).
<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$T_1$ lb./inch</th>
<th>$T_2$ lb./inch</th>
<th>$S$ lb./inch</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Exact</td>
<td>0.9</td>
<td>-1.6</td>
<td>-0.5</td>
</tr>
<tr>
<td>Computed</td>
<td>12.8</td>
<td>39.0</td>
<td>13.2</td>
</tr>
<tr>
<td>15°</td>
<td>10.9</td>
<td>39.8</td>
<td>11.6</td>
</tr>
<tr>
<td>Exact</td>
<td>23.8</td>
<td>76.2</td>
<td>27.4</td>
</tr>
<tr>
<td>Computed</td>
<td>24.0</td>
<td>73.9</td>
<td>27.5</td>
</tr>
<tr>
<td>30°</td>
<td>110.4</td>
<td>43.8</td>
<td>31.0</td>
</tr>
<tr>
<td>Exact</td>
<td>113.4</td>
<td>44.0</td>
<td>30.4</td>
</tr>
<tr>
<td>Computed</td>
<td>141.1</td>
<td>64.1</td>
<td>32.0</td>
</tr>
<tr>
<td>45°</td>
<td>142.0</td>
<td>63.7</td>
<td>31.6</td>
</tr>
<tr>
<td>60°</td>
<td>59.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>75°</td>
<td>166.7</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3
Clamped Hemisphere with Wind Load
Finally this structure was subjected to a load of second mode:

\[ q_n = \sin \theta \cos 2 \phi \]  
\[ \text{(9.4)} \]

A solution was obtained with 32 displacement coefficients - 10 A, 10 C, and 12 E - and 19 points. The standard deviation of computed loads from applied loads was 0.0485 psi.

A last case to be discussed is that of a spherical shell with tangential edge loading only. A solution to this case is given by Flügge (20) for membrane forces only as follows:

\[ T_{1m} = - T_{2m} = \frac{1}{\sin^2 \theta} \left( A_m \cot^m \theta \frac{\Theta}{2} + B_m \tan^m \theta \frac{\Theta}{2} \right), \]
\[ S_m = \frac{1}{\sin^2 \theta} \left( A_m \cot^m \theta \frac{\Theta}{2} - B_m \tan^m \theta \frac{\Theta}{2} \right), \]  
\[ \text{(9.5)} \]

where \( A_m \) and \( B_m \) are constants to be determined.

Rewrite these equations:

\[ T_{1m} = - T_{2m} = \frac{\sin^2 \Theta_0}{\sin^2 \theta} \left( A_m \cot^m \frac{\Theta_0}{2} + B_m \tan^m \frac{\Theta_0}{2} \right), \]
\[ S_m = \frac{\sin^2 \Theta_0}{\sin^2 \theta} \left( A_m \cot^m \frac{\Theta_0}{2} - B_m \tan^m \frac{\Theta_0}{2} \right), \]  
\[ \text{(9.6)} \]

where \( \Theta = \Theta_0 \) at the loaded boundary. Assume that direct force \( (T_{1m}) \) only is applied at the edge. Then we can say that

\[ S_m (\Theta_0) = 0, \]

\[ A_m = B_m \]  
\[ \text{(9.7)} \]
and simplify further to

\[ T_{lm} = -T_{2m} = A_m \frac{\sin^2 \theta_o}{\sin^2 \theta} \left( \cot^m \frac{\theta}{2} + \tan^m \frac{\theta}{2} \right), \]

\[ S_m = A_m \frac{\sin^2 \theta_o}{\sin^2 \theta} \left( \cot^m \frac{\theta}{2} - \tan^m \frac{\theta}{2} \right), \quad (9.8) \]

If we investigate the case of axisymmetric edge loading \( (m = 0) \) we have

\[ T_{lm} = -T_{2m} = 2A_m \frac{\sin^2 \theta_o}{\sin^2 \theta}, \]

\[ S_m = 0 \quad (9.9) \]

A doubly truncated shell, with edges at \( \theta_1 = 45^\circ \) and \( \theta_2 = 120^\circ \), was analyzed for edge load. A constant edge load \( T_1 = 100 \text{ lb.}/\text{inch} \) was applied at \( \theta_1 \) and the shell clamped at \( \theta_2 \). The expression for direct force becomes, according to membrane theory,

\[ T_{10} = -T_{20} = \frac{50}{\sin^2 \theta} \quad (9.10) \]

This indicates a symmetry of values about \( \theta = 90^\circ \) with a minimum value \( T_{10} = 50 \) at \( \theta = 90^\circ \).

The displacements were assumed to contain 20 parameters - 4 A, 4B, 6E, 6F - and equations for load were written at 31 points - 2.5° intervals. With six boundary conditions fourteen coefficients were available to fit the applied load; of which the regression selected ten. Results are depicted in Figures 16 to 24. It is seen that
Figure 16

Truncated Sphere - Constant Edge Load
Tangential Pressure Comparison
Figure 17

Truncated Sphere - Constant Edge Load
Normal Pressure Comparison
Figure 18
Truncated Sphere - Constant Edge Load
Meridional Deflection
Figure 19

Truncated Sphere - Constant Edge Load
Normal Deflection
Figure 20

Truncated Sphere - Constant Edge Load
Meridional Direct Force
Figure 21

Truncated Sphere - Constant Edge Load
Circumferential Direct Force
Figure 22

Truncated Sphere - Constant Edge Load
Meridional Bending Moment
Figure 23

Truncated Sphere - Constant Edge Load
Circumferential Bending Moment
Figure 24

Truncated Sphere - Constant Edge Load
Meridianal Transverse Shear
membrane loads have been modified somewhat by edge effects at the bottom. The minimum value is 38 lbs./inch, not 50, and appears at about $\theta = 100^\circ$, not $\theta = 90^\circ$. $T_1$ and $T_2$ can be seen to be nearly equal in magnitude, with opposite sign, down to $\theta = 100^\circ$.

The small errors in edge forces are balanced by residual normal and tangential pressures on the surface, which ensure overall equilibrium.

The case of $m = 1$ is a special case and will not be considered here. For higher modes we find that closing the shell at top or bottom necessitates the discarding of $A_m$ or $B_m$, respectively, to assure finite solutions at the poles. If the shell is doubly truncated, however, both terms remain in the expressions for membrane force. With $m = 2$ we have, from (9.8),

$$T_{12} = - T_{22} = A_2 \frac{\sin^2 \theta}{\sin^2 \frac{\theta}{2}} \left( \frac{\cot^2 \frac{\theta}{2}}{\cot^2 \frac{\theta}{2} + \tan^2 \frac{\theta}{2}} \right),$$

$$S_2 = A_2 \frac{\sin^2 \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}} \left( \frac{\cot^2 \frac{\theta}{2}}{\cot^2 \frac{\theta}{2} - \tan^2 \frac{\theta}{2}} \right). \quad (9.11)$$

For the same structure as above ($\theta = 45^\circ$) this becomes

$$T_{12} = - T_{22} = \frac{A_2}{2 \sin^2 \frac{\theta}{2}} \left( 0.17157 \cot^2 \frac{\theta}{2} + 5.82836 \tan^2 \frac{\theta}{2} \right),$$

$$S_2 = \frac{A_2}{2 \sin^2 \frac{\theta}{2}} \left( 0.17157 \cot^2 \frac{\theta}{2} - 5.82836 \tan^2 \frac{\theta}{2} \right). \quad (9.12)$$
The values of T_{12} and S_{2} for different values of $\theta$ are listed in Table 4.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>45°</th>
<th>60°</th>
<th>75°</th>
<th>90°</th>
<th>105°</th>
<th>120°</th>
</tr>
</thead>
<tbody>
<tr>
<td>T_{12}</td>
<td>2</td>
<td>1.638</td>
<td>2.00</td>
<td>3.0</td>
<td>5.359</td>
<td>11.695</td>
</tr>
<tr>
<td>S_{2}</td>
<td>0</td>
<td>-0.952</td>
<td>-1.683</td>
<td>-2.828</td>
<td>-5.250</td>
<td>-11.619</td>
</tr>
</tbody>
</table>

Table 4

Relative Values of Internal Force by Membrane Theory

The table indicates that forces increase with increasing distance from the loaded edge. This is directly contrary to the St. Venant principle of elasticity. Obviously membrane theory is inadequate in this case.

A loading of second mode ($T_{1} = 100 \cos 2 \phi$) was applied to the edge $\theta_{1} = 45^\circ$ and apparently realistic results obtained. Plotted values are shown in Figures 25 to 38. Values of $q_{2}, v, S, H,$ and $V_{2}$ are plotted at $\phi = 90^\circ$. All other values are plotted at $\phi = 0^\circ$. Here the direct force $T_{1}$ decreases continually to the fixed edge. The shearing force $S$ rises to a maximum value at about $\theta = 90^\circ$ and then decreases until the edge effect at $\theta = 120^\circ$ becomes predominant. It would appear that analysis of a shell with
Figure 25

Truncated Sphere - Statically Null Edge Load
Meridional Pressure Comparison
Figure 26

Truncated Sphere - Statically Null Edge Load
Circumferential Pressure Comparison
Figure 27

Truncated Sphere - Statically Null Edge Load
Normal Pressure Comparison
Figure 28

Truncated Sphere - Statically Null Edge Load
Meridional Deflection
Figure 29

Truncated Sphere - Statically Null Edge Load
Circumferential Deflection
Figure 30

Truncated Sphere - Statically Null Edge Load
Normal Deflection
Figure 31

Truncated Sphere - Statically Null Edge Load
Meridional Direct Force
Figure 32

Truncated Sphere - Statically Null Edge Load
Circumferential Direct Force
Figure 33

Truncated Sphere - Statically Null Edge Load
Meridional Bending Moment
Figure 34

Truncated Sphere - Statically Null Edge Load
Circumferential Bending Moment
Figure 35

Truncated Sphere - Statically Null Edge Load
In-Surface Shear
Figure 36

Truncated Sphere - Statically Null Edge Load
Twisting Moment
Figure 37

Truncated Sphere - Statically Null Edge Load
Meridianal Transverse Shear
Figure 38

Truncated Sphere - Statically Null Edge Load
Circumferential Transverse Shear
a greater range between $\theta_1$ and $\theta_2$ would demonstrate more completely the phenomenon of complete disappearance of stress with increased distance from the edge. Alternatively, all edge forces at $\theta_2$ can be set equal to zero since the applied load at $\theta_1$ is statically null.
Chapter 10

Conclusions

It has been demonstrated, by comparison with so-called "exact" solutions, that the method of assumed displacement functions is feasible for the analysis of spherical shells under arbitrary loading. Additional accuracy in the computation of internal forces and displacements should be forthcoming with increased knowledge of the behavior of the computer solutions. It is anticipated that increasing the number of terms in the displacement series would tend to reduce the error. Much remains yet to be learned about the behavior of the solutions as the distribution of the terms among \( A, B, \ldots, F \) is varied.

A simple extension of the theory to vibration analysis can be made. Consider equation (7.1)

\[
[\psi']\{x\} = \{Q\},
\]

where

\[
\{Q\} = -[m\omega^2] \{\delta\},
\]

and

\[
\{\delta\} = \text{displacements}.
\]

\( Q_i \) is the dynamic load at point \( i \) and \( x_i \) is the \( i \)th displacement coefficient. The vector \( \{x\} \) has been reduced to reflect boundary conditions. Displacements, also modified by boundary conditions,
may be expressed as

$$\begin{bmatrix} \delta \end{bmatrix} = \begin{bmatrix} \zeta \end{bmatrix} \begin{bmatrix} x \end{bmatrix} \quad (10.2)$$

Substituting (10.2) into (10.1) produces

$$\begin{bmatrix} \psi \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = - \begin{bmatrix} m & \omega^2 \end{bmatrix} \begin{bmatrix} \zeta \end{bmatrix} \begin{bmatrix} x \end{bmatrix},$$

or

$$\left( \begin{bmatrix} \psi \end{bmatrix} + \begin{bmatrix} m & \omega^2 \end{bmatrix} \begin{bmatrix} \zeta \end{bmatrix} \right) \begin{bmatrix} x \end{bmatrix} = 0 \quad (10.3)$$

This can be rewritten

$$\begin{bmatrix} D \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = 0. \quad (10.4)$$

The matrix \([D]\) is not square but it can be premultiplied by its transpose to give

$$\begin{bmatrix} D \end{bmatrix}^T \begin{bmatrix} D \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = 0 \quad (10.5)$$

of

$$\begin{bmatrix} E \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = 0$$

The result is a classical eigenvalue problem. Equation (10.5) may be solved readily for the eigenvalues, \(\omega^2\), by various methods. Most feasible for computer solution are matrix iteration or Jordan rotation to obtain a diagonal matrix. For each eigenvalue the corresponding eigenvector \(\{x\}\) may be computed by assuming a value of, say \(x_1\), and solving for the others. The actual mode shapes may be established by substituting \(\{x\}\) into equation (10.2).

In Chapter 5 equations were derived for general shells of revolution, with particular application to cylindrical and conical shells, as well as spherical shells. The cylinders and cones especially can be programmed for computer solution with ease. Less
usual configurations may be analyzed directly or may be approximated as combinations of cylindrical, conical, and spherical shell segments. In any case the extension is, conceptually speaking, trivial.

Many shells encountered in practice are not isotropic, as was assumed. Frequently their construction is orthotropic, primarily due to the presence of stiffeners. If the stiffeners are oriented in the coordinate directions we have principal elastic characteristics coinciding with principal curvatures and coordinates, a convenient situation. In this case four elastic constants \( E_{\alpha}, E_{\beta}, \nu_{\alpha\beta}, G \) are required instead of two \( E \) and \( \nu \). The writer has derived equations for an orthotropic sphere. Their complexity is not inordinately greater than those of Chapter 5. Extension to this case appears a reasonable next step.

In Chapter 4 the discussion was restricted to the case of rotationally closed shells of revolution. Assume that this is relaxed to permit boundaries along meridians, \( \phi = \text{constant} \). Each such edge will have four boundary conditions, making a total of sixteen when considering a segment of shell having four edges. Now the expressions for displacement and loading can no longer be written as in equations (5.1) and (5.2), thus permitting separation of variables.
The problem of a shell of revolution having four edges can be attacked by an extension of the methods of Part 2. Briefly, one must consider all of the modes \( m \) of equations (5.1) and (5.2), together with all of the surface of the shell segment. The method of Part 2 could be likened to a single integration along the zero meridian. Now a complete integration over the surface is necessary, requiring a simultaneous solution for all displacement coefficients.

The trigonometric displacement functions of equations (5.4) appear to be suitable for solution of all problems encountered to date. It appears likely that some displacement function could be assumed such that the system of vectors \( \{ \psi \} \) of equation (7.1) would be more nearly mutually orthogonal. In such a case the determination of the displacement coefficients would be greatly simplified. A quest for such a set of functions seems to be worthwhile.

An additional area for future investigation consists of thin shells not possessing the property of axial symmetry. In this case, of course, the matrices \( [L] \) and \( [N] \) of Chapter 4 would be much more complex. Even so the method of solution would correspond to that set forth above for the axi-symmetric shell segment having four edges. An orthogonal coordinate system would be required with boundaries defined by one coordinate value only.
Ultimately one would hope that this method could be extended to the solution of problems in three-dimensional elasticity.
REFERENCES


135


Appendix

Regularity Conditions for a Spherical Shell

Consider the displacement-equilibrium equations (5.6) for a sphere at the points $\theta = 0$ and $\theta = \pi$. All values for $\sin i \theta$, $\sin j \theta$, $\sin k \theta$ are zero at these points. A frequently occurring indeterminate form is $\frac{\sin i \theta}{\sin \theta}$. This may be evaluated by l'Hospital's rule.

\[
\frac{\sin i \theta}{\sin \theta} \to \frac{i \cos i \theta}{\cos \theta}, \quad (A.1)
\]

At $\theta = 0$,

\[
\frac{\sin i \theta}{\sin \theta} \to i, \quad (A.2-a)
\]

At $\theta = \pi$,

\[
\frac{\sin i \theta}{\sin \theta} \to -i (-1)^i, \quad (A.2-b)
\]

Another simplification is, for $\theta = 0$ or $\theta = \pi$,

\[
\cot^2 \theta \frac{\cos^2 \theta}{\sin^2 \theta} \to \frac{1}{\sin^2 \theta}, \quad (A.3)
\]

Now rewrite equations (5.6) in light of the above.

\[
\frac{1}{R^2} \left\{ -\sum \left[ \frac{m^2(1-U)}{2 \sin \theta} + 2 \frac{A_{\text{im}} \sin i \theta}{\sin \theta} + \left( i^2 + \frac{m^2(1-U)}{2 \sin^2 \theta} + U \right) \right] 
\right. 
\]

$B_{\text{im}} \cos i \theta$

138
\[-i \cot \theta \quad A_{im} \cos i \theta + i \cos \theta \quad B_{im} \quad \frac{i \theta}{\sin \theta} \]

\[-m \sum_j \left[ \frac{(3-\nu) \cot \theta}{2} \left( C_{jm} \frac{\sin j \theta}{\sin \theta} + D_{jm} \frac{\cos j \theta}{\sin \theta} \right) \right. \]

\[-\left( \frac{1+\nu}{2} \right) \left( C_{jm} \frac{\cos j \theta}{\sin \theta} - D_{jm} \frac{\sin j \theta}{\sin \theta} \right) \]

\[+ K \sum_k \left[ \frac{(k^2 - 2m^2)}{\sin^2 \theta} \right] \cos \theta \left( E_{km} \frac{\sin k \theta}{\sin \theta} + F_{km} \frac{\cos k \theta}{\sin \theta} \right) \]

\[+ k \left( \frac{1+\nu}{K} + \frac{k^2 + 1}{\sin \theta} + \frac{(1-\nu^2)}{2} + \frac{m^2 + 1}{\sin^2 \theta} \right) E_{km} \cos k \theta \quad (A.4-a) \]

\[- \left( k \frac{(m^2 + 1)}{\sin \theta} + k(1+\nu) \right) F_{km} \frac{\sin k \theta}{\sin \theta} \right] = - \frac{(1-\nu^2)}{\text{Et}} q_{2m} \]

\[\frac{1}{R^2} \left\{ -m \sum_i \left[ \frac{(3-\nu) \cot \theta}{2} \left( A_{im} \frac{\sin i \theta}{\sin \theta} + B_{im} \frac{\cos i \theta}{\sin \theta} \right) \right. \]

\[+ i \left( \frac{1+\nu}{2} \right) \left( A_{im} \frac{\cos i \theta}{\sin \theta} - B_{im} \frac{\sin i \theta}{\sin \theta} \right) \]

\[+ \sum_j \left[ -\frac{(2m^2 + 1 - \nu^2)}{2 \sin \theta} C_{jm} \frac{\sin j \theta}{\sin \theta} + \frac{(1-\nu^2)}{2} \right] \left( 1 - \cot^2 \theta \right. \]

\[\left. - \frac{j^2}{\sin^2 \theta} \right) \left( 1 - \cot^2 \theta \right) \]

\[+ \frac{(1-\nu)}{2} \cos \theta \quad j \left( C_{jm} \frac{\cos j \theta}{\sin \theta} - D_{jm} \frac{\sin j \theta}{\sin \theta} \right) \]

\[-m \sum_k \left[ \frac{(k^2 + \frac{m^2}{\sin^2 \theta} + \frac{1+\nu}{K}) \left( E_{km} \frac{\sin k \theta}{\sin \theta} + F_{km} \frac{\cos k \theta}{\sin \theta} \right) \right. \]

\[+ \cot k \left( \frac{E_{km} \cos k \theta}{\sin \theta} - \frac{F_{km} \sin k \theta}{\sin \theta} \right) \left. \right\} \]

\[= - \frac{(1-\nu^2)}{\text{Et}} q_{2m} \quad (A.4-b) \]
Now the values of \( q_{1m}, q_{2m}, q_{nm} \) are given at \( \theta = 0 \) and \( \theta = \pi \). In any real problem they have finite magnitudes at all points.

Hence we must require the coefficients in equations (A.4) to obey such relations that the loads do not become infinite. For example, in the
first term of equation (A.4-a) $1/ \sin \theta$ becomes infinite at $\theta = 0$ or $\theta = \pi$. Therefore, considering this term alone leads one to require

$$\sum_i A_{im} \frac{\sin i \theta}{\sin \theta}$$

to be zero at these points.

Examine each set of coefficients in equation (A.4-a).

We have

$$A_{im} \cos i \theta \left[ \frac{m^2(1 - \frac{1}{\sin^2 \theta}) - 2 \cos \theta}{2 \sin \theta \cos \theta} - \cot \theta \right] = A_{im} \frac{\cos i \theta}{\cos \theta} \frac{m^2(1 - \frac{1}{\sin^2 \theta})}{2 \sin \theta},$$

$$B_{im} \cos i \theta \left[ \frac{i^2 + \nu + m^2(1 - \nu) + 2}{2 \sin^2 \theta} + i^2 \right],$$

$$C_{jm} j \cos j \theta \left[ \frac{3 - \nu}{2} \frac{\cot \theta}{\cos \theta} - \frac{(1 + \nu)}{2} \frac{1}{\sin \theta} \right] = C_{jm} j \cos j \theta \frac{(1 - \nu)}{\sin \theta},$$

$$D_{jm} \cos j \theta \left[ \frac{(3 - \nu)}{2} \frac{\cot \theta}{\sin \theta} + \frac{(1 + \nu)}{2} \frac{j^2}{\cos \theta} \right],$$

$$E_{km} k \cos k \theta \left[ \frac{k^2 - 2m^2}{\sin^2 \theta} + 1 + \frac{(1 + \nu)}{K} + k^2 + \frac{m^2 + 1}{\sin^2 \theta} \right]$$

$$= E_{km} k \sin k \theta \left[ \frac{1 - m^2}{\sin^2 \theta} + 2k^2 + 1 + \frac{1 + \nu}{K} \right], \quad (A.5)$$

$$F_{km} \cos k \theta \left[ \frac{k^2 - 2m^2}{\sin^2 \theta} \right] \cos \theta \cos \theta$$

$$= F_{km} \cos k \theta \cos \theta \left[ \frac{k^2 (1 + \nu)}{\sin \theta} - \frac{k^2 m^2}{\sin^2 \theta} - \frac{2m^2}{\sin^3 \theta} \right].$$

Regularity requires the following relations:

$$\sum_i A_{i1} i \cos i \theta \cos \theta + 2 \sum_j C_{j1} j \cos j \theta = 0, \quad m = 1,$$

$$\sum_i A_{im} i \cos i \theta = 0, \quad m > 1.$$
\[ \sum_{i} B_{im} \cos i \theta = 0, \quad m \neq 1, \]
\[ \sum_{i} B_{i1} \cos i \theta + \sum_{j} D_{ij} \cos j \theta \cos \theta = 0, \quad m = 1, \]
\[ \sum_{j} C_{jm} j \cos j \theta = 0, \quad m > 1, \]
\[ \sum_{j} D_{jm} \cos j \theta = 0, \quad m > 1, \]
\[ \sum_{k} E_{km} k \sin k \theta = 0, \quad m \neq 1, \quad \text{(A.6)} \]
\[ \sum_{k} F_{km} \cos k \theta = 0, \quad m > 0, \]
\[ \sum_{k} F_{km} k^2 \cos k \theta = 0, \quad m > 0. \]

If equations (A.6) are satisfied then \( q_{lm} \) can be evaluated at the singularity points as follows:

\[ \sum_{i} 2i^2 B_{im} \cos i \theta + \sum_{j} (1+\nu) j^2 D_{ij} \cos j \theta \cos \theta = (1-\nu^2) \frac{R^2}{Et} q_{lm}. \quad \text{(A.7)} \]

To this expression the term \( \sum_{i} B_{i1} \cos i \theta \) is added when \( m = 1 \).

Now inspect the coefficients for equation (A.4-b). We have

\[ - \sum_{i} A_{im} i \cos i \theta \left[ \frac{3-\nu}{2} \cot \frac{\ell}{2} + \frac{1+\nu}{2} \frac{1}{\sin \ell} \right] + \frac{1}{\sin \ell} \]
\[ = - \sum_{i} A_{im} i \cos \frac{\ell}{2} \frac{2}{\sin \ell}, \]
The following regularity equations result:

\[ 2 \sum_i A_{ij} i \cos i \theta + \sum_j C_{jm} j \cos j \theta \cos \ell = 0, \quad m = 1, \]
\[ \sum_i A_{im} i \cos i \theta = 0, \quad m \neq 1, \]
\[ \sum_i B_{ij} \cos i \theta \cos \ell + \sum_j D_{ij} \cos j \ell = 0, \quad m = 1, \]
\[ \sum_i B_{im} \cos i \theta = 0, \quad m \neq 1, \]
\[ \sum_j C_{jm} j \cos j \theta = 0, \quad m \neq 1, \]
\[ \sum_j D_{jm} \cos j \theta = 0, \quad m \neq 1, \]
\[ \sum_k E_{km} k \cos k \theta = 0, \quad m \neq 1, \]
\[ \sum_k F_{km} \cos k \theta = 0, \]
\[ \sum_k F_{km} k^2 \cos k \ell = 0. \]

The first of equations (A.6) and the first of equations (A.9) are contradictory. Hence we must say
\[ \sum_i A_{im} i \cos i \theta = 0, \]
\[ \sum_j C_{jm} j \cos j \ell = 0. \]

With the satisfaction of equations (A.9) we have, at \( \theta = 0 \) and \( \theta = \pi \):
\[ m \sum_i B_{im} i^2 \cos i \theta \left( \frac{1+\nu}{2} \right) \cos \theta - \sum_j D_{jm} 2 j^2 \cos j \theta \left( \frac{1-\nu}{2} \right) \]
\[ -m K \sum_k E_{km} k^3 \cos k \ell \cos \ell = - \left( 1-\nu^2 \right) \frac{R^2}{E \ell} q_{2m}. \quad (A.10) \]

Add to this value, for \( m = 1 \)
\[ \sum_j D_{jm} \cos j \theta \left( \frac{1-\nu}{2} \right) + \sum_k E_{km} k \cos k \theta \left( 1+\nu \right) \cos \theta. \]

Now consider each term from equation (A.4-c):
\[ \sum_i A_{im} i \cos i \theta \left[ 1 + \nu + K \left( 2 i^2 + \frac{m^2 (3+\nu)}{2 \sin^2 \theta} - 2 \right) \sin \theta \right] \]
\[ - 1 + \nu + K \left( i^2 + \frac{2 + m^2 (3-\nu)}{2 \sin^2 \theta} \right), \]
\[
\frac{m^4}{\sin^4 \theta} \left( - k^2 \left\{ 2 k^2 - \nu + \frac{2m^2 - 1}{\sin^2 \theta} \right\} \right)
= \sum_k F_{km} \cos k \ell \left[ 2(1 + \nu) + K \left\{ -k^4 + k^2 \right\} \right.
- m^2 \left\{ 1 + \nu + \frac{5}{\sin^2 \theta} \right\} + \frac{m^4}{\sin^4 \theta} \right].
\]

For regularity the following relations must hold:

\[
\sum_i A_{im} \cos i \ell = 0, \quad m > 0,
\]
\[
\sum_i B_{im} \cos i \ell \cos \ell + \sum_j D_{jm} \cos j \ell = 0, \quad m = 1,
\]
\[
\sum_i B_{im} \cos i \theta = 0, \quad m \neq 1,
\]
\[
\sum_i B_{im} i^2 \cos i \ell = 0,
\]
\[
\sum_j C_{jm} j \cos j \theta = 0, \quad (A.12)
\]
\[
\sum_j D_{jm} \cos j \theta = 0, \quad m > 1,
\]
\[
\sum_k E_{km} k \cos k \ell = 0,
\]
\[
\sum_k E_{km} k^3 \cos k \ell = 0,
\]
\[
\sum_k F_{km} \cos k \ell = 0, \quad m > 0.
\]
The last of equations (A. 6) and (A. 8) provide one additional relation.

It is

\[ \sum_k F_{km} k^2 \cos k \theta = 0. \]  
\[ \text{(A. 13)} \]

Now rewrite the equations for applied load at the poles.

They become:

\[ \sum_j D_{jm} j^2 \cos j \left( \frac{1 + j^2}{2} \right) \cos \theta = \left( 1 - \nu^2 \right) \frac{R^2}{E_t} q_{1m}, \]
\[ \sum_j D_{jm} j^2 \cos j \left( 1 - \nu \right) = (1 - \nu^2) \frac{R^2}{E_t} q_{2m}, \]  
\[ \text{(A. 14)} \]

\[ \sum_i A_{im} 3 i^3 \cos i \theta - \sum_k F_{km} k^4 \cos k \theta = \frac{R^2(1 - \nu^2)}{E_t K} q_{nm}. \]

Finally the internal loads must be evaluated at the poles.

With the imposition of equations (A. 12) and (A. 13) equations (5. 9) become:

\[ T_{1m} = \frac{E_t}{R(1 - \nu^2)} \left[ \sum_i A_{io} i \cos i \theta + \sum_k F_{km} \cos k \theta \right] (1 + \nu), \]
\[ m = 0, \]
\[ T_{2m} = T_{1m} \]
\[ S_m = 0 \]
\[ M_{1m} = \frac{E_t K}{1 - \nu^2} \left[ \sum_i A_{im} \cos i \theta \right] (1 + \nu), \]
\[ M_{2m} = M_{1m} \]  
\[ \text{(A. 15)} \]
In the process of computing $V_{2m}$ at the poles one additional relationship is found.

It is

$$\sum_j D_{jm} j^2 \cos j \ell = 0.$$  \hspace{1cm} (A.16)