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THE APPLICATION OF SEQUENTIAL
DETECTION TO PULSED RADAR

DISSERTATION

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the Degree Doctor of Philosophy in the Graduate
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By

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The Ohio State University
1965

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CHAPTER I
THE APPLICATION OF STATISTICAL DECISION THEORY TO PULSED RADAR

The mathematical model of the radar signal

The radar system which will be investigated in this thesis is shown in block diagram form in Figure 1. The transmitter output consists of gigacycle sine waves which are modulated by a pulse train as shown in Figure 2. The antenna transmits this pulse train and also receives the reflected pulse trains from targets which are within the antenna beam. The antenna two way voltage gain pattern is given by $G(\beta)$ where $\beta =$ angular displacement from axis of beam. A typical form of $G(\beta)$ is shown in Figure 3. The antenna will be assumed to rotate in either discrete steps or at a constant angular rate. The function of the radar is assumed to be detection of all targets which appear in a circle centered on the radar. The antenna is rotated continuously in order to maintain surveillance of this area.

The reflected pulse trains received by the antenna are prepared for processing in the radar receiver by heterodyning the signal to the intermediate frequency (IF) as shown in Figure 1. The signal together with receiver thermal noise is amplified by the IF amplifier and detected. The detection process
antenna

\[ S(t) \cos ((\omega + \omega_0) t + \theta) \]

\[ y(t) = \cos ((\omega + \omega_0) t + \mu) \]

\[ S(t) = \sqrt{2P} \cdot g(\beta(t) - \phi) \cdot p(t - \frac{2r(t)}{C}) \]

\[ \otimes = \text{multiplier} \]

**Figure 1** — The basic components of a pulsed radar system.
Figure 2 — Radar transmitter output

\[ p(t) \cos(\omega_c t + \alpha) \]
Figure 3 — The radar antenna voltage gain pattern
removes the sinusoidal oscillations from the signal and noise and presents the resulting pulse train to the decision device. The decision device determines whether or not a signal is embedded in the noise. This process of transmission, reception, and decision will be described by a mathematical model. The design of a decision device to detect a signal in noise can then be formulated as a statistical decision theory problem. The transmitter output is

\[ \sqrt{2}Ap(t) \cos \left( \frac{\omega_c}{c} t + \alpha \right) \]

where

\[ p(t) = \text{the modulating pulse train of Figure 2} \]
\[ \omega_c = \text{the carrier angular frequency} \]
\[ \alpha = \text{arbitrary phase reference} \]
\[ A = \text{power level of transmission}. \]

The signal after reflection from a target, reception and heterodyning is

\[ \sqrt{2}P g(\beta(t)-\phi) p(t - \frac{2r(t)}{c}) \cos \left( \omega_o + \omega_d \right) t + \theta) \]

where

\[ P = \text{received signal power} \]
\[ \beta(t) = \text{angular position of the antenna} \]
\[ \phi = \text{bearing of the target} \]
\[ r = \text{range of the target} \]
\[ \omega_o = \text{intermediate frequency} \]
\[ \omega_d = \text{doppler shift of carrier-frequency} = 2 \frac{r(t)}{c} \omega_c/c \]
\[ c = \text{velocity of light} \]
\[ g(\beta) = \text{normalized antenna gain} = \frac{G(\beta)}{G(\phi)}. \]

The received signal power \( P \) is assumed to be constant although there is a very slow variation in \( P \) due to the change in \( r \). It will also fluctuate if the apparent radar cross section of the target fluctuates. The target will be assumed non-fluctuating unless otherwise specified. The function \( g(\beta(t)) \) represents the modulation of the signal by the antenna pattern. The factor \( \frac{2r}{c} \) is the time delay between transmission and reception for the target at range \( r \).

If the antenna scans at a constant angular rate, the resulting signal modulation can be represented as the product of the pulse train \( p(t-2r/c) \) and the antenna gain pattern \( g(\beta(t)-\phi) \) as shown in Figure 4. Therefore, the signal is a finite length pulse train of length proportional to the antenna beam width. If the antenna scans in discrete steps, the signal modulation is a rectangular pulse of length \( T_{D_1} \) and amplitude \( g(\Delta-\phi) \) times \( p(t-2r/c) \) as shown in Figure 4. The "dwell time" of the antenna at the target position determines the length of the modulation \( T_{D_1} \). The dwell time may be a constant or may be a random variable so the signal is a finite pulse train of constant or random length.

**The representation of thermal noise in pulsed radar**

In addition to the signal, the input to the receiver contains thermal noise which can be described as very wide band and gaussian in its amplitude statistics. The noise spectrum is restricted by the filter characteristic of the IF amplifier so that the IF output is narrow-band, gaussian noise. Assume
Figure 4 — The received signal pulse train with antenna pattern modulation for a constant angular rate scan, $g(\beta(t) - \phi)$, and for a step scan, $g(\Delta - \phi)$.
that the noise is periodic with period $\gamma$. This noise may be represented as a Fourier series on the interval $0 < t < \gamma$. The sample function $X(t)$ of this random process is represented as follows:

$$X(t) = \sum_{n=1}^{\infty} X_{cn} \cos \left( \frac{2\pi t}{\gamma} \right) + X_{sn} \sin \left( \frac{2\pi t}{\gamma} \right)$$

where

$$X_{cn} = \frac{2}{\gamma} \int_{0}^{\gamma} X(t) \cos \left( \frac{2\pi t}{\gamma} \right) \, dt$$

$$X_{sn} = \frac{2}{\gamma} \int_{0}^{\gamma} X(t) \sin \left( \frac{2\pi t}{\gamma} \right) \, dt.$$

The coefficients $X_{cn}$ and $X_{sn}$ are gaussian random variables which can be shown to be uncorrelated because of the periodicity assumed above.\footnote{W. B. Davenport and W. L. Root, "Random Signals and Noise", McGraw-Hill, New York, 1958, p. 94.} The noise is not actually periodic and expansion in a Fourier series of non-periodic noise results in coefficients $X_{cn}$ and $X_{sn}$ which are correlated. However, if the period $\gamma$ is increased without limit the coefficients become uncorrelated.\footnote{W. B. Davenport and W. L. Root, page 96.} Since $X_{cn}$ and $X_{sn}$ are gaussian, they are statistically independent if they are uncorrelated.

The Fourier series representation and the fact that $X_{cn}$ and $X_{sn}$ are independent gaussian random variables can be used to obtain a very useful
representation of narrow band gaussian noise. Note that

\[
\cos (n 2\pi t/\gamma) = \cos (\frac{n 2\pi t}{\gamma} - (\omega_0 + \omega_d) t - \theta + (\omega_0 + \omega_d) t + \theta)
\]

\[
= \cos (\frac{n 2\pi}{\gamma} - (\omega_0 - \omega_d) t - \theta) \cos ((\omega_0 + \omega_d) t + \theta) - \sin (\frac{n 2\pi}{\gamma} - (\omega_0 - \omega_d) t - \theta) \sin ((\omega_0 + \omega_d) t + \theta)
\]

and

\[
\sin (n 2\pi t/\gamma) = \sin (\frac{n 2\pi}{\gamma} - (\omega_0 - \omega_d) t - \theta) \cos ((\omega_0 + \omega_d) t + \theta) + \cos (\frac{n 2\pi}{\gamma} - (\omega_0 - \omega_d) t - \theta) \sin ((\omega_0 + \omega_d) t + \theta)
\]

Then, we can write \( X(t) \) as

\[
X(t) = X_c(t) \cos ((\omega_0 + \omega_d) t + \theta) - X_s(t) \sin ((\omega_0 + \omega_d) t + \theta)
\]

where

\[
X_c(t) = \sum_{n=1}^{\infty} X_{cn} \cos (\frac{n 2\pi}{\gamma} - (\omega_0 + \omega_d) t + \theta)
\]

\[
+ X_{sn} \sin (\frac{n 2\pi}{\gamma} - (\omega_0 - \omega_d) t - \theta)
\]

\[
X_s(t) = \sum_{n=1}^{\infty} X_{cn} \sin (\frac{n 2\pi}{\gamma} - (\omega_0 - \omega_d) t - \theta) - X_{sn} \cos (\frac{n 2\pi}{\gamma} - (\omega_0 - \omega_d) t - \theta).
\]
Now $X_c(t)$ and $X_s(t)$ are gaussian since they are sums of the gaussian random variables $X_{cn}$ and $X_{sn}$. Also, it can be shown that:

$$E(X_c(t)) = 0$$

$$\sigma^2(X_c(t)) = \sigma^2(X(t)) = N_o$$

$$R_x(\tau) = R_c(\tau) \cos ((\omega_o + \omega_d) \tau)$$

where

$$E(f(u)) = \int_{-\infty}^{\infty} f(u) p(u) \, du = \text{Expectation of } f(u)$$

$$\sigma^2(X) = E(X^2) - (E(X))^2$$

$$R_x(\tau) = E(X(t) X(t-\tau))$$

$$R_c(\tau) = E(X_c(t) X_c(t-\tau)).$$

Therefore, the $X_c(t)$ process has mean zero and variance $= N_o$ where $N_o$ is the noise power at the IF amplifier output. Also, the autocorrelation function of $X_c(t)$ process is the same as the envelope of the autocorrelation function of the IF noise $X(t)$.

The signal plus noise at the output of the IF amplifier can be written using Equations 1-1 and 1-2 as

$$(S(t) + X_c(t)) \cos ((\omega_o + \omega_d) t + \theta) - X_s(t) \sin ((\omega_o + \omega_d) t + \theta)$$

---

3 W.B. Davenport and W.L. Root, Ch. 8.
where

\[ S(t) = \sqrt{2P} \ g(\beta(t) - \phi) \ p(t - \frac{2r(t)}{C}). \]

**Signal and noise after detection**

The signal and noise are now detected as shown in Figure 1 in order to remove the sinusoidal oscillation. The detector may take variety of forms such as,

a) the linear envelope detector the output of which is \([S(t) + X_c(t)]^2 + X_s^2(t)]^{1/2},

b) the square law envelope detector which has the output \((S(t) + X_c(t))^2 + X_s^2(t),

c) the coherent phase detector which multiplies the IF output by \(D \cos ((\omega_0 + \omega_d t + \theta))\) and passes the resulting signal through a low pass filter.

Assuming the coherent phase detector as shown in Figure 5 is used, the signal of Equation 1-4 becomes at the output of the multiplier

\[
\frac{D (X_c(t) + S(t))}{2} (1 + \cos (2 (\omega_0 + \omega_d t + 2\theta))) - \frac{D X_s(t) \sin(2(\omega_0 + \omega_d t + 2\theta))}{2} 1-5
\]

and the low pass filter output is

\[
\frac{D(X_c(t) + S(t))}{2}. 1-6
\]
Figure 5 — The coherent phase detector
Here the double frequency components of Equation 1-5 are assumed to be removed by the low pass filter. This is realistic since the spectra of these components are centered at \(2(\omega_0 + \omega_d)\) and can therefore be greatly attenuated without affecting the detected signal \(D/2(X_c(t) + S(t))\). It will be convenient to normalize by choosing \(D = \frac{2}{\sqrt{N_o}}\).

The detected signal is then

\[
y = \frac{X_c(t)}{\sqrt{N_o}} + \frac{S(t)}{\sqrt{N_o}}. \tag{1-7}
\]

It has been shown that \(X_c(t)\) is gaussian with (Equation 1-3) mean zero and variance \(N_o\). Then, the probability density of \(y\) is

\[
p(y) = \frac{\exp \left(-\frac{(y - S(t)/\sqrt{N_o})^2}{2}\right)}{\sqrt{2\pi}}. \tag{1-8}
\]

The mean value of \(y\) is plotted in Figure 6 as a function of time. It can be seen that the effect of the signal \(S(t)\) is to introduce the pulse train \(\frac{S(t)}{\sqrt{N_o}}\) into \(y\) during the time the antenna is directed at the target. Now assume that \(S(t)\) is known and that \(y(t)\) is sampled in synchronism with \(S(t)\); that is, a sample of \(y\) is taken at the time \(2r(t)/C\) after each transmitted pulse. These sample values are random variables with the mean value \(m\) where \(m\) is the known value of \(S(t)/\sqrt{N_o}\) at the sampling time. Then, the probability density for the \(i\)-th sample is

\[
p(y_i) = \frac{\exp \left(-\frac{(y_i - m)^2}{2}\right)}{\sqrt{2\pi}}. \tag{1-9}
\]
Figure 6 — The mean value of the detected signal $y$ as a function of time.
The autocorrelation function of $X_c(t)$ (Equation 1-3) is shown in Figure 7 for the case when the IF amplifier is a matched filter. It is true in general that $R_c(\tau) \approx 0$ for $\tau$ greater than about $1/2$ pulse width. The sample values are therefore independent and the joint probability density function of $n$ samples is

$$\begin{align*}
    p(y_1, y_2, \ldots, y_n) &= \frac{\exp - \frac{1}{2} \sum_{i=1}^{n} (y_i - m)^2}{(2\pi)^{n/2}}
\end{align*}$$

As shown in Figure 6, the value of $m = 0$ except when the antenna is pointed at the target. If $n$ samples are taken at each antenna position, the presence of the target can be detected by deciding on the basis of these $n$ samples whether the mean of $y$ is zero of $S(t)\sqrt{N_o}$. The detection of targets is therefore a statistical problem. The remainder of this dissertation will be devoted to the investigation of problems which arise in the design of statistical tests for the detection of radar targets.

Outline of statistical decision theory

Statistical decision theory is concerned with the optimum way to make decisions from statistical data. By applying the results of this theory to the decision process involved in the detection of radar targets the optimum radar design can be obtained. The general results of statistical decision theory and their application to the radar problem will be outlined in order to orient the subsequent work.
\[ R_c(\tau) = E(X_c(t) \cdot X_c(t + \tau)) \]

\[ t_p = \text{pulse width} \]

Figure 7 — The autocorrelation function of the gaussian random variable \( X_c(t) \).
The major contributor to the development of statistical decision theory was Abraham Wald. He introduced the concepts of cost and risk as a basis for arriving at optimum statistical tests.

Assume that a decision about the value of a parameter must be made on the basis of statistical data. For example it may be required to decide which of two hypotheses is true, that the true parameter value is equal to zero $H_0$ or that it is greater than zero $H_1$. In general the decision will be subject to error due to the random character of the data. An error can be made by choosing the hypothesis $H_0$ when $H_1$ is true or by choosing $H_1$ when $H_0$ is true. The cost referred to above is the cost of making a decision which is in error and the risk is the cost of an error times the probability of making an error.

If the costs of making errors and the a priori probabilities of the hypotheses are known, then the Bayes criterion for choosing the optimum decision strategy is to minimize the expected risk. As shown in Helstrom, this procedure results in the following test, the Bayes solution of the decision problem:

\[ \text{if } \lambda(X) = \frac{p(X/H_1)}{p(X/H_0)} \frac{\rho (C_{10} - C_{00})}{(1-\rho) (C_{01} - C_{11})} \text{ choose } H_0. \]

and

\[ \text{if } \lambda(X) = \frac{\rho (C_{10} - C_{00})}{(1-\rho) (C_{01} - C_{11})} \text{ choose } H_1. \]

---


where

\( H_1 \) and \( H_0 \) are two (simple) hypotheses

\( \rho \) and \( 1 - \rho \) are the a priori probabilities of \( H_0 \) and \( H_1 \) respectively

\( C_{ij} \) is the cost of deciding \( H_1 \) when \( H_j \) is true

\( \lambda (X) \) is the likelihood ratio

\( p(X/H_i)^6 \) is the probability density function of \( X \) under the hypothesis \( H_i \).

The optimum method of making the decision is, therefore, to compare \( \lambda (X) \) which is called the likelihood ratio, to a threshold constant.

In the detection of radar targets the probability density function of the samples from which the decision is to be made is given by Equation 1-10. Here the hypotheses \( H_0 \) and \( H_1 \) are \( m = 0 \) and \( m = S(t)/\sqrt{N_0} \). The Bayes solution of the basic radar detection problem is then to compare the likelihood ratio \( \lambda \) to the threshold where

\[
\lambda = \frac{p(y_1, y_2, \cdots, y_n/S(t)/\sqrt{N_0})}{p(y_1, y_2, \cdots, y_n/0)}
\]

The notation \( p(X/\alpha, \beta, \gamma \cdots) \) will be used extensively in the following chapters. Here \( X \) denotes the random variable and the \( \alpha, \beta, \gamma \), etc. refer either directly or indirectly to parameters of the probability density function. Thus, \( p(X/H_1) \) means the probability density function of \( X \) with the parameters given the values implied by the hypothesis \( H_1 \).
or

\[
\lambda = \frac{\exp - \frac{1}{2} \sum_{i=1}^{n} \left( y_i - \frac{S(t)}{\sqrt{N_0}} \right)^2}{(2\pi)^{n/2}}
\]

\[
= \exp \sum_{i=1}^{n} \left( y_i \frac{S(t)}{\sqrt{N_0}} - \frac{S_i^2(t)}{2 N_0} \right).
\]

In many cases the a priori probabilities of \( H_0 \) and \( H_1 \) are unknown and a solution for the decision problem is then found by the minimax criterion. Here the a priori probability \( \rho \) which results in the maximum value for the minimum expected risk is used to compute the threshold constant. The minimax test compares the likelihood ratio to this threshold.

If the costs are also unknown, a different approach is necessary to obtain the optimum test. This approach is the Neyman-Pearson criterion which minimizes one probability of error while holding the other probability of error constant for a given sample size. The optimum test resulting from this criterion compares the likelihood ratio to a threshold constant where the constant is chosen to satisfy the specified probability of error.

The three criteria discussed above all result in optimum tests which compare the likelihood ratio to a constant.
If the parameter about which the decision is to be made can assume a range of values, the hypothesis is called composite. When the test for the composite hypothesis yields smaller error probabilities for any value of the parameter than any other test, the test is termed uniformly most powerful (UMP) for this parameter. The lack of knowledge about the value of the parameter does not detract from the performance of the test if the test is UMP for that parameter.

It often happens that the test is not UMP for the parameter about which the decision is to be made. That is, when the test is designed to be optimum for one value of the parameter, it is not optimum for another value. The problem then is to obtain a test which represents the best compromise over the range of values of the parameter. If the probability density function of the parameter is known as well as the relevant costs and a priori probabilities, then the Bayes criterion leads to the following form of the test,

\[
\lambda(X) = \frac{\int p(X/H_1, \theta) Z(\theta) d\theta}{p(X/H_0)} < \frac{\rho (C_{10} - C_{00})}{(1-\rho)(C_{01} - C_{11})}, \text{ choose } H_0
\]

and if

\[
\lambda(X) > \frac{\rho (C_{10} - C_{00})}{(1-\rho)(C_{01} - C_{11})}, \text{ choose } H_1,
\]

where the hypotheses are \(H_0: \theta = \theta_0\) and \(H_1: \theta \neq \theta_0\). Here \(Z(\theta)\) is the a priori probability density function of the unknown parameter \(\theta\). We are assuming that
the costs are independent of \( \theta \) and that \( H_0 \) is a simple hypothesis. This test compares the average of the likelihood ratio over \( Z(\theta) \) to the threshold constant.

If the prior probabilities and \( Z(\theta) \) are not known but the costs are, the approach is to choose the "worst" \( Z(\theta) \) and \( \rho \) and then proceed to minimize the average risk. This is the "least favorable distribution" principle and is obviously an extension of the minimax approach.

The Neyman–Pearson criterion may also be used for composite hypothesis tests and here again if the decision level is independent of the unknown parameter the test is UMP. If the test is not UMP for the unknown parameter, the average likelihood ratio is computed either from the known \( Z(\theta) \) or from the least favorable distribution.

In many cases there are unknown parameters for which the above tests are not UMP and for which the concept of the least favorable distribution is not applicable. Here the method of maximum likelihood ratio detection is often found to be a reasonable approach. The numerator of the likelihood ratio, \( p(X/H_1, \theta) \) is a function of the unknown parameter while the denominator, \( p(X/H_0) \) is not. This corresponds to a composite hypothesis \( H_1 \) and a simple hypothesis \( H_0 \) which is generally the case of interest in radar applications. If \( \theta \) may take on any value in the set \( \omega \), the value of \( \theta \) is chosen as that value \( \theta_1 \) which maximizes the likelihood ratio over \( \omega \). This is the value of \( \theta \) which maximizes \( p(X/H_1, \theta) \). From the theory of estimation of statistical parameters the value of \( \theta \) which maximizes \( p(X/H_1, \theta) \) is called the maximum
likelihood estimate. Therefore, we can look on this test as one where a
maximum likelihood estimate of the unknown parameter is obtained and this
parameter value is then used in the likelihood ratio.

It can be seen that the state of knowledge concerning the costs of making
eerrors, the a priori probabilities of the hypotheses and the values of para-

ters of the density functions determines which approach to the statistical
decision problem is useful. It will be informative to review the radar problem
from this aspect. In most radar environments the costs of making errors and
the a priori probabilities of target or no target are not obtainable. The false
alarm probability \( P_{FA} \) is generally dictated by exterior considerations and it
is important to maintain it at the specified level. The Neyman-Pearson crit-
tion in which the \( P_{FA} \) is fixed is therefore used. The parameters of the proba-
bility density function of Equation 1-9 under the target hypothesis are found in
Equation 1-1, the equation for the received signal. It can be seen that the perti-
nent parameters include signal power \( P \), the target azimuth and range \( \phi \) and \( R \),
respectively, the doppler shift \( \omega_D \) and the phase angle, \( \theta \). Other parameters
such as the target acceleration may become important in special cases. Many
of these signal parameters are unknown. The Neyman-Pearson test is not UMP
for all of these parameters and for many of them the least favorable distribution
is not applicable. Therefore, the maximum likelihood ratio approach is

\[ \text{E. J. Kelly, "The Radar Measurement of Range, Velocity and Acceleration,"}
\text{IEEE Trans. on Military Electronics, April, 1961.} \]

\[ \text{E. J. Kelly, I. S. Reed, W. L. Root, "The Detection of Radar Echoes in Noise,}
\text{I and II", Journal of SIAM, June and September, 1960.} \]
frequently used in radar problems to deal with the unknown signal parameters.

Instead of a single parameter the probability density now depends on a vector representing the many signal parameters. The specific application of maximum likelihood ratio tests to the radar problem will be discussed in detail in Chapter III.

The sequential method of making decisions

Up to this point the radar problem has been shown to be a statistical decision theory problem and the achievement of the optimum solution through this theory has been discussed. A basic assumption in all the previous discussion was that the tests are fixed in length or equivalently in number of samples. That is, the radar designer must decide first how many samples will be available and the decision thresholds can then be derived. In the Neyman–Pearson criterion for test design, for example, the number of samples and the false alarm probability $P_{FA}$ are fixed and the probability of false dismissal $P_{FD}$ is minimized.

Another class of tests may be obtained if $P_{FA}$ and $P_{FD}$ are fixed and the number of samples is allowed to vary. This is called the sequential test. In this test sampling continues until the $P_{FA}$ and $P_{FD}$ have reached the required levels so that the number of samples is a random variable. The optimum sequential test minimizes the average number of samples for fixed $P_{FA}$ and $P_{FD}$.

Wald\textsuperscript{9} has developed the theory of sequential tests and has shown that the

average length of the optimum sequential test is always less than the length of
the optimum fixed sample test of equal $P_{\text{FA}}$ and $P_{\text{FD}}$. The form of the optimum
sequential test which will be discussed in detail in the next chapter involves the
formation of the "probability ratio" and comparison to two thresholds, upper
and lower. The probability ratio is analogous to the likelihood ratio of Equation
1-11. The various criteria for optimum tests such as the Bayes, minimax, etc.
can be applied to the sequential test and the resulting test is always superior in
performance to the corresponding fixed sample test. The problem of unknown
parameters in sequential tests can also in principle be handled by the approaches
outlined for fixed sample tests. In the radar case many parameters may be un-
known as discussed above and it is found that the optimum sequential test when
a parameter is unknown is often exceedingly complex. An example of this is
the sequential test for a radar where the target range is unknown. Marcus
and Swerling point out that the unknown target range leads to a sequential test
with $2^K$ alternative hypotheses where $K$ may be of the order of $10^3$. The re-
sulting complexity in the form of the test makes it impractical to perform the
required computations on a digital computer. The problem of applying sequen-
tial tests to pulsed radar is thus equivalent to that of obtaining the optimum un-
known parameter sequential test.

10 M. B. Marcus and P. Swerling, "Sequential Detection in Radar with Multiple
The sequential test is theoretically always superior to the fixed sample test but the decision theory which is so useful in the fixed sample case fails to give useful results for the sequential test. In the following chapters various approaches to the design of the unknown parameter sequential test are investigated which avoid the complexity of the decision theory derived test.
CHAPTER II
THE SIMPLE HYPOTHESIS SEQUENTIAL TEST
IN PULSED RADAR

The simple hypothesis sequential test

As the brief discussion in the previous chapter indicated, the problem under consideration is the application of sequential statistical tests to pulsed radar. It has been shown that statistical decision theory can be successfully applied to pulsed radar when the number of samples is fixed and that the corresponding approach breaks down for the sequential case when there are unknown parameters. The sequential theory is, however, adequate when the signal is completely known so that only simple hypotheses are involved. The approach to the design of a sequential detector when the target range is unknown will be based on modifications of the simple hypothesis detector. Therefore, the simple hypothesis sequential test will be examined in detail to provide a foundation for future work.
The sequential test for simple hypotheses $H_0$ and $H_1$ is given by the following decision rules:

1. if $\lambda_k(X) = \frac{p(X_1, X_2, ..., X_k/H_1)}{p(X_1, X_2, ..., X_k/H_0)} < B$, choose $H_0$

2. if $\lambda_k(X) > A$, choose $H_1$

3. if $B < \lambda_k(X) < A$, take another sample.

Here $A$ and $B$ are chosen so that the desired probabilities of error $P_{FA}$ and $P_{FD}$ are obtained. The likelihood ratio $\lambda_k(X)$ at the $k$th stage of the test is the ratio of the probability density of the $k$ samples under the hypothesis $H_1$ to the probability density of the $k$ samples under $H_0$. This sequential test will now be applied to the problem of deciding whether the detected signal in the radar (Equation 1-7) has the mean value zero or $S(t)/\sqrt{N_0}$. The probability density function of the samples is given by Equation 1-10 so that the sequential test is:

\[ B < \frac{p(y_1, y_2, ..., y_k/a_1)}{p(y_1, y_2, ..., y_k/0)} < A \]

or

\[ B < \exp -1/2 \sum_{i=1}^{k} (y_i - a_1)^2 / \exp -1/2 \sum_{i=1}^{k} y_i^2 < A \]

or

\[ \log B < \sum_{i=1}^{k} (a_1 y_i - a_1^2/2) < \log A \]

where $a_1 = S(t)/\sqrt{N_0}$, $H_1$ is $m = a_1$ or signal present and $H_0$ is $m = 0$ or signal is absent. This is the sequential test for detection of a target for the radar of Figure 1 whereas Equation 1-12 represents the fixed-sample test for the same case.
Probability of error

If the probabilities of error $P_{FA} = \alpha$ and $P_{FD} = \beta$ are given, the thresholds $\log A$ and $\log B$ are determined. Wald\textsuperscript{1} has shown that the thresholds which correspond to the probabilities of error $\alpha$ and $\beta$ satisfy the inequalities $A \leq \frac{1 - \beta}{\alpha}$ and $B \geq \frac{\beta}{1 - \alpha}$. The equality would hold if the test were a continuous function of time rather than discrete samples. When discrete samples are taken, the threshold may be exceeded rather than just equalled which results in the inequality. If the equality is used for the discrete case, the true probabilities of error $\alpha'$ and $\beta'$ satisfy the inequalities $\alpha' \leq \frac{\alpha}{1 - \beta}$ and $\beta' \leq \frac{\beta}{1 - \alpha}$. If many samples are needed on the average for decision, the above relations will be almost equalities. It will be assumed hereafter that the thresholds are given by:

$$A = \frac{1 - \beta}{\alpha}, \quad B = \frac{\beta}{1 - \alpha}$$

2-3

The operating characteristic function

If the value of $S(t)$ is unknown, the test of Equation 2-2 is often used with a design value of $a_1 = \frac{S(t)}{\sqrt{N_0}}$. For example, $a_1$ may be chosen as the smallest likely value. Thus, the sequential test designed for the hypotheses $m = a_1$ versus $m = 0$ is often used when $m = s$ where $s \neq a_1$. The probability of error

\textsuperscript{1}A. Wald, "Sequential Analysis", page 45.
is then needed for the sequential test which is designed for \( m = a_1 \) but which is used with \( m = s \). An error is made in this case if \( H_0 \) is accepted when \( m = s \). Therefore, the probability of error is equal to the probability of accepting \( H_0 \).

The probability of accepting \( H_0 \) is called the operating characteristic function OCF by Wald. The OCF is derived by Wald\(^2\) as follows. From Equation 2-2 we can form a new sequential test as follows:

\[
B^h < \frac{p^*(y_1, y_2, \ldots, y_k/s)}{p(y_1, y_2, \ldots, y_k/s)} < A^h
\]

where

\[
p^*(y_1, y_2, \ldots, y_k/s) = \left[ \frac{p(y_1, y_2, \ldots, y_k/a_1)}{p(y_1, y_2, \ldots, y_k/0)} \right]^h p(y_1, y_2, \ldots, y_k/s)
\]

and where

\[
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p^*(y_1, y_2, \ldots, y_k/s) \, dy_1 \cdots dy_k = 1.
\]

This is by definition a sequential probability ratio test of the hypothesis \( H^* \) that \( p^*(y_1, y_2, \ldots, y_k/s) \) is the true density function versus the hypothesis \( H \) that

\(\text{---}\)

\( p(y_1, y_2, \ldots, y_k/s) \) is the true density function. Now Equation 2-4 can also be written as

\[
B^h < \left( \frac{p(y_1, y_2, \ldots, y_k/a_1)}{p(y_1, y_2, \ldots, y_k/0)} \right)^h < A^h
\]

or

\[
B < \frac{p(y_1, y_2, \ldots, y_k/a_1)}{p(y_1, y_2, \ldots, y_k/0)} < A.
\] 2-5

It can be seen that when the test of Equation 2-4 accepts \( H \) the test of Equation 2-2 accepts \( H_0 \). This can be seen by comparing Equation 2-4 and Equation 2-5. When \( m = s \), the probability of accepting \( H^* \) in Equation 2-4 is \( \alpha' \) by definition and therefore the probability of accepting \( H \) when \( m = s \) is \( 1 - \alpha' \). But we have seen that the test of Equation 2-4 accepts \( H \) whenever Equation 2-2 accepts \( H_0 \). Therefore, the probability of accepting \( H_0 \) when \( m = s \) is \( 1 - \alpha' \). But we have

\[
A^h = (1 - \beta')/\alpha' \quad \text{and} \quad B^h = \beta'/(1-\alpha') \quad \text{from Wald's results on the probabilities of error for a sequential test.}
\]

Solving for \( \alpha' \) we have

\[
\alpha' = \frac{1 - B^h}{A^h - B^h}.
\]

Then, the OCF is

\[
L(s) = 1 - \alpha' = \frac{A^h - 1}{A^h - B^h}
\] 2-6

where \( L(s) \) is Wald's symbol for the OCF.
The value of \( h \) is found from the requirement that \( p^*(y_1, y_2, \ldots, y_k/s) \) be a probability density function. Then \( h \) must be chosen so that from Equation 2-4

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{p(y_1, y_2, \ldots, y_k/a_j)}{p(y_1, y_2, \ldots, y_k/0)} h \left[ p(y_1, y_2, \ldots, y_k/s) \right] dy_1 \cdots dy_k = 1
\]

which becomes by virtue of the independence of the samples

\[
\int_{-\infty}^{\infty} \left( \frac{p(y/a_j)}{p(y/0)} \right) h \left( p(y/s) \right) dy = 1 \tag{2-7}
\]

Thus, for the sequential test of Equation 2-2 the probability of error when \( m = s \) is given by Equation 2-6 with the value of \( h \) determined from Equation 2-7. It is shown in Appendix I, Equation I-5, that for the sequential test of Equation 2-2

\[
h(s) = 1 - \frac{2s}{a_1} \tag{2-8}
\]

For \( s = 0 \) and \( s = a_1 \) we have from Equations 2-3, 2-6 and 2-8

\[
L(0) = \frac{A - 1}{A - B} \approx 1 - \alpha,
\]

\[
L(a_1) = \frac{A^{-1} - 1}{A^{-1} - B^{-1}} \approx \beta. \tag{2-9}
\]

These are approximations since the thresholds Equation 2-3 are approximate.

When \( \alpha \) and \( \beta \) are small, the approximation is very good.
The average number of samples

The number of samples, \( n \), required for the sequential test to terminate is a random variable. Wald has shown that the mean value of \( n \) is given approximately by

\[
E_n \sim \frac{L(s) \log B + (1 - L(s)) \log A}{E_z} \tag{2-10}
\]

where

\[
z = \log \frac{p(y/a_1)}{p(y/0)},
\]

\[
E_z = \int_0^\infty z w(z) \, dz,
\]

\[
w(z) = \text{probability density of } z,
\]

\[
E_n = \sum_{i=1}^\infty i \, p(i),
\]

\[
p(i) = \text{probability that the test ends on the } i\text{-th sample.}
\]

Note that the logarithm of the likelihood ratio, \( \log \frac{p(y_1, y_2, \ldots, y_k/a_1)}{p(y_1, y_2, \ldots, y_k/0)} \), may be written as \( \sum_{n=1}^k \log \frac{p(y_n/a_1)}{p(y_n/0)} \) since the samples \( y_i \) are independent. The sequential test can be written as \( B < \sum_{n=1}^k z_n < A \). Now \( z \) is a function of the random variable \( y \). For example, in the sequential test of Equation 2-2

\[
z = a_1 y - a_1^2/2. \text{ Here if } E_y = s \text{ then}
\]

\[
E_z = sa_1 - a_1^2/2 \tag{2-11}
\]
From Equation 2-10 for \( s = 0 \) and \( s = a_1 \) we have:

\[
E_n \bigg|_{s=0} = \frac{L(0) \log B + (1 - L(0)) \log A}{z(0)}
\]

where

\[
E_z \bigg|_{s=0} = z(0),
\]

or,

\[
\bar{n}(0) = \frac{(1-\alpha) \log \frac{\beta}{1-\alpha} + \alpha \log \frac{1-\beta}{\alpha}}{-a_1^{2/2}} \quad (2-12)
\]

and,

\[
\bar{n}(a_1) = \frac{\beta \log \frac{\beta}{1-\alpha} + (1-\beta) \log \frac{1-\beta}{\alpha}}{a_1^{2/2}}.
\]

When \( \beta \) and \( \alpha < 1 \), \( \bar{n}(0) \sim \frac{\log 1/\beta}{a_1^{2/2}} \) and \( \bar{n}(a_1) \sim \frac{\log 1/\alpha}{a_1^{2/2}} \). Using typical values \( \alpha = 10^{-6} \), \( \beta = 10^{-1} \), and \( a_1 = 1 \), we have \( \bar{n}(0) \approx 4.6 \) and \( \bar{n}(a_1) = 27.6 \). A representation of the sequential test of Equation 2-2 is given in Figure 8, which illustrates the reason for the difference in magnitude of \( \bar{n}(0) \) and \( \bar{n}(a_1) \) by plotting the test as a function of time.

**Probability density function of n**

In addition to the probability of error \( L(s) \) and the average sample number \( \bar{n}(s) \), some results are given in the literature on the probability density of the
Figure 8 — The sequential test for a signal in gaussian noise.
sample number, $p(n)$. Wald\(^3\) shows that when $z$ is gaussian and when $\beta \to 0$, $\alpha$ finite, and $\bar{z} > 0$ (Equation 2-11) the probability density of the normalized sample number $t$ is

$$p(t) \sim c/2 \frac{\exp c - t - c^2/4t}{\sqrt{\pi} t^{3/2}}$$  \hspace{1cm} 2-13

while for $\alpha \to 0$, $\beta$ finite, and $\bar{z} < 0$

$$p(t) \sim a/2 \frac{\exp a - t - a^2/4t}{\sqrt{\pi} t^{3/2}}$$  \hspace{1cm} 2-14

where

$$t = \frac{n \bar{z}^2}{2 \sigma_z} , \quad c = \frac{\bar{z}}{\sigma_z} \log A, \quad a = \frac{\bar{z}}{\sigma_z} \log B,$$

and

$$\sigma_z^2 = \mathbb{E} z^2 - (\mathbb{E} z)^2.$$

Note that $n$ is an integer so that $p(n)$ is actually a discrete probability. However, in the derivation of $p(t)$ Wald assumed that when the sequential test of Equation 2-1 ends the value of $\lambda_k$ exactly equals the threshold rather than exceeds it.\(^4\) As a result of this assumption, a probability density function is obtained for $t$ whereas $t$ is a discrete random variable. The probability distribution of $n$ is therefore found as $\sum_{i=0}^{N} p(n) \, dn$ instead of $\sum_{i=1}^{N} p(i)$. When the

\(^3\)A. Wald, "Sequential Analysis," page 193.
\(^4\)A. Wald, "Sequential Analysis," page 195.
average number of samples is large, the continuous distribution is a good
approximation. The assumption that $\lambda_k$ exactly equals the threshold is identical
to the approximation used in deriving the thresholds Equation 2–3. From this
point on the probability density approximation will be used for $p(n)$ unless other-
wise stated. Bussgang and Middleton\(^5\) give a result for $p(t)$ for the case when
$\alpha = \beta$. The probability distributions for Equations 2–13 and 2–14, respectively,
are given by Helstrom\(^6\) as

\[
P(t < g) = \frac{1}{2} \text{erfc} \left( \frac{\gamma^2}{\sqrt{g}} - \sqrt{g} \right) + \frac{1}{2} \alpha \left( \frac{\gamma^2}{\sqrt{g}} + \sqrt{g} \right)
\]

2–15

and

\[
P(t < g) = \frac{1}{2} \text{erfc} \left( \frac{\gamma^2}{\sqrt{g}} - \sqrt{g} \right) + \frac{1}{2} \beta \left( \frac{\gamma^2}{\sqrt{g}} + \sqrt{g} \right).
\]

2–16

Comparison of sequential test and fixed-sample test

It will be instructive at this point to interpret the performance of the
simple hypothesis sequential detector from a radar point of view. Using typical
radar parameters, the sequential detector can be compared to the fixed-sample
detector.

Perhaps the biggest difference between a radar application of the sequen-
tial test and usual statistical application such as checking for defective parts

\(^5\) J. J. Bussgang and D. Middleton, "Sequential Detection of Signals in Noise,"
Cruft Laboratory Technical Report No. 175, 1955, page 75.

Trans. on Information Theory, January 1962.
lies in the magnitude of the allowable error probabilities. In the typical radar case the probability of a false alarm, which is $\alpha$ in our previous discussion, must be of the order of $10^{-6}$ and the probability of missing a target $\beta$ is usually $10^{-1}$. Typical statistical applications might require $\alpha = \beta = 10^{-2}$.

Another characteristic of the radar problem is the sparsity of targets. It is often assumed that the radar signal is almost always noise and only occasionally does a target signal appear.

The results for the simple hypothesis test (which is equivalent to the completely known target) can be applied to the typical radar described above in order to illustrate what these radar characteristics mean in terms of sequential test performance. We will assume that the radar is scanned in discrete steps and that the dwell time of the antenna at any particular step is determined by the time required to reach a decision with the sequential test at that azimuth position. In other words, the antenna dwell time is determined by and is equal to the sequential test length. Then the average time the radar requires to complete one 360 degree scan or scan time is proportional to the average sequential test length. Since the radar is assumed to be operating in a sparse target environment, the average sequential test length for noise input determines the scan time of the sequential pulsed radar. Since the radar scan time is a good measure of the efficiency of a radar, the average test length will be used as a figure of merit for sequential tests in the subsequent work.
The sequential test of Equation 2-2 can now be compared to a fixed-sample test by specifying the same error probabilities $\alpha$ and $\beta$ for both tests and then comparing the average sample number $\bar{n}(0)$ Equation 2-12 to the number of samples required by the fixed sample test. The fixed-sample test is assumed to be a Neyman-Pearson test where the likelihood ratio of Equation 1-12 is compared to a threshold. The fixed-sample test for detecting the radar signal is therefore given by:

\[
\begin{align*}
\text{if } & \sum_{i=1}^{q} (a_1 y_i - a_1^2/2) \geq v, \text{ decide } H_1; \\
&& \\
\text{if } & \sum_{i=1}^{q} (a_1 y_i - a_1^2/2) < v, \text{ decide } H_0, \\
\end{align*}
\]

where

\[a_1 = S(t)/\sqrt{N}.
\]

As shown in Appendix I, Equations I-8 and I-9, the probabilities of error of the fixed sample test of Equation 2-17 are given by:

\[
\begin{align*}
\alpha_{FS} &= 1/2 \left(1 - \text{erf} \left( \frac{d/2}{\sqrt{j}} + \sqrt{j} \right) \right) \quad \text{I-8} \\
\beta_{FS} &= 1/2 \left(1 + \text{erf} \left( \frac{d/2}{\sqrt{j}} - \sqrt{j} \right) \right) \quad \text{I-9}
\end{align*}
\]

where \(\alpha_{FS}\) = probability of accepting \(H_1\) when \(m = 0\),

\(\beta_{FS}\) = probability of accepting \(H_0\) when \(m = a_1\),

\[d = v \bar{z}/\sigma_z^2, \quad j = q \bar{z}^2/2\sigma_z^2, \quad \bar{z} = (sa_1 - a_1^2/2), \]

\[\sigma_z^2 = a_1^2, \quad z = a_1 y - a_1^2/2.\]
The number of samples $q$ for various values of $\alpha$ is plotted versus $\beta$ with $a_1 = 1$ in Figure 9. Then, for $\alpha = 10^{-6}$, $\beta = 10^{-1}$ and $a_1 = 1$ we find $q = 37$ from this plot while from Equation 2-12 we obtain $\bar{n}(\theta) = 4.6$. This shows that in a sparse target environment the sequential radar can search at 8 times the rate of the fixed sample radar or it could operate at a lower value of $a_1^2$ by a factor of 8 and still maintain an average search rate equal to that of the fixed sample radar. It can now be seen that the performance of the sequential radar is much better than that of the fixed-sample because of the large difference in $\alpha$ and $\beta$ which results in very fast decisions when noise only is present and because of improvement that makes sequential detection look attractive for radar applications.

In the next chapter the application of the sequential test in the more practical case of unknown target range will be investigated.
Figure 9 — The number of samples $q$ required for the fixed sample test with error probabilities $\beta_{FS}$ and $\alpha_{FS}$. 
The application of the sequential test to a practical radar

The structures of the fixed-sample test and of the sequential test have been examined for the case of the known signal. We would now like to consider the practical radar situation and apply the fixed-sample test and the sequential test when some of the signal parameters are unknown. Among these parameters as given in Equation 1-1 are the range, azimuth, signal power, and doppler frequency shift. It was pointed out in Chapter I that in most cases the optimum fixed-sample test when signal parameters are unknown is the maximum likelihood ratio test. However, due to the complexity of the resulting test, the maximum likelihood approach cannot be used for the sequential test when signal parameters are unknown. A different approach is therefore needed in order to apply the sequential test to the practical radar situation. This approach will be to construct a non-optimum multiple hypothesis sequential test which is based on the optimum fixed sample test. This non-optimum sequential test will then be modified to improve its performance.
The fixed-sample test for a signal with unknown parameters

The structure of the fixed-sample test when range, azimuth, signal power, and doppler frequency are unknown will now be outlined. The maximum likelihood ratio test requires that the maximum likelihood estimate of the unknown parameter be used as the value of the parameter in the test. First, consider the fact that the range of the target or targets is unknown. In Figure 6 the variation of $y$ with time is plotted and it can be seen that $y = m$ for an interval equal to the pulse width $\tau$ and is zero otherwise. Then, the likelihood ratio Equation 1-12 will be maximized if $y$ Equation 1-7 is sampled in this interval of width $\tau$. In other words, the range estimate for the target must fall within this interval and if it does the likelihood ratio is maximized. Since one or more targets may appear anywhere from zero range to the maximum range of the radar, $R_{max}$, samples should be taken in each range interval of width $\frac{c\tau}{2}$ from zero to $R_{max}$. We therefore have $K = \frac{R_{max}}{c\tau/2}$ sets of samples each of which must be compared to the threshold as in Equation 2-17. The unknown doppler frequency through similar reasoning requires that samples be taken in each resolvable interval $\Delta f$ in the frequency domain. The fixed-sample test or detector for the practical radar, which will hereafter be denoted by the FS test, must then take the form shown in Figure 10. Each frequency interval $\Delta f$ of every range interval $\tau$ must be sampled $n$ times and the decision made as in Equation 2-17. The unknown azimuth of the target can also be estimated by the maximum likelihood method but it is found that it makes little difference in
This is repeated for \( \omega_d \)'s at intervals \( \Delta \omega \) through the doppler band of interest.

**Figure 10** — The fixed-sample test for detection of targets with a pulsed radar.
the performance of the detector. It is therefore almost always ignored in the structure of the test. The FS test is found to be UMP for the signal power so that the form of the test is unchanged when the signal power is unknown.

The sequential test for a signal with unknown parameters

We must now construct a sequential test for the case of unknown range, azimuth, signal power, and doppler frequency. The form of the FS test suggests that the detected signal \( y \) Equation 1-7 be sampled in each \( \tau \) interval. The sample values from each interval should by analogy to the FS test be fed to a sequential decision device Equation 2-2 in order to decide whether or not a target is present in the interval. Likewise, if the doppler frequency is unknown, we would expect that each \( \Delta f \) interval in the frequency domain should be sampled. At this point we can see a number of problems in this sequential test. First of all, the decision for each \( \tau \) interval is made by a sequential test which we will call a sub-test. This means a random number of samples will be taken in each interval. Some of the subtests will, therefore, end after a few samples while other subtests may require many samples. Now in the FS test all decisions are made at the same time and the antenna then is free to move to the next azimuth position. In the sequential case there is no fixed decision time so that the problem of when to move the antenna to the next position arises. The requirement that all subtests reach a decision before the antenna moves is obviously wasteful since the short sub-tests will be idle after a few samples. On
the other hand, if the antenna is moved before the last sub-test ends, a means for making the decision to move must be provided. It can be seen that the decision to move the antenna should involve the state of all of the K subtests in some manner. The optimum way to make this decision is to perform a multiple hypothesis sequential test as discussed in Chapter I. We reject this approach because of complexity. Instead, we will first assume that all sub-tests are required to reach a decision before the antenna moves and we will determine what kind of performance is obtained with this test. Next, various modifications will be made to the sub-tests in order to decrease the waste of time due to waiting for the slower sub-tests. Then, means of basing the decision to move the antenna on the states of all the K subtests will be investigated.

The sampling of each resolvable interval $\Delta f$ because of unknown doppler frequency also presents a problem in the sequential test. The random number of samples which are used for the sub-test means the resolvable frequency interval is random in width. There is, therefore, a problem of how to physically implement the test for the unknown doppler frequency as well as the multiple hypothesis problem arising from the additional sub-tests. The unknown doppler problem will not be considered further although the multiple hypothesis problem will be studied in connection with the unknown range parameter.
When the sequential test of Equation 2-2 which is designed to decide whether \( m = 0 \) or \( a_1 \) is used with \( m = s \), the probability of error is given by \( L(s) \) Equation 2-6 and the average sample number \( \overline{n(s)} \) is found from Equation 2-12. If a new sequential test is designed with \( a_1 \) replaced by \( s \) and with the probability of error equal to \( L(s) \), the new average sample number \( \overline{n'(s)} \) will be less than \( \overline{n(s)} \). Therefore, the sequential test of Equation 2-2 is not a "uniformly best" test\(^1\). A "uniformly best" test is a sequential test for which the average sample number is minimized for all values of \( m \). This is the property of the sequential test which is analogous to the UMP property of FS tests. The fact that the sequential test of Equation 2-2 is not uniformly best can be ignored because, as discussed in Chapter II, the radar operation will be judged on the basis of the average sample number for noise since targets are sparse.

The multiple hypothesis sequential test

The sequential test which we will use as a starting point for our investigation has now been outlined. As shown in Figure 11, the detected signal \( y \) Equation 1-7 is sampled in each of the \( K \) intervals of width \( \tau \) and a sequential test Equation 2-2, hereafter called a sub-test, is performed on each of the \( K \) sets of samples. As discussed above, the unknown azimuth of the target is ignored, the doppler is assumed known, and the unknown signal power is ignored. The basic problem that remains is to decide on the basis of the outputs of

\(^1\)A. Wald, "Sequential Analysis," page 34.
\[ Z_k^n = \sum_{i=1}^{k} a_i y_i^n - a_1^{2/2} \]

\[ Z_k^j > \log A \]
\[ \log B < Z_k^j < \log A \]
\[ Z_k^j < \log B \]

\[ Z_k^n > \log A \]
\[ \log B < Z_k^n < \log A \]
\[ Z_k^n < \log B \]

\[ \delta(1) \quad \delta(j) \quad \delta(n) \]
\[ \sum_{n=1}^{K} \delta(n) \]

move antenna to next position when
\[ \sum_{n=1}^{K} \delta(n) = K \]

Figure 11 — The basic multiple hypothesis sequential test for detection of targets with a pulsed radar.
the K sub-tests when to move the antenna to the next position. The simplest approach as mentioned above is to require all K sub-tests to decide before moving the antenna. This sequential test will be called the M test.

The M test can be stated as follows:

1. if $Z_{k}^{n} = \sum_{i=1}^{k} z_{i}^{n} \geq \log A$, decide $H_{1}$ and set $\delta(n) = 1$  \hspace{1cm} 3-1

2. if $Z_{k}^{n} \leq \log B$, decide $H_{0}$ and set $\delta(n) = 1$,

3. if $\log B < Z_{k}^{n} < \log A$, take another sample and set $\delta(n) = 0$,

4. if after the k-th sample $\sum_{n=1}^{K} \delta(n) = K$, terminate the test and move the antenna to the next position, where $z_{i}^{n} = a_{1} y_{i}^{n} - a_{2}^{1/2}$

and where the superscript $n$ refers to the n-th of the K intervals.

The average sample number of the M test

The M test ends when all the K sub-tests have come to a decision. Since the sub-tests are independent of one another, the length of the M test will be equal to the longest of the K sub-tests. The length of the M test is therefore a random variable which is the largest of K random variables. It is shown in Appendix II Equation II-1 that if $p(n)$ is the probability density of the sample number $n$ of the sub-test then the probability density of the sample number $n_{1}$ of the longest of K subtests is
\[ g(n_1) = K (P(n < n_1))^{K-1} p(n_1) \]  \hspace{1cm} (II-1)

where

\[ p(n < n_1) = \int_0^{n_1} p(n) \, dn \]

\[ K = \text{no. of sub-tests} \]

\[ p(n) = \text{probability density of the sample number for the sub-test} \]

\[ n_1 = \text{largest of } K \text{ random variables.} \]

Also, the probability distribution of \( n_1 \) is

\[ G(n_1 < N) = (P(n < N))^K. \]  \hspace{1cm} (II-2)

The probability density and distribution of the sequential test sample number \( n_1 \) can now be calculated, at least in principle. However, in the typical radar case \( K \) will be large (of the order of 100 to 1000) so that an asymptotic form for the density and distribution function for the largest of \( K \) random variables may be used. In Appendix II it is shown that when \( K \) is large the \( g(n_1) \) and \( G(n_1 < N) \) approach the following forms:

\[ g(n_1) = e_n \exp \left[ -e_n (n_1 - \mu_n) - \exp \left( -e_n (n_1 - \mu_n) \right) \right] \]  \hspace{1cm} (II-3)

and

\[ G(n_1 < N) = \exp \left[ - \exp \left( -e_n (N - \mu_n) \right) \right] \]  \hspace{1cm} (II-4)
where $\epsilon_n$ and $\mu_n$ are found from

$$P(n < \mu_n) = 1 - 1/K$$  \hspace{1cm} \text{(II-5)}

and where the $K$ independent random variables have the identical density and distribution functions $p(n)$ and $P(n < N)$, respectively.

The moments of $n_1$ can be found as shown in Appendix II Equation II-9. In particular, we find that

$$\mu_n = K p(\mu_n)$$

where $\gamma = \text{(Euler's constant)}$ = .5772

The performance of the $M$ test can now be compared to the performance of the FS test. For equal probabilities of error and for $S(t)/\sqrt{N_0} = a_1$ in both tests, the average sample number of the $M$ test $n_1(0)$, Equation 3-2, can be compared to the number of samples of the FS test $q$, Equation 2-17. If we assume that $\alpha = 10^{-6}$ and $\beta = 10^{-1}$, then Equations 2-14 and 2-16 can be used in Equation II-5 to evaluate the $\epsilon_n$ and $\mu_n$ needed in Equation 3-2. The $n_1(0)$ is plotted versus $K$ in Figure 12 and the ratio of $n_1(0)/q$ is plotted versus $K$ in Figure 13. In Chapter II it was found that when the signal is known the sequential detector average sample number $\overline{n(0)}$ Equation 2-12 was 4.6 while $q$ was 37. Figure 12 shows that the $K$ subtests of the $M$ test require an average
Figure 12 — The average sample number $n_1(0)$ of the M test as a function of the number of sub-tests $K$. 
Figure 13 — The ratio of the number of samples \( q \) required for the FS test to the average sample number \( n_1(0) \) of the M test versus the number of sub-tests \( K \).
sample number $n_1(0) = 24.8$ for $K = 100$ and $n_1(0) = 38.4$ for $K = 1000$. The advantage of the sequential test over the FS test in requiring a smaller number of samples is thus diminished considerably when the target range is unknown. Since the M test is the simplest possible sequential test which can be used when the target range is unknown, it would be expected that improved performance could be obtained with a more sophisticated test. In the next chapter we will begin the study of various modifications of the M test so as to minimize the loss in performance relative to the FS test. The tools needed to evaluate the effect of the various modifications will be developed in connected with each special case.
CHAPTER IV

THE TRUNCATED SEQUENTIAL TEST

The truncated M test

It has been shown in Chapter III that the M test Equation 3-1, although simple and practical, has a larger average sample number than the SH test Equation 2-2. This was due to the large number of samples taken by a few sub-tests. We would like to modify the M test in such a way as to improve the performance. As discussed in Chapter III, the sequential test when target range is unknown is basically a multiple hypothesis test. The K sub-tests can either be allowed to reach decisions independently of each other or an overall decision can be made to end the multiple hypothesis sequential test as a function of the state of each sub-test. In this and the next chapter modifications of independent subtests will be investigated.

A simple type of modification suggested by the need to eliminate the effect of the large number of samples taken by a few sub-tests is truncation. Truncation of a sequential test consists of arbitrarily stopping it at a predetermined
sample number \( n_0 \) if it does not terminate prior to \( n_0 \). The truncated M test, hereafter called the TM test, is stated as follows:

1. If \( Z^k_n = \sum_{i=1}^{k} z_i^n \geq \log A \) and \( k \leq n_0 \), decide \( H_1 \)

and set \( \delta(n) = 1 \),

2. If \( Z^k_n < \log B \) and \( k \leq n_0 \), decide \( H_0 \) and set \( \delta(n) = 1 \),

3. If \( \log B < Z^k_n < \log A \) and \( k < n_0 \), take another sample and set \( \delta(n) = 0 \),

4. If \( k < n_0 \) and \( \sum_{n=1}^{K} \delta(n) = K \), terminate the test,

5. If \( k = n_0 \), terminate the test, deciding \( H_0 \) if \( Z^n_{n_0} < C \) and \( H_1 \) if \( Z^n_{n_0} > C \),

where \( \log B < C < \log A \), \( z_i^n = a_1 y_i^n - a_1^2/2 \)

and where the superscript \( n \) refers to a quantity from the \( n \)-th of the \( K \) sub-tests.

The block diagram of the test is given in Figure 14 and a plot of a typical sub-test versus time is shown in Figure 15. It can be seen that the TM test eliminates the excess samples taken by the longer sub-tests while at the same time it does not affect the normal length sub-test. The probabilities of error
Figure 14 — The truncated sequential test applied to pulsed radar.
Figure 15 — Representation of a typical sub-test of the truncated sequential test.
will also be affected since the truncation may result in some sub-tests deciding $H_0$ when they would have decided $H_1$ and vice versa. We must therefore calculate the average sample number and error probabilities of the TM test and make a comparison to the $M$ test to determine if truncation yields a net improvement in performance.

The truncated simple hypothesis sequential test

The truncation of a sequential test is interesting and deserves study not only as a possible modification to improve performance of the $M$ test but also because in any practical radar system a bound would be placed on the number of samples. It will be found that the study of truncation requires the development of many relations which will be needed in the following chapters. The first part of this analysis through Equation 4-11, except for Equation 4-5, is due to Bussgang and Middleton.  

We will first consider truncation of the SH test Equation 2-2. The truncated SH test, hereafter called the TS test, may be stated as follows:

1. if $\sum_{i=1}^{k} z_i > \log A$ and $k \leq n_o$, decide $H_1$,

2. if $\sum_{i=1}^{k} z_i \leq \log B$ and $k \leq n_o$, decide $H_0$

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1 J. J. Bussgang and D. Middleton, page 52.
3. if \( \log B < \sum_{i=1}^{k} z_i < \log A \) and \( k \leq n_o \), take another sample,

4. if \( k = n_o \), terminate the test deciding \( H_0 \) if \( \sum_{i=1}^{n} z_i < C \) and \( H_1 \) if \( \sum_{i=1}^{n} z_i \geq C \), where \( z_i = a_1 y_i - a_1^2/2 \).

The probability of error of the TS test

The errors which occur in the TS test can be divided into two mutually exclusive sets; the errors committed before truncation and those which are committed at truncation. Now consider the decisions which are made before truncation. These can be divided into decisions in which \( H_0 \) is chosen and in which \( H_1 \) is chosen. The probability that a decision is made when \( k < n \) is by definition \( \int_{0}^{n} p(n) \, dn \). But using conditional probabilities, we can write the probability density \( p(n) \) of the sample number \( n \) as follows:

\[
p(n/s) = L(s) \, p_0(n/s) + (1 - L(s)) \, p_1(n/s)
\]

where \( p_1(n/s) \) = conditional probability of making a decision on the \( n \)-th sample given that \( H_1 \) is chosen. Then, the probability that the decision \( H_0 \) is made when \( k \leq n_o \) is

\[\text{See footnote page 18; } p(n/s) \text{ indicates the density function has the parameter } s.\]

\[\text{From this point on a subscript on a probability density function will denote a conditional probability.}\]
\[ \int_{0}^{n} L(s) p_0(n/s) \, dn. \]  

(Note that \( p(n) \) is actually a discrete probability as discussed in connection with equations 2-13 and 2-14.)

If the test is truncated, \( H_0 \) is chosen if \( \sum_{i=1}^{n_0} z_i < C \). Then, the probability that \( H_0 \) is chosen if the test is truncated is the probability that \( \sum_{i=1}^{n_0} z_i < C \).

Now \( Z_{n_0} = \sum_{i=1}^{n_0} z_i \) is a random variable which satisfies the condition \( \log B < Z_{n_0} < \log A \). Then, the probability density function of \( Z_{n_0} \) when the test is truncated is

\[ y(Z_{n_0}) = \frac{g_*(Z_{n_0})}{\log A - \log B} \int_{\log B}^{\log A} y(Z_{n_0}) \, dZ_{n_0} \]  

where \( g_*(Z_{n_0}) \) is the probability density of \( Z_{n_0} \) under the condition (denoted by the subscript\(^*\)) that \( \log B < Z_{n_0} < \log A \) and \( y(Z_{n_0}) \) is the probability density of \( Z_{n_0} \) before \( Z_{n_0} \) is compared to the thresholds \( \log A \) and \( \log A \). It will be shown later in this chapter that \( y(Z_{n_0}) \) is given by

\[ y(Z_{n_0}) = \int_{-\infty}^{\infty} g_*(Z_{n_0} - 1) \mu(Z_{n_0} - Z_{n_0} - 1) \, dZ_{n_0} - 1, \]

where \( \mu(z) = \) probability density of \( z \).

---

The density functions $g_{\pi}(Z_{n_o})$ and $y(Z_{n_o})$ will be discussed in detail later in this chapter. The test must end on or before the $n_o$-th sample or else it is truncated. Then by definition the probability that the test is truncated is

$$1 - \int_0^{n_o} p(n/s) \, dn.$$ 4-6

The probability that the test is truncated and that $H_0$ is accepted is from Equations 4-4 and 4-5

$$\left(1 - \int_0^{n_o} p(n/s) \, dn\right) \frac{\int_{\log B}^{\frac{C}{\log B}} y(Z_{n_o}) \, dZ_{n_o}}{\log A} \cdot 4-7$$

The probability that $H_0$ is accepted either before or after truncation is the OCF of the TS test which we will denote by $L_T(s)$. From Equations 4-4 and 4-7 we obtain Buusgang and Middleton's result

$$L_T(s) = L(s) \int_0^{n_o} p_0(n/s) \, dn + \left(1 - \int_0^{n_o} p(n/s) \, dn\right) \frac{\int_{\log B}^{\frac{C}{\log B}} y(Z_{n_o}) \, dZ_{n_o}}{\log A} \cdot 4-8$$
Similarly, we find that

\[ 1 - L_T(s) = (1 - L(s)) \int_0^n p_1(n/s) \, dn + (1 - \int_0^n p(n/s) \, dn) \frac{\log A}{\log A} \int y(Z_{n_0}) \, dZ_{n_0} \]

The probability of error when \( s = 0 \) is from Equations 4-9 and 2-9

\[ \alpha_T = 1 - L_T(0) = \alpha \int_0^n p_1(n/0) \, dn + (1 - \int_0^n p(n/0) \, dn) \frac{\log A}{\log A} \int y(Z_{n_0}/0) \, dZ_{n_0} \]

and when \( s = a_1 \) from Equations 4-8 and 2-9

\[ \beta_T = L_T(a_1) = \beta \int_0^n p_0(n/a_1) \, dn + (1 - \int_0^n p(n/a_1) \, dn) \frac{\log B}{\log B} \int y(Z_{n_0}/a_1) \, dZ_{n_0} \]

The sample number probability distribution for arbitrary \( \alpha \) and \( \beta \)

It can be seen from Equations 4-8 to 4-11 that it is necessary to know the probability distribution of the sample number \( \int p(n/s) \, dn \) and the conditional probability distributions \( \int p_1(n/s) \, dn \) and \( \int p_0(n/s) \, dn \) in order to evaluate \( \alpha_T \) and \( \beta_T \). However, as discussed in Chapter II, these distributions have been evaluated only for the special cases where either \( \alpha \) or \( \beta \) were zero. We will
therefore derive $\int p(n/s) \, dn$, $\int p_1(n/s) \, dn$, and $\int p_0(n/s) \, dn$ for arbitrary $\alpha$ and $\beta$. We will begin by finding the probability density function of the sample number.

The probability density function of the normalized sample number $p(t/s)$ is derived for arbitrary $\alpha$ and $\beta$ in Appendix II Equation II-43 for the sequential test of Equation 2-2. The result is

$$p(t/s) = \frac{e^{-t}}{\sqrt{\pi t^3}} \left( -e^{-\frac{\mu}{t}} \sum_{k=\infty}^{\infty} a(k) e^{-\frac{a(k)^2}{t}} - e^{\nu} \sum_{k=\infty}^{\infty} c(k) e^{-\frac{c(k)^2}{t}} \right) \quad 4-12$$

where

$$a(k) = a(k - 1/2) + ck$$
$$c(k) = c(k - 1/2) + ak$$
$$t = n \frac{z^2}{2\sigma^2}$$
$$\mu = \frac{z}{\sigma} \log B$$
$$\nu = \frac{z}{\sigma} \log A$$
$$a = |\mu|$$
$$c = |\nu|$$

The conditional probability densities are also obtained in Appendix II Equations II-44 and II-45 for arbitrary $\alpha$ and $\beta$

$$p_0(t/s) = \frac{1}{L(s)} \frac{e^{-t}}{\sqrt{\pi t^3}} \left( -e^{-\frac{\mu}{t}} \sum_{k=\infty}^{\infty} a(k) e^{-\frac{a(k)^2}{t}} \right) \quad 4-13$$

and

$$p_1(t/s) = \frac{1}{1 - L(s)} \frac{e^{-t}}{\sqrt{\pi t^3}} \left( -e^{\nu} \sum_{k=\infty}^{\infty} c(k) e^{-\frac{c(k)^2}{t}} \right) \quad 4-14$$
where the terms are defined as in Equation 4-12. We must now obtain the probability distributions corresponding to these density functions. The probability densities of Equations 4-12, 4-13, 4-14 are integrated in Appendix II to obtain the probability distributions Equations II-57, II-58 and II-59. Due to the length of these equations they are listed on a separate sheet in Table I Equations 4-15, 4-16 and 4-17. The coefficients of the error functions in Equation 4-15 can be written using Equation 2-3 as

\[(B/A)^{k_H} = \left( \frac{\beta \alpha}{1 + \alpha \beta - \alpha - \beta} \right)^{k_H}\]

and

\[(A/B)^{k_H} = \left( \frac{1 + \alpha \beta - \alpha - \beta}{\beta \alpha} \right)^{k_H}.

When we use typical values of \(\alpha\) and \(\beta\), say \(\alpha = 10^{-6}\) and \(\beta = 10^{-1}\), then to a good approximation

\[(B/A)^{k_H} \approx (\beta \alpha)^{k_H} = (10^{-7})^{k_H}\]

and

\[(A/B)^{k_H} \approx \left( \frac{1}{\beta \alpha} \right)^{k_H} = (10^7)^{k_H}.

For \(s = 0\) or \(a_1 H = 1\) so that the coefficient \((B/A)^{k_H}\) decreases in size as powers of \(10^{-7}\). It can be seen that only a few terms of the summations which have the coefficient \((B/A)^{k_H}\) would be needed. The remaining summations which are multiplied by \((A/B)^{k_H}\) all have the form \(\phi \left( \frac{b(k)}{\sqrt{g}} \right) + \sqrt{g}\). It can be shown that \(\phi \left( \frac{b(k)}{\sqrt{g}} \right) + \sqrt{g}\) has a maximum value which is approximately
Table 1 — Probability distribution and conditional probability distributions of the sample number of the SH test

\[ \int_{0}^{g} p(t/s) \, dt = B^2 \sum_{k=0}^{\infty} (B/A)^k \phi \left( \frac{b(k)}{\sqrt{g}} - \sqrt{g} \right) + B^2 \sum_{k=0}^{\infty} (A/B)^k \phi \left( \frac{b(k)}{\sqrt{g}} + \sqrt{g} \right) \]

\[ - \frac{(H+h)}{2} \sum_{k=1}^{\infty} (B/A)^k \phi \left( \frac{a(k)}{\sqrt{g}} - \sqrt{g} \right) - B^2 \sum_{k=1}^{\infty} (A/B)^k \phi \left( \frac{a(k)}{\sqrt{g}} + \sqrt{g} \right) \]

\[ + A^2 \sum_{k=0}^{\infty} (B/A)^k \phi \left( \frac{d(k)}{\sqrt{g}} - \sqrt{g} \right) + A^2 \sum_{k=0}^{\infty} (A/B)^k \phi \left( \frac{d(k)}{\sqrt{g}} + \sqrt{g} \right) \]

\[ - A^2 \sum_{k=1}^{\infty} (B/A)^k \phi \left( \frac{c(k)}{\sqrt{g}} - \sqrt{g} \right) - A^2 \sum_{k=1}^{\infty} (A/B)^k \phi \left( \frac{c(k)}{\sqrt{g}} + \sqrt{g} \right) \]

\[ \int_{0}^{g} p_{0}(t/s) \, dt = \frac{1}{L(s)} \left[ B^2 \sum_{k=0}^{\infty} (B/A)^k \phi \left( \frac{b(k)}{\sqrt{g}} - \sqrt{g} \right) + B^2 \sum_{k=0}^{\infty} (A/B)^k \phi \left( \frac{b(k)}{\sqrt{g}} + \sqrt{g} \right) \right] \]

\[ - \frac{(H+h)}{2} \sum_{k=1}^{\infty} (B/A)^k \phi \left( \frac{a(k)}{\sqrt{g}} - \sqrt{g} \right) - B^2 \sum_{k=1}^{\infty} (A/B)^k \phi \left( \frac{a(k)}{\sqrt{g}} + \sqrt{g} \right) \]

\[ + A^2 \sum_{k=0}^{\infty} (B/A)^k \phi \left( \frac{d(k)}{\sqrt{g}} - \sqrt{g} \right) + A^2 \sum_{k=0}^{\infty} (A/B)^k \phi \left( \frac{d(k)}{\sqrt{g}} + \sqrt{g} \right) \]

\[ - A^2 \sum_{k=1}^{\infty} (B/A)^k \phi \left( \frac{c(k)}{\sqrt{g}} - \sqrt{g} \right) - A^2 \sum_{k=1}^{\infty} (A/B)^k \phi \left( \frac{c(k)}{\sqrt{g}} + \sqrt{g} \right) \]
Table 1 — Continued

\[
\int_0^G p_1(t/s) \, dt = \frac{1}{1-L(s)} \left[ -\frac{(H+h)}{2} \sum_{k=0}^{\infty} (B/A)^k H \phi \left( \frac{d(k)}{\sqrt{g}} - \sqrt{g} \right) + A \frac{H-h}{2} \sum_{k=0}^{\infty} (A/B)^k H \phi \left( \frac{d(k)}{\sqrt{g}} + \sqrt{g} \right) \right]
\]

\[
- A \frac{H-h}{2} \sum_{k=1}^{\infty} \phi(B/A)^k H \left( \frac{c(k)}{\sqrt{g}} - \sqrt{g} \right) - A \frac{H-h}{2} \sum_{k=1}^{\infty} (A/B)^k H \phi\left( \frac{c(k)}{\sqrt{g}} + \sqrt{g} \right)
\]

\[
a(k) = a(k - 1/2) + ck \\
b(k) = a(k) + a \\
c(k) = c(k - 1/2) + ak \\
d(k) = c(k) + c
\]

\[
a = \left| \frac{z}{\sigma^2} \log B \right| \\
c = \left| \frac{-z}{\sigma^2} \log A \right|
\]

\[
a = \frac{\overline{z}}{\sigma^2} \log B \\
c = \frac{-\overline{z}}{\sigma^2} \log A
\]
\[
\frac{(B/A)^2 H_k}{4 \sqrt{b(k)} \pi}.
\]

Therefore, the terms of the summations multiplied by \((A/B)^{kH}\) have a maximum value of approximately

\[
\frac{(B/A)^H H_k}{4 \sqrt{b(k)} \pi}.
\]

Thus, these terms also decrease in size as \((\beta \alpha)^{kH}\). If we assume \(\alpha = 10^{-6}\) and \(\beta = 10^{-1}\) and neglect terms which are smaller than about \(10^{-6}\), then the probability distributions of the sample number for \(s = 0\) and \(s = a_1\) are from Equation 4-15

\[
\int_0^G p(t/0) \, dt \approx \phi\left( \frac{a/2}{\sqrt{g}} - \sqrt{g} \right) + 1/\beta \, \phi\left( \frac{a/2}{\sqrt{g}} + \sqrt{g} \right)
\]

and

\[
\int_0^G p(t/a_1) \, dt \approx \beta \, \phi\left( \frac{a/2}{\sqrt{g}} - \sqrt{g} \right) + \phi\left( \frac{a/2}{\sqrt{g}} + \sqrt{g} \right) + \phi\left( \frac{c/2}{\sqrt{g}} - \sqrt{g} \right)
\]

\[
+ \frac{1-\beta}{\alpha} \, \phi\left( \frac{c/2}{\sqrt{g}} + \sqrt{g} \right) - \beta \, \phi\left( \frac{(2a+c)/2}{\sqrt{g}} - \sqrt{g} \right)
\]

\[
+ \frac{1-\beta}{\alpha\beta} \, \phi\left( \frac{(2a+c)/2}{\sqrt{g}} + \sqrt{g} \right).
\]

The form of Equations 4-18 and 4-19 may be compared to Equations 2-15 and 2-16. It can be seen that our choice of \(\alpha = 10^{-6}\) and \(\beta = 10^{-1}\) is essentially equivalent to \(\alpha \rightarrow 0\) and \(\beta\) finite so that Equation 4-18 is identical to Equation
2-16. Comparison of Equations 2-15 and 4-19 shows that \( \int_0^g p(t/a_1) \, dt \) cannot be obtained from Wald's results for \( p(t) \). Therefore, it is necessary to use the general equation for \( p(t) \) Equation 4-12 to obtain \( \int_0^g p(t/a_1) \, dt \).

The exact probability density function of \( Z_{n_0} \)

We now require expressions for \( g_{*}(Z_{n_0}) \) and \( y(Z_{n_0}) \) of Equation 4-5. Consider the TS test Equation 4-2 and assume that the first sample has been taken. Then, the density function of the first sample \( Z_1 \) is

\[
y(Z_1) = \mu(Z_1) = \exp(-z - \bar{z})^2/2 \sigma_z^2 / \sqrt{2\pi \sigma_z^2}
\]

where

\[
\mu(z) = \frac{\exp(-z - \bar{z})^2/2 \sigma_z^2}{\sqrt{2\pi \sigma_z^2}}
\]

while the density function of \( Z_1 \) under the condition that \( \log B < Z_1 < \log A \) is from Equation 4-5

\[
g_{*}(Z_1) = \frac{y(Z_1)}{\log A} \int_{\log B}^{\log A} y(Z_1) \, dZ_1
\]

If the condition \( \log B < Z_1 < \log A \) holds, a second sample is taken. Now \( Z_2 \)
is the sum of $Z_1$ and the sample value $Z_2$ so that the probability density of $Z_2$ is, using Equation 4-21,

$$y(Z_2) = \int_{-\infty}^{\infty} g_*(Z_1) \mu(Z_2 - Z_1) \, dZ_1$$

$$= \frac{\log A}{\log B} \int_{\log B}^{\log A} \mu(Z_1) \mu(Z_2 - Z_1) \, dZ_1.$$  \hspace{1cm} 4-22

The probability density of $Z_2$ under the condition that $\log B < Z_2 < \log A$ is

$$g_*(Z_2) = \frac{y(Z_2)}{\log A} \int_{\log B}^{\log A} y(Z_2) \, dZ_2,$$  \hspace{1cm} 4-23

Similarly, if $\log B < Z_2 < \log A$ holds, the third sample is taken and the probability density of $Z_3$ is

$$y(Z_3) = \int_{-\infty}^{\infty} g_*(Z_2) \mu(Z_3 - Z_2) \, dZ_2.$$  \hspace{1cm} 4-24
Then, using Equation 4-23 in Equation 4-24, we find

\[
y(Z_3) = \frac{\int_{\log B}^{\log A} \int_{\log A}^{\log A} \mu(Z_1) \mu(Z_2 - Z_1) dZ_1 \mu(Z_3 - Z_2) dZ_2}{\int_{\log B}^{\log A} \int_{\log A}^{\log B} \mu(Z_1) \mu(Z_2 - Z_1) dZ_1 dZ_2}
\]

In general, if \(\log B < \log Z_{k-1} < \log A\), the probability density of \(Z_k\) is

\[
y(Z_k) = \frac{\int_{\log B}^{\log A} \int_{\log A}^{\log A} \mu(Z_1) \mu(Z_2 - Z_1) dZ_1 \mu(Z_k - Z_{k-1}) dZ_{k-1}}{\int_{\log B}^{\log A} \mu(Z_1) dZ_1 \int_{\log B}^{\log A} y(Z_2) dZ_2 \cdots \int_{\log B}^{\log A} y(Z_{k-1}) dZ_{k-1}}
\]

which can be written as

\[
y(Z_k) = \frac{\int_{\log B}^{\log A} \int_{\log A}^{\log A} \mu(Z_1) \mu(Z_2 - Z_1) dZ_1 \mu(Z_k - Z_{k-1}) dZ_{k-1}}{\int_{\log B}^{\log A} \mu(Z_1) dZ_1 \int_{\log B}^{\log A} \mu(Z_2 - Z_1) dZ_2 \cdots \int_{\log B}^{\log A} \mu(Z_{k-2} - Z_{k-1}) dZ_{k-2} dZ_{k-1}}
\]

A formulation of the exact \(L_T(s)\) equations for digital computation

In Chapter VIII the exact probability density of the sample number \(p(n)\) is developed. A relation that will be useful in obtaining a convenient form of the
exact solution for $L_T(s)$ is contained in Equation 8-8. Rewriting Equation 8-8, we find

$$\log A \int_{\log B} y(Z_k) \, dZ_k = \frac{1 - \sum_{i=1}^{k} p(i)}{1 - \sum_{i=1}^{k-1} p(i)}.$$  

Using Equation 4-28 in Equation 4-26, we obtain an alternate expression for $y(Z_k)$

$$y(Z_k) = \frac{f(Z_k)}{\frac{1 - \sum_{i=1}^{2} p(i)}{1 - \sum_{i=1}^{3} p(i)} \left( \frac{1 - \sum_{i=1}^{2} p(i)}{1 - \sum_{i=1}^{3} p(i)} \right) \cdots \left( \frac{1 - \sum_{i=1}^{k-2} p(i)}{1 - \sum_{i=1}^{k-1} p(i)} \right)}.$$

$$= \frac{f(Z_k)}{1 - \sum_{i=1}^{k-1} p(i)}, \quad 4-29$$

where $f(Z_k)$ is defined by the numerator of Equation 4-26. Substituting Equations 4-28 and 4-29 in Equation 4-5 we find

$$g^*(Z_k) = \frac{f(Z_k)}{1 - \sum_{i=1}^{k} p(i)}, \quad \log B < Z_k < \log A.$$  

$$4-30$$
From Equations 8-9 to 8-15 and using Equation 4-29 it can be seen that

\[
(1 - L(s) p_1(n/s)) = (1 - \sum_{i=1}^{n-1} p(i)) \int_{\log A}^{\infty} y(Z_n) \, dZ_n
\]

and also,

\[
L(s) P_0(n/s) = (1 - \sum_{i=1}^{n-1} p(i)) \int_{-\infty}^{\infty} y(Z_n) \, dZ_n
\]

Using Equations 4-30 and 4-32, we find that \( L_T(s) \) Equation 4-8 can be written as

\[
1 - L_T(s) = \sum_{i=1}^{n_0} \log B \int f(Z_i) \, dZ_i + \int_{-\infty}^{C} f(Z_n) \, dZ_n
\]

and using Equations 4-30 and 4-31, \( 1 - L_T(s) \), Equation 4-9 becomes

\[
1 - L_T(s) = \sum_{i=1}^{n_0} \int_{\log A}^{\infty} f(Z_i) \, dZ_i + \int_{C}^{n_0} f(Z_n) \, dZ_n
\]
Approximate forms of $L_T(s)$

The results of Equations 4-33 and 4-34 are useful for the digital computer calculation of the exact error probabilities of the TS test. It would also be desirable to have approximate equations for $L_T(s)$ and $1 - L_T(s)$ which are simpler to evaluate. If $k_2$ and $k_4$ are small compared to $\log A$ and $\log B$, the limits $\log A$ and $\log B$ can be approximated by $-\infty$ and $+\infty$ in $y(Z_k)$, Equation 4-27. The multiple convolution integral in Equation 4-27 can be easily evaluated if the limits are replaced by $-\infty$ and $+\infty$ as it represents the probability density of the sum of $k$ gaussian random variables. If this assumption is made, $g_*(Z_k)$ becomes

$$g_*(Z_k) \approx \frac{\exp - (Z_k - k \bar{z})^2 / 2k \sigma_z^2}{\sqrt{2\pi} k \sigma_z^2}, \quad \log B < Z_k < \log A$$

$$\int_{\log B}^{\log A} \frac{\exp - (Z_k - k \bar{z})^2 / 2k \sigma_z^2}{\sqrt{2\pi} k \sigma_z^2}$$

The approximation of Equation 4-35 together with the expressions derived for the probability distribution of the sample number Equations 4-15, 4-16, and 4-17 enable us to readily calculate $\alpha_T$ and $\beta_T$. 
A somewhat different approximation is suggested by substituting Equation 4-28 into the expressions for $L_T(s)$ and $1 - L_T(s)$ Equations 4-8 and 4-9.

$$L_T(s) = L(s) \int_0^\infty p_0(n/s) \, dn + \int_0^\infty p(n/s) \, dn$$

$$1 - L_T(s) = \frac{\int y(Z_n) \, dZ_n}{\log B}$$

and $1 - L_T(s)$ Equation 4-9 becomes

$$1 - L_T(s) = (1 - L(s)) \int_0^\infty p_1(n/s) \, dn + \int_0^\infty p(n/s) \, dn \int C y(Z_n) \, dZ_n.$$  

The probabilities of error when $s = 0$ and when $s = a_1$, $\alpha_T$ and $\beta_T$, are from Equations 4-36 and 4-37.

$$\alpha_T = 1 - L_T(0) = \alpha \int_0^\infty p_1(n/0) \, dn + \int_0^\infty p(n/0) \, dn \int C y(Z_n/0) \, dZ_n.$$ 

$$\beta_T = L_T(a_1) = \beta \int_0^\infty p_0(n/a_1) \, dn + \int_0^\infty p(n/a_1) \, dn \int C y(Z_n/a_1) \, dZ_n.$$
Comparison of exact and approximate expressions for $L_T(s)$

The approximation of Equation 4-35 was used in Equations 4-8 and 4-9 to calculate the probabilities of error $\alpha_T$ and $\beta_T$ as a function of the truncation threshold level $C$. The approximate expressions of Equation 4-38 were also used to calculate $\alpha_T$ and $\beta_T$ for the same values of $C$. The results of these error probability calculations are compared to the error probabilities obtained by evaluating the exact expressions Equations 4-33 and 4-34. In Figures 16 and 17, respectively, $\alpha_T$ and $\beta_T$ from the two approximate expressions and the exact expression are plotted as a function of the truncation threshold level $C$ for the truncation sample number $n_o$ equal to 12. The gaussian approximation of Equation 4-35 appears to give better results than the approximation of Equation 4-38. The error in calculation of $\alpha_T$ due to the approximation of Equation 4-35 is not large compared to the precision with which thresholds can be maintained in actual practice. In general, it is also true that a factor of 2 or 3 in the false alarm probability of a test is equivalent to only a small percentage change in the signal to noise ratio required to achieve a given detection probability. We can therefore conclude that the gaussian approximation for $y(Z_{n_o})$ Equation 4-35 can be used in Equations 4-8 and 4-9 to compute the error probabilities of the truncated sequential test with acceptable accuracy. If exact results are needed, the exact expressions can be used in conjunction with a digital computer.
Figure 16 - The false alarm probability of a sequential test truncated at the $n_{th}$ sample as a function of the truncation threshold level $C$; calculated using the exact solution and two approximate solutions.
Figure 17 — The false dismissal probability of a sequential test truncated at the $n_0$th sample as a function of the truncation threshold level $C$; calculated using the exact solution and two approximate solutions.
Choice of the truncation threshold \( C \)

The TS test Equation 4–2 requires the specification of the usual upper and lower thresholds \( A \) and \( B \) but in addition the truncation sample number \( n_o \) and the truncation decision threshold \( C \) must also be given. If the threshold \( C \) is set at \( \log A \), Equations 4–10 and 4–11 become

\[
\alpha_T = (1 - \alpha) \int_0^{n_o} p_1(n/0) \, dn \tag{4-39}
\]

and

\[
\beta_T = \beta \int_0^{n_o} p_0(n/a_1) \, dn + (1 - \int_0^{n_o} p(n/a_1) \, dn) \tag{4-40}
\]

where \( \alpha \) and \( \beta \) are obtained from Equation 2–9. If \( C = \log B \), Equations 4–10 and 4–11 become

\[
\alpha_T = (1 - \alpha) \int_0^{n_o} p_1(n/0) \, dn + (1 - \int_0^{n_o} p(n/0) \, dn) \tag{4-41}
\]

and

\[
\beta_T = \beta \int_0^{n_o} p_0(n/a_1) \, dn. \tag{4-42}
\]

As \( C \) varies between the limits \( \log A \) and \( \log B \), \( \alpha_T \) varies between the values given by Equation 4–39 and Equation 4–41 while \( \beta_T \) varies between the values given by Equation 4–40 and 4–42. Since the choice of \( C \) equal to \( \log A \) or \( \log B \) eliminates the troublesome \( y(Z_{n_o}) \) in Equations 4–10 and 4–11, the values for \( \alpha_T \) and \( \beta_T \) of Equations 4–39 to 4–42 are exact except for the approximations.
used in obtaining \( p(n) \). If a value is specified for \( \alpha_T \), it can be seen that there is a range of values which could be chosen for \( A \) and \( C \) to meet this specification. In fact, if \( A = \infty \), then \( C \) will be minimum while the maximum value for \( C \) occurs when \( C = \log A \).

**Average sample number of the TS test**

When comparing the performance of the TS test Equation 4-2 to the performance of the SH test Equation 2-2, we will require that \( \alpha_T = \alpha \) and \( \beta_T = \beta \). We will then compare the moments of the sample number of the two tests as a criterion for judging the performance. The \( r \)-th moment of the normalized sample number of the sequential test \( t^F(s) \) is derived in Appendix II Equation II-63. Likewise, the \( r \)-th moment for the TS test \( t_T(s) \) is derived in Appendix II Equation II-73 and the equation for \( t_T(s) \) is obtained from it. When \( \alpha = 10^{-6} \) and \( \beta = 10^{-1} \), Equations II-63 and II-73 reduce to

\[
\overline{t}(0) \approx a/2 \quad 4-43
\]

and

\[
\overline{t}_T(0) \approx (a/2 - T) \phi \left( \frac{a/2}{\sqrt{T}} - \sqrt{T} \right) - 1/\beta \left( a/2 + T \right) \phi \left( \frac{a/2}{\sqrt{T}} + \sqrt{T} \right) + T, \quad 4-44
\]

respectively, where \( T \) is the truncation point = \( n_o \ 2 \sigma_z^2 \),

\[
\overline{t}_T(0) = \frac{n_T(0)}{n(0)} \overline{z}/2 \sigma_z^2 \quad a = \log \frac{\overline{z}/\sigma_z^2}{\log B}
\]

\[
\overline{t}(0) = \overline{n}(0) \overline{z}/2 \sigma_z^2
\]
In Figure 18 \( n_{T(0)} \) is plotted versus \( n_0 \) and is compared to \( n(0) \). It can be seen that \( n_{T(0)} \) is always larger than \( n(0) \) but approaches \( n(0) \) as \( n_0 \) increases. Note that the average length of the TS test \( n_{T(0)} \) does not become essentially equal to the average length of the equivalent SH test \( n(0) \) until \( n_0/n_{T(0)} > 8 \) even though \( n_{T(0)} \) is essentially constant after \( n_0/n_{T(0)} > 3.5 \).

The TM test

We are now prepared to calculate the performance of the TM test Equation 4-1. The TM test is the truncated version of the basic multiple hypothesis sequential test Equation 3-1 and thus consists of \( K \) independent sub-tests. Each sub-test is equivalent to a TS test Equation 4-2. Since each sub-test runs independently, the probabilities of error of the sub-test are the same as for the TS test as given by Equations 4-10 and 4-11. However, the TM test does not end until all of the \( K \) subtests have decided. Therefore, we need the average sample number of the longest of \( K \) TS tests. It is shown in Appendix II Equation II-16 that the average sample number of the TM test \( n_{1T} \) is

\[
\overline{n_{1T(s)}} = n_0 + \frac{1}{\epsilon_k} E_i \left[ -\exp\left( -\epsilon_k (n_o - \mu_k) \right) \right]
\]

where \( \epsilon_k, \mu_k \) are defined in Equation II-4,

\[
E_i(-x) = -\int_x^\infty e^{-t} t^{-1} dt,
\]

\( n_o = \) truncation point.
Figure 18 — The average sample number $n_T(0)$ of the TS test compared to the average sample $n(0)$ of an equivalent $SH$ test as a function of the truncation sample number $n_0$. 

Truncation threshold = 0.8
In order to compare the TM test to the M test, we set $\alpha = \alpha_T$ and $\beta = \beta_T$ and then compare $n_{1T}^{(0)}$ Equation 4-45 to $n_1^{(0)}$ Equation 3-2. The results of the calculations of $n_{1T}^{(0)}$ and $n_1^{(0)}$ with truncation threshold $C = 7.6$ and truncation sample number $n_0 = 24$ are plotted as functions of the number of sub-tests $K$ in Figure 19. It is apparent that truncation is an effective modification for improving the performance of a multiple hypothesis sequential test when the number of sub-tests is large.
Figure 19 — The average sample number \( \overline{n}_{1T}(0) \) of the TM test compared to the average sample number \( \overline{n}_1(0) \) of an equivalent M test as a function of the number of sub-tests \( K \).
CHAPTER V

MODIFICATIONS OF THE MULTIPLE HYPOTHESIS SEQUENTIAL TEST

Selection of several types of modifications

At this point we have developed methods of analyzing the performance with arbitrary \( \alpha \) and \( \beta \) of

1. the simple hypothesis sequential test, Equation 2-2,

2. the basic multiple hypothesis sequential test with independent sub-tests, Equation 3-1,

3. the truncated simple hypothesis sequential test, Equation 4-2,

4. the truncated multiple hypothesis test, Equation 4-1.

We have seen that the performance improvement of the multiple hypothesis sequential test, the M test, over the FS test declines as the number of sub-tests increases because of the increase of the average sample number \( n_1(0) \) of the M test. This is due to the high probability of at least one sub-test lingering and thereby lengthening the test. We will, therefore, inspect the probability density of the sample number for the M test \( p(n_1) \) to determine what parameters
of the sub-test affect \( p(n_1) \). We would hope to determine what modifications could be made to the sub-tests which would result in a decrease of \( n_1(0) \) of a modified M test compared to the M test. Now \( p(n_1) \) is given in Appendix II, Equation II-3 as

\[
p(n_1) = \epsilon_k \exp \left[ -\epsilon_k (n_1 - \mu_k) - \exp (-\epsilon_k (n_1 - \mu_k)) \right]
\]

where \( \epsilon_k \) and \( \mu_k \) are found from

\[
\int_0^1 p(n/s) \, dn = 1 - 1/K,
\]

\[
\epsilon_k = K p\mu_k /s).
\]

Here \( p(n) \) and \( \int p(n) \, dn \), Equations 4-12 and 4-15, are the density function and distribution of the sample number of the sub-test. The number of sub-tests \( K \) may be as large as 1000 so that \( 1 - 1/K \) is very close to unity. The parameters \( \epsilon_k \) and \( \mu_k \) of \( p(n_1) \) therefore depend on the sub-test only through the value of \( p(n) \) and \( \int p(n) \, dn \) at the point where \( \int p(n) \, dn \) is very close to unity. In Figure 20 \( p(t) \) and \( \int p(t) \, dt \) are plotted where \( t \) is the normalized sample number

\[
\frac{n \bar{X}}{2 \sigma_z^2}.
\]

When \( s = 0 \) or \( a_1 \), we have \( t = \frac{n a_1^2}{8} \). Therefore, the value of \( a_1^2 \) enters into \( p(n) \) and \( \int p(n) \, dn \) as a scale factor on \( n \). Also, from Equations 4-12 and 4-15 we see that \( p(n) \) and \( \int p(n) \, dn \) depend on the threshold values \( A \) and \( B \). Note that \( p(t) \) and \( \int p(t) \, dt \) are plotted in Figure 20 for several values of \( A \) and \( B \). It can be seen that an increase in the threshold level (corresponding to
Figure 20 — The probability density \( p(t/0) \) and probability distribution \( \int_0^t p(t/0) \, dt \) of the normalized sample number of a sequential test with thresholds \( \log A \) and \( \log B \).
smaller values of B and larger values of A) lengthens the sequential test and a
decrease in the threshold level shortens the sequential test.

Now, \( n_1(0) \) is from Equation 2-17

\[
\frac{n_1(0)}{n_1(0)} = \frac{\gamma}{\epsilon_k} + \mu_k \tag{5-3}
\]

where \( \epsilon_k \) and \( \mu_k \) are determined from Equation 5-2 with \( s = 0 \). Therefore, if
\( \mu_k \) is decreased, \( \epsilon_k \) will decrease as will \( n_1(0) \). We seek, therefore, modifi-
cations of \( a^2 \) and of A and B of the subtest which will decrease \( \mu_k \).

The variation of the design signal level

From Equation 5-2 and Figure 20 we can see that an increase of \( a^2 \) will
decrease \( \mu_k \). Since it is only necessary to change \( \int p(n) \, dn \) near the point
where it equals \( 1 - 1/K \), it seems reasonable to increase \( a^2 \) after the M test
has taken a number of samples, say \( N \) samples. In this way we will affect only
the tests which are longer than \( N \) samples. This is illustrated by the sketch of
a sequential test with a step increase of \( a^2 \) at \( N \) as shown in Figure 21. Now
\( a_1 = S(t)/\sqrt{N_0} \) and from Equation 1-4 we see that \( S(t) \) is proportional to the re-
ceived signal power. The step increase of \( a^2 \) is, therefore, equivalent to a
step increase in transmitter power. The modification of the simple hypothesis
sequential test Equation 2-2, hereafter called the SH test, to include the step in
\( a^2 \) will be called the SP test. The SP test may be stated as
Figure 21 — A modified sequential test with a step increase in the design signal level at the Nth sample.
1. if $Z_n < \log B$, choose $H_0$,

2. if $Z_n \geq \log A$, choose $H_1$,

3. if $\log B < Z_n < \log A$, take another sample,

where $Z_n = \sum_{i=1}^{n} (k a_1 y_i - (k a_1)^2/2)$, $n \leq N$

and $Z_n = \sum_{i=1}^{N} (k a_1 y_i - (k a_1)^2/2) + \sum_{i=N+1}^{n} (a_1 y_i - a_1^2/2)$, $n > N$, $0 < k < 1$.

The modification of the M test Equation 3-1 to include the step in $a_1^2$ will be called the MSP test. The MSP test may be stated as follows.

1. if $Z_n^j \geq \log A$, decide $H_1$ and set $\delta(j) = 1$,

2. if $Z_n^j \leq \log B$, decide $H_0$ and set $\delta(j) = 1$

3. if $\log B < Z_n^j < \log A$, take another sample and set $\delta(j) = 0$,

4. if after the $n$-th sample $\sum_{j=1}^{K} \delta(j) = K$, terminate the test and move the antenna to the next position,

where $Z_n^j = \sum_{i=1}^{N} (k a_1 y_i^j - (k a_1)^2/2)$, $n \leq N$

and $Z_n^j = \sum_{i=1}^{N} (k a_1 y_i^j - (k a_1)^2/2) + \sum_{i=N+1}^{n} (a_1 y_i^j - a_1^2/2)$, $n > N$. 

Variation of the thresholds

Consider the effect of changing the lower threshold log B of the sub-test. From Equation 5-2 and Figure 20 we can see that a decrease in the magnitude of log B will decrease $\mu_k$ when $s = 0$. By the same reasoning as for the SP test, we conclude that the lower threshold should be decreased at the N-th sample. This is illustrated by the sketch in Figure 22 of a sequential test with a step change in the lower threshold at the N-th sample. The modification of the SH test Equation 2-2 to include the step in the lower threshold will be called the ST test. The ST test may be stated as

1. if $Z_n \leq \log B_n$, choose $H_0$,

2. if $Z_n > \log A$, choose $H_1$,

3. if $\log B_n < Z_n < \log A$, take another sample,

where $B_n = B_{1n} \leq N$,

$B_n = B_{2n} > N$,

$Z_n = a_1 y_i - a_2^{1/2}$.

Likewise, the M test Equation 3-1 may be modified to include the step change in the lower threshold. This will be called the MST test and is given by
Figure 22 — A modified sequential test with a step change in the lower threshold at the Nth sample.
1. if $Z_n^j \geq \log A$, decide $H_1$ and set $\delta(j) = 1$,

2. if $Z_n^j \leq \log B_n$, decide $H_0$ and set $\delta(j) = 1$,

3. if $\log B_n < Z_n^j < \log A$, take another sample and set $\delta(j) = 0$,

4. if after the $n$-th sample $\sum_{j=1}^{K} \delta(j) = K$, terminate the test and move the antenna to the next position, where $B_n = B_1$, $n < N$ and $B_n = B_2$, $n > N$.

Performance equations of the SP test

As in the case of the truncated test, the above modifications will affect both the average sample number and the probability of error. We will first investigate the SP test to determine $n_{SP}$, $\alpha_{SP}$, and $\beta_{SP}$. Many of the results can then be applied to the MSP test. The probabilities of error, $\alpha_{SP}$ and $\beta_{SP}$ can be obtained from the OCF of the SP test. The OCF of the SP test may be derived by following exactly the derivation of the OCF of the SH test in Chapter II, Equations 2-4 through 2-7. For the SP test, however, the likelihood ratio is

$$\lambda_n = \frac{p(y_1, y_2, \ldots, y_N/k a_1)}{p(y_1, y_2, \ldots, y_N/0)} \cdot \frac{p(y_{N+1}, y_{N+2}, \ldots, y_n/a_1)}{p(y_{N+1}, y_{N+2}, \ldots, y_n/0)}$$

so that we must obtain $h$ from

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p^*(y_1, \ldots, y_N/k s) p^*(y_{N+1}, \ldots, y_n/s) \, dy_1 \ldots dy_n = 1$$
where
\[ p^* (y_1, \ldots, y_N/ks) = \frac{p(y_1, \ldots, y_N/ka_1)}{p(y_1, \ldots, y_N/ks)} \]
and
\[ p^*(y_{N+1}, \ldots, y_n/s) = \frac{p(y_{N+1}, \ldots, y_n/a_1)}{p(y_{N+1}, \ldots, y_n/0)} \]

Since the samples \( y_1, \ldots, y_n \) are independent, Equation 5-9 reduces to
\[
\left( \int_0^\infty \left( \frac{p(y/ka_1)}{p(y/0)} \right) p(y/ks) \, dy \right) ^h \int_0^\infty \left( \frac{p(y/a_1)}{p(y/0)} \right) p(y/s) \, dy = 1
\]

It is shown in Appendix I that
\[
\int_{-\infty}^\infty \left( \frac{p(y/ka_1)}{p(y/0)} \right) p(y/ks) \, dy = 1 \text{ if } h'(s) = 1 - \frac{2ks}{ka_1} \]

and that
\[
\int_{-\infty}^\infty \left( \frac{p(y/a_1)}{p(y/0)} \right) p(y/s) \, dy = 1 \text{ if } h(s) = 1 - \frac{2s}{a_1} \]

But \( h' \) from Equation 5-11 equals \( h \) from Equation 5-12 so that Equation 5-10 is satisfied by
\[
h(s) = 1 - \frac{2s}{a_1} \]
Then, as shown in Chapter II, Equation 2-6, the OCF of the SP test is

\[ L(s) = \frac{A}{A} \frac{h(s)}{h(s)} - 1 \frac{B}{B} h(s) \]  

where \( h(s) \) is given by Equation 5-13.

But this is exactly the result of Equations 2-6 and 2-8 so that we find the step in \( a_1^2 \) of the SP test does not change the probability of error as compared to the SH test with constant \( a_1^2 \).

In order to find the average sample number of the SP test \( n_{SP} \) we must first derive the probability density of the sample number of the SP test. We have shown that for the SH test Equation 2-2 the probability density of the normalized sample number \( t \) is given by \( p(t/s) \) Equation 4-12. But the SP test Equation 5-4 for \( n < N \) is identical to the test of Equation 2-2 except that the design signal level and true signal level are \( ka \) and \( ks \), respectively. Therefore, for the SP test the probability density of \( t \) for \( n < N \) is

\[ p(t/ks), \ n \leq N \]  

where

\[ t = \frac{n \bar{z}^2}{2\sigma^2} \]

For \( n < N \) we have from Equation 5-4 \( z = ka \bar{y} - k^2 a_1^2/2 \) and if \( \bar{y} = ks \), \( \sigma^2_y = 1 \) (Equation 1-8), we have
\[
\bar{z} = k^2 (sa_1 - a_1^2/2), \quad 5-16
\]

\[
\sigma_z^2 = k^2 a_1^2
\]

so that

\[
t = \frac{n k^2 a_1^2 h^2}{8} \quad \text{where} \quad h = (1 - \frac{2a}{a_1}).
\]

The probability density of \(n\), \(p(n/ks)\), is from Equations 5-15 and 5-16

\[
p(n/ks) \, dn = f \left( \frac{n a_1^2 h^2}{8} / ks \right) \left( \frac{k^2 a_1^2 h^2}{8} \right) \, dn, \quad n \leq N. \quad 5-17
\]

The probability density of \(n\) for the SP test for \(n > N\) is derived in Appendix III. From Equation III-13 we find

\[
p((n + k^2 N-N)/s) \, dn = f \left( \frac{n a_1^2 h^2}{8} / s \right) \left( \frac{a_1 h^2}{8} \right) \, dn, \quad n > N. \quad 5-18
\]

The probability density of \(n\) for the SP test as given by Equations 5-17 and 5-18 is plotted in Figure 23. We can now obtain the average sample number \(n_{SP}\) as

\[
\overline{n}_{SP(s)} = \int_0^N n \, p(n/ks) \, dn + \int_N^\infty np((n + k^2 N-N)/s) \, dn. \quad 5-19
\]

If we set \(v = k^2 n\) in the first integral of Equation 5-19 and \(\mu = n + k^2 N-N\) in the second, we have
Figure 23 — The probability density of the sample number of the SP test with a step in signal level of $1/k^2$ at the Nth sample number.
\[ n_{SP}(s) = \frac{N_k^2}{k^2} \int_0^N v p(v/s) \, dv + (N-Nk^2) \int_{Nk^2}^{\infty} p(\mu/s) \, d\mu + \int_{Nk^2}^{\infty} \mu p(\mu/s) \, d\mu \]  
5-20

or

\[ n_{SP}(s) = \frac{N_k^2}{k^2} \int_0^n n p(n/s) \, dn + \int_{Nk^2}^{\infty} n p(n/s) \, dn + (N-Nk^2) \int_{Nk^2}^{\infty} p(n/s) \, dn. \]

We would now like to compare \( n_{SP}(0) \) to \( n(0) \), Equation 2-12, but a difficulty arises in that the design signal level of the SP test is a random variable; that is, the design signal level is \( k^2 a_1^2 \) for a random number of samples up to \( N \) and is \( a_1^2 \) for a random number of samples after \( N \). We assume then that the SH test Equation 2-2 is equivalent to the SP test Equation 5-4 when the probabilities of error of the tests are equal and the design signal level of the SH test equals the average design signal level of the SP test. The design signal level of the SP test, which we will denote by \( P_{SP} \), is from the definition of the SP test Equation 5-4 given by

\[ P_{SP} = k^2 a_1^2 \text{ for } n \leq N \]  
5-21

and

\[ P_{SP} = a_1^2 \text{ for } n > N. \]

If a particular SP test ends with \( n \leq N \), the value of \( P_{SP} = k^2 a_1^2 \) throughout that test. On the other hand, if an SP test ends with \( n > N \), the value of \( P_{SP} = k^2 a_1^2 \) until \( n = N \) and \( P_{SP} = a_1^2 \) for \( n > N \). The average of \( P_{SP} \) can then be expressed in terms of the average lengths of SP tests which end with \( n \leq N \).
and those which end with \( n > N \). For the tests which end with \( n \leq N \), the contribution to the average \( P_{SP} \) is

\[
\frac{k^2 a_1^2}{\int_0^n n p(n/k s) \, dn}
\]

\[
\int_0^n n p(n/k s) \, dn + \int_0^n n p((n+k^2 N-N)/s) \, dn
\]

For the tests which end with \( n > N \), the contribution to the average \( P_{SP} \) is

\[
\frac{a_1^2}{\int_0^n n p((n+k^2 N-N)/s) \, dn - (N a_1^2 - N k^2 a_1^3) \int_0^n p((n+k^2 N-N)/s) \, dn}
\]

\[
\int_0^n n p(n/k s) \, dn + \int_0^n n p((n+k^2 N-N)/s) \, dn
\]

If we let \( v = k^2 n \) in Equation 5-22 and recognize the denominator as \( n_{SP} \)

Equation 5-18, Equation 5-22 becomes

\[
\frac{a_1^2}{\int_0^{n_{SP}} v p(v/s) \, dv}
\]

Also, let \( \mu = n + k^2 N-N \) in Equation 5-23, and using Equation 5-18, Equation 5-23 becomes

\[
\frac{a_1^2}{\int_0^{n_{SP}} \mu p(\mu/s) \, d\mu + (N a_1^2 - N k^2 a_1^3) \int_0^{n_{SP}} p(\mu/s) \, d\mu - (N a_1^2 - N k^2 a_1^3) \int_0^{n_{SP}} p(\mu/s) \, d\mu}
\]
The average $P_{SP}$ is the sum of the averages over the tests that end with $n < N$ and the tests that end with $n > N$ so that from Equations 5-24 and 5-25 we have

$$
\bar{P}_{SP}(s) = \frac{a_1^2 \int_{Nk^2}^{\infty} n p(n/s) \, dn + a_1^2 \int_{Nk^2}^{\infty} n p(n/s) \, dn}{n_{SP}}
$$

$$
= \frac{a_1^2 \int_{0}^{\infty} n p(n/s) \, dn}{n_{SP}}
$$

$$
= \frac{a_1^2 \bar{n}(s)}{n_{SP}(s)}
$$

where $\int_{0}^{\infty} n p(n/s) \, dn$ is by definition $\bar{n}(s)$. We have assumed that all comparisons of performance are made with $s = 0$ because of the sparse target assumption so that we will use

$$
\bar{P}_{SP}^{(0)} = \frac{a_1^2 \bar{n}(0)}{n_{SP}(0)}
$$
as the average design signal for the SP test. The equivalent SH test therefore
has error probability equal to those of the SP test and a design signal level
(which we denote by \( P \)) of

\[
P = \frac{a_1^2 \overline{n(0)}}{n_{SP}(0)}.
\]

We can now compare the performance of the SP test to the equivalent SH test
by comparing the average sample numbers of the two tests. From Equation
2-12 we have

\[
\frac{n(0)}{\overline{n(0)}} = \frac{(1-\alpha) \log \frac{\beta}{1-\alpha} + \alpha \log \frac{1-\beta}{\alpha}}{-a_1^{2/2}}
\]

where the design signal level of the test is \( a_1^2 \). The average sample number of
the equivalent SH test with a design signal level \( P \) is

\[
\frac{a_1^2 \overline{n(0)}}{P}.
\]

We then take the ratio of the average sample numbers of the SP test and
the equivalent SH test Equations 5-20 and 5-29 to obtain

\[
\frac{n_{SP}(0)}{\overline{n(0)}} \quad \frac{a_1^2 \overline{n(0)}}{P}.
\]
which upon substitution of Equation 5-27 becomes

\[
\frac{n_{SP}^{(0)}}{a_{1}^{2}n^{(0)}} \cdot \frac{a_{1}^{2}n^{(0)}}{n_{SP}^{(0)}} = 1.
\]

\[5-31\]

There is, therefore, no improvement in performance of the SP test Equation 5-4 over the SH test Equation 2-2.

**Performance of the MSP test**

We are now prepared to analyze the MSP test Equation 5-5. Each of the K sub-tests of the MSP test is an SP test and since the sub-tests terminate independently the probabilities of error for the MSP test are given by the OCF for the SP test Equation 5-14. The MSP test ends when all of the K sub-tests have come to a decision. Therefore, the sample number of the M test is a random variable which is the largest of the K sub-test sample numbers. Then, we find from Appendix II Equation II-3 that the probability density of \(n_1\) is given by

\[
\omega(n_1) = \epsilon_k \exp \left[ - \epsilon_k (n_1 - \mu_k) - \exp \left( - \epsilon_k (n_1 - \mu_k) \right) \right], \quad n_1 \leq N
\]

where

\[
k = \int_{0}^{Kp(\mu_k/ks)} \frac{p(n/ks) \, dn}{1 - 1/K},
\]

\[
\epsilon_k = \int_{0}^{Kp(\mu_k/ks)} \frac{p(n/ks) \, dn}{1 - 1/K}, \quad 5-32
\]

and

\[
\omega(n_1) = \epsilon \exp \left[ - \epsilon (n_1 - \mu) - \exp \left( - \epsilon (n_1 - \mu) \right) \right], \quad n_1 > N
\]
where
\[ \epsilon = K_p \left( (\mu + k^2 N - N)/s \right), \]

\[
\int_0^N p(n/ks) \, dn + \int_0^\mu p \left( (n + k^2 N - N)/s \right) \, dn = 1 - 1/K.
\]

Here we have used the probability density of \( n \) for \( n \leq N \) given by Equation 5-17 to obtain the parameters \( \epsilon_k \) and \( \mu_k \) and the probability density of \( n \) for \( n > N \) given by Equation 5-18 to obtain \( \epsilon \) and \( \mu \). The average sample number of the MSP test \( n_{SP} \) can now be obtained as follows

\[
\overline{n_{SP}} = \int_{-\infty}^{\infty} n_1 \omega(n_1) \, dn_1
\]

\[
= \int_{-\infty}^{N} n_1 \epsilon_k \exp \left[ -\epsilon_k (n_1 - \mu_k) - \exp \left( -\epsilon_k (n_1 - \mu_k) \right) \right] \, dn_1 5-34
\]

\[
+ \int_{N}^{\infty} n_1 \epsilon \exp \left[ -\epsilon (n_1 - \mu) - \exp \left( -\epsilon (n_1 - \mu) \right) \right] \, dn_1.
\]

The integrals in Equation 5-34 are evaluated in Appendix II Equations II-13 to II-16 so that we have

\[
\overline{n_{SP}} = N \exp \left[ -\exp \left( -\epsilon_k (N - \mu_k) \right) \right] + \frac{1}{\epsilon_k} E_1 \left[ -\exp \left( -\epsilon_k (N - \mu_k) \right) \right]
\]

\[
+ \gamma/\epsilon + \mu - N \exp \left[ -\exp \left( -\epsilon (N - \mu) \right) \right] - \frac{1}{\epsilon} E_1 \left[ -\exp \left( -\epsilon (N - \mu) \right) \right]
\]

5-35
where

\[ \gamma = \text{Euler's constant} \]

\[ E_i(-x) = - \int_x^\infty e^{-t} t^{-1} dt. \]

In Appendix III Equations III-19 and III-20 it is shown that

\[ \mu_k = 1/k^2 (\mu + k^2 N - N), \]

\[ \epsilon_k = k^2 \epsilon. \]

Using Equation 5-36, we can simplify Equation 5-35 and we find

\[ \overline{n_{1SP}} = \gamma /\epsilon + \mu + (1/k^2 - 1) 1/\epsilon E_i \left[ - \exp \left( \epsilon (N - \mu) \right) \right] \]

where the terms are defined in Equations 5-32 to 5-35. The average design

signal level of the MSP test \( \overline{P_{1SP}} \) can now be obtained by using the probability
density of \( n_1 \) Equations 5-32 and 5-33 in the equations for \( \overline{P_{SP}} \) Equations 5-22

and 5-23. The result is

\[ \overline{P_{1SP}} = \frac{k a_1^2 \int_0^N n_1 \omega(n_1) dn_1 + a_1^2 \int_n^\infty n_1 \omega(n_1) dn_1 - N a_1^2 (1-k^2) \int_0^\infty \omega(n_1) dn_1}{\int_0^N n_1 \omega(n_1) dn_1 + \int_0^\infty n_1 \omega(n_1) dn_1} \]

The denominator of Equation 5-38 is by definition \( \overline{n_{1SP}} \) while the integrals of
the numerator are evaluated in Appendix II Equations II-14 and II-15. Making
the substitutions and simplifying, we obtain

\[ P_{1SP} = a_1^2 \frac{(\gamma/\epsilon + \mu - N (1 - k^2))}{n_{1SP}} \]  

5-39

We would like to compare the performance of the MSP test Equation 5-5 to that
of the M test Equation 3-1. That is, we will compare \( n_{1SP} \) Equation 5-37 to
\( n_1 \) Equation 3-2 when the tests are equivalent. The M test will be considered
equivalent to the MSP test if the error probabilities of the two tests are equal
and if the design signal level of the M test \( P_M \) is equal to the average design
signal level of the MSP test \( P_{1SP} \). Then, from Equation 5-39 we have

\[ P_M = a_1^2 \frac{(\gamma/\epsilon + \mu - N (1 - k^2))}{n_{1SP}} \]  

5-40

From Equation 3-2 the average sample number of the M test is

\[ n_1 = \frac{\gamma/\epsilon_M + \mu_M}{n_{1SP}} \]  

5-41

In Appendix III it is shown that \( \epsilon_M \) and \( \mu_M \) are related to \( \epsilon_k \) and \( \mu_k \) Equation
5-32 as follows

\[ \epsilon_M = \frac{P_M}{k^2 a_1^2} \epsilon_k \]  

5-42

\[ \mu_M = \frac{k^2 a_1^2 \mu_k}{P_M} \]  

Using Equation 5-42 in 5-41, we find

$$\frac{n_1}{n_1} = \frac{k^2 a_1^2}{P_M} \left( \gamma/E_k + \mu_k \right).$$  \hspace{1cm} 5-43

If we solve Equation 5-40 for $\frac{n_{1SP}}{n_1}$ and take the ratio of $\frac{n_{1SP}}{n_1}$ to $n_1$ Equation 5-43 we obtain

$$\frac{n_{1SP}}{n_1} = \frac{a_1^2}{P_M} \left( \gamma/E_k + \mu - N \left( 1 - k^2 \right) \right)$$  \hspace{1cm} 5-44

and using Equation 5-36 we find

$$\frac{n_{1SP}}{n_1} = 1.$$  \hspace{1cm} 5-45

The MSP test thus results in no improvement in performance compared to the M test.

**Performance of the ST test**

We will now investigate the effect on performance of a step in the lower threshold. Consider first the ST test Equation 5-6.

If we consider the ST test at the time of the step in the lower threshold, it can be seen that it is very similar to the truncated test in that the test is ended by truncation when $\log B_1 < Z_N < \log B_2$. However, those tests for
which \( \log B_2 < Z_N < \log A \) are allowed to continue as sequential tests with the new lower threshold \( \log B_2 \). The test can end as a sequential test by crossing either the upper or lower thresholds for \( n < N \), by truncation at \( n = N \), or as a sequential test for \( n > N \). We will, therefore, obtain the error probabilities for the ST test in terms of the errors occurring before the step, errors which occur at time of the step \( n = N \), and errors which occur after the step \( n > N \). As shown for the TS test Equation 4-4, the probability of deciding \( H_0 \) before \( n = N \) is

\[
L_1(s) = \int_0^N p_0(n/s, \beta_1) \, dn
\]

where

\[
L_1(s) = \text{OCF for } n < N \text{ corresponding to the thresholds } A, B_1.
\]

\[
\beta_1 = L_1(a_1).
\]

Also, from the result for the TS test Equation 4-7, we find the probability of deciding \( H_0 \) at \( n = N \) is

\[
\frac{\log B_2}{\log B_1} \int_{\log A}^{\log B_1} \frac{y(Z_N/s)}{dZ_N} \, dZ_N
\]

The notation \( p_0(n/s, \beta_1) \) is used to make clear the threshold level to which this particular probability density refers.
or using Equation 4-37

\[
\log B_2 \left( \int_{N-1}^{\infty} p(n/s, \beta_1) \, dn \right) \left( \int \frac{y(Z_N/s)}{\log B_1} \, dZ_N \right) \tag{5-48}
\]

where \(y(Z_N/s)\) is the exact probability density of \(Z_N\) as given by Equation 4-27.

The tests which do not terminate at \(n < N\) and also do not decide \(H_0\) at the step \(n = N\) are allowed to continue as sequential tests. These tests will decide either \(H_0\) or \(H_1\). The probability that the test will decide \(H_0\) under the condition that the test continues past \(N\) is

\[
\frac{L_2(s)}{\int_{N}^{\infty} p(n/s, \beta_2) \, dn} \left( \int_{N}^{\infty} p(n/s, \beta_2) \, dn \right) \tag{5-49}
\]

where \(L_2(s) = \text{OCF for } n > N\) corresponding to thresholds \(A\) and \(B_2\)

\[\beta_2 = L_2(a_1)\]  

The probability that the test is allowed to continue past \(N\) is

\[
\left( \int_{N-1}^{\infty} p(n/s, \beta_1) \, dn \right) \left( \int \frac{\log A}{\log B_2} W(Z_N/s) \, dZ_N \right) \tag{5-50}
\]

where we have used Equation 5-48.
The probability of deciding $H_0$ after $N$ is the product of Equations 5-49 and 5-50

\[
\left( \int_{N-1}^{\infty} p(n/s, \beta_1) \, dn \right) \frac{\log A_2(s)}{p_0(n/s, \beta_2)} \left( \int_{N-1}^{\infty} w(Z_N/s) \, dZ_N \right) \frac{L_2(s)}{N} \frac{\int_{N}^{\infty} p_0(n/s, \beta_2) \, dn}{p(n/s, \beta_2)} - 5-51
\]

The probability of deciding $H_0$ for the ST test $Q(s)$ is the sum of the probabilities of deciding before the step $n < N$, at the step $n = N$, and after the step $n > N$. Therefore, from Equations 5-46, 5-48 and 5-51 we find

\[
Q(s) = L_1(s) \int_{0}^{N} p_0(n/s, \beta_1) \, dn + \left( \int_{N-1}^{\infty} p(n/s, \beta_1) \, dn \right) \frac{\log B_2}{p_0(n/s, \beta_2)} \left( \int_{N-1}^{\infty} w(Z_N/s) \, dZ_N \right)
\]

\[
+ \left( \int_{N-1}^{\infty} p(n/s, \beta_1) \, dn \right) \frac{\log B_2}{p_0(n/s, \beta_2)} \left( \int_{N-1}^{\infty} w(Z_N/s) \, dZ_N \right) \frac{L_2(s)}{N} \frac{\int_{N}^{\infty} p_0(n/s, \beta_2) \, dn}{p(n/s, \beta_2)} - 5-52
\]

Similarly, we can find the probability of deciding $H_1$ for the ST test $1 - Q(s)$ by determining the probability of deciding $H_1$ for $n < N$ and $n > N$. By the same reasoning as for Equations 5-46 and 5-51, we find
\[1 - Q(s) = (1 - L_1(s)) \int_0^N p_1(n/s, \beta_1) \, dn\]

\[= \int_0^\infty p_1(n/s, \beta_2) \, dn \int p(n/s, \beta_2) \, dn\]

\[\frac{\log A}{\log B_2} (\int p(n/s, \beta_1) \, dn) (\int w(Z_N/s) \, dZ_N) (1 - L_2(s))\]

In order to find the average sample number of the ST test \(n_{ST}\), we need the probability density of \(n\) for the ST test. Actually, we have used it in the formulation of Equations 5-52 and 5-53. From consideration of Equations 5-52 and 5-53 we can write the probability density of \(n\) for the ST test as follows:

\[p(n/s, \beta_1), \quad 0 < n < N\]

\[\frac{\log A}{\log B_2} (\int p(n/s, \beta_1) \, dn) (\int w(Z_N/s) \, dZ_N) \delta(n - N), \quad n = N\]

\[\frac{\log A}{\log B_2} \int w(Z_N/s) \, dZ_N \int p(n/s, \beta_2) \, dn \quad p(n/s, \beta_2), \quad n > N\]

where \(\delta(n - N) = 1\) for \(n = N\) and \(\delta(n - N) = 0\) for \(n \neq N\).
We can now find $n_{ST}$ from Equations 5-54, 5-55 and 5-56

$$n_{ST} = \int_0^N n \cdot p(n/s, \beta_1) dn + N \left( \int_{N-1}^{\infty} p(n/s, \beta_1) dn \right) \int \frac{w(Z_N/s)}{\log B_1} d Z_N$$

$$+ \left( \int_{N-1}^{\infty} p(n/s, \beta_1) dn \right) \int \frac{\log A}{\log B_2} \frac{w(Z_N/s)}{d Z_N} d Z_N \int_n^{\infty} n \cdot p(n/s, \beta_1) dn \right) \int \frac{\log A}{\log B_2} \frac{w(Z_N/s)}{d Z_N} d Z_N$$

Performance of the MST test

The MST test Equation 5-7 can now be analyzed. Since the sub-tests of the MST test are independent, the probabilities of error for each sub-test are the same as for the ST test Equations 5-52 and 5-53. In order to find the average sample number of the MST test $n_{1ST}$, the probability density of the MST test $p(n_1)$ must be found. Since the MST test does not end until all K sub-tests have decided, $n_1$ is the largest of the K sub-test sample numbers. Then, using Equation II-1 and also the probability density of $n$ for the ST test Equations 5-54, 5-55 and 5-56, we find

$$p(n_1) = K \left( \int_0^{n_1} p(n/s, \beta_1) dn \right)^{K-1} p(n_1/s, \beta_1), n_1 < N$$

$$p(n_1) = \left( \int_0^N p(n/s, \beta_1) dn + \Delta P(N) \right)^K - \left( \int_0^N p(n/s, \beta_1) dn \right)^K \delta (n_1 - N), n_1 = N$$

5-58

5-59
\[ p(n_1) = K \left( 1 - R \int_{n_1}^{\infty} p(n/s, \beta_2) \, dn \right)^{K-1} \int_{n_1}^{\infty} p(n/s, \beta_2) \, dn, \quad n_1 > N \] 5-60

where

\[ \Delta P(N) = \left( \int_{N-1}^{\infty} p(n/s, \beta_1) \, dn \right) \left( \int \frac{w(Z_N/s)}{\log B_1} dZ_N \right) \delta(n-N) \]

\[ R = \left( \int_{N-1}^{\infty} p(n/s, \beta_1) \, dn \right) \frac{\log A \int w(Z_N/s) \, dZ_N}{\log B_2 \int_{N}^{\infty} p(n/s, \beta_2) \, dn} \]

We can obtain the asymptotic approximation to \( p(n_1) \) for the MST test in the same manner as for the MSP test Equations 5-32 and 5-33. From Equations 5-58, 5-59 and 5-60 and using Equation II-3 we find

\[ p(n_1) = \epsilon_1 \exp \left[ -\epsilon_1 (n_1 - \mu_1) - \exp (-\epsilon_1 (n_1 - \mu_1)) \right], \quad n_1 < N \] 5-61

\[ p(n_1) = \left( \int_{0}^{N} p(n/s, \beta_1) \, dn + \Delta P(N) \right)^K - \left( \int_{0}^{N} p(n/s, \beta_1) \, dn \right)^K \delta(n_1-N), \quad n_1 = N \] 5-62

\[ p(n_1) = \epsilon_2 \exp \left[ -\epsilon_2 (n_1 - \mu_2) - \exp (-\epsilon_2 (n_1 - \mu_2)) \right], \quad n_1 > N \] 5-63
and where $R$ and $\Delta P(N)$ are defined in Equations 5-59 and 5-60. The average sample number for the MST test $n_{1ST}$ is from Equations 5-61, 5-62 and 5-63

$$n_{1ST} = \int_{0}^{\infty} n_1 p(n_1) \, dn_1$$

$$= \int_{-\infty}^{N} n_1 \epsilon_1 \exp \left( -\epsilon_1 (n_1 - \mu_1) - \exp \left( -\epsilon_1 (n_1 - \mu_1) \right) \right) \, dn_1$$

$$+ N \left( \int_{-\infty}^{N} p(n/s, \beta_1) \, dn + \Delta P(N) \right)^{K}$$

$$- \int_{N}^{\infty} \exp \left( -\epsilon_1 (n_1 - \mu_1) - \exp \left( \epsilon_1 (n_1 - \mu_1) \right) \right) \, dn_1 \quad 5-64$$

$$+ \int_{N}^{\infty} n_1 \epsilon_2 \exp \left( -\epsilon_2 (n_1 - \mu_2) - \exp \left( -\epsilon_2 (n_1 - \mu_2) \right) \right) \, dn_1.$$
The integrals in Equation 5-64 can be evaluated using Equation II-14 so that we find
\[
\frac{n_{1ST}}{\epsilon_1} = \frac{1}{\epsilon_1} E_1 \left[ - \exp (-\epsilon_1 (N-\mu_1)) \right] + \mu_2 + \gamma / \epsilon_2 - N \exp \left[ - \exp (-\epsilon_2 (N-\mu_2)) \right]
\]
\[
- \frac{1}{\epsilon_2} E_2 \left[ \exp (-\epsilon_2 (N-\mu_2)) \right] + N \left( \int p(n/s, \beta_1) \, dn + \Delta P(N) \right).
\]

Evaluation of the performance of the ST test and the MST test

We can now evaluate the performance of the ST test and the MST test.

The Gaussian approximation will be used for \( W(Z_N) \) for computational convenience since it was shown to be reasonably accurate in this type of calculation in Chapter IV. The ST test will be compared to the SH test by making the error probabilities equal and then comparing the average sample numbers of the two tests. From Equations 5-52 and 5-53 the error probabilities of the ST test, \( \alpha_{ST} \) and \( \beta_{ST} \), are

\[
\alpha_{ST} = 1 - Q(0) = \alpha_1 \int_0^N p_1(n/0, \beta_1) \, dn
\]
\[
+ \int_{N-1}^{\infty} \frac{\log A}{\log B_2} \left( \int w(Z_n/0) \, dZ_n \right) \frac{\alpha_2 \int_0^N p_1(n/0, \beta_2) \, dn}{\int_0^N p(n/0, \beta_2) \, dn}.
\]
\[
\beta_{ST} = Q(a_1) = \beta_1 \int_0^N p_0(n/a_1, \beta_1) \, dn + (\int N-1 p(n/a_1, \beta_1) \, dn) \left( \int \log B_1 \right)
\]

\[
+ \int N-1 p(n/a_1, \beta_1) \, dn \left( \int \log B_2 + \int w(Z_N/a_1) \, dZ_N \right) \int p(n/a_1, \beta_2) \, dn
\]

where

\[
\alpha_1 = 1 - L_1(0), \beta_1 = L_1(a_1), \alpha_2 = 1 - L_2(0), \beta_2 = L_2(a_1).
\]

We then set \( \alpha = \alpha_{ST} \) and \( \beta = \beta_{ST} \) and compare \( n_{ST}(0) \) Equation 5-57 to \( n(0) \) Equation 2-12. In Figure 24 \( n_{ST}(0) \) and the equivalent \( n(0) \) are plotted versus the threshold step sample number \( N \) for threshold levels \( \beta_1 = 0.1 \) and \( \beta_2 = 0.2 \).

It can be seen that the ST test is poorer in performance than the SH test. This is to be expected since the SH test is known to be optimum.

The MST test will be compared to the M test by making \( \alpha = \alpha_{ST} \) and \( \beta = \beta_{ST} \) and then comparing \( n_{ST}(0) \) Equation 5-65 to \( n(0) \) Equation 3-2. In Figure 25 \( n_{ST}(0) \) and the equivalent \( n(0) \) are plotted versus the number of sub-tests \( K \) for a threshold step sample number \( N = 8 \) and threshold levels \( \beta_1 = 0.1 \) and \( \beta_2 = 0.2 \). It can be seen that the step in the lower threshold has resulted
Figure 24 — The average sample number $\frac{n_{ST}(0)}{n(0)}$ of the ST test compared to the average sample number $\frac{n_{ST}(0)}{n(0)}$ of an equivalent SH test versus the threshold-step sample number $N$ with initial lower threshold $\log B_1$ and final lower threshold $\log B_2$. 
Figure 25 — The average sample number $\bar{n}_1^{ST(0)}$ of the MST test compared to the average sample number $\bar{n}_1(0)$ of an equivalent M test versus the number of sub-tests $K$ with a step in the lower threshold from $\log B_1$ to $\log B_2$ at the N-th sample.
in an improvement in the performance of the MST test. However, the difference between \( n_{1ST}^{(0)} \) and \( n_{1}^{(0)} \) is practically constant so that the performance of the MST test falls off as \( K \) increases at the same rate as the M test. This modification of the M test is therefore not very promising.
A more efficient multiple hypothesis sequential test

In Chapter III we derived a sequential test for a signal with unknown parameters by analogy to the FS test. We found that the unknown range of the target required the use of K sequential sub-tests to sample and make decisions for each \( \tau \) interval. The problem then arose how to decide when to move the antenna to the next position. Two possibilities were pointed out: (1) move the antenna only when all of the sub-tests have independently terminated or (2) move the antenna when some function of the state of all the sub-tests reaches the required value. In Chapters III, IV, and V we have investigated tests of the first type; that is, the antenna is moved only when all of the sub-tests have independently decided. We have attempted to make this simple approach more efficient by modifying the structure of the sub-tests. In this chapter we will investigate a test of the second type.
In the previous chapters we have noted that the lingering sub-tests caused a serious degradation of performance. The fact that the shorter sub-tests are idle a large share of the time waiting for the longer sub-tests to end is responsible for this loss. Our approach has been to force these long sub-tests to terminate more quickly by modifying the individual sub-tests. This reduces but does not eliminate the idle time of the shorter sub-tests.

Therefore, we will analyze several tests in which some or all of the sub-tests are allowed to continue until all of the sub-tests fulfill certain conditions. This class of tests will be called the forced continuation FC test. In the FC test, type A, all of the sub-tests are allowed to continue until they have all crossed either the upper or lower threshold. In the FC test, type B, the sub-tests which cross the upper threshold are terminated whereas those which cross the lower threshold are allowed to continue. The FC test is stated as follows:

1. if \( Z_{jn} \geq \log A \),
   a) for the type A test — take another sample and set \( \delta(j) = 1 \),
   b) for the type B test — choose \( H_1 \) and set \( \delta(j) = 1 \),

2. if \( \log B < Z_{jn} < \log A \), take another sample and set \( \delta(j) = 0 \),

3. if \( Z_{jn} \leq \log B \), take another sample and set \( \delta(j) = 1 \).
4. if after the M-th sample \( \sum_{j=1}^{K} \delta(j) = K \), choose \( H_0 \) if \( Z_M^j < D \)

and choose \( H_1 \) if \( Z_M^j \geq D \),

where 
\[
Z_M^j = \sum_{i=1}^{M} a_i y_i^j - a_1^{z/2}
\]

and the superscript \( j \) refers to quantities of the \( j \)-th \( \tau \) interval. In (1) and (2) of Equation 6-1 we mean that if \( \delta(j) \) is set equal to one on any sample it does not change thereafter even if on a later sample we find that \( \log B < Z_n^j < \log A \).

Thus, we count the first crossing of \( \log A \) or \( \log B \) of each sub-test in deciding when to end the FC test. A sketch of the FC test is shown in Figure 26.

The average sample number of the FC test

It can be seen that the FC test is exactly equivalent to the M test Equation 3-1 in that both tests end when all \( K \) sub-tests have crossed the upper or lower threshold. Therefore, the probability density of the sample number of the FC test is the same as that of the M test. Likewise, the average sample number of the FC test \( n_{1FC} \) is the same as that for the M test. Therefore, from Equation 3-2 we obtain

\[
\overline{n_{1FC}(0)} = \frac{\gamma}{\epsilon_F} + \mu_F
\]

where
\[
\int_{0}^{\infty} p(n/0) \, dn = 1 - 1/K,
\]

\[
\epsilon_F = K p(\mu_F/0) .
\]
log A

log B

$\log A$

$\log B$

$D$

$M$

$n_j = \text{first crossing of } \log A \text{ or } \log B \text{ for the } j\text{th sub-test}$

$M = \text{first crossing of } \log A \text{ or } \log B \text{ for the longest of the } K \text{ sub-tests}$

Figure 26 — The forced continuation sequential test in which the sub-tests continue until all have crossed one of the thresholds.
Conditions for an error in the FC-A test

The probabilities of error for the FC test, $\alpha_{FC}$ and $\beta_{FC}$, differ from those of the M test, $\alpha$ and $\beta$, because in the FC test the sub-tests are allowed to continue after crossing a threshold. We will consider first the FC-A test, in which the sub-tests continue after crossing either the upper or the lower threshold. We will find that the type B test in which the sub-test terminates if it crosses the upper threshold can be analyzed in a similar manner.

Consider a particular sub-test, the $j$-th. From Equation 6-1 we see that $Z_n^j$ must cross either $\log A$ or $\log B$ for the first time before the end of the FC test. Let $n_0$ be the sample number at which the $j$-th sub-test crosses a threshold and let $n_1$ be the sample number at which the last of the other $K-1$ sub-tests crosses a threshold. If the $j$-th sub-test is the last to cross a threshold, $N = n_1 - n_0$ will be negative. If the $j$-th sub-test crosses a threshold before the last of the $K-1$ sub-tests, $N = n_1 - n_0$ will be positive and the $j$-th sub-test continues for $N$ more samples. We will compute in the following the probability of error when $\overline{N} < 0$ and when $N > 0$; the probability of error for the FC test will then be the sum of these probabilities.

The probability of error in the FC-A test for $N > 0$

We will first consider the case $N > 0$. If we assume that at the $n$-th sample $Z_n^j$ exactly equals the threshold rather than exceeds it, then we can say
that when the FC-A test ends at the n+N-th sample

\[ Z_{n+N}^j = \log B + \sum_{i=n+1}^{n+N} z_i = \log B + Z_N^j \]  

6-3

\[ Z_{n+N}^j = \log A + \sum_{i=n+1}^{n+N} z_i = \log A + Z_N^j \]  

6-4

according to whether the sub-test crosses log B or log A. For any sub-test in

the FC-A test either Equation 6-3 or 6-4 is true. From equation 6-1 we see

that the sub-test will decide \( H_0 \) if \( Z_{n+N}^j < D \). There are, therefore, two ways

in which a type A sub-test can decide \( H_0 \): (1) Equation 6-3 is true and \( Z_{n+N}^j < D \)
or, (2) Equation 6-4 is true and \( Z_{n+N}^j < D \).

We will first assume that Equation 6-3 is true. Under the condition that

Equation 6-3 is true the probability of deciding \( H_0 \) is

\[ \int_{-\infty}^{D} X(Z_N^j + \log B) \, dZ_N^j \]  

6-5

where \( X(Z_N^j) \) is the density function of \( Z_N^j \). In the FC-A test \( Z_N^j \) is the sum

of \( N \) gaussian random variables since the sub-tests are allowed to continue

even if they cross one of the thresholds. Therefore, the density function of

\( Z_N^j \) is

\[ X(Z_N^j) = \exp - (Z_N^j - N \bar{Z})^2 / 2N \sigma_z^2 \sqrt{2\pi N \sigma_z^2} \]  

6-6
where \( z = a_1 y - a_1^2/2 , \overline{z} = s a_1 - a_1^2/2 \), \( \sigma_z^2 = a_1^2 \). The probability that a sub-test crosses \( \log B \) is by definition the OCF of the sub-test. Since each sub-test is equivalent to an SH test Equation 2-2, the OCF is given by Equation 2-6:

\[
L(s) = \frac{A^h - 1}{A^h - B^h}
\]

where from Equation 2-8 \( h(s) = 1 - \frac{2s}{a_1} \).

The product of Equations 6-5 and 6-7 is the probability of crossing \( \log B \) and deciding \( H_0 \) under the condition that the FC test ends \( N \) samples after the crossing occurs. We must average this conditional probability over \( N \) to finally obtain the probability of crossing \( \log B \) and deciding \( H_0 \).

The probability density of \( N \)

In order to find the probability function of \( N \) we observe that the random variable \( N \) is the difference between the random variables \( n_1 \) and \( n_0 \). Now \( n_1 \), \( n_0 \), and \( N \) are discrete random variables but, as shown in Chapter II Equations 2-13 to 2-16, the sequential test sample number can be approximated by a continuous random variable. This approximation is the result of Wald's assumption that the sequential test ends with \( Z_n \) exactly equal to the threshold; we have also assumed this in writing Equations 6-3 and 6-4. Therefore, we will use Equations 4-12 to 4-14 for the probability density of the sub-test sample number \( p(n_0/s) \). The probability density of the sample number of the longest of
the K-1 sub-tests can be approximated by \( p(n_1) \) Equation II-3. The probability density of \( N \), \( g(N/s) \), is therefore given by the convolution of these probability densities:

\[
g(N/s) = \int_{-\infty}^{\infty} p(n_0/s) \, \epsilon F \exp \left[ - \epsilon F (n_0 + N - \mu F) - \exp (-\epsilon F (n_0 + N - \mu F)) \right] \, dn_0. \tag{6-8}
\]

Note that according to Equation 4-3 we can write \( p(n_0/s) \) as

\[
p(n_0/s) = L(s) p_0(n_0/s) + (1 - L(s)) p_1(n_0/s)
\]

so that we have

\[
g(N/s) = L(s) g_0(N/s) + (1 - L(s)) g_1(N/s).
\]

Then, the probability density of \( N \) under the condition that the sub-test crosses the lower threshold is

\[
g_0(N/s) = \int_0^\infty \epsilon F \exp \left[ - \epsilon F (n+N - \mu F) - \exp (-\epsilon F (n+N - \mu F)) \right] p_0(n/s) \, dn \tag{6-9}
\]

where \( p_0(n/s) \) is given by Equation 4-13. The integral of Equation 6-9 is evaluated in Appendix IV Equation IV-18 so that we find

\[
g_0(N/s) = \frac{1}{L(s)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \epsilon F \exp \left[ -(k+1) \epsilon F (N - \epsilon F) \right] \frac{H_\epsilon(k)-h}{B^2} \frac{H_\epsilon(k)-h}{A^2} \frac{A^{-H(k)}}{1 - (\frac{B}{A})^k} \tag{6-10}
\]
where

\[
\epsilon(k) = \sqrt{(k+1)} \frac{\epsilon_F}{\epsilon_F + 1}, \quad A = \frac{1 - \alpha}{\beta}, \quad B = \frac{\beta}{1 - \alpha}, \quad h = 1 - \frac{2s}{a_1}, \quad H = \left| h \right|.
\]

The probability that the sub-test will cross log B and decide \( H_0 \) when \( N > 0 \) can now be found by integrating the product of Equations 6-5, 6-7 and 6-10 over \( N \) from \( N = 0 \) to \( \infty \):

\[
\int_0^\infty g_0(N/s) (L(s) \int_{-\infty}^D X(Z_N + \log B) dZ_N) dN. \tag{6-11}
\]

Substituting for \( X(Z_N) \) from Equation 6-6, Equation 6-11 may be written as

\[
L(s) \int_0^\infty g_0(N/s) (1/2 + 1/2 \text{erf} \left( \frac{D - \log B - N \bar{Z}}{\sqrt{2N} \sigma_z^2} \right)) dN. \tag{6-12}
\]

Now assume that Equation 6-4 is true. The probability of deciding \( H_0 \) under the condition that Equation 6-4 is true is

\[
\int_{-\infty}^D X(Z_N + \log A) dZ_N. \tag{6-13}
\]

The probability that the sub-test crosses log A is by definition \( 1 - \text{OCF} \) so that from Equation 2-6 we find

\[
1 - L(s) = \frac{1 - B^h}{A^h - B^h}, \tag{6-14}
\]

where \( h = 1 - \frac{2s}{a_1} \).
The product of Equations 6-13 and 6-14 is the probability of crossing log A and deciding $H_0$ under the condition that the FC test ends $N$ samples after the crossing occurs. In order to obtain the probability of crossing log A and deciding $H_0$ we must find the probability density of $N$ under the condition that the sub-test crosses the upper threshold:

$$g_1(N/s) = \int_0^\infty \epsilon_F \exp \left[ -\epsilon_F (n+N-\mu_F) - \exp (-\epsilon_F (n+N-\mu_F)) \right] p_1(n/s) \, dn \quad 6-15$$

where $p_1(n/s)$ is given by Equation 4-14. The integral of Equation 6-15 is evaluated in Appendix IV Equation IV-23:

$$g_1(N/s) = \frac{1}{1-L(s)} \sum_{k=0}^\infty \frac{(-1)^k}{k!} \epsilon_F \exp \left[ -(k+1) \epsilon_F (n-N-\mu_F) \right] \frac{A^2}{1 - \frac{2B}{A} \frac{A^2}{A^2 - (B/A)^h \epsilon(k)}}$$

$$6-16$$

where

$$\epsilon(k) = (k+1) \epsilon_F + 1, \quad A = \frac{1-\alpha}{\beta}, \quad B = \frac{\beta}{1-\alpha}, \quad h = 1 - \frac{2s}{a_1}, \quad H = h.$$  

We can now obtain the probability that the sub-test will cross log A and decide $H_0$ for $N > 0$ from Equations 6-13, 6-14, and 6-16:

$$\int_0^\infty g_1(N) \left( 1 - L(s) \right) \int_{-\infty}^D X(Z_N + \log A) \, dZ_N \, dN.$$  

$$6-17$$
Substituting for $X(Z_N^*)$ from Equation 6-6, we find that Equation 6-17 may be written as

$$
(1 - L(s)) \int_0^\infty g_1(N) \left(1/2 + 1/2 \text{erf} \left( \frac{D - \log A - N \bar{Z}}{\sqrt{2 N \sigma_z^2}} \right) \right) dN. \quad 6-18
$$

The probability that the type A sub-test will decide $H_0$ for $N > 0$ is the sum of Equations 6-12 and 6-18:

$$
L(s) \int_0^\infty g_0(N/s) \left(1/2 + 1/2 \text{erf} \left( \frac{D - \log B - N \bar{Z}}{\sqrt{2 N \sigma_z^2}} \right) \right) dN \quad 6-19
$$

$$
+ (1 - L(s)) \int_0^\infty g_1(N/s) \left(1/2 + 1/2 \text{erf} \left( \frac{D - \log A - N \bar{Z}}{\sqrt{2 N \sigma_z^2}} \right) \right) dN
$$

where $g_0(N)$ and $g_1(N)$ are given by Equations 6-10 and 6-16.

**The error probabilities for the FC-A test**

Now consider the case $N < 0$. If we assume that the threshold $D$ in Equation 6-1 is chosen so that $\log B < D$, then by definition the sub-test decides $H_0$ if $N < 0$ and it crosses $\log B$. Therefore, if $N < 0$, the probability that the $j$-th sub-test decides $H_0$ is

$$
L(s) \int_{-\infty}^0 g_0(N/s) dN = L(s) \left(1 - \int_0^\infty g_0(N/s) dN \right). \quad 6-20
$$
The probability that the type A sub-test decides $H_0$, $FA(s)$, is the sum of Equations 6-19 and 6-20 while the probability that the sub-test decides $H_1$ is $1 - FA(s)$. Therefore the error probabilities for the FC-A test, $\alpha_{FCA}$ and $\beta_{FCA}$, are from Equations 6-19 and 6-20

$$\alpha_{FCA} = 1 - FA(0) = 1 - (1-\alpha) \left[ 1 - \int_0^\infty g_0(N/0) \, dN \right] - (1-\alpha) \int_0^\infty g_0(N/0) \, dN$$

$$(1/2 + 1/2 \text{erf} \left( \frac{D - \log B}{\sqrt{2 Na_1^2}} + \frac{\sqrt{N a_1^2}}{8} \right)) \, dN - \alpha \left[ \int_0^\infty g_1(N/0) \, dN \right]$$

$$\beta_{FCA} = FA(a_1) = \beta \left[ \int_0^\infty g_0(N/a_1) \left( 1/2 + 1/2 \text{erf} \left( \frac{D - \log B}{\sqrt{2 Na_1^2}} - \frac{\sqrt{N a_1^2}}{8} \right) \right) dN \right]$$

$$+ (1 - \beta) \left[ \int_0^\infty g_1(N/a_1) \left( 1/2 + 1/2 \text{erf} \left( \frac{D - \log A}{\sqrt{2 Na_1^2}} - \frac{\sqrt{N a_1^2}}{8} \right) \right) dN \right]$$

$$+ \beta \left[ 1 - \int_0^\infty g_0(N/a_1) \, dN \right]$$

where we have substituted for $\bar{z}$ and $\sigma_z^2$ from Equation 6-6.
The error probabilities for the FC-B test

Now consider the FC-B test in which the sub-test terminates if log A is crossed and continues if log B is crossed. Consider a sub-test which crosses log B. It will either terminate by subsequently crossing log A and deciding $H_1$ or else it will be terminated when the last sub-test crosses a threshold. It is equivalent to a TS test Equation 4-1; that is, it terminates whenever it crosses log A and otherwise it is truncated at the N-th sample. If we assume that at the n-th sample $Z_n^j$ exactly equals log B, then the sub-test is the equivalent of a TS test truncated at N with upper threshold $log A_{eq} = log A - log B$, lower threshold of $-\alpha(B_{eq} \approx 0)$ and truncation threshold $C_{eq} = D - log B$. Then the probability of deciding $H_0$ under the condition that the sub-test crosses log B, N samples before the FC test terminates, is from Equation 4-8

$$L_{eq}(s) \int_0^N p_0(n/s) \, dn + (1 - \int_0^N p(n/s) \, dn) \frac{C_{eq}}{log B_{eq}} y(Z_N) \, d Z_N$$

where the subscript "eq" denotes the equivalent TS test parameter as outlined above. Using Equations 2-15, 4-17 and 4-28, setting $s = a_1$, and substituting
for the equivalent parameters, Equation 6-22 becomes

$$\left[ 1 - \frac{1}{2} \operatorname{erfc} \left( \frac{\log A - \log B}{\sqrt{\frac{N a_1^2}{8}}} \right) - \frac{1 - \beta}{\beta \alpha} \frac{1}{2} \operatorname{erfc} \left( \frac{\log A + \log B}{\sqrt{\frac{N a_1^2}{8}}} \right) \right]$$

$$\frac{1 + \operatorname{erf} \left( \frac{D - \log B}{\sqrt{\frac{N a_1^2}{8}}} \right)}{1 + \operatorname{erf} \left( \frac{\log A - \log B}{\sqrt{\frac{N a_1^2}{8}}} \right)}$$

6-23

The probability of deciding $H_1$ given that the sub-test crosses $\log B$, $N$ samples before the FC test terminates, is from Equation 4-9

$$(1 - L_{eq}(s)) \int_0^N p_1(n/s) \, dn + (1 - \int_0^N p(n/s) \, dn) \frac{\log A_{eq}}{\log A} \frac{\int_{C_{eq}} y(Z_N) \, d Z_N}{\int_{\log B_{eq}} y(Z_N) \, d Z_N}. 6-24$$

Using Equations 2-16, 4-17 and 4-28, setting $s = 0$, and substituting for the equivalent parameters, Equation 6-24 becomes
\[
\frac{\beta a}{1-\beta} \frac{1}{2} \text{erfc} \left( \frac{\log A - \log B}{\sqrt{2/N}} - \frac{\sqrt{N/a_1^2}}{8} \right) + \frac{1}{2} \text{erfc} \left( \frac{\log A - \log B}{\sqrt{2/N}} + \frac{\sqrt{N/a_1^2}}{8} \right)
\]

\[
1 - \frac{\alpha \beta}{1-\beta} \frac{1}{2} \text{erfc} \left( \frac{\log A - \log B}{\sqrt{2/N}} - \frac{\sqrt{N/a_1^2}}{8} \right) - \frac{1}{2} \text{erfc} \left( \frac{\log A - \log B}{\sqrt{2/N}} + \frac{\sqrt{N/a_1^2}}{8} \right)
\]

\[
\frac{\text{erf} \left( \frac{\log A - \log B}{\sqrt{2/N}} + \frac{\sqrt{N/a_1^2}}{8} \right) - \text{erf} \left( \frac{\log A - \log B}{\sqrt{2/N}} - \frac{\sqrt{N/a_1^2}}{8} \right)}{1 + \text{erf} \left( \frac{\log A - \log B}{\sqrt{2/N}} + \frac{\sqrt{N/a_1^2}}{8} \right)}
\]

6-25

The probability that the sub-test will cross log B and decide \( H_0 \) with \( s = a_1 \) and \( N > 0 \) is from Equations 6-7, 6-10 and 6-23:

\[
\int_0^\infty g_0(N/a_1) \left\{ \beta \int_N p_1(n/a_1) \, dn \right\} \left( \begin{array}{c} 1 + \text{erf} \left( \frac{D - \log B}{\sqrt{2/N}} - \frac{\sqrt{N/a_1^2}}{8} \right) \\ 1 + \text{erf} \left( \frac{\log A - \log B}{\sqrt{2/N}} + \frac{\sqrt{N/a_1^2}}{8} \right) \end{array} \right) \, dN
\]

6-26

where \( \int_N p_1(n/a_1) \, dn \) is defined by Equation 6-23.

The probability that the sub-test decides \( H_0 \) for \( N < 0, \ s = a_1 \), is given by Equation 6-20. Then the probability that the sub-test of the B type test
decides $H_0$ when $s = a_1$ is the sum of Equations 6-20 and 6-26:

$$\beta_{FCB} = FB(a_1) = \beta \left[ 1 - \int_{0}^{\infty} g_0(N/a_1) \, dN \right] + \beta \int_{0}^{\infty} g_0(N/a_1) \left[ \int_{N}^{\infty} p_1(n/a_1) \, dn \right]$$

$$= \frac{\left[ 1 + \text{erf} \left( \frac{D - \log B}{\sqrt{2} N a_1^2} - \frac{\sqrt{N} a_1^2}{8} \right) \right]}{\left[ 1 + \text{erf} \left( \frac{\log A - \log B}{\sqrt{2} N a_1^2} + \frac{\sqrt{N} a_1^2}{8} \right) \right]} \, dN \quad 6-27$$

The probability that the sub-test will cross log $B$ and decide $H_1$ with $s = 0$, $N > 0$, is from Equations 6-14, 6-15 and 6-25

$$p_1(n/0) \left\{ \left[ \frac{\alpha}{1-\beta} \int_{0}^{N} p_1(n/0) \, dn \right] + \left[ 1 - \frac{\alpha}{1-\beta} \int_{0}^{N} p_1(n/0) \, dn \right] \right\}$$

$$= \frac{\text{erf} \left( \frac{\log A - \log B}{\sqrt{2} N a_1^2} + \frac{\sqrt{N} a_1^2}{8} \right) - \text{erf} \left( \frac{D - \log B}{\sqrt{2} N a_1^2} + \frac{\sqrt{N} a_1^2}{8} \right)}{\left[ 1 + \text{erf} \left( \frac{\log A - \log B}{\sqrt{2} N a_1^2} + \frac{\sqrt{N} a_1^2}{8} \right) \right]} \, dN \quad 6-28$$

where $\int_{0}^{N} p_1(n/0) \, dn$ is defined by Equation 6-25.
The probability that the sub-test will decide $H_1$ when $s = 0$ is equal to the sum of Equation 6-28 and the probability that the sub-test crosses $\log A$ first:

\[
\alpha_{FCB} = \alpha \quad \text{Equation 6-28}
\]

Comparison of the FC test to the M test

The FC test can now be compared to the M test Equation 3-1 by making

\[
\alpha_{FC} = \alpha \text{ and } \beta_{FC} = \beta \text{ and then comparing } n_1^{FC(0)} \quad \text{Equation 6-2 to } n_1^{(0)}
\]

Equation 3-2.

In Figure 27 $\overline{n_1^{FC(0)}}$ and $\overline{n_1^{(0)}}$ for the FC-A test are plotted versus the number of sub-tests $K$ for $D = 4$, $A = 9 \times 10^5$, and $B = 10^{-1}$. The parameter $D$ is the decision threshold while $\log A$ and $\log B$ are the upper and lower thresholds. It can be seen that the FC-A test achieves the desired result of decreasing the loss in performance when the number of sub-tests is increased. The FC-B test results in almost the same performance as the FC-A test; that is, the $\overline{n_1^{(0)}}$ corresponding to $\alpha_{FC-B}$ and $\beta_{FC-B}$ are essentially equal to the $n_1^{(0)}$ for the FC-A test for a particular threshold $D$. The error probabilities for the FC-A and FC-B tests are plotted as a function of $K$ in Figure 28 for $D = 4$, $A = 9 \times 10^5$ and $B = 10^{-1}$. Note that the error probabilities vary with $K$ since the sub-tests are not independent.
Figure 27 — The average sample number $n_{FC(0)}$ of the FC test compared to the average sample number $n_{M(0)}$ of the M test versus the number of sub-tests $K$ for FC test thresholds $\log A$, $\log B$, and $D$. 
Figure 28 — The error probabilities $\alpha_{FC}$ and $\beta_{FC}$ for the FC test versus the number of sub-tests $K$ for thresholds $\log A$ and $D$. 
A fixed dwell time sequential test

Up to this point we have considered the application of sequential tests to a radar which had the capability of scanning in discrete steps with the dwell time at each position being random and equal to the sequential test length. Such a radar is feasible but obviously more complex than one with a more limited scan capability. In particular it is of interest to consider a radar with a fixed dwell time and to attempt to modify the sequential test for use with it so that the performance of the FS test can be improved upon even with the fixed dwell time limitation.

If the dwell time is fixed, the antenna is moved to the next position after a predetermined number of samples. Therefore, we cannot use the sequential approach to decrease the average number of samples. Instead, we will use a sequential type of test to decrease the probability of error when a target is present. This can be done by using the upper threshold log A to control a step increase in power. When log A is crossed, it is likely to be a target and so the
power is increased until all of the samples have been taken at that antenna position. A sketch of a typical sub-test is shown in Figure 29. This modification will be applied to the SH test Equation 2-2 and to the M test Equation 3-1. The modification of the SH test will be called the fixed dwell time FD test while the modification of the M test will be denoted the MFD test. The FD and MFD tests differ from the SP and MSP tests of Chapter V in that in the FD and MFD tests the step in power occurs only when the upper threshold log A is crossed instead of when the N-th sample is reached. It can be seen that the lower threshold is of no use in this application since we are constrained to terminate the test after a fixed number of samples. The FD test can now be stated as

1. if $Z_M < \log A$ decide $H_0$, 

2. if $Z_n > \log A$ where $n < M$, continue sampling but change the test to:

3. if $\log A + Z_{M-n} < V$ decide $H_0$, 

4. if $\log A + Z_{M-n} \geq V$ decide $H_1$, 

where $V > \log A$, 

$$z_1 = a_1 y_i - a_1 z_i / 2, \quad Z_n = \sum_{i=1}^{n} z_i,'$$

$$x_i = \sqrt{k} a_1 y_i - k a_i z_i / 2, \quad Z_{M-n} = \sum_{i=n+1}^{M} x_i,'$$

$M =$ termination sample number.
Figure 29 — An adaptation of sequential tests to a fixed-dwell time pulsed radar in which the design signal level is increased if the threshold $\log A$ is crossed.
The MFD test consists of $K$ sub-tests, each of which is an FD test. The test is ended after $M$ samples have been taken and the step increase in design signal level occurs when the $Z_n$ of any of the $K$ sub-tests crosses the upper threshold $\log A$. Then the MFD test can be stated as

1. if $Z_{M}^{j} < \log A$ decide $H_0$,

2. if in any of the $K$ sub-tests $Z_{n}^{j} > \log A$ where $n < M$, continue sampling but change all $K$ sub-tests to:

3. if $Z_{n}^{j} + Z_{M-n}^{j} < V$ decide $H_0$

4. if $Z_{n}^{j} + Z_{M-n}^{j} \geq V$ decide $H_1$

where $V > \log A$,

$$z_{i}^{j} = a_{1} y_{i}^{j} - a_{1} \frac{2}{2}, \quad Z_{n}^{j} = \sum_{i=1}^{n} z_{i}^{j},$$

$$x_{i}^{j} = \sqrt{k} a_{1} y_{i}^{j} - k a_{1} \frac{2}{2}, \quad Z_{M-n}^{j} = \sum_{i=n+1}^{M} x_{i}^{j},$$

superscript $j = j$-th of the $K$ sub-tests,

$M = \text{termination sample number},$

$k > 1.$

It can be seen that the MFD test is exactly like the FD test except that the step increase in design signal level occurs when any one of the $Z_{n}^{j}$ of the $K$ sub-tests exceeds $\log A$. 
Probability of error of the FD test

From Equation 7-1 it can be seen that the FD test can result in the decision \( H_0 \) in either of two ways

\[
Z_n < \log A \text{ for all } n, \ 0 < n < M, \quad 7-3
\]

or

\[
\log A + Z_{M-n} < V \quad 7-4
\]

where we have assumed that \( Z_n \) exactly equals \( \log A \) rather than exceeds it.

The probability that Equation 7-3 is true is easily obtained by noting that the probability that a sequential test crosses \( \log A \) at or before the \( M \)-th sample is by definition

\[
(1 - L(s)) \int_0^M p_1(n/s) \ dn. \quad 7-5
\]

Then, the probability that Equation 7-3 is true is

\[
1 - (1 - L(s)) \int_0^M p_1(n/s) \ dn. \quad 7-6
\]

From Equation 7-4 we see that if \( \log A \) is crossed at the \( n \)-th sample then the probability of deciding \( H_0 \) is

\[
\int_{-\infty}^V W(Z_{M-n} + \log A) \ d Z_{M-n} \quad 7-7
\]
where

\[ Z_{M-n} = \sum_{i=n+1}^{M} x_i. \]

Now \( n \) is a random variable which is the sample number of a sequential test. Therefore, the conditional probability density of \( n \) given that \( \log A \) is crossed is given by

\[ p_{1}(n/s) \]

where \( p_{1}(n/s) \) is obtained from Equation 4-14. The probability that the test crosses \( \log A \) is given by Equation 2-6

\[ 1 - L(s). \]

Then the probability that Equation 7-4 is true is from Equations 7-7, 7-8 and 7-9

\[
\int_{0}^{M} (1-L(s)) p_{1}(n/s) \left\{ \int_{-\infty}^{V} W(Z_{M-n} + \log A) \, dZ_{M-n} \right\} \, dn. \]

From Equation 7-2 we see that \( Z_{M-n} \) is the sum of \( M-n \) gaussian random variables, \( x_i \):

\[
W(Z_{M-n}) = \frac{\exp - (Z_{M-n} - (M-n) \bar{x})^2 / 2 (M-n) \sigma_x^2}{\sqrt{2\pi (M-n) \sigma_x^2}}. \]
Using Equation 7-11 in Equation 7-10, we find that the probability that Equation 7-4 is true is given by

\[
\int_0^M (1-L(s)) p_1(n/s) \left( \frac{1/2 + 1/2 \text{erf} \left( \frac{V - \log A - (M-n) \bar{x}}{\sqrt{2} (M-n) \sigma_x^2} \right)}{\sqrt{2} (M-n) \sigma_x^2} \right) \text{dn.} 7-12
\]

The probability that the FD test will decide \( H_0 \), \( D(s) \), is then the sum of Equations 7-6 and 7-12:

\[
D(s) = 1 - (1-L(s)) \int_0^M p_1(n/s) \text{dn} + \int_0^M (1-L(s)) p_1(n/s) \left( \frac{1/2 + 1/2 \text{erf} \left( \frac{V - \log A - (M-n) \bar{x}}{\sqrt{2} (M-n) \sigma_x^2} \right)}{\sqrt{2} (M-n) \sigma_x^2} \right) \text{dn.} 7-13
\]

where \( \bar{x} = \kappa a_1 - k a_1^{2/2}, \sigma_x^2 = k a_1^2 \).

The probabilities of error for the FD test, \( \alpha_{FD} \) and \( \beta_{FD} \), are from Equation 7-13

\[
\alpha_{FD} = 1 - D(0) = \int_0^M p_1(n/0) \frac{1}{2} \text{erfc} \left( \frac{V - \log A + (M-n) \kappa a_1^{2/2}}{\sqrt{2} (M-n) \kappa a_1^2} \right) \text{dn.} 7-14
\]

\[
\beta_{FD} = D(a_1) = 1 - \int_0^M p_1(n/a_1) \frac{1}{2} \text{erfc} \left( \frac{V - \log A - (M-n) \kappa a_1^{2/2}}{\sqrt{2} (M-n) \kappa a_1^2} \right) \text{dn.} 7-15
\]

where \( \log A = \log 1/a \).
Averag e design signal level of the FD test

In order to compare the FD test to the FS test Equation 2-7 we will make $\alpha_{FD}$, $\beta_{FD}$ and $M$ equal to $\alpha_{FS}$, $\beta_{FS}$ and $q$ Equations 1-8 and 1-9. We will then compare the design signal level of the FS test $P_{FS}$ to the average design signal level of the FD test $P_D$. The average design signal level of the FD test can be obtained by dividing the tests into two groups, those which cross log A before $n = M$ and those which do not cross log A. For the tests which do not cross log A the design signal level is $a_1^2$ for the $M$ samples duration of the test. Then, the average design signal level, considering only the tests which do not cross log A, is

$$a_1^2 (1 - \int_0^M \alpha p_1(n/0) \, dn). \tag{7-16}$$

For the tests which cross log A the design signal level is $a_1^2$ for $n$ samples and $k a_1^2$ for $M-n$ samples. We can therefore find the average for the tests which cross log A by averaging the function

$$\frac{n a_1^2 + (M-n) k a_1^2}{M} :$$

$$\int_0^M \left( \frac{n a_1^2 + (M-n) k a_1^2}{M} \right) \alpha p_1(n/0) \, dn. \tag{7-17}$$

Then, the average design signal level $P_D$ is the sum of Equations 7-16 and 7-17:

$$P_D = a_1^2 + a_1^2(k-1) \int_0^M \alpha p_1(n/0) \, dn - \frac{a_1^2(k-1)}{M} \int_0^M \alpha n p_1(n/\theta) \, dn. \tag{7-18}$$
Probability of error of the MFD test

From Equation 7-2 it can be seen that for the MFD test the step increase in design signal level will occur when one of the K sub-tests crosses log A. If more than one sub-test crosses log A, the step increase occurs at the first crossing. Now the conditional probability density of $n$, given that log A is crossed, is from Equation 7-8

$$p_1(n/s).$$

7-19

Suppose that $j$ sub-tests cross log A. The sample number $n_a$ of the sub-test which crosses log A first of the $j$ sub-tests is a random variable which is the smallest of $j$ random variables. Therefore, the conditional probability density of the sample number $n_a$ of the first sub-test out of $j$ to cross log A, given that $j$ sub-tests have crossed log A, is

$$n_a \left[ j (1 - \int_0^{p_1(n/s)} dn) j-1 \right]^1 \ p_1(n_a/s).$$

7-20

The probability that exactly 1 sub-test out of the K crosses log A is

$$\alpha (1 - \alpha)^{K-1}$$

7-21

while the probability that exactly $j$ sub-tests out of the K cross log A is

$$\binom{K}{j} \alpha^j (1 - \alpha)^{K-j}.$$  7-22

---

The probability density of the sample number of the sub-test that crosses log $A$ first out of $j$ is the product of Equations 7-20 and 7-22:

$$\binom{K}{j} a^j (1-\alpha)^{K-j} j (1 - \int_0^n p_1(n/s) \, dn)^{j-1} p_1(n/s).$$  

7-23

Finally, the probability density of the sample number of the sub-test that crosses first is the sum of Equation 7-23 with $j = 1, 2, 3, \ldots K$:

$$\sum_{j=1}^{K} \binom{K}{j} a^j (1-\alpha)^{K-j} j [1 - \int_0^n p_1(n/s) \, dn]^{j-1} p_1(n/s).$$  

7-24

Expanding Equation 7-24 we have for the first 3 terms

$$(K \alpha (1-\alpha)^{K-1}) p_1(n/s) (1 + \frac{(K-1) \alpha}{1-\alpha} + \frac{(K-1) (K-2) \alpha^2}{2 (1-\alpha)^2}$$

$$- (K(K-1) \alpha^2 (1-\alpha)^{K-2}) p_1(n/s) \int_0^n p_1(n/s) \, dn .$$  

7-25

Typical values for $K$ and $\alpha$ are $10^3$ and $10^{-6}$, respectively, so that the second term of Equation 7-25 is approximately $10^{-3}$ times the first term. We therefore approximate Equation 7-24 by

$$(K\alpha + K(K-1) \alpha^2) p_1(n/s)$$  

7-26

Note that in Equations 7-19 through 7-26 we have assumed that the true signal level is $s$ in all of the $K$ sub-tests. When the signal is not present, this will be true and $s = 0$. However, when the signal is present, one sub-test will have $s = a_1$ before the step while the other $K-1$ tests will have $s = 0$. Now the
conditional probability density of \( n \), given that \( \log A \) is crossed when \( s = a_1 \), is from Equation 7-19

\[ p_1(n/a_1). \] 7-27

Also, the probability that the sub-test, where \( s = a_1 \), crosses \( \log A \) is from Equation 7-9

\[ 1 - L(a_1). \] 7-28

But, since the lower threshold is at \(-\infty\), the probability of crossing the lower threshold \( L(a_1) \) is zero so that Equation 7-28 reduces to unity. Therefore, the probability density of \( n \) for the sub-test where \( s = a_1 \) is given by Equation 7-27.

Comparing Equation 7-27 to Equation 7-26 we see that the contribution of the \( K-1 \) sub-tests where \( s = 0 \) to the probability density of \( n \) is negligible compared to the sub-test where \( s = a_1 \). Therefore, when \( s = a_1 \) in one of the \( K \) sub-tests, the probability density of the sample number of the sub-test that crosses \( \log A \) first is given by Equation 7-27:

\[ p_1(n/a_1). \] 7-29

From Equation 7-2 it can be seen that the MFD test can result in the decision \( H_0 \) in either of two ways:

\[ Z_n^j < \log A \text{ for all } n, \ 0 < n < M, \text{ and for all } j, 1 \leq j \leq K, \] 7-30

or

\[ Z_n^j + Z_{M-n}^j < V. \] 7-31
The probability that Equation 7-30 is true is just the probability that \( \log A \) was not crossed for \( 0 < n < M \) so that from Equations 7-26 and 7-29 we have

\[
1 - \int_{0}^{M} (K\alpha + K(1)\alpha^2) \ p_1(n/0) \ dn \quad 7-32
\]

when \( s = 0 \) in all \( K \) sub-tests,

\[
1 - \int_{0}^{M} p_1(n/a_1) \ dn \quad 7-33
\]

when \( s = a_1 \) in one sub-test and \( s = 0 \) in the other \( K-1 \) sub-tests.

From Equation 7-31 we see that \( H_0 \) will also be decided if \( \log A \) is crossed at \( n \) and if \( X_M < V \). The probability that \( X_M < V \) is

\[
\int_{-\infty}^{V} W(X_M) \ dX_M \quad 7-34
\]

where

\[
X_M = Z_n + Z_{M-n}, \quad \text{when } s = 0 \text{ in all } K \text{ sub-tests,}
\]

\[
= \log A + Z_{M-n}, \quad \text{when } s = a_1 \text{ in one sub-test.}
\]

But \( Z_n \) is the sum of \( n \) gaussian random variables \( z_i \) so that the probability density of \( Z_n \) is given by

\[
\frac{\exp\left(-\frac{(Z_n - n\bar{z})^2}{2n\sigma^2}ight)}{\sqrt{2\pi n\sigma^2}} \quad 7-35
\]
Also, $Z_{M-n}$ is the sum of $M-n$ gaussian random variables $x_i$ so that the probability density of $Z_{M-n}$ is given by

$$
\frac{\exp - \left(\frac{(Z_{M-n} - (M-n) \bar{x})^2}{2(M-n)} \right)}{\sqrt{2\pi (M-n) \sigma_x^2}}.
$$

From Equation 7-1 $z_i = a_1 y_i - a_1^2/2$ so that $\bar{z} = (s a_1 - a_1^2/2)$, $\sigma_x^2 = a_1^2$

while from Equation 7-2 $x_i = k a_1 y_i - k a_1^2/2$ so that $\bar{x} = (k s a_1 - k a_1^2/2)$, $\sigma_x^2 = k a_1^2$. Finally, from Equations 7-35 and 7-36 we find that the probability density $W(X_M)$ is given by

$$
W(X_M) = \frac{\exp - \left(\frac{(X_M - (n \bar{z} + (M-n) \bar{x}))^2}{2(n\sigma_z^2 + (M-n) \sigma_x^2)}\right)}{\sqrt{2\pi (n\sigma_z^2 + (M-n) \sigma_x^2)}}.
$$

when $s = 0$ in all $K$ sub-tests or,

$$
W(X_M) = \frac{\exp - \left(\frac{(X_M - (M-n) \bar{x} - \log A)^2}{2 (M-n) \sigma_x^2}\right)}{\sqrt{2\pi (M-n) \sigma_x^2}}
$$

when $s = a_1$ in one sub-test, where $\bar{z} = s a_1 - a_1^2/2$, $\bar{x} = k \bar{z}$, $\sigma_z^2 = a_1^2$, $\sigma_x^2 = k a_1^2$.

From Equations 7-26, 7-34, and 7-37 the probability that Equation 7-31 is true when $s = 0$ in the $K$ sub-tests is found to be

$$
\int_{0}^{M} \left(1/2 + 1/2 \text{erf} \left(\frac{V + M k a_1^2/2 - n (k-1) a_1^2/2}{\sqrt{2 (Mk a_1^2 - n (k-1) a_1^2)}}\right)\right) dn
$$

7-38
From Equations 7-29, 7-34, and 7-37 the probability that Equation 7-31 is true when \( s = a_1 \) in one of the sub-tests and \( s = 0 \) in the other \( K-1 \) sub-tests is given by

\[
\int_0^M p_1(n/a_1) \left[ 1/2 + 1/2 \text{erf} \left( \frac{V - \log A - (M-n) k a_1^{2/2}}{\sqrt{2 (M-n) k a_1^{2}}} \right) \right] \text{d}n. \tag{7-39}
\]

The probability \( D_M(0) \) that the MFD test decides \( H_0 \) is the sum of the probabilities that Equations 7-30 and 7-31 are true. Therefore, when \( s = 0 \) in \( K \) sub-tests, the probability that the MFD test decides \( H_0 \) is the sum of Equations 7-32 and 7-38

\[
D_M(0) = 1 - \int_0^M (K\alpha + K(K-1) \alpha^2) p_1(n/0) \text{d}n
\]

\[
+ \int_0^M (K\alpha + K(K-1) \alpha^2) p_1(n/0) \left[ 1/2 + 1/2 \text{erf} \left( \frac{V + M k a_1^{2/2} - n(k-1) a_1^{2/2}}{\sqrt{2 (M k a_1 - n(k-1) a_1^{2})}} \right) \right] \text{d}n. \tag{7-40}
\]

The probability of false alarm for the MFD test \( \alpha_{MFD} \) is from Equation 7-40

\[
\alpha_{MFD} = 1 - D_M(0)
\]

\[
= \int_0^M (K\alpha + K(K-1) \alpha^2) p_1(n/0) \frac{1}{2} \text{erfc} \left( \frac{V + M k a_1^{2/2} - n(k-1) a_1^{2/2}}{\sqrt{2 (M k a_1 - n(k-1) a_1^{2})}} \right) \text{d}n. \tag{7-41}
\]
When $s = a_1$ in one of the sub-tests and $s = 0$ in the other K-1 sub-tests, the probability that the MFD test decides $H_0$ is the sum of Equations 7-25 and 7-31:

$$D_M(a_1) = 1 - \int_0^M p_1(n/a_1) \, dn + \int_0^M p_1(n/a_1) \left( \frac{1}{2} + \frac{1}{2} \text{erf} \left( \frac{V - \log A - (M-n) k a_1^2/2}{\sqrt{2(M-n) k a_1^2}} \right) \right) \, dn$$  \hspace{1cm} 7-42

or

$$\beta_{MFD} = 1 - \int_0^M p_1(n/a_1) 1/2 \text{erfc} \left( \frac{V - \log A - (M-n) k a_1^2/2}{\sqrt{2(M-n) k a_1^2}} \right) \, dn.$$  \hspace{1cm} 7-43

**Average design signal level of the MFD test**

The average design signal level of the MFD test $\overline{P_D}$ is obtained in the same manner as $\overline{P_D}$ Equation 7-18. The average design signal level for those tests which do not cross $\log A$ is

$$a_1^2 (1 - \int_0^M (K\alpha + K(K-1) \alpha^2) p_1(n/0) \, dn).$$  \hspace{1cm} 7-44

The average design signal level for those tests which cross $\log A$ is

$$\int_0^M \left( \frac{a_1^2 n + k a_1^2 (M-n)}{M} \right) (K\alpha + K(K-1) \alpha^2) p_1(n/0) \, dn.$$  \hspace{1cm} 7-45
Then, $P_{\text{MD}}$ is the sum of Equations 7-44 and 7-45

$$P_{\text{MD}} = a_1^2 + a_1^2(k-1) (K\alpha + K(K-1) \alpha^2) \int_0^M p_1(n/0) \, dn$$

$$- \frac{a_1^2 (k-1)}{M} (K\alpha + K(K-1) \alpha^2) \int_0^M n p_1(n/0) \, dn.$$

\textit{Comparison of FD and MFD tests to the FS test}

In order to compare the FD and MFD tests to the FS test they must be equivalent. We will therefore require that the error probabilities achieved be equal and also that the test lengths be the same. The average signal levels required to achieve this performance can then be compared.

The FD test is compared to the FS test in Figures 30 and 31 where the ratio $P_{\text{FS}}/P_{\text{D}}$ is plotted versus $V$ for various values of $k$ and $A$. In Figure 30 $P_{\text{FS}}/P_{\text{D}}$ is plotted for $M = 16$, $A = 10^3$, and $k = 2, 4$ and 6. In Figure 31 $P_{\text{FS}}/P_{\text{D}}$ is plotted for $M = 16$, $k = 2$ and $A = 10^2, 10^3, 10^4$ and $10^5$. The parameters, $P_{\text{FS}}, P_{\text{D}}, V, k, A$ and $n$ are the signal level of the FS test, the average signal level of the FD test, the FD test decision threshold, the ratio of signal levels before and after the step increase, the power step decision threshold, and the test length. It can be seen that the FD test yields an improvement in performance over the FS test even though it is constrained to the same length
Figure 30 — The ratio of the signal level $P_{FS}$ required by the FS test to the average signal level $P_D$ of an equivalent FD test versus the FD test decision threshold level $V$ for various magnitudes of step in signal level $k$. 
Figure 31 — The ratio of the signal level $P_{FS}$ required by the FS test to the average signal level $P_D$ of an equivalent FD test versus the decision threshold $V$ for various values of the signal level threshold $\log A$. 
of test. This is attained at the cost of providing the capability of the step in signal level.

The MFD test is compared to the FS test in Figures 32 and 33 where \( \frac{P_{FS}}{P_{MD}} \) is plotted versus \( V \) for various values of \( k, A \) and \( K \). Here \( K \) represents the number of sub-tests in the MFD test. In Figure 32 \( \frac{P_{FS}}{P_{MD}} \) is plotted for \( M = 16, A = 10^4, K = 500 \) and \( k = 2, 4 \) and 6. In Figure 33 \( \frac{P_{FS}}{P_{MD}} \) is plotted for \( M = 16, A = 10^4, k = 2 \) and \( K = 100 \) and 1000. The MFD test is shown to represent a considerable improvement in performance over the FS test. The improvement slowly declines as \( K \) increases but it is still considerable at \( K = 1000 \).
Figure 32 — The ratio of the signal level $P_{FS}$ of the FS test to the average signal level $P_{MD}$ of the MFD test versus the decision threshold $V$ for various magnitudes of signal level step $k$. 

Test length = $M = 1K$
Number of sub-tests = $K = 001$
Signal level threshold = $\log A = 3.2$
Figure 33 — The ratio of the signal level $P_{FS}$ of the FS test to the average signal level $\overline{P_{MD}}$ of the MFD test versus the decision threshold $V$ for several values of the number of sub-tests $K$. 

Test length = $M = 16$

Signal level step = $k = 2$

Signal level threshold = $\log A = 0.3$
CHAPTER VIII

EXACT ANALYSIS OF SEQUENTIAL TESTS

Exact performance equations of sequential tests

The methods used in Chapter IV to derive Equations 4-27 and 4-30 can be used to obtain the exact probability density of the sample number and the exact probabilities of error for any sequential type test. Consider for example a sequential test as follows:

1. if $Z_n > \log A$ decide $H_1$

2. if $Z_n \leq \log B$ decide $H_0$

3. if $\log B < Z_n < \log A$ take another sample

where $Z_n = \sum_{i=1}^{n} z_i$ and $z_i$ is a function of the $i$-th sample of a random process.

If the condition $\log B < Z_{n-1} < \log A$ holds, the $n$-th sample $z_n$ is taken and $Z_n = Z_{n-1} + z_n$ is formed. Therefore, $Z_n$ is the sum of the random variables $Z_{n-1}$ and $z_n$. The probability density of $Z_n$ under the condition that $\log B < Z_{n-1} < \log A$ is
\[
y(Z_n) = \int_{-\infty}^{\infty} g^*(Z_{n-1}) \omega(Z_n - Z_{n-1}) \, dZ_{n-1}
\]

where

\[g^*(Z_{n-1}) = \text{probability density of } Z_{n-1} \text{ under the condition that}
\]

\[\log B < Z_{n-1} < \log A,
\]

\[\omega(z_n) = \text{probability density of } z_n.
\]

Now, the sequential test will end on the n-th sample if \( Z_n < \log B \) or \( Z_n > \log A \) and if \( \log B < Z_{n-1} < \log A \). The probability that \( \log B < Z_{n-1} < \log A \) is equal to the probability that the sequential test has not ended on or before the n-1st sample:

\[
1 - \sum_{i=1}^{n-1} p(i)
\]

where \( p(i) \) is the probability that the sequential test ends on the i-th sample.

The probability \( p(n) \) that \( Z_n < \log B \) or \( Z_n > \log A \) and that \( \log B < Z_{n-1} < \log A \) is from Equations 8-2 and 8-3

\[
p(n) = \left(1 - \sum_{i=1}^{n-1} p(i) \right) \left(1 - \int_{\log B}^{\log A} y(Z_n) \, dZ_n \right).
\]

When the first sample \( z_1 \) is taken, the probability density of \( Z_1, y(Z_1) \), is equal to \( \omega(z_1) \). The probability that the sequential test ends on the first
sample is then
\[
p(1) = 1 - \log A \int_{\log B} y(Z_1) \, dZ_1. \tag{8-5}
\]

From Equations 8-4 and 8-5 the probability that the sequential test ends on the second sample is
\[
p(2) = (1 - p(1)) \left(1 - \int_{\log B} y(Z_2) \, dZ_2\right) \tag{8-6}
\]
\[
= (1 - \mu_1) \mu_2
\]

where
\[
\mu_1 = 1 - \int_{\log B} y(Z_1) \, dZ_1
\]
and
\[
\mu_k = 1 - \int_{\log B} y(Z_k) \, dZ_k.
\]

Also,
\[
p(3) = (1 - p(1) - p(2)) \left(1 - \int_{\log B} y(Z_3) \, dZ_3\right)
\]
\[
= (1 - \mu_1 - (1 - \mu_1) \mu_2) \mu_3 \tag{8-7}
\]
\[
= (1 - \mu_1) (1 - \mu_2) \mu_3
\]
and
\[
p(n) = (1 - \sum_{i=1}^{n-1} P(i)) (1 - \int \log B y(Z_n) d Z_n).
\]

\[= \pi (1 - \mu_i) \mu_n.\]

The sequential test must end by deciding either \(H_0\) or \(H_1\). The sequential test decides \(H_0\) on the \(n\)-th sample if \(Z_n < \log B\) and if \(\log B < Z_{n-1} < \log A\). Then, from Equations 8-2 and 8-3 the probability of deciding \(H_0\) on the \(n\)-th sample is
\[
(1 - \sum_{i=1}^{n-1} P(i)) \int_{-\infty}^{\log B} y(Z_n) d Z_n. \tag{8-9}
\]

Using Equation 8-7, this can be written
\[
\pi (1 - \mu_i) b_n. \tag{8-8}
\]

Similarly, the probability of deciding \(H_1\) on the \(n\)-th sample is
\[
(1 - \sum_{i=1}^{n-1} P(i)) \int \log A y(Z_n) d Z_n. \tag{8-10}
\]

From Equation 8-8 it can be seen that Equations 8-9 and 8-10 can be written as
\[ n-1 \sum_{i=1}^{\infty} (1 - \mu_i) b_n \] 
\[ \pi \]

where

\[ a_n = \int_{\log A}^{\infty} y(Z_n) \, dZ_n \]
\[ b_n = \int_{-\infty}^{\log B} y(Z_n) \, dZ_n \]

The probability \( L(s) \) of deciding \( H_0 \) for the sequential test is the probability of deciding \( H_0 \) on the first sample or the second or the third, etc. so that from Equation 8-11 we find

\[ L(s) = \sum_{n=1}^{\infty} \sum_{i=1}^{n-1} \pi (1 - \mu_i) b_n \]

Also, from Equations 8-4, 8-10 and 8-11 it can be seen that

\[ p(n) = \sum_{i=1}^{n-1} \pi (1 - \mu_i) b_n + \sum_{i=1}^{n-1} \pi (1 - \mu_i) a_n \]

where

\[ a_n = \int_{\log A}^{\infty} y(Z_n) \, dZ_n \]

\[ b_n = \int_{-\infty}^{\log B} y(Z_n) \, dZ_n \]
But \( p(n) \) can be expressed in terms of conditional probability densities as shown in Equation 4-3

\[
p(n/s) = L(s) p_0(n/s) + (1 - L(s)) p_1(n/s).
\]

From Equations 8-11, 8-12, and 8-13 we find that the conditional probability, \( p_0(n/s) \), of deciding on the \( n \)-th sample given that \( H_0 \) is decided is

\[
p_0(n/s) = \sum_{n=1}^{\infty} \left( \prod_{i=1}^{n-1} (1 - \mu_i) b_n \right) \frac{n-1}{\pi} (1 - \mu_i) a_n.
\]

Similarly, the conditional probability \( p_1(n/s) \) of deciding on the \( n \)-th sample given that \( H_1 \) is decided is

\[
p_1(n/s) = \prod_{i=1}^{n-1} (1 - \mu_i) a_n / \sum_{n=1}^{\infty} \left( \prod_{i=1}^{n-1} (1 - \mu_i) a_n \right) \frac{n-1}{\pi} (1 - \mu_i) a_n.
\]

**Derivation of exact performance equations for a practical sequential detector**

With Equations 8-8, 8-11, 8-14 and 8-15 we have exact equations from which the performance of any sequential test of the type given by Equation 8-1 can be obtained. Note that there is no requirement that \( Z_n \) represent a likelihood ratio as in Wald's test Equation 2-1. It often happens in practice that the likelihood ratio is a complicated function so that a test is constructed using an approximation to the Wald test. In such a case Wald's equations for the
probability density of the sample number would be approximations since they assume that the exact likelihood ratio is used.

A common problem in practical radar systems is to determine the effect of a randomly fluctuating signal on the performance of the radar. This occurs because the equivalent "radar cross section" of the target fluctuates randomly due to small target motions. We will apply the exact equations derived above to the analysis of a sequential test where an approximation of the likelihood ratio must be used and where the signal is fluctuating randomly.

When the doppler frequency \( \omega_D \) and phase angle \( \theta \) of the received signal Equation 1-1 are unknown, a square law envelope detector is often used. Referring to Equation 1-4, the envelope detector forms the function

\[
R^2 = (S(t) + X_c(t))^2 + X_s(t)^2
\]

and from Equation 1-3 we find that \( X_c(t) \) and \( X_s(t) \) are gaussian random variables with

\[
E(X_c(t)) = E(X_s(t)) = 0
\]

\[
\sigma^2(X_c(t)) = \sigma^2(X_s(t)) = N_0.
\]

The probability density of \( R \) is given by Rice

\[
p(R) = \frac{R}{N_0} \exp \left( - \frac{R^2 + S^2}{2N_0} \right) I_0 \left( \frac{RS}{N_0} \right), \quad R > 0
\]

where $I_0$ is the Bessel function with imaginary argument. If we let

$$ x = \frac{S^2}{2N_0} \quad \text{and} \quad v = \frac{R^2}{2N_0}, $$

we obtain

$$ p(v/x) = \exp(-v - x) I_0(2\sqrt{vx}), \quad v > 0 \quad 8-19 $$

$$ = 0, \quad v \leq 0. $$

Note that $x = \frac{S^2}{2}$ in Equation 2-2. Also when $x = 0$, we find

$$ p(v/0) = \exp(-v), \quad v > 0 \quad 8-20 $$

$$ = 0, \quad v \leq 0. $$

We can now form the probability ratio using Equations 8-19 and 8-20 and thus obtain an optimum sequential probability ratio test Equation 2-1 of the detector output:

$$ \lambda_n = \left( \frac{p(v_n/\mu)}{p(v_n/0)} \right) \cdots \left( \frac{p(v_n/\mu)}{p(v_n/0)} \right) \quad 8-21 $$

$$ = (e^{-\mu} I_0(2\mu v_n)) \cdots (e^{-\mu} I_0(2\sqrt{\mu} v_n)) $$

or

$$ \log \lambda_n = \sum_{i=1}^{n} -\mu + \log I_0(2\sqrt{\mu} v_n) \quad 8-22 $$

where $\mu$ is the design value of $\frac{S^2}{2N_0}$, the equivalent of $a_{1/2}^2$ in Equation 2-2.
It isn't practical to implement the test of Equation 8-22 because of the complexity of the \( \log I_0 \) function. Therefore, we seek a suitable approximation to Equation 8-22. We will follow Bussgang and Middleton's analysis of a similar problem. Expanding \( \log I_0 \) in a series, we find

\[
z = -\mu + \log I_0 (2 \sqrt{\mu} v) \tag{8-23}
\]

\[
= -\mu + \log (1 + \mu v + \frac{\mu^2 v^2}{4} + \cdots)
\]

\[
= -\mu + (\mu v + \frac{\mu^2 v^2}{4}) - 1/2 (\mu^2 v^2 + \frac{\mu^3 v^3}{2} + \frac{\mu^4 v^4}{16}) + \cdots
\]

\[
= -\mu + \mu v - \frac{\mu^2 v^2}{4} + \cdots \tag{8-24}
\]

Now, assume that we approximate \( z_1 \) by

\[
z_1 = -\mu + \mu v_1 \tag{8-25}
\]

Then, if the actual signal level is \( x \), the mean value of \( z \) is

\[
\overline{z(x)} = -\mu + \mu (1 + x). \tag{8-26}
\]

When \( x = 0 \), we find

\[
\overline{z(0)} = 0. \tag{8-27}
\]

\[\text{\textsuperscript{2}}\text{J. J. Bussgang and D. Middleton, page 22.}\]

\[\text{\textsuperscript{3}}\text{J. I. Marcum, "Studies of Target Detection by Pulsed Radar," IEEE Trans. on Information Theory, April 1960, page 175.}\]
From Wald's relation for \( n(0) \) Equation 2-10 we see that this approximation for \( z \) would result in an infinite \( n(0) \). Instead, assume that we use as the approximation for \( z_1 \)

\[
z_1 = -\mu + \mu v_i - \frac{\mu^2 v_i^2}{4}.
\]  

8-28

Then, the mean value of \( z \) is

\[
z(x) = -\mu + \mu (1 + x) - \frac{\mu^2}{4} (x^2 + 4x + 2)
\]  

8-29

which for \( x = 0 \) is

\[
z(0) = -\frac{\mu^2}{2}.
\]  

8-30

Therefore, a better approximation to \( z_1 \) can be obtained by adding Equation 8-30 to Equation 8-25 so that we have

\[
z_1 = -\mu - \frac{\mu^2}{2} + \mu v_i.
\]  

8-31

The sequential test which approximates the Wald sequential probability ratio test is, therefore

\[
\log B < \sum_{i=1}^{n} \frac{-\mu + \mu^2 / 2 + \mu v_i}{\log A}.
\]  

8-32

Assume now that the received signal is fluctuating because of fluctuations of the equivalent radar cross section of the target. Therefore, the signal level \( x \) is a random variable. We will assume that the target fluctuation is Swerling
Type II\(^4\); that is, the fluctuations are independent between adjacent samples
and the probability density of \(x\) is given by

\[
\omega(x/x) = \frac{\exp(-x/\bar{x})}{\bar{x}}.
\]

The probability density of \(v\) is found from Equations 8-19 and 8-33 as shown by
Swerling\(^5\):

\[
p(v/x) = \int_0^\infty \omega(x/x) p(v/x) \, dx
\]

\[
= \frac{\exp(-v/(1+\bar{x})}{1+\bar{x}}, \quad v > 0.
\]

Now using Equations 8-2, 8-4, 8-11 and 8-34 we can compute the exact probability of error and also the exact \(p(n)\) for the sequential test of Equation 8-32.

First, let us rewrite the test of Equation 8-32 as follows:

\[
B_n < Z_n < A_n
\]

where

\[
B_n = \log B + n (\mu + \mu^2/2) \frac{1}{\mu (1 + \bar{x})},
\]

\[
A_n = \log A + n (\mu + \mu^2/2) \frac{1}{\mu (1 + \bar{x})},
\]

on Information Theory, April 1960, page 274.

\(^5\)P. Swerling, page 285.
\[ Z_n = \sum_{i=1}^{n} z_i, \]

\[ z_i = \frac{v_i}{1 + x}. \]

Assume that \( B = e^{-2}, A = e^6, \mu = 1, \bar{x} = 1 \) so that we have

\[ B_n = -1 + .75n, A_n = 3 + .75n \] 8-36

The probability density of \( z_i \) is from Equations 8-34 and 8-35

\[ f(z_1) = e^{-z_1}, \ z_1 > 0. \] 8-37

The probability density of \( Z_1 \) when the first sample is drawn is

\[ y(Z_1) = e^{-Z_1}, \ Z_1 > 0. \] 8-38

The probability that the test ends after the first sample is

\[ p(1) = 1 - \int_{B_1}^{A_1} y(Z_1) \, dZ_1 = e^{-A_1}. \] 8-39

The probability density of \( Z_1 \) under the condition that \( B_1 < Z_1 < A_1 \) is

\[ g^*(Z_1) = \frac{y(Z_1)}{\int_{0}^{A_1} y(Z_1) \, dZ_1}, \quad 0 < Z_1 < A_1. \] 8-40
The probability density of $Z_2 = Z_1 + z_2$ when the second sample is drawn is

$$y(Z_2) = \int_{-\infty}^{\infty} g_*(Z_1) f(Z_2 - Z_1) \, dZ_1.$$  

From Equations 8-37 and 8-40 we find that

$$f(Z_2 - Z_1) = 0, \, Z_2 - Z_1 < 0$$  

$$g_*(Z_1) = 0, \, Z_1 < 0 \text{ or } Z_1 > A_1.$$  

Then, using Equations 8-39, 8-42, and 8-43 in 8-41, we find

$$y(Z_2) = \int_0^{Z_2} e^{-Z_1} \frac{e^{-Z_2}}{1 - p(1)} \, dZ_1$$  

$$= \frac{e^{-Z_2}}{1 - p(1)}, \quad 0 < Z_2 < A_1$$  

$$y(Z_2) = \frac{e^{-Z_2}}{1 - p(1)}, \quad A_1 < Z_2.$$  

The probability density of $Z_2$ under the condition that $B_2 < Z_2 < A_2$ is

$$g_*(Z_2) = \frac{y(Z_2)}{\int_{B_2}^{A_2} y(Z_2) \, dZ_2}, \quad B_2 < Z_2 < A_2.$$
Note from Equation 8-4 that

\[
1 - \int_{B_n}^{A_n} y(Z_n) \, dZ_n = \frac{p_n}{\sum_{i=1}^{n-1} p(i)}
\]

or

\[
\int_{B_n}^{A_n} y(Z_n) \, dZ_n = \frac{1 - \sum_{i=1}^{n} p(i)}{\sum_{i=1}^{n-1} p(i)} \quad \text{(8-48)}
\]

The probability density of \(Z_3 = Z_2 + z_3\) is

\[
y(Z_3) = \int_{-\infty}^{\infty} g_\star(Z_2) \, f(Z_3 - Z_2) \, dZ_2 \quad \text{(8-49)}
\]

where \(B_2 < Z_2 < A_2\) and also \(Z_3 - Z_2 > 0\).

Then from Equations 8-45, 8-46, 8-47, 8-48 and 8-49 we find

\[
y_1(Z_3) = \int_{B_2}^{Z_3} Z_2 e^{-Z_2 - (Z_3 - Z_2)} \, \frac{e^{-(Z_3 - Z_2)}}{(1 - \sum_{i=1}^{2} p(i))} \, dZ_2, \quad B_2 < Z_3 < A_1
\]

\[
y_1(Z_3) = e^{-Z_3} \left( \frac{Z_3^2}{2} - \frac{B_2^2}{2} \right) \quad \text{(8-50)}
\]
\[
y_2(Z_3) = y_1(Z_3 = A_1) + \int_{A_1}^{Z_3} \frac{A_1 e^{-Z_2} e^{-(Z_3 - Z_2)}}{1 - \sum_{i=1}^{2} p(i)} \, dZ_2, \quad A_1 < Z_3 < A_2
\]

\[
y_2(Z_3) = y_1(Z_3 = A_1) + e^{-Z_3} \frac{(A_1 Z_3 - A_1^2)}{1 - \sum_{i=1}^{2} p(i)} \quad 8-51
\]

\[
y_3(Z_3) = y_2(Z_3 = A_2), \quad A_2 < Z_3. \quad 8-52
\]

Similarly, \( y(Z_4) \), \( y(Z_5) \), \( \ldots \), \( y(Z_n) \) can be calculated exactly and from these density functions the exact \( p(n) \) and \( L(s) \) equations 8-4 and 8-11 can be calculated.

**Comparison of theoretical and exact analysis of sequential tests**

The exact equations for \( y(Z_k) \) and \( g_k(Z_k) \) equations 4-29 and 4-30 were used to compute the exact \( p(n) \) and \( L(s) \) equations 8-4 and 8-11 for the SH test equation 2-2. The \( p(n) \) was then used to compute the probability distribution

\[
\sum_{i=1}^{n} p(i), \quad n(\theta) \quad \text{and} \quad n(a_1).
\]

The signal level and the upper and lower thresholds were chosen so that \( a_1^2 = 2.0 \), \( \log A = 13.8 \) and \( \log B = -2.4 \), respectively. These threshold levels when substituted in equation 2-3 yield error probabilities of \( \beta = 0.09 \) and
\( \alpha = 0.92 \times 10^{-6} \). The average sample numbers \( \Xi(0) \) and \( \Xi(a^1) \) when calculated from Equation 2-12 are found to be \( \Xi(0) = 2.4 \) and \( \Xi(a^1) = 12.35 \). The actual values obtained are \( \Xi(0) = 3.22 \) and \( \Xi(a^1) = 14.15 \). The exact probability distribution of the sample number for \( s = 0 \) is plotted in Figure 34 together with the distribution calculated using Equation 4-15. The exact and the calculated probability distributions of the sample number for \( s = a^1 \) are plotted in Figure 35. It can be seen that the calculated distributions are good approximations of the exact distributions, particularly in regard to the shapes of the curves.

The exact equations for \( y(Z_k) \) developed in the previous section for the square law approximation sequential test of Equation 8-32 were used to compute the exact \( p(n) \) and \( L(s) \). The values of the error probabilities calculated from Equation 2-3 are \( \beta = 0.136 \) and \( \alpha = 0.002 \) whereas the actual error probabilities were found to be \( \beta = 0.384 \) and \( \alpha = 0.0097 \). The exact probability distributions for \( s = 0 \) and \( s = a^1 \) are plotted in Figure 36.
Figure 34 — The probability distribution of the sample number of the SH test from exact equations and from approximate equations with signal level of zero.
Figure 35 — The probability distribution of the sample number of the SH test from exact equations and from approximate equations with signal level of two.
Figure 36 — The exact probability distribution of the sample number of the square law sequential test with a randomly fluctuating signal.
CHAPTER IX
SUMMARY AND CONCLUSIONS

The capability of a pulsed radar to detect a distant target is limited by the interference of thermal noise with the desired signal in the radar receiver. The problem of detecting a signal immersed in noise can be solved in an optimum manner by the application of statistical decision theory. In the past, statistical tests which use a fixed number of samples of signal and noise have been successfully applied to radar signal detection. Sequential tests which use a random number of samples are capable of more efficient performance than the fixed sample tests. Optimum sequential tests, when compared with fixed sample tests of equal error probabilities, are found to use a smaller number of samples on the average. The optimum sequential test for the pulsed radar case derived from statistical decision theory is too complicated for practical implementation. As a result, it is desirable to seek practical sequential tests for radar use among the many types of non-optimum sequential tests.

Consideration of the optimum fixed sample test suggests a non-optimum sequential test designated the M (for multiple sub-test) test, which utilizes a group of simple sequential sub-tests running concurrently but independently. The M test can be easily designed, but it has poor efficiency. The main goal of
this dissertation is to investigate the efficiency of various modifications of the 
M test. In this way, insight is gained into how an efficient and practical se-
quential test might be constructed for pulsed radar signal detection. The bulk 
of the work assumes the signal is to be detected in gaussian noise although an 
exact analysis is made of a typical case where the noise is non-gaussian.

An analysis of the performance of the M test shows that its inefficiency 
is due to the fact that, while most of the sub-tests terminate promptly, some 
few sub-tests use a large number of samples to arrive at a decision. A 
straightforward way to limit the length of the sub-tests is to truncate all the 
sub-tests at a predetermined number of samples.

The truncated M test, designated the TM test, has been considered by a 
number of writers. However, it was found that the exact performance of the 
truncated sequential test has never been determined and that a number of 
analytical expressions necessary to this task were also not available. There-
fore, the probability distribution of the sample number of the sequential test 
with arbitrary thresholds was derived. Also, the average sample number and 
exact expressions for the error probabilities of the TM test were obtained. 
When the performance of the TM test was analyzed and compared to the per-
formance of the M test, it was found that the former was more efficient and that 
the margin of efficiency increased as the number of sub-tests increased. In 
the analysis of the TM test both approximate and exact expressions for the 
error probabilities were derived and the accuracy of the simple approximate
equations was investigated.

Next, two modifications of the M test were investigated in which the sub-test design was altered during the course of the test. In the first case, designated the MSP (for stepped power) test, the design signal level was increased at the Nth sample and in the second, designated the MST (for stepped threshold) test, the lower threshold was raised at the Nth sample. The probability distribution of the sample number, average sample number, error probabilities, and the average signal level were derived for these two tests so that performance calculations could be made. It was then shown that the MSP test had exactly the same performance as the M test and therefore could not be used to secure increased detection efficiency. The MST test showed some improvement in efficiency compared to the M test but the difference between the two tests decreased as the number of sub-tests increased. It was therefore concluded that modifications which change the sub-test signal level or threshold levels as a function of time while the sub-tests terminate independently do not alleviate the cause of inefficiency found in the M test.

A modification of the M test called the FC (forced continuation) test, which forces all sub-tests to terminate simultaneously, was then considered. The FC test requires all sub-tests to continue until the last sub-test has crossed one or the other of two thresholds. The test is then terminated and the detection decisions made by comparison to a third threshold. The major analytical problem which was solved in the evaluation of this test was the
derivation of the error probabilities. It was found that the efficiency of the FC test increased relative to the efficiency of the M test as the number of sub-tests increased. It was concluded that modifications of the M test which base the termination decision on the state of all of the sub-tests are capable of improving the efficiency of the M test.

The fact that a sequential test uses a random number of samples is a drawback since the radar antenna is required to move at random intervals. Therefore, the (fixed-dwell time) FD test, which uses a fixed number of samples but in which the signal level is controlled by sequential sub-tests, was investigated. In the MFD test the signal level is increased if a threshold is crossed at any time before the fixed number of samples have been taken. The analysis of the performance of this test required the derivation of the average signal level and the error probabilities. The efficiency of the MFD test was found to be considerably improved in comparison to the fixed sample test. The performance improvements obtained from the FC test and the MFD test show that sequential tests can be adapted to radar detection with significant gains in efficiency.

The exact equations for the probability distribution of the sample number of a sequential type test were also derived and were used to analyze the performance of several sequential tests. The sequential test of a known signal in gaussian noise was investigated as well as an approximate sequential test of a randomly fluctuating signal. The performance of all the tests studied in this
This dissertation demonstrates methods of analyzing the performance of several new sequential tests which are adapted to the requirements of pulsed radar. Results are obtained in terms of both approximate expressions which are convenient to evaluate and exact expressions which can be used to verify the accuracy of the approximations. All of the significant results in this dissertation were programmed in the Fortran II language for numerical evaluation on the 7094 computer at the Computer Services Laboratory of the Ohio State University.
In this appendix we will evaluate the OCF Equation 2-6 of the sequential test of Equation 2-2 and also the probabilities of error of the FS test Equation 2-17.

The OCF of the sequential test is the probability that the test will decide $H_0$. It is given by Equation 2-6:

$$L(s) = \frac{A^h - 1}{A^h - B^h}$$  \hspace{1cm} \text{I-1}

where $h$ is a function of $s$ which is obtained from Equation 2-7

$$\int_{-\infty}^{\infty} \left[ \frac{p(y/a_1)}{p(y/0)} \right]^{h(s)} p(y/s) \ dy = 1. \hspace{1cm} \text{I-2}$$

Using Equation 1-8 in I-2, we have

$$\int_{-\infty}^{\infty} \left[ \frac{\exp - (y-a_1)^2/2}{\exp y^2/2} \right]^{h(s)} \frac{1}{\sqrt{2\pi}} \exp - (y-s)^2/2 \ dy = 1 \hspace{1cm} \text{I-3}$$

or

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp - (y^2 - 2(s + h a_1) y + (h a_1^2 + s^2))/2 \ dy = 1. \hspace{1cm} \text{I-4}$$
Then, from 1-4 it can be seen that

\[ h(s) = 1 - \frac{2s}{a_1}. \quad \text{I-5} \]

The FS test Equation 2-17 results in an error if \( \sum_{i=1}^{q} z_i > V \) when \( m = 0 \), \( \sum_{i=1}^{q} z_i < V \) when \( m = a_1 \). Then, the probabilities of error for the FS test are

\[ \alpha_{FS} = \int_{V}^{\infty} \omega(Z_q/0) \, dZ_q, \quad \text{I-6} \]

\[ \beta_{FS} = \int_{-\infty}^{V} \omega(Z_q/a_1) \, dZ_q \]

where \( Z_q = \sum_{i=1}^{q} z_i \), \( z_i = a_1 y_i - a_1^2/2 \). Then using Equations 1-9 and 1-6, we find

\[ \alpha_{FS} = \int_{V}^{\infty} \frac{\exp\left(-(Z_q - q \, \overline{z(\theta)})^2/2q \, \sigma_z^2\right)}{\sqrt{2 \pi q \, \sigma_z^2}} \, dZ_q \quad \text{I-7} \]

\[ = \frac{1}{2} \left(1 - \text{erf}\left(\frac{V - q \, \overline{z(0)}}{\sqrt{2q \, \sigma_z^2}}\right)\right). \]

Now, \( \overline{z} = a_1 \overline{y} - a_1^2/2 \) so that when \( \overline{y} = 0 \), \( \overline{z} = -a_1^2/2 \). Therefore, \( \alpha_{FS} = 1/2 \left(1 - \text{erf}\left(\frac{d/2}{\sqrt{v_j}} + \sqrt{v_j}\right)\right) \quad \text{I-8} \)
where

\[ j = \frac{q \overline{(z(0))^2}}{2 \sigma_z^2}, \quad d = \frac{V \overline{|z(0)|}}{\sigma_z^2}. \]

When \( y = a_1, \quad z = a_1^2/2 \) and we find

\[ \beta_{FS} = \frac{1}{2} \left( 1 + \text{erf} \left( \frac{d/2}{\sqrt{j}} \right) \right) \]

\[ (1-9) \]

where

\[ j = \frac{q \overline{(z(a_1))^2}}{2 \sigma_z^2}, \quad d = \frac{V \overline{z(a_1)}}{\sigma_z^2}. \]
APPENDIX II

In this appendix we find the probability density, probability distribution, and mean value of the sample number for the M test Equation 3-1 and the TM test Equation 4-1. Also, we obtain the probability density, probability distribution, and moments of the sample number of the SH test Equation 2-2 as well as the moments of the sample number of the TS test Equation 4-2.

The M test Equation 3-1 ends when the longest of the K sub-tests ends. Therefore, the sample number \( n_1 \) of the M test is the largest of the K sub-test sample numbers or, in other words, \( n_1 \) is the largest of K random variables.

The probability density of \( n_1 \), \( g(n_1) \), is obtained using the relation given by Gumbel:

\[
g(n_1) = K \left( P(n < n_1) \right)^{K-1} p(n_1)
\]

where

\[
P(n < n_1) = \int_0^{n_1} p(n) \, dn
\]

\( p(n) = \) probability density of the sub-test sample number.

\[\text{---}\]

\(1\) E. J. Gumbel, page 19.
Also, the probability distribution of $n_1$ is

$$G(n_1 < N) = \left( P(n < N) \right)^K. \tag{II-2}$$

Gumbel derives an asymptotic probability density and probability distribution of $n_1$ which are much easier to work with. The asymptotic approximations are

$$g(n_1) = \epsilon_M \exp \left[ - \epsilon_M (n_1 - \mu_M) - \exp - \epsilon_M (n_1 - \mu_M) \right] -\infty < n_1 < \infty \tag{II-3}$$

and

$$G(n_1 < N) = \int_{-\infty}^{N} g(n_1) \, dn_1 = \exp \left[ - \exp - \epsilon_M (N - \mu_M) \right] \tag{II-4}$$

where $\epsilon_M$ and $\mu_M$ are found from

$$P(n < \mu_M) = 1 - 1/K \tag{II-5}$$

$$\epsilon_M = K \, p(\mu_M).$$

The average sample number $\bar{n}_1$ can now be obtained using II-3:

$$\bar{n}_1 = \int_{-\infty}^{\infty} n_1 \, \epsilon_M \exp \left[ - \epsilon_M (n_1 - \mu_M) - \exp - \epsilon_M (n_1 - \mu_M) \right] \, dn_1 \tag{II-6}$$

Now let $\epsilon_M (n_1 - \mu_M) = x$ in II-6 and we obtain

$$\bar{n}_1 = \int_{-\infty}^{\infty} \left( \frac{x}{\epsilon_M} + \mu_M \right) \exp \left[ - x - \exp (-x) \right] \, dx \tag{II-7}$$

$$= \mu_M + \frac{1}{\epsilon_M} \int_{-\infty}^{\infty} x \exp \left[ - x - \exp (-x) \right] \, dx. \tag{II-8}$$
If we make the substitution \( t = e^{-x} \) in (2.8), we have

\[
\bar{n}_1 = \mu_M + \frac{1}{\epsilon_M} \left( -\int_0^\infty \ln t e^{-t} dt \right)
\]

\[
= \mu_M + \frac{\gamma}{\epsilon_M} \tag{2.9}
\]

where \( \gamma = \text{Euler's constant}^2 \)

\[= 0.5772.\]

The TM test Equation 4-1 ends when the longest of the \( K \) sub-tests ends except if the test reaches the \( n_0 \)-th sample it is truncated. The probability that the test will end after the \( n_0 \)-the sample is

\[
1 - \int_{-\infty}^{n_0} g(n_1) dn_1. \tag{2.10}
\]

Then, using (2.3) and (2.10), the average sample number of the TM test is

\[
\bar{n}_{IT} = \int_{-\infty}^{n_0} n g(n_1) dn_1 + n_0 \left( 1 - \int_{-\infty}^{n_0} g(n_1) dn_1 \right) \tag{2.11}
\]

\[
= \int_{-\infty}^{n_0} n g(n_1) \exp \left[ -\frac{\epsilon_M (n_1 - \mu_M)}{\epsilon_M} - \exp - \frac{\epsilon_M (n_1 - \mu_M)}{\epsilon_M} \right] dn_1
\]

\[
+ n_0 \left( 1 - \int_{-\infty}^{n_0} \exp \left[ -\frac{\epsilon_M (n_1 - \mu_M)}{\epsilon_M} - \exp - \frac{\epsilon_M (n_1 - \mu_M)}{\epsilon_M} \right] dn_1 \right).
\]

If we let \( x = \epsilon M (n_1 - \mu_M) \) and \( G = \epsilon M (n_0 - \mu_M) \), Equation II-11 becomes

\[
\frac{n_{1T}}{n_1 T} = \int_{-\infty}^{\infty} \left( \frac{x}{\epsilon M} + \mu_M \right) \exp \left( -x - \exp(-x) \right) dx + \left( \frac{G}{\epsilon M} + \mu_M \right) \left( 1 - \int_{-\infty}^{\infty} \exp \left( -x - \exp(-x) \right) dx \right)
\]

II-12

\[
= \frac{1}{\epsilon M} \int_{-\infty}^{\infty} x \exp \left( -x - \exp(-x) \right) dx + \left( \frac{G}{\epsilon M} + \mu_M \right) - \frac{G}{\epsilon M} \int_{-\infty}^{\infty} \exp \left( -x - \exp(-x) \right) dx.
\]

II-13

The first integral in II-13 can be evaluated as follows. Let \( t = e^{-x} \) and we have

\[
\frac{-1}{\epsilon M} \int_{-\infty}^{\infty} \ln t \ e^{-t} \ dt
\]

and integrating by parts we obtain

\[
\frac{-1}{\epsilon M} \left[ - \ln t \ e^{-t} \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-t} \ t^{-1} \ dt
\]

\[
= - \frac{1}{\epsilon M} \left[ - \ G \exp \left( -\exp \left( -G \right) \right) + \int_{-\infty}^{\infty} e^{-t} \ t^{-1} \ dt \right]
\]

\[
= \frac{G}{\epsilon M} \exp \left( -\exp \left( -G \right) \right) + \frac{1}{\epsilon M} \ E_1(-e^{-G})
\]

II-14
where \( E_1(-x) \) is the exponential integral.  

Using II-14 in II-13, we find

\[
\frac{n_{1T}}{\epsilon M} = \frac{G}{\epsilon M} \exp \{-\exp(-G)\} + \frac{1}{\epsilon M} E_1(-e^{-G}) + \frac{G}{\epsilon M} \frac{e^{-G}M + \mu}{\epsilon M} \left[ \exp(-\exp(-G)) \right]
\]

or letting

\[
N = \frac{G}{\epsilon M} + \mu M,
\]

\[
\frac{n_{1T}}{\epsilon M} = N + \frac{1}{\epsilon M} E_1[\exp(-\mu M(N - \mu M))]
\]

The probability density for the sample number \( n \) of the sequential test of Equation 2-1 can be obtained as follows. Wald has shown that the characteristic function of \( n \) under the condition that \( H_1 \) is chosen is

\[
E_1 e^{i\tau n} = \frac{t_1}{B - B} \frac{t_2}{(1 - L(s))} \frac{t_1}{A - A} \frac{t_2}{B - B}
\]

---

5. The expectation operator with a subscript \( E_\alpha \) is the conditional expectation under the condition \( \alpha \).
and under the condition that \( H_0 \) is chosen is

\[
E_{0} e^{j\tau n} = \frac{t_2 - t_1}{L(s) \left( A_2 \frac{t_2}{A} - A_1 \frac{t_1}{B} - B \frac{t_2}{B} \right)}.
\]

where \( t_1 \) and \( t_2 \) are functions of \( \tau \) which are defined below and \( L(s) \) is the OCF of the sequential test Equation 2-6.

Using Equations II-17 and II-18, the characteristic function of \( n \) is found

\[
E_{0} e^{j\tau n} = L(s) E_{0} e^{j\tau n} + (1 - L(s)) E_{1} e^{j\tau n}
\]

\[
= \frac{t_2 - A_1 + B_1 - B}{A_2 B_1 - A_1 B_2}.
\]

The functions \( t_1(\tau) \) and \( t_2(\tau) \) are obtained as follows. Set the moment generating function of \( z \) equal to \( e^{-j\tau} \) and solve this equation for the two roots \( t_1(\tau) \) and \( t_2(\tau) \). In particular, for the sequential test of Equation 14, \( z = a_1 y - a_1^2/2 \).

Also, using Equation 1-8 the probability density of \( z \) is

\[
\frac{\exp - (z - \bar{z})^2/2 \sigma_z^2}{\sqrt{2\pi \sigma_z^2}}
\]

where \( \bar{z} = a_1 \bar{y} - a_1^2/2 \), \( \sigma_z^2 = a_1^2 \).

\[\text{II-20}\]

\[\text{II-19}\]

\[\text{II-18}\]

\[\text{II-17}\]

\[\text{II-16}\]

\[\text{II-15}\]

\[\text{II-14}\]

\[\text{II-13}\]

\[\text{II-12}\]

\[\text{II-11}\]

\[\text{II-10}\]

\[\text{II-9}\]

\[\text{II-8}\]

\[\text{II-7}\]

\[\text{II-6}\]

\[\text{II-5}\]

\[\text{II-4}\]

\[\text{II-3}\]

\[\text{II-2}\]

\[\text{II-1}\]

\[\text{I-20}\]

\[\text{I-19}\]

\[\text{I-18}\]

\[\text{I-17}\]

\[\text{I-16}\]

\[\text{I-15}\]

\[\text{I-14}\]

\[\text{I-13}\]

\[\text{I-12}\]

\[\text{I-11}\]

\[\text{I-10}\]

\[\text{I-9}\]

\[\text{I-8}\]

\[\text{I-7}\]

\[\text{I-6}\]

\[\text{I-5}\]

\[\text{I-4}\]

\[\text{I-3}\]

\[\text{I-2}\]

\[\text{I-1}\]

\[\text{I}\]

\[\text{H}\]

\[\text{G}\]

\[\text{F}\]

\[\text{E}\]

\[\text{D}\]

\[\text{C}\]

\[\text{B}\]

\[\text{A}\]

\[\text{Wald, "Sequential Analysis", page 186.}\]
The moment generating function of $z$ is

$$E e^{zt} = e^{zt} + \sigma_z^2 t^2/2.$$  

The functions $t_1(\tau)$ and $t_2(\tau)$ for the SH test Equation 2-2 are therefore found as the roots of the following equation

$$E e^{zt} = e^{-j\tau}$$

or

$$e^{zt} + \sigma_z^2 t^2/2 = e^{-j\tau}.$$  

The equation II-22 is satisfied if

$$\frac{\sigma_z^2}{2} t^2 + zt + j\tau = 0$$

so that

$$t_1(\tau) = -\frac{z}{\sigma_z^2} (1 - (1 - \frac{2 \sigma_z^2 j\tau}{z^2})^{1/2})$$

and

$$t_2(\tau) = \frac{z}{\sigma_z^2} (1 + (1 - \frac{2 \sigma_z^2 j\tau}{z^2})^{1/2}).$$

If we let $h = -\frac{2z}{\sigma_z^2}$ and $g = (1 + \frac{4j\tau}{h^2})^{1/2}$, Equations II-23 and II-24 become

$$t_1(\tau) = h/2 (1 - g)$$

$$t_2(\tau) = h/2 (1 + g).$$
Now, using Equations II-25 and II-26 we have

\[ t_2^2 = \frac{A}{2} \left( h/2 (1 + g) \right) \]

\[ t_2^2 = \frac{B}{2} \left( h/2 (1 + g) \right) \]

And if we write \( e^{\log A} \) and \( e^{\log B} \) for \( A \) and \( B \) in II-27 and II-23 we have

\[ t_2^2 = e^{h/2 \log A} \left( 1 + g \right) = e^{c} \left( 1 + g \right) \]

\[ t_2^2 = e^{h/2 \log B} \left( 1 + g \right) = e^{-a} \left( 1 + g \right) \]

Where \( c = \frac{h}{2} \log A, \quad a = \frac{h}{2} \log B \).

Using Equations II-29 and II-30 in II-17, II-18, and II-19 we find

\[ E_1 e^{\tau n} = \frac{1}{(1 - L(s))} \frac{\sinh ag}{e^{c+ \sinh (c+a) g}} \]

\[ E_0 e^{\tau n} = \frac{1}{L(s)} \frac{\sinh cg}{e^{-a} \sinh (c+a) g} \]

\[ E e^{\tau n} = \frac{e^{a} \sinh cg + e^{-c} \sinh ag}{\sinh (c+a) g} \]

The sign of \( a \) is positive if \( h/2 \log B \) is negative; the sign of \( c \) is positive if \( h/2 \log A \) is negative. But, \( h/2 \log B \) is negative if \( s < a_1/2 \) and \( h/2 \log A \) is negative if \( s > a_1/2 \). Therefore, use +a and -c if \( s < a_1/2 \); use +c and -a if
s > a_1/2. The characteristic function of n is given by

\[ E e^{i \tau n} = \psi(\tau) = \int_{-\infty}^{\infty} p(n) \, e^{i \tau n} \, dn. \]  

II-34

Then, the probability density of the sample number n is

\[ p(n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(\tau) \, e^{-j \tau n} \, d\tau. \]  

II-35

From Equations II-23, II-24, and II-33 we find the characteristic function of n is

\[ \psi(\tau) = \frac{e^{\pm a} \sinh c (1 + \frac{4j\tau}{hz})^{1/2} + e^{-c} \sinh a (1 + \frac{4j\tau}{hz})^{1/2}}{\sinh (c+a) (1 + \frac{4j\tau}{hz})}. \]  

II-36

Let \( \mu = 1 + \frac{4j\tau}{hz} \) and substitute II-36 into II-35:

\[ p(n) = -\frac{hz}{4} e^{+n \frac{hz}{4}} \frac{1}{2\pi j} \int_{1-j\infty}^{1+j\infty} \frac{e^{\pm a} \sinh c \vu + e^{-c} \sinh a \vu}{\sinh (c+a) \vu} e^{-un \frac{hz}{4}} \, du. \]  

II-37

The integral of II-37 can be evaluated using Erdelyi page 258 equation 31:

\[ \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \frac{\sinh x \sqrt{p}}{\sinh y \sqrt{p}} e^{pz} \, dp = \frac{1}{y} \frac{\alpha x}{\alpha x + \theta} \left( \frac{x}{2y} \frac{\pi g}{y^2} \right) \]  

II-38

---

where \(-y < x < y\) and \(\theta_4\) is the theta function defined by Erdelyi page 387 as

\[
\theta_4(v/w) = \frac{1}{(-1)^{v/2}} \sum_{k=-\infty}^{\infty} \exp \left( -\frac{i \pi (v + k - 1/2)^2}{w} \right).
\] II-39

Comparing Equations II-37 and II-38 we find that

\[
y = c + a = \left| \frac{h}{2} \log \frac{1 - \beta}{\alpha} \right| \quad \text{and} \quad x = a = \left| \frac{h}{2} \log \frac{1 - \beta}{\alpha} \right| \quad \text{or} \quad x = c = \left| \frac{h}{2} \log \frac{1 - \beta}{\alpha} \right|.
\] II-40

Therefore, \(y < x < y\) as required. In Equation II-39, \(v = \frac{x}{2y}\) and \(w = \frac{i \pi g}{y^2}\) so that II-38 becomes

\[
\frac{1}{y} \frac{\alpha}{\alpha x} \theta_4 \left( \frac{x}{2y} \frac{i \pi g}{y^2} \right) = \frac{1}{y} \frac{\alpha}{\alpha x} \frac{1}{\sqrt{i \pi g}} \sum_{k=-\infty}^{\infty} \exp \left( -\frac{(\frac{x}{2y} + k - \frac{1}{2})^2}{g} \right)
\]

\[
= \frac{-1}{\sqrt{i \pi g}} \sum_{k=-\infty}^{\infty} \exp \left( -\frac{x/2 + y(k - 1/2))^2}{g} \right) \frac{(x/2 + y(k - 1/2))}{g}.
\]

Let \(t = -n h \frac{\pi}{4}\) in II-37 and then using II-38 and II-41 we find

\[
p(t) = \frac{e^{-t}}{\sqrt{\pi t}^3} (-e^{a} + a) \sum_{k=-\infty}^{\infty} \exp \left( -\frac{(c/2 + (c+a)(k-1/2))^2}{t} \right)
\]

\[
(c/2 + (c+a)(k-1/2)) - e^{\frac{c}{t}} \sum_{k=-\infty}^{\infty} \exp \left( -\frac{(a/2 + (c+a)(k-1/2))^2}{t} \right)
\]

\[(a/2 + (c+a)(k-1/2)). \] II-42
From Equations II-25, II-26, II-29, II-30, and II-42 we have

\[ a = \frac{h}{2} \log B = \frac{z}{\sigma^2 z} \log B, \quad c = \frac{h}{2} \log A = \frac{z}{\sigma^2 z} \log A \]

\[ t = -\frac{n h z}{4} = \frac{n z^2}{2 \sigma^2}. \]

Also, from Equation II-33 we find that \( e^a \) and \( e^{-c} \) are used in II-42 if \( s < a_1/2 \) while \( e^{-a} \) and \( e^c \) are used if \( s > a_1/2 \). Since \( h/2 \log B \) is positive and \( h/2 \log A \) is negative when \( s < a_1/2 \), we can simplify II-42 by letting \( \mu = \frac{z}{\sigma^2 z} \log B \) and \( v = \frac{z}{\sigma^2 z} \log A \). Then we find

\[ p(t) = \frac{e^{-t}}{\sqrt{\pi} t^r} \left( -e^e \sum_{k=-\infty}^{\infty} a(k) e^{-\frac{a(k)^2}{t}} - e^v \sum_{k=-\infty}^{\infty} c(k) e^{-\frac{c(k)^2}{t}} \right) \quad \text{II-43} \]

where \( a(k) = a (k-1/2) + ck, \ c(k) = c(k-1/2) + ak \)

\[ t = \frac{n z^2}{2 \sigma^2 z}, \quad \mu = \frac{z}{\sigma^2 z} \log B, \quad v = \frac{z}{\sigma^2 z} \log A, \]

\[ a = \left\lvert \mu \right\rvert, \quad c = \left\lvert v \right\rvert. \]

The conditional probability densities of the sample number \( p_1(n) \) and \( p_0(n) \) are found in the same way as \( p(n) \) using Equations II-31 and II-32 in II-35. Then, in terms of the normalized sample number \( t \), we find

\[ p_1(t) = \frac{1}{(1-L(z))} - \frac{e^{-t+v}}{\sqrt{\pi} t^3} \sum_{k=-\infty}^{\infty} c(k) e^{-\frac{c(k)^2}{t}} \quad \text{II-44} \]
\[
p_0(t) = \frac{1}{L(s)} \sum_{k=-\infty}^{\infty} a(k) e^{-\frac{a(k)^2}{t}}.
\]

The probability distribution of \( n \) for the SH test Equation 2-2 is found by integrating Equation II-43. The result will be an infinite series which has the general term

\[
\int_0^\infty \frac{\nu}{2\sqrt{\pi}} \exp\left(-t - \frac{\nu^2}{4t}\right) dt.
\]

where \( \nu = a(2k-1) + 2a(k) \) or \( \nu = c(2k-1) + 2ka = 2c(k) \). The integral of II-46 may be evaluated as follows. The Fourier transform of an integral is given by \(^8\) (using Campbell and Foster's notation)

\[
\int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(t) dt \right) e^{-pg} dg = \frac{\int_{-\infty}^{\infty} f(t) e^{-pt} dt}{p}.
\]

The integral may therefore be evaluated by taking the inverse Fourier transform of the right side of II-47

\[
\int_{-\infty}^{\infty} f(t) dt = \int_{-\infty}^{\infty} \left( \frac{\int_{-\infty}^{\infty} f(t) e^{-pt} dt}{p} \right) e^{pg} dg.
\]

From C & F 817 we find

\[
\int_{-\infty}^{\nu/2} \frac{\exp\left(-t - \frac{\nu^2}{4t}\right)}{\sqrt{\pi} t^{3/2}} \, dt = \exp(-\nu \sqrt{p+1})
\]

where \(0 < \nu\). Then, from II-48 and II-49 we have

\[
\int_{0}^{\nu/2} \frac{\exp\left(-t - \frac{\nu^2}{4t}\right)}{\sqrt{\pi} t^{3/2}} \, dt = \int_{-\infty}^{\infty} \frac{\exp(-\nu \sqrt{p+1})}{p} \, e^{pg} \, df.
\]

Using C & F 819 in II-50, we find

\[
\int_{0}^{\nu/2} \frac{\exp\left(-t - \frac{\nu^2}{4t}\right)}{\sqrt{\pi} t^{3/2}} \, dt = \frac{e^{-\nu}}{2} \text{erfc}\left(\frac{\nu/2}{\sqrt{g}} - \sqrt{g}\right) + \frac{e^\nu}{2} \text{erfc}\left(\frac{\nu/2}{\sqrt{g}} + \sqrt{g}\right)
\]

where \(0 < \nu\).

From Equation II-46 we have when \(k\) is positive,

\[
\nu/2 = a(k-1/2) + c \, k = a(k) \text{ or } \nu/2 = c(k-1/2) + a \, k = c(k).
\]

When \(k\) is negative in II-46, we have

\[
\nu/2 = -(a(k + 1/2) + c \, k) = -(a(k) + a) = -b(k)
\]

or

\[
\nu/2 = -(c(k) + c) = -d(k).
\]
It will be convenient to rewrite II-43 in terms of $a(k)$, $b(k)$, $c(k)$, and $d(k)$, Equations II-52 and II-53, so that the condition of II-51 that $0 < \nu$ will be fulfilled

\[
p(t) = \frac{e^{-t}}{\sqrt{\pi t}} \left( e^{\mu} \sum_{k=0}^{\infty} \frac{b(k)}{t} - e^{\mu} \sum_{k=1}^{\infty} \frac{a(k)}{t} \right)
+ e^\nu \sum_{k=0}^{\infty} \frac{d(k)}{t} - e^{\mu} \sum_{k=1}^{\infty} \frac{c(k)}{t}.
\]

II-54

Also using II-43, II-52, II-53, and II-54 and letting $H = \frac{2Z}{\sigma^2}$ we can evaluate the terms $e^{-\nu}$ and $e^\nu$ in II-51 as follows

\[
e^{-a(k)} = \exp (-a(2k-1) - 2kc)
= B^{kH - H/2} A^{kH}
\]

\[
e^{a(k)} = B^{-kH + H/2} A^{kH}
\]

\[
e^{-c(k)} = \exp (-c(2k-1) + 2ka)
= A^{kH - H/2} B^{-kH}
\]

\[
e^{c(k)} = A^{kH-H/2} B^{-kH}
\]

II-55
The terms $e^{-d(k)}$ and $e^{-d(k)}$ in II-51 can be written using II-43 and letting $h = \frac{-2\bar{z}}{\sigma z}$ as

$$e^{-d(k)} = B^{-kH + H/2} A^{-kH},$$

$$e^{-d(k)} = A^{-kH - H/2} B^{kH}.$$

Therefore, using Equations II-51 to II-56, we find the probability distribution of the sample number $n$. This result is given in Table 1 of Chapter 4 as Equation 4-15 together with the conditional probability distributions Equations 4-16 and 4-17.

The $r$-th moment of the normalized sample number $t^F$ of the SH test Equation 2-2 is by definition

$$t^F = \int_0^\infty t^F p(t/s) \, dt$$
where \( p(t/s) \) is given by Equation II-54. The result will be an infinite series which has the general term

\[
\int_0^\infty \frac{\nu t^{r-3/2}}{2\sqrt{\pi}} \exp\left(-t - \frac{\nu^2}{4t}\right) dt
\]

where \( \nu = 2a(k), 2b(k), 2c(k), \) or \( 2d(k). \)

The integral of II-61 may be evaluated using Erdelyi\(^9\) Equation 29 page 146:

\[
\int_0^\infty \frac{\nu t^{r-3/2}}{2\sqrt{\pi}} \exp\left(-t - \frac{\nu^2}{4t}\right) dt = \left(\frac{\nu}{2}\right)^{r+1/2} 2^{\frac{r}{\sqrt{\pi}}} K_{r-1/2}(\nu)
\]

where

\[
K_{r-1/2}(\nu) = \frac{\sqrt{\pi}}{\sqrt{2\nu}} e^{-\nu} \sum_{m=0}^{r-1} \frac{\Gamma(r+m)}{(2\nu)^m m! \Gamma(r-m)}
\]

\[=\] the modified Bessel function.

Then, using Equations II-54, II-55, II-56, and II-62 in II-60, we find

\[
t^r(s) = B 2^{-\frac{H-h}{2}} \sum_{k=0}^{\infty} (B/A)^k b(k)^r S_b(k) - B 2^{-\frac{(H+h)}{2}} \sum_{k=1}^{\infty} (B/A)^k a(k)^r S_a(k)
\]

\[
-\frac{(H+h)}{2} \sum_{k=0}^{\infty} (B/A)^k d(k)^r S_d(k) - A 2^{-\frac{H-h}{2}} \sum_{k=1}^{\infty} (B/A)^k C(k)^r S_c(k)
\]

\[=\]

\[A.~Erdelyi,~et~al.,~"Tables~of~Integral~Transforms,~Volume~I."\]
where

\[ S_{b}(k) = \sum_{m=0}^{r-1} \frac{\Gamma(r+m)}{(2b(k))^m m! \Gamma(r-m)} \]

\[ h(s) = -2z/s^2 - \frac{2s}{a_1} \]

If we assume that \( \alpha = 10^{-6} \), \( \beta = 10^{-1} \), and \( s = 0 \), then II-63 reduces to

\[ t^r(0) = \frac{a}{2} \sum_{m=0}^{r-1} \frac{\Gamma(r+m)}{a^m m! \Gamma(r-m)} \]  \hspace{1cm} \text{II-64} \]

where

\[ a = \frac{z}{\sigma^2} \log B = \frac{(s a_1 - a_1^{2/2})}{a_1^2} \log \frac{\beta}{1-\alpha} = 1/2 \log \frac{1-\alpha}{\beta} \].

The mean value of the normalized sample number \( t \) for the SH test is from II-64

\[ \bar{t}(0) = \frac{a}{2} \] \hspace{1cm} \text{II-65} \]

while the mean value of the sample number \( n \) is

\[ \bar{n}(0) = \frac{2 \sigma^2}{z^2} \bar{t}(0) = \frac{\log \frac{1-\alpha}{\beta}}{a_1^{2/2}} \] \hspace{1cm} \text{II-66} \]

The average sample number of the TS test Equation 4-2 is

\[ \bar{t}_{T}(\theta) = \int_{0}^{T} t \ p(t/s) \ dt + T \int_{T}^{\infty} \ p(t/s) \ dt \] \hspace{1cm} \text{II-67} \]
where \( p(t/s) \) is given by Equation II-54 and \( T \) is the truncation sample number.

The first integral in II-67 will result in an infinite series which has the general term

\[
\int_0^T \frac{\nu}{2 \sqrt{\pi} t} \exp \left( -t - \frac{\nu^2}{4t} \right) dt.
\]  

II-68

This integral can be evaluated by the method used for Equation II-46. That is,

\[
\int_0^T f(t) dt = \int_{-\infty}^\infty \left( \frac{\nu}{p} \right) e^{\nu t} df
\]

II-69

where

\[
f(t) = \frac{\nu}{2 \sqrt{\pi} t} \exp \left( -t - \frac{\nu^2}{4t} \right).
\]

Using C & F 823 we find

\[
\int_{-\infty}^\infty \frac{\nu}{2 \sqrt{\pi} t} \exp \left( -t - \frac{\nu^2}{4t} \right) e^{-\nu t} dt = \nu/2 \frac{\exp \left( -\nu (p+1)^{1/2} \right)}{(p+1)^{1/2}}
\]

so that from II-69 and II-70 we have

\[
\int_0^T \frac{\nu}{2 \sqrt{\pi} t} \exp \left( -t - \frac{\nu^2}{4t} \right) dt = \int_{-\infty}^\infty \frac{\nu/2}{\frac{\exp \left( -\nu (p+1)^{1/2} \right)}{(p+1)^{1/2}}} e^{\nu T} df.
\]

II-71

\[\text{---}\]

\[10\] G.A. Campbell and R. M. Foster.
Then using II-54, II-55, II-56, and II-72 together with the results of II-57 we can evaluate Equation II-67, the average sample number of the TS test. The result is given in Table 2 Equation II-73.

If we assume \( \alpha = 10^{-6}, \beta = 10^{-1} \) and \( s = 0 \), II-73 reduces to

\[
\bar{t}_{T(0)} \approx T + \frac{a/2 - T}{2} \text{erfc} \left( \frac{a/2}{\sqrt{T}} - \sqrt{T} \right) - \left( \frac{1 - \alpha}{\beta} \right) \frac{a/2 + T}{2} \text{erfc} \left( \frac{a/2}{\sqrt{T}} + \sqrt{T} \right)
\]

The higher moments of the sample number of the TS test can be obtained by differentiating the characteristic function of the sample number. The characteristic function of the normalized sample number \( t \) for the TS test is

\[
E e^{xt} = \int_0^T p(t/s) e^{xt} dt + e^{xT} \int_T^\infty p(t/s) dt
\]

where \( p(t/s) \) is given by Equation II-54 and \( T \) is the truncation sample number. The first integral in Equation II-75 will result in an infinite series which has the general term
Table 2. The normalized average sample number of the TS test

\[
\bar{t}_{T(s)} = T - (B \frac{H-h}{2} \sum_{k=0}^{\infty} (B/A)^k H (T-b(k)) \phi(\frac{b(k)}{\sqrt{T}} - \sqrt{T}) + B \frac{-(H+h)}{2} \sum_{k=0}^{\infty} (A/B)^k H (T+b(k)) \phi(\frac{b(k)}{\sqrt{T}} + \sqrt{T})
\]

\[
- \frac{-(H+h)}{2} \sum_{k=1}^{\infty} (B/A)^k H (T-a(k)) \phi(\frac{a(k)}{\sqrt{T}} + T) - \frac{H-h}{2} \sum_{k=1}^{\infty} (A/B)^k H (T+a(k)) \phi(\frac{a(k)}{\sqrt{T}} + \sqrt{T})
\]

\[
+ A \frac{-(H+h)}{2} \sum_{k=0}^{\infty} (B/A)^k H (T-d(k)) \phi(\frac{d(k)}{\sqrt{T}} - \sqrt{T}) + A \frac{H-h}{2} \sum_{k=0}^{\infty} (A/B)^k H (T+d(k)) \phi(\frac{d(k)}{\sqrt{T}} + \sqrt{T})
\]

\[
- A \frac{H-h}{2} \sum_{k=1}^{\infty} (B/A)^k H (T-c(k)) \phi(\frac{c(k)}{\sqrt{G}} - \sqrt{T}) - A \frac{-(H+h)}{2} \sum_{k=1}^{\infty} (A/B)^k H (T+c(k)) \phi(\frac{c(k)}{\sqrt{T}} + \sqrt{T}) \quad \text{II-73}
\]

\[
\frac{2\sigma^2}{\bar{t}_{T(s)}} \bar{t}_{T(s)}
\]

See Table 1, Chapter 4 for definitions of terms.
\[ \int_0^{\nu/2} \frac{\exp\left(-t - \frac{\nu^2}{4t}\right)}{\sqrt{\pi} t^3} \, e^{\nu t} \, dt \quad \text{II-76} \]

or

\[ \int_0^{\nu/2} \frac{\exp\left(-(-x + 1) t - \frac{\nu^2}{4t}\right)}{\sqrt{\pi} t^3} \, dt \quad \text{II-77} \]

But

\[ \int_0^{\nu/2} \frac{\exp\left(-(-x + 1) t - \frac{\nu^2}{4t}\right)}{\sqrt{\pi} t^3} \, dt = \int_0^\infty \frac{\int_0^{\nu/2} f(t) \, e^{-pt} \, dt}{p} \, e^{pT} \, df. \quad \text{II-78} \]

From C & F 817 we find

\[ \int_{-\infty}^{\nu/2} \frac{\exp\left(-(-x+1) t - \frac{\nu^2}{4t}\right)}{\sqrt{\pi} t^3} \, e^{-pt} \, dt = \exp(-\nu (p-x+1)). \quad \text{II-79} \]

Therefore, from Equations II-77, II-78, and II-79

\[ \int_0^{\nu/2} \frac{\exp\left(-(-x+1) t - \frac{\nu^2}{4t}\right)}{\sqrt{\pi} t^3} \, dt = \int_{-\infty}^{\infty} \frac{\exp(-\nu (p-x+1))}{p} \, e^{pT} \, df. \quad \text{II-80} \]

Using C & F 819 we find

\[ \int_0^{\nu/2} \frac{\exp\left(-(-x+1) t - \frac{\nu^2}{4t}\right)}{\sqrt{\pi} t^3} \, dt = e^{-\nu \sqrt{1-x}} \frac{\nu}{2} \text{erfc} \left( \frac{\nu}{2\sqrt{T}} - \sqrt{1-x} \sqrt{T} \right) \]

\[ + \frac{e^\nu \sqrt{1-x}}{2} \text{erfc} \left( \frac{\nu}{2\sqrt{T}} + \sqrt{1-x} \sqrt{T} \right). \quad \text{II-81} \]
If we assume $\alpha = 10^{-6}$, $\beta = 10^{-1}$, and $s = 0$, we find from II-54 and II-81

$$E e^{xt} = \frac{\sqrt{B}}{2} \left\{ \exp \left( -a \sqrt{1-x} \right) \text{erfc} \left( \frac{a/2}{\sqrt{T}} - \sqrt{(1-x)T} \right) \right. + \left. \exp \left( a \sqrt{1-x} \right) \text{erfc} \left( \frac{a/2}{\sqrt{T}} + \sqrt{(1-x)T} \right) \right\} + \text{II-82}$$

$$e^{xT} (1 - 1/2 \text{erfc} \left( \frac{a/2}{\sqrt{T}} - \sqrt{T} \right) - \frac{1}{2B} \text{erfc} \left( \frac{a/2}{\sqrt{T}} + \sqrt{T} \right)) \cdot$$

$$t(0) = \frac{d}{dx} E e^{xt} \bigg|_{x = 0}$$

$$= T - (T - a/2) \frac{1}{2} \text{erfc} \left( \frac{a/2}{\sqrt{T}} - \sqrt{T} \right) - (T + a/2) \frac{1}{2B} \text{erfc} \left( \frac{a/2}{\sqrt{T}} + \sqrt{T} \right) \cdot$$

**II-83**

Higher moments of $t$ for the TS test can be found by repeated differentiation of the characteristic function.
APPENDIX III

In this appendix we will derive the probability density function of \( n \) for the SP test Equation 5-4 and also some relations among the parameters of the probability density function of \( n_1 \) for the MSP test Equation 5-5.

From Equation 5-4 it can be seen that for \( n < N \) the SP test is equivalent to the SH test Equation 2-2 with a design signal level of \((k_a)^2\) and a true signal level of \((k_1)^2\). Therefore, the probability density of \( n \) for \( n < N \) is

\[
p(n/ks) = \frac{n}{k^2 s a_1^2 - (k_1)^2/2}^{2 (k_1)^2} = \frac{n}{k^2 s a_1^2 - (k_1)^2/2}^{2 (k_1)^2}
\]

where we use Equation 4-12 with

\[
t = \frac{n z^2}{2 \sigma^2} = \frac{n (k^2 s a_1 - (k_1)^2/2)^2}{2 (k_1)^2}
\]

\[
h = 1 - \frac{2s}{a_1}.
\]

For \( n > N \) the SP test takes the form

\[
\log B < \sum_{i=1}^{N} z_i^1 + \sum_{i=N+1}^{n} z_i < \log A
\]
where
\[ z'_1 = k a_1 y_1 - (ka_1)^2/2 \]
\[ z_1 = a_1 y_1 - a_1^2/2. \]

This is equivalent to an SH test with a design signal level of \( a_1^2 \) and a true signal level of \( s^2 \) except for the first \( N \) samples. Therefore, the probability density of \( n \) for the SP test for \( n > N \) is apparently given by Equation 4-12 except for a correction which accounts for the first \( N \) samples. We can determine what this correction should be by reviewing the basic steps Wald used to derive the probability density of \( n \). Wald begins his derivation of \( p(n) \) by writing his fundamental identity\(^1\)

\[ E(e^{Z_n^t} (\phi_{Z_n} (t))^{-1}) = 1 \]

or

\[ E(e^{Z_n^t} (\phi(t))^{-n}) = 1 \]

where

\[ Z_n = \sum_{i=1}^{n} z_i \]

\[ \phi_{Z_n} (t) = \text{moment generating function of } Z_n \]

\[ = \int_{-\infty}^{\infty} \omega(Z_n) e^{Z_n^t} dZ_n \]

\(^1\)A. Wald, "Sequential analysis," p. 159.
$\phi(t) = \int_{-\infty}^{\infty} \omega(z) e^{zt} \, dz.$

In our case we have from III-2

$$Z_n = \sum_{i=1}^{N} z'_i + \sum_{i=N+1}^{n} z'_i$$

and since the samples $z'_i$ and $z'_j$ are independent

$$\phi_{Z_n}(t) = (\phi_{z'_i}(t))^N (\phi_{z'_j}(t))^{n-N}$$

where

$$\phi_{z'_i}(t) = \int_{-\infty}^{\infty} \omega(z'_i) e^{z'_it} \, dz'_i$$

$$\phi_{z'_j}(t) = \int_{-\infty}^{\infty} \omega(z) e^{zt} \, dz.$$

From Equations I-9 and III-2 we find

$$\omega(z'_i) = \frac{\exp - (z'_i - \bar{z})^2 / 2}{\sqrt{2\pi} \sigma_z^2}$$

$$\omega(z) = \frac{\exp - (z - \bar{z})^2 / 2}{\sqrt{2\pi} \sigma_z^2}$$
where

\[ z' = (k^2 s a_1 - k^2 a_1^2/2) \]

\[ \sigma_{z'}^2 = k^2 a_1^2 \]

\[ z = (s a_1 - a_1^2/2) \]

\[ \sigma_z^2 = a_1^2. \]

Then, using III-6 and III-7, we can find \( \phi_{z'}(t) \) and \( \phi_z(t) \):

\[ \phi_{z'}(t) = \exp \left( (k^2 s a_1 - k^2 a_1^2/2) t + k^2 a_1^2 t^2/2 \right) \quad \text{III-8} \]

\[ \phi_z(t) = \exp \left( (s a_1 - a_1^2/2) t + a_1^2/2 \right) \quad \text{III-9} \]

or

\[ \phi_{z'}(t) = (\phi_z(t))^{k^2}. \quad \text{III-10} \]

Comparison of Equations III-3, III-5 and III-10 shows that we should write the moment generating function of \( Z_n \) for the SP test \( n > N \) as follows:

\[ \psi_{Z_n} = (\phi(t))^{n-N+k^2N} \quad \text{III-11} \]

The fundamental identity Equation III-3 is then

\[ E(e^{Z_n t} (\phi(t))^{-n+N-k^2N}) = 1. \quad \text{III-12} \]
We could now follow Wald's derivation exactly and obtain the characteristic function of the sample number equivalent to Equation II-19 except that we would replace n by n - N + k² N. Therefore, we conclude that the probability density of the sample number for the SP test n > N is

\[ p(\frac{n - N + k^2 N}{s}) \]

where we use Equation 4-12 with

\[ t = \frac{(n - N + k^2 N)}{2 \sigma^2 z} \]

The probability density of n for the SP test is, therefore, from III-1 and III-13

\[ p(n/ks), \quad n < N \]

\[ p(\frac{n - N + k^2 N}{S}), \quad n > N. \]

For the MSP test Equation 5-5 the probability density of the sample number \( n_1 \) is given by Equations 5-32 and 5-33:

\[ \omega(n_1) = \epsilon_k \exp \left[ - \epsilon_k (n_1 - \mu_k) - \exp (- \epsilon_k (n_1 - \mu_k)) \right], \quad n_1 < N \]

where

\[ \epsilon_k = K \frac{\mu_k}{p(\mu_k/ks)}, \quad \int_0^{\mu_k} p(n/ks) \, dn = 1 - 1/K, \]

\[ \omega(n_1) = \epsilon \exp \left[ - \epsilon(n_1 - \mu) - \exp (- \epsilon(n_1 - \mu)) \right], \quad n_1 > N \]

\[ ^2 \text{A. Wald, "Sequential Analysis," Appendices A-5 and A-6.} \]
where

\[ \epsilon = K \left( (\mu - N + k^2 N)/s \right) \]

\[ \int_0^N p(n/k) \, dn + \int_0^\mu p((n - N + k^2 N)/s) \, dn = 1 - 1/K. \]

In order to evaluate Equation 5-34 and Equation 5-38 we need the relations between \( \epsilon \) and \( \epsilon_k \) and between \( \mu \) and \( \mu_k \). Now from Equations III-15 and III-16 we can write

\[ \int_0^N p(n/k) \, dn + \int_0^\mu p((n - N + k^2 N)/s) \, dn = 1 - 1/K \]

or

\[ \int_0^N p(n/k) \, dn + \int_{N-N+k^2N}^{\mu-N+k^2N} p((n - N + k^2 N)/s) \, dn = 1 - 1/K \]

or

\[ \int_0^{\mu-N+k^2N} p(n) \, dn = 1 - 1/K \] \hspace{1cm} \text{III-17} \]

and

\[ \int_0^{\mu_k} p(n/k) \, dn = 1 - 1/K \]

or

\[ \int_0^{\mu_k k^2} p(n) \, dn = 1 - 1/K \] \hspace{1cm} \text{III-18} \]

Therefore, \( \mu_k k^2 = \mu - N + k^2 N \). \hspace{1cm} \text{III-19} \]
Also, from III–15 and III–16 and using III–19 we find

\[ \epsilon_k = K p(\mu_k/k \lambda) = K k^2 p(k^2 \mu_k/s) \]

and

\[ \epsilon = K p(\mu + k^2 N - N)/s = K p(k^2 \mu_k/s) \]

so that

\[ k^2 \epsilon = \epsilon_k \].
It is desired to evaluate the integrals of Equations 6-9 and 6-16. Consider first the integral of Equation 6-9:

\[ g_0(N/s) = \int_0^\infty p(n+N) p_0(n/s) \, dn \]  

\[ = \int_0^\infty \epsilon_F \exp \left\{-\epsilon_F(n+N-\mu_F) - \exp(-\epsilon_F(n+N-\mu_F))\right\} p_0(n/s) \, dn. \]  

where \( p_0(n/s) \) may be obtained from Equation 4-13 by setting \( n = \frac{2 \sigma^2 t}{z} \).

First, let us make the change of variable \( t = \frac{n z}{2 \sigma^2} \) in IV-1 and also let \( G = \frac{N z}{2 \sigma^2} \). Then \( p_0(t/s) \) is given by Equation 4-13 and IV-1 becomes

\[ g_0(G/s) = \int_0^\infty \epsilon(t + G) p_0(t/s) \, dt \]

\[ = \int_0^\infty \epsilon_f \exp \left\{-\epsilon_f(t+G-\mu_f) - \exp(-\epsilon_f(t+G-\mu_f))\right\} p_0(t/s) \, dt \]  

where

\[ \epsilon_f = \frac{2 \sigma^2}{z^2} \quad \epsilon_F, \mu_F = \frac{-z^2}{2 \sigma^2} \mu_F. \]
In IV-2 and \( \mu_f \) can also be obtained from

\[
\int_0^\mu f p(t/s) \, dt = 1 - 1/K \quad \text{and} \quad \epsilon_f = K p(\mu_f/s).
\]

The general term of \( p_0(t/s) \) is from Equation 4-13

\[
\frac{-e^{-\mu}}{L(s) \sqrt{\pi} t^3} \exp - (t - \frac{a(k)^2}{t}). \quad \text{IV-3}
\]

Also, we can write \( \omega(t+G) \) Equation IV-2 as

\[
\omega(t+G) = \epsilon_f \left[ \exp - \epsilon_f(G-\mu_f) \right] \left[ \exp - \epsilon_f t \exp - (\exp - \epsilon_f(G-\mu_f)(\exp - \epsilon_f)) \right]
\]

\[
= b d \exp - (d \exp - bt) \exp - (d \exp - bt) \quad \text{IV-4}
\]

where \( b = \epsilon_f, d = \exp - \epsilon_f(G - \mu_f). \)

Now we can expand \( \exp - (d \exp - bt) \) in an infinite series of powers of 

\( (d \exp - bt) \) so that IV-4 becomes

\[
\omega(t+G) = b(d \exp (-bt) - d^2 \exp (-2bt) + \frac{d^3 \exp (-3bt)}{2!} - \frac{d^4 \exp (-4bt)}{3!} + \ldots
\]

\[
= b \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (d \exp - bt)^{n+1} \right). \quad \text{IV-5}
\]
Then, the product of $\omega(t+G)$ and $p_\theta(t/s)$, Equations IV-3 and IV-5, has the general term as follows:

$$-\frac{e^{-\mu} a(k) b}{L(s) \sqrt{\pi t^3}} \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} d^{n+1} \exp\left(-\frac{(b(n+1)+1)t - \frac{a(k)^2}{t}}{t}\right) \right). \tag{IV-6}$$

Therefore the integral of Equation IV-2 can be written as an infinite series which has the general term

$$-\frac{e^{-\mu} b}{L(s)} \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} d^{n+1} \int_{0}^{\infty} \frac{a(k)}{\sqrt{\pi t^3}} \exp\left(-\frac{(b(n+1)+1)t - \frac{a(k)^2}{t}}{t}\right) dt. \right. \tag{IV-7}$$

Now from Campbell and Foster\textsuperscript{1} pair 817 we find

$$\int_{0}^{\infty} \frac{\alpha}{\sqrt{\pi g^3}} \exp\left(-\frac{\rho g - \alpha^2/g}{g} e^{-Pg} \right) dg = \exp\left(-2\alpha(p+\rho)^{1/2}\right) \tag{IV-8}$$

so that setting $p = 0$ we have

$$\int_{0}^{\infty} \frac{\alpha}{\sqrt{\pi g^3}} \exp\left(-\frac{\rho g - \alpha^2/g}{g} \right) dg = \exp\left(-2\alpha \sqrt{\nu}\right). \tag{IV-9}$$

Using IV-9 in IV-7 we find that the general term of $g_0(G/s)$ Equation IV-2 is

$$\frac{e^{-\mu} b}{L(s)} \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} d^{n+1} \exp\left(-2a(k) \sqrt{b(n+1)+1}\right) \right). \tag{IV-10}$$

\textsuperscript{1}G.A. Campbell and R. M. Foster.
Now from Equation 4-13 we see that \( p_0(t/s) \) is an infinite series of terms given by IV-3 so that the complete solution for \( g_0(G/s) \) is from IV-10

\[
g_0(G/s) = \frac{\mu b}{L(s)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} d^{n+1} \left( \sum_{k=0}^{\infty} \exp \left( -2(a(k) + a)\sqrt{b(n+1) + 1} \right) \right)
\]

The summation on \( k \) in IV-11 can also be written as

\[
\sum_{k=0}^{\infty} \left( \exp - \sqrt{b(n+1) + 1} \right) + (2a(k) + a) - \sum_{k=1}^{\infty} \left( \exp - \sqrt{b(n+1) + 1} \right) 2a(k).
\]

From Equation 4-13 we find \( a(k) = a(k-1/2) + ck \) so that we can write Equation IV-12 as follows

\[
\sum_{k=0}^{\infty} \left( \exp - (2a \sqrt{b(n+1) + 1} + 2c \sqrt{b(n+1) + 1}) \right)^k e^{-a\sqrt{b(n+1) + 1}}
\]

The summations of IV-13 are geometric series which can be summed so that

IV-13 can be written as

\[
\frac{e^{-a\sqrt{b(n+1) + 1}} - e^{a\sqrt{b(n+1) + 1}}}{1 - \exp - (2a\sqrt{b(n+1) + 1} + 2c\sqrt{b(n+1) + 1})}
\]

\[
\frac{(e^{-2(a+c)\sqrt{b(n+1) + 1}} - 1)}{(e^{-2(a+c)\sqrt{b(n+1) + 1}} + 1)}.
\]
From Equation 4-13 we have \( a = \frac{z}{\sigma z} \log B \), \( c = \frac{z}{\sigma z} \log A \) or if we

let \( H(s) = \frac{2z}{\sigma z} \) we find

\[
a = \frac{H}{2} \log 1/B
\]

\[
c = \frac{H}{2} \log A.
\]

Using IV-15 and IV-16 in IV-14 we find that the summations of IV-11 can be written as

\[
\frac{H \varepsilon(n)}{2} - \frac{B}{2} \frac{H \varepsilon(n)}{A} \frac{-H \varepsilon(n)}{1 - (B/A) H \varepsilon(n)}
\]

where\[\epsilon(n) = \sqrt{b(n+1)} + 1\]

Using IV-17 in IV-11 and recognizing from Equation 4-13 that \( e^\mu = B^{-h/2} \), we find that

\[
g_0(G/s) = \frac{1}{\mathcal{L}(s)} \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!} \varepsilon f \exp(-(n+1)\varepsilon f(G-\varepsilon f)) \left( \frac{H \varepsilon(n)-h}{2} - \frac{B}{2} \frac{H \varepsilon(n)-h}{A} - \frac{H \varepsilon(n)}{1 - (B/A) H \varepsilon(n)} \right).
\]

We can now easily evaluate the integral of Equation 6-16 which we shall write as

\[
g_1(G/s) = \int_0^{\infty} \omega(t+G) p_1(t/s) \, dt
\]

IV-19
where \( p_1(t/s) \) is given by Equation 4-14. In the same manner as for \( g_0(G/s) \) we find that the general term of \( g_1(G/s) \) can be written by analogy to Equation IV-10 as follows:

\[
\frac{e^v b}{1 - L(s)} \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} d^{n+1} \exp \left( -2 c(k) \sqrt{b(n+1) + 1} \right) \right).
\]  

From Equation 4-14 we find \( c(k) = c(k - 1/2) + ak \) so that we obtain by analogy to Equation IV-14:

\[
\frac{e^{-c\sqrt{b(n+1) + 1}}}{1 - \exp (-2(a+c)\sqrt{b(n+1) + 1})}.
\]

Using Equations IV-15 and IV-16 in IV-21, we find that the summations of IV-20 can be written as:

\[
\frac{A^{-H/2} \epsilon(n) - A^{-H/2} \epsilon(n) - B \epsilon(n)}{1 - (B/A)^H \epsilon(n)}.
\]

Finally, we substitute Equation IV-22 into Equation IV-20 and, recognizing from Equation 4-14 that \( e^v = A^{-h/2} \), we find that:

\[
g_1(G/s) = \frac{1}{1 - L(s)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \epsilon \exp(-(n+1)\epsilon) \left( \frac{-h(n) + b}{B \epsilon(n)} \right)\left( \frac{-h(n) + b}{B \epsilon(n)} \right).
\]
BIBLIOGRAPHY


