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DISSERTATION

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A measure space is a system \( \{S, \mathcal{M}, \mu\} \) consisting of a set \( S \), a \( \sigma \)-field \( \mathcal{M} \) of subsets of \( S \), and a \( \sigma \)-finite complete measure \( \mu \) defined on \( \mathcal{M} \). By a transformation from a measure space \( \{S, \mathcal{M}, \mu\} \) onto a measure space \( \{S', \mathcal{M}', \mu'\} \) we mean a single valued function \( T \) with domain \( S \) and range \( S' \) under which the measurability properties of the two spaces are related; the precise hypotheses are set forth as H1-H8 by Reichelderfer in [5] and summarized in Chapter II. A weight function for \( T \) is a function associating with each point \( s' \) in \( S' \) and every set \( D \) in a certain subset \( \mathcal{D} \) of \( \mathcal{M} \) a non-negative extended real number \( W'(s', T, D) \) subject to the following conditions: \( W'(s', T, D) \) is zero at points \( s' \) not in \( TD \); for every \( s' \) in \( S' \), \( W'(s', T, D) \) is under additive and inner continuous on \( \mathcal{D} \); and for each \( D \) in \( \mathcal{D} \), \( W'(s', T, D) \) is measurable \( \mathcal{M}' \) on \( S' \). The value of the integral of \( W'(s', T, D) \) with respect to the measure \( \mu' \) on the set \( S' \) is termed the weight \( WD \) attached to \( D \). The transformation \( T \) is said to be of bounded variation with respect to the weights \( WD \)--briefly \( BVWD \)--if \( WD \) is a finite real number for
every $D$ in $\mathcal{D}$. The transformation $T$ is said to be absolutely continuous with respect to the weights $W$--briefly ACWD--if $T$ is $BV$ and there exists a non-negative extended real valued function $f$ defined on $\mathcal{S}$ and measurable $\mathcal{M}$ such that

$$\int f \, d\mu = \int W'(s', T, D) \, d\mu',$$

for every $D$ in $\mathcal{D}$. The function $f$ is termed a greatest lower bound function for the weights $W$--briefly a glbfWD--and is unique in the sense that two functions satisfying the above relations may differ only on a set of measure zero. For transformations $T$ which are ACWD Reichelderfer has shown in [5] that a transformation theory is always available in the following sense. Let $H'$ be a real valued function defined on $\mathcal{S}'$ and measurable $\mathcal{M}'$. Then for every $D$ in $\mathcal{D}$, $H'(s')W'(s', T, D)$ is measurable $\mathcal{M}'$ on $\mathcal{S}'$ and $H'T(s)f(s)$ is measurable $\mathcal{M}$ on $D$. If either $H'(s')W'(s', T, D)$ is integrable $\mu'$ on $\mathcal{S}'$ or $H'T(s)f(s)$ is integrable $\mu$ on $D$ then both are integrable and the transformation formula,

$$\int H'T(s)f(s) \, d\mu = \int H'(s')W'(s', T, D) \, d\mu',$$

holds.

In this paper we let $T$ be a transformation from the measure space $\{\mathcal{S}, \mathcal{M}, \mu\}$ onto the space $\{\mathcal{S}', \mathcal{M}', \mu'\}$, we assume that $T$ is ACWD with the glbfWD $f(s)$. We let $T'$ be a transformation from the measure space $\{\mathcal{S}', \mathcal{M}', \mu'\}$ onto a measure
space \([\mathbb{S}^n, \mathbb{M}^n, \mu^n]\); we assume that \(T'\) is \(ACW'D'\) and that \(g'(s')\) is a \(g_{lbf}W'D'\). We let the transformation \(T''\) be the product, \(T'T\), of \(T\) and \(T'\), and we define a weight function \(W''(s'', T'', D)\), \(s''\) in \(\mathbb{S}^n\), \(D\) in \(\mathcal{D}\), for \(T''\) in terms of the weight functions for \(T\) and \(T'\). The main purpose of this paper is to find conditions under which the product \(T''\) will be absolutely continuous with respect to the weights \(W''D\).

In Chapter VI we consider the case in which \(T'\) is one-to-one; in Chapter VII we consider the case in which \(T'\) is essentially locally one-to-one; in Chapter VIII we consider the case in which \(T\) is essentially locally one-to-one; and in Chapter IX we assume that the weight functions for both \(T\) and \(T'\) belong to a certain special class of weight functions described by Reichelderfer in [5]. In each of these cases we obtain under suitable conditions the following result: A necessary and sufficient condition for \(T''\) to be \(ACW'D\) is that \(T''\) be \(BVW''D\); \(T''\) is \(BVW''D\) if and only if the product, \(f(s)g'T(s)\), of the greatest lower bound functions for the weights for \(T\) and \(T'\) is integrable \(\mu\) on \(\mathbb{S}\); moreover, if \(T''\) is \(ACW''D\) then \(f(s)g'T(s)\) is a greatest lower bound function for the weights \(W''D\).

Helsel and Levine [3] have established that the product of two plane transformations, each absolutely continuous in the sense of Banach, is absolutely continuous in the sense of Banach if and only if the product of the
derivatives is summable over the domain of the first trans­formation, and that if the product transformation is abso­lutely continuous then its derivative is the product of the derivatives of the factor transformations almost everywhere. Reichelderfer [6] has given a theorem which is a generali­zation of this result to functions defined in Euclidean n-space with values in n-space, and in Chapter X of the present paper we indicate how this theorem may be obtained as a special case of the results given in Chapter IX.

In this paper in addition to discussing the product of transformations we also develop in Chapter III an exten­sion of the transformation theory given in [5]. In the extended transformation theory weights are assigned to all of the sets in a σ-field which contains the class $\mathcal{F}$ as a subclass. The relationship between the extended transfor­mation theory and the original transformation theory is described. We make use of our extension of the transfor­mation theory in obtaining certain of our results for the product transformation.

In this paper certain of the functions will be extended real valued. Throughout the paper we will use the convention $0 \cdot \infty = 0$. 
CHAPTER II

THE TRANSFORMATION THEORY

In this chapter we describe the transformation theory which is the basis for the results obtained in this paper. Each statement in this chapter has been taken from Reichelderfer's paper, *A transformation theory for measure space* [5], and has been labeled with its number in that paper. Some changes in notation have been made in order to distinguish statements concerning the transformation $T$ from statements to be made in succeeding chapters concerning transformations $T'$ and $T''$.

**Standard hypotheses for $T$.** The following hypotheses describe the transformation $T$.

- **H1T.** $[S, \mathcal{M}, \mu]$ is a $\sigma$-finite complete measure space.
- **H2T.** $[S', \mathcal{M}', \mu']$ is a $\sigma$-finite complete measure space.
- **H3T.** $T$ is a single valued transformation from $S$ onto $S'$.
- **H4T.** $\mathcal{D}$ is a collection of subsets $D$ of $S$ having the following properties: The empty set, $\emptyset$, and the space $S$ belong to $\mathcal{D}$. $\mathcal{D}$ is a subset of $\mathcal{M}$, and $T\mathcal{D}$—that is,
\{TD;D \text{ in } \mathcal{D}\} -- is a subset of \(\mathcal{M}'\). If \(1D\) and \(2D\) belong to \(\mathcal{D}\) then there is a countable number of pairwise disjoint sets \(D_1\) in \(\mathcal{D}\) such that \(1D \cap 2D = UD_1\). \(\mathcal{S}\) can be expressed as the union of a countable number of sets \(\mathcal{D}_j\) in \(\mathcal{D}\) such that \(\mu \mathcal{D}_j\) is finite for every \(j\), and if \(M\) in \(\mathcal{M}\), \(M \subset \mathcal{D}_j\) for some \(j\), then for every positive real number \(\varepsilon\) there is a countable number of pairwise disjoint sets \(D_1\) in \(\mathcal{D}\) such that \(M \subset UD_1\) and \(\Sigma uD_1 < \mu M + \varepsilon\).

Definition 1.1T. An element \(D_0\) in \(\mathcal{D}\) is said to be of type \(\gamma T\) if it is one of a countable number of pairwise disjoint sets \(D_1, i \geq 0\), in \(\mathcal{D}\) for which there are two subsets \(Y, Z\) of \(\mathcal{S}\) such that

\[
\mathcal{S} = Y \cup Z \cup U \mathcal{D}_1, \ i \geq 0;
\]

\(Y\) in \(\mathcal{M}\), \(\mu Y = 0\); \(TZ\) in \(\mathcal{M}'\), \(\mu' TZ = 0\).

H5T. If \(D\) is an element in \(\mathcal{D}\) there is a countable sequence of sets \(\mathcal{D}_j\) in \(\mathcal{D}\) such that each \(\mathcal{D}_j\) is of type \(\gamma T\), \(\mathcal{D}_j \subset \mathcal{D}_{j+1}\) for every \(j\), and \(UD_j = D\).

H6T. \(\mathcal{S}'\) is a \(\sigma\)-field of subsets \(B'\) of \(\mathcal{S}'\) having the following properties. \(\mathcal{S}'\) is a subset of \(\mathcal{M}'\) and \(T^{-1}\mathcal{S}'\) is a subset of \(\mathcal{M}\). For each element \(M'\) in \(\mathcal{M}'\) there are sets \(B'_1\) and \(B'_2\) in \(\mathcal{S}'\) such that \(B'_1 \subset M' \subset B'_2\) and \(\mu B'_1 = \mu B'_2\).

Definition 1.2T. Denote by \(\mathcal{O}'\) the class of subsets \(O'\) of \(\mathcal{S}'\) for each of which there is a countable number of pairwise disjoint sets \(D_1\) in \(\mathcal{D}\) such that \(T^{-1}O' = UD_1\).

H7T. \(\mathcal{S}'\) can be expressed as the union of a countable
number of sets $\star 0_j$ in $S'$ such that $\mu' \star 0_j$ is finite for every $j$, and if $M'$ in $M'$, $M' \subseteq \star 0_j$ for some $j$, $\mu' M' = 0$, then for every positive real number $\varepsilon$ there is a set $0'$ in $S'$ such that $M' \subseteq 0'$ and $\mu' 0' < \varepsilon$.

Definition 1.3T. An element $0'$ in $S'$ is said to be of type $\gamma'T$ if it is one of a countable number of pairwise disjoint sets $0_j$, $i \geq 0$, in $S'$ for which there are two subsets $Y'$, $Z'$ of $S'$ such that

$$S' = Y' \cup Z' \cup \cup 0_j, i \geq 0;$$

$$T^{-1} Y' \mbox{ in } M, \mu T^{-1} Y' = 0; Z' \mbox{ in } M', \mu' Z' = 0.$$  

Definition 1.4T. An element $M'$ in $M'$ is said to be of type $\iota'T$ if there is a countable sequence of sets $0_j$ in $S'$ such that each $0_j$ is of type $\gamma'T$, $0_j \subseteq 0_{j+1}$ for every $j$, and $\cup 0_j = M'$.

H8T. If $B'$ is an element of $S'$ there is a countable sequence of sets $M_j$ in $M'$ and two sets $U'$, $V'$ in $M'$ such that each $M_j$ is of type $\iota'T$, $M_j \supseteq M_{j+1}$ for every $j$, $\mu' U' = 0$, $\mu' V' = 0$, and $\cap M_j \cup U' = B' \cup V'$.

H9T. $W'(s', T, D)$, $s'$ in $S'$, $D$ in $D$, is a non-negative extended real valued function satisfying:

i) If $D$ is in $D$ and $s'$ is in $C'TD$ then $W'(s', T, D)$ is equal to zero, where $C'$ denotes complementation in $S'$.

ii) For each $s'$ in $S'$, $W'(s', T, D)$ is under additive on $D$—that is, if a set $D$ contains a countable number of pairwise disjoint sets $D_j$ then $\Sigma W'(s', T, D_j) \leq W'(s', T, D)$. 

iii) For each $s'$ in $\mathcal{S}'$, $W'(s', T, D)$ is inner continuous on $\mathcal{S}$, that is, if a set $D$ is the union of a countable number of sets $D_j$ such that $D_j \subseteq D_{j+1}$ for every $j$, then 
\[ \lim_{j \to \infty} W'(s', T, D_j) = W'(s', T, D). \]

iv) For each $D$ in $\mathcal{S}$, $W'(s', T, D)$, $s'$ in $\mathcal{S}'$, is measurable $\mu'$.

A function having these properties is termed a weight function for $T$.

**Bounded variation with respect to the weights.** The statements in this section are from Chapter 2 of [5]. $H_{IT-H_{9T}}$ are assumed throughout.

**Definition 2.1T.** For each $D$ in $\mathcal{S}$ set 
\[ WD = \int_{\mathcal{S}'} W'(s', T, D) d\mu'. \]

Thus, to each set $D$ in $\mathcal{S}$ there is assigned a non-negative extended real number $WD$, termed the weight attached to $D$.

**Definition 2.2T.** The transformation $T$ is said to be of bounded variation with respect to the weights $WD$, briefly, BVWD, if $W_D$ is a finite real number. Equivalently, $T$ is BVWD if and only if for each $D$ in $\mathcal{S}$, $W'(s', T, D)$, $s'$ in $\mathcal{S}'$, is integrable $\mu'$.

**Definition 2.3T.** Let $f(s)$ be a non-negative extended real valued function defined on $\mathcal{S}$, measurable $\mu$, and such that for each $D$ in $\mathcal{S}$,
\[ \int_D f(s) d\mu \leq WD = \int_{\mathcal{S}'} W'(s', T, D) d\mu'. \]
Such an \( f(s) \) is termed a lower bound function for the weights \( WD \)--briefly, a \( lbfWD \).

**Lemma 2.7T.** Assume that \( T \) is BVWD, and \( f(s) \) is a \( lbfWD \). Suppose that \( E \) and \( F \) are subsets of \( \mathbb{S} \) satisfying the conditions: \( E \in \mathcal{M}, \mu E = 0, \mu' TF = 0, \) and \( EUF \) in \( \mathcal{M} \). Then \( f(s) = 0 \) a.e. \( \mu \) on \( EUF \), and \( \int_{EUF} f(s) \, d\mu = 0 \).

**Theorem 2.13T.** Assume that \( T \) is BVWD, and \( f(s) \) is a \( lbfWD \). Suppose that \( H'(s'), s' \in \mathbb{S}' \), is real valued and measurable \( \mathcal{M}' \). Then, for each \( D \in \mathcal{S}', H'(s')W'(s', T, D), s' \in \mathbb{S}' \), is measurable \( \mathcal{M}' \) and \( H'(s')f(s), s \in D, \) is measurable \( \mathcal{M} \).

**Absolute continuity with respect to the weights.**
The definition in this section is from Chapter 3 of [5]. \( H1T-H9T \) are assumed.

**Definition 3.4T.** Assume that \( T \) is BVWD and there exists a non-negative extended real valued function \( f \) defined on \( \mathbb{S} \) which is a \( lbfWD \) such that

\[
\int_{\mathbb{S}} f(s) \, d\mu = W_\mathbb{S} = \int_{\mathbb{S}'} W'(s', T, \mathbb{S}) \, d\mu'.
\]

Then the transformation \( T \) is said to be absolutely continuous with respect to the weights \( WD \)--briefly, ACWD; and \( f \) is said to be a greatest lower bound function for the weights--briefly, a \( glbfWD \).

**The transformation formula.** The statements in this section are from Chapter 4 of [5]. \( H1T-H9T \) are assumed.
Definition 4.2T. Assume that $T$ is ACWD, and that $f$ is a glbfWD. Let $H'$ be a real valued function defined on $\mathcal{S}'$ and measurable $\mathcal{M}'$. Fix $D$ in $\mathcal{A}$. By Theorem 2.13T the function $H'(s')W'(s', T, D)$, $s'$ in $\mathcal{S}'$, is measurable $\mathcal{M}'$ and the function $H'T(s)f(s)$, $s$ in $D$, is measurable $\mathcal{M}$. Denote by $\mathcal{H}(D)$ the set of such functions $H'$ for which the following conditions also hold:

1) $H'(s')W'(s', T, D)$, $s'$ in $\mathcal{S}'$, is integrable $\mu'$;

2) $H'T(s)f(s)$, $s$ in $D$, is integrable $\mu$;

3) $\int_D H'T(s)f(s)d\mu = \int_{\mathcal{S}'} H'(s')W'(s', T, D)d\mu'$.

The formula in 3) is termed a transformation formula for the function $H'$.

Theorem 4.10T. Assume that $T$ is ACWD, and $f$ is a glbfWD. Fix $D$ in $\mathcal{A}$. Suppose that $H'$ is a real valued function defined on $\mathcal{S}'$ and measurable $\mathcal{M}'$ for which either $H'(s')W'(s', T, D)$, $s'$ in $\mathcal{S}'$, is integrable $\mu'$ or $H'T(s)f(s)$, $s$ in $D$, is integrable $\mu$. Then $H'$ is in $\mathcal{H}(D)$.

A class of weight functions $W'$. The statements in this section are from Chapter 7 of [5]. H1T-H8T are assumed, but H9T is not assumed.

Definition 7.1T. With each point $s$ in $\mathcal{S}$ associate a subset $S_s$ of $\mathcal{S}$ by the formula

$$S_s = \cap D, \ s \ in \ D \ in \ \mathcal{A}.$$ 

Since $s$ in $\mathcal{S}$ in $\mathcal{A}$ by H4T these sets are well defined. Obviously $s$ in $S_s$. Let $\mathcal{S}$ denote the set of all subsets $S$
of $S$ for each of which there is a point $s$ in $S$ such that $S = S_s$.

**H11T.** If $s_1$ and $s_2$ are points in $S$ for which there is a set $D_0$ in $\mathcal{D}$ such that $s_1$ in $D_0$, $s_2$ not in $D_0$, then there are sets $D_1$ and $D_2$ in $\mathcal{D}$ such that $s_1$ is in $D_1$, and $s_2$ is in $D_2$, and $D_1 \cap D_2$ is empty.

**Lemma 7.2T.** Assume H11T. The sets in $\mathcal{D}$ constitute a partition of $S$—that is, the sets in $\mathcal{D}$ are non-empty, pairwise disjoint, and their union is $S$. If $S$ in $\mathcal{D}$ and $D$ in $\mathcal{D}$ are such that $S \cap D$ is not empty then $S \subseteq D$. For every $S$ in $\mathcal{D}$ one has $S = \cap D$, $S \subseteq D$ in $\mathcal{D}$.

**Definition 7.6T.** Assume H11T. Denote by $\mathcal{E}$ the set all subsets $E$ of $S$ which have the following property: If $S$ in $\mathcal{D}$ is such that $S \cap E$ is not empty then $S \subseteq E$.

It is readily seen that $\mathcal{E}$ has these properties. If $E$ is in $\mathcal{E}$ then $CE$ is in $\mathcal{E}$. The union and the intersection of the sets belonging to any subset of $\mathcal{E}$ are in $\mathcal{E}$. From Lemma 7.2T it is clear that $\mathcal{D} \subseteq \mathcal{E}$ and $\mathcal{E} \subseteq \mathcal{E}$. In particular $\mathcal{E}$ is a $\sigma$-field.

**H12T.** For each $S$ in $\mathcal{D}$ the set $TS$ consists of a single point in $S'$.

**Definition 7.7T.** Assume H11T, H12T. Let $w$ be a non-negative extended real valued function defined on $\mathcal{D}$. Denote by $\mathcal{D}^+$ the set consisting of those sets $S$ in $\mathcal{D}$ such that $wS > 0$, and put $S^+ = \cup S$, $S$ in $\mathcal{D}^+$. Clearly, $S^+$ in $\mathcal{D}$.
For each point $s'$ in $\mathcal{S}'$ and each set $E$ in $\mathcal{S}$ define

$$W^*_s(s', T, E) = \sum_{S \in \mathcal{S}} w_S, S \subseteq E, TS = s',$$

where the expression on the right stands for the lub of the set of numbers \( \{0, \sum_{k=1}^{p} w_{S_k}, S_k \in \mathcal{S}, S_k \subseteq E, TS_k = s'\} \).

Evidently $W^*_s(s', T, E)$, $s'$ in $\mathcal{S}'$, $E$ in $\mathcal{S}$, is a non-negative extended real valued function. Clearly, $W^*_s(s', T, E) = W^*_s(s', T, E \cap S^+)$, and $W^*_s(s', T, E) \geq 0$ if and only if $s'$ is in $T(E \cap S^+)$. 

**Lemma 7.8T.** Assume H11T, H12T, and let $w_S, S$ in $\mathcal{S}$, and $W^*_s(s', T, E)$, $s'$ in $\mathcal{S}'$, $E$ in $\mathcal{S}$, be as in Definition 7.7T. Then the following statements are valid.

1) If $E$ is in $\mathcal{S}$ and $s'$ is in $C'TE$ then $W^*_s(s', T, E)$ is equal to zero.

11) For each $s'$ in $\mathcal{S}'$, $W^*_s(s', T, E)$ is strongly additive on $\mathcal{S}$--that is, if $E$ is the union of any number of pairwise disjoint sets $E_a$ then $W^*_s(s', T, E) = \sum W^*_s(s', T, E_a)$.

1ii) For each $s'$ in $\mathcal{S}'$, $W^*_s(s', T, E)$ is both under additive and over additive on $\mathcal{S}$. By over additive we mean that if a set $E$ is contained in the union of a countable number of sets $E_j$ then $W^*_s(s', T, E) \leq \sum W^*_s(s', T, E_j)$.

iv) For each $s'$ in $\mathcal{S}'$, $W^*_s(s', T, E)$ is inner continuous on $\mathcal{S}$.

v) For each $s'$ in $\mathcal{S}'$, $W^*_s(s', T, E)$ is outer continuous on $\mathcal{S}$ provided the values involved are finite--that is, if a set $E$ is the intersection of a countable number of...
sets \( E_j \) such that \( E_j \supseteq E_{j+1} \) for every \( j \) and \( W'_*(s',T,E_j) < \infty \) for some \( j \) then \( \lim_{j \to \infty} W'_*(s',T,E_j) = W'_*(s',T,E) \).

Remark 7.9T. Assume \( H11T, H12T \), and let \( w_S, S \in \mathcal{J} \), be as in Definition 7.7T. Recall that \( \mathcal{J} \neq \emptyset \). Thus \( W'_*(s',T,D), s' \in S', D \in \mathcal{B} \), is a non-negative extended real valued function satisfying, according to Lemma 7.8T, conditions 1), 11), 11i) in \( H9T \). Thus \( W'_*(s',T,D), s' \in S, D \in \mathcal{B} \), is a weight function for \( T \) provided only that \( W'_*(s',T,D), s' \in S', \) is measurable \( \mathcal{M}' \) for every \( D \in \mathcal{B} \).

Definition 7.10T. Assume \( H11T, H12T \). Suppose that \( W'*(s',T,D), s' \in S', D \in \mathcal{B} \), is a weight function for \( T \) as defined in \( H9T \) for which there exists a non-negative extended real valued function \( w_S, S \in \mathcal{J} \), such that \( W'*(s',T,D) = W'_*(s',T,D), s' \in S', D \in \mathcal{B} \), where \( W'_*(s',T,E), s' \in S', E \in \mathcal{E} \), is as defined in Definition 7.7T. Then the weight function \( W'*(s',T,D), s' \in S', D \in \mathcal{B} \), is said to be generated by \( w_S, S \in \mathcal{J} \).

H13T. \( W'*(s',T,D), s' \in S', D \in \mathcal{B} \), is a weight function for \( T \) which is generated by a non-negative extended real valued function \( w_S, S \in \mathcal{J} \).

Theorem 7.24T. Assume \( H1T-H8T, H11T-H13T \), and that \( T \) is BVWD. Then \( T \) is ACWD if and only if for every set \( M \) in \( \mathcal{M} \) such that \( M \subseteq S^+ \) and \( \mu M = 0 \) it is true that \( TM \) is in \( \mathcal{M}' \) and \( \mu'TM = 0 \).
CHAPTER III

AN EXTENSION OF THE TRANSFORMATION THEORY

Throughout this chapter $H1T - H8T$, $H11T$, and $H12T$ are always assumed; but $H13T$ is assumed only where explicitly stated.

Recall that the class $\mathcal{G}$ as described in Definition 7.6T is a $\sigma$-field containing $\mathcal{G}$. Since $\mathcal{G} \subseteq \mathcal{M}$ it follows that the class $\mathcal{G} \cap \mathcal{M}$ is a $\sigma$-field containing $\mathcal{G}$. Consider the weight function $W'(s', T, D)$, $s'$ in $\mathcal{S}'$, $D$ in $\mathcal{D}$, and the key role which this weight function plays in the transformation theory. In the present chapter our purpose is to describe an extended transformation theory in which a corresponding role is played by an extended weight function $W'_*(s', T, E)$, $s'$ in $\mathcal{S}'$, $E$ in $\mathcal{G} \cap \mathcal{M}$.

1 Definition. Let $W'_*(s', T, E)$, $s'$ in $\mathcal{S}'$, $E$ in $\mathcal{G} \cap \mathcal{M}$, be a non-negative extended real valued function; we shall term $W'_*$ an extended weight function for the transformation $T$ in case the following conditions are satisfied:

1) If $E$ is in $\mathcal{G} \cap \mathcal{M}$ and $s'$ is in $\mathcal{G}' \cup \mathcal{E}$ then $W'_*(s', T, E) = 0$.

iii) For each $s'$ in $\mathcal{S}'$, $W'_*(s', T, E)$ is under additive on $\mathcal{G} \cap \mathcal{M}$. 

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iii) For each $s'$ in $S'$, $W_*(s', T, E)$ is inner continuous on $\delta \cap m$.

iv) For each $E$ in $\delta \cap m$, $W_*(s', T, E)$, $s'$ in $S'$, is measurable $m'$.

Remark. If we let $W_*(s', T, E)$, $s'$ in $S'$, $E$ in $\delta$, denote the function which is described in Definition 7.7T, then by Lemma 7.8T conditions i), ii), iii), of the definition in .1 are satisfied; and hence to show that $W_*(s', T, E)$, $s'$ in $S'$, $E$ in $\delta \cap m$, is an extended weight function for $T$ it is sufficient to show that for every $E$ in $\delta \cap m$, $W_*(s', T, E)$, $s'$ in $S'$, is measurable $m'$.

.2 Definition. Let $W_*(s', T, E)$, $s'$ in $S'$, $E$ in $\delta \cap m$, be an extended weight function for $T$. Let

$$ W_*E = \int_{S'} W'_*(s', T, E) . $$

We term $W_*E$ an extended weight attached to $E$.

.3 Definition. The transformation $T$ is said to be of bounded variation with respect to the extended weights, $W_*E$—briefly, $T$ is BVW$_*E$—if $W_*S$ is a finite real number.

.4 Definition. Let $f$ be a non-negative extended real valued function defined on $S$, measurable $m$, and such that for each $E$ in $\delta \cap m$,

$$ \int_E f(s) dm \leq W_*E = \int_{S'} W'_*(s', T, E) dm' . $$

Such an $f$ is termed a lower bound function for the extended weights—briefly a lbfW$_*E$.
.5 Definition. Assume that $T$ is $BV_{\#E}$ and that there exists a real valued function $f$ defined on $\mathcal{S}$ which is a $lbf_{\#E}$ such that

$$\int_{\mathcal{S}} f(s) \, du = W_{\#E} \mathcal{S} = \int_{\mathcal{S}'} W'_{\#}(s', T, \mathcal{S}) \, du'.$$

Then the transformation $T$ is said to be absolutely continuous with respect to the extended weights $W_{\#E}$—briefly, $ACW_{\#E}$—and $f$ is said to be a greatest lower bound function for the weights $W_{\#E}$—briefly, a $glbf_{\#E}$.

Throughout the remainder of this chapter we let $W'_{\#}(s', T, \mathcal{E}), s' \in \mathcal{S}', \mathcal{E} \in \mathcal{E}$, denote the function which is described in Definition 7.7T. In what follows we need further definitions and results from Chapter 7 of [5], and for convenience these are restated below.

**Definition 7.12T.** Let $\mathcal{E}^*$ denote the class of all sets $\mathcal{E}^*$ in $\mathcal{E}$ such that $W'_{\#}(s', T, \mathcal{E}^*), s' \in \mathcal{S}'$, is measurable $\mathcal{M}'$. Let $W_{\#E^*} = \int_{\mathcal{S}'} W'_{\#}(s', T, \mathcal{E}^*) \, du'$.

**Lemma 7.13T.** Assume H13T. The following statements are valid for the class of sets $\mathcal{E}^*$:

1) For any countable number of pairwise disjoint sets $\mathcal{E}^*_i$ in $\mathcal{E}$ it is true that $\cup \mathcal{E}^*_i$ is in $\mathcal{E}$ and $W_{\#\cup \mathcal{E}^*_i}$ is equal to $\Sigma W_{\#\mathcal{E}^*_i}$.

2) For any countable sequence of sets $\mathcal{E}^*_j$ in $\mathcal{E}$ such that $\mathcal{E}^*_j \subset \mathcal{E}^*_{j+1}$ for every $j$ it is true that $\cup \mathcal{E}^*_j$ is in $\mathcal{E}$ and $W_{\#\cup \mathcal{E}^*_j} = \lim W_{\#\mathcal{E}^*_j}$. 
Lemma 7.15T. Assume $H_{13T}$. Assume that $T$ is BVWD. Then these statements are valid.

i) If $E^*$ is in $\mathcal{E}^*$ then so is $CE^*$, and $W_#CE^*$ is equal to $WS - W_*E^*$.

ii) For any countable sequence of sets $E^*_j$ in $\mathcal{E}^*$ such that $E^*_j \supseteq E^*_j+1$ for every $j$ it is true that $\cap E^*_j$ is in $\mathcal{E}^*$ and also, we have $W_#E^*_j = \lim W_#E^*_j$.

Lemma 7.17T. Given a double sequence of sets $D_{k1}$ in $\mathcal{D}$ such that for each $k$ the sets $D_{k1}^*, \mathfrak{l} = 1$, are pairwise disjoint, there exists a double sequence of sets $D_{k1}$ in $\mathcal{D}$ such that for each $k$ the sets $D_{k1}, \mathfrak{l} = 1$, are pairwise disjoint, $U_1D_{k1} \supseteq U_1D_{k+1}$ for each $k$, and $\cap_k U_1D_{k1} = \cap_k U_1D_{k1}$.

Lemma 7.18T. Assume $H_{13T}$. Assume that $T$ is BVWD. Then for every double sequence of sets $D_{k1}$ in $\mathcal{D}$ such that for each $k$ the sets $D_{k1}, \mathfrak{l} = 1$, are pairwise disjoint, the set $\cup_k U_1D_{k1}$ is in $\mathcal{E}^*$.

Definition 7.19T. Assume $H_{13T}$. Let $\mathcal{E}^{**}$ denote the set of all subsets $E^{**}$ of $\mathcal{S}$ for each of which there exists a double sequence of sets $D_{k1}$ in $\mathcal{D}$ such that for each $k$ the sets $D_{k1}, \mathfrak{l} = 1$, are pairwise disjoint and $E^{**} = \cap_k U_1D_{k1}$.

According to the lemma in 7.17T one may also assume that $U_1D_{k1} \supseteq U_1D_{k+1}$ for every $k$. Clearly $\mathcal{E} \subseteq \mathcal{E}^{**}$. If $T$ is BVWD then $\mathcal{E}^{**} \subseteq \mathcal{E}^*$ by the lemma in 7.18T.

6 Lemma. Let $E_1^*$ and $E_2^*$ be sets in $\mathcal{E}^*$ such that $E_1^* \subseteq E_2^*$, then $W_#E_1^* \leq W_#E_2^*$. 
Proof. From Definition 7.7T it is clear that for
$s'$ in $\mathcal{S}'$ and $E_1, E_2$ in $\mathcal{E}$ such that $E_1 \subseteq E_2$, we have

$$W^*(s', T, E_1) \leq W^*(s', T, E_2).$$

Hence

$$W^*(E_1) = \int_{S'} W^*(s', T, E_1) \, d\mu' \leq \int_{S'} W^*(s', T, E_2) \, d\mu' = W^*(E_2).$$

7 Lemma. Assume H13T. Assume that $T$ is BVWD with $lbfWD f(s)$. Then for each $E^{**}$ in $\mathcal{E}^{**}$ we have

$$\int f(s) \, d\mu \leq W^*(E^{**}) = \int_{S'} W^*(s', T, E^{**}) \, d\mu'.$$

Proof. Let $E^{**}$ in $\mathcal{E}^{**}$. By Definition 7.19T

$$E^{**} = \cap_{k} U_1 D_{k1}$$

where $D_{k1}, i \geq 1, k \geq 1$, is a sequence of sets of $\mathcal{E}$ such that for each $k$ the sets $D_{k1}, i \geq 1$, are pairwise disjoint.

Since $\mathcal{E} \subseteq \mathcal{M}$ it follows that $E^{**}$ is in $\mathcal{M}$, so that

$$\int f(s) \, d\mu$$

exists. According to the observation following Definition 7.19T, $E^{**}$ is in $\mathcal{E}^*$ so that $W^*(E^{**})$ is defined.

Clearly each $D_{k1}$ is in $\mathcal{E}^{**} \subseteq \mathcal{E}^*$, and for each $k$

$$U_1 D_{k1}$$

is in $\mathcal{E}^{**} \subseteq \mathcal{E}^*$. According to the observation following Definition 7.19T we may also assume that for each $k$

$$U_1 D_{k1} = U_1 D_{k+11},$$

and hence we may apply Lemma 7.15T ii) to obtain

$$W^*(E^{**}) = W^*[\cap_{k} (U_1 D_{k1})] = \lim_{k} W^*(U_1 D_{k1}).$$

Also, by the lemma in .6, $W^*$ is monotone on $\mathcal{E}^*$ so that

$$W^*(E^{**}) = \lim_{k} W^*(U_1 D_{k1}) = \text{glb}\{W^*(U_1 D_{k1}) : k \geq 1\}.$$

For each $k$, $E^{**} \subseteq U_1 D_{k1}$; from $T$ BVWD with $lbfWD f(s)$ and H13T we have

$$\int f(s) \, d\mu \leq WD_{k1} = W^*D_{k1}$$

for each $k, i$; and by

$$D_{ki}$$
Lemma 7.13T 1) \( W_* \) is additive on \( \delta^* \); therefore for each \( k \),
\[
\int_{E^{**}} f(s) \, du \leq \int_{\bigcup D_{ki}} f(s) \, du
\]
\[
= \Sigma_i \int_{D_{ki}} f(s) \, du
\]
\[
\leq \Sigma_i W_D_{ki}
\]
\[
= \Sigma_i W_* D_{ki}
\]
\[
= W_* (\bigcup D_{ki}).
\]
Thus \( \int_{E^{**}} f(s) \, du \) is a lower bound for the set \( \{ W_* (\bigcup D_{ki}) : k \geq 1 \} \).

Combining this result with (1) we obtain
\[
\int_{E^{**}} f(s) \, du \leq W_* E^{**} = \int_{E^{**}} W_* (s', T, E^{**}) \, du'.
\]

3 Lemma. Assume H13T. Assume that \( T \) is ACWD and \( f(s) \) is a glbfWD. Then for each \( E \) in \( \delta \cap \mathcal{M} \), we have \( E \in \delta^* \); and
\[
\int_{E} f(s) \, du \leq W_* E = \int_{E'} W_* (s', T, E) \, du'.
\]

Proof. Case 1) Recall the countable family of sets of finite measure, \( *D_j, j \geq 1 \), described in H4T. Let \( E \) in \( \delta \cap \mathcal{M} \) be such that for some \( j \), \( E \subseteq *D_j \). Let \( k \) be a positive integer. By H4T we may choose a sequence of pairwise disjoint sets \( D_{ki} \), \( i \geq 1 \), in \( \delta \) such that \( E \subseteq \bigcup D_{ki} \) and \( \Sigma_i \mu D_{ki} < \mu E + 1/k \). Define the set \( E^{**} \) by \( E^{**} = \bigcap_k \bigcup D_{ki} \). It follows that \( E^{**} \) is in \( \mathcal{M} \), \( E \subseteq E^{**} \), \( \mu E = \mu E^{**} \), and since the sets are of finite measure \( \mu (E^{**} \cap CE) = 0 \).

Note that \( E^{**} \) is in the class \( \delta^{**} \) and recall that \( \delta^{**} \subseteq \delta^* \), so that \( W_* (s', T, E^{**}) \), \( s' \) in \( S' \), is measurable \( \mathcal{M}' \). Observe that since \( \delta \) is a \( \sigma \)-field the set \( E^{**} \cap CE \) is in \( \delta \),
and recall that $W_\ast$ is additive on $\mathcal{S}$. Consequently, we have

$$W_\ast(s',T,E) + W_\ast(s',T,E\#\cap CE) = W_\ast(s',T,E\#).$$

Let $S^+ = US$, $S$ in $\mathcal{F}$, $wS > 0$, as in Definition 7.7T. Recall that by Theorem 7.24T, $T$ is ACWD if and only if for every set $M$ in $\mathcal{M}$ such that $M \subset S^+$ and $\mu M = 0$ it is true that $\mu' TM = 0$. The set $E\#\cap CE\#S^+$ is in $\mathcal{M}$ by HIT; $E\#\cap CE\#S^+$ is a subset of $S^+$; clearly $\mu(E\#\cap CE\#S^+) = 0$; and consequently $\mu'(E\#\cap CE\#S^+) = 0$. Since $W_\ast(s',T,E\#\cap CE) > 0$ if and only if $s'$ is in $T(E\#\cap CE\#S^+)$, we conclude that $W_\ast(s',T,E\#\cap CE) = 0$, a.e. $\mu'$ on $S'$. Using this result and (1) above we have a.e. $\mu'$ on $S'$,

$$W_\ast(s',T,E) = W_\ast(s',T,E\#).$$

Consequently $W_\ast(s',T,E)$, $s'$ in $S'$, is measurable $\mathcal{M}'$, i.e., $E$ is in the class $\mathcal{S}^\#$. Moreover,

$$W_\ast E = \int_{S'} W_\ast(s',T,E) d\mu' = \int_{S'} W_\ast(s',T,E\#) d\mu' = W_\ast E\#.$$

Finally, from $E$, $E\#$ in $\mathcal{M}$, $E \subset E\#$, and $f(s)$ non-negative on $S$, we obtain, by applying the lemma in .7,

$$\int_E f(s) d\mu \leq \int_{E\#} f(s) d\mu \leq W_\ast E\# = W_\ast E.$$

Case 11) Let $E$ be in $\mathcal{S} \cap \mathcal{M}$. Let $S$ be expressed as the union of a countable number of sets $\ast D_j$ with the properties described in H4T. Define a sequence of pairwise disjoint sets $E_j$, $j \geq 1$, whose union is $E$, by $E_1 = \ast D_1 \cap E$, $E_2 = \ast D_2 \cap CE_1$, ..., $E_j = \ast D_j \cap CE_1 \cap CE_2 \cap \ldots \cap CE_{j-1}$. $\mathcal{S} \cap \mathcal{M}$ is a $\sigma$-field containing $\mathcal{S}$, so that for each $j$ the set $E_j$ satisfies the
hypothesis of Case 1) above, so that $E_j$ is in $\mathfrak{A}^*$ for every $j$. By Lemma 7.13T, $E = \sum E_j$, $j \geq 1$, is in $\mathfrak{A}$ and also $W^*E = \sum W^*E_j = W^*(\sum E_j)$, $j \geq 1$. Combining these observations with the conclusions for Case 1), we have

$$\int f(s) d\mu = \int f(s) d\mu = \sum \int f(s) d\mu \leq \sum W^*E_j = W^*E.$$

9 Remark. Let $W^*_{s'}(s',T,E)$, $s'$ in $\mathfrak{S}'$, $E$ in $\mathfrak{A}$, denote the non-negative extended real valued function described in Definition 7.7T. Assume H13T. From the remark following the definition in .1 and from the lemma in .8, it follows that if $T$ is ACWD then the function $W^*_{s'}(s',T,E)$, $s'$ in $\mathfrak{S}'$, $E$ in $\mathfrak{A}$, is an extended weight function for $T$. From H13T, $W^*_{s'}(s',T,E) = W^*(s',T)$, $s'$ in $\mathfrak{S}'$, since $\mathfrak{S}$ in $\mathfrak{A} = \mathfrak{A}$, so that $W^*\mathfrak{S} = \mathfrak{S}$; and hence whenever $T$ is BVWD, $T$ is also BVW$^*E$. By the lemma in .8, if $T$ is ACWD and $f(s)$ is a glbfWD then $f(s)$ is also a lbfW$^*E$.

10 Theorem. Let $W^*_{s'}(s',T,E)$, $s'$ in $\mathfrak{S}'$, $E$ in $\mathfrak{A}$, be as described in Definition 7.7T. Assume H13T. Assume that $T$ is ACWD and that $f(s)$ is a glbfWD. Then the function $W^*_{s'}(s',T,E)$, $s'$ in $\mathfrak{S}'$, $E$ in $\mathfrak{A}$, is an extended weight function for $T$; also $T$ is ACW$^*E$ and $f(s)$ is a glbfW$^*E$.

Proof. According to the remark in .9, the first part of the conclusion has been established. It has also been established that $T$ is BVW$^*E$ and that $f(s)$ is a lbfW$^*E$.

Let $E$ be in $\mathfrak{A}$. CE is also in $\mathfrak{A}$, and $\mathfrak{A} \subset \mathfrak{A}$. Recall that $W^*_{s'}$ is additive on $\mathfrak{A}$, and that $W^*\mathfrak{S} = \mathfrak{S}$. Thus,
\[ \int_{E} f(s) \, du = \int_{\Omega} f(s) \, du + \int_{\Gamma} f(s) \, du \leq W_\#E + W_\#CE = W_\# = \int_{\Omega} f(s) \, du. \]

Hence \[ \int_{E} f(s) \, du + \int_{\Omega} f(s) \, du = W_\#E + W_\#CE. \] But \[ \int_{E} f(s) \, du \leq W_\#E, \] \[ \int_{\Omega} f(s) \, du \leq W_\#CE, \] and all values involved are non-negative and finite, therefore \[ \int_{E} f(s) \, du = W_\#E. \] Thus we have established that \( T \) is ACW\#E and \( f(s) \) is a glbfW\#E.

**Remark.** Let \( W_\#(s', T, E) \), \( s' \) in \( S' \), \( E \) in \( \mathcal{C} \), be as in Definition 7.7T. It follows that if \( W_\#(s', T, E) \), \( s' \) in \( S' \), \( E \) in \( \mathcal{C} \cap \mathcal{M} \), is an extended weight function for \( T \) then H13T holds for the associated weight function \( W'(s', T, D) = W_\#(s', T, D) \), \( s' \) in \( S' \), \( D \) in \( \mathcal{D} \); if \( T \) is BVW\#E with lbffW\#E \( f(s) \) then \( T \) is BVWD with lbffWD \( f(s) \); and if \( T \) is ACW\#E with glbfW\#E \( f(s) \) then \( T \) is ACWD with glbfWD \( f(s) \). In particular, \( T \) is ACW\#E and \( f(s) \) is a glbfW\#E if and only if H13T holds for the associated weight function and \( T \) is ACWD with glbfWD \( f(s) \).

**Definition.** Assume H13T. Assume that \( T \) is ACWD and that \( f \) is a glbfWD. It follows from .10 that \( T \) is ACW\#E and that \( f \) is a glbfW\#E. Let \( H' \) be a real valued function defined on \( S' \) and measurable \( \mathcal{M}' \). Fix \( E \) in \( \mathcal{C} \cap \mathcal{M} \). By Theorem 2.13T in Chapter II, the function \( H'T(s)f(s) \), \( s \) in \( S \), is measurable \( \mathcal{M} \), hence \( H'T(s)f(s) \), \( s \) in \( E \), is measurable \( \mathcal{M} \); and by the lemma in .8, \( W_\#(s', T, E) \), \( s' \) in \( S' \), is measurable \( \mathcal{M}' \), so that the function \( H'(s')W_\#(s', T, E) \), \( s' \) in \( S' \), is measurable \( \mathcal{M}' \). Denote by \( \bar{H}'(E) \) the set of all
such functions $H'$ for which the following conditions also hold:

1) $H'(s')W'(s', T, E)$, $s'$ in $S'$, is integrable $\mu'$;

2) $H'T(s)f(s)$, $s$ in $E$, is integrable $\mu$;

3) $\int_E H'T(s)f(s)\,d\mu = \int_{S'} H'(s')W'(s', T, E)\,d\mu'$.

The formula in iii) is termed an extended transformation formula for the function $H'$.

.13 Lemma. Assume $H13T$. Assume that $T$ is ACWD with glbfWD $f(s)$. Fix $E$ in $\\delta M$. Let $H'$ be a bounded non-negative real valued function defined on $S'$ and measurable $\mu'$. Then $H'$ is in $H'(E)$.

Proof. Since $T$ is ACWD with glbfWD $f(s)$ and $H'$ is bounded it is clear that conditions 1) and 2) of the definition in .12 are satisfied. It remains to show that condition iii) is also satisfied.

Let $S$ be expressed as the union of a countable number of sets $*D_j$ with the properties described in H4T. We consider two cases.

Case 1) $E$ is such that $E \subset *D_j$ for some $j$. By H4T $*D_j$ has finite measure. By H4T we may choose, for every positive integer $k$, a sequence of pairwise disjoint sets $D_{k1}$, $i \geq 1$, in $\\delta$ such that $E \subset \bigcup_i D_{k1}$ and $\sum_i \mu(D_{k1}) < \mu E + 1/k$. We may also assume that for every $k$, $\bigcup_i D_{k1} \supset \bigcup_i D_{k+1}$.

For each $k$, define the set $M_k$ by $M_k = \bigcup_i D_{k1} \cap CE$. 
For each $k$, \( \int_{E} H'(T(s)f(s))du + \int_{\mathbb{M}_k} H'(T(s)f(s))du = \int_{U_1 D_k} H'(T(s)f(s))du. \)

Clearly, \( \lim_{k \to \infty} M_k = 0; \) \( H'(T(s)f(s)) \), \( s \) in \( E \), is integrable as a consequence of \( T \) ACWD with \( \text{glb}fWD \) \( f(s) \) and the fact that \( H' \) is bounded; and hence it follows that \( \lim_{k \to \infty} \int_{M_k} H'(T(s)f(s))du = 0. \)

From these observations we have

(1) \( \int_{E} H'(T(s)f(s))du = \lim_{k \to \infty} \int_{U_1 D_k} H'(T(s)f(s))du. \)

In view of \( H_1 \Theta T, \) \( T \) ACWD, and \( H' \) bounded, we have for each \( k, i \), by the transformation formula of Chapter II,

\[
\int_{U_1 D_k} H'(T(s)f(s))du = \int_{\mathbb{S}'} H'(s') W_*(s', T, D_k)du'.
\]

Recall also that \( \mathcal{D} \subseteq \mathcal{D} \) and that \( W_*(s', T, E) \), \( s' \) in \( \mathbb{S}' \), is additive on \( \mathcal{D} \), and that for \( k \) fixed the sets \( D_k \), \( i \geq 1 \), are pairwise disjoint.

From these observations it follows that for each \( k \),

(2) \( \int_{U_1 D_k} H'(T(s)f(s))du = \sum_{T \mathbb{D}_k} H'(s') W_*(s', T, D_k)du'. \)

Define the set \( \mathbb{E}^{**} \) by \( \mathbb{E}^{**} = \cap_k U_1 D_k \). Note that \( T \) ACWD implies \( W_*(s', T, E) \) finite a.e. \( u' \), and we have assumed that \( U_1 D_k \supseteq U_1 D_{k+1} \) for every \( k \), so that Lemma 7.8T v) of Chapter II gives \( \lim_{k} W_*(s', T, U_1 D_k) = W_*(s', T, \mathbb{E}^{**}) \), a.e. \( u' \).

Furthermore, by the same reasoning used in the proof of the
Lemma in 8, \( W_*(s', T, E^{**}) = W_*(s', T, E) \), a.e. \( u' \) on \( S' \), and hence \( H'(s') W_*(s', T, E^{**}) = H'(s') W_*(s', T, E) \), a.e. \( u' \) on \( S' \). Using these observations and (2), we have

\[
\lim_{k} \int_{S'} H'(s') W_*(s', T, E^{**}) \, du' = \lim_{k} \int_{S'} H'(s') W_*(s', T, E) \, du'.
\]

since all of the functions are non-negative, and all of the integrals have finite values. From (1) and (3) we have

\[
\int_{E} H'(s') f(s) \, du = \int_{S'} H'(s') W_*(s', T, E) \, du'.
\]

Case ii) \( E \) is arbitrary in \( S^{\infty} \). The set \( E \) may be expressed as the union of a countable family of pairwise disjoint sets \( E_j, E_j \) in \( S^{\infty} \), \( E_j = \#D_j, j \geq 1 \). Then by case i) and the additivity of \( W_*(s') \), we have

\[
\int_{E} H'(s') f(s) \, du = \sum_{E_j} \int_{S'} H'(s') W_*(s', T, E_j) \, du'.
\]

Lemma. Assume \( H_{13} T \). Assume that \( T \) is ACWD with
glbfWD $f(s)$. Fix $E$ in $\mathcal{M}$. Let $H'$ be a non-negative real valued function defined on $\mathbb{R}^d$ and measurable $\mathcal{M}$. Then we have

$$\int_E H'(s')W_\alpha(s',T,E)du = \int_{\mathbb{R}^d} H'(s')W_\alpha(s',T,E)du'. $$

Thus, if either $H'(s')W_\alpha(s',T,E)$, $s'$ in $\mathbb{R}^d$, is integrable $\mu'$ or $H' f(s')W_\alpha(s',T,E)$, $s$ in $E$, is integrable $\mu$ then $H'$ is in $\mathbb{H}'(E)$.

Proof. Let $n$ be a positive integer. Define the function $H_n$ on $\mathbb{R}^d$ by

$$H_n(s') = \begin{cases} H'(s'), & H'(s') \leq n, \\ n, & H'(s') > n. \end{cases}$$

For each $n$, $H_n(s')$, $s'$ in $\mathbb{R}^d$, is a bounded non-negative real valued function which is measurable $\mathcal{M}$; therefore, by the lemma in .13 we have

$$\int_E H_n'(s')W_\alpha(s',T,E)du = \int_{\mathbb{R}^d} H_n'(s')W_\alpha(s',T,E)du'. $$

It is clear that for each $n$

$$H_n'(s')W_\alpha(s',T,E) \leq H_{n+1}'(s')W_\alpha(s',T,E), \text{ } s' \text{ in } \mathbb{R}^d;$$

$$H_n'(s')f(s) \leq H_{n+1}'(s')f(s), \text{ } s \text{ in } E.$$

Moreover, it follows from the properties of the functions involved that

$$\lim_n H_n'(s')W_\alpha(s',T,E) = H'(s')W_\alpha(s',T,E), \text{ } s' \text{ in } \mathbb{R}^d;$$

$$\lim_n H_n'(s')f(s) = H'(s')f(s), \text{ } s \text{ in } E.$$

Therefore,

$$\int_E H'(s')f(s)du = \lim_n \int_E H_n'(s')f(s)du.$$
\[ \lim_{n \to \infty} \int_{S'} H_n(s') W^*(s', T, E) d\mu' = \int_{S'} H'(s') W^*(s', T, E) d\mu'. \]

15 Theorem. Assume $H \geq T$. Assume $T$ is ACWD with $g \leq f(s)$. Fix $E$ in $\mathcal{F}$. Suppose that $H'$ is a real valued function defined on $S'$ and measurable $\mathcal{M}'$ for which either $H'(s') W^*(s', T, E), s' \in S'$, is integrable $\mu'$ or $H'T(s)f(s), s \in E$, is integrable $\mu$. Then $H'$ is in $\mathcal{H}'(E)$.

Proof. Since $H'$ is measurable $\mathcal{M}'$ it follows that $|H'|$ and $|H'| + H'$ are non-negative real valued functions defined on $S'$ and measurable $\mathcal{M}'$. If $H'(s') W^*(s', T, E), s' \in S'$, is integrable $\mu'$ then so are $|H'(s') W^*(s', T, E)| = |H'(s') W^*(s', T, E)|, s' \in S'$, and $[|H'(s')| + H'(s')] W^*(s', T, E), s' \in S'$. Thus $|H'|$ and $|H'| + H'$ are in $\mathcal{H}'(E)$ by the preceding lemma. It is easily verified that the difference of two functions in $\mathcal{H}'(E)$ is also in $\mathcal{H}'(E)$, and one concludes that $H'$ is in $\mathcal{H}'(E)$. On the other hand, if $H'T(s)f(s), s \in E$, is integrable $\mu$, a similar reasoning leads to the desired conclusion.
CHAPTER IV

THE TRANSFORMATION T'

In this chapter we describe a second transformation, T'. T' will be a transformation from the measure space \([\mathcal{S}',\mathcal{M}',\mu']\) onto a measure space \([\mathcal{S}'',\mathcal{M}'',\mu'']\). We shall require T' to satisfy all of the conditions satisfied by T; therefore, results analogous to those stated in Chapters II and III will be valid for T'. We shall not restate these results in terms of T' since each statement may be obtained formally by making appropriate changes in notation. In order to indicate that a particular result applies to T' we shall place the symbol T' after the number of the result.

H1T'. \([\mathcal{S}',\mathcal{M}',\mu']\) is the \(\sigma\)-finite complete measure space described in H2T.

H2T'. \([\mathcal{S}'',\mathcal{M}'',\mu'']\) is a \(\sigma\)-finite complete measure space.

H3T'. T' is a single valued transformation from \(\mathcal{S}'\) onto \(\mathcal{S}''\).

H4T'. \(\mathcal{S}'\) is a collection of subsets \(\mathcal{D}'\) of \(\mathcal{S}'\) having the following properties. The empty set \(\emptyset\) and the space \(\mathcal{S}'\) belong to \(\mathcal{S}'\). \(\mathcal{D}'\) is a subset of \(\mathcal{M}'\). T'\(\mathcal{D}'\) is a
subset of $\mathcal{M}''$. If $D'_1$ and $D'_2$ belong to $\mathcal{B}'$ then there is a countable number of pairwise disjoint sets $D'_i$ in $\mathcal{B}'$ such that $D'_1 \cap D'_2 = U D'_i$. $\mathcal{S}'$ can be expressed as the union of a countable number of sets $D'_j$ in $\mathcal{B}'$ such that $\mu' D'_j$ is finite for every $j$, and if $M'$ is in $\mathcal{M}'$, $M' \subseteq D'_j$ for some $j$, then for every positive real number $\varepsilon$ there is a countable number of pairwise disjoint sets $D'_i$ in $\mathcal{B}'$ such that $M' \subseteq U D'_i$ and $\sum \mu' D'_i < \mu' M' + \varepsilon$.

Definition 1.1T'. An element $D'_o$ in $\mathcal{B'}$ is said to be of type $\gamma'T'$ if it is one of a countable number of pairwise disjoint sets $D'_j$, $j \geq 0$, in $\mathcal{B}'$ for which there are two subsets $Y'$, $X'$ of $\mathcal{S}'$ such that

$$\mathcal{S}' = Y'U X'U U D'_1, \; j \geq 0;$$

$Y'$ in $\mathcal{M}'$, $\mu' Y' = 0$; $T'Z'$ in $\mathcal{M}'$, $\mu'' T'Z' = 0$.

H5T'. If $D'$ is an element in $\mathcal{B}'$ there is a countable sequence of sets $D'_j$ in $\mathcal{B}'$ such that each $D'_j$ is of type $\gamma'T'$, $D'_j \subseteq D'_{j+1}$ for every $j$, and $U D'_j = D'$.

H6T'. $\mathcal{B}$ is a $\sigma$-field of subsets $B''$ of $\mathcal{S}$ having the following properties. $\mathcal{B}$ is a subset of $\mathcal{M}$ and $T'^{-1} \mathcal{B}$ is a subset of $\mathcal{M}'$. For each element $M''$ in $\mathcal{M}$ there are sets $B'_1$ and $B'_2$ in $\mathcal{B}$ such that $B'_1 \subseteq M'' \subseteq B'_2$ and $\mu'' B'_1 = \mu'' B'_2$.

Definition 1.2T'. Denote by $\mathcal{O}_T'$ all the subsets $\mathcal{O}'$ of $\mathcal{S}$ for each of which there is a countable number of pairwise disjoint sets $D'_1$ in $\mathcal{B}'$ such that $T'^{-1} \mathcal{O}' = U D'_1$.

H7T'. $\mathcal{S}$ can be expressed as the union of a
countable number of sets $O_j$ in $C"T"$ such that $\mu"O_j$ is finite for every $j$, and if $M$ is in $m"$, $M" \subseteq O_j$ for some $j$, $\mu"M" = 0$, then for every positive real number $\varepsilon$ there is a set $O"$ in $C"T"$ such that $M" \subseteq O"$ and $\mu"O" < \varepsilon$.

Definition 1.3T*. An element $O_j"$ of $C"T"$ is said to be of type $\gamma"T"$ if it is one of a countable number of pairwise disjoint sets $O_j$, $i \geq 0$, in $C"T"$ for which there are two subsets $Y", Z"$ of $S"$ such that

$S" = Y"UZ"UO_j", i \geq 0$;

$T"-1Y"$ in $m", \mu"T"-1Y"=0$; $Z"$ in $m", \mu"Z"=0$.

Definition 1.4T*. An element $M$ in $m"$ is said to be of type $\iota"T"$ if there is a countable sequence of sets $O_j"$ in $C"T"$ such that each $O_j$ is of type $\gamma"T", O_j" \subseteq O_{j+1}"$ for every $j$, and $UO_j = M$.

$HST^*$. If $B$ is an element of $B"$ there is a countable sequence of sets $M_j$ in $m"$ and two sets $U", V"$ in $m$ such that each $M_j$ is of type $\iota"T", M_j \supseteq M_{j+1}$ for every $j$, $\mu"U" = 0$, $\mu"V" = 0$, and $\cap M_j U" = B"UV".$

$H9T^*$. $W"(s" ,T", D")$, $s"$ in $S", D"$ in $B", \text{is a non-negative extended real valued function satisfying the following conditions}$:

1) If $D$ is in $B$ and $s$ is in $C"T"D$ then $W"(s",T",D") = 0$.

11) For each $s$ in $S", W"(s" ,T",D")$ is under additive on $B$. 


iii) For each \( s'' \) in \( S'' \), \( W''(s'', T', D') \) is inner continuous on \( B' \).

iv) For each \( D' \) in \( B' \), \( W''(s'', T', D') \), \( s'' \) in \( S'' \), is measurable \( \mathbb{M}'' \).

A function having these properties is termed a weight function for the transformation \( T' \).

**Definition 2.1T'**. Assume \( H1T' - H9T' \). Set

\[
W'D' = \int_{S''} W''(s'', T', D') du''.
\]

\( W'D' \) is termed the weight attached to \( D' \).

**Definition 2.2T'**. Assume \( H1T' - H9T' \). The transformation \( T' \) is said to be of bounded variation with respect to the weights \( W'D' \)--briefly, \( BVW'D' \)--if \( W'S' \) is a finite real number.

**Definition 2.3T'**. Assume \( H1T' - H9T' \). Let \( g'(s') \), \( s' \) in \( S' \), be a non-negative extended real valued function, measurable \( \mathbb{M}' \), and such that for each \( D' \) in \( B' \),

\[
\int_{B'} g'(s') du' \leq W'D' = \int_{S''} W''(s'', T', D') du''.
\]

Such a function is termed a lower bound function for the weights \( W'D' \)--briefly, a \( lbfW'D' \).

**Definition 3.4T'**. Assume \( H1T' - H9T' \). Assume that \( T' \) is \( BVW'D' \) and that there exists a non-negative extended real valued function \( g'(s') \), \( s' \) in \( S' \), which is a \( lbfW'D' \) such that

\[
\int_{S'} g'(s') du' = W'S' = \int_{S''} W''(s'', T', S') du''.
\]
Then the transformation $T'$ is said to be absolutely continuous with respect to the weights $W'D'$—briefly, $ACW'D'$—and $g'(s')$ is said to be a greatest lower bound function for the weights $W'D'$—briefly, a glbf$W'D'$.

Definition 4.2$T'$. Assume $H1\leq T' \leq H9T'$. Assume that $T'$ is $ACW'D'$, and that $g'(s')$, $s'$ in $S'$, is a glbf$W'D'$. Let $H''$ be a real valued function defined on $S''$ and measurable $\mathcal{M}''$. Fix $D'$ in $\mathcal{D}'$. By Theorem 2.13$T'$ the function $H''(s'')W''(s'',T',D')$, $s''$ in $S''$, is measurable $\mathcal{M}''$ and the function $H''T'(s')g'(s')$, $s'$ in $D'$, is measurable $\mathcal{M}'$. Denote by $\mathcal{H}''(D')$ the set of such functions $H''$ for which the following conditions also hold:

1) $H''(s'')W''(s'',T',D')$, $s''$ in $S''$, is integrable $\mu''$;
2) $H''T'(s')g'(s')$, $s'$ in $D'$, is integrable $\mu'$;
3) $\int_{S''} H''T'(s')g'(s')du' = \int_{S''} H''(s'')W''(s'',T',D')du''$.

The formula in iii) is termed a transformation formula for the function $H''$.

Definition 7.1$T'$. Assume $H1\leq T' \leq H8T'$. With each point $s'$ in $S'$ associate a subset $S'_s$, by the formula

$S'_s = \cap D'$, $s'$ in $D'$ in $\mathcal{D}'$.

Let $\mathcal{J}$ denote the set of all subsets $S'$ of $S'$ for which there exists an $s'$ in $S'$ such that $S' = S'_s$.

$H1\leq T'$. If $s'_1$ and $s'_2$ are points in $S'$ for which there is a set $D'_o$ in $\mathcal{D}'$ such that $s'_1$ in $D'_o$ and $s'_2$ is not in
D_0', then there are sets D_1' and D_2' in S' such that s_1' is in D_1' and s_2' is in D_2' and D_1' \cap D_2' = \emptyset'.

Definition 7.6T'. Assume H1T'-H8T', and H11T'.

Denote by \delta' the set of all subsets E' of S' which have the following property: if S' is in \delta' and is such that E' \cap S' \neq \emptyset' then S' \subset E'.

H12T'. For each S' in \delta' the set T'S' consists of a single point in S'.

Definition 7.7T'. Assume H1T'-H8T', H11T', and H12T'. Let w' be a non-negative extended real valued function defined on \delta'. For each point s'' in S'' and each set E' in \delta' define

W'_*(s'',T',E') = \Sigma w'S', S' in \delta', S' \subset E', T'S'=s''.

Definition 7.10T'. Assume H1T'-H8T', H11T', and H12T'. Suppose that W''(s'',T',D'), s'' in S'', D' in \delta', is a weight function for T' as defined in H9T' for which there exists a non-negative extended real valued function w'S', S' in \delta', such that W''(s'',T',D') = W'_*(s'',T',E'), s'' in S'', D' in \delta', where W'_*(s'',T',E'), s'' in S'', E' in \delta', is as defined in Definition 7.7T'. Then the weight function W''(s'',T',D'), s'' in S'', D' in \delta', is said to be generated by w'S', S' in \delta'.

H13T'. W''(s'',T',D'), s'' in S'', D' in \delta', is generated by a non-negative extended real valued function w'S', S' in \delta'.
CHAPTER V

THE PRODUCT TRANSFORMATION T''

If we assume H3T and H3T' then there arises naturally the single valued transformation T'' = T'T from \( S \) onto \( S'' \). When conditions H1T''-H8T'' analogous to H1T-H8T, or to H1T'-H8T', are satisfied we have a transformation theory for the transformation T'' from the measure space \( \{ S', \mathcal{M}', \mu' \} \) onto the measure space \( \{ S'', \mathcal{M}'', \mu'' \} \). But it is clear that conditions H1T''-H8T'' are by no means independent of H1T-H8T and H1T'-H8T'. Below we state explicitly conditions H1T''-H8T'', H11T''-H13T'' for T'' = T'T and indicate how these are related to and determined by the corresponding conditions for T and T' given in Chapters II and IV.

H1T''. \( \{ S, \mathcal{M}, \mu \} \) is the \( \sigma \)-finite complete measure space described in H1T.

H2T''. \( \{ S'', \mathcal{M}'', \mu'' \} \) is the \( \sigma \)-finite complete measure space described in H2T'.

H3T''. The single valued transformation T'' from \( S \) onto \( S'' \) is the product T'T of the transformations T and T'.

H4T''. \( \mathcal{D} \) is a class of subsets \( D \) of \( S \) satisfying H4T and T''\( \mathcal{D} \) is a subset of \( \mathcal{M}'' \).
Remark. Recall $H^4T$. All of the properties of $\mathcal{B}$ are independent of the transformation $T$ except the requirement that $T\mathcal{B}$ is a subset of $\mathcal{M}'$, hence $H^4T''$ gives a condition for $T''$ analogous to the condition for $T$ given by $H^4T$.

Definition 1.1T''. An element $D_0$ in $\mathcal{B}$ is said to be of type $\gamma T''$ if it is one of a countable number of pairwise disjoint sets $D_1, i \geq 0$, in $\mathcal{B}$ for which there are two subsets $Y, Z$ of $\mathcal{S}$ such that

\[ S = YUZU\cup D_1, i \geq 0, \]

$Y$ in $\mathcal{M}$, $uY = 0$; $T''Z$ in $\mathcal{M}'$, $u''T''Z = 0$.

H5T''. If $D$ is an element in $\mathcal{B}$ there is a countable sequence of sets $D_j$ in $\mathcal{B}$ such that each $D_j$ is of type $\gamma T''$, $D_j \subset D_{j+1}$ for every $j$, and $UD_j = D$.

H6T''. $\mathcal{B}''$ is a class of subsets $\mathcal{B}''$ of $\mathcal{S}''$ satisfying $H6T'$ and $T''^{-1}\mathcal{B}'$ is a subset of $\mathcal{M}$.

Remark. Recall $H6T'$. All of the properties of $\mathcal{E}''$ are independent of the transformation $T'$ except the property that $T'^{-1}\mathcal{E}''$ is a subset of $\mathcal{M}'$, hence $H6T''$ gives a condition for $T''$ analogous to the condition for $T$ given by $H6T$.

Definition 1.2T''. Denote by $\mathcal{C}''T''$ the class of subsets $\mathcal{C}''$ of $\mathcal{S}''$ for each of which there is a countable number of pairwise disjoint sets $D_j$ in $\mathcal{B}$ such that $T''^{-1}\mathcal{C}'' = UD_j$.

H7T''. $\mathcal{S}''$ can be expressed as the union of a countable number of sets $*0_j$ in $\mathcal{C}''T''$ such that $u''*0_j$ is finite.
for every j and if M'' in ℳ'', M'' ⊆ ∪₀⁰ j for some j, u''M'' = 0, then for every positive real number ε there is a set Ṡ'' in ℰ'''' such that M'' ⊆ Ṡ'' and u''Ṡ'' < ε.

3 Remark. It is interesting to observe that H7ᵀ'' may be obtained as a consequence of H7ᵀ', H3ᵀ'', and the following additional condition: The class Ṣ' described in H4ᵀ'' is a subclass of the class Ṣ' described in Definition 1.2ᵀ. We verify this statement as follows: Let Ṡ'' in ℰ''''ᵀ'. By Definition 1.2ᵀ' there exists a family of pairwise disjoint sets D_k', k≥1, in Ṣ' such that T'⁻¹Ṡ'' = ∪D_k'. From Ṣ' ⊆ Ṣ' and Definition 1.2ᵀ it follows that for each k there exists a family of pairwise disjoint sets D_k₁', i≥0, in Ṣ such that T'⁻¹D_k' = U_iD_k₁'. Thus,

T''⁻¹Ṡ'' = T⁻¹T'⁻¹Ṡ'' = T⁻¹∪D_k' = U_kT⁻¹D_k' = U_k∪_iD_k₁';

therefore, Ṡ'' is in ℰ''''ᵀ''. Clearly, H7ᵀ'' is a consequence of H7ᵀ' and ℰ''''ᵀ' ⊆ ℰ''''ᵀ''.

Definition 1.3ᵀ''. An element Ṡ''₀ in ℰ''ᵀ'' is said to be of type Ṣ''''ᵀ'' if it is one of a countable number of pairwise disjoint sets Ṡ''_i', i≥0, in ℰ''ᵀ'' for which there are two subsets Y'', Z'' of ℳ'' such that

Ṡ'' = Y''∪Z''∪∪₀ⁿ Ṡ''_i', i≥0,
T''⁻¹Y'' in ℳ, uT''⁻¹Y''=0; Z'' in ℳ'', u''Z''=0.

Definition 1.4ᵀ''. An element M'' in ℳ'' is said to be of type Ṣ''''ᵀ'' if there is a countable sequence of sets Ṡ''_j in
such that each $O_j^T$ is of type $\gamma_j^T$, $O_j = O_{j+1}$ for every $j$, and $UO_j^T = M^T$.

H8T". If $B^T$ is an element of $\mathcal{S}^T$ there is a countable sequence of sets $M_j^T$ in $\mathcal{M}$ and two sets $U^T$, $V^T$ in $\mathcal{M}$ such that each $M_j^T$ is of type $\iota_j^T$, $M_j^T \supseteq M_{j+1}^T$ for every $j$, $U^T \cap Y = 0$, $U^T \cap Y_1 = 0$, and $\cap M_j^T U^T = B^T U V^T$.

H9T". $W^T(s'',T'',D)$, $s''$ in $\mathcal{S}$, $D$ in $\mathcal{F}$, is a non-negative extended real valued function satisfying the following conditions:

1) If $D$ is in $\mathcal{F}$ and $s''$ is in $C^T D$, then $W^T(s'',T'',D) = 0$.

2) For each $s''$ in $\mathcal{S}$, $W^T(s'',T'',D)$ is under additive on $\mathcal{F}$.

3) For each $s''$ in $\mathcal{S}$, $W^T(s'',T'',D)$ is inner continuous on $\mathcal{F}$.

4) For each $D$ in $\mathcal{F}$, $W^T(s'',T'',D)$, $s''$ in $\mathcal{S}$, is measurable $\mathcal{M}$.

A function having these properties is termed a weight function for the transformation $T''$.

Definition 2.1T". For each $D$ in $\mathcal{F}$ set

$$W''D = \int_{\mathcal{S}} W^T(s'',T'',D) d_{\mathcal{M}}.$$ 

$W''D$ is termed the weight attached to $D$ by $T''$ and $W''$.

Definition 2.2T". $T''$ is said to be of bounded variation with respect to the weights $W''D$--briefly, BV$W''D$--if $W''S$ is a finite real number.
Definition 2.3T". Let f be a non-negative extended real valued function defined on \( S \), measurable \( \mathcal{M} \), and such that for each \( D \) in \( \mathcal{D} \)
\[
\int_D f(s) \, d\mu \leq W^"D = \int_S^\" W^"(s" , T" , D) \, d\mu" \,.
\]
Such an \( f \) is termed a lower bound function for the weights \( W^"D \)--briefly, a \( \text{lbfW}^"D \).

Definition 3.4T". Assume \( H^\uparrow T"-H^\downarrow T" \). Assume \( T" \) is \( \text{BVW}^"D \) and that there exists a non-negative extended real valued function \( f \) defined on \( S \) which is a \( \text{lbfW}^"D \) such that
\[
\int_S f(s) \, d\mu = W^"S = \int_S^\" W^"(s" , T" , S) \, d\mu" \,.
\]
Then \( T" \) is said to be absolutely continuous with respect to the weights \( W^"D \)--briefly, \( \text{ACW}^"D \)--and \( f \) is said to be a greatest lower bound function for the weights \( W^"D \)--briefly, a \( \text{glbW}^"D \).

Definition 4.2T". Assume \( H^\uparrow T"-H^\downarrow T" \). Assume that \( T" \) is \( \text{ACW}^"D \), and that \( f(s), \ s \) in \( S \), is a \( \text{glbW}^"D \). Let \( H" \) be a real valued function defined on \( S" \) and measurable \( \mathcal{M}" \). Fix \( D \) in \( \mathcal{D} \). By Theorem 2.13T" the function
\[
H"(s") W^"(s", T", D), \ s" \ in \ S", is measurable \( \mathcal{M}" \) and the function \( H"T"(s)f(s), \ s \ in \ D, is measurable \( \mathcal{M}. \) Denote by \( H"(D) \) the set of such functions \( H" \) for which the following conditions also hold:

1) \( H"(s") W^"(s", T", D), \ s" \ in \ S", is integrable \( \mu" \);

11) \( H"T"(s)f(s), \ s \ in \ D, is integrable \( \mu; \)
iii) \[ \int_{\mathbb{S}''} \mathbb{H}''(s'' \mid T'', E) \, d\mu = \int_{\mathbb{S}''} \mathbb{H}''(s'') \mathbb{W}''(s'', T'', D) \, d\mu''. \]

The formula in iii) is termed a transformation formula for the function \( \mathbb{H}'' \).

**Definition 7.1T.** Identical with Definition 7.1T.

**H11T.** Identical with H11T.

**Definition 7.6T.** Identical with Definition 7.6T.

**H12T.** For each \( s \) in \( \mathbb{S}'' \) the set \( T'' S \) consists of a single point in \( \mathbb{S}'' \).

Note that H12T. is a consequence of H12T, H3T', and H3T''.

**Definition 7.7T.** Assume H1T"-H8T", H11T", and H12T". Let \( \mathbb{W}'' S, \ S \) in \( \mathbb{S}'' \), be a non-negative extended real valued function. For each point \( s'' \) in \( \mathbb{S}'' \) and each \( E \) in \( \mathfrak{E} \) define

\[ \mathbb{W}''(s'', T'', E) = \Sigma \mathbb{W}'' S, \ S \) in \( \mathfrak{N} \), \( S \in E \), \( T'' S = s'' \).

**Definition 7.10T.** Assume H1T"-H8T", H11T", and H12T". Suppose that \( \mathbb{W}''(s'', T'', D), \ s'' \) in \( \mathbb{S}'' \), \( D \) in \( \mathfrak{D} \), is a weight function for \( T'' \) as defined in H9T" for which there exists a non-negative extended real valued function \( \mathbb{W}'' S, \ S \) in \( \mathfrak{N} \), such that \( \mathbb{W}''(s'', T'', D) = \mathbb{W}''(s'', T'', D), \ s'' \) in \( \mathbb{S}'' \), \( D \) in \( \mathfrak{D} \). Then the weight function is said to be generated by \( \mathbb{W}'' S, \ S \) in \( \mathfrak{N} \).

**H13T.** \( \mathbb{W}''(s'', T'', D), \ s'' \) in \( \mathbb{S}'' \), \( D \) in \( \mathfrak{D} \), is generated by a non-negative extended real valued function \( \mathbb{W}'' S, \ S \) in \( \mathfrak{N} \).
Throughout this chapter $H_{1T}$-$H_{9T}$, $H_{1T'}$-$H_{9T'}$, and $H_{1T''}$-$H_{8T''}$ are always assumed.

We introduce an additional hypothesis for the transformation $T'$.

$H_{F1}$. $T'$ is one-to-one from $S'$ onto $S''$.

1 Definition. Assume $H_{F1}$. Let $W''(s', T, D)$ and $W''(s'', T', D')$ be weight functions for $T$ and $T'$ as described in $H_{9T}$ and $H_{9T'}$ respectively. As a consequence of $H_{F1}$, $T'^{-1} s''$, $s''$ in $S''$, is a single point of $S'$.

For each point $s''$ in $S''$ and each set $D$ in $\mathcal{D}$ define

$W''(s'', T'', D) = W'(T'^{-1} s'', T, D) W''(s'', T', S'')$.

2 Lemma. Assume $H_{F1}$. Let $W''(s'', T'', D)$ be as defined in .1. If $D$ is in $\mathcal{D}$ and $s''$ is in $C'' T'' D$ then $W''(s'', T'', D) = 0$.

Proof. Fix $D$ in $\mathcal{D}$. Let $s''$ in $C'' T'' D$. Then $T'^{-1} s''$ is in $C' T D$, so that from $H_{9T'}$, $W'(T'^{-1} s'', T, D) = 0$. Hence $W''(s'', T'', D) = 0$.

3 Lemma. Assume $H_{F1}$. Let $W''(s'', T'', D)$ be as defined in .1. For each $s''$ in $S''$, $W''(s'', T'', D)$ is under additive on $\mathcal{D}$. 40
Proof. Let \( D \in \mathcal{D} \) contain a countable number of pairwise disjoint sets \( D_1 \) in \( \mathcal{D} \). Then for \( s'' \) in \( \mathcal{S}'' \),

\[
\sum W''(s'',T'',D_1) = \sum W'(T'^{-1}s'',T,D_1)W''(s'',T'',\mathcal{S}') = W''(s'',T'',\mathcal{S}')\sum W'(T'^{-1}s'',T,D_1) \leq W''(s'',T'',\mathcal{S}')W'(T'^{-1}s'',T,D) = W''(s'',T'',D),
\]

since \( W'(T'^{-1}s'',T,D) = W'(s'',T,D) \) is under additive by H9T.

.4 Lemma. Assume HFl. Let \( W''(s'',T'',D) \) be as defined in .1. For each \( s'' \) in \( \mathcal{S}'' \), \( W''(s'',T'',D) \) is inner continuous on \( \mathcal{D} \).

Proof. Let \( D \in \mathcal{D} \) be the union of a countable number of sets \( D_j \) in \( \mathcal{D} \) such that \( D_j \subseteq D_{j+1} \) for every \( j \). Let \( s'' \) be fixed in \( \mathcal{S}'' \), so that \( W''(s'',T'',\mathcal{S}') \) is constant. By H9T, \( \lim W'(T'^{-1}s'',T,D_j) = W'(T'^{-1}s'',T,D) \). Therefore,

\[
\lim W''(s'',T'',D_j) = \lim W'(T'^{-1}s'',T,D_j)W''(s'',T'',\mathcal{S}') = W''(s'',T'',\mathcal{S}')\lim W'(T'^{-1}s'',T,D_j) = W''(s'',T'',\mathcal{S}')W'(T'^{-1}s'',T,D) = W''(s'',T'',D).
\]

.5 Theorem. Assume HFl. Let \( W''(s'',T'',D) \) be as defined in .1. Assume that \( W'(T'^{-1}s'',T,D) \), \( s'' \) in \( \mathcal{S}'' \), is measurable \( \forall s'' \) for every \( D \) in \( \mathcal{D} \). Then \( W''(s'',T'',D) \), \( s'' \) in \( \mathcal{S}'' \), \( D \) in \( \mathcal{D} \), is a weight function for \( T'' \).

Proof. \( W''(s'',T'',D) \), \( s'' \) in \( \mathcal{S}'' \), \( D \) in \( \mathcal{D} \), is a non-negative extended real valued function satisfying, according to the lemmas in .2, .3, and .4, conditions 1), 11),
and iii) in $H9T"$. By $H9T'$, $W"(s", T', S')$, $s"$ in $S"$, is measurable $\mathfrak{M}"$. Fix $D$ in $\mathcal{D}$; if $W'(T'^{-1}s", T, D)$, $s"$ in $S"$, is measurable $\mathfrak{M}"$ then $W"(s", T', D)$, $s"$ in $S"$, is the product of two measurable functions, and hence is measurable $\mathfrak{M}"$ so that iv) of $H9T"$ is also satisfied.

.6 Remark. Chaney [2] has observed that the following conditions are sufficient (but not necessary) for $W'(T'^{-1}s", T, D)$ to be measurable $\mathfrak{M}"$ on $S"$ for each $D$ in $\mathcal{D}$:

1) The weight function $W'(s', T, D)$, $s'$ in $S'$, is measurable $\mathfrak{B}'$ where the class $\mathfrak{B}'$ is as described in $H6T$;

11) The class $\mathfrak{B}'$ has the additional property that $T'\mathfrak{B}'$ is a subclass of $\mathfrak{M}"$.

.7 Theorem. Assume $Hf1$. Assume that $T$ is ACWD with $\text{glbf}_W f(s)$ and that $T'$ is ACW'D' with $\text{glbf}_W'D'$ $g'(s')$. Assume that $W'(T'^{-1}s", T, D)$, $s"$ in $S"$, is measurable $\mathfrak{M}"$ for each $D$ in $\mathcal{D}$. Let the weight function for $T''$ be the function $W"(s", T'', D)$, $s"$ in $S"$, $D$ in $\mathcal{D}$, defined in .1. Then $T''$ is ACW"D if and only if $T''$ is BVW"D. $T''$ is BVW"D if and only if $f(s)g'T(s)$, $s$ in $S$, is integrable $\mu$. Further, if $T''$ is ACW"D then $f(s)g'T(s)$ is a $\text{glbf}_W"D$.

Proof. First, assume that $T''$ is BVW"D, that is, $W"(s", T'', D)$, $s"$ in $S"$, is integrable $\mu"$ for each $D$ in $\mathcal{D}$. Fix $D$ in $\mathcal{D}$. By hypothesis $T'$ is ACW'D' with $\text{glbf}_W'D'$ $g'(s')$, and $W'(T'^{-1}s", T, D)$, $s"$ in $S"$, is measurable $\mathfrak{M}"$. Therefore, by Theorem 4.10$T'$, we have
where each integral is finite valued. Observe that as a consequence of HFl we have for each s' in Σ',

\[ W'(T'^{-1}T's',T,D) = W'(s',T,D). \]

By hypothesis T is ACWD with glbfWD f(s). By definition 2.3T', g'(s') is measurable \( M'. \) Therefore, by Theorem 4.10T, we have

(2) \[ \int_{\Sigma'} W'(T'^{-1}T's',T,D)g'(s')d\mu' = \int_{\Sigma'} W'(s',T,D)g'(s')d\mu' = \int_D f(s)g'T(s)d\mu, \]

where each integral is finite valued. Since \( \Sigma \) is in \( \mathcal{F} \) it follows that \( f(s)g'T(s), s \in \Sigma, \) is integrable \( \mu. \)

Next, assume that \( f(s)g'T(s), s \in \Sigma, \) is integrable \( \mu. \) Let \( D \) be in \( \mathcal{F}. \) Using the same reasoning we obtain first (2) and then (1), so that we have

\[ \int_D f(s)g'T(s)d\mu = \int_{\Sigma''} W''(s'',T'',D)d\mu'', \]

where each integral is finite valued. Thus \( T'' \) is ACW''D and moreover, \( f(s)g'T(s) \) is a glbfW''D.

Finally, by definition, \( T'' \) ACW''D implies \( T'' \) BVW''D. This last statement completes the proof.
CHAPTER VII

T' IS ESSENTIALLY LOCALLY ONE-TO-ONE

Throughout this chapter $H^1T-H^9T$, $H^1T'-H^9T'$, and $H^1T''-H^9T''$ are always assumed.

The discussion in this chapter is an extension of the discussion in Chapter VI. In Chapter VI we assume that the transformation $T'$ is one-to-one; for the purposes of the present chapter we introduce the following hypothesis:

HF2. There exists a countable number of pairwise disjoint sets $D_i$, $i \geq 0$, in $\beta'$ and subsets $X'$, $Y'$ of $\beta'$ such that

$$\beta' = X' \cup Y' \cup \bigcup D_i, \; i \geq 0;$$

$X'$ in $\beta'$, $\mu'X'=0$; $T'Y'$ in $\beta''$, $\mu''T'Y'=0$;

and for each $i$ the function $T'$ is one-to-one from $D_i$ onto $T'D_i$; $X'$, $Y'$, and $D_i$, $i \geq 0$, are pairwise disjoint.

.1 Definition. Assume HF2. Then for $i \geq 0$, $s''$ in $\beta''$, either $T'^{-1}s'' \cap D_i$ is a single point, $s'$, of $D_i$ or else $T'^{-1}s'' \cap D_i$ is empty. Let $W'(s', T, D)$ and $W''(s'', T', D')$ be weight functions for $T$ and $T'$. For $s''$ in $\beta''$, $D$ in $\beta$, $i \geq 0$, define

$$H_i''(s'', T'', D) = \begin{cases} W'(s', T, D), & s' = T'^{-1}s'' \cap D_i; \\ 0, & s' = T'^{-1}s'' \cap D_i. \end{cases}$$

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For each point $s^*$ in $S^*$ and each set $D$ in $\mathcal{D}$ define

$$W''(s^*, T^*, D) = \sum W''(s^*, T', D'_k)H''(s^*, T^*, D), \quad i \geq 0.$$ 

Evidently $W''(s^*, T^*, D)$, $s^*$ in $S^*$, $D$ in $\mathcal{D}$, is a non-negative extended real valued function.

.2 Lemma. Assume HF2. Let $W''(s^*, T^*, D)$ be as defined in .1. If $D$ is in $\mathcal{D}$ and $s^*$ is in $C''T''D$ then

$$W''(s^*, T^*, D) = 0.$$ 

Proof. Fix $D$ in $\mathcal{D}$. Let $s^*$ in $C''T''D$. Let $i \geq 0$. If $i$ is such that $T'^{-1}s'' \cap D'_i = \emptyset'$, then $H''(s^*, T^*, D) = 0$ by definition. If $i$ is such that $T'^{-1}s'' \cap D'_i = s'$ for some $s'$ in $S'$, then $H''(s'', T''D) = W'(s', T, D)$ and $s'$ is in $T'^{-1}s'' \subseteq T'^{-1}C''T''D \subseteq C'T'D$; by H9T, $W'(s', T, D) = 0$ and hence $H''(s'', T'', D) = 0$. Therefore,

$$W''(s'', T'', D) = \sum W''(s'', T', D'_i)H''(s'', T'', D), \quad i \geq 0$$

$$= 0.$$

.3 Lemma. Assume HF2. Let $W''(s'', T'', D)$ be as defined in .1. For each $s''$ in $S''$, $W''(s'', T'', D)$ is under additive on $\mathcal{D}$.

Proof. Fix $s''$ in $S''$ and $D$ in $\mathcal{D}$. Let $D$ contain a countable number of pairwise disjoint sets $D_k$, $k \geq 0$, of $\mathcal{D}$. Fix $i \geq 0$. If $i$ is such that $T'^{-1}s'' \cap D'_i = \emptyset'$, then from the definition in .1, $H''(s'', T'', D) = 0$, and also for each $k$, $H''(s'', T', D_k) = 0$, so that trivially, we have

$$\sum_k H''(s'', T'', D_k) = H''(s'', T'', D).$$

If $i$ is such that $T'^{-1}s'' \cap D'_i$ is a unique point $s'$ of $S'$, then by the definition in .1,
\( H''_i(s'', T'', D) = W'(s', T, D) \), and for each \( k \), \( H''_i(s'', T'', D_k) = W'(s', T, D_k) \); thus, by H9T, \( \Sigma_k H''_i(s'', T'', D_k) \leq H''_i(s'', T'', D) \). Therefore,
\[
\Sigma_k W''(s'', T'', D_k) = \Sigma_k \sum_j W''(s'', T', D'_1) H''_i(s'', T'', D_k)
\]
\[
= \sum_j W''(s'', T'', D'_1) \Sigma_k H''_i(s'', T'', D_k)
\]
\[
\leq \sum_j W''(s'', T'', D'_1) H''_i(s'', T'', D)
\]
\[
= W''(s'', T'', D).
\]

4. Lemma. Assume HF2. Let \( W''(s'', T'', D) \) be as defined in 1. For each \( s'' \) in \( \mathbb{S}'' \). \( W''(s'', T'', D) \) is inner continuous on \( \mathcal{D} \).

Proof. Let \( D \) in \( \mathcal{D} \) be the union of a countable number of sets \( D_j \) in \( \mathcal{D} \) such that \( D_j \subseteq D_{j+1} \) for every \( j \). Let \( s'' \) be fixed in \( \mathbb{S}'' \).

Let \( i \geq 0 \) be fixed. If \( i \) is such that \( T'' - 1 s'' \cap D'_1 \) is empty, then from the definition in 1, \( H''_i(s'', T'', D) = 0 \), and for each \( j \), \( H''_i(s'', T'', D_j) = 0 \), so that trivially,
\[
\lim_j H''_i(s'', T'', D_j) = H''_i(s'', T'', D).
\]
If \( i \) is such that \( s' = T'' - 1 s'' \cap D'_1 \) for some \( s' \) in \( \mathbb{S}' \), then from the definition in 1, \( H''_i(s'', T'', D) = W'(s', T, D) \) and for each \( j \),
\[
H''_i(s'', T'', D_j) = W'(s', T, D_j); \quad \text{thus, by H9T,} \quad \lim_j H''_i(s'', T'', D_j) = H''_i(s'', T'', D).
\]
Therefore,
\[
\lim_j W''(s'', T'', D_j) = \lim_j \sum_j W''(s'', T', D'_1) H''_i(s'', T'', D_j)
\]
\[
= \sum_j \lim_j W''(s'', T', D'_1) H''_i(s'', T'', D_j)
\]
\[
= \sum_j W''(s'', T', D'_1) \lim_j H''_i(s'', T'', D_j)
\]
Theorem. Assume HF2. Let \( W''(s'', T'', D) \) be as defined in .1. Assume that \( H_1''(s'', T'', D) \), \( s'' \) in \( \mathcal{S}'' \), is measurable \( \mathcal{M}'' \) for each \( i \geq 0 \), and each \( D \) in \( \mathcal{B} \). Then \( W''(s'', T'', D) \), \( s'' \) in \( \mathcal{S}'' \), \( D \) in \( \mathcal{B} \), is a weight function for \( T'' \).

Proof. \( W''(s'', T'', D) \), \( s'' \) in \( \mathcal{S}'' \), \( D \) in \( \mathcal{B} \), is a non-negative extended real valued function satisfying, according to the lemmas in .2, .3, and .4, conditions i), ii), and iii) in H9T''. The function \( W''(s'', T'', D) \), \( s'' \) in \( \mathcal{S}'' \), is measurable \( \mathcal{M}'' \) by H9T''; thus, for each \( i \), \( H_1''(s'', T'', D) \), \( s'' \) in \( \mathcal{S}'' \), measurable \( \mathcal{M}'' \) implies that \( W''(s'', T'', D)H_1''(s'', T'', D) \), \( s'' \) in \( \mathcal{S}'' \), is measurable \( \mathcal{M}'' \). From this it follows that \( W''(s'', T'', D) \), \( s'' \) in \( \mathcal{S}'' \), is measurable \( \mathcal{M}'' \) so that condition iv) of H9T'' is satisfied.

Observe that the weight function used in Chapter VI is a special case of this function.

The following lemma gives a set of conditions which are sufficient (but not necessary) to ensure that for each \( i \geq 0 \) and each \( D \) in \( \mathcal{B} \), \( H_1''(s'', T'', D) \) is measurable \( \mathcal{M}'' \) on \( \mathcal{S}'' \).

Lemma. Assume HF2. Assume the following:

1) \( W'(s', T, D) \), \( s' \) in \( \mathcal{S}' \), is measurable \( \mathcal{B}' \) for each \( D \) in \( \mathcal{B} \), where \( \mathcal{B}' \) is as described in H6T;

2) The class of sets \( \mathcal{B}' \) has the additional property that \( T'S' \) is a subclass of \( \mathcal{M}'' \);
iii) The class $\mathcal{B}'$ described in $\mathcal{H}^4T'$ is a subclass of the class $\mathcal{B}'$.

Let $i \geq 0$, let $D$ in $\mathcal{B}$, and let $H_i'(s'',T'',D)$, $s''$ in $\mathcal{S}''$, be as defined in .1; then $H_i'(s'',T'',D)$ is measurable $\mathcal{M}''$.

Proof. Let $i$ be fixed. Let $r$ real. Let $D$ in $\mathcal{B}$. 

Case 1) $r < 0$. Then, since $H_i'(s'',T'',D)$ is non-negative, 

$\{s'' \in \mathcal{S}'' : H_i'(s'',T'',D) > r\} = \mathcal{S}''$, which is in $\mathcal{M}''$. 

Case ii) $r \geq 0$. If $s''$ is in $E'' = \{s'' \in \mathcal{S}'' : H_i'(s'',T'',D) > r\}$ then there exists a unique point $s' = T' - s'' \cap D_i'$, and by the definition in .1, $H_i'(s'',T'',D) = W'(s',T,D)$. Hence, $s'$ is in the set $E' = \{s' \in \mathcal{S}': W'(s',T,D) > r\}$. As a result of this observation we have $E''$ is a subset of $T'(D_i' \cap B')$. On the other hand if $s''$ is in $T'(D_i' \cap B')$ then there is a unique point $s'$ in $\mathcal{S}'$ such that $T' - s'' \cap D_i' = s'$, and $H_i'(s'',T'',D) = W'(s',T,D) > r$, so that $s''$ is in $E''$. Thus, $E'' = T'(D_i' \cap B')$.

As a consequence of conditions 1) and iii), $D_i'$ and $B'$ are in $\mathcal{B}'$; and since $\mathcal{B}'$ is a σ-field by $H6T$, $D_i' \cap B'$ is in $\mathcal{B}'$.

By condition ii), $T'(D_i' \cap B')$ is in the class $\mathcal{M}''$, so that $E'' = T'(D_i' \cap B')$ is in $\mathcal{M}''$. Thus, for each $i \geq 0$ and each $D$ in $\mathcal{B}$, $H_i'(s'',T'',D)$, $s''$ in $\mathcal{S}''$, is measurable $\mathcal{M}''$.

7. Theorem. Assume $HF2$. Assume that $T$ is ACWD with $glbfwd f(s)$ and that $T'$ is ACW'D' with $glbfwd' g'(s')$. Assume that $H_i''(s'',T'',D)$, $s''$ in $\mathcal{S}''$, as defined in .1, is measurable $\mathcal{M}''$ for each $i \geq 0$ and each $D$ in $\mathcal{B}$. Let the weight function for $T''$ be the function $W''(s'',T'',D)$, $s''$
in $S''$, $D$ in $\mathcal{D}$, defined in .1. Then $T''$ is ACW''D if and only if $T''$ is BVW''D. $T''$ is BVW''D if and only if $f(s)g''T(s)$, $s$ in $S$, is integrable $\mu$. Further, if $T''$ is ACW''D then $f(s)g''T(s)$ is a glbW''D.

Proof. First, assume that $T''$ is BVW''D, that is, $W''(s'',T,D)$, $s''$ in $S''$, is integrable $\mu''$ for each $D$ in $\mathcal{D}$. Fix $D$ in $\mathcal{D}$. By hypothesis $T'$ is ACW'D with glbW'D $g'(s')$, and $H''_1(s'',T'',D)$, $s''$ in $S''$, is measurable $\mathcal{M}$ for each $i \geq 0$. Therefore, by Theorem 4.10T', we have

\begin{align*}
&\int_{S''} W''(s'',T'',D) d\mu'' = \int_{S''} \sum_{\mathcal{D}_i} W''(s'',T',D_i) H''_1(s'',T'',D) d\mu'' \\
&\quad = \sum_{\mathcal{D}_i} \int_{S''} W''(s'',T',D_i) H''_1(s'',T'',D) d\mu'' \\
&\quad = \sum_{\mathcal{D}_i} \int_{S''} H''_1(T's',T'',D) g'(s') d\mu',
\end{align*}

where each integral is finite valued. From HF2, we have for each $i$, $s'$ in $\mathcal{D}_i$ implies that $s' = T'-1T's' \cap \mathcal{D}_i$; hence, from the definition in .1, $H''_1(T's',T'',D) = W'(s',T,D)$. By HF2, the sets $\mathcal{D}_i$, $i \geq 0$, $X'$, and $Y'$ are pairwise disjoint and their union is $S'$. From the properties of $X'$ and $Y'$ given in HF2 and Lemma 2.7T' we can observe that $g'(s') = 0$ a.e. $\mu'$ on $X' \cup Y'$, and hence that $\int_{X' \cup Y'} W'(s',T,D) g'(s') d\mu' = 0$. Therefore, we have

\begin{align*}
&\sum_{\mathcal{D}_i} \int_{S''} H''_1(T's',T'',D) g'(s') d\mu' = \sum_{\mathcal{D}_i} \int_{S''} W'(s',T,D) g'(s') d\mu' \\
&\quad = \int_{\mathcal{D}_i} W'(s',T,D) g'(s') d\mu' \\
&\quad = \int_{S''} W'(s',T,D) g'(s') d\mu',
\end{align*}
where each integral is finite valued. By hypothesis \( T \) is ACWD with \( \text{glbfWD} f(s) \). By Definition 2.3\( T' \), \( g'(s') \) is measurable \( \mathcal{M}' \). Therefore by Theorem 4.10\( T \), we have

\[
\int_{S'} W'(s', T, D) g'(s') d\mu' = \int_{D} f(s) g'(s') T(s) d\mu,
\]

where each integral is finite valued. Since \( S \) is in \( \mathcal{D} \) it follows that \( f(s) g'(s') T(s), s \in S \), is integrable \( \mu \).

Next, assume that \( f(s) g'(s') T(s), s \in S \), is integrable \( \mu \). Let \( D \) be in \( \mathcal{D} \). Using the same arguments we obtain (3), then (2), and finally, (1); therefore, we have

\[
\int_{D} f(s) g'(s') T(s) d\mu = \int_{S''} W''(s'', T'', D) d\mu'',
\]

where each integral is finite valued. Thus, \( T \) is ACW''D and moreover, \( f(s) g'(s') T(s) \) is a \( \text{glbfW''D} \).

Finally, by definition, if \( T'' \) is ACW''D then \( T'' \) is BVW''D.
CHAPTER VIII

T IS ESSENTIALLY LOCALLY ONE-TO-ONE

Throughout this chapter we assume $H_1T-H_8T$, $H_{11}T-H_{13}T$, $H_1T'-H_9T'$, and $H_1T''-H_8T''$.

In this chapter we shall assume that $T$ is essentially locally one-to-one in the sense described in the following hypothesis:

$\text{HF3}$. There exists a countable family of pairwise disjoint sets $E_k$, $k \geq 1$, $E_k$ in $\delta \cap \mathcal{M}$, and a subset $Z$ of $S$, $Z \cap E_k = \emptyset$, $k \geq 1$, such that

1) \( S = Z \cup \bigcup_{k \geq 1} E_k \);
2) $D$ in $\mathcal{D}$ implies that $T(D \cap E_k)$ is in $\mathcal{D}'$;
3) $S_1$, $S_2$ in $\mathcal{J}$, $S_1$, $S_2 \subseteq E_k$, $k \geq 1$, and $S_1 \neq S_2$ imply that $TS_1 \neq TS_2$;

where $\mathcal{D}$ and $\mathcal{D}'$ are as described in $H_{4T}$ and $H_{4T}'$, $\mathcal{J}$ is as described in Definition 7.1T, and $\delta$ is as described in Definition 7.6T.

Remark. Recall from Definition 7.6T that if $S$ is in $\mathcal{J}$, $E$ is in $\delta$, and $S \cap E$ is not empty then $S \subseteq E$. For each $k$, if $s'$ is in $TE_k$ then HF3 implies that $T^{-1}s' \cap E_k$ is a single element $S$ of $\mathcal{J}$. Thus, for $s'$ in $\mathcal{S}'$ and $k$ fixed
either $T^{-1}s' \cap E_k$ is a unique $S$ of $\mathcal{J}$ or else $T^{-1}s' \cap E_k = \emptyset$.

Further, by H12T, $S$ in $\mathcal{J}$ implies that $TS$ is a single point of $S'$. Hence, for the function $T|_{E_k}$, $k \geq 1$, we have a one-to-one correspondence between the elements $S$, $S$ in $\mathcal{J}$, $S \subseteq E_k$, and the points $s'$, $s'$ in $S'$, $s'$ in $T|_{E_k}$.

By H13T, the weight function $W'(s',T,D)$, $s'$ in $S'$, $D$ in $\mathcal{J}$, is generated by a non-negative extended real valued function $w_S$, $S$ in $\mathcal{J}$. The following hypothesis describes a condition to be satisfied by the function $w_S$, $S$ in $\mathcal{J}$.

HF4. There exists a non-negative extended real valued sequence $r_k$, $k \geq 1$, such that $S$ in $\mathcal{J}$, $S \subseteq E_k$ implies that $w_S = r_k$. Also $S$ in $\mathcal{J}$, $S \subseteq Z$ implies that $w_S = 0$. The sets $E_k$, $k \geq 1$, and $Z$ are as described in HF3.

.2 Remark. Recall the function $W^*_k(s',T,E)$, $s'$ in $S'$, $E$ in $\mathcal{J}$, described in Definition 7.7T; namely,

$$W^*_k(s',T,E) = \sum w_S, S \in \mathcal{J}, S \subseteq E_k, TS = s'.$$

Assume HF3 and HF4, and let $k$ be fixed. Since $\mathcal{J}$ is a $\sigma$-field containing $\mathcal{J}$ it follows that $D \cap E_k$ is in $\mathcal{J}$ for each $D$ in $\mathcal{J}$. Let $s'$ in $S'$ and let $D$ in $\mathcal{J}$. If $s'$ is in $C(T(D \cap E_k))$ then $T^{-1}s' \cap E_k = \emptyset$, and $W^*_k(s',T,D \cap E_k) = 0$; if $s'$ is in $T(D \cap E_k)$ then, by the remark in .1, there is a unique $S$ such that $S$ in $\mathcal{J}$, $S \subseteq E_k$, $TS = s'$, and we have $W^*_k(s',T,D \cap E_k) = w_S = r_k$. Also it is clear that $Z = \bigcup E_k$, $k \geq 1$, is in $\mathcal{J}$, and that $W^*_k(s',T,Z) = 0$, $s'$ in $S'$.

.3 Definition. Assume HF3 and HF4. For $D$ in $\mathcal{J}$,
k ≥ 1, set D′_k = T(D∩E_k); by HF3 ii) D′_k is in J for each k.
Let W''(s'', T'D'), s'' in S'', D' in J, be the weight function for T' described in H9T'. For k ≥ 1, s'' in S'', D in J, we define

W''_k(s'', T'', D) = W''(s'', T', D_k) r_k.

For s'' in S'', D in J, we define

W''(s'', T'', D) = Σ W''_k(s'', T'', D), k ≥ 1.

Lemma. Assume HF3 and HF4. Let W''(s'', T'', D) be as defined in .3. If D is in J and s'' is in C''T''D then W''(s'', T'', D) = 0.

Proof. Let D in J. Let s'' in C''T''D. Let k ≥ 1, then s'' is in C''T'T'(D∩E_k) = C''T'D_k', so that by H9T' W''(s'', T'', D_k') is zero. It follows that W''(s'', T'', D) = Σ W''_k(s'', T'', D) = Σ W''_k(s'', T'', D_k') r_k = 0.

Lemma. Assume HF3 and HF4. Let W''(s'', T'', D) be as defined in .3. For each s'' in S'', W''(s'', T'', D) is under additive on J.

Proof. Let D in J contain a countable number of pairwise disjoint sets D_1, i ≥ 1, in J. Let s'' in S''. Let k be fixed. The sets D_1∩E_k, i ≥ 1, are pairwise disjoint subsets of D∩E_k ≤ E_k. Also, T(D_1∩E_k) ≤ T(D∩E_k), for each i. From HF3 the sets T(D_1∩E_k), i ≥ 1, are pairwise disjoint; which is verified as follows: s' in T(D_1∩E_k)∩T(D_j∩E_k) ≤ T E_k implies that there exists an S in J such that S = T^{-1}s'∩E_k, and
\[ T^{-1}s' \cap E_k = T^{-1}[T(D_1 \cap E_k) \cap T(D_j \cap E_k)] \cap E_k \]
\[ = (D_1 \cap E_k) \cap (D_j \cap E_k), \]
contrary to \((D_1 \cap E_k)\) pairwise disjoint, \(i \geq 1\).

Let \(D_k' = T(D \cap E_k)\) and \(D_{k1} = T(D_1 \cap E_k), \ k \geq 1, \ i \geq 1; \) by
HF3 ii) these sets are in \(\mathcal{B}'\). By H9T' ii), for each \(k, \)
\[ \Sigma_1 W''(s'', T'', D_{k1}) \leq W''(s'', T'', D_k'), \] since \(D_{k1} = T(D_1 \cap E_k), \ i \geq 1, \)
are a countable number of pairwise disjoint sets contained
in \(D_k' = T(D \cap E_k)\). Using the definition in 3, we have
\[ \Sigma_1 W''(s'', T'', D_1) = \Sigma_1 \Sigma_k W''_k(s'', T'', D_1) \]
\[ = \Sigma_1 \Sigma_k r_k W''(s'', T'', D_{k1}) \]
\[ = \Sigma_k r_k \Sigma_1 W''(s'', T'', D_{k1}) \]
\[ \leq \Sigma_k r_k W''(s'', T'', D_k') \]
\[ = \Sigma_k W''_k(s'', T'', D) \]
\[ = W''(s'', T'', D). \]

.6 Lemma. Assume HF3 and HF4. Let \(W''(s'', T'', D)\) be
as defined in 3. For each \(s''\) in \(\mathcal{S}''\), \(W''(s'', T'', D)\) is inner
continuous on \(\mathcal{B}\).

Proof. Let \(D\) in \(\mathcal{B}\) be the union of a countable num­
ber of sets \(D_j\) in \(\mathcal{B}\) such that \(D_j \subseteq D_{j+1}\) for every \(j\). Let
\(s''\) in \(\mathcal{S}''\) be fixed.

Let \(k\) be fixed; the sets \(D_j \cap E_k, \ k \geq 1, \) are subsets of
\(E_k\) such that their union is \(D \cap E_k\) and \(D_j \cap E_k \subseteq D_{j+1} \cap E_k\) for
every \(j\). From HF3 ii) the sets \(T(D \cap E_k)\) and \(T(D_j \cap E_k), j \geq 1, \)
are in \(\mathcal{B}'\). Put \(D_k' = T(D \cap E_k)\) and \(D_{k1} = T(D_j \cap E_k), j \geq 1; \) then,
by H9T' iii), we have \( \lim_j W''(s'',T',D_{kj}) = W''(s'',T',D_k) \).

All values involved are non-negative; therefore, we have, by the definition in .3,

\[
\lim_j W''(s'',T'',D_j) = \lim_j \sum_k W''_k(s'',T'',D_j)
= \lim_j \sum_k r_k \lim_j W''(s'',T',D_{kj})
= \sum_k r_k \lim_j W''(s'',T',D_{kj})
= \sum_k W''_k(s'',T'',D)
= W''(s'',T'',D).
\]

.7 Lemma. Assume HF3 and HF4. Let \( W''(s'',T'',D) \) be as defined in .3. For each \( D \) in \( \mathcal{D} \), \( W''(s'',T'',D) \), \( s'' \) in \( S'' \), is measurable \( \mathcal{M}'' \).

Proof. Let \( D \) in \( \mathcal{D} \). By H9T' iv), \( W''(s'',T',D') \), \( s'' \) in \( S'' \), is measurable \( \mathcal{M}'' \) for each \( D' \) in \( \mathcal{D}' \). It follows from the definition in .3 that for each \( k \), \( W''_k(s'',T'',D) \), \( s'' \) in \( S'' \), is measurable \( \mathcal{M}'' \), and hence that \( W''(s'',T'',D) \), \( s'' \) in \( S'' \), is measurable \( \mathcal{M}'' \), since all values involved are non-negative.

.8 Theorem. Assume HF3 and HF4. The function \( W''(s'',T'',D) \), \( s'' \) in \( S'' \), \( D \) in \( \mathcal{D} \), as defined in .3, is a weight function for \( T'' = T'T \).

Proof. This is a consequence of the lemmas in .4, .5, .6, and .7.

.9 Theorem. Assume HF3 and HF4. Let \( T \) be ACWD with \( \text{glbfWD} f(s) \) and let \( T' \) be ACW'D' with \( \text{glbfW'D'} g'(s') \). Let
the weight function for $T''$ be the function $W''(s'',T'',D)$, $s''$ in $S''$, $D$ in $B$, defined in .3. Then $T''$ is ACW''D if and only if $T''$ is BVW''D. $T''$ is BVW''D if and only if $f(s)g'T(s)$, $s$ in $S'$, is integrable $\mu$. Further, if $T''$ is ACW''D then $f(s)g'T(s)$ is a glbfW''D.

Proof. First, assume that $T''$ is BVW''D; that is,

$$W''(s'',T'',D) = \sum_k r_k W''(s'',T'',D_k)$$

is integrable $\mu$ for each $D$ in $B$. By hypothesis $T'$ is ACW'D with glbfW'D $g'(s')$. For each $k$, $r_k$ is constant. Fix $D$ in $B$. By Theorem 4.10T', we have

$$\int_{S''} W''(s'',T'',D)d\mu'' = \int \sum_k r_k W''(s'',T'',D_k)d\mu''$$

$$= \sum_k \int_{S''} r_k W''(s'',T'',D_k)d\mu''$$

$$= \sum_k \int_{D_k} r_k g'(s')d\mu'',$$

where each integral is finite valued. For each $k$, $s'$ in $D_k' = T(D\cap E_k)$ implies $r_k = W_*(s',T,D\cap E_k)$ and $s'$ in $C'D_k'$ implies $0 = W_*(s',T,D\cap E_k)$. Therefore,

$$\sum_k \int_{D_k'} r_k g'(s')d\mu' = \sum_k \int_{T(D\cap E_k)} W_*(s',T,D\cap E_k) g'(s')d\mu'$$

$$= \sum_k \int_{S''} W_*(s',T,D\cap E_k) g'(s')d\mu'',$$

where each integral is finite valued. By hypothesis, $T$ is ACWD with glbfWD $f(s)$; therefore, by the theorem in .10 of Chapter III, $T$ is ACW_4E with glbfW_4E $f(s)$. By HF3, $E_k$ is in $\delta\cap m$, and hence $D\cap E_k$ is in $\delta\cap m$, $k \geq 1$; by definition 2.3T',
\( g'(s'), s' \text{ in } \mathcal{S}', \) is measurable \( \mathcal{M}' \), so that by the theorem in .15 of Chapter III,

\[
\Sigma_k W'_k(s', T, D \cap E_k) g'(s') \, du' = \Sigma_k f(s) g' T(s) \, du,
\]

where each integral is finite valued. It follows from HF3 that \( D = D \cap Z \cup \bigcup_k D \cap E_k \). It is noted in .2 that \( W'_k(s', T, Z) = 0 \). It follows that \( \int_Z f(s) \, du = \int_{\mathcal{S}'} W'_k(s', T, Z) = 0 \). Since \( f(s) \) is non-negative this gives \( f(s) = 0 \) a.e. \( \mu \) on \( Z \); hence \( f(s) g' T(s) = 0 \) a.e. \( \mu \) on \( Z \). Therefore,

\[
\Sigma_k f(s) g' T(s) \, du = \int_D f(s) g' T(s) \, du
\]

where each integral is finite valued. Since \( \mathcal{S} \) is in \( \mathcal{D} \) it follows that \( f(s) g' T(s), s \text{ in } \mathcal{S}, \) is integrable \( \mu \).

Next, assume that \( f(s) g' T(s), s \text{ in } \mathcal{S}, \) is integrable \( \mu \). Let \( D \) in \( \mathcal{D} \). Using the same arguments we obtain (4), (3), (2), and finally, (1), and hence

\[
\int_D f(s) g' T(s) \, du = \int_{\mathcal{S}''} W''(s'', T'', D) \, du'',
\]

where each integral is finite valued. Thus \( T'' \) is ACW''D and moreover, \( f(s) g' T(s) \) is a glbfW''D.

Finally, by definition, if \( T'' \) is ACW''D then \( T'' \) is BVW''D.
CHAPTER IX

A GENERATED WEIGHT FUNCTION FOR T" 

Throughout this chapter H1T-H8T, H11T-H13T, H1T'-H8T', H11T'-H13T', H1T''-H8T'', H11T'', and H12T'' are always assumed.

1 Definition. Let the sets S of the class J be as described in Definition 7.1T, and let the sets S' of the class J' be as described in Definition 7.1T'. By H13T the weight function \( W'(s', T, D) \), s' in \( S' \), D in \( J' \), is generated by a non-negative extended real valued function \( w_S \), S in \( J \). By H13T' the weight function \( W''(s'', T', D') \), s'' in \( S'' \), D' in \( J' \), is generated by a non-negative extended real valued function \( w'S' \), S' in \( J' \). By H12T the image TS of each S in \( J \) consists of a single point s' in \( S' \). By Lemma 7.2T' the sets in \( J' \) constitute a partition of \( S' \). For each s' in \( S' \) we define

\[ w's' = w'S' \], s' in \( S' \) in \( J' \).

For each S in \( J \) we define

\[ w'S = wSw's' \], s'=TS.

For each s'' in \( S'' \) and each D in \( J \) we define

\[ W''(s'', T'', D) = \sum w'S \], S in \( J \), S=D, T''S=s''.

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where the expression on the right stands for the least upper bound of \( \{0, \sum_{k=1}^{p} w^S_{S_k} : S_k \text{ in } S, S_k \subseteq D, T"S_k = s"\} \).

2 Remark. The function \( w^S \) is a non-negative extended real valued function defined for \( S \) in \( J \) so that, as pointed out in Remark 7.9T", a sufficient condition for \( w^S \) to be a weight function for \( T" \) is that \( w^S(s", T", D), s" \) in \( S" \), is measurable \( m" \) for each \( D \) in \( J \).

3 Lemma. Let \( D \) in \( J \) and \( s" \) in \( S" \) be given, and let \( W^S(s", T", D) \) be as defined in .1. Then

\[
W^S(s", T", D) = \sum w^S_w'(s', T, D), \quad s' \text{ in } TD, \quad T's'=s",
\]

where the expression on the right stands for the least upper bound of \( \{0, \sum_{k=1}^{p} w^S_{s_k}'(s_k', T, D) : s_k' \text{ in } TD, \quad T's_k'=s"\} \).

Proof. Fix \( D \) in \( J \), \( s" \) in \( S" \). Let \( S_k, k=1,2,...,p \), be sets such that \( S_k \) is in \( J \), \( S_k \subseteq D \), and \( T S_k = s" \). It follows that there exists a set of points \( s_j', j=1,2,...,m \), having the following properties: \( s_j' \) in \( S', s_j' \) in \( TD \), \( T's_j' = s" \), and for each \( k \) there is a unique \( j \) such that \( T S_k = s_j' \). Renumber the sets \( S_k, k=1,2,...,p \), as \( S_j1, j=1,2,...,m, i=1,2,...,r_j \), in such a manner that we have, for every \( j, i, TS_j1 = s_j'. \) Then we may write

\[
\sum_{k=1}^{p} w^S_{S_k} = \sum_{j=1}^{m} \sum_{i=1}^{r_j} w^S_{S_j1} = \sum_{j=1}^{m} \sum_{i=1}^{r_j} w^S_{S_j1} W^S_{s_j'} = \sum_{j=1}^{m} w^S_{S_j'} \sum_{i=1}^{r_j} w^S_{S_j1}.
\]

From H13T we have, for each \( s' \) in \( S' \),
\[ W'(s', T', D) = \text{lub}\{0, \Sigma_{k=1}^{n} wS_k: S_k \text{ in } \mathcal{S}, S_k \subseteq D, TS_k = s'\}; \]
hence, for each \( j \), \( \Sigma_{j=1}^{m} w's_j \Sigma_{j=1}^{n} wS_j \leq W'(s', T, D) \). Consequently,
\[
\Sigma_{j=1}^{m} w's_j \Sigma_{j=1}^{n} wS_j \leq \Sigma_{j=1}^{m} w's_j W'(s', T', D)
\leq \Sigma w's_j W'(s', T', D), s' \text{ in TD, } Ts' = s'.
\]
Thus, \( \Sigma w's_j W'(s', T', D), s' \text{ in TD, } Ts' = s' \), is an upper bound for \( \{0, \Sigma_{k=1}^{p} w'S_k: S_k \text{ in } \mathcal{S}, S_k \subseteq D, T'S_k = s'\} \); hence,
\[ W''(s'', T'', D) = \Sigma w'S_j W'(s', T', D), s' \text{ in TD, } Ts' = s'.
\]
On the other hand, we can also show that
\[ \Sigma w'S_j W'(s', T', D), s' \text{ in TD, } Ts' = s'' \leq W''(s'', T'', D), \]
which with (1) gives the conclusion of the lemma. In view of the definition of \( \Sigma w'S_j W'(s', T', D), s' \text{ in TD, } Ts' = s'' \), we may prove (2) by noting that \( 0 \leq W''(s'', T'', D) \), and showing
that any number of the form
\[ \Sigma_{j=1}^{n} w's_j W'(s', T', D), s_j \text{ in TD, } Ts_j = s'' \]
is \( \leq W''(s'', T'', D) \). We show this last statement as follows: Consider any sum of the form (3). Without loss of generality we may assume that each term is positive. Choose real numbers \( r_j, j=1,2,...,n \), such that \( 0 < r_j < w's_j W'(s', T', D) \). Fix \( j \). From \( W'(s', T', D) = \text{least upper bound} \{0, \Sigma_{k=1}^{p} wS_k: S_k \text{ in } \mathcal{S}, S_k \subseteq D, TS_k = s'\} \) it follows that
\[ w's_j W'(s', T', D) = \text{lub}\{0, w's_j \Sigma_{k=1}^{p} wS_k: S_k \text{ in } \mathcal{S}, S_k \subseteq D, TS_k = s'\}. \]
Hence there exists a family of sets \( S_{j_1}, i=1,...,p_j \), such that \( r_j \leq w's_j^2 \Sigma_{i=1}^{p_j} wS_{j_i} \) and \( S_{j_i} \text{ is in } \mathcal{S}, S_{j_1} \subseteq D, TS_{j_1} = s' \);
moreover, \( Ts_{j_1} = s'' \), so that, for each \( i \), \( T'S_{j_1} = s'' \).
Therefore, we have

\[ \sum_{j=1}^{n} r_j \leq \sum_{j=1}^{n} w's_j \sum_{i=1}^{p_j} w'S_{j_i} \]
\[ = \sum_{j=1}^{n} \sum_{i=1}^{p_j} w'S_{j_i} w's_j \]
\[ = \sum_{j=1}^{n} \sum_{i=1}^{p_j} w'S_{j_i} \]
\[ \leq \text{lub} \{0, \sum_{k=1}^{p} wS_k : S_k \in \mathcal{J}, S_k \subseteq D, T''S_k = s'' \} \]
\[ = W''(s'', T'', D). \]

From the manner in which the \( r_j, j=1, \ldots, n, \) were chosen it follows that any sum of the form (3) is \( \leq W''(s'', T'', D). \)

Next, we introduce a restriction on the weight function \( W'(s', T, D), \ s' \in \mathcal{S}', \ D \in \mathcal{D}. \)

\( \text{HF5.} \) For each \( D \in \mathcal{D} \) \( W'(s', T, D), \ s' \in \mathcal{S}', \) has a countable range \( \{a_k\}, \ k=0,1,2,\ldots. \)

Observe that the value 0 may be assigned as \( a_0. \) The set of values may depend upon the particular set \( D. \) A condition sufficient to ensure \( \text{HF5} \) is, for example, the requirement that \( wS, S \in \mathcal{J}, \) is integral valued.

Next, we introduce a restriction on the sets \( \mathcal{S}' \) of the class \( \mathcal{J}'. \)

\( \text{HF6.} \) Let \( \mathcal{J}' \) be as described in Definition 7.1T'. Then each \( S' \in \mathcal{J}' \) consists of a single point.

Remark. Assume \( \text{HF6}. \) Let \( w'S' \) be defined on \( \mathcal{J}'. \) Let \( s' \) be a point of \( \mathcal{S}'; \) by Lemma 7.2T', there exists a unique \( S' \in \mathcal{J}' \) such that \( s' \) is in \( S'. \) Recall \( w's' = w'S', \)
\( s' \in \mathcal{S}', \ s' \in S' \in \mathcal{J}'. \) From \( \text{HF6} \) \( S' = \{s'\}, \) and we have,
for each \( S' \) in \( \mathcal{S}' \), \( w'S' = w's' \), \( S' = \{ s' \} \). Let the class \( \mathcal{D}' \) be as described in Definition 7.6T'. From HF6 the class \( \mathcal{D}' \) consists of all subsets \( E' \) of \( \mathcal{S}' \).

By H13T W*(s'', T', D'), \( s'' \) in \( \mathcal{S}'' \), \( D' \) in \( \mathcal{D}' \), is generated by a non-negative extended real valued function \( w'S' \), \( S' \) in \( \mathcal{S}' \). By Definition 7.6T' we have, for each \( s'' \) in \( \mathcal{S}'' \) and each \( E' \) in \( \mathcal{D}' \),

\[
W''(s'', T', E') = \Sigma w's', \ S' \text{ in } \mathcal{S}', \ S' \subseteq E', \ T'S' = s''.
\]

From HF6 and \( w's' = w'S' \), \( \{ s' \} = S' \) in \( \mathcal{S}' \), we have, for each \( s'' \) in \( \mathcal{S}'' \) and each subset \( E' \) of \( \mathcal{S}' \),

\[
W''(s'', T', E') = \Sigma w's', \ s' \text{ in } \mathcal{S}', \ s' \text{ in } E', \ T's's''.
\]

5 Definition. Assume HF5 and HF6. Let \( D \) in \( \mathcal{D} \) be fixed. Let \( \{ a_k \}, k=0,1,2,\ldots, \) be the set of values of \( W'(s', T, D) \). For each \( k \) define

\[
E_k' = \{ s': W'(s', T, D) = a_k \}.
\]

By H13T \( W'(s', T, D) \), \( s' \) in \( \mathcal{S}' \), is measurable \( \mathcal{M}' \); therefore, for each \( k \), \( E_k' \) is in \( \mathcal{M}' \). Further, the sets \( E_k' \), \( k \geq 0 \), are pairwise disjoint and their union is \( \mathcal{S}' \).

6 Lemma. Assume HF5 and HF6. Let \( D \) in \( \mathcal{D} \) and \( s'' \) in \( \mathcal{S}'' \) be given; let \( W''(s'', T'', D) \) be as defined in .1; and let \( E_k' \), \( k \geq 0 \), be as defined in .5. Then

\[
W''(s'', T'', D) = \Sigma a_k W''(s'', T', E_k').
\]

Proof. By H13T \( W'(s', T, D) = 0 \) for \( s' \) in \( C'TD \); therefore, from the lemma in .3, we have

\[
W''(s'', T'', D) = \Sigma w's' W'(s', T, D), \ s' \text{ in } TD, \ T's's''
\]
The sets $E_k^s$, $k \geq 0$, are pairwise disjoint and their union is $S'$; by the remark in .4 we have, for each $k$, $E_k^s$, $s'$ in $E_k^s$, $T's'=s''$, is equal to $W'(s'',T',E_k^s)$. Thus,

$$W''(s'',T'',D) = \sum_k \sum_w s''W'(s',T,D), s' in E_k^s, T's'=s''$$

$$= \sum_k \sum_k W''(s'',T',E_k^s).$$

**Lemma.** Assume HF5 and HF6. Assume that $T'$ is ACW'D' and that $g'(s')$ is a glbfW'D'. Let $D$ in $\mathcal{D}$ be fixed. Let the sets $E_k^s$, $k \geq 0$, be as defined in .5. Then for each $k$

$$W'(s'',T',E_k^s), s'' in S'', is measurable in $\mathcal{M}'$, and$$

$$\int_{E_k^s} g'(s')d\mu' = \int_{S''} W'(s'',T',E_k^s)d\mu''.$$

**Proof.** Let $k$ be fixed. Recall that $W'(s',T,D), s'$ in $S'$, is measurable in $\mathcal{M}'$; hence it follows from the definition in .5 that $E_k^s$ is in $\mathcal{M}'$. Thus $E_k^s$ is in the class $S' \cap \mathcal{M}'$. From the lemma in .8 of Chapter III, as applied to $T'$, $W'(s'',T',E_k^s), s'' in S'', is measurable in $\mathcal{M}'$. From the theorem in .10 of Chapter III, as applied to $T'$,

$$\int_{E_k^s} g'(s')d\mu' = \int_{S''} W'(s'',T',E_k^s)d\mu''.$$

**Theorem.** Assume HF5 and HF6. Assume that $T'$ is ACW'D' and that $g'(s')$ is a glbfW'D'. Let $W''(s'',T'',D), s'' in S'', D$ in $\mathcal{D}$, be as defined in .1. Then $W''(s'',T'',D), s'' in S'', D$ in $\mathcal{D}$, is a weight function for $T''$.

**Proof.** As remarked in .2 it is sufficient to show
that \( W''(s'', T'', D) \), \( s'' \) in \( S'' \), is measurable \( \mathcal{M}'' \) for each \( D \) in \( \mathcal{B} \). Let \( D \) in \( \mathcal{B} \) be fixed. Let the sets \( E_k^i \), \( k \geq 0 \), be as defined in .5. By the lemma in .6, \( W''(s'', T'', D) = \sum_k a_k W''(s'', T'', E_k^i) \), where the \( a_k \), \( k \geq 0 \), are constants. By the lemma in .7, \( W''(s'', T'', E_k^i) \), \( s'' \) in \( S'' \), is measurable \( \mathcal{M}'' \). It now follows, in view of the fact that all values are non-negative, that \( W''(s'', T'', D) \), \( s'' \) in \( S'' \), is measurable \( \mathcal{M}'' \).

.9 Theorem. Assume HF5 and HF6. Assume that \( T \) is ACWD with glbfWD \( f(s) \), and that \( T' \) is AC\'W'D' with \( \text{glbfW'D'} g'(s') \). Let the weight function for \( T'' \), \( W''(s'', T'', D) \), \( s'' \) in \( S'' \), \( D \) in \( \mathcal{B} \), be as defined in .1. Then \( T'' \) is AC\"W'D if and only if \( T'' \) is BVW'D. \( T'' \) is BVW'D if and only if \( f(s)g'T(s) \), \( s \) in \( S \), is integrable \( \mu \). Further, if \( T'' \) is AC\"W'D then \( f(s)g'T(s) \) is a glbfW'D.

Proof. First, assume that \( T'' \) is BVW'D. In view of the definition in .5 and the lemmas in .6 and .7, we have, for each \( D \) in \( \mathcal{B} \),

\[
(1) \quad \int_{S''} W''(s'', T'', D)d\mu'' = \int_{S''} \sum_k a_k W''(s'', T'', E_k^i)d\mu'' = \sum_k \int_{E_k^i} a_k W''(s'', T'', E_k^i)d\mu'' = \sum_k \int_{E_k^i} a_k g'(s')d\mu' = \sum_k \int_{E_k^i} W'(s', T, D)g'(s')d\mu',
\]

where each integral has finite value. By Definition 2.3, \( g'(s') \) is measurable \( \mathcal{M}' \). By hypothesis \( T \) is ACWD with
glbfWD $f(s)$. Therefore, by Theorem 4.10T and the fact that the sets $E_k'$, $k \geq 0$, are a collection of pairwise disjoint sets whose union is $S'$, we have

$$\sum_{k} \int_{E_k'} W'(s', T, D) g'(s') d\mu' = \int_{S'} W'(s', T, D) g'(s') d\mu'$$

$$= \int_{D} f(s) g'T(s) d\mu,$$

where each integral has finite value. Since $S$ is in $\mathcal{J}$, $f(s) g'T(s), s \in S$, is integrable $\mu$.

Next, assume that $f(s) g'T(s), s \in S$, is integrable $\mu$. Fix $D$ in $\mathcal{J}$. Then by the same arguments we obtain (2) and then (1); therefore,

$$\int_{D} f(s) g'T(s) d\mu = \int_{S''} W''(s'', T'', D) d\mu''.$$

Thus $T''$ is ACW$''D$ and $f(s) g'T(s), s \in S$, is a glbfW$''D$.

Finally, by definition, $T''$ ACW$''D$ implies that $T''$ is BVW$''D$. 
CHAPTER X

AN APPLICATION

No a priori assumptions are made in this chapter. Instead we describe a situation in which $H_{1T}-H_{8T}$, $H_{11T}-H_{13T}$, $H_{1T'}-H_{8T'}$, $H_{11T'}-H_{13T'}$, $H_{1T''}-H_{8T''}$, $H_{11T''}$, $H_{12T''}$, $HF_5$, and $HF_6$ are satisfied, and hence to which the theorem in .9 of Chapter IX applies.

It is noted in Chapter 9 of [5] that $H_{1T}-H_{8T}$, $H_{11T}$, and $H_{12T}$ are satisfied under the following conditions:

1) Let $\mathbb{S}$ be a non-empty domain in Euclidean $n$-space, $\mathbb{R}^n$. Let $\mathcal{M}$ denote the class of all subsets $M$ of $\mathbb{S}$ which are $n$-dimensionally Lebesgue measurable, and let $\mu_M$ denote the $n$-dimensional Lebesgue measure of $M$. Then $[\mathbb{S}, \mathcal{M}, \mu]$ is a $\sigma$-finite complete measure space. Thus $H_{1T}$ is satisfied.

2) Let $T$ be a continuous mapping whose domain of definition is $\mathbb{S}$ and whose range $\mathbb{S}'$ is a subset of $\mathbb{R}^n$. Recall that $\mathbb{S}'$ is a Borel set. Then $T$ transforms $\mathbb{S}$ onto $\mathbb{S}'$. Thus $H_{3T}$ is satisfied.

3) Let $\mathcal{M}'$ denote the set of all subsets $M'$ of $\mathbb{S}'$ which are $n$-dimensionally Lebesgue measurable, and let $\mu'M'$
be the n-dimensional Lebesgue measure of $M'$; \( \{S', m', \mu'\} \) is a \( \sigma \)-finite complete measure space. Thus $H2T$ is satisfied.

iv) Let $\mathcal{D}$ denote the set of all domains in $S$. Then $\mathcal{D}$ possesses all the properties listed in $H4T$.

v) Let $D$ be a domain in $\mathcal{D}$ whose boundary relative to $S$, $BD$, is such that $\mu BD = 0$. Then $D$ is of type $\gamma T$. For every domain $D$ in $\mathcal{D}$ there exists a countable sequence of domains $D_j$ in $\mathcal{D}$ such that $\mu BD_j = 0$ for each $j$, $D_j \subseteq D_{j+1}$ for each $j$, and $\cup D_j = D$. Thus $H5T$ is satisfied.

vi) Let $B'$ denote the set of all Borel sets $B'$ which are subsets of $S'$. Thus $H6T$ is satisfied.

vii) If $O'$ is a set open in $S'$ then from the continuity of $T$ it follows that $O'$ belongs to the class $O'$ described in Definition 2.1T. From this fact one concludes that $O'$ bears the relation to $M'$ required in $H7T$.

viii) Let $O'$ be a set open in $S'$ whose boundary relative to $S'$, $B'O'$, is such that $\mu B'O' = 0$. Then $O'$ is of type $\gamma'T$. For every set $O'$ open in $S'$ there exists a countable sequence of sets $O'_j$ open in $S'$ such that $\mu B'O'_j = 0$ for each $j$, $O'_j \subseteq O'_{j+1}$ for each $j$, and $\cup O'_j = O'$. Hence every set $O'$ open in $S'$ is of type $\gamma'T$. Now for any set $M'$ in $M'$ there exists a sequence of sets $O'_j$ in $O'$ such that each $O'_j$ is open in $S'$, $O'_j \supseteq O'_{j+1}$ for every $j$, and $\mu'(\cap O'_j \cap C'M') = 0$. Thus $H8T$ is satisfied.

ix) In view of the choice of $\mathcal{D}$ it is clear that
each set \( S \) of the class \( J \) described in Definition 7.1T consists of a single point, and that \( H11T \) and \( H12T \) are satisfied.

2 Consider the remaining hypotheses except \( H13T \), \( H13T' \) and \( HF5 \) which concern the weight functions.

The measure space \( \{S', m', \mu'\} \) described in .1 iii) satisfies \( H1T' \) and \( \{S, m, \mu\} \) of .1 i) satisfies \( H1T'' \).

We let \( S^* \) be a domain in \( R^n \) such that \( S' \subset S^* \). Let \( T^* \) be a continuous mapping from \( S^* \) into \( R^n \). Let \( T' = T^*|S' \). Let \( T'' = T^*T = T'T \). Let \( S'' \) be the Borel set \( T'S' = T''S' \). \( T'' \) is is a continuous mapping from the domain \( S \) onto \( S'' \). Thus \( H3T' \) and \( H3T'' \) are satisfied.

Let \( M'' \) denote the set of all subsets \( M'' \) of \( S'' \) which are \( n \)-dimensionally Lebesgue measurable, and let \( \mu'' M'' \) be the \( n \)-dimensional Lebesgue measure of \( M'' \). Then \( \{S'', m'', \mu''\} \) is a \( \sigma \)-finite complete measure space. Thus \( H2T' \) and \( H2T'' \) are satisfied.

Let \( D' \) be the class of sets \( D' \) such that \( D' \) is the intersection of \( S' \) with a domain in \( R^n \). It follows that \( D' \) satisfies \( H4T' \). The class \( D \) described in .1 iv) satisfies \( H4T'' \). Let \( D' \) be a set in \( D' \) which is the intersection of \( S' \) with a domain in \( R^n \) whose boundary has measure zero. It follows that \( D' \) is of type \( \gamma'T' \) and that \( H5T' \) is satisfied. It follows from .1 v) that sets of type \( \gamma'T \) are also of type \( \gamma'T'' \) and that \( H5T'' \) is satisfied.
Let $\mathcal{B}^n$ denote the set of all Borel sets $B^n$ in $\mathbb{R}^n$ which are subsets of $\mathbb{R}^n$. It follows that $H_{T'}$ and $H_{T''}$ are both satisfied.

If $O^n$ is a set open in $\mathbb{R}^n$ then from the continuity of $T'$ and $T''$ it follows that $O^n$ belongs to the classes $O^nT'$ and $O^nT''$ as described in Definitions 2.1T' and 2.1T''. It follows that $H_{T'}$ and $H_{T''}$ are satisfied. Let $O^n$ be a set open in $\mathbb{R}^n$ whose boundary relative to $\mathbb{R}^n$, $B^nO^n$, is such that $\mu^nB^nO^n = 0$. Then $O^n$ is of type $\gamma^nT'$ and $O^n$ is of type $\gamma^nT''$. As in .1 viii) it follows that $H_{T'}$ and $H_{T''}$ both are satisfied.

From the choice of $S'$ it is clear that each set $S'$, as described in Definition 7.1T', is a single point, so that $H_{11T'}, H_{12T'},$ and $H_{F6}$ are satisfied. Also $H_{11T''}$ is identical with $H_{11T'}$ and $H_{12T''}$ follows from $H_{12T}$ and $H_{3T'}$.

3 Next, letting $S$ and $T$ be as described in .1 i) and ii) and assuming further that $S$ and $TS$ are bounded, we restate, with a modification in the notation, the following definitions and results from Chapter V.4 of [6]. For each point $x$ in $\mathbb{R}^n$ and each subset $E$ of $\mathbb{R}^n$, let $N(x,T,E)$ be the number of points in the set $T^{-1}x \cap E$. Whenever $E$ is a Borel set $N(x,T,E)$ is a Lebesgue measurable function of $x$. $T$ is of bounded variation in $\mathbb{R}^n$ in the Banach sense—briefly BVB—if and only if $N(x,T,S)$ is Lebesgue summable on $\mathbb{R}^n$. If $T$ is BVB then the function of domains $\int_{\mathbb{R}^n} N(x,T,D) dL$, $D \subseteq S$, 


where \( L \) denotes Lebesgue measure, has a derivative \( D_B(s,T) \) a.e. in \( S \) and \( D_B(s,T) \) is Lebesgue summable on \( S \). \( T \) is absolutely continuous in \( S \) in the Banach sense—briefly ACB—if and only if \( T \) is BVB and \( T \) satisfies condition (\( N \)), where condition (\( N \)) is the following: Every subset of \( S \) of measure zero is transformed by \( T \) into a set of measure zero. If \( T \) is ACB, \( H(x) \) is a finite real valued Lebesgue measurable function on \( \mathbb{R}^n \), and \( M \) is a Lebesgue measurable subset of \( S \) then,

\[
\int_{M} H(Ts)DB(s,T)dL = \int_{\mathbb{R}^n} H(x)N(x,T,M)dL
\]
as soon as one of the two integrals involved exists.

Recall from .1 ix) above that each \( S \) in \( \mathcal{J} \) consists of a single point, and hence the class of sets \( \mathcal{J} \) consists of all subsets of \( S \). Define a non-negative extended real valued function \( w \) on \( \mathcal{J} \) by \( wS = 1, S \in \mathcal{J} \). Then, for \( E \) in \( \mathcal{J} \), \( s' \) in \( S' = TS \in \mathbb{R}^n \), we have \( N(s',T,E) = EwS, S \in \mathcal{J}, S \subseteq E, TS = s' \). Since for Borel sets \( E \) the function \( N(x,T,E), x \in \mathbb{R}^n \), is Lebesgue measurable, the function \( W'(s',T,D) = N(s',T,D), s' \in S', D \in \mathcal{J}, \) is a weight function for \( T \) which satisfies \( H13T \) and, moreover, \( HF5 \). It is clear that \( T \) is BVWD if and only if \( T \) is BVB. By comparing condition (\( N \)) with Theorem 7.24T, we see that \( T \) is ACWD if and only if \( T \) is ACB, since from \( wS = 1, S \in \mathcal{J} \), it follows that \( S^+ = \emptyset \). Using (3) with \( H(x) = 1, x \in \mathbb{R}^n \), we have, when \( T \) is ACB and hence ACWD,
\[ \int_{S} D_B(s, T) \, d\mu' = \int_{R^n} N(x, T, S) \, dL = \int_{S'} W'(s', T, S) \, d\mu' \]

recalling that \( \mu \) and \( \mu' \) are n-dimensional Lebesgue measure, and noting that \( N(x, T, S) = 0 \) for \( x \) not in \( S' = TS \). In view of the uniqueness of the glbf WD it follows that the derivative \( D_B(s, T) \) is equal to a glbf WD a.e. \( \mu \) on \( S \).

Recall \( S^* \) and \( T^* \) described in \( \ref{6.2} \) and assume further that \( S^* \) and \( T^* S^* \) are bounded. The statements of Chapter V.4 of [6] may be applied to \( T^* \). For \( E^* \subset S^* \), let \( N(x, T^*, E^*) \), \( x \) in \( R^n \), denote the number of points in \( T^{-1} x \cap E^* \). Let \( D_B(s^*, T^*) \), \( s^* \) in \( S^* \), denote the derivative.

Recall from \( \ref{6.2} \) that \( S' = S^* \), \( T' = T^* \mid S' \), and \( S'' = T' S' = T^* S^* \). Also recall that each \( S' \) in \( \delta' \) is a single point. Put \( w'S' = 1 \), \( S' \) in \( \delta' \). Then for each \( E' \) in \( \delta' \), where \( \delta' \) is as described in Definition \( 7.6T' \), and each point \( s'' \) in \( S'' = T^* S' \subset R^n \), we have \( N(s'', T^*, E') = \Sigma w'S' \), \( S' \) in \( \delta' \), \( S' \subset E' \), \( T^* S' = T'S = s'' \). It follows that \( W''(s'', T', D') = N(s'', T^*, D') \), \( s'' \) in \( S'' \), \( D' \) in \( \delta' \), is a weight function for \( T' \) which satisfies \( H1T' \). Also it follows that if \( T^* \) is ACB, then \( T' \) is ACW'D', and since \( S' \) is a Lebesgue measurable subset of \( S^* \), we have

\[ \int_{S'} D_B(s', T^*) \, d\mu' = \int_{R^n} N(x, T^*, S') \, dL = \int_{S''} W''(s'', T', S') \, d\mu'' \]

recalling that \( \mu' \) and \( \mu'' \) are n-dimensional Lebesgue measure, and that \( N(x, T^*, S') = 0 \) for \( x \) not in \( S'' = T^* S' \). In view of the uniqueness of the glbf WD, it follows that the
derivative $D_B(s', T^*)$ is equal to a $\text{glbw}'D'$ almost everywhere on $\mathcal{S}'$.

5 The transformation $T'^* = T^*T = T'T$ is a continuous transformation from a bounded domain $\mathcal{S}$ in $\mathbb{R}^n$ into a bounded portion $\mathcal{S}''$ of $\mathbb{R}^n$; therefore, the statements from Chapter V.4 of [6] also apply to $T''$.

Let $w''S, S$ in $\mathcal{J}$, and $W''(s'', T'', D), s''$ in $\mathcal{S}''$, $D$ in $\mathcal{J}$, be as defined in 1 of Chapter IX. Then for each $S$ in $\mathcal{J}$, $w''S = 1$, and $W''(s'', T'', D) = N(s'', T^*T, D)$, $s''$ in $\mathcal{S}''$, $D$ in $\mathcal{J}$. $W''(s'', T'', D)$, $s''$ in $\mathcal{S}''$, $D$ in $\mathcal{J}$, is a weight function for $T''$, and $T''$ is ACW''D if and only if $T'' = T*T$ is ACB and moreover, whenever $T''$ is ACW''D then $D_B(s, T^*T)$ is equal to a $\text{glbw}'D'$ a.e. $\mu \mathcal{S}$.

6 The following theorem, quoted from Chapter V.4 of [6] may be seen to be a consequence of Theorem .9 in Chapter IX of the present paper. Also, it is pointed out in [6] that for the case of the Euclidean plane, Theorem V.4 of [6] coincides with the result given in [3] for the product of two plane transformations, each absolutely continuous in the sense of Banach.

Theorem [6]V.4. Let $\mathcal{S}$ and $\mathcal{S}^*$ be bounded domains in $\mathbb{R}^n$, $T$ a continuous transformation from $\mathcal{S}$ into $\mathcal{S}^*$, and $T^*$ a continuous transformation from $\mathcal{S}^*$ into a bounded portion of $\mathbb{R}^n$. Assume that both $T$ and $T^*$ are ACB. Then a necessary and sufficient condition in order that $T^*T$ be ACB is that
the product $D_B^o(T_s,T^*)D_B(s,T)$ be Lebesgue summable on $S$, where $D_B(s',T^*), D_B(s,T)$ are the derivatives of the functions of domains $\int N(x,T^*,D)dL, D \subseteq S^*$, $\int N(x,T,D)dL, D \subseteq S$, respectively and $D_B^o(s',T^*)$ is the function obtained by extending the definition of $D_B(s',T^*)$ to all of $R^n$ by assigning it the value zero at points where it is not naturally defined. If all the transformations involved are ACB then the relation

$$D_B(s,T^*T) = D_B^o(T_s,T^*)D_B(s,T)$$

holds almost everywhere on $S$. 
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