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By
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INTRODUCTION

The problem of designing a system which will provide a satisfactory set of output quantities when appropriate inputs are applied is frequently encountered by engineers. In order to derive quantitative information about the outputs, quantitative information about the inputs is required. Sometimes all of the inputs can be represented as specific functions of time; often, however, the time variation of one or more of the inputs cannot be predicted. This is especially true of inputs which disturb the system, for example, thermal noise in an amplifier. In the case of communication receivers, even the desired inputs cannot be predicted. If they could be predicted, there would be no need to send the messages. When the unpredictable inputs can be regarded as unspecified members of ensembles of time functions for which statistical information is available, the statistical properties of the outputs can be derived. On the basis of these statistics, the set of outputs can be judged satisfactory or not satisfactory.

An ensemble of time functions which is characterized statistically is known as a stochastic or random process, and the individual


time functions are known as members of the process. At any point in time at which the process exists, the process can be regarded as a random variable. If several points are considered jointly, the process can be considered as a set of random variables. If the process is defined for a continuum of points in an interval, and all points are considered, the process can be regarded as an infinite set of random variables. A joint probability distribution of infinite order completely characterizes a process statistically. For many purposes, a complete characterization is not required, and first or second-order distributions suffice, or even finite-order moments of these distributions. By extending the order of the statistical characterization (distribution or moments), several processes can be considered jointly. If a process can be realized over a continuum of points, and if the probability of realization at every point is zero, the process can be characterized by probability density functions of various orders.

This dissertation is concerned with the problem of deriving joint probability density functions for a stochastic process and various low-order derivatives of the process. General expressions are developed and applied to specific distributions.

There are several motivations for dealing with this problem:

1. Formulas for deriving statistics of zero-crossings and extrema of stochastic processes involve joint probability density functions for the process and some of its low-order derivatives.

2. The outputs of some physical devices can be represented as derivatives of their inputs, for example, a tachometer, or a high-pass resistor-capacitor network over a limited frequency range. Since the input is an integral of the output, an indication of the effect of integration on probability distribution can be obtained.
The operation of some systems can be characterized by a differential equation of the form \(X(t) = a_0 Y(t) + a_1 Y'(t) + \cdots + a_n Y^{(n)}(t)\), in which the \(a_i\) are constants. If \(Y(t)\) is a stochastic process, and if the joint probability density function for the process and its derivatives is specified, then the probability density functions for \(X(t)\) can be derived by a standard procedure. In some cases \(X(t)\) might be an input process, and in other cases, an output process. If \(X(t)\) is an input process, then an indication of the effect of lowpass systems on probability distribution can be obtained.

As implied by item 1, problems of deriving joint probability density functions for a process and its derivatives arise in determining statistics of zero crossings or maxima per unit time.\(^6,7,8,9\) The formula for the expected number of times the stochastic process \(X\) crosses zero in a unit time interval is

\[
E(N_0) = \int_{-\infty}^{\infty} dx |\dot{x}| g(0,\dot{x}), \quad (1)
\]

where \(g(x,\dot{x})\) is the joint probability density function for \(X(t)\) and \(X'(t)\), \(x\) and \(\dot{x}\) are the distribution variables for \(X(t)\) and \(X'(t)\) respectively, and \(E\) is the expectation operator. The second moment of \(N_0\) is

\[
E(N_0^2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\dot{x}_1 d\dot{x}_2 |\dot{x}_1| |\dot{x}_2| g(0,\dot{x}_1,0,\dot{x}_2), \quad (2)
\]

where \(g(x_1,\dot{x}_1,\dot{x}_2,\dot{x}_2)\) is the joint probability density function for \(X(t_1)\),

---


The expected number of maxima is

$$E(N_m) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx \, d\xi \, g(x, \tau, \xi),$$

(3)

where \(g(x, \tau, \xi)\) is the joint probability density function for \(X(t), X'(t),\) and \(X''(t)\).

Equations (1), (2), and (3) have been applied to the case of a gaussian process, and different methods have been employed to derive the required joint probability density functions. Rice obtained the required distributions by demonstrating that the joint distribution of a gaussian process and its derivatives is gaussian and then deriving the joint correlation matrix. One procedure was based on the trigonometric series representation for shot noise current, and a second employed differentiation of the definition of the autocorrelation function\(^{10}\). The latter procedure can be readily modified for deriving second moment functions for mixed derivatives from the second moment function for the process. The second moment function for the stochastic process \(X(t)\) is

$$M(t, s) = E[X(t)X(s)],$$

(4)

where \(t\) and \(s\) are time scales, and \(E\) is the expectation operator.

If (4) is differentiated \(m\) times with respect to \(t\), and \(n\) times with respect to \(s\), the result is

$$\frac{\partial^{m+n}}{\partial t^m \partial s^n} M(t, s) = E[X^{(m)}(t)X^{(n)}(s)].$$

(5)

Thus the joint moment function for the \(m^{th}\) and \(n^{th}\) derivatives of a process can be expressed in terms of the second moment function for the process. For stationary processes\(^*\), a different form can be obtained.

\(^{10}\) Rice, op. cit., p. 54 and p. 65.

\(^*\) A process is stationary if its statistics are invariant with a shift of the time origin.
\[ M(T - S) = E[X(t + S)X(t + T)] \]  

(6)

Differentiating (6) \( m \) times with respect to \( S \) gives

\[ (-1)^m M^{(m)}(T - S) = E[X^{(m)}(t + S)X(t + T)], \]  

(7)

and differentiating (7) \( n \) times with respect to \( T \) gives

\[ (-1)^m M^{(m + n)}(T - S) = E[X^{(m)}(t + S)X^{(n)}(t + T)]. \]  

(8)

If \( S \) is zero in (8), the result is

\[ (-1)^m M^{(m + n)}(T) = E[X^{(m)}(t)X^{(n)}(t + T)]. \]  

(9)

Higher-order joint moment functions can be derived by employing similar procedures.

In some cases, the derivatives of the second moment function \( M(T) \) or the corresponding covariance function \( K(T) \) will not be defined at \( T = 0 \). For such cases, the covariance of the derivatives is not defined. A necessary condition for the existence of a particular derivative of \( K(T) \) at \( T = 0 \) can be stated in terms of the spectral density function \( S(w) \), where \( w \) is the frequency in radians per second. One of the Wiener-Khinchin relations\(^{11}\) is

\[ K(T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dw \, S(w) \exp iwT. \]

Differentiating \( n \) times with respect to \( T \) gives

\[ K^{(n)}(T) = \frac{i^n}{2\pi} \int_{-\infty}^{\infty} dw \, w^n S(w) \exp iwT, \]

and evaluating at \( T = 0 \) gives

\[ K^{(n)}(0) = \frac{4^n}{2\pi} \int_{-\infty}^{\infty} dw \, w^n S(w). \]

\( K^{(n)}(0) \) does not exist unless the integral is convergent. If \( S(w) \) is a ratio of polynomials in \( w \), then the integral converges if the degree of

\(^{11}\) Bendat, op. cit., pp. 67-68.
the denominator exceeds that of the numerator polynomial by at least \( n + 2 \).\(^{12}\)

The method employed by Middleton\(^{13}\) to obtain the probability density function for the first derivatives of a gaussian process proceeds by first expanding the associated characteristic function\(^*\) \( C(U) \) as a Taylor series.

\[
C(U) = E\{\exp[iUX'(t)]\} = \sum_{m=0}^{\infty} \frac{(iU)^m}{m!} E[X'(t)^m] = \sum_{m=0}^{\infty} \frac{(iU)^m}{m!} \left\{ \frac{\partial^m}{\partial t_1 \cdots \partial t_m} E[X(t_1), \ldots, X(t_m)] \right\}_{t_j = t, j=1,2,\ldots,m},
\]

since expectation and limiting operations commute\(^{14}\). To this point, the procedure applies to any distribution. However, in general, expressions for the \( m^{th} \) moment functions cannot be summed in closed form. For the gaussian process, higher-order moment functions can be expressed in terms of second moment functions:\(^{15}\)

\[
E(z_1 z_2 \ldots z_{2n}) = \sum_{\text{all pairs}} \prod \limits_{j \neq k} E(z_jz_k) \quad E(z_1 z_2 \ldots z_{2n+1}) = 0, \quad n = 1,2,3,\ldots; \quad E(z_1) = 0,
\]

where the number of pairs is \((2n!)2^n n!\). Substituting these expressions


\(^{13}\) Middleton, op. cit., pp. 370-371.

\(^{14}\) Ibid., pp. 67-68.

\(^{15}\) Ibid., p. 343.

* The characteristic function is the Fourier transform of the probability density function.
in the previous equation gives

\[ C(u) = \sum_{n=0}^{\infty} \frac{(iu)^{2n}}{(2\pi)^{2n}} \left[ \frac{\partial^2}{\partial t_j \partial t_k} \sum_{j \neq k}^{n} \sum_{\text{all pairs}} E(z_j z_k) \right] t_j=t_1=t_k \]

\[ = \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n}}{(2\pi)^n n!} \frac{2^n}{2^n} \left[ \frac{\partial^2}{\partial t_1 \partial t_2} M_x(t_1, t_2) \right] \]

\[ = \exp \left[ -\frac{1}{2} \frac{u^2}{\partial t_1 \partial t_2} M(t_1, t_2) \right]. \]

The procedure can be extended to find higher-order characteristic functions for the derivatives of a Gaussian process.

Lebedev\(^\text{16}\) derived the probability density function for the first derivatives of a stationary Gaussian process by a different procedure. In the joint Gaussian probability density function for \(X(t)\) and \(X(t+h)\), the variables are transformed as follows: \(x(t) = x, x(t+h) = x + \Delta x\). The resulting expression is integrated over \(x\) to obtain the probability density function for \(\Delta x\), which is a Gaussian random variable since it is the difference of variables whose joint distribution is Gaussian. Lebedev concludes that the distribution of the derivatives is Gaussian since the distribution of the differential is Gaussian. The standard deviation of the derivatives is derived by finding the limit of the standard deviation of \(\Delta x\) divided by \(h\) as \(h\) approaches zero. Some of the disadvantages of

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this procedure are (1) the integration over $x$ required a table of definite integrals and (2) the joint probability density of the process and its derivatives was not obtained.

Bartlett$^{17}$ presents a derivation for the joint characteristic function for $X(t)$ and $X'(t)$. The procedure is applicable to both gaussian and nongaussian processes. If the joint characteristic function for $X(t)$ and $X(t + h)$ is

$$C(U,V) = E\left\{\exp\left[iUX(t) + iVX(t + h)\right]\right\},$$

then the joint characteristic function for $X(t)$ and $X'(t)$ is

$$\lim_{h \to 0} E\left\{\exp\left[iUX(t) + iV\frac{X(t + h) - X(t)}{h}\right]\right\} = \lim_{h \to 0} C(U - V/h, V/h).$$

Bartlett applies the result to a stationary normal process. The method is straight-forward and should be especially useful when the characteristic function is desired as a final result or will be involved in further operations. If the probability density function is required, the result would be Fourier transformed.

Procedures have been developed for deriving characteristic functions for

$$Y(t) = \int_0^t \! du \, k(t - u)N[X(u)],$$

in which $k(t)$ is the impulse response of a linear system, $N$ represents a zero-memory operation and $X(t)$ is a stochastic process with known distributions. If $k(t)$ is a unit doublet, then $Y(t)$ is a process comprised of the derivatives of the process $N[X(t)],$ and it appears that such

$^{17}$ Bartlett, op. cit., pp. 156-157.


procedures could be employed to derive characteristic functions for process derivatives.

The Kac-Siegert procedure applies only to the case in which \( X(t) \) is a gaussian process, and \( N \) is a squaring operation. The characteristic function is found by expanding \( X(t) \) in a series of the eigenfunctions of an integral equation involving the impulse response function and the covariance function for \( X(t) \). The Fortet-Siegert approach applies to the case in which \( X(t) \) is a Markoff process or a component of a multidimensional Markoff process, and \( N \) is any zero-memory operation. The characteristic function is obtained as the solution of an integral equation.

For the purpose of deriving characteristic functions for derivatives, the Kac-Siegert approach is of limited interest since it applies only to chi-square processes. The Fortet-Siegert approach applies only to Markoff processes or components of multidimensional Markoff processes. Stationary Markoff processes have covariance functions of the form \( R(t) = R(0) \exp(-c|t|), c > 0 \). The derivatives for such processes are not defined since \( R''(0) \) is not defined. (See discussion on pages 5 and 6.) Hence the Fortet-Siegert approach cannot be employed for deriving characteristic functions for derivatives. In view of the simplicity and general applicability of the procedure presented by Bartlett (discussed on page 8), there seems to be little reason to employ any other procedure if characteristic functions for derivatives are desired.

In the next section, a procedure for deriving joint probability density functions for a process and its derivatives from multiple-order probability density functions for the process is developed. The development proceeds from the definitions of the first and higher-order derivatives.

\[^{20}\text{Doob, op. cit., pp. 233-234.}\]
In this section, various probability density functions involving process derivatives are derived from probability density functions of the process and are applied for processes with specific distributions.

**Expressions for Derivatives**

Derivatives of any order can be expressed as limits of linear combinations of the function evaluated at different values of the variable. The first derivative of a function $x(t)$ is defined as

$$x'(t) = \lim_{h \to 0} \frac{1}{h} \left[ x(t + h) - x(t) \right]$$

over the range of $t$ in which the function is defined and where the limit exists. The $n$th derivative is defined as the first derivative of the $n-1$th derivative. By using these two definitions, expressions can be developed for second and higher-order derivatives.

$$x''(t) = \lim_{h \to 0} \frac{1}{h} \left[ x'(t + h) - x'(t) \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left\{ \lim_{h \to 0} \frac{1}{h} \left[ x(t + 2h) - x(t + h) \right] - \lim_{h \to 0} \left[ x(t + h) - x(t) \right] \right\}$$

$$= \lim_{h \to 0} \frac{1}{h^2} \left[ x(t + 2h) - 2x(t + h) + x(t) \right]. \quad (11)$$

In Appendix I, it is shown by mathematical induction that

$$x^{(n)}(t) = \lim_{h \to 0} \frac{1}{h^n} \left\{ x(t + nh) - nx[t + (n - 1)h] + \frac{n(n - 1)}{2!} x[t + (n - 2)h] \right.$$

$$- \frac{n(n - 1)(n - 2)}{3!} x[t + (n - 3)h] + \cdots$$

$$\left. + (-1)^r \frac{n!}{(n - r)!r!} x[t + (n - r)h] + \cdots \right\}. \quad (10)$$
Joint Probability Density Function for a Process and its First Derivatives

The derivative of a function is defined by (10). Let
\[ X = X(t) \]
\[ Y = \frac{1}{h} \left[ -X(t) + X(t + h) \right] \]

It is seen that \( \lim_{h \to 0} Y = X'(t) \). The inverse transformation is
\[ X(t) = X \]
\[ X(t + h) = X + hY. \]

Since the transformations are one-to-one, the distribution functions are related by the Jacobian of the transformation.\(^1\) Thus
\[ g(x, y) = hf(x, x + hy), \]
where \( g \) is the joint probability density function of \( X \) and \( Y \), \( f \) is the second order probability density function of \( X \), and the time variable has been suppressed. Taking the limit of each side as \( h \to 0 \) yields
\[ g(x, \dot{x}) = \lim_{h \to 0} hf(x, x + hx). \quad (12) \]

The first-order probability density function for the derivatives is the appropriate margin of \( g(x, \dot{x}) \).

Example - Stationary Gaussian Process

The joint probability density function for a stationary gaussian process and its derivatives will be derived by employing (12).

\(^1\)
The second-order probability density function for a stationary gaussian process is

\[
  f(x,y) = \frac{\exp \left[ -\frac{x^2 - 2r(h)xy + y^2}{2c^2[1 - r^2(h)]} \right]}{2\pi c^2\sqrt{1 - r^2(h)}},
\]

where \( x = x(t), y = x(t+h) \), \( r(h) \) is the autocovariance coefficient function, and \( c^2 \) is the variance of the distribution. Replacing \( y \) by \( x + hx \) in (13) and substituting the result in (12) gives

\[
  g(x,x) = \lim_{h \to 0} \frac{1}{2\pi c^2} \frac{h}{\sqrt{1 - r^2(h)}} \exp \left[ -\frac{x^2 - 2r(h)x(x + hx) + (x + hx)^2}{2c^2[1 - r^2(h)]} \right].
\]

The limit of a product is the product of the limits. Let \( L_1 \) be the limit of square of the second factor on the right side of (14), i.e.

\[
  L_1 = \lim_{h \to 0} \frac{h^2}{1 - r^2(h)}.
\]

Since \( r(0) = 1 \), this is an indeterminate form, and L'Hopital's rule is applied.

\[
  L_1 = \lim_{h \to 0} \frac{h}{-r(h)r'(h)}
\]

If \( r'(h) \) is continuous at \( h = 0 \), then \( r'(0) = 0 \), because \( r(h) \) is an even function of \( h \). Applying L'Hopital's rule again gives

\[
  L_1 = \lim_{h \to 0} \frac{1}{-r(h)r''(h) - r'(h)r'(h)} = \frac{1}{-r''(0)}.
\]

Let \( L_2 \) be the limit of the argument of the exponential in (14), i.e.

\[
  L_2 = \lim_{h \to 0} \frac{x^2 - 2r(h)x(x + hx) + (x + hx)^2}{1 - r^2(h)}.
\]
Since this is an indeterminate form, L'Hospital's rule is applied:

\[
I_2 = \lim_{h \to 0} \frac{-2r(h)x\dot{x} - 2r'(h)x(x + h\dot{x}) + 2(x + h\dot{x})\dot{x}}{-2r(h)r'(h)}
\]

This is another indeterminate form.

\[
I_2 = \lim_{h \to 0} \frac{r'(h)x\dot{x} - r''(h)x\dot{x} - r''(h)x(x + h\dot{x}) + x^2}{-r(h)r''(h) - r'(h)r'(h)}
\]

\[
= \frac{-r''(0)x^2 + x^2}{-r''(0)}
\]

Using (15) and (16), the limit indicated by (14) is

\[
g(x, \dot{x}) = \frac{1}{2\pi c^2} \exp \left( -\frac{x^2}{2c^2} \right) \exp \left( -\frac{\dot{x}^2}{-2c^2r''(0)} \right).
\]

The right side of (17) can be expressed as a product of two factors, one involving only \(x\), and the other involving only \(\dot{x}\); hence, the process and its derivatives are statistically independent. The derivatives constitute a process with a Gaussian first-order distribution with variance \(-c^2r''(0)\).

Since \(r(h)\) is an even function, and \(r(h)\) is maximum for \(h = 0\), \(r''(0)\) is a negative quantity when it is defined. Because linear functions of Gaussian processes are Gaussian processes, and differentiation is the limit of a linear function, the form of the distribution of the derivative was anticipated.

**Example - Absolute Values of a Gaussian Process**

The joint probability density function for the absolute values of a stationary Gaussian process and their first derivatives will be derived by employing (12). In Appendix II, the second-order probability density function for the absolute values of a stationary Gaussian process is
derived. Replacing \( u \) by \( x \) and \( v \) by \( x + h \)\( x \) in (49) and substituting the result in (12) gives

\[
g(x, x) = \begin{cases} 
\lim_{h \to 0} \frac{1}{\pi c^2 \sqrt{1 - r''(h)}} \left[ \exp \left( - \frac{x^2 - 2r(h)x(x + h) + (x + h)^2}{2c^2[1 - r^2(h)]} \right) \right] , & x, x + hx \geq 0 \\
0, \text{ elsewhere.}
\end{cases}
\]

The limits of the second factor and of the first term of the third factor were evaluated previously, and the limit of the second term of the third factor is zero. The final result is

\[
g(x, \dot{x}) = \begin{cases} 
\frac{1}{\pi c^2 \sqrt{- r''(0)}} \exp \left( -\frac{x^2}{2c^2} \right) \exp \left( -\frac{x^2}{2c^2 r''(0)} \right) , & x \geq 0 \\
0, \ x < 0,
\end{cases}
\]

for all \( \dot{x} \). It is seen that the process and its derivatives are statistically independent, and that the derivatives are distributed normally with variance \(-c^2 r''(0)\). The fact that the first-order probability density of the derivative is gaussian does not necessarily mean that the derivative process is gaussian. It is known that linear transformations of gaussian processes produce gaussian processes. Since integration is a linear transformation, and the integral of this derivative process is not gaussian, the derivative process cannot be gaussian.

Example - Rayleigh Process

The joint probability density function for a stationary Rayleigh process and its derivatives will be derived by employing (12). The
second-order probability density function for a stationary Rayleigh process is

\[ f(x,y) = \begin{cases} \frac{xy}{c^4[1-r^2(h)]} \exp \left( -\frac{x^2+y^2}{2c^2[1-r^2(h)]} \right) I_0 \left( \frac{r(h)xy}{c^2[1-r^2(h)]} \right)^2, & x,y \geq 0 \\ 0, \text{ elsewhere} \end{cases} \]

where \( I_0 \) is the modified Bessel function of the first kind of order zero.

Replacing \( y \) by \( x + h\hat{x} \) in (19) and substituting the result into (12) gives

\[ g(x,\hat{x}) = \lim_{h \to 0} \begin{cases} \frac{hx(x + h\hat{x})}{c^4[1-r^2(h)]} \exp \left( -\frac{x^2 + (x + h\hat{x})^2}{2c^2[1-r^2(h)]} \right) I_0 \left( \frac{r(h)x(x + h\hat{x})}{c^2[1-r^2(h)]} \right), & x,x + h\hat{x} \geq 0 \\ 0, \text{ elsewhere} \end{cases} \]

For greater ease in evaluating the limit, (20) can be written as

\[ g(x,\hat{x}) = \lim_{h \to 0} \begin{cases} \frac{x(x + h\hat{x})}{c^4 \sqrt{1-r^2(h)}} \exp \left( -\frac{x^2 - 2r(h)x(x + h\hat{x}) + (x + h\hat{x})^2}{2c^2[1-r^2(h)]} \right), & x,x + h\hat{x} \geq 0 \\ 0, \text{ elsewhere}. \end{cases} \]

The limit of the first factor in the first domain is \( x^2/c^4 \), and the limits of the second and third factors were evaluated in the gaussian example.

---

For large values of \( z \),
\[
I_0(z) = \frac{\exp z}{\sqrt{2\pi z}} \left( 1 + \frac{1}{8z} + \frac{9}{128z^2} + \ldots \right)^3.
\]

Thus the limit of the last factor of the first domain of (21) is
\[
\lim_{h \to 0} \frac{\exp \left( \frac{r(h)x(x + hx)}{c [1 - r^2(h)]} \right)}{\sqrt{2\pi} \sqrt{r(h)x(x + hx)}} \left( 1 + \frac{1}{8} r(h)x(x + hx) + \ldots \right) = \frac{c}{\sqrt{2\pi}x}.
\]

The final result is
\[
g(x,\dot{x}) = \begin{cases} \frac{x}{c^2} \exp \left( \frac{-x^2}{2c^2} \right) \frac{1}{\sqrt{2\pi} c} \frac{1}{\sqrt{-r''(0)}} \exp \left( \frac{-x^2}{-2c^2 r''(0)} \right), & x \geq 0, \\ 0, & x < 0 \end{cases}
\]
for all \( \dot{x} \). The process and its derivatives are statistically independent and the derivatives are distributed normally with variance \(-c^2 r''(0)\).

**Example - Chi-Square Process**

The joint probability density function for a stationary chi-square process (the square of a stationary gaussian process with zero mean) and its derivatives will be derived by employing (12). The second-order probability density function for a chi-square process is derived in Appendix III. Replacing \( u \) by \( x \) and \( v \) by \( x + hx \) in (53) and substituting the result in (12) gives

---

\[ g(x, \dot{x}) = \lim_{h \to 0} \left\{ \frac{1}{4\pi^2 c^2} \frac{h}{\sqrt{1 - r^2(h)}} \frac{1}{\sqrt{x(x + hx)}} \begin{align*} &\exp \left( -\frac{x - 2r(h)\sqrt{x(x + hx)} + x + hx}{2c^2[1 - r^2(h)]} \right) \\
&\quad + \exp \left( -\frac{x + 2r(h)\sqrt{x(x + hx)} + x + hx}{2c^2[1 - r^2(h)]} \right) \right\}, x, x + hx \geq 0 \\
\] 0, elsewhere. \tag{23}

The limit of the second factor of the first domain was determined previously, and the limit of the third factor is \(1/x\). For evaluating the limit of the first term of the fourth factor, let \(L_1\) be the limit of the argument of the exponential, i.e.

\[ L_1 = \lim_{h \to 0} \frac{2x + hx - 2r(h)\sqrt{x(x + hx)}}{1 - r^2(h)}. \]

The form is indeterminate, so L'Hopital's rule is applied.

\[ L_1 = \lim_{h \to 0} \frac{x - r(h) \frac{\ddot{x}}{\sqrt{x(x + hx)}} - 2r'(h)\sqrt{x(x + hx)}}{-2r(h)r'(h)}. \]

This form is also indeterminate, so L'Hopital's rule is applied again.

\[ L_1 \lim_{h \to 0} \left( \frac{r'(h)\sqrt{x(x+hx)} - r(h)\frac{\dddot{x}}{2\sqrt{x(x+hx)}} - r'(h)\frac{\ddot{x}}{\sqrt{x(x+hx)}}}{x(x + hx)} - 2r''(h)\sqrt{x(x+hx)} \right) = \frac{\ddot{x}^2}{-4r''(0)x} + x. \]

The limit of the second term of the fourth factor of the first domain of (23) is zero. Combining this result with \(L_1\) and \(L_2\) gives
for all \( \hat{x} \). In this case, the process and its derivatives are not statistically independent.

**Probability Density Function for the Derivatives of a Chi-Square Process.** The marginal probability density function for the chi-square process derivatives is the integral of (24) over all \( x \):

\[
g(\hat{x}) = \frac{1}{4\pi c^2\sqrt{-r''(0)}} \int_0^{\infty} \frac{dx}{x} \exp\left(\frac{-x}{2c^2}\right) \exp\left(\frac{-\hat{x}^2}{8c^2r''(0)x}\right).
\]

Replacing \( x \) by \( 2c^2 u \) gives

\[
g(\hat{x}) = \frac{1}{4\pi c^2\sqrt{-r''(0)}} \int_0^{\infty} \frac{du}{u} \exp(-u) \exp\left(\frac{-\hat{x}^2}{16c^2r''(0)u}\right),
\]

where \( Q = 1/4c^2\sqrt{-r''(0)} \). Now let

\[
m(\hat{x}) = \frac{1}{\pi} \int_0^{\infty} \frac{du}{u} \exp(-u) \exp\left(\frac{Q\hat{x}}{u}\right); \text{ then}
\]

\[
g(\hat{x}) = Q m(Q\hat{x}).
\]

A differential equation involving \( m(\hat{x}) \) can be obtained by differentiating (25) twice with respect to \( \hat{x} \).
Integration by parts gives
\[ m'(\dot{x}) = -\frac{2}{\pi T} \left\{ \left[ \exp(-u) \exp \frac{\dot{x}^2}{u} \right]_0^\infty + \int_0^\infty du \exp(-u) \exp \frac{\dot{x}^2}{-u} \right\} \]
\[ = \frac{-2}{\pi \dot{x}} \int_0^\infty du \exp(-u) \exp \frac{\dot{x}^2}{-u}. \]  

(27)

\[ m''(x) = \frac{4}{\pi} \int_0^\infty \frac{du}{u} \exp(-u) \exp \frac{\dot{x}^2}{-u} + \frac{2}{\pi \dot{x}^2} \int_0^\infty du \exp(-u) \exp \frac{\dot{x}^2}{-u} \]
\[ = 4m(\dot{x}) - \frac{1}{\dot{x}} \dot{m}(\dot{x}) \]

Rearranging the terms gives
\[ 0 = m''(\dot{x}) + \frac{1}{\dot{x}} m'(\dot{x}) - 4m(\dot{x}) \]

(28)

Equation (28) is a homogeneous second-order linear differential equation with a regular singular point at $\dot{x} = 0$. The solution is found by substituting
\[ m(\dot{x}) = \dot{x}^r \sum_{n=0}^\infty a_n \dot{x}^n \]
in (28). From the resulting indicial equation it is determined that $r^2 = 0$. One linearly independent solution is
\[ m_1(\dot{x}) = \sum_{n=0}^\infty a_n \dot{x}^n \]
where $a_n = \begin{cases} 0, & n \text{ odd} \\ \left( \frac{n!}{2^{n/2}} \right)^2, & n \text{ even} \end{cases}$.

The second linearly independent solution is
\[ m_2(\dot{x}) = m_1(\dot{x}) \log \dot{x} + \sum_{n=0}^\infty b_n \dot{x}^n, \]
where \( b_0 = 1, b_1 = 0 \), and the recursion formula for the \( b_n \) is

\[
b_n + 2 = \frac{4}{(n + 2)^3} \left[ (n + 2) b_n - 2a_n \right].
\]

The general solution of \((28)\) is then

\[
m(\dot{x}) = A m_1(\dot{x}) + B \left[ m_1(\dot{x}) \log x + \sum_{n=0}^{\infty} b_n x^n \right], \tag{29}
\]

and its derivative is

\[
m'(\dot{x}) = A m'_1(\dot{x}) + B \left[ m'_1(\dot{x}) + m'_1(\dot{x}) \log x + \sum_{n=0}^{\infty} n b_n x^n - 1 \right], \tag{30}
\]

where \( A \) and \( B \) are undetermined constants.

If \((27)\) is substituted for \( m'(\dot{x}) \) in \((30)\), and both sides are multiplied by \( \dot{x} \), the result is

\[
-\frac{2}{\pi} \int_{0}^{\infty} du \exp(-u) \exp -\frac{x^2}{u}
\]

\[
= Axm'_1(\dot{x}) + B \left[ m'_1(\dot{x}) + m'_1(\dot{x}) \log x + \dot{x} \sum_{n=0}^{\infty} n b_n x^n - 1 \right].
\]

Taking the limit as \( \dot{x} \to 0 \) gives

\[
-\frac{2}{\pi} \int_{0}^{\infty} du \exp(-u) = B;
\]

thus \( B = -2/\pi \).

If \((29)\) is evaluated at \( \dot{x} = 1 \), the result is

\[
m(1) = A m_1(1) + B \sum_{n=0}^{\infty} b_n, \text{ and}
\]

\[
A = \frac{1}{m'_1(1)} \left[ m(1) - \frac{2}{\pi} \sum_{n=0}^{\infty} b_n \right].
\]
The quantity \( m(1) \) was found by evaluating (25) for \( x = 1 \) by numerical integration. The final result is

\[
m(\dot{x}) = \frac{1}{\pi} \left[ (0.8457 - \log \dot{x}^2) \sum_{n=0}^{\infty} a_n x^n - 2 \sum_{n=0}^{\infty} b_n x^n \right].
\]  

(31)

The first seven nonzero coefficients for (31) are listed in Table 1.

### Table 1

**Power Series Coefficients for Equation (32)**

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Value of Coefficients for Following Values of n</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>( a_n )</td>
<td>1</td>
</tr>
<tr>
<td>( b_n )</td>
<td>1/8</td>
</tr>
</tbody>
</table>

The function \( m(\dot{x}) \) is graphed in Figure 1. To summarize, \( g(\dot{x}) \), the probability density function for the derivatives of a chi-square process is given by (26), in which \( m(\dot{x}) \) is defined by (31).

**Joint Probability Density Function for a Process and its First and Second Derivatives**

Equations (10) and (11) are expressions for the first and second derivatives of a function in terms of the function. Let

\[
X = X(t)
\]

\[
Y = \frac{1}{h} \left[ -X(t) + X(t + h) \right]
\]

\[
Z = \frac{1}{h^2} \left[ X(t) - 2X(t + h) + X(t + 2h) \right].
\]
FIGURE 1 PROBABILITY DENSITY FUNCTION FOR FIRST DERIVATIVES OF CHI-SQUARE PROCESS

Derived from Normal (0,1) Process
With $r''(0) = -1/16$
It is seen that \( \lim_{h \to 0} Y = X'(t) \), and \( \lim_{h \to 0} Z = X''(t) \). The inverse transformation is

\[
X(t) = X \\
X(t + h) = X + hY \\
X(t + 2h) = X + 2hY + h^2Z.
\]

Since the transformations are one-to-one,

\[
g(x, y, z) = h^3 f(x, x + hy, x + 2hy + h^2y).
\]

Taking the limit of each side as \( h \to 0 \) yields

\[
g(x, \dot{x}, \ddot{x}) = \lim_{h \to 0} h^3 f(x, x + hx, x + 2hx + h^2\ddot{x}). \tag{32}
\]

**Example - Absolute Values of a Stationary Gaussian Process**

In Appendix IV, the outline of the derivation of the joint probability density function for the absolute values of a stationary gaussian process and their first and second derivatives is presented. The final result of that derivation is

\[
g(x, \dot{x}, \ddot{x})
\]

\[
= \begin{cases} 
2 \exp \left\{ -\frac{1}{2} \frac{-r''(0)r^{(4)}(0)x^2 + [r^{(4)}(0) - r''^2(0)]\dot{x}^2 - r''(0)\ddot{x}^2 + 2r''^2(0)x\ddot{x}}{-r''(0)c^2[r^{(4)}(0) - r''^2(0)]} \right\}, & x \geq 0 \\
\frac{c^3\sqrt{2\pi}^3}{\sqrt{-r''(0)[r^{(4)}(0) - r''^2(0)]}}, & \text{all } x, \dot{x}, \ddot{x} \\
0, & x < 0 \\
\end{cases} \tag{33}
\]

It is seen that the first derivatives are statistically independent of the process and its second derivatives, but that the process and its second derivatives are not statistically independent.

Expression (33) is not defined for certain autocorrelation coefficient functions. For example, if \( r(s) = \cos ws \), then \( r''(s) = -w^2 \cos ws \), \( r^{(4)}(s) = -w^4 \cos ws \), and \( r^{(4)}(0) - r''^2(0) = 0 \). This autocorrelation
coefficient function is associated with an ensemble of sinusoidal functions with the same frequency. For such an ensemble, the second derivatives of the member functions are perfectly correlated with the member functions, so the joint probability density tends to infinity.

Joint Probability Density Function for $X(t_1), X(t_2), X'(t_1)$ and $X'(t_2)$

Equation (10) defines the first derivative of a function. Consider the following set of linear transformations:

\[ Y_1 = X(t_1) \]
\[ Y_2 = X(t_2) \]
\[ Y_3 = -X(t_1)/h + X(t_1 + h)/h \]
\[ Y_4 = -X(t_2)/h + X(t_2 + h)/h. \]

It is seen that $\lim_{h \to 0} Y_3 = X'(t_1)$, and $\lim_{h \to 0} Y_4 = X'(t_2)$.

The set of inverse transformations is

\[ X(t_1) = Y_1 \]
\[ X(t_2) = Y_2 \]
\[ X(t_1 + h) = Y_1 + hY_3 \]
\[ X(t_2 + h) = Y_2 + hY_4. \]

Since the transformations are one-to-one,

\[ g(y_1, y_2, y_3, y_4) = h^2 f(y_1, y_2, y_1 + hy_3, y_2 + hy_4), \]

where $f$ is the joint probability density function for $X(t_1), X(t_2), X(t_1 + h)$, and $X(t_2 + h)$.

Taking the limit of each side of the equation as $h \to 0$ gives

\[ g(x_1, x_2, \dot{x}_1, \dot{x}_2) = \lim_{h \to 0} h^2 f(x_1, x_2, x_1 + h\dot{x}_1, x_2 + h\dot{x}_2), \quad (34) \]
where $x_1 = x(t_1)$, and $x_2 = x(t_2)$.

**Example - Absolute Values of a Stationary Gaussian Process**

In Appendix V, the outline of the derivation of $g(x_1, x_2, \dot{x}_1, \dot{x}_2)$ for the absolute values of a stationary gaussian process is presented. The final result of that derivation is

$$g(x_1, x_2, \dot{x}_1, \dot{x}_2) = \begin{cases} \frac{2}{(2\pi)^2 c^4 \sqrt{R(T)}} \left\{ \exp \left[ -\frac{1}{2} \frac{Q(T;x_1, x_2, \dot{x}_1, \dot{x}_2)}{c^2 R(T)} \right] + 2 \right\} \right. & \text{if } x_1, x_2 \neq 0, \text{ all } \dot{x}_1, \dot{x}_2 \\ 0, \text{ elsewhere} & \right. \end{cases} \tag{35}$$

where

$$R(T) = \left[ r^2(0) - r^2(T) \right] \left[ 1 - r^2(T) \right] + 2 r^2(T) \left[ r''(0) - r(T) r''(T) \right] + r^2(T)$$

$$Q(T; x_1, x_2, \dot{x}_1, \dot{x}_2) = \left[ r^2(0) - r^2(T) + r''(0) r^2(T) \right] \left( x_1^2 + x_2^2 \right) - 2 \left( r(T) \left[ r''(0) - r''(T) \right] + r'{}^2(T) r''(T) \right) x_1 x_2$$

$$+ 2 r'(T) \left[ r''(0) r(T) - r''(T) \right] (x_1 \dot{x}_1 - x_2 \dot{x}_2)$$

$$+ 2 r'(T) \left[ r''(0) - r(T) r''(T) + r'{}^2(T) \right] (x_1 \dot{x}_2 - x_2 \dot{x}_1)$$

$$- \left\{ r''(0) \left[ 1 - r^2(T) \right] + r'{}^2(T) \right\} \left( \dot{x}_1^2 + \dot{x}_2^2 \right)$$

$$+ 2 \left( r(T) \left[ 1 - r^2(T) \right] + r(T) r'{}^2(T) \right) \dot{x}_1 \dot{x}_2.$$
The operation of some systems can be characterized by a differential equation of the form

\[ Y(t) = a_0 X(t) + a_1 X'(t) + \cdots + a_n X^{(n)}(t) \]  

(36)

in which the \( a_i \) are constant. If \( X(t) \) is a stochastic process, and if the joint probability density function for the process and its first \( n \) derivatives is specified, then the first-order probability density for \( Y(t) \) is readily derived. The joint probability density functions for a process and its first derivatives which were derived for specific distributions in the previous section will be employed. An expression for the first-order probability density function for \( Y(t) \) for the case \( n = 2 \) is also developed.

In (36), \( Y(t) \) could be either the input process or the output process. If \( Y(t) \) is the input process, then an indication of the effect of lowpass systems on probability distribution can be obtained. Since this is the case of greater interest, \( Y(t) \) will be considered as the input process.

**First-Order Lowpass Systems**

For the case \( n = 1 \) and \( a_0 = 1 \), (36) becomes \( Y = X + aX' \).

Consider the transformation \( X = X \)

\[
\dot{X} = -\frac{X}{a} + \frac{Y}{a}.
\]

Since the transformation is one-to-one, the distributions of the sets of variables are related by the Jacobian of the transformation. Thus

\[
h(x, y) = \frac{1}{a} g(x, \frac{y - X}{a})
\]  

(37)
where \( h(x,y) \) is the joint probability density function of the output and the input, and \( g(x,x) \) is the joint probability density of the output process and its first derivatives. The distribution of the input is the appropriate margin of \( h(x,y) \). If desired, \( h(x,y) \) can be expressed in terms of the second-order probability density function of the output by replacing \( x \) by \( (y - x)/a \) in (12) and substituting the result into (37):

\[
h(x,y) = \frac{1}{a} \lim_{h \to 0} hf(x,x + h \frac{y - x}{a}).
\]

Example - Gaussian Output Process from First-Order Systems

The distribution of the inputs of first-order stationary systems whose outputs constitute a stationary gaussian process was obtained by replacing \( x \) by \( (y - x)/a \) in (17), substituting the result in (37), and integrating over \( x \). The result is

\[
h(y) = \frac{\exp\left(-\frac{y^2}{c^2[1 - a^2 r''(0)]}\right)}{\sqrt{2\pi c} \sqrt{1 - a^2 r''(0)}}.
\]

As expected, the input is gaussian. It is seen that the input variance increases with \( a \), the system time constant. Thus, the larger the time constant, the smaller the variance of the output. This is compatible with the known filtering action of lowpass networks.

Example - Absolute Values of a Gaussian Process From First-Order Systems

The distribution of the inputs of first-order stationary systems whose outputs are the absolute values of the members of a stationary gaussian process will be obtained by replacing \( x \) by \( (y - x)/a \) in (18), substituting the result in (37), and integrating over \( x \).
The transformation \( x = -u + \frac{y}{1 - a^2r''(0)} \) leads to

\[
h(y) = \frac{1}{c^2a\sqrt{-r''(0)}} \int_0^\infty dx \exp \left( \frac{a^2r''(0)y^2 - (y - x)^2}{-2a^2c^2r''(0)} \right).\]

where \( n(z) \) is the normal (0,1) probability density function, and \( N(z) \) is the cumulative normal (0,1) distribution. In Figure 2, (39) is graphed for the case \(-a^2r''(0) = 1, c^2 = 1/2\).

Figure 2 shows that the input can be either positive or negative, but that the output can only be positive. These conditions occur in the operation of rectifiers, which are nonlinear devices, and their occurrence with the operation of a linear system is noteworthy. The same conditions would obtain when integrating the derivatives of any process which assumes values of only one sign. (See (18), (22), and (24).) The same phenomena can be exhibited by linear discrete-time systems, for which output distributions can readily be derived from the input distributions.

Consider a system whose output is given by \( Z = X + Y \), where \( Z = Z(nT) \), \( X \) is the input for \( t = nT \), and \( Y \) is the input for \( t = (n + 1)T \).

Now \( P(Z \leq 0) = P(X + Y \leq 0) = P(Y \leq -X) \), so the phenomenon occurs if \( P(Y \leq -X) = 0 \). Bivariate distributions can be constructed to satisfy this
FIGURE 2  PROBABILITY DENSITY FUNCTIONS FOR THE INPUTS AND OUTPUTS OF SIMPLE LOWPASS SYSTEMS WHEN THE OUTPUT PROCESS IS THE ABSOLUTE VALUE OF A GAUSSIAN PROCESS
condition and the requirement that $X$ and $Y$ be distributed on both sides of zero. As one example, consider

$$f(x,y) = \begin{cases} 
\frac{1}{2}, & -x \leq y \leq 1, \quad -1 \leq x \leq 1 \\
0, & \text{elsewhere}
\end{cases}. $$

The distribution is symmetrical in $x$ and $y$, and the margins are distributed on both sides of zero:

$$f(y) = \int_{-\infty}^{\infty} dx \ f(x,y) = \begin{cases} 
\int_{-\infty}^{\frac{1}{2}} dx/2, & -1 \leq y \leq 1 \\
(1 + y)/2, & -1 \leq y \leq 1 \\
\int_{-\infty}^{0} dx/2, & 0 \leq y \leq 1 \\
0, & \text{elsewhere}
\end{cases}. $$

The distribution of the output is

$$g(z) = \int_{-\infty}^{\infty} dx \ f(x,z - x) = \begin{cases} 
\int_{-\infty}^{z-1} dx/2, & 0 \leq z \leq 2 \\
(2 - z)/2, & 0 \leq z \leq 2 \\
\int_{-\infty}^{0} dx/2, & 0 \leq z \leq 2 \\
0, & \text{elsewhere}
\end{cases}. $$

It is seen that the output is distributed over positive values only.

In a similar fashion, more general examples could be constructed in which the output is a linear combination of a larger set of the sequence of input values. The operation of a continuous-time linear system can be represented by a convolution integral, which is the limit of a sequence of linear combinations of input samples; thus in principle, a continuous-time input process could be constructed which exhibits this phenomenon when applied to a continuous-time linear system.

**Example - Rayleigh Output From Simple Lowpass Systems**

The distribution of the inputs of simple lowpass systems whose outputs constitute a Rayleigh process will be obtained by replacing $x$ by
\[(y - x)/a \text{ in (22), substituting the result in (37), and integrating over x.}\]

\[h(y) = \frac{1}{\sqrt{2\pi ac^3\sqrt{-r''(0)}}} \int_{-\infty}^{\infty} dx \exp \left\{ \frac{a^2r''(0)x^2 - (y - x)^2}{-2a^2c^2r''(0)} \right\}\]

If the substitution \(x = -u + \frac{y}{1 - a^2r''(0)}\) is made, the result is,

\[h(y) = \frac{\exp \left\{ \frac{y^2}{-2c^2[1 - a^2r''(0)]} \right\}}{\sqrt{2\pi ac^3\sqrt{-r''(0)}}} \int_{-\infty}^{\infty} du \left\{ -u + \frac{y}{1 - a^2r''(0)} \right\} \exp \left\{ \frac{1 - a^2r''(0)}{2a^2c^2r''(0)} u^2 \right\}.
\]

The right side of (40) consists of two terms. The first term is

\[I(y) = \frac{\exp \left\{ \frac{y^2}{-2c^2[1 - a^2r''(0)]} \right\}}{\sqrt{2\pi ac^3\sqrt{-r''(0)}}} \int_{-\infty}^{\infty} du u \exp \left\{ \frac{1 - a^2r''(0)}{2a^2c^2r''(0)} u^2 \right\}.
\]

If the substitution \(w = u^2\) is made in this integral, then

\[I(y) = \frac{\exp \left\{ \frac{y^2}{-2c^2[1 - a^2r''(0)]} \right\}}{\sqrt{2\pi ac^3\sqrt{-r''(0)}}} \int_{0}^{\infty} dw \exp \left\{ \frac{1 - a^2r''(0)}{2a^2c^2r''(0)} w \right\}.
\]

\[= \frac{\exp \left\{ \frac{y^2}{-2c^2[1 - a^2r''(0)]} \right\}}{\sqrt{2\pi ac^3\sqrt{-r''(0)}}} a^2c^2r''(0) \exp \left\{ \frac{y^2}{2a^2c^2r''(0)[1 - a^2r''(0)]} \right\}
\]

\[= \frac{a^{\sqrt{-r''(0)}}}{2 c[1 - a^2r''(0)]} \exp \left\{ \frac{y^2}{2a^2c^2r''(0)} \right\}
\]

\[= \frac{a^{\sqrt{-r''(0)}}}{c[1 - a^2r''(0)]} n \left( \frac{y}{ac\sqrt{-r''(0)}} \right).\]
The second term in (40) is

\[
J(x) = \frac{y \exp \left( -\frac{y^2}{2c^2[1 - a^2r''(0)]} \right)}{\sqrt{2\pi} \ ac^3 \sqrt{-r''(0)} \ [1 - a^2r''(0)]} \int_{-\infty}^{y/\left[1 - a^2r''(0)\right]} du \exp \left( \frac{1 - a^2r'(0)u^2}{2a^2c^2r''(0)} \right).
\]

This integral is converted to the integral of a normal variable with the result

\[
J(x) = \frac{y \exp \left( -\frac{y^2}{2c^2[1 - a^2r''(0)]} \right)}{c^2[1 - a^2r''(0)]^{3/2}} N \left( \frac{y}{c\sqrt{1 - a^2r''(0)}} \right) \frac{y}{ac\sqrt{-r''(0)} \sqrt{1 - a^2r''(0)}}
\]

Adding (41) and (42) gives

\[
h(x) = \frac{a\sqrt{-r''(0)}}{c[1 - a^2r''(0)]} \ n \left( \frac{y}{ac\sqrt{-r''(0)}} \right)
\]

\[
+ \frac{\sqrt{2\pi}y}{c^2[1 - a^2r''(0)]^{3/2}} \ n \left( \frac{y}{c\sqrt{1 - a^2r''(0)}} \right) \frac{y}{ac\sqrt{-r''(0)} \sqrt{1 - a^2r''(0)}}
\]

(43)

In Figure 3, (43) is graphed for the case \(-a^2r''(0) = 1\), \(c^2 = 1/2\).

Example - Chi-Square Outputs from Simple Lowpass Systems

The distribution of the inputs of simple lowpass systems whose outputs constitute a chi-square process will be obtained by replacing \(\dot{x}\) by \((y - x)/a\) in (24), substituting the result in (37), and integrating over \(x\).
FIGURE 3  PROBABILITY DENSITY FUNCTIONS FOR THE INPUTS AND OUTPUTS OF SIMPLE LOWPASS SYSTEMS WHEN THE OUTPUT PROCESS IS A RAYLEIGH PROCESS
\[ h(y) = \frac{1}{4\pi ac^2\sqrt{-r''(0)}} \int_0^\infty \frac{dx}{x} \exp \left( -\frac{x}{2c^2} \right) \exp \left( -\frac{(y - x)^2}{8a^2c^2r''(0)x} \right) \]

\[ \exp \left( -\frac{2y}{8a^2c^2r''(0)} \right) \int_0^\infty \frac{dx}{x} \exp \left( \frac{1 - 4a^2r''(0)}{8a^2c^2r''(0)} x \right) \exp \left( -\frac{y^2}{8a^2c^2r''(0)x} \right) \]

Let \( w = \frac{1 - 4a^2r''(0)}{8a^2c^2r''(0)} x \); then

\[ h(y) = \frac{\exp \left( -\frac{2y}{8a^2c^2r''(0)} \right)}{4ac^2\sqrt{-r''(0)}} \cdot \frac{1}{\pi} \int_0^\infty \frac{dw}{w} \exp(-w) \exp \left( \frac{[1 - 4ar''(0)]y^2}{[-8a^2c^2r''(0)]^2 (-w)} \right) \]

By reference to (25) it is seen that

\[ h(y) = \frac{\exp \left( -\frac{2y}{8a^2c^2r''(0)} \right)}{4ac^2\sqrt{-r''(0)}} \cdot \left( \frac{\sqrt{1 - 4ar''(0)}}{-8a^2c^2r''(0)} y \right). \quad (44) \]

In Figure 4, (44) is graphed for the case \(-a^2r''(0) = 1, c^2 = \sqrt{5}/8\).

Second-Order and Nonlinear Systems

For the case \( n = 2 \) and \( a_0 = 1 \), (36) becomes \( Y = X + a\dot{X} + b\ddot{X} \), and

\[ h(x,\dot{x},y) = \frac{1}{b} g(x,\dot{x}, \frac{y - x - a\dot{x}}{b}). \quad (45) \]

The distribution of the input is the appropriate margin of \( h(x,\dot{x},y) \). If desired, \( h(x,\dot{x},y) \) can be expressed in terms of the third-order probability density function of the output by replacing \( \dot{x} \) by \( (y - x - a\dot{x})/b \) in (32) and substituting the result into (45):

\[ h(x,\dot{x},y) = \frac{1}{b} \lim_{h \to 0} h^3 f(x,x, + h\dot{x}, x \otimes 2h\dot{x} + h^2 \frac{y - x - a\dot{x}}{b}). \]
FIGURE 4  PROBABILITY DENSITY FUNCTIONS FOR THE INPUTS AND OUTPUTS OF SIMPLE LOWPASS SYSTEMS WHEN THE OUTPUT PROCESS IS A CHI-SQUARE PROCESS
The procedure need not be limited to linear systems. Consider a system whose operation is characterized by

\[ Y = X^2 + a \dot{X}, \]

and let \( H(y) \) be the cumulative distribution of the input.

\[
H(y) = P(Y \leq y) = P(X^2 + a \dot{X} \leq y)
\]

\[
= P[\dot{X} \leq (y - X^2)/a]
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{(x - y^2)/a} \text{d}x \text{d}\dot{x} \, g(x, \dot{x})
\]

\[
h(y) = \frac{dH}{dy}
\]

\[
= \frac{1}{a} \int_{-\infty}^{\infty} \text{d}x \, g(x, \frac{y - x^2}{a})
\]
A procedure has been developed for deriving various joint probability density functions for a stochastic process and its derivatives. In the procedure, a change of variables in a multiple-order probability density function for the process is followed by a limiting operation. General expressions for $g(x,\dot{x})$, $g(x,\dot{x},\ddot{x})$, and $g(x_1,x_2,\dot{x}_1,\dot{x}_2)$ were derived and applied to both gaussian and nongaussian processes. In the previous section, these results were applied to find first-order probability density functions for inputs of first-order linear lowpass systems.

If multiple-order characteristic functions for the process are available, and joint characteristic functions for the process and its derivatives are desired, the procedure presented by Bartlett can be employed. This procedure is described in the Introduction.

The general expressions for $g(x,\dot{x})$, $g(x,\dot{x},\ddot{x})$, and $g(x_1,x_2,\dot{x}_1,\dot{x}_2)$ were derived quite readily, and it appears that general expressions for higher-order joint densities could be derived with reasonable effort. Furthermore, expressions for $g(x,\dot{x})$ for specific distributions were obtained without excessive manipulation. However, obtaining expressions for $g(x,\dot{x},\ddot{x})$ and $g(x_1,x_2,\dot{x}_1,\dot{x}_2)$ for a particular process required extensive manipulation for differentiation of third and fourth order determinants.

Several interesting phenomena were indicated by the results obtained for specific processes:

1. The first-order distribution of the first derivatives of a Rayleigh process is gaussian, but the derivatives do not comprise a gaussian process. This is also true for the derivatives of the absolute values of a gaussian process. This serves to emphasize that in general, a process is characterized by a hierarchy of probability functions.
2. With certain kinds of input processes, a linear system appears to behave as a rectifier, i.e., the output assumes only values of a single polarity even though the input assumes values of both polarities. It was shown that the phenomenon occurs if the joint probability density functions for the input process are zero over domains bounded by hyperplanes.

The Procedure developed and the results obtained can be applied or extended in several ways:

1. Formulas for statistics of zero crossings and maxima of a stochastic process involve various joint probability density functions for the process and its derivatives. The general expressions developed for \( g(x,\dot{x}) \), \( g(x,\dot{x},\ddot{x}) \), and \( g(x_1,x_2,\dot{x}_1,\dot{x}_2) \) can be applied to obtain the required joint probability density functions for specific processes. This application was mentioned in the Introduction as a motivation for developing the procedure.

2. By applying the general expression for \( g(x_1,x_2,\dot{x}_1,\dot{x}_2) \) to specific distributions and integrating the result over \( x_1 \) and \( x_2 \), second-order probability density functions for stochastic processes can be generated. The fourth-order probability density functions required by the general expression can be generated by zero-memory transformations of a gaussian process.

3. In several cases, it was found that the process was statistically independent from its first derivatives. This suggests that the general expression for \( g(x,\dot{x}) \) might be useful for evaluation of the margins of second-order probability density functions. In the case of the chi-square process, the process and its derivatives were not statistically independent. However, in similar cases, it might still be easier to evaluate the integral after the transformation.
4. Beginning with the definitions of partial derivatives, a procedure could be developed for deriving joint probability density functions for a multiparameter random process and its partial derivatives. These functions could be employed in the determination of statistical properties of random surfaces.¹

APPENDIX I

EXPRESSION FOR N\textsuperscript{TH} ORDER DERIVATIVE

It will be shown by mathematical induction that the expression for the n\textsuperscript{th} order derivative of a function \( x(t) \) is given by

\[
x^{(n)}(t) = \lim_{h \to 0} \frac{1}{h^n} \left\{ x(t + nh) - nx[t + (n - 1)h] + \frac{n(n - 1)}{2!} x[t + (n - 2)h] - \frac{n(n - 1)(n - 2)}{3!} x[t + (n - 3)h] + \cdots \right. \\
+ \left. (-1)^r \frac{n!}{(n - r)! r!} x[t + (n - r)h] + \cdots \right\},
\]

where the index \( r \) runs from zero to \( n \).

For \( n = 1 \), (47) yields \( x'(t) = \lim_{h \to 0} \frac{1}{h} \left[ x(t + h) - x(t) \right] \), which is the definition of the first order derivative.

Assume that (47) is true for \( n = k \), and prove that it is also true for \( n = k + 1 \). For \( n = k \),

\[
x^{(k)}(t) = \lim_{h \to 0} \frac{1}{h^k} \left\{ x(t + kh) - kx[t + (k - 1)h] + \frac{k(k - 1)}{2!} x[t + (k - 2)h] - \frac{k(k - 1)(k - 2)}{3!} x[t + (k - 3)h] + \cdots \right. \\
+ \left. (-1)^r \frac{k!}{(k - r)! r!} x[t + (k - r)h] + \cdots \right\}.
\]

It must be shown that

\[
x^{(k + 1)}(t) = \lim_{h \to 0} \frac{1}{hk + 1} \left\{ x[t + (k + 1)h] - (k + 1)x(t + kh) + \frac{(k + 1)k}{2!} x[t + (k - 1)h] - \frac{(k + 1)k(k - 1)}{3!} x[t + (k - 2)h] + \cdots \right. \\
+ \left. (-1)^r \frac{(k + 1)!}{(k + 1 - r)! r!} x[t + (k + 1 - r)h] + \cdots \right\}.
\]

(48)
The \((k+1)\)th derivative is defined as the first derivative of the \(k\)th derivative:

\[
x^{(k+1)}(t) = \lim_{h \to 0} \frac{1}{h^{k+1}} \left[ x^{(k)}(t + h) - x^{(k)}(t) \right].
\]

Expression (17) for \(n = k\) is substituted in this expression:

\[
x^{(k+1)}(t) = \lim_{h \to 0} \left( \lim_{n \to 0} \frac{1}{h} \left\{ x[t + (k + 1)h] - kx(t + kh) \right. \right.
\]
\[
+ \frac{k(k - 1)}{2!} x[t + (k - 1)h]
\]
\[
- \frac{k(k - 1)(k - 2)}{3!} x[t + (k - 2)h] + \cdots
\]
\[
+ (-1)^r \frac{k!}{(k-r)!r!} x[t + (k + 1 - r)h] + \cdots \right\}
\]
\[
\lim_{h \to 0} \frac{1}{h^k} \left\{ x(t + kh) - kx[t + (k - 1)h] \right.
\]
\[
+ \frac{k(k - 1)}{2!} x[t + (k - 2)h]
\]
\[
- \frac{k(k - 1)(k - 2)}{3!} x[t + (k - 3)h] + \cdots
\]
\[
+ (-1)^r \frac{k!}{(k-r)!r!} x[t + (k - r)h] + \cdots \right\}
\]

Collecting the terms with the same arguments gives

\[
x^{(k+1)}(t) = \lim_{h \to 0} \frac{1}{h^{k+1}} \left\{ x[t + (k + 1)h] - (k + 1)x(t + kh) \right.
\]
\[
+ \frac{k(k - 1) + 2k}{2!} x[t + (k - 1)h]
\]
\[
- \frac{k(k - 1)(k - 2) + 3k(k - 1)}{3!} x[t + (k - 2)h] + \cdots
\]
\[
+ \left[\frac{(-1)^{r-1} k!}{(k-r)!r!} \frac{(-1)^{r-1} k!}{(k-r+1)(r-1)!} \right] x[t + (k + 1 - r)h] + \cdots \right\}
\]
\[
x^{(k+1)}(t) = \lim_{h \to 0} \frac{1}{h^{k+1}} \left\{ x \left[ t + (k+1)h \right] - (k+1)x(t + kh) \right. \\
+ \frac{k(k+1)}{2!} x \left[ t + (k-1)h \right] \\
- \frac{(k+1)k(k-1)}{3!} x \left[ t + (k-2)h \right] + \cdots \\
+ (-1)^{r} \frac{r!}{k! \cdots} \frac{(k-r+1) + r}{(k-r+1)!} x \left[ t + (k+1-r)h \right] + \cdots \right\},
\]

In the coefficient of the last term of this expression, \(k!(k+1)\) gives \((k+1)!\), making this expression identical to \((h^k)^\prime\). Thus, if \((h^7)\) is correct for \(n = k\), it is correct for \(n = k+1\). It has been shown that the expression is correct for \(n = 1\); hence it is true for \(n = 2,3,4,\ldots\).
APPENDIX II

PROBABILITY DENSITY FUNCTIONS FOR THE ABSOLUTE VALUES OF A STATIONARY GAUSSIAN PROCESS

Consider a process $U$ derived from a process $X$ with known distribution functions, and let each member of $U$ be the absolute value of a member of $X$. Let $U = U(t_1)$, $V = U(t_2)$, $X = X(t_1)$, and $Y = X(t_2)$. The transformation between the members of the processes is symbolized by $U = |X|$, $V = |Y|$.

$$G(u,v) = \mathbb{P}(U=\pm u, V=\pm v)$$

$$= \begin{cases} 
\mathbb{P}(-u < X < u, -v < Y < v), & u, v \geq 0 \\
0, & \text{otherwise} 
\end{cases}$$

$$= \begin{cases} 
\int_{-u}^{u} \int_{-v}^{v} dx \, dy \, f(x,y), & u, v \geq 0 \\
0, & \text{otherwise} 
\end{cases}$$

where $G(u,v)$ is the second-order distribution function of the process $U$, and $f(x,y)$ is the second-order probability density function of the process $X$.

$$g(u,v) = \frac{\partial^2 G}{\partial u \partial v}$$

$$= \begin{cases} 
\frac{\partial}{\partial u} \int_{-u}^{u} dx \left[ f(x,v) + f(x,-v) \right], & u, v \geq 0 \\
0, & \text{otherwise} 
\end{cases}$$

$$= \begin{cases} 
f(u,v) + f(u,-v) + f(-u,v) + f(-u,-v), & u, v \geq 0 \\
0, & \text{otherwise} 
\end{cases}$$
If \( X \) is a gaussian process, \( f(u,v) = f(-u,-v) = n(u,v) \), and \( f(u,-v) = f(-u,v) = n(u,-v) \). Thus

\[
 g(u,v) = \begin{cases} 
 2[n(u,v) + n(u,-v)], & u,v \geq 0 \\
 0, & \text{elsewhere} 
\end{cases} 
\]

\[
 g(u,v) = \begin{cases} 
 \frac{1}{\pi c^2 \sqrt{1 - r^2(T)}} \left[ \exp \left( \frac{u^2 - 2r(T)uv + v^2}{-2c^2[1 - r^2(T)]} \right) + \exp \left( \frac{u^2 + 2r(T)uv + v^2}{-2c^2[1 - r^2(T)]} \right) \right], & u,v \geq 0 \\
 0, & \text{elsewhere} 
\end{cases} 
\]

(49)

\[
 g(u,v) = \begin{cases} 
 \frac{2}{\pi c^2 \sqrt{1 - r^2(T)}} \exp \left( -\frac{u^2 + v^2}{2c^2[1 - r^2(T)]} \right) \cosh \left( \frac{r(T)uv}{c^2[1 - r^2(T)]} \right), & u,v \geq 0 \\
 0, & \text{elsewhere} 
\end{cases} 
\]

(50)

A similar procedure is employed to derive the third-order probability density function. Let \( U = U(t_1), V = U(t_2), W = U(t_3), X = X(t_1), Y = X(t_2), Z = X(t_3), U = |X|, V = |Y|, W = |Z| \).

\[
 G(u,v,w) = P(U = u, V = v, W = w) 
\]

\[
 = \begin{cases} 
 P(-u < X < u, -v < Y < v, -w < X < w), & u,v,w \geq 0 \\
 0, & \text{elsewhere} 
\end{cases} 
\]

\[
 g(u,v,w) = \int_{-u}^{u} dx \int_{-v}^{v} dy \int_{-w}^{w} dz f(x,y,z), u,v,w \geq 0 
\]

(49)

\[
 g(u,v,w) = \begin{cases} 
 \frac{3}{\partial x \partial y \partial z} \
 \frac{\partial^2 F}{\partial x \partial y} \int_{-u}^{u} dx \int_{-v}^{v} dy [f(x,y,w) + f(x,y,-w)], & u,v,w \geq 0 \\
 0, & \text{elsewhere} 
\end{cases} 
\]

(50)
\[
\frac{\partial F}{\partial x} \int_{-u}^{u} dx \left[ f(x,v,w) + f(x,v,-w) 
+ f(x,-v,w) + f(x,-v,-w) \right], \quad u,v,w \geq 0
\]
0, elsewhere

\[
G(u,v,w,s) = P(U \leq u, \ V \leq v, \ W \leq w, \ S \leq s)
= \int_{-u}^{u} \int_{-v}^{v} \int_{-w}^{w} \int_{-s}^{s} dt f(x,y,z,t), \quad u,v,w,s \geq 0
\]
0, elsewhere

The same procedure is employed to derive the fourth-order probability density function. Let \( U = U(t_1), \ V = U(t_2), \ W = U(t_3), \ S = U(t_4), \ X = x(t_1), \ Y = x(t_2), \ Z = x(t_3), \ T = x(t_4), \)
\( U = |X|, \ V = |Y|, \ W = |Z|, \ S = |T|. \)

If \( f(x, y, z) = n(x, y, z), \) then

\[
g(u,v,w) = \begin{cases} 
2\left[n(u,v,w) + n(u,v,-w) + n(-u,v,-w) + n(-u,-v,-w)\right], & \text{if } u,v,w \geq 0 \\
0, & \text{elsewhere}
\end{cases}
\]

(51)
\[ g(u,v,w,s) = \frac{\partial^4 G}{\partial x \partial y \partial z \partial t} \]

\[
\begin{aligned}
&= \begin{cases} 
 f(u,v,w,s) + f(-u,v,w,s) + f(u,-v,w,s) + f(-u,-v,w,s) \\
+ f(u,v,-w,s) + f(-u,v,-w,s) + f(u,-v,-w,s) + f(-u,-v,-w,s) \\
+ f(u,v,w,-s) + f(-u,v,w,-s) + f(u,-v,w,-s) + f(-u,-v,w,-s) \\
+ f(u,v,-w,-s) + f(-u,v,-w,-s) + f(u,-v,-w,-s) + f(-u,-v,-w,-s), \ u,v,w,s \geq 0 \\
0, \text{ elsewhere}
\end{cases}
\end{aligned}
\]

If \( f(x,y,z,t) = n(x,y,z,t) \), then

\[
\begin{aligned}
g(u,v,w,s) &= \begin{cases} 
 2[n(u,v,w,s) + n(-u,v,w,s) + n(u,-v,w,s) \\
+ n(u,v,-w,s) + n(u,v,-w,-s) + n(u,v,w,-s) \\
+ n(-u,v,-w,s) + n(-u,v,w,-s) + n(-u,v,w,-s)], \ u,v,w,s \geq 0 \\
0, \text{ elsewhere.}
\end{cases}
\end{aligned}
\]

(52)
APPENDIX III

SECOND-ORDER PROBABILITY DENSITY FUNCTION FOR THE SQUARE OF A STATIONARY GAUSSIAN PROCESS

Consider a process U derived from a process X with known probability density functions, and let each member of U be the square of a member X. Let U = U(t₁), V = U(t₂), X = X(t₁), Y = X(t₂). The transformation between the members of the process is U = X², V = Y².

\[ G(u,v) = P(U < u, V < v) \]

\[ = \begin{cases} 
P(-\sqrt{u} < X < \sqrt{u}, -\sqrt{v} < Y < \sqrt{v}), u,v \geq 0 \\
0, \text{ elsewhere} 
\end{cases} \]

\[ = \begin{cases} 
\frac{\sqrt{u}}{2\sqrt{u}} \int_{-\sqrt{v}}^{\sqrt{v}} \int_{-\sqrt{u}}^{\sqrt{u}} f(x,y) \, dx \, dy, u,v \geq 0 \\
0, \text{ elsewhere} 
\end{cases} \]

where \( G(u,v) \) is the second-order probability distribution function of the process U, and \( f(x,y) \) is the second-order probability density function of the process X.

\[ g(u,v) = \frac{\partial^2 G}{\partial u \partial v} \]

\[ = \begin{cases} 
\frac{1}{2\sqrt{v}} \int_{-\sqrt{u}}^{\sqrt{u}} \left[ f(x, v) + f(x, -v) \right] \, dx, u,v \geq 0 \\
0, \text{ elsewhere} 
\end{cases} \]
\[
\frac{1}{4\sqrt{uv}} \left[ f(\sqrt{u}, \sqrt{v}) + f(\sqrt{u}, -\sqrt{v}) + f(-\sqrt{u}, \sqrt{v}) + f(-\sqrt{u}, -\sqrt{v}) \right], u, v \geq 0
\]
\[
0, \text{ elsewhere}.
\]

If X is a gaussian process, \( f(\sqrt{u}, \sqrt{v}) = f(-\sqrt{u}, -\sqrt{v}) = n(\sqrt{u}, \sqrt{v}) \), and \( f(\sqrt{u}, -\sqrt{v}) = f(-\sqrt{u}, \sqrt{v}) = n(\sqrt{u}, -\sqrt{v}) \).

Thus
\[
g(u, v) = \begin{cases} 
\frac{1}{2\sqrt{uv}} \left[ n(\sqrt{u}, \sqrt{v}) + n(\sqrt{u}, -\sqrt{v}) \right], & u, v \geq 0 \\
0, \text{ elsewhere} 
\end{cases}
\]

\[
\left( \frac{\exp \left( -\frac{1}{2} \frac{u + 2r(T)\sqrt{uv}}{c^2[1 - r^2(T)]} \right) + \exp \left( -\frac{1}{2} \frac{u - 2r(T)\sqrt{uv}}{c^2[1 - r^2(T)]} \right)}{2\pi c^2[1 - r^2(T)] 2\sqrt{uv}} \right), \quad u, v \geq 0
\]
\[
0, \text{ elsewhere}.
\] (53)

\[
\left( \frac{1}{2\pi c^2[1 - r^2(T)] \sqrt{uv}} \exp \left( -\frac{1}{2} \frac{u + v}{c^2[1 - r^2(T)]} \right) \cosh \left( \frac{r(T)\sqrt{uv}}{c^2[1 - r^2(T)]} \right) \right), \quad u, v \geq 0
\]
\[
0, \text{ elsewhere}.
\] (54)
APPENDIX IV

THE JOINT PROBABILITY DENSITY FUNCTION FOR THE ABSOLUTE VALUES OF A STATIONARY GAUSSIAN PROCESS AND ITS FIRST AND SECOND DERIVATIVES

The joint probability density function for the absolute values of a gaussian process and their first and second derivatives will be derived by employing Equation (32). The third-order probability density function for the absolute values of a gaussian process is (51) which is derived in Appendix II. Replacing $v$ by $x + hx$ and $w$ by $x + 2hx + h^2 x$ in (51), and substituting the result in (32) gives

$$g(x,\dot{x},\ddot{x}) = \lim_{h \to 0} h^3 \left\{ 2 \left\{ n(x,x + hx,x + 2hx + h^2 x) \right. \right.$$

$$\left. + n\left[x, x + hx, - (x + 2hx + h^2 x)\right] \right.$$\n
$$\left. + n\left[x, - (x + hx), x + 2hx + h^2 x\right] \right.$$\n
$$\left. + n\left[-x, x + hx, x + 2hx + h^2 x\right], \right.$$\n
$$x \leq 0, \text{ all } \dot{x} \text{ and } \ddot{x},$$

$$0, x < 0, \text{ all } \dot{x} \text{ and } \ddot{x},$$

where $n$ is the third order gaussian probability density function for $X(t), X(t + h), X(t + 2h)$. The domains of definition are determined from the third order probability density function. For the first domain it requires $0 \leq x < \infty, 0 \leq x + hx < \infty, 0 \leq x + 2hx + h^2 x < \infty$, and this implies $-x \leq hx < \infty, \text{ and } -x - 2hx \leq h^2 x < \infty$.

The first term of the first domain of (55) is

$$Q = \lim_{h \to 0} h^3 \left\{ \frac{h^3}{(2\pi)^{3/2} c^3 A^{1/2}} \exp \left\{ - \frac{1}{2} \frac{B(x,\dot{x},\ddot{x})}{c^2 A} \right\} \right.$$

$$\left. \right\},$$
\[
\frac{1}{(2\pi)^{3/2}c_3} \lim_{h \to 0} \frac{\hbar^3}{A^{1/2}} \lim_{h \to 0} \exp \left\{ -\frac{1}{2} \frac{B(x',x',x')}{c_2 A} \right\} , \quad (56)
\]

where
\[
A = \begin{vmatrix}
1 & r(T) & r(2T) \\
r(T) & 1 & r(T) \\
r(2T) & r(T) & 1 \\
\end{vmatrix}
\]

\[
B(x',x',x') = A_{11} x^2 + A_{22} (x + h\dot{x})^2 + A_{33} (x + 2h\dot{x} + h^2\ddot{x})^2
\]
\[
+ 2A_{12} x(x + h\dot{x}) + 2A_{13} x(x + 2h\dot{x} + h^2\ddot{x})
\]
\[
+ 2A_{23} (x + h\dot{x})(x + 2h\dot{x} + h^2\ddot{x})
\]

is the cofactor of the element in the \(i^{th}\) row and \(j^{th}\) column of \(A\).

In evaluating the second factor of (56), there is no ambiguity in taking the square-root of \(\lim_{h \to 0} \frac{\hbar^6}{A}\). In determining this limit, L'Hopital's rule must be applied six times. It is convenient to use the rule for differentiating determinants to find the derivatives of \(A\). Let
\[
A = (a_1, a_2, a_3), \quad \text{where} \quad a_i \quad \text{is the} \quad i^{th} \quad \text{column of} \quad A. \quad \text{Then}
\]
\[
A' = (a_1', a_2', a_3') + (a_1, a_2', a_3) + (a_1', a_2, a_3').
\]

Since the manipulations for the higher derivatives are quite lengthy, a more abbreviated notation is employed:
\[
A' \quad 100 \quad 010 \quad 001,
\]
in which plus and equal signs are deleted, and the numbers denote the order of the derivative of the corresponding column. For the second derivative,
\[
A'' \quad 200 \quad 110 \quad 101
\]
\[
110 \quad 020 \quad 011
\]
\[
101 \quad 011 \quad 002.
\]
Combining like expressions gives

\[ A'' \times 1 \quad 200 \quad 020 \quad 002 \]
\[ \times 2 \quad 110 \quad 101 \quad 011. \]

The same procedure is employed to find the higher order derivatives.

\[ A''' \times 1 \quad 300 \quad 030 \quad 003 \]
\[ \times 3 \quad 210 \quad 201 \quad 120 \quad 102 \quad 021 \quad 012 \]
\[ \times 6 \quad 111. \]

\[ A^{(4)} \times 1 \quad 400 \quad 040 \quad 004 \]
\[ \times 4 \quad 310 \quad 301 \quad 130 \quad 031 \quad 103 \quad 013 \]
\[ \times 6 \quad 220 \quad 202 \quad 022 \]
\[ x12 \quad 211 \quad 121 \quad 112. \]

\[ A^{(5)} \times 1 \quad 500 \quad 050 \quad 005 \]
\[ \times 5 \quad 410 \quad 401 \quad 140 \quad 041 \quad 104 \quad 014 \]
\[ x10 \quad 320 \quad 302 \quad 230 \quad 032 \quad 203 \quad 023 \]
\[ x20 \quad 211 \quad 211 \quad 113 \]
\[ x30 \quad 221 \quad 212 \quad 122 \]

\[ A^{(6)} \times 1 \quad 600 \quad 060 \quad 006 \]
\[ \times 6 \quad 510 \quad 501 \quad 150 \quad 051 \quad 105 \quad 015 \]
\[ x15 \quad 420 \quad 402 \quad 240 \quad 042 \quad 204 \quad 024 \]
\[ x30 \quad 411 \quad 141 \quad 114 \quad (57) \]
\[ x20 \quad 330 \quad 303 \quad 033 \]
\[ x60 \quad 321 \quad 312 \quad 132 \quad 213 \quad 231 \quad 123 \]
\[ x90 \quad 222. \]
For evaluating the limits of these determinants it is convenient to write out the following determinants which are evaluated at $T = 0$:

\[
\begin{array}{ccc}
000 &=& \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} \\
111 &=& \begin{vmatrix} 0 & r''(0) & 4r''(0) \\ r''(0) & 0 & r''(0) \\ 4r''(0) & r''(0) & 0 \end{vmatrix} \\
333 &=& \begin{vmatrix} 0 & r^{(4)}(0) & 16r^{(4)}(0) \\ r^{(4)}(0) & 0 & r^{(4)}(0) \\ 16r^{(4)}(0) & r^{(4)}(0) & 0 \end{vmatrix}
\end{array}
\]

From these determinants it is seen that any determinant with either two columns from 000, or one column from 111, 333, or 555 will be zero. Thus, most of the determinants can be evaluated by inspection. The limit as $h \to 0$ of $A^{(5)}$ were found to be zero. Most of the determinants comprising $A^{(6)}$ are found to be zero by inspection. The determinants comprising $A^{(6)}$ not found to be zero are evaluated below.

\[
\begin{array}{ccc}
420 &=& \begin{vmatrix} 0 & r''(0) & 1 \\ r^{(4)}(0) & 0 & 1 \\ 16r^{(4)}(0) & r''(0) & 1 \end{vmatrix} = 16r^{(4)}(0)r''(0) \\
402 &=& \begin{vmatrix} 0 & 4r''(0) \\ r^{(4)}(0) & r''(0) \\ 16r^{(4)}(0) & 1 \end{vmatrix} = -44r^{(4)}(0)r''(0) \\
240 &=& \begin{vmatrix} 0 & r^{(4)}(0) & 1 \\ r''(0) & 0 & 1 \\ 4r''(0) & r^{(4)}(0) & 1 \end{vmatrix} = 4r^{(4)}(0)r''(0)
\end{array}
\]
\[
042 = \begin{vmatrix}
1 & r^{(4)}(0) & 4r''(0) \\
0 & 1 & r''(0) \\
0 & r^{(4)}(0) & 0
\end{vmatrix} = 4r^{(4)}(0)r''(0)
\]

\[
204 = \begin{vmatrix}
0 & 1 & 16r^{(4)}(0) \\
r''(0) & 1 & r^{(4)}(0) \\
4r''(0) & 1 & 0
\end{vmatrix} = 44r^{(4)}(0)r''(0)
\]

\[
024 = \begin{vmatrix}
1 & r''(0) & 16r^{(4)}(0) \\
0 & 1 & r^{(4)}(0) \\
0 & r''(0) & 0
\end{vmatrix} = 16r^{(4)}(0)r''(0)
\]

\[
222 = 8r''^3(0)
\]

Multiplying these expressions by the coefficients indicated by (57) gives

\[
A^{(6)}(0) = 15(-48)r''(0)r^{(4)}(0) + 90x8r''^3(0) = -720r''(0)\left[r^{(4)}(0) - r''^2(0)\right],
\]

and

\[
\lim_{h \to 0} \frac{h^3}{A^{1/2}} = \frac{1}{\sqrt{-r''(0)}\sqrt{r^{(4)}(0) - r''^2(0)}}. 
\]

Next, the third factor of (56) will be evaluated by finding the limit of the argument of the exponential. The derivatives of the covariance determinant \( A \) have already been found, and the derivatives of \( B(x, \dot{x}, \ddot{x}) \) must be found and evaluated at \( h = 0 \). \( B(x, \dot{x}, \ddot{x}) \) is the sum of products of the form \( C(h) = D(h)V(h) \). The derivatives of these terms are

\[
\begin{align*}
C' &= DV' + D'V \\
C'' &= DV'' + 2D'V' + D''V \\
C''' &= DV + 3D'V'' + 3D''V' + DDV
\end{align*}
\]
\[ c^{(4)} = D V^{(4)} + 4 D' V + 6 D'' V + 4 D V' + D^{(4)} V \]
\[ c^{(5)} = D V^{(5)} + 5 D' V^{(4)} + 10 D'' V + 10 D V'' + 5 D^{(4)} V' + D^{(5)} V \]
\[ c^{(6)} = D V^{(6)} + 6 D' V^{(5)} + 15 D'' V^{(4)} + 20 D V'' + 15 D^{(4)} V'' + 6 D^{(5)} V' + D^{(6)} V. \]

The V-functions and their nonzero derivatives are listed below:

\[ V_1 = x^2 \]
\[ V_2 = (x + h\dot{x})^2 \]
\[ V'_2 = 2(x + h\dot{x})\dot{x} \]
\[ V''_2 = 2\dot{x}^2 \]
\[ V_3 = (x + 2h\dot{x} + h^2\ddot{x})^2 \]
\[ V'_3 = 4(x + 2h\dot{x} + h^2\ddot{x})(\dot{x} + h\dddot{x}) \]
\[ V''_3 = 4(x + 2h\dot{x} + h^2\ddot{x})\dddot{x} + 8(\dot{x} + h\dddot{x})^2 \]
\[ V''''_3 = 24(\dot{x} + h\dddot{x})\dddot{x} \]
\[ V^{(4)}_3 = 24\dddot{x}^2 \]
\[ V_4 = 2x(x + h\dot{x}) \]
\[ V'_4 = 2x\dddot{x} \]
\[ V_5 = 2x(x + 2h\dot{x} + h^2\ddot{x}) \]
\[ V'_5 = 4x(\dot{x} + h\dddot{x}) \]
\[ V''_5 = 4x\dddot{x} \]
\[ V_6 = 2(x + h\dot{x})(x + 2h\dot{x} + h^2\ddot{x}) \]
\[ V' = 4(x + h\dot{x})(\dot{x} + h\ddot{x}) + 2\dot{x}(x + 2h\dot{x} + h^2\ddot{x}) \]

\[ V'' = 4(x + h\dot{x})\ddot{x} + 4\dot{x}(\dot{x} + h\ddot{x}) + 4\ddot{x}(\dot{x} + h\ddot{x}) \]

\[ V''' = 12\dddot{x} \]

The limit of these \( V \)-functions and their derivatives as \( h \to 0 \) is tabulated below.

**TABLE 2**

LIMITS OF THE \( V \)-FUNCTIONS AND THEIR DERIVATIVES

<table>
<thead>
<tr>
<th>Derivative Order</th>
<th>Number of the Term of ( B(x, \dot{x}, \ddot{x}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>( x^2 )</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
</tr>
</tbody>
</table>

The \( D \)-functions are the cofactors of the covariance determinant \( A \).

Using the same notation that was employed in deriving the derivatives of \( A \),

\[ D = 00 \quad \text{(all zero)} \]

\[ D' = 10 \quad 01 \quad \text{(all zero)} \]

\[ D'' = 20 \quad 02 \]

\[ x = 2 \quad 11 \quad \text{(all zero)} \]
\[
\begin{array}{ccc}
D'' & 30 & 03 \\
x & 3 & 21 & 12
\end{array} (\text{all zero})
\]

\[
\begin{array}{ccc}
D^{(4)} & 40 & 04 \\
x & 4 & 31 & 13 \\
x & 6 & 22
\end{array} (\text{all zero})
\]

\[
\begin{array}{ccc}
D^{(5)} & 50 & 05 \\
x & 5 & 41 & 14 \\
x & 10 & 32 & 23
\end{array} (\text{all zero})
\]

\[
\begin{array}{ccc}
D^{(6)} & 60 & 06 \\
x & 6 & 51 & 15 \\
x & 15 & 42 & 24 \\
x & 20 & 33
\end{array} (\text{all zero})
\]

Again, many of the determinants are zero when evaluated at \( h = 0 \). Furthermore, all \( D, D', D'', \) and \( D^{(5)} \) are zero, and the expressions for the terms of \( B(x,\dot{x},\ddot{x}) \) can be simplified:

\[
\begin{align*}
C &= 0 \\
C' &= 0 \\
C'' &= D''V \\
C''' &= 3D''V' \\
C^{(4)} &= 6D''V'' + D^{(4)}V \\
C^{(5)} &= 10D''V''' + 5D^{(4)}V' \\
C^{(6)} &= 15D''V^{(4)} + 15D^{(4)}V'' + D^{(6)}V
\end{align*}
\]

Next, the nonzero limits of the cofactors and their derivatives are computed.
\[ A_{11}'' = \begin{vmatrix} 0 & 1 \\ r''(0) & 1 \end{vmatrix} + \begin{vmatrix} 1 & r''(0) \\ 1 & 0 \end{vmatrix} = -2r''(0) \]

\[ A_{11}^{(4)} = \begin{vmatrix} 0 & 1 \\ r^{(4)}(0) & 1 \end{vmatrix} + \begin{vmatrix} 1 & r^{(4)}(0) \\ 1 & 0 \end{vmatrix} + 6 \begin{vmatrix} 0 & r''(0) \\ r'(0) & 0 \end{vmatrix} = -2r^{(4)}(0) - 6r''^2(0) \]

\[ A_{11}^{(6)} = \begin{vmatrix} 0 & 1 \\ r^{(6)}(0) & 1 \end{vmatrix} + \begin{vmatrix} 1 & r^{(6)}(0) \\ 1 & 0 \end{vmatrix} + 15 \begin{vmatrix} 0 & r^{(4)}(0) \\ r''(0) & 0 \end{vmatrix} = -2r^{(6)}(0) - 30r''r^{(4)}(0) \]

\[ A_{22}'' = \begin{vmatrix} 0 & 1 \\ 4r''(0) & 1 \end{vmatrix} + \begin{vmatrix} 1 & 4r''(0) \\ 1 & 0 \end{vmatrix} = -8r''(0) \]

\[ A_{22}^{(4)} = \begin{vmatrix} 0 & 1 \\ 16r^{(4)}(0) & 1 \end{vmatrix} + \begin{vmatrix} 1 & 16r^{(4)}(0) \\ 1 & 0 \end{vmatrix} + 6 \begin{vmatrix} 0 & 4r''(0) \\ 4r'(0) & 0 \end{vmatrix} = -32r^{(4)}(0) - 96r''^2(0) \]

\[ A_{22}^{(6)} = \begin{vmatrix} 0 & 1 \\ 64r^{(6)}(0) & 1 \end{vmatrix} + \begin{vmatrix} 1 & 64r^{(6)}(0) \\ 1 & 0 \end{vmatrix} + 15 \begin{vmatrix} 0 & 16r^{(4)}(0) \\ 4r''(0) & 0 \end{vmatrix} = -128r^{(6)}(0) - 30 \times 64r''r^{(4)}(0) \]

\[ A_{33}'' = \begin{vmatrix} 0 & 1 \\ r''(0) & 1 \end{vmatrix} + \begin{vmatrix} 1 & r''(0) \\ 1 & 0 \end{vmatrix} = -2r''(0) \]

\[ A_{33}^{(4)} = \begin{vmatrix} 0 & 1 \\ r^{(4)}(0) & 1 \end{vmatrix} + \begin{vmatrix} 1 & r^{(4)}(0) \\ 1 & 0 \end{vmatrix} + 6 \begin{vmatrix} 0 & r''(0) \\ r'(0) & 0 \end{vmatrix} = -2r^{(4)}(0) - 6r''^2(0) \]
\[
A_{33}^{(6)} = \begin{vmatrix} 0 & 1 & 1 & r^{(6)}(0) & +15 & r^{(4)}(0) & 0 & r''(0) \\ r^{(6)}(0) & 1 & 1 & +15 & r^{(4)}(0) & 0 \\ +15 & 0 & r^{(4)}(0) & r''(0) & 0 \\ \end{vmatrix} = -2r^{(6)}(0) - 30r''(0)r^{(4)}(0)
\]

\[
A_{12}^{(6)} = -\begin{vmatrix} r''(0) & 1 & +1 & 0 \\ 4r''(0) & 1 & 1 & r''(0) \\ \end{vmatrix} = 4r''(0)
\]

\[
A_{12}^{(4)} = -\begin{vmatrix} r^{(4)}(0) & 1 & +1 & r^{(4)}(0) \\ 16r^{(4)}(0) & 1 & 1 & 0 \\ \end{vmatrix} = 16r^{(4)}(0) + 24r''^2(0)
\]

\[
A_{12}^{(6)} = -\begin{vmatrix} r^{(6)}(0) & 1 & +1 & r^{(6)}(0) \\ 64r^{(6)}(0) & 1 & 1 & 0 \\ \end{vmatrix} = 64r^{(6)}(0) + 15 \times 20r''(0)r^{(4)}(0)
\]

\[
A_{13}^{(4)} = -\begin{vmatrix} r^{(4)}(0) & 1 & +1 & 0 \\ 16r^{(4)}(0) & 1 & 1 & r^{(4)}(0) \\ \end{vmatrix} = -14r^{(4)}(0) + 6r''^2(0)
\]

\[
A_{13}^{(6)} = -\begin{vmatrix} r^{(6)}(0) & 1 & +1 & 0 \\ 64r^{(6)}(0) & 1 & 1 & r^{(6)}(0) \\ \end{vmatrix} = -62r^{(6)}(0) + 30r''(0)r^{(4)}(0)
\]

\[
A_{23}^{(6)} = \begin{vmatrix} 0 & 1 & +1 & r''(0) \\ 4r''(0) & 1 & 1 & r''(0) \\ \end{vmatrix} = 4r''(0)
\]
\[
A_{23}^{(4)} = \begin{vmatrix} 0 & 1 \\ 16r^{(4)}(0) & 1 \end{vmatrix} - \begin{vmatrix} 1 & r^{(4)}(0) \\ 1 & r^{(4)}(0) \end{vmatrix} = 16r^{(4)}(0) + 24r''(0)
\]

\[
A_{23}^{(6)} = \begin{vmatrix} 0 & 1 \\ 64r^{(6)}(0) & 1 \end{vmatrix} - \begin{vmatrix} 1 & r^{(6)}(0) \\ 1 & r^{(6)}(0) \end{vmatrix} = 64r^{(6)}(0) + 20 \times 15r''(0)r^{(4)}(0)
\]

These nonzero limits of cofactors are summarized in Table 3.

<table>
<thead>
<tr>
<th>TABLE 3</th>
</tr>
</thead>
</table>

**NONZERO LIMITS OF COFACTORS**

<table>
<thead>
<tr>
<th>Order of the Derivative</th>
<th>2</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cofactor</td>
<td>r''(0)</td>
<td>r''2(0)</td>
<td>r''4(0)</td>
</tr>
<tr>
<td>A11</td>
<td>-2</td>
<td>-6</td>
<td>-2</td>
</tr>
<tr>
<td>A22</td>
<td>-8</td>
<td>-96</td>
<td>-32</td>
</tr>
<tr>
<td>A33</td>
<td>-2</td>
<td>-6</td>
<td>-2</td>
</tr>
<tr>
<td>A12</td>
<td>4</td>
<td>24</td>
<td>16</td>
</tr>
<tr>
<td>A13</td>
<td>-2</td>
<td>6</td>
<td>-14</td>
</tr>
<tr>
<td>A23</td>
<td>4</td>
<td>24</td>
<td>16</td>
</tr>
</tbody>
</table>
### TABLE 4

**TERMS FOR B^n(0;x,x,x)**

<table>
<thead>
<tr>
<th>Order of Derivative</th>
<th>Multiply Entries by</th>
<th>Number of the Term</th>
<th>Sum of Entries</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>r''(0)x^2</td>
<td>-2</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-8</td>
<td>-2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-8</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>3r''(0)xx</td>
<td>0</td>
<td>-16</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-16</td>
<td>-8</td>
</tr>
<tr>
<td></td>
<td>6r''(0)x^2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0</td>
<td>-8</td>
</tr>
<tr>
<td></td>
<td>6r''(0)xx</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>r''(0)x^2</td>
<td>-6</td>
<td>-96</td>
</tr>
<tr>
<td></td>
<td>r''(4)(0)x^2</td>
<td>-2</td>
<td>-32</td>
</tr>
<tr>
<td>4</td>
<td>10r''(0)xx</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0</td>
<td>-192</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0</td>
<td>-64</td>
</tr>
<tr>
<td></td>
<td>5r''(4)(0)xx</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0</td>
<td>-64</td>
</tr>
<tr>
<td></td>
<td>5r''(4)(0)xx</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0</td>
<td>-192</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0</td>
<td>-64</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0</td>
<td>-24</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>r''(0)r''(4)(0)x^2</td>
<td>-30</td>
<td>-64</td>
</tr>
<tr>
<td></td>
<td>r''(6)(0)x^2</td>
<td>-2</td>
<td>-128</td>
</tr>
</tbody>
</table>

Examination of Table 4 shows that the second through the fifth derivatives of B(h;x,x,x) evaluated for h = 0 are zero. The sixth derivative evaluated at zero is

\[-720 \left\{ r''(0)r''(4)(0)x^2 + \left[ r''(0) - r''(4)(0) \right] x^2 + r''(0)x^2 - 2r''(0)x\dot{x} \right\} .\]

With the information already obtained, the limits of the second, third, and fourth terms in the first domain of (55) may be evaluated.
From the Expression for $B(h; x, \dot{x}, \ddot{x})$ below Equation (56) it is seen that if the signs of the last three terms are changed appropriately, the expression becomes suitable for incorporation in the second, third and fourth terms in the first domain of (55). Making the appropriate changes of sign for the second-derivative entries of Table 4 yields the results listed in Table 5.

**TABLE 5**

**EVALUATION OF SECOND DERIVATIVES**

<table>
<thead>
<tr>
<th>Term Number of (55)</th>
<th>Multiply Entries by $r''(0)x^2$</th>
<th>Number of the Term</th>
<th>Sum of Entries</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>-2</td>
<td>-8</td>
<td>-2</td>
</tr>
<tr>
<td>3</td>
<td>-2</td>
<td>-8</td>
<td>-2</td>
</tr>
<tr>
<td>4</td>
<td>-2</td>
<td>-8</td>
<td>-2</td>
</tr>
</tbody>
</table>

In each case $B''(h; x, \dot{x}, \ddot{x})$ is a positive quantity since $r''(0)$ is a negative quantity. Now the sign of $A''(h)$ in the neighborhood of $h = 0$ must be determined. It was shown that the first five derivatives of $A(h)$ were zero and the sixth was positive at $h = 0$. The first term of a power series expansion around $h = 0$ would be the product of a positive constant and the sixth power of $h$. The constant is positive because $A$ is the covariance determinant. Thus in the neighborhood of $h = 0$, the second derivative of $A$ would be positive. Hence, the limit as $h \to 0$ of the second, third, and fourth terms of the first domain of (55) is zero.

Combining all of the results obtained gives the result (33) in the text.
APPENDIX V

JOINT PROBABILITY DENSITY FUNCTION FOR X(t), X(t + T), X'(t), AND X'(t + T) FOR THE ABSOLUTE VALUES OF A STATIONARY GAUSSIAN PROCESS

Equation (34) will be employed to derive the joint probability density function for X(t), X(t + T), X'(t), and X'(t + T) for the absolute values of the members of a stationary gaussian process. The fourth-order probability density function of this process is given by (52) in Appendix II. Replacing u by x₁, v by x₂, w by x₁ + hₓ₁, and s by x₂ + hₓ₂ in (52), and substituting the result in (34) gives

\[
g(x_1, x_2, \dot{x}_1, \dot{x}_2) = \lim_{h \to 0} h^2 \left\{ \begin{array}{c}
2 \left[ n(x_1, x_2, x_1 + h\dot{x}_1, x_2 + h\dot{x}_2) + n(-x_1, x_2, x_1 + h\dot{x}_1, x_2 + h\dot{x}_2) \\
+ n(x_1, -x_2, x_1 + h\dot{x}_1, x_2 + h\dot{x}_2) + n(x_1, x_2, -x_1 - h\dot{x}_1, x_2 + h\dot{x}_2) \\
+ n(x_1, x_2, x_1 + h\dot{x}_1, -x_2 - h\dot{x}_2) + n(x_1, x_2, -x_1 - h\dot{x}_1, -x_2 - h\dot{x}_2) \\
+ n(-x_1, x_2, -x_1 - h\dot{x}_1, x_2 + h\dot{x}_2) + n(-x_1, x_2, x_1 + h\dot{x}_1, -x_2 - h\dot{x}_2) \right] \\
0, \text{ otherwise}
\end{array} \right.
\]

where \( x_1 = x(t) \), and \( x_2 = x(t + T) \). Except for sign differences in the arguments, the terms of (59) are similar. The limit of each term will be evaluated by considering the first term in detail and allowing for sign differences.

62
The first term of the first domain of (59) is

\[ J = \lim_{h \to 0} 2\hbar^2 n(x_1, x_2, x_1 + h\hat{x}_1, x_2 + h\hat{x}_2) \]

\[ = \lim_{h \to 0} \frac{2\hbar^2}{(2\pi)^2 c^4 \sqrt{A(h)}} \exp \left( -\frac{1}{2} \frac{B(h)}{c^2 A(h)} \right) , \tag{60} \]

where \( A(h) = \begin{vmatrix} 1 & r(T) & r(h) & r(T + h) \\ r(T) & 1 & r(T - h) & r(h) \\ r(h) & r(T - h) & 1 & r(\emptyset) \\ r(T + h) & r(h) & r(T) & 1 \end{vmatrix} \)

\[ B(h) = A_{11} x_1^2 + 2A_{12} x_1 x_2 + 2A_{13} x_1 (x_1 + h\hat{x}_1) + 2A_{14} x_1 (x_2 + h\hat{x}_2) \]

\[ + A_{22} x_2^2 + 2A_{23} x_2 (x_1 + h\hat{x}_1) + 2A_{24} x_2 (x_2 + h\hat{x}_2) \]

\[ + A_{33} (x_1 + h\hat{x}_1)^2 + 2A_{34} (x_1 + h\hat{x}_1) (x_2 + h\hat{x}_2) \]

\[ + A_{44} (x_2 + h\hat{x}_2)^2 \]

and \( A_{ij} \) is the cofactor of the element in the \( i \)th row and \( j \)th column of \( A(h) \). The second through seventh terms of (59) are similar to \( B(h) \) except for sign differences. The limit of a product is the product of the limits; hence

\[ J = \frac{2}{(2\pi)^2} \left( \lim_{h \to 0} \frac{\hbar^2}{\sqrt{A(h)}} \right) \left[ \lim_{h \to 0} \exp \left( -\frac{1}{2} \frac{B(h)}{c^2 A(h)} \right) \right] . \tag{61} \]

In evaluating the second factor of (61), it is convenient to find \( \sqrt{\lim_{h \to 0} \frac{\hbar^4}{A(h)}} \). To do so, L'Hospital's rule must be applied four times. The procedure for determining the derivatives of \( A(h) \) is similar to that employed in Appendix IV. Table 6 lists the determinants comprising the
first through fourth derivatives of a fourth order determinant. The notation is the same as that employed in Appendix IV.

### TABLE 6
DERIVATIVES OF A FOURTH ORDER DETERMINANT

<table>
<thead>
<tr>
<th>Order of Derivative</th>
<th>Multiply Line by</th>
<th>Terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1000 0100 0010 0001</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1100 1010 1001 0110 0101 0011</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2100 2010 2001 1200 0210 0201</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>1020 0120 0021 1002 0102 0012</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>1110 1101 1011 0111</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>2200 2020 2002 0220 0202 0022</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>2110 2101 2011 1210 1201 0211</td>
</tr>
<tr>
<td></td>
<td>24</td>
<td>1111</td>
</tr>
</tbody>
</table>

By employing Table 6 and the definition of \( A(h) \), it is determined that \( A'(0), A''(0), \) and \( A'''(0) \) are zero, and that
\[ A^{(4)}(0) = 24 \left\{ \left[ r''^2(0) - r''^2(T) \right] \left[ 1 - r^2(T) \right] \\
+ 2r'^2(T) [r''(0) - r(T)r''(T)] + r'^4(T) \right\}. \] (62)

Since the fourth derivative of \( h^4 \) with respect to \( h \) is \( 24 \), then

\[
\lim_{h \to 0} \frac{h^2}{\sqrt{A(h)}} = \frac{1}{\sqrt{\left[ r''^2(0) - r''^2(T) \right] \left[ 1 - r^2(T) \right] + 2r'^2(T) [r''(0) - r(T)r''(T)] + r'^4(T)}}. \] (63)

Next, the third factor of (61) is evaluated.

\[
\lim_{h \to 0} \exp \left( -\frac{1}{2} \frac{B(h)}{c^2 A(h)} \right) \exp \left( -\frac{1}{2} \lim_{h \to 0} \frac{B(h)}{c^2 A(h)} \right) \] (64)

The limits of the terms of \( B(h) \) (see (60)) are of the form

\[ C = (2 - \delta_{ij}) A_{ij}(h) F_{ij}(h) \]

where \( \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \)

and \( F_{ij} \) are the factors containing \( x_1, x_2, \dot{x}_1, \) and \( \dot{x}_2 \).

The limits of the derivatives of the terms of \( B(h) \) with respect to \( h \) are

\[ C' = (2 - \delta_{ij})(A_{ij} F'_{ij} + A'_{ij} F_{ij}) \]
\[ C'' = (2 - \delta_{ij})(A_{ij} F''_{ij} + 2A'_{ij} F'_{ij} + A''_{ij} F_{ij}) \]
\[ C''' = (2 - \delta_{ij})(A_{ij} F'''_{ij} + 3A'_{ij} F''_{ij} + 3A''_{ij} F'_{ij} + A'''_{ij} F_{ij}) \]
\[ C^{(4)} = (2 - \delta_{ij})(A_{ij} F^{(4)}_{ij} + 4A'_{ij} F'''_{ij} + 6A''_{ij} F''_{ij} + 4A'''_{ij} F'_{ij} + A^{(4)}_{ij} F_{ij}). \]

It is readily determined that \( \lim_{h \to 0} A_{ij}(h) = 0 \), and \( \lim_{h \to 0} A'_{ij}(h) = 0 \).

Thus, \( C \) and \( C' \) are zero, and
\[ c'' = (2 - \delta_{ij})A''_{ij}F_{ij} \]

\[ c''' = (2 - \delta_{ij})(3A''_{ij}F'_{ij} + A'''_{ij}F_{ij}) \]

\[ c^{(4)} = (2 - \delta_{ij})(6A''_{ij}F'''_{ij} + 4A'''_{ij}F''_{ij} + A^{(4)}_{ij}F_{ij}) \]

Table 7 lists \( F_{ij} \), \( F'_{ij} \), and \( F''_{ij} \) for the terms of \( B''(0) \), \( B'''(0) \), and \( B^{(4)}(0) \).

<table>
<thead>
<tr>
<th>Number of Term</th>
<th>Order of Derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>( x_1^2 )</td>
</tr>
<tr>
<td>2</td>
<td>( x_1x_2 )</td>
</tr>
<tr>
<td>3</td>
<td>( x_1^2 )</td>
</tr>
<tr>
<td>4</td>
<td>( x_1x_2 )</td>
</tr>
<tr>
<td>5</td>
<td>( x_2^2 )</td>
</tr>
<tr>
<td>6</td>
<td>( x_1x_2 )</td>
</tr>
<tr>
<td>7</td>
<td>( x_2^2 )</td>
</tr>
<tr>
<td>8</td>
<td>( x_1^2 )</td>
</tr>
<tr>
<td>9</td>
<td>( x_1x_2 )</td>
</tr>
<tr>
<td>10</td>
<td>( x_2^2 )</td>
</tr>
</tbody>
</table>
The procedure described in Appendix IV was employed to evaluate the limits of the derivatives of the cofactors of B(h). The expressions for $A_{ij}''$ and $A_{ij}'''$ are listed in Table 8, and those for $A_{ij}^{(4)}$ in Table 9.

The results listed in Tables 7 and 8 are combined to obtain expressions for $B''(0)$ for each term of (59). In the upper part of Table 10, the coefficients for each term of $B''(0)$ are listed, together with the signs appropriate for each term of (59) (for clarity, only minus signs are shown). In Table 11, the coefficients are summed for each term of (59). It is seen that for the first and seventh terms of (59), $B''(0)$ is zero, and L'Hopital's rule must be applied again. In the other terms of (59), $B''(0)$ is not identically zero. The ratio $B''(0)/A''(0)$ appears in the argument of an exponential. Since $A''(0) = 0$, the magnitude of the ratio approaches infinity. If the ratio approaches positive infinity, then the exponential approaches zero, but if the ratio approaches minus infinity, the exponential approaches plus infinity, and the probability density function is not defined. Now $A^{(4)}(0)$ is a positive multiple of the covariance determinant for two gaussian variates and their derivatives; hence, it is positive for any value of $T$. Since $A''(0)$ is zero, then $A''(T)$ is positive in the neighborhood of $T = 0$. Thus $B''(0)$ must be a positive quantity for (59) to be defined. Table 11 shows that $B''(0)$ is $-8\{r''(0)[1 - r^2(T)] + r^2(T)\}x^2$ in the second through fifth terms of (59). For these terms, $B''(0)$ is positive if

$$r^2(T) < -r''(0)[1 - r^2(T)] .$$

(65)

Table 11 also shows that in the sixth and eighth terms of (59), $B''(0)$ is the quadratic form
<table>
<thead>
<tr>
<th>((1) j_{(0)} \mu_{-1}(1)<em>{\mu</em>{-1}} )</th>
<th>((1)<em>{1} \mu</em>{-1} )</th>
<th>((\mu_{-1})_j )</th>
<th>((1)_{j-1} )</th>
<th>((1)_{j+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>([([1]<em>{j-1}] (0)</em>{\mu_{-1}} + ([1]<em>{j+1})</em>{\mu_{-1}} )</td>
<td>((1)<em>{\mu</em>{-1}} )</td>
<td>((1)_{j-1} )</td>
<td>((1)_{j+1} )</td>
</tr>
<tr>
<td>([([1]<em>{j-1}] (0)</em>{\mu_{-1}} + ([1]<em>{j+1})</em>{\mu_{-1}} )</td>
<td>((1)<em>{\mu</em>{-1}} )</td>
<td>((1)_{j-1} )</td>
<td>((1)_{j+1} )</td>
<td></td>
</tr>
<tr>
<td>([([1]<em>{j-1}] (0)</em>{\mu_{-1}} + ([1]<em>{j+1})</em>{\mu_{-1}} )</td>
<td>((1)<em>{\mu</em>{-1}} )</td>
<td>((1)_{j-1} )</td>
<td>((1)_{j+1} )</td>
<td></td>
</tr>
<tr>
<td>([([1]<em>{j-1}] (0)</em>{\mu_{-1}} + ([1]<em>{j+1})</em>{\mu_{-1}} )</td>
<td>((1)<em>{\mu</em>{-1}} )</td>
<td>((1)_{j-1} )</td>
<td>((1)_{j+1} )</td>
<td></td>
</tr>
<tr>
<td>([([1]<em>{j-1}] (0)</em>{\mu_{-1}} + ([1]<em>{j+1})</em>{\mu_{-1}} )</td>
<td>((1)<em>{\mu</em>{-1}} )</td>
<td>((1)_{j-1} )</td>
<td>((1)_{j+1} )</td>
<td></td>
</tr>
<tr>
<td>([([1]<em>{j-1}] (0)</em>{\mu_{-1}} + ([1]<em>{j+1})</em>{\mu_{-1}} )</td>
<td>((1)<em>{\mu</em>{-1}} )</td>
<td>((1)_{j-1} )</td>
<td>((1)_{j+1} )</td>
<td></td>
</tr>
<tr>
<td>([([1]<em>{j-1}] (0)</em>{\mu_{-1}} + ([1]<em>{j+1})</em>{\mu_{-1}} )</td>
<td>((1)<em>{\mu</em>{-1}} )</td>
<td>((1)_{j-1} )</td>
<td>((1)_{j+1} )</td>
<td></td>
</tr>
<tr>
<td>([([1]<em>{j-1}] (0)</em>{\mu_{-1}} + ([1]<em>{j+1})</em>{\mu_{-1}} )</td>
<td>((1)<em>{\mu</em>{-1}} )</td>
<td>((1)_{j-1} )</td>
<td>((1)_{j+1} )</td>
<td></td>
</tr>
</tbody>
</table>
### TABLE 9

**Expressions for \((2-\delta_{ij})A_{ij}^{(h)}\)**

<table>
<thead>
<tr>
<th>(B(n)) Term</th>
<th>((2-\delta_{ij})A_{ij})</th>
<th>((2-\delta_{ij})A_{ij}^{(h)})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (A_{11})</td>
<td>(-2r_{(1)}^{(h)}(0) \left[ 1-r^2(T) \right] -6r^2(0)+12r(0)r(T)r(T)-6r^2(T)-8r'(T)r''(T))</td>
<td>(-2r_{(1)}^{(h)}(0) \left[ 1-r^2(T) \right] -6r^2(0)+12r(0)r(T)r(T)-6r^2(T)-8r'(T)r''(T))</td>
</tr>
<tr>
<td>2 (2A_{12})</td>
<td>(4r_{(1)}^{(h)}(0) \left[ 1-r^2(T) \right] -12r^2(0)r(T)+24r(0)r(T)-12r(T)r^2(T)+16r(T)r'(T)r''(T))</td>
<td>(4r_{(1)}^{(h)}(0) \left[ 1-r^2(T) \right] -12r^2(0)r(T)+24r(0)r(T)-12r(T)r^2(T)+16r(T)r'(T)r''(T))</td>
</tr>
<tr>
<td>3 (2A_{13})</td>
<td>(4r_{(1)}^{(h)}(0) \left[ 1-r^2(T) \right] +36r^2(0)-24r(0)r(T)-24r(0)r(T)-12r^2(T)-16r'(T)r''(T))</td>
<td>(4r_{(1)}^{(h)}(0) \left[ 1-r^2(T) \right] +36r^2(0)-24r(0)r(T)-24r(0)r(T)-12r^2(T)-16r'(T)r''(T))</td>
</tr>
<tr>
<td>4 (2A_{14})</td>
<td>(-4r_{(1)}^{(h)}(0) \left[ 1-r^2(T) \right] -12r^2(0)r(T)-24r(0)r(T)+36r(T)r^2(T)-24r^2(T)r(T)-16r(T)r'(T)r''(T))</td>
<td>(-4r_{(1)}^{(h)}(0) \left[ 1-r^2(T) \right] -12r^2(0)r(T)-24r(0)r(T)+36r(T)r^2(T)-24r^2(T)r(T)-16r(T)r'(T)r''(T))</td>
</tr>
<tr>
<td>5 (A_{22})</td>
<td>(-2r_{(1)}^{(h)}(0) \left[ 1-r^2(T) \right] -6r^2(0)+12r(0)r(T)r(T)-6r^2(T)-8r'(T)r''(T))</td>
<td>(-2r_{(1)}^{(h)}(0) \left[ 1-r^2(T) \right] -6r^2(0)+12r(0)r(T)r(T)-6r^2(T)-8r'(T)r''(T))</td>
</tr>
<tr>
<td>6 (2A_{23})</td>
<td>(-4r_{(1)}^{(h)}(0) \left[ 1-r^2(T) \right] -12r^2(0)r(T)-24r(0)r(T)+36r(T)r^2(T)-24r^2(T)r(T)-16r(T)r'(T)r''(T))</td>
<td>(-4r_{(1)}^{(h)}(0) \left[ 1-r^2(T) \right] -12r^2(0)r(T)-24r(0)r(T)+36r(T)r^2(T)-24r^2(T)r(T)-16r(T)r'(T)r''(T))</td>
</tr>
<tr>
<td>7 (2A_{24})</td>
<td>(4r_{(1)}^{(h)}(0) \left[ 1-r^2(T) \right] +36r^2(0)-24r(0)r(T)-24r(0)r(T)-12r^2(T)-16r'(T)r''(T))</td>
<td>(4r_{(1)}^{(h)}(0) \left[ 1-r^2(T) \right] +36r^2(0)-24r(0)r(T)-24r(0)r(T)-12r^2(T)-16r'(T)r''(T))</td>
</tr>
<tr>
<td>8 (A_{33})</td>
<td>(-2r_{(1)}^{(h)}(0) \left[ 1-r^2(T) \right] -6r^2(0)+12r(0)r(T)r(T)-6r^2(T)-8r'(T)r''(T))</td>
<td>(-2r_{(1)}^{(h)}(0) \left[ 1-r^2(T) \right] -6r^2(0)+12r(0)r(T)r(T)-6r^2(T)-8r'(T)r''(T))</td>
</tr>
<tr>
<td>9 (2A_{34})</td>
<td>(+4r_{(1)}^{(h)}(0) \left[ 1-r^2(T) \right] -12r^2(0)r(T)+24r(0)r(T)-12r(T)r^2(T)+16r(T)r'(T)r''(T))</td>
<td>(+4r_{(1)}^{(h)}(0) \left[ 1-r^2(T) \right] -12r^2(0)r(T)+24r(0)r(T)-12r(T)r^2(T)+16r(T)r'(T)r''(T))</td>
</tr>
<tr>
<td>10 (A_{44})</td>
<td>(-2r_{(1)}^{(h)}(0) \left[ 1-r^2(T) \right] -6r^2(0)+12r(0)r(T)r(T)-6r^2(T)-8r'(T)r''(T))</td>
<td>(-2r_{(1)}^{(h)}(0) \left[ 1-r^2(T) \right] -6r^2(0)+12r(0)r(T)r(T)-6r^2(T)-8r'(T)r''(T))</td>
</tr>
<tr>
<td>Number of Term of B(h)</td>
<td>$r^n(0)[1-r^2(T)]x_1^2$</td>
<td>$r^n(0)[1-r^2(T)]x_2^2$</td>
</tr>
<tr>
<td>-----------------------</td>
<td>-------------------------</td>
<td>-------------------------</td>
</tr>
<tr>
<td>1</td>
<td>-2</td>
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</tr>
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<td>2</td>
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<td>-2</td>
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<td>4</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>-2</td>
<td>-2</td>
</tr>
</tbody>
</table>
### TABLE 11

**Coefficients for $B''(0)$ for each term of (59)**

<table>
<thead>
<tr>
<th>Number of Term of (59)</th>
<th>${r''(0)\left[1-r^2(T)\right]}$</th>
<th>${r''(0)\left[1-r^2(T)\right]}$</th>
<th>${r''(T)\left[1-r^2(T)\right]}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$+r^2(T)x_1^2$</td>
<td>$+r^2(T)x_2^2$</td>
<td>$+r(T)r^2(T)x_1x_2$</td>
</tr>
<tr>
<td>1</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>2</td>
<td>$-8$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>$-8$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$-8$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
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<td></td>
<td>$16$</td>
</tr>
<tr>
<td>6</td>
<td>$-8$</td>
<td>$-8$</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>$-8$</td>
<td>$-8$</td>
<td>$-16$</td>
</tr>
</tbody>
</table>
This quadratic form divided by eight has the discriminant

\[
D = \pm \left\{ r''(0)[1 - r^2(T)] + r'^2(T) \right\} x_1^2 - 8 \left\{ r''(0)[1 - r^2(T)] + r'^2(T) \right\} x_1 x_2^2.
\]

The principal minor of the discriminant is

\[
D_1 = - \left\{ r''(0)[1 - r^2(T)] + r'^2(T) \right\}.
\]

The quadratic form is positive definite if \( D > 0 \), and \( D_1 > 0 \). The requirement \( D_1 > 0 \) is simply (65) which applies to terms 2, 3, 4, and 5 of (59).

For values of \( T \) such that \( 1 - r^2(T) > 0 \), the requirement \( D > 0 \) is identical to that which must be met for (63) to be defined. If \( D > 0 \), and \( D_1 > 0 \), then all the terms of (59) are zero excepting the first and seventh.

Next \( B''(0) \) will be evaluated for the first and seventh terms of (59). The nonzero terms of \( B''(0) \) were shown to be of the form \( (2 - \delta_{ij}) \)

\[
(3A_{ij}F_{ij} + A_{ij}^2F_{ij}).
\]

The results listed in Tables 7 and 8 are combined in Tables 12 and 13. It is seen that the sum of each column is zero.
<table>
<thead>
<tr>
<th>Number of Term of B(h)</th>
<th>{r^n(0) [1-r^2(T)] } x_1 \dot{x}_1 + {r^n(T) [1-r^2(T)] } x_2 \dot{x}_2 + {r^n(T) [1-r^2(T)] } x_1 \dot{x}_1 + {r^n(0) [1-r^2(T)] } x_2 \dot{x}_2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<tr>
<td>2</td>
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<td>9</td>
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<td>10</td>
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</tr>
</tbody>
</table>
TABLE 13

EXPRESSIONS FOR $A_{ij}$ $F_{ij}$

<table>
<thead>
<tr>
<th>Number of Term of $B(h)$</th>
<th>$[r''(T) - r''(0)r(T)]$</th>
<th>$[r''(T) - r''(0)r(T)]$</th>
<th>$[r''(0) - r(T)r''(t) + r'^2(T)]$</th>
<th>Signs for Term 7 of (59)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>0</td>
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</tr>
<tr>
<td>3</td>
<td>4</td>
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<tr>
<td>4</td>
<td></td>
<td>12</td>
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<tr>
<td>5</td>
<td>-2</td>
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<tr>
<td>6</td>
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<td>-12</td>
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<td>7</td>
<td>4</td>
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<tr>
<td>8</td>
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<tr>
<td>9</td>
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<tr>
<td>10</td>
<td>-2</td>
<td></td>
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</tr>
</tbody>
</table>
for both the first and seventh terms of (59). Thus \( B''(0)/A''(0) \) for these terms are indeterminate forms, and L'Hospital's rule is applied again.

Next \( B(4)(0) \) will be evaluated for the first and seventh terms of (59). The nonzero terms of \( B(4)(0) \) were shown to have the the form 
\[
(2 - \delta_{ij})(6A''_{ij}F''_{ij} + 4A''_{ij}F'_{ij} + A''_{ij}F_{ij}).
\]
The results listed in Tables 7 and 8 are combined in Tables 14 and 15 to obtain expressions for 
\[
(2 - \delta_{ij})A''_{ij}F''_{ij} \quad \text{and} \quad (2 - \delta_{ij})A''_{ij}F'_{ij}
\] respectively.

### TABLE 14

**Expressions for \((2 - \delta_{ij})A''_{ij}F''_{ij}\)**

<table>
<thead>
<tr>
<th>Number of Term of ( B(h) )</th>
<th>( r''(0)[1-r^2(T)] )</th>
<th>( r''(0)[1-r^2(T)] )</th>
<th>( r''(0)[1-r^2(T)] )</th>
<th>Signs for Term 7 of ((59))</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>( +r^2(T))</td>
<td>( +r^2(T))</td>
<td>( +r^2(T))</td>
<td>( +r^2(T))</td>
</tr>
<tr>
<td>9</td>
<td>( -4 )</td>
<td>( 8 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>( -4 )</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

From Table 7 it is seen that \( x_1^2 \) occurs with terms 1, 3, and 8 of \( B(4)(0) \), that \( x_2^2 \) occurs with terms 5, 7, and 10, and that \( x_1x_2 \) occurs with terms 2, 4, 6, and 9. For term 7 of (59), the signs for terms 2, 4, 6, and 9 are minus. With this information, terms of the form 
\[
(2 - \delta_{ij})A^{(4)}_{ij}F
\] can be formed using the information in Table 9. Following this procedure and employing the results from Tables 14 and 15 gives
### TABLE 15

**EXPRESSIONS FOR** \((2 - s_{ij})A_{ij}F_{ij}\)

<table>
<thead>
<tr>
<th>Number of Term of (B(h))</th>
<th>(A_{11}x_1x_1)</th>
<th>(2A_{14}x_1x_2)</th>
<th>(2A_{14}x_2x_1)</th>
<th>(A_{11}x_2x_2)</th>
<th>Signs for Term 7 of (59)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
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<td>4</td>
<td>1</td>
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<td>7</td>
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<tr>
<td>9</td>
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<td>0</td>
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<td>10</td>
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<td>2</td>
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</tbody>
</table>

**Sum, Term 1** 1 2 1 -1 2

**Sum, Term 7** 2 -1 1 2
\[ B^{(4)}(0) = \left[ 24r''^2(0) - 24r''^2(T) + 24r''(0) r'(T) \right] (x_1^2 + x_2^2) \]

\[ + \left[ -48r''^2(0) r(T) + 48r(T) r''^2(T) - 48r'^2(T) r''(T) \right] x_1 x_2 \]

\[-48r'(T) \left[ r''(T) - r''(0) r(T) \right] (x_1 \dot{x}_1 - x_2 \dot{x}_2) \]

\[ + 48r'(T) \left[ r''(0) - r(T) r''(T) + r'^2(T) \right] (x_1 \dot{x}_2 + x_2 \dot{x}_1) \]

\[ - 24 \left[ r''(0) \left[ 1 - r^2(T) \right] + r'^2(T) \right] (x_1^2 + x_2^2) \]

\[ + 48 \left[ r''(T) \left[ 1 - r^2(T) \right] + r(T) r'^2(T) \right] \dot{x}_1 \dot{x}_2 \]

Combining this result with those of (62) and (63) gives the final result (35) in the text.
BIBLIOGRAPHY


AUTOBIOGRAPHY

I, Magnus Moll, was born in East Orange, New Jersey, July 23, 1928. I received my secondary-school education in the public schools of West Orange, New Jersey. After serving as an electronics technician in the U. S. Navy, I attended Purdue University and received the Bachelor of Science degree in electrical engineering in 1953. Following service as an electronics maintenance officer in the U. S. Navy, I attended the University of Illinois and received the Master of Science degree in electrical engineering in 1956. Since then, I have been employed in the Systems Engineering Division of Battelle Memorial Institute while completing the requirements for the degree Doctor of Philosophy.