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THE ISSUE OF COLLEGE MATHEMATICS IN GENERAL EDUCATION

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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*****

The Ohio State University
1965

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CHAPTER I
INTRODUCTION

The subject of mathematics has taken on an air of renewed importance during the past ten years, at least in the public mind. Numerous articles have been written concerning what should be taught and how it should be taught. Educators and laymen have bemoaned the inept manner in which many mathematics classrooms have been handled. They have criticized teachers for presenting mathematics as a dead and boring subject, rather than as the alive and ever expanding subject which it really is. Recent Russian scientific advances have somehow brought these criticism into focus, and have forced educators to renew their efforts to "modernize" the mathematics curriculum. Although efforts had been directed toward this goal prior to the Sputnick era, these Russian achievements did serve to bring to public attention the fact that instruction in mathematics had not kept up with the tremendous advances made within the field.

With few exceptions mathematics was being taught just as it had been taught for a hundred years or more,
even though more mathematics had been developed during this period than in any other time in history. Consequently numerous commissions have been formed to develop a curriculum which it is hoped will be a fitting partner to the ever unfolding field of mathematics. And in order to insure a supply of teachers capable of teaching the new curriculum, foundations have been established which provide the means necessary to enable teachers to return to college for further training. And at the collegiate level, various colleges have revised their graduation requirements to include mathematics as a part of general education. The movement is on. We must produce a mathematically literate populace. For many college students this means enrolling in a one-or two-term course in mathematics during their freshman year.

During a period of time extending from 1955 to 1963 this writer was engaged in teaching college mathematics at The Ohio State University. Much of this teaching was at the freshman level, and to a large extent to freshmen who were required or elected to complete one or at most two courses in mathematics. The mathematics course designated to serve the purposes of general education is often the same course designed for students who wish to major in the subject. Such was the case at Ohio State during this writer's experience at that institution. During this teaching stint the nature of that course offered
to general education students was changed, by way of textbook assignment, five times. At first the course was of an integrated-survey type covering many of the usual topics found in algebra, trigonometry, analytic geometry, and calculus. Although the text used for this purpose was satisfactory to the instructors involved it proved too difficult for unselected freshmen. The text was abandoned, and in its place were chosen two traditional texts; one of algebra, and one of trigonometry. Experiences with this combined text were unsatisfactory to instructors as well as to students. Students echoed the time-honored complaint, "What is all of this good for?", and instructors were hard put for an answer. The third and fourth texts chosen for this course were of the more "modern" type, each emphasizing certain fundamental ideas of mathematics, tracing these ideas throughout the various topics of the course, keeping always in mind a logical development. For various reasons, not the least of which was student difficulty, both of these texts were dropped. The fifth approach and that now being used resulted from specially prepared material written by a professor in the mathematics department. This material has been written in the main to illustrate the nature of mathematics, and in such a way that it tends to parallel the new high school curriculums.
Of all students studying mathematics, those who take one or two courses to satisfy general education requirements represent the most numerous group. These students, if for no other reason than the largeness of their number, deserve a course of the very best possible caliber, designed to meet most nearly the demands of general education. Presumably these students have already exhibited minimal knowledge of the rudiments of mathematics, enough at least to enable them to do what little mathematics is required for everyday living. On the other hand their mathematical talents are usually not of the sort to enable them to handle a course filled to the brim with rigorous, abstruse mathematics, that type of course usually designed for honors students. The student of general mathematics is in need of just that, general mathematics.

Judging by the number of times even one university has altered the content of that course designated to serve the need of general education, there seems to be no real consensus concerning what mathematics should form the core of such a course.

The purpose of this paper is to review the question of general education and especially the role of mathematics in general education, and to present the design of a course appropriate to the needs of college students seeking a general education in mathematics. Chapter II deals with
the background of the general education movement. Certain of its characteristics are noted in order to evolve a philosophy applicable to the development of a general mathematics course. Chapter III samples authoritative opinion relative to the nature and content of such a course, in an effort to develop a set of objectives to be applied to its design. Chapter IV is concerned with the choice of content and methods of presentation for the course, each consistent with a set of guiding assumptions which have been gleaned from the general discussions of the preceding two chapters. Chapter V is an evaluation in terms of student opinions concerning the success of the course as taught to two groups of freshmen at The Ohio State University.
CHAPTER II

DEVELOPMENT OF THE GENERAL EDUCATION MOVEMENT

It is common to associate the beginnings of liberal education with the Greek civilization of Socrates, Plato, and Aristotle. Theirs was a civilization based upon the existence of slavery, and it was this institution which in large part dictated the nature of Greek education. The Greeks despised any and every chore of a mechanical sort; such work was felt to be slavish in kind, and was thus to be performed by slave labor. To be sure all Greeks, slave or not, were to be trained well enough to perform their tasks efficiently and to live happily in whatever station of life happened to be their fate, but such training was not thought of as education. The free man in the Greek social order was to be trained in such a manner that his privileges and leisure resulted in general human happiness throughout the entire social strata. This latter type of training was labeled liberal education. Note, however, that this education was not merely an accumulation of inconsequential information to be put on parade every now and then to amuse the not so well educated. It was an education of the
leisure class, to be sure, but it had as its primary purpose the training of responsible social leaders.

Life in the Athens of Socrates was marked by an intellectual freedom never before heard of and never since practiced. Any branch of knowledge was fair play to the Athenian student, and in fact there are persons today who believe that science and philosophy were first created as branches of knowledge in the days of the Greek empire. Students were free to select any teacher who could teach them the most concerning any topic about which they wished to learn. Credits and degrees were unheard of; it was an education entirely unprescribed (9:20). Individualism and freedom were so rampant as to cause concern for the existence of any orderly society. Because of Plato's concern he desired to discover certain standards of truth, reality, and justice which were beyond the tampering of ordinary man. The real aim of liberal education came to be the "contemplation and discovery" of these eternal truths. And thus Plato prescribed for his philosopher-kings a set pattern of studies which were destined to become the backbone of liberal education for many years to come.

The studies which Plato prescribed consisted of grammar, arithmetic, geometry, astronomy, music, and philosophy. These studies were to comprise the whole of
liberal education for several hundred years, at least until the second century B.C., at which time the Roman Varo saw fit to enlarge the list to include grammar, rhetoric, logic, astronomy, arithmetic, geometry, music, architecture, and medicine. This list was later reduced to seven by omitting architecture and medicine. The monk, Cassiordus, in writing on the subject of a proper curriculum for monastic studies, lent his authority to this list of studies which in due course became the curriculum of the monastic and cathedral schools (9:24).

In the Middle Ages these same seven liberal arts gained a permanent stature in higher education when they comprised the curriculum of the newly founded universities. University education in the Middle Ages, although still an education of the leisure class, lost some of the ideal of social responsibility so characteristic of Greek education. The ideal of medieval education became principally scholastic.

The good life had become one of pious contemplation. . . . The life of reason became one of skill in the formal logic with which a given system of life and thought was elaborated. (19:32)

Criticism has been leveled at the educational scheme of the medieval university for its seeming lack of intellectual curiosity, and its attitude of reverence for things past. However numerous its defects may have been, the medieval
university presented to its students an education convincingly characterized by its own distinctive conception of unity. And it is an equivalent unity which some representations of the general education movement would restore to higher education today.

Unity Within the Medieval University

In order to understand how universities of the Middle Ages achieved unity, one might very well look at their course of studies, the administrative setup within the schools, and, above all, take into account the cultural and intellectual climate of the society in which the university functioned. The curriculum, although theoretically consisting of the seven liberal arts, was largely a study of the works of Aristotle. Such books as Old Logic, New Logic, Ethics, Metaphysics, Astronomy, Psychology, and Natural Philosophy were to be read by every student and lectured on by every teacher (23:3). The curriculum was thus heavily weighted in favor of philosophy. Today the study of philosophy in many instances is left to chance. Students in the medieval university were presented an integrated philosophy of life; today the professor of philosophy is generally free to choose any topic on which to lecture, and quite often various lecturers present entirely different views of life.
A second factor contributing to unity of higher education concerned the manner in which teachers were licensed. The church, desirous of orthodoxy within its teaching faculty, prescribed a pattern through which a young scholar could work toward the degree which would permit him to teach. And in addition to this procedure, textbooks were often prescribed, and many times even teaching techniques were regulated. But far and away the most important factor contributing to educational unity was the unity which existed within the medieval culture itself. Theirs was a culture basically Catholic. Students and teacher alike brought to their study a common interpretation of life and man's place in that life. The purpose of education was to mold the student's mind to a predetermined view of the nature of things (23:6). In effect, a teacher in one university was a potential teacher in every university, and there did seem to be a movement of teachers from university to university. If one characterizes the medieval university as unified, it might then be appropriate to characterize the modern university as chaotic. What has happened during the past six or seven hundred years to destroy so effectively that concept of unity which many educators today would have us restore?
The Breakdown of Educational Unity

The study of philosophy, that is, the inquiry into the nature and causes of all human, physical, and metaphysical phenomena, constituted the mainspring of Greek education. In medieval times the chief liberal study was once again philosophy, and it was in the main scholastically oriented; it was based upon the writings of Aristotle as interpreted by St. Thomas. The dogmas of the church were the adhesive that bonded together medieval thought and culture. Whatever there was in the way of natural or social science had to be related in some manner to the proper study of philosophy. However, the time was ripe for expanding the number of subjects offered within the university curriculum. The Renaissance signaled a new era in higher education, for with its onset there emerged the desire for new knowledge. Subjects such as chemistry, botany, and mineralogy were introduced into the curriculum, and there found permanent status. Scholastic philosophers, rather than attempt to assimilate these new subjects into their own philosophy, instead turned their backs to these new disciplines and thus seemingly performed a disservice to the cause of philosophy in higher education.

A second factor related to disunity can be traced to the proliferation of religious sects resulting from the Reformation. This influence can be seen quite clearly in
the early development of colleges in America. During the period of time between the Revolutionary War and the Civil War a number of permanent colleges were born. Tewksbury (28:84) lists one-hundred and eighty colleges founded in the period from 1780 to 1861. These colleges functioned primarily to meet the spiritual needs of a new nation. Tewksbury relates that in 1855 approximately one-fourth of the 40,000 college graduates became ministers. A similar statement could be made concerning the role of the medieval university and the professions to which their graduates aspired. However, there is one major difference between the religious instruction of the medieval university and that of the colleges of America during its formative years. Rather than represent one religious philosophy the American college represented many.

The one-hundred and eighty colleges established by 1860 were supported by 54,745 churches. These churches could accommodate a combined congregation of approximately 19 million people, or in other words, over half of the nation's total population. The colleges founded by religious sects included: Methodist, 34; Baptist, 25; Presbyterian, 49; Catholic, 14; Congregational, 21; Episcopal, 11; Lutheran, 6; Christian, 5; German Reformed, 4; Friends, 2; Universalist, 4; Dutch Reformed, 1; and Unitarian, 2 (28:69). Students in the medieval universities
were educated, and indeed brought to their education a philosophy of life prescribed by the then only existent religion, Catholicism. The picture in early American education was far different. It is only natural to assume that these many sectarian colleges attempted to educate their students according to their own religious beliefs. And even though the curriculums in all of these colleges consisted of the same liberal arts, each religious sect had its own interpretation of life, and thus to each college the differing denominations tended to impart a different philosophy. Hence, denominationalism contributed to the breakdown in educational unity.

However damaging to unity early denominationalism might have been, the picture of higher education today is even less unified. Even though institutions of higher learning one hundred years ago represented many different religious sects, these institutions were united to the extent that religion served as a basic framework for their curriculum, and imparted a sense of direction to their teaching. Colleges and universities today are by contrast largely secular. To be sure a large number of colleges are still controlled by various religious organizations; however, enrollment in these schools seems to be dwindling in comparison to enrollment in public institutions and other private
schools. Cunninggim noted in 1942 that 695 colleges out of 1,243 were church related, and yet enrollment in church-related schools was far less than the combined enrollment of public and other private institutions. He goes on to say, "The tone of higher education is secular, and the total impact upon the majority of students is, if not anti-, at least, non-religious" (11:250).

A third factor contributing to disunity, and by all odds the most publicized, is the free election system which obtained its momentum from the curricular revisions at Harvard under President Eliot (1869-1909). From the beginnings of our country certain sociological, psychological, and philosophical forces have been at work, all of which have tended to fashion change in our social institutions; none, perhaps, more than education. As a larger proportion of the population received elementary school education, even more citizens desired such schooling; and those who received it were desirous of an education beyond that of the elementary school. The American people as a whole have desired freedom of all sorts, including the desire to be free from ignorance. Parents with few exceptions want their children to have more education than they themselves had. And finally, the American dream includes the ideal of the continual betterment in the world's, as well as the nation's,
standard of living. Social forces such as these have resulted in several manifestly evident educational conditions which have caused us to consider and reconsider the very nature of higher education.

School populations have grown beyond the wildest dreams of early proponents of universal education. The Harvard Report (14:7) states that between 1870 and 1940, while the general population was increasing three times over, the enrollment of high schools was being multiplied about ninety times and that of the colleges about thirty times. College and university enrollment increased even more rapidly following the Second World War, this perhaps as a result of the G.I. Bill of Rights and its tuition benefits. In 1940 approximately one and a half million students attended college--by 1947 this figure rose to include two million three hundred and fifty thousand students. After a short reversal, enrollment figures once again began to climb. This increase is well illustrated at The Ohio State University where the academic year count in 1952 was 21,337; in 1957, 22,179; and in 1962, 32,228. In attempting to understand why so many people are seeking education, one must first look at the nineteenth-century compulsory education laws. These laws in part explain the large enrollment figures in elementary and secondary schools,
but one must look further to explain college and university enrollments, for certainly no one is required to attend these institutions. The reason for these latter enrollment figures can in part be traced to the attitudes of the American people, who having once been exposed to the advantages of education desire more of it, and desire it for more people. But whatever the reasons, the tremendous number of persons attending our colleges represent part of a social change of such magnitude that it literally demands the attention of college teachers and administrators.

Medieval universities, and indeed those colleges representing the early days of this country, had a rather clear picture of the society into which their students would matriculate. This is no longer true; in fact it would seem that the reverse of this statement would more nearly characterize our present society. The stable image once so characteristic of society is no longer valid, either with respect to the student body and its reasons for seeking education, or with respect to the society from which comes that student body and to which it will eventually return. Society is in a continual state of flux; no one can predict what society will be like four years from any given date. The question for educators is, "What
type of education should we offer these masses of students in such a society?"

The first impulse of many educators in addressing themselves to this problem was to look for more and different subjects for these students to study, a quest which led eventually to the system of free election. College education, thus altered, became uncoordinated; and although resulting in a system in which teachers could pursue their specialties, worked quite possibly to the disadvantage of students. Cultural training which once formed the core of liberal education became diluted to the point of being practically nonexistent. The age of specialism was upon us; but so too was an age of confusion. Students could earn the baccalaureate degree and yet possess no common body of information with other students.

Beyond the curriculum of the elementary school, the most pervasive facts concerning the curriculums of our schools, and especially of our colleges and universities, are the ever increasing number of subject matter fields available for study and the fragmentation of courses offered within particular fields of study. In mathematics, for example, it was once possible, granted it was several thousand years ago, for individuals to possess a comprehensive knowledge of all that was then known in the field of mathematics. Since that time the study of mathematics
has grown so that today a man may spend a lifetime specializing in a narrow phase of a single branch of mathematics, perhaps linear algebra. Even though the field of mathematics has grown continually during the past two thousand years, it didn't really begin to become immense until the stultifying influence of Euclid was shaken by the non-Euclidean techniques of Riemann and Lobachevski. Once these techniques became known, understood, and accepted, powers of human curiosity turned mathematics into a truly wonderous, self-perpetuating, and self-generating field of study. However rapid the growth of mathematics has been, other fields of study have also been spectacular in their growth, and indeed as a result of ever increasing school enrollment and unbridled intellectual curiosity, entirely new fields of study have been developed and continue to be developed.

The results of this ever expanding curriculum have been just as spectacular as the curricular growth itself. It has provided man with an incalculable advantage in his search of knowledge about man, and his mastery of the world in which he lives. No one would have us retreat to that time in which the curriculum was static and completely prescribed, and yet the general education movement has received a great deal of favorable comment and indeed action. Curricular expansion during the latter part of the
nineteenth century and thereafter, coupled with the free elective system, although playing a vastly important role in improving our civilization, has also had negative effects. The elective system has tended to destroy the ideal of breadth and coherence which college students of an earlier time could enjoy; even the meaning of the Bachelor of Arts degree has lost some of its significance. The elective system has played an important role in the creation of specialists, and no one will deny the need for specialists in our present day society. However, the system has also tended to create college graduates woefully weak in the ability to understand those problems common to persons of all specialties, or to see and act upon relationships existing between peoples of all nations. The age of specialism has created a unique vocabulary for each specialty now existent. It has become increasingly difficult for persons to communicate with one another. This is true even among people who speak the same language and who have achieved the same educational level.

But the mere multiplication of courses has not been the sole culprit in this educational loss. The proliferation of courses tended to create the pattern of departmentalization so characteristic of an age of specialism. There has been a tendency for academic departments to
design courses which function primarily and often solely for the purpose of training students for more advanced courses in the same discipline. There is little attempt made to organize and present a course in such a manner as to reveal its meaning and significance for that general education to which many educators believe all students should be exposed. And it seems that science and mathematics departments are especially vulnerable to such criticism.

The free elective system as practiced at Harvard under President Eliot did not spread in its purest form to other colleges and universities, and yet its effects have been felt in all parts of the country. The traditional four-year prescribed curriculum will perhaps never exert the influence it once did. Nearly every institution of higher learning has adopted the elective system in one form or other. However, since its inception many educators have looked askance at the elective system, believing that some core of knowledge should be prescribed for all students. Trouble arises though when these educators attempt to agree on those courses which are essential to a general education. With the student body being as large as it is, and heterogeneous as it must be, what courses of study should form the basis of information common to all students?
The general education movement has revealed itself, in part, as a quest for universal education. Those educators who are concerned with the problems of college dropouts express the opinion that all students should partake of certain common intellectual experiences and yet they lament the high mortality rate of college students. They regard liberal education as education for the academically elite, and hence undesirable and even unobtainable for the masses of our population who require education. Mathematics has certainly done its share in this questionable business of keeping the non-academically oriented student from obtaining a general education. These educators have joined the battle against over-specialization in the hope that unity might be realized in the form of universal education which meets the needs and abilities of the ordinary student.

Most educators agree that students should master those skills which enable them to be articulate in both speech and writing and to be able to think rationally and objectively, and to know the difference between right and wrong. But in addition to these abilities there is widespread opinion that general education should be concerned with the everyday problems of life. It has been a common occurrence in the experience of this writer to
hear students after completing their segment of general mathematics education ask the question, "What has all of this got to do with my life?" In a well-organized course student might not feel compelled to ask such a question.

The general education movement then has gained importance as a device to restore unity and to counterbalance overspecialization, and it is thought to be that education suitable for universal education. The Harvard Report declares that general education is "training in what unites, rather than in what divides, modern man. . . ." (14:37). "It is not merely the imparting of knowledge, but the cultivation of certain aptitudes and attitudes in the mind of the young . . . to think effectively, to communicate thought, to make relevant judgments, to discriminate among values" (14:64). Many educators agree that a too specialized curriculum is educationally bad, and that those elements of life common in the experiences of everyone should receive greater attention. Disagreement crops up when these educators attempt to discover what studies should be prescribed for all, and further how those studies, once decided upon, should be taught.

Attempts at General Education

The approaches to general education, although by no means universal, can nevertheless be classified as
belonging to one of five major categories which can be distinguished in the various attempts to secure general education. These categories include (1) distribution requirements, (2) survey courses, (3) functional subject matter, (4) great books, and (5) individualized curriculum (14:181).

Distribution Requirements

Since general education in one form or another has as a major goal the unification of education, it seems only natural that one of the first attempts at general education involves prescribed studies. However, educators were not eager to part with that system of specialism largely responsible for so many great cultural gains, and so a compromise was effected between a completely elective curriculum and that curriculum of the medieval university in which all courses of study were prescribed. In the distributive plan certain courses of study are grouped within divisions, the criterion for such grouping consisting of either related content or method of investigation. Thus, for example, one division might be labeled Natural Science, and include such courses as Physics, Chemistry, and Mathematics. Students are required to distribute their curriculum so as to include courses of study in each of the major divisions of knowledge. Thus students in addition to becoming proficient in some particular field of knowledge, i.e.,
their specialty, are required to become somewhat acquainted with other important fields of knowledge.

The individual college typically includes three or four major divisions of knowledge, either (1) natural sciences, (2) humanities, and (3) social studies, or (1) natural sciences, (2) humanities, (3) social studies, and (4) philosophy and religion. Due most likely to its ease of administration the distribution plan has become quite popular. Of one hundred seventy-four college and university catalogues sampled by this writer, only eight indicate no required courses of study other than perhaps physical education and/or English, while seventeen indicate complete or partial prescription, nine for the first two years of education and five for the entire four-year curriculum. On the other hand, eighteen schools emphasize pure division requirements in addition to a major field, and one hundred thirteen schools combine basic requirements, usually physical education and English, and in increasing numbers mathematics, along with division requirements. Of the catalogues sampled, 39 out of 174 indicate mathematics as a required course of study. The number of students enrolled in mathematics seems to be increasing. At The Ohio State University, for example, 33 per cent of all students took a mathematics course in the academic year 1952-1953; this figure was 57 per cent in 1957-1958, and in 1962-1963 it jumped to 62 per cent. And of those
mathematics courses taken within the year 1962-1963, 25 per cent were either mathematics 416 or 417, at that time the usual sequential courses designed for prospective mathematics majors as well as for those students who take mathematics to fulfill requirements of one sort or another. And it might be well to note that of the 3,639 students enrolled in mathematics 416, only 1,458, or 40 per cent, went on to take mathematics 417, the follow-up course. In other words 60 per cent of those students who had begun study in the freshman mathematics sequence stopped after the first course. Although a number of these students undoubtedly failed mathematics 416, or even dropped out of school altogether, these figures would seem to indicate that a sizeable number of students take mathematics solely to satisfy general education requirements.

Divisional organization does in some measure open up broader vistas of knowledge to the faculty as well as to the students. Faculty members are forced to view their own subject within the framework of a comprehensive whole. Students are forced to study subjects which seemingly have nothing at all to do with their own specialty. Even so the typical student often fails to gain the divisional perspective sought after. This failure may be due to the particular choice of study within a division; however a more likely
explanation may be found within the content of the courses specified to meet divisional requirements.

No particular curriculum changes are necessary for the distribution plan to function, and in fact quite often the course chosen for study within a particular division is precisely that course designed for prospective majors in the subject. A student is thus introduced only to the facts and definitions in a field of knowledge, and never really gets the chance to see how these bits of information fit into the scheme of the entire field, let alone how that field fits into the culture in which he lives. In a foundational field of knowledge, such as mathematics, such criticism is particularly relevant. Mathematics educators have become increasingly aware of the inadequacies of the usual beginning mathematics course for those students not majoring in mathematics or any allied subjects, and have during the past ten years attempted to find or write text materials which will serve as the basis for a proper course for these students. To the extent that students fail to see the relevance of a particular course of study to an entire field of knowledge or division, the distributive plan fails.

It seems to be especially true in the field of mathematics that students who study its content simply to satisfy divisional requirements enroll by necessity in that course
designed for mathematics majors. This course often boils
down to the traditional college algebra-trigonometry sequence,
and as Morris Kline (16:297) descriptively puts it, this is
"teaching bricklaying instead of architecture, and color
mixing instead of painting." Any students who complete such
a course often leave its study with a distaste for mathematics,
and in some cases a distaste for all learning. Surely such a
practice does more harm than good to the cause of general edu­
cation. Consequently considerable attention has been focused
on the task of designing mathematics courses appropriate for
general education. Many of these new courses are designed
around concepts such as the theory of sets, topics from ab­
stract algebra, topics from topology and the like. And with
respect to the manner in which these topics are taught, atten­
tion is focused upon mathematical rigor and deductive struc­
ture. But these new courses often fail in the same respect
as do the more traditional college algebra and trigonometry
courses, that is, they fail to provide appropriate motivation
and application, and once again the liberal arts student is
inclined to question the good of it all. These new courses
often seem to be as much out of context as are the more
traditional courses.

Survey Courses

A second approach to general education, namely, the
survey course, is in the main a device to help the student
gain that perspective he might otherwise miss by choosing an improper course within a particular division. The survey course has traditionally been a prescribed course within a particular division, a course offering a sampling of the basic facts and principles of the entire division. For example, all beginning students at the University of Florida enroll for their first two years in University College which offers comprehensive courses in each of the broad divisions of knowledge. The purpose of these courses is to "acquaint the student with the principal concepts in the biological and physical sciences, with the history and development of human society, with a knowledge of the heritage of Western civilization as expressed in its art and literature, and to improve his communicative skills" (29:126).

Survey courses have also been developed within individual fields of knowledge. Educators, having recognized that beginning courses for prospective majors may not be entirely satisfactory for students of general education, have attempted to design courses which will provide such students with a proper perspective of the entire field. Survey texts have been written to meet the needs of orientation courses and integrated courses. Among the most obvious purposes of survey courses in mathematics can be listed: improvement in communicative skills, knowledge of truth
and validity, an historical-cultural perspective, and an idea of what mathematics is really about. Such courses aim to play down drill techniques, stressing instead basic principles. The desire is to present mathematics not merely as a tool subject, but rather in such a manner that its true nature becomes evident.

Survey courses have been criticized for their seeming superficiality. It is too often the case that an overwhelming mass of material is presented, giving students too little time to digest any substantial part of it. Students are able to pass such a course by memorizing a long list of seemingly unrelated facts, facts easily memorized but soon forgotten. It might be better in such a course to deal with a relatively small number of topics, examining these from all sides so as to better insure proper understanding. A recent text entitled A Survey of Basic Mathematics by H. G. Apostle illustrates the temptation to cover a vast array of topics. In this text an extensive array of material is presented, including topics from algebra, geometry, logic, arithmetic (including progressions), statistics (including probability), plane geometry, plane trigonometry, vector analysis, solid geometry, and calculus. It hardly seems possible that a group of students, at least unselected students, can handle such a mass
of material with any sort of real understanding in one or two semesters of study.

**Functional Subject Matter**

The functional subject matter approach to general education is not at all like either of the first two. The curriculum here is not based upon traditional subject matter but rather is constructed to meet the surveyed needs of students as they go about the business of living. Stephens College pioneered this approach when certain research studies were designed to determine the skills needed by a group of graduates in their daily lives. These skills were categorized into seven curricular areas of study. Courses within each of these categories were also constructed in terms of surveyed needs (23:84). The University of Minnesota also constructed its curriculum (for the General College) according to the needs of its students.

The functional approach resulted in a number of technique courses in the field of mathematics, for example, business mathematics, vocational mathematics, and agricultural mathematics. As late as 1957 a course was designed for general mathematics students in California junior colleges (26:42). This course consisted of a number of
"important" topics, so ascertained by questionnaires sent to mathematics instructors and recent graduates. Topics agreed on as being important included a review of fundamental topics of arithmetic including percentage and its applications, business and consumer concepts, budgeting, elementary algebra, graphs, units of measurement, estimation and approximation, and the role of mathematics in the world today. Courses designed in such a manner although popular at one time have all but disappeared from the scene. Possibly the most tenacious of these, Mathematics of Finance, has only recently been dropped from the curriculum of several large universities. The feeling seems to be that if a student is well grounded in the fundamentals of mathematics he will be able to apply this knowledge to any particular branch of applied mathematics. The tool aspects of mathematics are certainly important but to say that mathematics is merely a tool is to do mathematics a grave injustice.

Great Books Approach

The great books of the past and present constitute the basis for the fourth approach to general education. All students regardless of interests are to study a common core of subjects, and these are thought to be best exemplified in the classic books of Western thought. Advocates of the Great Books approach would have the curriculum consist
of certain prescribed subjects each based upon one of the traditional seven liberal arts, thus precluding any possibility of student electives. According to these educators, these liberal arts can be best obtained by reading the great books. And since these seven liberal arts include the study of logic, and since general education is most certainly concerned with training the mind for intelligent action, then certainly the study of logic must be included in the curriculum of all students. And how can logic be learned best? Some would say the answer is simple; read the first and best example of a mathematical science, namely, the Elements of Euclid. Once again these educators proclaim that nowhere can logic be learned better than by studying mathematics, and nowhere can mathematics be learned better than by studying the masters of that science whose works are set down in the great books. Such an education, although enabling students to become proficient in the skills of the traditional liberal arts, yet runs the danger of leaving them deficient in the ability to integrate all of knowledge. And the ability to integrate knowledge into a workable philosophy of life is certainly one of the major objectives of general education.

The Individualized Approach

A fifth approach to general education, that is, individualized instruction, is really a retreat from the
ideal of unified knowledge. Various methods have been used to foster individualized instruction. These include the assignment of tutors whose task it is to help their charges in the matter of studying and preparing for examinations; independent study plans designed to enable the able student to proceed more rapidly than the average class; attendance regulations which allow students to miss class in order that they might spend more time concentrating on their own studies; and group discussions and seminar sessions. In addition to these practices the curriculum itself is often tailored to meet the needs and abilities of the individual student. In such a situation the problem of motivation is practically nonexistent, and the task of sustaining student interest is certainly minimized. But at the same time little if any attention is given to the task of providing a common core of knowledge for all students, and this is certainly contrary to the ideal of general education.

Summary

The general education movement is characterized chiefly as a quest for unity, as a counterbalance to overspecialization, as that education suitable for universal education, as that education directed toward improving communicative skills, and as that education oriented
toward everyday activities of man. Even though the free elective system of President Elliot of Harvard has been abandoned, students in many colleges and universities are still given a great deal of freedom in their choice of studies. There has been a tendency for many years for the curriculum to grow and multiply, and as course offerings increase there is usually a corresponding dilution of cultural content in college education. Today the tendency is toward greater prescription. In order for communication on an advanced level to be possible there should exist a body, larger than there was during the past fifty years, of shared experiences among those who wish to communicate with one another. Another characteristic of the general education movement is its concern with everyday activities of life. That is, the movement in the minds of many constitutes a reaction against the abstract systematic instruction usually characteristic of courses of study within particular academic disciplines. Those interested in general education believe that instruction should begin with problems which concern those students not interested in pursuing the subject into its upper reaches.
CHAPTER III
MATHEMATICS IN GENERAL EDUCATION

The history of the role played by mathematics in the unfolding pattern of general education has been anything but smooth. In recent years a number of colleges and universities have altered their general education curriculums so as to require the study of mathematics by every liberal arts graduate; but this has not always been the case. Educators in the 1930's seemed to have the idea that mathematics offers no potential educational value to the liberal arts student. W. L. Schaaf reported, "There is a growing movement throughout the country to eliminate mathematics as a required subject for students in liberal arts colleges" (27:445). Schaaf, D. E. Smith, Arnold Dresden, and others spoke out in favor of mathematics and its contributions to civilization.

The Second World War provided a tremendous impetus to the belief that mathematics is, after all, of inestimable worth in the education of our citizenry. Draftees were found to be woefully lacking in even the most simple mathematical skills, and with the game of war becoming increasingly technical such ignorance could prove
disastrous. So there was a tendency in the post-war era to stress the technical and skill aspects of mathematics, and to neglect its contributions to general education.

In 1942, K. E. Brown, specialist for mathematics, United States Department of Health, Education, and Welfare, conducted a survey of general mathematics in the colleges and concluded that there was a trend toward including a course in general mathematics for the terminal student (6). However, there was also a word of caution in that report: it was feared that the national crisis might encourage students to enroll in the more traditional courses to the neglect of a general course. In 1947, Brown once again conducted a survey, and this time was forced to conclude that general mathematics was on its way out (7:158). The main reasons advanced to explain this negative trend were (1) absence of suitable text material, (2) increased college enrollment, especially of returning veterans, and (3) a faculty largely overworked and unable to maintain several different offerings of freshman mathematics.

Available textbooks were criticized as being too difficult for the average student who wished or was required to take only one course in mathematics. These texts rushed from topic to topic, not stopping long enough to give the student a thorough idea of any one topic; and
further, these texts were criticized for being uninteresting from the student's standpoint. The balance between college enrollment and faculty size became somewhat more stable during the fifties, enough so that criticism of the practice of forcing terminal students of mathematics to enroll in the algebra-trigonometry sequence once again became fashionable. Reviewing research in the teaching of mathematics in 1955 and 1956, Brown noted that between 1948 and 1955 the number of colleges offering general mathematics increased. And, in general, the cultural type of general mathematics course received the greatest amount of attention from research workers (5:5). Yet he went on to say that although much experimenting is going on with respect to course content, it is usually the case that at least half of the content of general mathematics courses consists of topics chosen from arithmetic, algebra, geometry, and trigonometry. The remainder of the topics are chosen from analytic geometry, statistics, higher mathematics, and recreational mathematics.

As to the kind of course available to the student who is required to complete at most one or two semesters of mathematics, essentially three different types exist. There are the traditional separate courses in college algebra, trigonometry, and analytic geometry; and certainly,
a large number of texts have been written for courses of this type. A number of writers, however, are of the opinion that it would be better to interweave these topics into a unified course; and a number of integrated texts have been written. Then there are those educators who favor the survey type course which scans the entire field of mathematics. Normally, such a course emphasizes fundamental principles along with cultural and historical aspects, while the usual technique and drill work either is left out altogether or is at least minimized. And finally, there are those educators who believe that a course for general mathematics should strive to impart an idea of the nature of mathematics, and they contend that this can be done only by having the students actually do mathematics rather than merely talk about mathematics. The number of books written to emphasize the "real" aspects of mathematics has increased during the past several years. These texts emphasize logical structure usually beginning with rigorous definitions and proceeding to develop an airtight mathematical structure. Generally, the topic of algebra is presented in its axiomatic trappings, and every other subject treated is tied in with this algebraic development.

Much has been written during the past twenty years concerning what type of mathematics should be offered in a
terminal mathematics course designed for the liberal arts student. There have been outspoken critics of the traditional algebra-trigonometry sequence as well as of the more modern cultural-survey type course. In recent years that type of course which, as indicated above, deals with "real" mathematics has received a great deal of attention. It has been argued that such attention is merited since many college students today have received their mathematical training in one of the new secondary school programs, either S.M.S.G., Illinois, or some other; and thus, should be accommodated by a collegiate program which carries on in the same spirit. (24:Preface)

Speaking out against that type of course designed around the survey text, H. P. Evans, professor of mathematics, University of Wisconsin, maintained:

In such a course few direct demands are made on the student's previous training in mathematics and but little time is devoted to the development of the techniques which are necessary for further work in mathematics. This last consideration is a very important one in view of the fact that many entering freshmen expect to take but one year of mathematics and later, through choice or necessity, take additional courses in the subject. (13:13)

Morris Kline, professor of mathematics at New York University, seemed to be speaking directly to this point ten
years later when he answered criticisms to his proposed cultural mathematics course.

Then there are "practical" objections. Some teachers point out that a student can't go on from a course such as I have described to more advanced mathematics. My answer is this. If a student who has indicated at the outset that he does not intend to pursue mathematics or use it in later life changes his mind at the end of the course, then he should be willing to take a technical course such as college algebra and trigonometry before going on to the higher courses. In any case the 99 per cent who won't go on should not be sacrificed to the 1 per cent who might. (16:305)

Evans definitely favors the traditional algebra-trigonometry sequence, or at least a slight modification of it for all college students who take mathematics:

At risk of being trite we reaffirm the tenet that the first collegiate course in mathematics should aim to acquaint all students of the subject with the nature of precise mathematical definitions and proofs of theorems and, also, to help them to acquire those basic skills without which further work in mathematics or its applications would be unthinkable. The purpose of the course should not be to entertain the student, or to impress him with the diverse applications of mathematics, or to attempt to increase the popularity of the subject by watering it down so as to make it easy for the indolent to grasp. It seems entirely possible that many of the brilliant men and women who have avoided mathematics after their first year in college have done so because of a lack of interest in a diluted first course which consisted largely of an embellishment of their high school work in the subject. (13:14)
... In the light of the foregoing discussion, and for other reasons, it seems erroneous to assume that different groups of students with the same mathematical preparation should have different types of beginning courses in college mathematics. (13:14)

The usual course in college algebra deals with a rather lengthy list of topics including quadratic functions and equations, progressions, the binomial theorem, inequalities, complex numbers, logarithms, theory of equations, ratio and variation, permutations and combinations, determinants, et cetera. Evans would alter such a course to include fewer topics namely, linear and quadratic functions, ratio and variation, and the binomial theorem, these topics to be tied together by the function concept. He would leave to the latter half of the sequence a discussion of logarithms needed for computational work in trigonometry. As to the point Evans makes concerning good students being turned away from mathematics, Kline agrees but for different reasons.

What have we been feeding the liberal arts students? The most universal diet has been college algebra and trigonometry. I believe that these courses are a complete waste of time. What educational values are there really in exponents, radicals, logarithms, Horner's method, partial fractions, binomial theorem, the trigonometric identifies, and the law of tangents, just to mention a few conventional topics? ... The truth about college algebra and trigonometry is that these subjects comprise nothing but a series of dry, boring, unmotivated, disconnected,
and, to the student, unimportant techniques. The subjects are taught as techniques and the students are expected to master and reproduce them in parrot-like fashion. The entire body of material is of no value to the non-specialist and no argument for it such as discipline will ever make such material palatable or pregnant with significance. (16:295)

The experiences of this writer would tend to support the views of Kline. There seems nearly always to be an underlying feeling that students studying mathematics for its contributions to general education have in reality gotten little out of its study. At the end of the course, they seem to be exceedingly happy that it is all over, and sometimes express the opinion that never again will they be caught in a mathematics classroom. And, what seems to be even worse, they tend to leave its study with a decided distaste for mathematics, and possibly a distaste for all learning.

Many widely respected educators who have expressed an opinion believe that algebra and trigonometry should not be the required course of study for the student of general education, and yet courses of this type continue to be offered. Several reasons are advanced to explain this seeming paradox. (1) It is difficult to obtain teachers for any course other than the well known, (2) few if any good text-books exist for courses other than the time-honored one, and
(3) it is simply easier to continue the status quo rather than design a new course. As Lyle Dixon put it,

There is a growing feeling of dissatisfaction in the traditional program of mathematics and its contributions to general education. There are many indications that this feeling is justified, but teachers, and it seems particularly true of mathematics teachers, are reluctant to change. (12:206)

That a course intended especially for general education should be designed has been echoed from all sides. B. H. Brown relates,

At Dartmouth, nearly three-quarters of the freshman class elect the calculus. Of the remaining quarter, some lack ability, many have plenty of ability, but lack interest. . . . Should this remaining 25 per cent, or any substantial part of it elect any mathematics? If it is a technical course, my answer is no. Specifically, I have no faintest interest in patching together assorted topics from college algebra, trigonometry, solid geometry, and analytic geometry. But this 25 per cent must never be sold short; it contains at least 50 per cent of the creative genius of the entering class. Can a non technical course be designed for these men which is not a travesty on mathematics or science? (4:286)

In discussing a project in which he undertook to discover what type of mathematics is offered to freshmen in most institutions of higher learning, E. A. Cameron concluded that "The traditional algebra and trigonometry are still the usual freshmen courses in most institutions." And yet he goes on, "Most of the people with whom I talked
agree that much of the trigonometry and some of the algebra in the traditional freshman course can well be replaced by mathematics that is more interesting and of greater significance. Especially is this true for the student who will not take any more mathematics" (10:152). In asserting that a cultural course does possess a definite potential, Harriet Montague and Phyllis Henery assert,

We believe that this general education, cultural course [in the main an historical course] in mathematics has value and can be studied profitably and successfully by students regardless of their preparation or ability in mathematics or their attitudes toward mathematics. We believe that such a course gives the layman a sounder and broader idea of mathematics than skill courses in arithmetic, algebra, or similar subjects. (20:104)

Oystein Ore echoes the same sentiment,

On which principles should a single mathematical course be organized so that it serves as well as possible the educational interests of those students for whom it will most likely represent the end of their mathematics studies? At present many institutions make no particular provision for these men. They are simply offered one or two courses which are identical with those given to the beginning science and engineering students. For many students this may be too difficult, but a much more important deficiency lies in the fact that the students after such a course are left with an impression confirming their previous school experiences that mathematics is mainly a technical tool which requires some special mental skill. They are presented with no broader view of mathematics as a method and as an important branch of human thought and philosophy developed through centuries. (22:455)
One of the most pronounced effects of specialism is that through its influence subjects have been taught in such a manner as to preserve their own internal logic rather than exhibiting their usefulness to students. If the word "mathematics" is substituted for "science" in the following statement from the Harvard Report, a fair picture of the state of mathematics in general education results.

From the viewpoint of general education the principal criticism to be leveled at much of present college instruction in science is that it consists of courses in special fields, directed toward training the future specialist and making few concessions to the general student. Most of the time in such courses is devoted to developing a technical vocabulary and technical skills and to a systematic presentation of the accumulated fact and theory which the science has inherited from the past. Comparatively little serious attention is given to the examination of basic concepts, the nature of the scientific enterprise, the historical development of the subject, its great literature, or its interrelationships with the other areas of interest and activity. What such courses frequently supply are only the bricks of the scientific structure. The student who goes on into more advanced work can build something from them. The general student is more likely to be left with bricks. (14:220)

It would seem that a terminal course in mathematics for general education should be of a different stripe than the technical courses designed for future mathematicians, engineers, or more generally, those persons who need mathematics as a tool course. But the question remains as to
what the content of such a course should be. The discussion of this question is so basic in some instances as to revolve about the notion of whether such a course should contain "real" mathematics, or should it contain material about mathematics? In speaking to this point, Harriet Montague and Phyllis Henry relate,

Dr. Edward Teller recently remarked with reference to a course in physics for the student not going on in the subject that a person can learn to appreciate music to a greater degree by listening to it and learning about it than by setting down to play the scales. This is our point of view with respect to mathematics. College students can learn about mathematics and the work done by mathematicians without learning how to perform the skills of mathematics. (20:104)

On the other hand Cameron contends, "It is to be emphasized that it is not thought that this [the fuller realization of the educational values long believed to be inherent in the discipline of mathematics] can be accomplished by use of descriptive material about the subject. Serious mathematics must be the background of the course" (10:152).

John Riner is of the same opinion:

If a person is to use the subject as a tool or if a person is required to study it as a cultural course, he should be somewhat aware of its nature. It is unfortunate that an awareness of the nature of mathematics cannot be given quickly to an interested person. It is only by doing
Lyle Dixon is somewhat an arbiter between these two positions.

One should be aware that a mathematics course designed for general education must have real mathematics as its core. Courses in which one talks about mathematics do not insure that the desired objectives will be achieved. It has been noted many times that one does not learn what mathematics is like by talking about it. On the other hand, too much emphasis on skills and technique will also destroy the effectiveness of a course in mathematics. Techniques are necessary but they must be in the right proportion.

There is some doubt concerning the meaning of the phrase "techniques must be in the right proportion," but it will be assumed that Dixon meant a correct mixture of techniques and what might be called appreciation.

To this writer it makes a great deal of sense to design a general mathematics course in which there is balance between material which exemplifies mathematics and material which talks about mathematics. When a student enrolls in college and elects or is required to study mathematics as part of his general education, it is supposed that he has already successfully completed a certain amount of high school mathematics, enough at least to enable him to do the ordinary amount of mathematics required in everyday
living. In fact, many colleges and universities require the student to exhibit a minimum knowledge of mathematics as shown by examination, or successful completion of a remedial course, before being permitted to enroll in the general mathematics course. The typical proficiency exam includes questions from the subjects of arithmetic, algebra, geometry, and trigonometry. Remedial courses cover the same material. There seems little point then in stuffing the required general mathematics course with the same material with which the student is, or is assumed to be, familiar. If one accepts this point of view, then the traditional college algebra-trigonometry sequence has little or no justification for being, at least as a general mathematics course.

However, if one believes that a general mathematics course should contain real mathematics, then certain topics should be found to replace the traditional algebra and trigonometry. And, in addition, if general mathematics is to be appreciated, these topics should be presented in a manner to exemplify an historical perspective and a cultural significance, and indeed some effort might be made to shed light on the intuitive aspects of mathematics. To further complicate matters, if such a course is to be limited to one or at most two semesters of study,
one must, it seems to this writer, be careful not to present too many topics.

Pupils are bewildered by a multiplicity of detail, without apparent relevance either to great ideas or to ordinary thought. . . . Mathematics, if it is to be used in general education, must be subjected to a rigorous process of selection and adaptation. . . . The science as presented to young pupils must lose its aspects of redundancy. It must, on the face of it, deal directly and simply with a few general ideas of far-reaching importance. (30:85)

There seems to be a tendency on the part of textbook writers to include a great deal more material than can be studied in the time allotted to a particular course. Or perhaps the tendency should be stated the other way around, that is, teachers choose texts which include more material than they can possibly use. Or again, and this seems to be more nearly the case, there exists no text known to this writer of appropriate length which at the same time contains real mathematics portrayed in a setting which can be appreciated by the liberal arts student.

Those students who elect or are required to study one or at most two semesters of college mathematics probably represent the largest group of students taking mathematics. And indeed due to the largeness of
their number, this group in all likelihood will include many of the ablest of students on any college campus.

These reasons constitute a sufficient case for designing the best possible course for a general mathematics education, but they do not exhaust all possible reasons. There exist several selfish motives for designing such a course. These students someday will marry and have families, and as parents will influence the attitudes of "like and dislike" of their children. If the parents recall with bitter distaste their own experiences with mathematics, there is every reason to believe that this distaste will be transferred to their children. No one would be so optimistic as to believe that a single course in mathematics will engender a positive liking of the subject in the case of every student enrolled, but it might be hoped at least that the course would leave with its students a neutral attitude toward mathematics. Then these students, as future parents, might pass on to their children a more tolerant attitude toward mathematics, so that we as teachers might have a less difficult task in "selling" our subject. In the same selfish vein we might note that these students will in the future occupy seats on boards of education, and indeed will control the purse strings of our schools. It is not too difficult to imagine the fate of general mathematics, if its case be presented to a board of education whose
members despise mathematics. In this writer's opinion we must be concerned with the attitudes concerning mathematics, as well as the knowledge of mathematics which our students take away from its study.

Selected General Mathematics Courses

During the past half century a great deal of interest (although rather sporadic) has been generated in the development of a course in mathematics suitable for general education. Certain of these have been selected to illustrate the various types of courses which have been used specifically for general education. Several of these which seem best described as survey courses will be outlined along with others which seem to be of the cultural-historical type and still others which aim to present the nature of mathematics. The courses described here certainly do not exhaust all possible choices which could have been made. However they do seem to be fairly typical of the courses which might be classified under these three main headings. It is well to note that these three classifications are not mutually exclusive, and that a course which is listed as of the survey type might very well have in it elements which seem better described as historical.

F. L. Griffin described a survey-type mathematics course given at Reed College in 1915. Its purpose was to
give a bird's-eye view to the non-specialist student, but at the same time be effective as an introductory course for the specialist student. This course consisted of the following topics: (1) practical uses of graphs, (2) important limit concepts, (3) differentiation, (4) integration, (5) trigonometric functions of acute angles, (6) logarithms, (7) use of rectangular coordinates, (8) solution of equations, (9) polar coordinates and trigonometric functions in general, (10) trigonometric analysis, (11) definite integrals, (12) progressions and series, and (13) probability and least squares. Such a course certainly touches upon a large portion of the field of mathematics, enough so that it could justifiably be termed a survey course. The number of topics is so large in fact that it would seem that only a superficial coverage of each could be insured, and that no real understanding of the nature of mathematics would result from their study (15:325). However, a mere listing of topics is insufficient to fairly describe the effectiveness of a general mathematics course.

The manner in which a topic is taught is perhaps of more importance than the topic itself. An understanding of the nature and significance of mathematics as a whole is of prime importance to general education. A particular topic is relevant only to the extent that this objective is realized. Basic concepts are important, certainly they are,
but so too are the logical processes used in the development of the subject; and one should realize that proper motivation and relevant applications tend to make the subject meaningful and well accepted. In an effort to guarantee a mathematically literate student body, Oklahoma A. and M. College in the early 1940's required all Arts and Science students to complete a mathematics course (33:246). The course designated to meet this requirement consisted essentially of two different types of subject matter; one emphasizing the historical development of certain aspects of the subject and the other emphasizing the foundations of certain mathematical concepts. Historical materials might include, for example, an account of the development of our numeral system, tracing its history from the earliest of numerals used by Egyptians and Babylonians to the Hindu-Arabic system used today. Foundational concepts might include a logical development of the meaning of numbers along with the elementary operations of arithmetic.

In 1953 "honors students" (more than half of those who participated were from the Humanities and Social Science areas) at Harvard were given a course in pure mathematics designed to bring out its relevance to general education (25:406). The course consisted of a series of lectures based for the most part upon the text, *What Is Mathematics,*
by Richard Courant and Herbert Robbins. As its text would indicate, the course, was highly mathematical and probably not appropriate for a group of unselected freshmen at the typical public college or university. The lectures included: two, devoted to proofs of the irrationality of 2, the infinitude of primes, and the unique factorization of the integers; one, devoted to a discussion of the axiomatic method; one, to a critique of the proofs in Euclidean geometry based upon original axioms; one, to a presentation of finite geometries illustrating the relation existing between a theory and applied models illustrating that theory; four, to the Hilbert axiomatization of Euclidean geometry; one, to an axiomatized treatment of algebra; and finally, one to the theory of transfinite numbers.

Such a course would in all likelihood satisfy those educators desirous of a course in which students do mathematics, and certainly this course would illustrate the nature of mathematics. However, little attention, if any, is given to the task of illustrating the role played by mathematics in the unfolding drama of our culture. Neither does there seem to be any effort to illustrate the tremendous applicability of mathematics (especially of the calculus and the theory of differential equations) to our modern world.
The University of Chicago, long a leader in the general education movement, has also seen fit to design a mathematics course suitable for general education (21:201). The College of the University of Chicago was established in 1931 and given the "responsibility for doing the work of the university in general higher education." The program which it offered in 1950 was the result of a great deal of thought, discussion, and experimentation carried on during the twenty years of its existence. As of 1950 the program of general education at The University of Chicago was not elective. In order to become eligible for the bachelors degree a student had to pass examinations testing his competence in the "basic principles, the major concepts and methods, and the salient facts in the natural sciences, the social sciences, the humanities, and mathematics, and his ability to express himself clearly." (21:11) Mathematics was included as part of the student's general education because educators responsible for formulating the curriculum believed that precision in formulating definitions and assumptions, and rigor in reasoning could be more simply studied in mathematics than in any other discipline. They argued that these traits of mind are pertinent to every intellectual activity of man.

The course at Chicago, given in 1950, was divided into three parts, each corresponding to one quarter of
study. The first part is concerned with those concepts basic to the needs of elementary mathematics. Sets, subsets, and ordered pairs are dealt with in order to precisely define the concepts of function and relation; these in turn lead to propositional functions and the meaning of quantification; finally, the ideas underlying the structure of a deductive science are discussed including the relation between the structure itself and the various models of that structure. The structure of a commutative group is thought to be ideally suited to achieve these purposes. Part two extends the commutative group to include those axioms necessary to define the field concept. The system of rational numbers, a noteworthy model of the field, is studied at length and this study includes a discussion of rational equations. The field is then extended to include those axioms necessary to define the ordered field, and this latter concept is extended to include all those axioms necessary to discuss the real numbers as a model of a postu-lational science. In this connection the theory of real equations and inequalities is fully discussed. Part three of this course is designed primarily to illustrate certain important examples of real valued functions including those of an algebraic and trigonometric nature. In order that the behavior of these functions might be pictured, certain concepts of analytic geometry are developed. That such a
course is mathematically sound there can be no doubt, and certainly no one can deny that the nature of mathematics is well portrayed; however, once again there is no reference to the historical-cultural significance of the subject, nor is any mention made of applications outside of the field of mathematics.

To meet the needs of its freshmen class, i.e., that 25 per cent of them who do not elect calculus, educators at Dartmouth have designed a general mathematics course (4:285). Theirs is a course designed specifically to meet the needs of Dartmouth students, and consequently, no text material has been published. However Dartmouth educators feel that "any college of liberal arts does well to consider that its students not interested in science may profit from some unconventional course based largely on scientific curiosity."

The course as originally conceived was given first in 1958. This course features three main topics; namely, number, time, and space. Topics which have been studied under the heading number include our numeral system, numbers written in other bases, Fermat's Minor Theorem, Fechner's law, and the paradoxes of scoring systems. Various calendar systems have been studied with reference to time, while space topics consist primarily of cartography. Other topics which have been studied at various times include: the application of Kepler's laws to Sputniks, Euler's theory of unicursal networks, and
a study of professional gambling. Any and every topic included for study is treated in such a manner as to emphasize the nature of the problem and then to invent and apply appropriate mathematical technique to its solution. Here the historical-cultural aspects of mathematics are brought to the fore, and most likely the intuitional processes involved in mathematics receive due attention, but there seems little opportunity in such a course to discuss the postulational nature of the subject.

Purdue University has expressed its concern that the student should have some idea of the manner in which the development of mathematics influenced the development of civilization. Hence, all freshmen are required to complete a course designed for non-science, non-mathematics students, which is supposed to impart such knowledge. Mimeographed notes, written by W. V. Ayres, G. G. Fry, and H. F. S. Jonah (2) specifically for this course, were eventually published under the title, General College Mathematics. Much of the book concerns topics already familiar to many college freshmen; ratio, proportion and variation, linear and quadratic equations, the geometry of triangles, trigonometric functions and their applications, and exponents and logarithms. These topics, however, are here taught for understanding rather than for drill or technique. There are, in
fact, few drill exercises in the entire book. A number of topics not ordinarily included in a freshman course are found here; namely, finance (simple and compound interest and simple annuities) growth problems, probability, statistics, theory of numbers, and topology, with each of the latter topics developed only to an elementary level. The book surveys the field of mathematics and seemingly does it well, and there is running commentary throughout relating the historical development of the subject. However, the book never really stops long enough to take a good look at the nature of mathematics. Students are led from topic to topic so quickly it would seem that they never really get the chance to see and understand the relation existing between a particular topic and all the rest of mathematics.

Summary

Mathematics considered as a required course in general education has had an uneven history during the past thirty years. However a number of colleges and universities in recent years have added its study to their list of general education requirements. Essentially three different types of mathematical content have been incorporated in courses designed for general education purposes. Traditionally topics from college algebra and trigonometry have been used; however, many educators today have expressed dissatisfaction
with such courses. A number of these educators have expressed the belief that such courses should be concerned with historical and cultural aspects of mathematics. Still others believe that the real nature of mathematics should form the backbone of such a course, and they contend that this can only be done by presenting "real" mathematics in such a manner that its rigorous, precise, and deductive nature become evident.

Courses of each type have been designed and taught to various groups of college students. Evaluations of these courses have ranged from quite satisfactory to something less than desirable.

Students have been heard to question the good of that course they were required to study; instructors have evaluated certain courses as too difficult for groups of unselected freshmen. A number of educators have questioned the social applicability of those courses designed to impart insight into the nature of mathematics. Still others maintain that such courses, although satisfactory for honors programs, are too difficult and abstruse for unselected freshmen. On the other hand, numerous educators believe that mathematics is studied in general education precisely for the reason that educated persons should be made aware of the nature of mathematics, and that this will
not come about in courses in which one only talks about mathematics. They maintain that one must do mathematics in order to understand mathematics. Opinions concerning what content should be taught in general mathematics, and what methods of presentation are best for such a course are so varied that it would seem that still other types of courses might be designed. And specifically, it should be possible to design a course in which students are required to do mathematics but in a setting which emphasizes its applications to life, and at the same time is pictured in an historical-cultural perspective.
CHAPTER IV

GENERAL MATHEMATICS - COURSE CONTENT

The previous chapters of this paper have investigated the issue of general education from several viewpoints. First, the general education movement was examined historically in an effort to pinpoint certain of its chief characteristics. These characteristics in turn serve to form the basis of a philosophy of general education to aid in the design of a mathematics course suitable for general education. Second, in the immediately preceding chapter, a survey of authoritative opinion revealed certain objectives of general education specifically related to the subject of mathematics. All of these objectives have been used at one time or another to form the bases of various mathematics courses designed to implement general education.

In the present chapter certain topics from mathematics will be chosen to form the core of a general mathematics course, and various methods of presenting this course will be analyzed. The course as designed and taught by this writer is presented in detail in the appendix of this paper. Methods of presentation and
course content have both been chosen in such a manner as to be consistent, it is hoped, with a set of principles chosen from those set down in the preceding chapters. In what follows, these principles will assume the status of postulates, and will form the basis of the course as presented in the Appendix.

The set of principles which this writer assumes important and which he believes should form the basis of a general mathematics course include:

A. Principles of general education

1. What is taught should relate to life.

2. All educated persons should share a common knowledge of a wide array of the terminology of various academic disciplines.

3. Whatever is taught should be made available to students even though they are not of the academically elite.

4. The attempt should be made to expand upon the cultural pool of knowledge common to all educated persons.

B. Principles of general education specifically applicable to mathematics
1. The nature of mathematics should be emphasized.

2. Definitions and assumptions should be made clear and precise.

3. Mathematics should be seen as an art in that the mathematician is free to construct any structure at all simply by assuming different sets of postulates.

4. The course should not consist of too many topics; those which are taught should be taught thoroughly.

5. Concepts should be introduced and developed in a manner which will portray their historical setting and cultural significance.

6. Proper motivation should be given for the introduction of any mathematical concept.

7. Relevant applications, when possible, should round out the student's experience.

8. Whenever possible students should be encouraged to experiment, to use their intuitive powers when developing a mathematical concept.

9. The various concepts developed in a particular course should, whenever possible, be related to each other, i.e., a thread of continuity should appear to tie
the entire course into one unified science.

10. Students should be aware of the various bases of human knowledge; one of these consists of the methods of calculus.

A course in mathematics properly designed to further the cause of general education should consist of an appropriate number of topics carefully chosen, and taught in such a manner that the desired outcomes result. It should not contain too much material, and yet that which it does contain must be adequate. It must not leave students with the idea that mathematics is merely an abstract, internally consistent field of knowledge, but that it has in fact exercised great influence over thought in all fields. Nor should students believe that mathematics is solely an exercise in deductive reasoning. Much mathematics has been discovered by noting its relation with the physical world, and only later did its systematic validation become a concern. Neither should its study leave the impression that there is only one true model of any particular mathematical structure, nor indeed that there is only one mathematical structure sufficient to explain a particular physical phenomena. It should be pointed out that mathematics is an art and that the mathematician is an artist in the sense that he is free to construct any number of structures that
please him simply by assuming different sets of postulates. And yet mathematics is not all structure, important as this aspect may be, for a significant portion of mathematics has sprung from profound intuition. Finally, of ever increasing importance to students living in a civilization largely characterized by its advanced state of science and technology is the ability to communicate with specialists in all walks of life. The scientific terminology so plentiful in speech today demands a citizenry aware of its meaning.

That mathematics suitable for the student of general education should be of such a nature that it satisfies a two-fold purpose. The instructor of such a course should begin the consideration of new material by basing it upon social situations familiar to his students. Mathematics should be presented as a great and beneficial construct of the human mind, a foundation piece basic to culture as it is known today. But at the same time this material should be presented in such a manner that the real science of the subject become evident. Mathematics must cease to appear merely as a bag of tricks, and instead be presented in the truly remarkable garments of deductive science. However, to make mathematics really meaningful one must do more than explain
the rationale of its deductive structure. The mathematician relies heavily upon experimental processes and powers of intuition in bridging the gap between mathematics for social application and mathematics characterized solely by its internal consistency. The one real hope for making mathematics meaningful to the student of general education lies in the utilization of motivational and experimental processes understood and practiced by the students. They must somehow discover new truths for themselves. Now certainly many topics in mathematics are far beyond the scope of the average student of general education. The final form of a mathematical definition may seem obscure to him; but it should at least be possible to provide enough experimental background to make evident the need of such a definition. New experiences should be related to old; each new concept should be developed gradually; and the impression that mathematics is a bag of tricks should be avoided at all costs.

Of fundamental concern to the student of mathematics for general education is the nature of the subject itself. Just what is mathematics?

While talking to a young student several years ago, the topic of plane geometry was mentioned. It so happened that she had successfully completed (she earned the grade of "A" in each six weeks period) a course in
plane geometry the year before as a sophomore in high school. She mentioned several axioms that the class had used during the year, and unhesitatingly revealed that she believed these axioms to be true without question. Further questions made it evident that she felt that the theorems to which these axioms led describe in detail and without error the world in which we live.

A discussion like this was repeated twice this past year; each time the students in a required freshman mathematics course were asked the same questions concerning the axioms of the plane geometry course they studied as high school students. Without exception the students who had studied plane geometry gave the same responses as had the girl in the earlier discussion. It would not be in error to conclude that these students have acquired a narrow, and in fact an incorrect, impression of the nature of mathematics. That such an impression does result from our teaching is sad considering the fact that so much has been said and written during the past fifty years concerning the nature of mathematics.

In the eleventh yearbook of the National Council of Teachers of Mathematics, E. T. Bell says,

Let us see what the dictionary has to say about axioms. In the unabridged Webster's New International "axioms" is defined in "Logic and Mathematics" as
follows: A self evident truth, or a proposition whose truth is so evident that no reasoning or demonstration can make it plainer; a proposition which is necessary to take for granted; as, "The whole is greater than a part," or "A thing cannot, at the same time, be and not be." This gives a fairly comprehensive summary of the major misconceptions which have been held in the past regarding the mathematical status of axioms. Detailed comment seems unnecessary. The first example, about the whole and the part, is particularly unfortunate; for it is neither necessary nor true in a vast region of mathematics. This definition contradicts the whole history of modern mathematics. It might have satisfied Plato or Euclid, but it should satisfy no one who was born later than 1826. (3:139)

R. L. Wilder contrasts geometry as seen by the Greeks and the modern view of geometry.

For, in the Greek culture, geometry was considered either an idealistic description of real space dictated by natural phenomena, or a doctrine imposed by a philosophy of absolutes; whereas in our culture the analogue of what the Greeks called geometry is only one of several co-existing geometries, each of which embodies only a special mathematical concept. . . . In its modern axiomatic form it is something quite different from what the Greeks considered it. (31:271)

The mathematician is really quite free to construct any world which meets his fancy, being restricted only by the set of postulates which he is able to conceive. If he cannot find a set of physical phenomena in the real world which obey the logical scheme of his structure it simply means that he has not been able to apply his mathematics, but this in no way invalidates his mathematical scheme.
Mathematics, perhaps more than any other subject, has been misunderstood with respect to its true nature. Of all the sciences, mathematics alone is credited with revealing absolute truth. And yet mathematics has very little to do with meaning and truth, its only real concern being that of validity. Mathematics is confused with computing wizardry and sometimes accounting; and all too often mathematics is equated with one or more of its tools: formulas, equations, sentences involving "x," graphs, and charts. Mathematical laws are often considered as unchangeable and fixed, when in reality the mathematician changes these laws to meet the needs of whatever topic is being discussed. There is widespread belief that mathematics has reached its full development, that there is no longer a frontier to explore, but actually mathematics is expanding so rapidly that it is impossible for any individual, however gifted, to keep abreast of all of its developments. All of these misconceptions, and others too, lead to serious problems in communication.

A course in general mathematics might very well begin with a discussion of the nature of mathematics. Specifically the statement of some theorem might be mentioned, preferably one which may not be accepted as
"true" by the students, and one which will serve in other capacities throughout the course. It must be kept in mind that the number of topics discussed in such a course should be limited so that students have a chance to digest whatever is discussed, and, consequently, that the material which is chosen must be carefully screened. The Pythagorean theorem is of extreme importance, and, in the experience of this writer, many students do not accept it. If this is the case a second statement is made (chosen by hindsight as one which will eventually be used in a deductive chain of reasoning) and is put to the test of acceptance by the students. If this statement is not accepted a third statement is produced. And so it goes, a sequence of statements is produced, each somehow more primitive than its immediate predecessor and thus more believable; but each statement is chosen so that if it were accepted then its predecessor would likewise have to be accepted. Thus the doubter of the original statement is backed into a corner if he does indeed finally accept some statement as "true." For by accepting this statement he must accept any statement implied by it until he must eventually accept the originally doubted statement. But what about the more primitive statements, from where do they come?
The manner in which the mathematician arrives at his primitive statements, called axioms or postulates, is an excellent opportunity to discuss the intuitional aspects of mathematics. Even though the role of deduction is supreme in mathematics, the mathematician relies heavily on his powers of intuition and experimentation, and his wide range of experiences. The final publishable form of mathematics is nearly always in the deductive style, beautifully elegant but hinting not at all at the intuitional, experimental basis behind its picturesque facade. It is really dishonest to portray mathematics in its deductive trappings alone, leaving to the imagination a mystic aura surrounding the particulars of its birth.

Much of today's mathematics was discovered through experimental processes and what might be termed intuition. Mathematics is not divorced from the real world, but rather derives much of its motivation from nature. To teach mathematics only as a deductive science and to neglect the role played by experience, intuition, and experiment is to neglect a valuable source of insight. As Morris Kline put it, "The current emphasis on deductive structure is at the very least overdone. Mathematics is a series of great intuitions supported by deductive proof" (17:68).
To experimentally invent certain axioms concerning parallel lines, for example, set down on paper two line segments of fixed length which meet each other at a small angle. Connect the two free ends by a third line of appropriate length. Rotate the two given line segments in stages, correcting the length of the third line so that it always connects the free ends of the two given segments. Continue to rotate until the angle becomes nearly a straight angle. At this point draw an auxiliary line from the point of intersection of the two given lines so that it is parallel to the third line. Now rotate the given lines in reverse direction so that the angle becomes smaller, but adjust the auxiliary line so that it remains parallel to the third line. For further information concerning this construction refer to the Appendix. Students will see all kinds of relationships here and will eventually come up with those axioms concerning angles formed when two parallel lines are cut by a third line. The same figure can be used to see the relations between the areas of rectangles, triangles, and parallelograms. With these axioms as ammunition it is now possible to prove the Pythagorean theorem. At this time it would be well to discuss the meaning of proof; when should a person be satisfied that a proof has indeed been constructed? And again what is the meaning of truth and validity? It has
been asserted by many educators that a major goal of general education is the training of the mind for intelligent action. If this be the case then surely the study of logic must be part of that education, and logic in its purest form is mathematics.

The fact that different geometries result from different sets of axioms might come as a shock to some students, but nevertheless the risk of shock is well worth the dividend paid in mathematical understanding. The world of Poincare (32:14-25), a world in which the shortest path between two points is curved, a world in which distances grow shorter the further one moves away from the world's center, is not too difficult to discuss in a general mathematics classroom and, in the experience of this writer, certainly creates enthusiasm. This enthusiasm is rewarded in mathematical understanding when one realizes that either Euclid's or Poincare's geometry may be applicable to our world; or for that matter that other geometries might work as well.

Mortimer Adler spoke of the liberally educated man as the one "who manifests all the goods which belong to the intellect." (1:15) And by goods he meant "the truth and various ways of getting at the truth." Mathematics above all things is a unique style of thinking,
characterized by three features: (1) the method of reasoning called deduction, (2) insistence upon rigor, and (3) the quest for generalizations (27:445). Mathematics has been described and defined many times and in many different ways, but in nearly all attempts at definition there is stated or implied the goal of utmost generality. Many educators believe that the ideal of universalized thinking is realized more easily in mathematics than in any other subject. It is in mathematics that one may most easily divorce logical structure from those objects to which the structure might apply. The study of group theory is typical of many in which great generality is achieved, and yet it is simple enough to be understood by the average student in a class of unselected college freshmen. Simple and interesting examples of groups can be constructed, and postulational systems may be illustrated by finite groups in which each of the group axioms may be checked in all cases. But most important, the group idea may be extended to the idea of a field which is tied in with our familiar number systems. Thus algebra, as taught in high school, may be put on an axiomatic basis, and the rules and gimmicks once merely taken for granted are encountered as theorems in a highly organized mathematical science.
In keeping with the notion that new concepts should be developed from old, one would be ill advised to begin the study of groups by simply stating the necessary axioms. Some sort of groundwork which will lead to the desired axioms should be set down first. One good way to achieve this groundwork is through an informal discussion of the real number system. The axioms of the real numbers are known to many students; and, if they are not known, they are readily accepted as being reasonable. Certain of these axioms are then selected as forming the framework of the group concept. At this point the group concept is entirely abstract, its only relation with real numbers being that the real number system, or rather certain parts of it, are applications of the group concept. But there are many other applications which can be mentioned, and many of these have nothing at all to do with what is usually considered as mathematics. The study of groups is admirably suited to perform the task of portraying the postulational aspects of mathematics as well as providing an example of a mathematical structure. In addition to these advantages it makes it possible to view high school algebra as a mathematical science rather than as merely a collection of tricks.
Besides portraying the nature of mathematics, a course designed for general education has other promises to fulfill. It was mentioned earlier that such a course should present mathematics in its cultural-historical perspective, should improve students' communicative skills, and should give them some idea of the tremendous range of applications of mathematics. In presenting the material chosen to illustrate the nature of mathematics certain communicative skills are bound to accrue, and if the material is properly taught an historical setting will be evident. However, there are further possibilities.

For thousands of years man has been intensely concerned with the study of nature. Of primary importance in this study is the ability to use and to understand mathematics. For the purpose of applying mathematics to the study of nature, two aspects of the subject are immediately apparent. An understanding of mathematics has enabled man to build bridges and satellites; but, of even more importance, mathematics sheds its light on those cardinal questions concerning the nature of the universe and man's position in it, questions such as How old is the universe? How do the planets move? and How far away are the stars?
There is alive today the belief that one is able to make experiences more meaningful and positive if only he has some understanding of the sciences. This belief is ages old but it gains credence as scientific achievements advance; and no one will deny that this is an age of science. Every well educated person needs to understand the various sources of human knowledge, and certainly one of these sources is the method of science. "Our urgent need today is for an American culture in which science and technology would play their role as two of the many parts of a harmonious whole. To this end, a genuine knowledge of the basic sciences must be the possession of every man" (18:234). But to understand science at all one must come to grips with certain phases of mathematics. As science is basic to our civilization, the elements of calculus are basic to science.

It would appear that a subject of such scope and magnitude as calculus should be known to all educated persons. This is not to say that it is possible nor even desirable to give all college students a thorough understanding of the calculus. But it is possible and it is desirable that some of the time allotted to a general mathematics course be devoted to enlarging the students' vocabulary to enable them to effectively communicate with those persons who do specialize in science.
One of the really major problems we face today, which can, in part, be traced to overspecialization, is that of communication. It is all but impossible for individual groups of learned men to communicate with one another; this is especially true between scientists and non-scientists. In the days of the totally prescribed curriculum, nearly every educated person shared a common culture, and this included a modicum of scientific knowledge. But today there exist very few experiences shared in common by educated laymen and professional scientists including mathematicians.

Certain mathematical terms appear quite frequently in the literature of science, and these same terms are used frequently in several of the forms of mass-communicative media, and in everyday speech as well. There is a need in our society of what might be termed communicative closure. However, knowledge of terminology itself will not guarantee this closure; what is needed is an understanding of the concepts to which the terms refer. The understanding of mathematical terminology and correlated ideas should surely play a major role in any course designed for general education. The terminology of calculus appears with ever increasing frequency in scientific literature. Terms such as limit, derivative, differential equation, and integral are used to such an extent, and the ideas to which they refer
are so seminal to science, as to make at least a passing acquaintance with them imperative for all educated persons.

In keeping with the practice that mathematics should, whenever possible, be presented within a framework of an experimental development, the study of calculus might well begin with a discussion of certain experiences common to every student. Most everyone has at one time or another calculated the average velocity of an automobile trip from one city to another. But if you were to ask them to calculate their speed of travel at a specific instant of time during that trip they would be hard pressed for an answer. They could, however, calculate the average velocity for that interval of the trip between the specific instant in question and an instant either preceding or succeeding the given instant. For example, if the two instants represent the times when they traveled through two cities along the route of travel, the average velocity between these two cities can be calculated. And if a third instant of time representing a third city even closer to the original city is chosen, an average velocity can once again be calculated. And thus a number of average velocities are obtained, each succeeding one calculated over a shorter interval of time. If the distance traveled is a function of time, and if average velocities are calculated on either side of the given
instant of time, two sets of average velocities will result. Students almost without exception will see a pattern between these two sets of average velocities, in that both sets appear to be getting close to a common number. They will guess the value of that seemingly common number and probably call it instantaneous velocity. The problem for the mathematician arises when one attempts to ascertain what is meant by "getting close to."

This problem, although hinted at for many centuries, waited upon the careful definition of the limit concept for its solution. Before attempting to define limit it might be well to discuss one of the paradoxes of Zeno. Consider, for example, the situation in which the fleet footed Achilles is to run a foot race with a very slow tortoise. To be fair the tortoise is given a head start. It is argued that while Achilles runs to the position the tortoise first occupies the tortoise runs on, and then while Achilles runs to the new position of the tortoise, the tortoise once again will run to a new position. The race continues in this way making it impossible for Achilles to catch the tortoise. This situation is easily visualized and quite puzzling to students. That this paradox contradicts everyday experience is apparent and would seem to necessitate an explanation of its statement. The explanation involves a discrepancy between our idea of
continuous motion and the mathematical language used to express the logical significance of that motion. It was not until the French mathematician Cauchy clarified the limit concept that this paradox was finally understood. We must disregard any prior intuitive feeling for the situation and rely instead upon a mathematical construct. Intuitively we think of a runner passing from one position to another by successively passing through all intermediate points. In mathematics however there is no such thing as a next point in the so-called continuum. The mathematical explanation replaces the appearance of "continuous" motion by a static non-moving explanation. With the introduction of the limit concept, not only were Zeno's paradoxes clarified, but mathematics and science in general were the recipients of an abstract concept which paved the way for development of calculus and all of its many applications.

Once the concept of limit is defined it may be used to shed light on several of the topics already discussed, namely the nature of real numbers, and the idea of instantaneous velocity. It is a good practice in the teaching of mathematics to run the thread of particular topics throughout the entire term of the course. Each succeeding time a particular topic is met its treatment becomes increasingly sophisticated and usually better understood. With respect
to real numbers the limit concept makes it possible to prove that any repeating decimal represents a rational number, a result only hinted at before.

The concepts of average and instantaneous velocities coupled with the definition of limit lead naturally to the the abstract concept of derivative. In the case of the foot race between Achilles and the tortoise the limit concept involved dividing an interval into a discrete number of points, the corresponding limit concept may be thought of as a "leap frog limit." In the case of instantaneous velocity we allow the time interval to squeeze down on a particular instant, but surely it does so in a continuous manner, and thus we here think of a "continuous limit." Thus it is quite natural to define instantaneous velocity as the limit of a number of average velocities each calculated over an interval which is allowed to continuously shrink.

In our development of calculus thus far we rely on concepts familiar to every student, but we quickly find our experiences too narrowing, that is, we have trouble when attempting to determine instantaneous velocity. However, by experimenting with average velocities we see the direction in which to proceed in order to come up with a plausible concept of instantaneous velocity. Using this knowledge as a lever and departing from all physical meaning, we define the concept of limit. Using this well-defined abstract
mathematical concept along with our knowledge of the manner
in which average velocity is calculated, we define the
equally abstract concept "derivative." The derivative
having received its initial birthright from everyday ex-
periences may now be seen as a tool to explain various
phenomena of nature through its many and diverse applica-
tions. The cycle is once again complete. Out of experi-
ence is developed a concept so general that it can be
applied not only to the experiences from which it was con-
ceived but also to other seemingly unrelated experiences.
Most students of general mathematics will understand the
derivative as here presented, and nearly all of them will
be able to use the method of definition to calculate deriva-
tives from simple polynomial expressions. After several
examples of obtaining derivatives of polynomials of various
degrees, brighter students will guess at and come up with
the power and sum rules of differentiation. It may not be
possible nor even desirable here to develop these formulas
with any degree of rigor. The important point in this
entire development is that a mathematical concept is
defined in conjunction with physical experiences, that is,
students must first see a reason for defining the concept.
In this way they are able to participate in its develop-
ment and appreciate the significance of its meaning. After
the concept is defined it is important to return to
experiences exhibiting its usefulness in a variety of situations.

With this knowledge of the derivative as background it would be well now to give the students a particular derived function, asking them to find the function from which it was derived. Again the brighter students will probably come up with the correct function, and if polynomial expressions are used, they will most likely recognize the "gimmick" which will give the correct answer. Several students might even recognize the fact that an additive constant does not change the value of the derivative of a function. By naming the latter type of problem as an example of a differential equation, one more link in the case for communicative closure is secured. Aside from noting the vast importance of differential equations to various problems in science and engineering, nothing more need be said of them.

The second major concept of calculus, the integral, can also be introduced to these students, and, like the derivative, it too can be developed in an experimental manner. This can be accomplished in one of several different ways, the choice depending upon the background of the students involved. If the students seem inclined towards problems from physics, then it might be well to introduce
the integral by discussing a work problem, and specifically that of stretching a spring. If they are not so inclined then it might be best to discuss the problem of calculating the areas of regions bounded by curved lines. The latter approach has the advantage of being quite prominent in the course of history. The Greeks were interested in the problem of calculating the areas of non-polygonal regions such as circles and regions bounded by parabolic arcs. The only areas they knew how to compute were of shapes bounded by straight line segments. To apply this knowledge to the problem of finding the area of a non-polygonal region, they placed within the region a number of polygons (rectangles for example), and concluded that the desired area is approximated by the sum of the areas of the inscribed polygons. It was obvious to them and to most students that the approximation can be made better if a larger number of rectangles is placed within the region in such a way that the largest rectangle becomes smaller. We have then a set of numbers, each succeeding one a better approximation than its predecessor. It is a natural step, although it wasn't natural for the Greeks, to think of the desired area as a limit of the approximating areas, the limit considered as the number of rectangles placed within the region becomes large without bound, and such that the area of the smallest rectangle becomes small without bound. The abstract concept of
integral is then defined using the above experimental procedure as a springboard. Since it is an abstract concept it can be applied to any situation in which physical entities can be found which are not contrary to the mathematical scheme of the concept, although no physical applications need exist at all. Such applications as area, volume, work, and gravitational problems can be mentioned, and the simpler ones illustrated. Thus the integral, like the derivative, is defined to answer a physical question, that is, to solve a problem. But once defined it is entirely abstract and can be applied to any problem with the proper qualifications.

A course incorporating each of these ideas was designed and was taught to two groups of college freshmen who were taking mathematics solely to satisfy graduation requirements. The course included four major topics of discussion, Non-Euclidean geometry, group theory, the derivative, and the integral. With regard to the selection of content for a general mathematics course there is no unanimity of opinion. It is hoped that the subject matter represented in the course as given in the Appendix is consistent with the opinions of the various educators who have been quoted and reviewed in the foregoing pages of this paper. The attempt has been made to design a course which will incorporate those topics which seem to be most important
to a large number of educators. Because of their small number these topics can be discussed with a thoroughness impossible to attain in the usual survey type course. The term thorough is not to be confused with comprehensive. Considering a topic of such magnitude as integral, for example, one would be unrealistic to attempt any sort of comprehensive coverage of it in one semester. Even if it were physically possible to do such a thing, students in the usual general mathematics class do not have the background to appreciate the effort. However, thoroughness of instruction is achieved in that those topics which are discussed are thrown into many different combinations, and certain of these topics appear again and again throughout the course, each time becoming somehow better understood, and certainly seen in a different light and related to different applications.

As important as the selection of topics might be, the manner in which a topic is taught is really just as important. Students should not be allowed to remain ignorant of the manner in which much of mathematics is developed. It is too often the case that mathematics is presented exclusively in its deductive trappings. Students should be allowed to participate in the development of the topics which have been chosen for discussion. They should participate in
the formulation of postulates, or at least be made aware of their origin. That which is taught should not be taught exclusively in the manner in which it might be written for publication. The several topics chosen for the course as presented in the Appendix were taught in this spirit, and it is hoped with some success.

Many mathematics educators have expressed the opinion that the nature of mathematics should be made known to students of general education. To this end the course begins with a discussion of what it means to prove something. The essence of what comprises a mathematical system is studied at some length. The question of whether it is possible to begin with different sets of postulates is examined, and to this end a geometry different from that of Euclid's is presented. For homework students were asked questions which required answers quite different from those required in the usual mathematics assignment. Students were asked for example to discuss conditions under which two quadrilaterals may be congruent. They were asked to discuss terms such as valid, true, and deduction, and they were asked to determine if conclusions obtained by deductive reasoning were always true. They were asked to make up examples in which different sets of postulates imply the same conclusion. Other types of homework problems may be found in the Appendix.
To illustrate the abstract nature of a mathematical system, the theory of groups was discussed. And here, it would be well to mention, is precisely the spot which gave most students their most difficult moments. They found it difficult to prove even the most elementary theorems. One might conjecture and probably rightly so that this difficulty can be located in the abstract nature of the topic. Students became somewhat better acquainted with these theorems after interpreting them in terms of algebraic techniques they learned in high school. Even though the topic is abstract students eventually caught on to the various methods of proof, and some of them eventually constructed proofs of their own. The bonus of teaching group theory came about through the many and seemingly unrelated examples of groups. Modulus systems were introduced for example, but were received by a somewhat less than enthusiastic welcome. Many students were skeptical of the value of studying such seemingly artificial systems. They were not quite ready to study mathematics as a deductive system quite apart from its many applications. However they were eager to learn of the possibility of constructing coding systems based upon modular arithmetic, and were somewhat amazed to learn of the breaking of the Japanese code which was based on such a system. Here students were asked to code certain phrases which they had made up, and then other
students were asked to decode the messages. Without excep-
tion the students enjoyed doing this, however, it is not so
certain that all of them appreciated the theory of groups.

The system of real numbers was developed somewhat
rigorously although no mention was made of Dedekind cuts
or of nested intervals. Intuition and experimentation were
employed wherever possible. It was felt that in a terminal
course in mathematics it would be better for the students
to gain a "feeling" for the development of a topic rather
than allow them to become bogged down in a rigorous discus-
sion which would at best be understood by only a small number
of students. For homework students were asked to analyze the
why behind the fact that a rational number can be expressed
as a repeating decimal, to write the rational number cor-
responding to a repeating decimal, and to discuss the
statement \( \sqrt{2} \approx 1.414 \). Once again other homework examples
may be found in the Appendix.

It has been supported by quotations from several
different sources that our college graduates must be made
aware of the many bases from which knowledge is garnered.
One of these is science, and the language and methods of
calculus are certainly basic to scientific investigation.
The derivative and integral were included in the course,
and once again were developed in an experimental manner.
It is true that the discussion of calculus provided the highlight of the course for most students. This was due most likely to the many practical applications of the subject, but in addition most students have heard of calculus from one source or another and it was probably true that their curiosity had been aroused. Students did well here and toward the end of the discussion were capable of handling many of the problems found in second semester calculus texts. It should be pointed out that students were not given a rigorous development of the subject, but such is not the goal of a terminal course. Homework problems for this section may be found in the Appendix.

The final examination for the course will be analyzed in the next chapter along with student evaluations of the course.
CHAPTER V

COURSE EVALUATION

The course as it appears in the Appendix of this paper was presented twice, each time to a group of Ohio State University freshmen who were required to complete only one five quarter hour course in college mathematics. Each time the course was given, a group of students was chosen from those who had enrolled in the regular freshmen mathematics sequence, and who had expressed the desire to enroll in a special mathematics course. Of those who said yes they would like to enroll in such a course, twenty-six were chosen for the first quarter's offering, and thirty-eight enrolled the second time the course was offered. At the conclusion of each quarter's work, students were invited to submit a short evaluation of the course. They were instructed to leave their evaluations unsigned. Of the fifty-nine students who completed the course, fifty-two submitted evaluations.

In general these evaluations were favorable, the major criticisms being directed toward the lack of a textbook, and the scarcity of homework assignments. Students
felt that they were handicapped by not having a textbook to refer to when studying. They expressed concern over their inability to properly take notes; and without a text, classroom notes became all important. Several students expressed the belief that too many topics were presented during the quarter's work. Considering that only four major topics were included, such criticism would seem to indicate that the "usual" introductory college mathematics courses include too many topics. Most students expressed either a positive dislike or at least a lack of interest in mathematics when they began the course. Several students attributed their dislike to experiences in high school mathematics. Yet, these same students, after completing the special mathematics course, admitted that mathematics is not as "terrible" as they once thought, and, in fact, appears to be quite interesting. Several of these evaluations will be presented here in part to illustrate the change in feeling toward mathematics.

After transferring from the College of Arts to the College of Commerce and finding out a need to take a math course, I found myself looking back to my previous high school mathematics. In High school my academic life included Algebra I, II, III; Trig.; Plane Geometry. Thus I felt the first math course I took here would be just as boring as those in high school. However, I do feel my predetermined judgments were unjust. Truthfully I have found this experimental course in mathematics very educating and also very interesting.
Math has not been one of my favorite subjects and I even dreaded taking this course. I thought it would be one of those courses where the professor said do it this way and I would do it that way and not learn anything. The manner in which this was taught not only gave me a greater interest in mathematics but I feel that some of it can be applied in my life. This course was thoroughly enlightening and I would advise the teaching of Math 416 like this all of the time.

This course has changed my outlook on mathematics. I have always disliked math and have done poorly in it because I did not usually understand what it was.

I personally have found the course very intriguing; it was enjoyable in the sense of enlightenment; and it most surely provides the student with a "liberal" approach to mathematics. All in all, I got quite a lot out of the course except a good grade.

When I first signed up for Math 416, I dreaded the day when I would have to go to the first class. I had a bad experience with a math course in high school and ever since then I have dreaded the thought of taking another math course. But, now, after taking your "experimental" course, my interest in math has suddenly increased. I never thought that math could be so interesting and have so many practical purposes.

I still can't believe it. I can say that this is the most interesting, best performed, best taught course I have taken during my five quarters here at Ohio State. My only regret is that the quarter is over. You have given me a positive approach to math.

Since I was in no way required to take math at Ohio State (and was thanking my lucky stars), it came as a blow when my major adviser suggested pointedly that nobody should go through four years of college without some math. I am afraid I entered the quarter with the wrong attitude, like "Let's get it over with." It has turned out to be my most enjoyable course this quarter. I had to admit this when I found myself engrossed in a world where the shortest distance between two points was a curved line, and the further out in a spaceship one traveled, the smaller he got, till he disappeared. It caught my imagination and put all sorts of new ideas into my head. At times I have been tempted to delve into a layman's translation
of some of Einstein's theory of relativity, etc., something which I just might do someday. A course which can prompt this sort of reaction is definitely serving a worthy purpose.

I did not realize math could be so interesting until I took the special Math 416. I have found out math is not all adding and subtracting.

I have been able to obtain a new and clearer meaning of many things that I have had in former math courses which at the time seemed to be only words and/or numbers.

Experimental Math 416 has been one of the most interesting math courses I have ever taken, probably because I could understand it. It cleared up many facts that weren't very clear in my math background, and gave me plenty of room for thought. To help in the beginning of each section you present the math in English and you make sure you get the point across before going into figures."

The last evaluation illustrates one of the basic assumptions used in designing the course, namely, that whatever is taught in a general mathematics class should be taught so that it is understandable to the student of average intelligence; especially is this true at a public university. The student whose evaluation is quoted above considers the course interesting because he can understand it. Another student commented that "explanations were extensive enough for everyone's understanding." Another student notes, "Thank heavens it didn't start "If A then B, B if and only if A" like my last dry, dry math class. Rather, we attempted a survey of mathematics. . . . This is
the way I think math should be examined, first in a broad way to see its importance, application, and get a small glance of its epic proportions." To this writer, such comments get right to the heart of the matter. In order to learn a concept students must first become aware of the need for such a concept. It seems that students are in the habit of prejudging that which they are asked to learn. If they see in their own mind some reason for learning the concept they at least open their minds to its reception. However if a topic is introduced in its minutest detail, without first pausing to discuss the relevance of such a concept, it is likely that any further teaching concerning this concept will fall on deaf ears.

Another objective of the course included the desire to make known to students the meaning of certain important mathematical terminology. Here too, student evaluations indicate that the course was somewhat successful.

"Group theory," "sequences," "functions," "calculus," were all words dropped wickedly by math and physics majors, to impress the poor liberal arts student. Not any more. I at least have some basis for nodding acquaintance.

This course has fulfilled one purpose of which I know. It has given me a much broader conception of mathematics and its application to other fields. I have come to realize the great importance of mathematics in our society today and in years gone by. I had no idea what calculus was before I had this course. It seemed to me to be just some higher form of
mathematics. I now have some idea of what calculus is, and what a few of its applications are.

On the whole I was very interested in the class (I don't think I've ever had such regular class attendance in a course) and I feel it was very worthwhile. And besides that, it gave me something to write to my roommate from last year (a math major).

I never learned about functions, sequences, non-Euclidean geometry, or calculus in my previous math study and presently, I now know enough about these topics to act somewhat learned when others mention them. In other words, I have broadened my intellectual mind in the field of mathematics.

Now I am able to converse with engineering students somewhat and actually know a little of what they mean when they talk of velocity and so forth. This course has given me some intellectual power for conversation.

Many of these evaluations indicate that the course was somewhat successful with respect to objectives other than those for which the evaluations were specifically chosen to illustrate. For example, in many of them is expressed either explicitly or implicitly, the opinion that the nature of mathematics has been meaningfully portrayed and understood by the students. And certainly these evaluations reveal that at least several students are now aware of the methods of calculus as important bases of human knowledge. The following reveal evaluations of a more personal nature.

You showed me how mathematics can be used in our everyday thinking, and how it can be used to solve practical, down-to-earth problems.

The course showed me how to attack problems of every nature, not just problems in mathematics.
Thanks a lot for opening a few mathematical doors for me.

According to the students these evaluations would indicate that the course was successful in meeting its intended objectives. For many of these students it proved to be an eye opener. They began its study with a dislike of mathematics, a dislike nurtured in several cases by high school courses in which mathematics was presented as a bagfull of tricks with little more use than solving a list of boring drill exercises. No one had ever bothered to present it as an exciting creation of man, pregnant with meaning and utterly irreplaceable in man's quest for knowledge of nature as well as of constructs of the human mind. It is hoped that the course as here presented does in some measure present mathematics in a manner which does justice to the nature of the subject.

No formal comparison was made between student achievement in this course and student achievement in any other course. In fact it is probably true that an effective teacher can teach anything he wishes to teach, and his students will probably do well on examination questions designed to test competence in those skills and areas of knowledge he has sought to teach. Even so a final examination was given each time the course was taught and certain of the results on that examination might be of interest in determining the effectiveness of the course.
The final examination (found in the Appendix) was taken by twenty-two students the first time the course was offered and by thirty-seven students the second time the course was offered. The median score was 74 per cent for the first group, and 77 per cent for the second. Top scores were 91 per cent for the first, and 98 per cent for the second. Low scores were 30 per cent for the first class and 39 per cent for the second. The low of 30 in the first class is somewhat misleading since the second low score in the class was 51 per cent. Although no formal comparison of examination scores was made between these groups and any other groups, these scores are quite respectable in terms of similar examination questions given by this writer to other groups of students. The second group of students scored somewhat better on the examination than did the first. Several reasons might be advanced to explain this difference including: (1) The examination was the same for each group and even though exams were not returned to the students, and one quarter elapsed before the course was offered for the second time, there may have been some carry over, and (2) certain parts of the course which seemed to give students the most trouble were given more attention the second time through. The same examination was given both times to see
what effect certain changes might have on student achievement.

The results of the examination were very encouraging, and when coupled with student evaluations would seem to indicate a certain amount of success for the course. If the course were to be taught again certain changes would be in order. For instance more time should be spent on the function concept, not necessarily a more rigorous development, but more practice in evaluating functions and writing the algebraic or graphic rule for a function given a verbal description of it. And after the topic of inequalities is introduced it might be well to graph the solution sets of systems of inequalities, and finally to introduce problems from linear programming.

Concerning the manner in which the course was taught certain changes are in order. The course as it is presented in the Appendix represents day by day teaching notes. These notes were presented and discussed each day of class. The students were expected to take their own notes. They were then asked to go over their notes at home and put them in good order. If they did not understand any portion of their notes they were instructed to ask questions the following class period. In this way it was hoped that each student would create his own text for the
course. It turned out that some students were unable to take good notes and were thus somewhat handicapped. If this course were repeated it would be well to mimeograph the notes and distribute them each day. This procedure, however, will lose its effectiveness if students do not study these notes day by day. When taking their own notes they are at least forced to reread them in attempting to put them in good order. In order to make mimeographed notes effective, homework assignments would have to be made quite regularly. Of all suggestions made by students, that one suggesting more homework assignments was most prevalent.

All things considered the course does seem to have been successful. Certain parts of it need improvement, and it may be that the selection of topics is not the best. However the idea of the course does seem to be sound. A terminal course in mathematics designed for students of general education need not consist solely of cultural-historical material, nor must it be restricted solely to the portrayal of the nature of mathematics. It can do both of these things at once, and at the same time preview several of the important topics of mathematics without assuming the stigma of a survey course. For better or for worse, that is what this writer has attempted. The result of his effort may be found in the Appendix.
APPENDIX
Mathematics has been described as a game in which one player forces his opponent to accept a statement he doesn't really want to accept by skillfully driving him into a corner from which he cannot retreat. To accomplish this victory a series of statements is made, each of a more primitive nature than its immediate predecessor. This goes on until a statement is made with which the opponent does not disagree, indeed a statement which is immediately accepted. From this humble beginning the die is cast, the opponent is doomed to accept his fate. For upon this statement is reared a logical structure from which there is no escape, and ultimately the statement which was first denied must now be accepted.

Consider the statement that the square drawn on the hypotenuse of any right triangle equals the sum of the squares drawn on the other two sides of the triangle. This statement known as the theorem of Pythagoras may be put into algebraic form: if c represents the length of the hypotenuse, and a and b the lengths of the other two sides then $c^2 = a^2 + b^2$. Although this theorem bears the name of
Pythagoras it was known to the Babylonians and perhaps to the Egyptians centuries before his life began. It may be fairly said that the theorem is called Pythagorean because of the Greeks' attitude toward mathematics, and indeed toward all of knowledge. The cultures of the Babylonians and Egyptians differed in at least one respect from that of the Greeks. The former were builders, their business to get the job done, their mathematical tools of trade were not questioned. It was enough for these people that their "tools" worked, gave correct results, helped to build lasting monuments. The Greeks on the other hand were an idle people, and as such were free to devote their time to speculation. They were lovers of knowledge; it was not enough for them that a tool worked, they had to know why it worked. They devoted their time and energy, not to the use of tools, but to the art of proving their universality.

The Greeks were responsible for giving mathematics its definitive form. As philosophers, they sought only after truths, and what better way to discover the truth than by deductive reasoning. They were wise enough to realize that reasoning by both induction and analogy often lead to error. One is reminded of the Chinaman who reasoned that all mankind possesses yellow skin; his reasoning based on the coloring of four hundred million Chinese.
The Babylonians were content to arrive at the theorem known now as the Pythagorean Theorem by an experimental process. They might have, for example, noticed the pattern of floor tiles laid in one of their temples, and within observational limitations conclude that the area of the square drawn on the hypotenuse equals the sum of the areas of the squares drawn on the two sides.

But this is not mathematics as perceived by the Greeks. The Greeks although initially introduced to the theorem through contact with other peoples, preferred not to accept this rule as the ultimate of experience. They desired to establish its truth on a more firm foundation constructed of simpler rules, and from these simpler rules called axioms or postulates deduce the more complicated consequences called theorems.
Deductive reasoning for the Greeks, and for today's mathematicians is the essence of mathematics. Formalized geometry got its start by abstracting a set of postulates from experience, but it soon left experience and traveled the way of deductive logic into a world of undreamed of theorems. The role of reason is supreme, but one must not forget the part played by imagination and intuition in the construction of a mathematical science.

The final publishable form of mathematics is to be sure of a deductive nature. Deduction lends the air of certainty one expects from mathematics, but the logically concise proof one sees in print is not the sum total of all that the mathematician has done in accomplishing this proof. There are essentially two processes which must be dealt with before a proof is forthcoming. First one must know what is to be proved, and secondly some method must be discovered which will result in its proof. In coming to grips with both of these processes the mathematician utilizes experimentation, observation, intuition, imagination, and certainly a great deal of "human creativity."

To illustrate the experimental, imaginative nature of mathematics, consider three lines situated as follows. One of these lines, call it the base, is to have a given
length; from one end of this base line a second line also of given length is drawn. Place the second line in such manner as to make a small angle with the base. Connect the two free ends of these lines with a third line of such length as to insure the formation of a triangle. Now successively rotate the second line increasing the aforementioned angle; make appropriate adjustments in the length of the third line in order to maintain the triangular form.

Note that the size and shape of the triangle is fixed as soon as the lengths of the two sides are fixed, along with the size of the angle between them. This might lead us to postulate that two triangles are equal in size and shape if both have two sides respectively equal, along with the angles included between them. Such triangles are termed congruent in plane geometry texts.

Continue rotating the second line until it nearly assumes the position of an extension of the base line, complete the triangle. As auxiliary lines, from the intersection of base and second line draw a line parallel to third side; also draw the base line extension. Label all angles. Rotate the second line again, this time in
the opposite direction, keeping the auxiliary line always parallel.

Observe a certain similarity in the sizes of certain pairs of angles as the second line continues its rotation. Angles 2 and 4 seem to remain equal as do angles 1 and 5. Generalizing this observation, we might postulate that when two parallels are cut by a third line angles interior to the parallels but on opposite sides of the third line are equal. Now since the sum of angles 3, 4 and 5 equals a straight angle (180°), and since angle 3 is one of the angles of the triangle, and since angle 1 equals angle 5 and 2 equals 4, we conclude that the sum of the angles of any triangle is 180°.

By drawing other auxiliary lines it is possible to discover other useful information. From the intersection of the second and third lines draw a line parallel to the base, and such that it intersects the first parallel. The figure formed by the base, third line, and both auxiliary
parallels is called a parallelogram. In this case the parallelogram is cut into two triangles by a diagonal. The second side is common to both triangles; angles 2 and 4 one from each triangle are equal from parallel postulate; the third side and the parallel to it from the other triangle are equal if one allows that parallel segments cut from parallels are equal. But these are precisely the conditions for congruent triangles. All of this indicates that the area of a parallelogram equals twice that of a triangle formed by a diagonal. Now by dropping perpendiculars to the base and its extension from the upper two vertices of the parallelogram a rectangle will have been formed. The area of this rectangle is the same as that of the original parallelogram (apply congruence test to small triangles), but since the parallelogram has the same base and height as the rectangle, its area formula must be identical to that of the rectangle. Note further that the original triangle also has these same dimensions (base and height), and since its area is half that of the parallelogram its area formula must be half that of the formula for the area of the parallelogram.
We are now in a position to prove the Pythagorean theorem. In this theorem and throughout the remainder of this course we shall make use of certain properties of equality (=), properties which in all likelihood are already familiar to you. These include the reflexive property, \( a = a \); the symmetric property, if \( a = b \) then \( b = a \); the transitive property, if \( a = b \) and \( b = c \), then \( a = c \); the substitution property, if \( a = b \) then \( a \) may be replaced by \( b \) in any sentence without changing the truth of the sentence; and the property which allows us to add equals to equals and once again have equals.

Statement of the theorem: in right angled triangles the square on the hypotenuse is equal to the sum of the squares on the other two sides. Consider the right triangle ABC, with appropriate squares drawn on each of the three sides.
Construct AL parallel to BD
Join F to C, and A to D

Now angle DBC equals angle FBA, since each is right; let angle ABC be added to each thus making angle DBA equal angle FBC, and since FB = BA, and BD = BC all conditions for congruent triangles are met, thus triangle ABD is equal in size to triangle FBC. Triangle ABD equals one-half of parallelogram BDLP, since both have the same base and height. By a similar argument triangle FBC equals one-half of square ABFG. If one accepts that doubles of equals are equal we may conclude that square ABFG equals parallelogram BDLP. By a similar argument it may be shown that square ACKH equals parallelogram CELP. By adding equals to equals we finally conclude that square BDEC equals the sum of squares ABFG and ACKH. (8)

We will not attempt to define mathematics, for that matter such a definition may be impossible to make, however, we will characterize mathematics as that science which deals with a set of statements, such that each may be derived from any or all of the preceding statements. Any body of statements satisfying this criterion will be termed a mathematical science; but what characterizes the body of statements? Certainly if each statement is a consequence of preceding
statements there must be at least one statement which
does not follow logically from others. In other words there
must be at least one unproved statement. Any mathematical
science then must contain at least one assumed statement or
postulate. Similarly any mathematical science must contain
undefined words, for in order to avoid cyclical definitions
certain primitive terms must be agreed upon as a starting
place. The question which confronts us is the nature of
these unproved statements (postulates), and primitive
terms. Are the postulates obvious, self evident, a-priori
truths? Would it be possible to think logically without
them? Are the primitive terms perfectly clear in meaning,
or is it possible that under different circumstances they
would conjure up different meanings?

In an effort to shed light on these questions, we
will look briefly at the first model of a mathematical
science, namely Euclid's Elements of Geometry written some
twenty-three hundred years ago. Euclid began his work by
attempting to define certain terms:

A point is that which has no parts.
A line is length without breadth.
A straight line is a line which lies evenly between
two of its points.
It might have been better had he left point, line, and straight line undefined, for in order to understand their meaning by the above definitions one must understand the meaning of the terms parts, length, et cetera. Euclid also incorporated postulates in his geometry.

I. It shall be possible to draw a straight line joining any two points.

II. A terminated straight line may be extended without limit in either direction.

III. It shall be possible to draw a circle with given center and through a given point.

IV. All right angles are equal.

V. If two straight lines in a plane meet another straight line in the plane so that the sum of the interior angles on the same side of the latter straight line is less than two right angles, then the two straight lines will meet on that side of the latter straight line.

Other postulates of a more universal nature were:

VI. Thing equal to the same thing are equal to each other.

VII. If equals be added to equals, the results are equal.
VIII. If equals be subtracted from equals, the remainders are equal.
IX. The whole is greater than any of its parts.
X. Things that coincide are equal.

Do these postulates form a foundation from which the entire science of Euclidean geometry can be deduced? Euclid throughout the development of his geometry made other implicit assumptions, e.g., the shortest distance between two points is a straight line. One may then conclude that this set of postulates is not sufficient to develop the entire geometry. Are these postulates absolutely true or would others work as well? Are the terms such as point, line, distance, et cetera, really self-evident as to meaning? In order to shed light on these questions Poincare created a hypothetical world, a world described by Young in *Fundamental Concepts of Algebra and Geometry* (32:14-25).

Imagine a world surrounded by a large sphere, and further imagine that the temperature within this sphere varies from point to point achieving its maximum at the sphere's center, and diminishing from there in direct proportion to the distance from the center until a temperature of absolute zero is reached at the sphere's rim. To symbolize this law of temperature change let $R$ be the
radius of the large sphere, and \( r \) be the distance from the point in question to the sphere's center. Then if \( t \) represents the temperature at the point and \( c \) is the constant of proportionality we may write:

\[
t = c(R^2 - r^2)
\]

We will also assume that as a body moves from point to point (points of different temperature) its size also changes being at an instant in direct proportion to its temperature. Thus as a body moves toward the spheres rim it becomes indefinitely small. Assume further that a body is immediately put into proper size for a given temperature.

Such a world is certainly within the realm of the imaginable, and further there is nothing contradictory in its postulates. But consider now an inhabitant of such a world, what type of geometry might he develop? Would his understanding of the term straight line be equivalent to Euclid's understanding? And what about the notion of shortest distance? Would his geometry be finite or infinite?

This world would certainly appear to be finite when viewed from the outside, but what about the viewpoint of the inhabitant? He would view his world as if it were
infinite in extent, for as he moves toward the rim of his world he would become cooler and his body smaller so that his steps would become so small that he could never reach the limiting sphere. His world would seem just as infinite to him as ours does to us. What about the property "shortest distance"? In our world we generally consider the shortest distance to be that path which requires the least number of steps to traverse, so to in our imaginary world. However there is a very definite difference, for in the imaginary world steps become longer as the individual gets closer to the world's center. It would seem reasonable then that in this world the shortest path is not that of a straight line but rather that of a curved line. In fact it can be shown that the shortest path between two points in this world is along a circle through the two points which meets the limiting sphere at right angles.

One further assumption will be made, namely that light does not travel along straight lines but along the shortest lines just described. Now if a man were to view an object, light from that object would reach the man's eyes by traveling along the shortest path. If the man were to move toward the object, always keeping the object directly in front of his eyes, he would walk with the least number of steps toward the object. So that in his geometry these
shortest lines serve the same purpose as do straight lines in our geometry. In fact these shortest lines would appear straight to him.

Concerning Euclid's Fifth Postulate:

Consider \(a\) and \(b\) as the two given lines and \(c\) as a third line cutting across both \(a\) and \(b\).

![Diagram](image)

If \(Q\) moves along \(a\) in either direction \(c\) will revolve about \(P\). As \(Q\) moves to the right a limiting position of \(c\) is reached; likewise if \(Q\) moves to the left. The fifth postulate implies that these two limiting positions of \(c\) are the same. That is through a point \(P\) there is one and only one parallel to a given line \(a\). What about parallels in the new world?

Corresponding to \(a\), we would have a circle which meets the boundary sphere at right angles. Pick a point \(Q\) on circle \(a\), and corresponding to \(c\) draw circle \(c\). Now for each position of \(Q\) on \(a\), there would be a definite circle \(c\) cutting the boundary sphere at right angles.
The question is "what are the limiting positions of circle c as Q moves in either direction along circle a"? Q would appear to approach R when moving to the right, but since this is an infinite world, Q would never reach R and indeed circle PR would be a limiting position. Likewise, circle PS would be a limiting position if Q were to move to the left. This situation in general leaves an angle θ between the two limiting circles. Now if we define the two shortest lines in the same plane as being parallel if they do not meet no matter how far extended, we see that there exists an infinite number of shortest lines through P all parallel to a.

Note: This discussion has been confined to a plane surface through the sphere's center. Corresponding to plane surfaces of Euclidean geometry we have spherical surfaces in the new world which meet the world's rim everywhere in right angles. A plane through the center can be thought of as a sphere of an infinite radius.
Is this new world geometry applicable to our own familiar world? Consider the boundary sphere as having an immense radius but still finite, and consider the earth as being quite near the center. Now shortest lines through the sphere's center are straight lines in our sense of the word straight, and lines near the center are nearly straight; in fact they may be made as straight as we wish by making them close enough to the center. With respect to the angle θ spoken of earlier, its size depends upon the nearness of a and P to the sphere's center. Or another way of looking at it, θ becomes indefinitely small the larger the boundary sphere becomes. Now if our earth were placed in such a sphere, a sphere with radius so large that the angle θ is so small as to be immeasurable, we would then have no way of knowing which geometry is applicable. If we were to postulate the existence of more than one parallel to a given line through a given point, it would not contradict any observation we might make.

Postulates are not self evident truths as once suspected; in fact a particular set of postulates may be such as to defy our being able to tell whether they are true or false. But even so a perfectly sound mathematical science may be reared upon these postulates.
The role of deduction as seen in geometry is familiar to many students, since Plane Geometry is a course taken in high school by many college bound students. Not so familiar, however, is the field of algebra studied as a mathematical science. But it is possible, and mathematically advisable, to elevate algebra to the status of deductive logic; too often algebra is taught as if it consisted only of a bag of tricks, easily learned but soon forgotten. To begin such a study we will assume a familiarity with the positive integers (1, 2, 3, 4, 5, ...). With these as building blocks an attempt will be made to trace the development of our present day number system. Our development will certainly not emulate the actual historical evolution of numbers, indeed such a task may be impossible, but rather we propose to examine the salient features of the positive integers, those features (postulates?) upon which a mathematical science may be reared.

The act of counting must surely be of pre-historic origin even though our sophisticated system of numeration was developed only recently in terms of all recorded time. Out of his early experiences with counting, man soon developed the idea of addition. In adding two sets of objects together, he began with the integer representation of the first set, and then by the process of "and one more"
he counted until the second set had been recorded. The final number counted was said to be the sum of the numbers of both sets. Moreover he noted that it made no difference to the final sum which set he began with. Now even though there be an infinite number of counting numbers and thus of pairs of counting numbers it seems reasonable to postulate the existence of a unique positive integer as the sum for any pair of positive integers. That is, if \( a \) and \( b \) represent any two positive integers, their sum, represented by \( a + b \), exists as a unique positive integer. And further it seems natural to assume that it makes no difference in which order the two numbers are added, i.e., \( a + b = b + a \).

When adding three numbers such as \( 4 + 5 + 6 \), we note that \( 4 + 5 = 9 \), and \( 9 + 6 = 15 \). We might also proceed \( 5 + 6 = 11 \), and \( 4 + 11 = 15 \), in either case the answer is the same positive integer 15. This would seem to indicate a third postulate; if \( a \), \( b \), and \( c \) represent any three natural numbers, then: \( (a + b) + c = a + (b + c) \). The parentheses here indicate the order in which the operations are to be performed.

Multiplication arose quite naturally as a short cut method to do addition problems. If one thinks of multiplication as repeated addition then surely the result of multiplying any two positive integers yields a product which is once again a positive integer. And it seems safe to assume
that any two positive integers may be multiplied in either
direction to obtain the same result \((a \cdot b = b \cdot a)\), and indeed
we may postulate that \(a \cdot (b \cdot c) = (a \cdot b) \cdot c\). To tie the opera-
tions of addition and multiplication together note for ex-
ample \(2 \cdot (3 + 4) = (3 + 4) + (3 + 4) = 14\), but \(2 \cdot (3 + 4)\) can
also be written \(2 \cdot 3 + 2 \cdot 4 = 6 + 8 = 14\), in general
\(a \cdot (b + c) = a \cdot b + a \cdot c\). If \(a, b,\) and \(c\) represent any three
positive integers, the above postulates may be summarized
as:

1) \(a + b = a\) unique positive integer (property of
closure)
2) \(a + b = b + a\) (property of commutivity for
addition)
3) \(a + (b + c) = (a + b) + c\) (property of associ-
ativity for addition)
4) \(a \cdot b = a\) unique positive integer (property of
closure)
5) \(a \cdot b = b \cdot a\) (property of commutivity for multi-
plication)
6) \(a \cdot (b \cdot c) = (a \cdot b) \cdot c\) (property of associativity
for multiplication)
7) \(a \cdot (b + c) = a \cdot b + a \cdot c\) (property of distribu-
tion)
Note the role played by 1: \(1 \cdot a = a\) for any positive integer \(a\). One is said to be the identity element with respect to multiplication.

In solving the equation \(a + x = b\), i.e., find the number \(x\) which represents the difference between \(b\) and \(a\), we note that there is at most one solution value \(x\), but in general the equation has no solution in the set of positive integers. In other words the equation \(a + x = b\) makes sense only if \(b > a\), where \(b > a\) is read \(b\) is greater than \(a\). Similarly the equation \(a - x = b\) has at most one solution but in general has no solution in the set of positive integers. Because of the lack of solutions to many problems within the set of positive integers, the set was expanded, first to include all integers (both positive and negative and zero) to insure a solution to the equation \(a + x = b\).

Note the role played by zero: \(0 + a = a\) for all integers \(a\). Zero is called the identity element with respect to addition. To insure a solution to the equation \(a \cdot x = b\) \((a \neq 0)\), the set was further expanded to include now all rational numbers. The list of postulates may be expanded to include:

8) \(a + x = b\) has a unique solution

9) \(a \cdot x = b\) \((a \neq 0)\) has a unique solution
Notation: The unique solution to the equation $a \cdot x = b$
will be denoted by $x = b/a$, where $a$ and $b$
represent integers. Numbers of the form
$b/a$ are termed rational numbers.

A positive rational number $b/a$ may, by the divi-
sion algorithm be reduced to either a terminating decimal
or a decimal which repeats itself cyclically. Since $a$ and
$b$ represent integers, division by $a$ will yield a remainder
of $0, 1, 2, 3, 4, \ldots, a-1$. If zero is eventually
achieved as a remainder the decimal terminates, if zero
is never achieved as a remainder then certainly a non-
zero remainder from $1, 2, 3, \ldots, a-1$ must repeat eventu-
ally so that from that point on the preceding remainders
repeat themselves in that cycle. The converse is also
true, i.e., every terminating or repeating decimal can
be represented as a rational number. This last statement
will be proved later. We might note here that between any
two rational numbers, no matter how close together, there
exists at least one other rational. The immediate example
of an intermediate rational is seen as the arithmetic
average of the two original rationals. This property is
meant when the rationals are described as a dense set.
In other words the rationals are tightly packed on the
number line, between any two of them is a third one.
If the rationals are tightly packed on the number line do they fill every possible point of the line, or are there still holes to be filled with other numbers?

Consider the following construction: On the number line on which all of the infinitely many rationals have been placed, construct a square one unit on a side, so that one of its sides coincides with the unit length from 0 to 1. Using the length of a diagonal as radius and the point 0 as center swing an arc cutting the number line. Does this intersection point correspond to a rational number? By the Pythagorean Theorem we know that the square of the hypotenuse equals the sum of the squares of the two sides of any right triangle. In our case, since each side of the triangle is one unit long, we know that the square of the hypotenuse is 2. A number whose square is 2 will be denoted by $\sqrt{2}$. In other words the point of intersection is represented by the number $\sqrt{2}$, but is $\sqrt{2}$ a rational number?

We will prove that $\sqrt{2}$ is not rational (and therefore irrational) by making use of the method of proof termed reductio ad absurdum. This method is based on the principle that a statement is either true or false. To prove the statement A true we will assume that the contrary of it is true, that is A is false. Then by a series
of deductive arguments, starting with the postulate, A is false, we hope to reach an absurd conclusion. At which point, assuming our reasoning to be correct, we conclude that the original postulate is incorrect, and indeed the Statement A is correct.

Now \( \sqrt{2} \) is either rational or irrational; consider the statement that \( \sqrt{2} \) can be expressed as a rational number \( a/b \) which is reduced to lowest terms, i.e., there is no factor other than one common to both \( a \) and \( b \).

\[
\sqrt{2} = \frac{a}{b}
\]
\[
2 = \frac{a^2}{b^2}
\]
\[
2b^2 = a^2
\]

Definition: An even number is one which can be written \( 2 \cdot x \), where \( x \) is an appropriate integer. Thus \( 8 = 2 \cdot 4 \) and \( 42 = 2 \cdot 21 \)

Now \( b \) is an integer, hence \( b^2 \) is an integer and \( 2b^2 \) is an even integer; therefore \( a^2 \) is an even integer.

Lemma 1: If \( n \) is an odd integer (an even integer plus 1) the \( n^2 \) is odd.

\[
n = 2 \cdot x + 1
\]
\[
n^2 = (2 \cdot x + 1)^2 = 4x^2 + 4x + 1 = 2(2x^2 + 2x) + 1
\]
\[ 2x^2 + 2x \text{ is an integer, } 2(2x^2 + 2x) \text{ is an even integer, } 2(2x^2 + 2x) + 1 \text{ is an odd integer.} \]

**Lemma 2:** If \( n^2 \) is even then \( n \) is even.

\( n \) is either even or odd, if odd then \( n^2 \) must be odd, but since \( n^2 \) is even, \( n \) itself must be even.

We had concluded above that \( a^2 \) is even, hence \( a \) is even and may be written as \( 2c \), where \( c \) is an appropriate integer.

\[
2b^2 = a^2 = (2c)^2 = 4c^2 \\
2b^2 = 4c^2 \\
b^2 = 2c^2
\]

Now \( 2c^2 \) is certainly even, and thus \( b^2 \) is even as is \( b \). Since \( a \) and \( b \) are now both even the rational number \( a/b \) can be reduced by at least the factor 2, but this is contrary to the assumption that \( a/b \) is reduced to lowest terms. We have thus come to a contradiction; \( a/b \) cannot at the same time be reduced to lowest terms and also possess the common factor 2. Our reasoning seems to be correct, hence the original assumption must be in error. There is no rational representation of \( \sqrt{2} \).

The rational numbers have been exhibited to distribute themselves densely upon the number line, and yet
the number line contains at least one hole. Irrational numbers (numbers not expressible as a ratio of two integers) were invented to fill such holes. We noted earlier that rational numbers could be written as either a terminating or cyclically repeating decimal expansion; what about the irrational numbers? \( \sqrt{2} \) may be approximated as a decimal by the following process: \( \sqrt{2} \) represents some number whose square equals 2, thus try various numbers until squares are obtained which bracket 2 from above and from below. Following this procedure note that \( 1^2 = 1 \), \( 2^2 = 4 \), hence:

\[
1 < \sqrt{2} < 2. \text{ Similarly } 1.1^2 = 1.21, 1.2^2 = 1.44, 1.3^2 = 1.69, 1.4^2 = 1.96, 1.5^2 = 2.25, \text{ hence;}
\]

\[
1.4 < \sqrt{2} < 1.5. \text{ We thus have successive approximations to } \sqrt{2}. \]

\[
1.41 < \sqrt{2} < 1.42
\]

\[
1.414 < \sqrt{2} < 1.415
\]

\[
\ldots \ldots \ldots \ldots \ldots \ldots
\]

Radical 2 is thus bracketed between pairs of terminating decimals which become progressively close to each other the further we carry the approximating process. In fact the difference between these pairs may be made smaller
than any preassigned positive number. In other words \( \sqrt{2} \) may be approximated to any desired degree of accuracy. From the above argument can it be said that \( \sqrt{2} \) can never be expressed as a terminating or periodic decimal? The answer is no, for no matter how convincing the argument may be, there is always the possibility that if we carry the process just a little further a terminating decimal would result. That this is not the case has been noted earlier, when it was stated that any terminating or repeating decimal may be expressed as a rational number. The point we are trying to make here is that no matter how far the approximating scheme is carried there will always exist a gap, no matter how small, between pairs of approximating rationals. This gap will never be filled by a rational number, for, no matter what rational is picked for this purpose it will eventually be "pushed" out of the gap by either the approximating rational from the left or from the right. Clearly the number meant to fill this gap is the tailor-made \( \sqrt{2} \). We have spoken of only one irrational number, but there are many more, in fact there are at least as many irrationals as there are rationals. For each rational we might for example add to it an irrational and once again create an irrational. As an exercise you might prove this statement in general terms, and, for example, in the specific
case $3 + \sqrt{2}$. There are other types of irrationals too, for example the number $\pi = 3.1415\ldots$. The set of numbers consisting of all rational numbers and all real numbers is termed the set of real numbers.

The Greeks contributed more than the use of deductive reasoning to mathematics. They were concerned with proofs of a general nature. For example, the Pythagorean theorem was proved in the general case by using as sides of the triangle the variables $a$, $b$, and $c$, rather than specific lengths as 3, 4, and 5. If specific lengths are used in a proof then the theorem can be accepted as true for only those particular values and no others, and it cannot be concluded from the proof for these specific values that the theorem is true for other values. The eventual goal of mathematics is utmost generality.

To illustrate this we will construct a mathematical science which may, but not necessarily, be applicable to the real numbers just described. Recall that a mathematical science contains undefined terms, and a set of postulates concerning these terms. In the present case consider the terms "set" (of elements), and an operation as undefined. As for postulates, the most easily accepted
arise from experience, however they need not do so. Our discussion of real numbers will suffice to provide the postulates for the mathematical science now under discussion.

For convenience designate the set of elements by "G," its elements by a, b, c, ..., and the operation by "•". The first postulate concerning positive integers was that of closure, so with the set G we will assume the property of closure.

1) If a and b are any two elements of G, then 
a • b is also an element of G.

2) If a, b, and c are elements of G then 
(a • b) • c = a • (b • c), i.e., associativity is assumed for the operation •.

Recall the real number zero and its property of neutrality (0 + a = 0), zero was called the identity element with respect to addition. We will assume the existence of an identity element for G.

3) There exists an element i of G such that for any element a of G a • i = a.
And lastly recall that any real number when added to its negative counterpart yields zero.

4) For any element \( a \) of \( G \) there exists an element \( a' \) (called the inverse of \( a \)) such that \( a \cdot a' = 1 \)

Note that postulate 4 implies the uniqueness of \( 1 \), a property which will be proved as a theorem. Any set of elements satisfying these four postulates is termed a group. In addition to the above postulates, if it is true that for any elements \( a \) and \( b \) of \( G \), \( a \cdot b = b \cdot a \) then \( G \) is termed an Abelian or commutative group.

Examples:

I: The manner in which the postulates were chosen gives rise to an immediate example, namely the set of all integers coupled with the operation of addition. Here zero is the identity element, and the negative of any integer is its inverse.

II: Let \( G \) consist of the digits 0, 1, 2, 3, and 4. Let the operation \( \cdot \) be defined as follows: with the five elements equally spaced about a circle, \( a \cdot b \) will mean to start on the circle at position \( a \), and then move clockwise \( b \) positions. The position thus moved to will be considered as the result of such operation.

thus; \( 2 \cdot 1 = 3, \ 2 \cdot 2 = 4, \ 2 \cdot 3 = 0, \ 2 \cdot 4 = 1, \ldots \)
Does such a system satisfy the group postulates? Clearly the system is closed since the operation \( \cdot \) must yield one of the positions 1, 2, 3, 4, or 0. Postulate 2 can be easily verified in all cases (only a finite number of them). For example:

\[
(2 \cdot 3) \cdot 4 = 0 \cdot 4 = 4
\]

\[
2 \cdot (3 \cdot 4) = 2 \cdot 2 = 4
\]

The identity element is 0, and the inverse of any element \( a \) is that element which when coupled to \( a \) by the operation \( \cdot \) yields 0.

\[
4 \cdot 1 = 0, \text{ the inverse of 4 is 1}
\]

\[
3 \cdot 2 = 0, \text{ the inverse of 3 is 2}
\]

The system then is a group, in fact it is a commutative group. As an application of this particular group, consider the following coding procedure. Assign a letter to each group symbol, as: 0 - a, 1 - b, 2 - c, 3 - d, 4 - e. Devise a coding equation such as \( y = x \cdot 3 \), in which \( x \) stands for a message letter, while \( y \) stands for the corresponding coded letter.

Now to code a letter, for example "c", assign the value 2 to \( x \) and solve the resulting equation, \( y = 2 \cdot 3 = 0 \), thus the coded letter is a. The coded form of the message "dead bad cad" then is, "bcdb edb adb". To decode a message proceed as follows: if the coded letter is b, what is the
corresponding message letter? b corresponds to 1, to find the message letter we must determine what symbol when combined by $\cdot$ to 3 will yield 1. Starting at 3, we must move three places clockwise to arrive at 1, hence the message letter corresponding to b is d. For coding systems in which we would like a greater latitude in our choice of letters, we would simply add more symbols to the group. If our coding system is found out it is a simple matter to change the coding equation.

III: Consider an equilateral triangle ABC;
The set of elements of \( G \) will here be thought of as the various rotations of the triangle, either about its center or about one of its median lines. A rotation will be considered legitimate so long as the triangle returns to its original position with the possible exception of its vertices being permuted. There are six such rotations:

- About \( S \) through 120° counterclockwise \( R_1 \)
- About \( S \) through 240° counterclockwise \( R_2 \)
- About \( S \) through 360° counterclockwise \( R_3 \)
- About \( a \) through 180° \( R_4 \)
- About \( b \) through 180° \( R_5 \)
- About \( c \) through 180° \( R_6 \)

If it be assumed that these are the only rotations which will transform the triangle into itself, and if the operation \( \circ \) is taken to mean followed by, then these six elements (rotations) and the defined operation \( \circ \) do form a group. Surely two operations performed in succession are equivalent to one of the original six operations, i.e., the property of closure is evident. Associativity could be demonstrated in all cases:

\[
(R_1 \circ R_2) \circ R_4 = R_3 \circ R_4 = R_4
\]

\[
R_1 \circ (R_2 \circ R_4) = R_1 \circ R_5 = R_4
\]
The identity element is represented by the rotation labeled \( R_3 \). As for inverse elements, since \( R_1 \cdot R_2 = R_3 \), \( R_1 \) is the inverse of \( R_2 \), and similarly in other cases. This system of rotations does constitute a group, note, however, that it is not commutative. \( R_1 \cdot R_4 = R_6 \), but \( R_4 \cdot R_1 = R_5 \). Several theorems:

I: If \( a, b, \) and \( c \) are elements of \( G \), and

If \( a \cdot c = b \)

Then \( c = a' \cdot b \)

**Proof** \( a' \cdot b = a' \cdot b \)

\[
(a' \cdot a) \cdot c = a' \cdot b
\]

\[
i \cdot c = a' \cdot b
\]

\[
c = a' \cdot b
\]

II: If \( a, b, \) and \( c \) are elements of \( G \) and

If \( a \cdot c = b \cdot c \)

Then \( a = b \)

**Proof** \( (a \cdot c) = (b \cdot c) \)

\[
(a \cdot c) \cdot c' = (b \cdot c) \cdot c'
\]

\[
a \cdot (c \cdot c') = b \cdot (c \cdot c')
\]

\[
a \cdot i = b \cdot i
\]

\[
a = b
\]

This theorem is known as the right hand cancellation law.
III: The commutativity of $i$;  
For any element $a$ of $G$, $a \ast i = i \ast a$  

Proof  
\[ i \ast (a \ast a') = i \ast (a \ast a') \]
\[ (i \ast a) \ast a' = i \ast (a \ast a') \]
\[ (i \ast a) \ast a' = i \ast i \]
\[ (i \ast a) \ast a' = i \]
\[ (i \ast a) \ast a' = a \ast a' \]
\[ i \ast a = a \]

But by postulate 3 we may write $a \ast i = a$
\[ i \ast a = a \ast i \]

IV: The identity element $i$ is unique.  
Proof  
assume that $i$ and $j$ are two different identity elements.  
\[ i \ast j = i \]
and  
\[ j \ast i = j \]
but  
\[ i \ast j = j \ast i \]
hence  
\[ i = j \]

V: The commutativity of the inverse element;  
for every $a$ and $a'$ of $G$ it is true that $a \ast a' = a' \ast a$  
\[ (a' \ast a) \ast a' = (a' \ast a) \ast a' \]
\[ (a' \ast a) \ast a' = a' \ast (a' \ast a') \]
\[ (a' \ast a) \ast a' = a' \ast i \]
(a' ° a) ° a' = i ° a'
(a' ° a) = i

But by postulate 4, a' ° a' = i
hence a' ° a = i = a ° a'

VI: The left hand cancellation law:
If a, b, and c are elements of G, and if c ° a = c ° b
then a = b.
    c ° a = c ° b
    c' ° (c ° a) = c' ° (c ° b)
    (c' ° c) ° a = (c' ° c) ° b
    i ° a = i ° b
    a' i = b ° i
    a = b

VII: An inverse element is unique.
For each a of G the inverse a' is unique.
Assume the existence of two inverses a' and b' for the element a.

now a ° a' = i and a ° b' = i
hence a ° a' = a ° b'
and a' = b'

VIII: For each element a of G, (a')' = a
a' ° (a')' = i = a ° a' = a' ° a
thus (a')' = a
We have thus far talked about a set of elements, an operation under which the elements satisfied certain postulates. After giving several examples of such a system we proved several theorems. The postulates chosen were suggested by the real number system. However the real number system consists of not one but two operations, addition and multiplication, and further the real number system contains properties other than those assumed as group postulates. It would be well then to discuss systems which hold promise of explaining more specific sets of elements such as the real numbers.

Consider a set of elements $R$ on which two operations designated by $+$ and $\cdot$ are defined which satisfies the following list of postulates:

1) The sum or product of any two elements of $R$ is once again an element of $R$ (closure). If $a$, $b$, and $c$ are elements of $R$ then;

2) $a + b = b + a$
   $a \cdot b = b \cdot a$ \hspace{2cm} \text{commutivity}

3) $(a + b) + c = a + (b + c)$
   $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ \hspace{2cm} \text{associativity}
4) \( a \cdot (b + c) = a \cdot b + a \cdot c \) 

5) \( a + x = b \) a unique solution exists in the set \( R \)

Note: the set of all integers is certainly an example of the above system.

Does this system contain an identity element with respect to addition, and if so is such element unique?

The equation \( a + x = a \) has a unique solution \( \omega \) such that \( a + \omega = a \). Pick any other element \( a_1 \), \( a_1 + x = a_1 \) has a unique solution \( \omega_1 \). Now does \( \omega = \omega_1 \)?

\[
\begin{align*}
a_1 + \omega &= b \\
a + a_1 + \omega &= a + b = c \\
a_1 + a + \omega &= c \\
a_1 + a &= c, \text{ or } a + a_1 = c \\
a + a_1 &= c, \text{ and } a + b = c, \text{ but since } a + x = c \text{ has a unique solution } a_1 = b, \text{ which means that } a_1 + \omega_1 = a_1, \text{ and } a_1 + \omega = a_1 \text{ hence } \omega_1 = \omega
\end{align*}
\]

Is there an element \( \omega \) in \( R \) such that \( a \cdot \omega = \omega \), for every \( a \) in \( R \)? Agree to write \( a^2 \) for \( a \cdot a \).

\[
\begin{align*}
\text{now } a^2 + a \cdot \omega &= a(a + \omega) = a^2 \text{ (since } a + \omega = a) \\
\text{but since } a^2 + x = a^2 \text{ has a unique solution, } a \cdot \omega &= \omega
\end{align*}
\]
When studying algebra the student is often required to accept rules on faith, and one of the most notorious of these is the rule governing the multiplication of a negative by a negative. The rules of signs apply to real numbers, but the set of real numbers is an example of the mathematical science just described. To prove theorems (rules) for the real numbers then it is enough to prove the theorems for the system in general, the theorems will hold true for any specific example of the system. With respect to the real numbers, \( \omega \) turns out to be 0. Notationally we agree that \( a + \_a = \omega \), and \( b + \_b = \omega \), i.e., in the real number system \( \_a \) is the negative of \( a \), and \( \_b \) is the negative of \( b \). Now to prove that a negative times a negative is positive, we will prove that \( \_a \cdot \_b = a \cdot b \).

\[
\begin{align*}
  a + \_a &= \omega & b + \_b &= \omega \\
  \text{multiply by } b & & \text{multiply by } a \\
  (a + \_a) \cdot \_b &= \omega \_b & (b + \_b) \cdot a &= \omega a \\
  a \cdot \_b + \_a \cdot \_b &= \omega & b \cdot a + \_b \cdot a &= \omega
\end{align*}
\]

hence \( a \cdot \_b + \_a \cdot \_b = b \cdot a + \_b \cdot a \)

and finally \( a \cdot \_b = a \cdot b \)

In the real number system: \((-a)(-b) = a \cdot b\)

To prove in the real number system that \((-a)(b) = (a)(-b)\), we will prove that \(a \cdot \_b = a \cdot \_b = \_a \cdot b\).
now $a + a = \omega$ and $b + b = \omega$
multiply by $b$  
$\omega + a = \omega b$
multiply by $a$
$\omega + a = \omega a$

$a \cdot b + a \cdot b = \omega$

$b \cdot a + b \cdot a = \omega$

$a \cdot b + a \cdot b = b \cdot a + b \cdot a = b \cdot a + a \cdot b = a \cdot b + a \cdot b$

hence $a \cdot b = a \cdot b$

Note: $a \cdot b + x = \omega$ has a unique solution $x = \frac{a \cdot b}{b}$, so that
$a \cdot b + a \cdot b = \omega$. But $a \cdot b + a \cdot b = \omega$, and $a \cdot b + a \cdot b = \omega$, so
that $a \cdot b = a \cdot b = a \cdot b$. To prove that $-(-a) = a$, we will
prove that $a = a$. But what is meant by $a$? $a + x = \omega$ has
a unique solution $x = a$, so that $a + a = \omega$.

$a + a = \omega$ and $a + a = \omega$, thus $a + a = a + a$, hence $a = a$.

The above system will be extended so as to include solutions for every such equation $a \cdot x = b$.

Postulate (6) if $a \neq \omega$ then $a \cdot x = b$ has a unique solution in the system.

Notation: the unique solution to the equation $a \cdot x = b$ will be denoted by $x = b/a$.

Question: Does $b_1/a_1 \cdot b_2/a_2 = b_1 \cdot b_2/a_1 \cdot a_2$?
We must assume that \( a_1 \neq \omega, a_2 \neq \omega, \) and \( a_1 \cdot a_2 \neq \omega. \) But if \( a_1 \neq \omega, \) and \( a_2 \neq \omega, \) is it implied that \( a_1 \cdot a_2 \neq \omega? \)

assume that \( a_1 \cdot a_2 = \omega \)

now \( a_1 \cdot x = \omega \) has a unique solution, but \( \omega \) has the property that \( a_1 \cdot \omega = \omega. \) Thus \( a_2 = \omega, \) contrary to original assumption, i.e., it is not necessary to assume that \( a_1 \cdot a_2 \neq \omega. \)

To show: \( b_1/a_1 \cdot b_2/a_2 = b_1 \cdot b_2/a_1 \cdot a_2, a_1 \neq \omega, a_2 \neq \omega. \)

\( a_1 \cdot x_1 = b_1, \) unique solution \( x_1 = b_1/a_1 \)

\( a_2 \cdot x_2 = b_2, \) unique solution \( x_2 = b_2/a_2 \)

multiply \( (a_1 \cdot x_1) \cdot (a_2 \cdot x_2) = b_1 \cdot b_2, \) \( x_1 \cdot x_2 = b_1/a_1 \cdot b_2/a_2 \)

associative law \( (a_1 \cdot a_2) \cdot (x_1 \cdot x_2) = b_1 \cdot b_2 \)

unique solution \( x_1 \cdot x_2 = b_1 \cdot b_2/a_1 \cdot a_2 \)

hence \( b_1 \cdot b_2/a_1 \cdot a_2 = b_1/a_1 \cdot b_2/a_2 \)

To show: \( b_1/a_1 + b_2/a_2 = b_1 \cdot a_2 + b_2 \cdot a_1/a_1 \cdot a_2 \)

(1) \( a_1 \cdot x_1 = b_1, \) unique solution \( x_1 = b_1/a_1 \)

(2) \( a_2 \cdot x_2 = b_2, \) unique solution \( x_2 = b_2/a_2 \)

multiply (1) by \( a_2, \) \( a_1 \cdot a_2 \cdot x_1 = b_1 \cdot a_2 \)

multiply (2) by \( a_1, \) \( a_1 \cdot a_2 \cdot x_2 = b_2 \cdot a_1 \)
add \( a_1 \cdot a_2 \cdot x_1 + a_1 \cdot a_2 \cdot x_2 = b_1 \cdot a_2 + b_2 \cdot a_1 \)

distributive \( a_1 \cdot a_2 \cdot (x_1 + x_2) = b_1 \cdot a_2 + b_2 \cdot a_1 \)

unique solution \( x_1 + x_2 = b_1 \cdot a_2 + b_2 \cdot a_1 / a_1 \cdot a_2 \)

thus \( b_1 / a_1 + b_2 / a_2 = b_1 \cdot a_2 + b_2 \cdot a_1 / a_1 \cdot a_2 \)

We have thus shown the rules of fractions (usually given as rules with little or no understanding) to be the result of a mathematical science.
LIMITS

Some time ago the Greek philosopher Zeno confounded his contemporaries by stating that in a race the faster participant could never catch the slower if the latter were given a head start. Achilles being fleet of foot was to run a race with the notoriously slow tortoise. Fairness of play dictated that the tortoise be given a headstart of some preassigned distance. Zeno then argued that no matter how long the race continued Achilles could never catch the tortoise. His argument ran as follows: in order to catch the tortoise Achilles would first have to traverse the distance decided upon as initial handicap, but by the time he reached the spot from which the tortoise began, the tortoise moved on. Achilles would have to cover this second handicap, but while doing so the tortoise would surely move on to a third position, and thus the race proceeded, Achilles forever doomed to second place in his race with the tortoise. Now certainly Zeno realized that in point of fact Achilles would overtake the tortoise, but how to explain this seeming paradox?

To make the paradox explicit, let's look at a numerical example. Suppose the initial handicap to be
one unit, with Achilles being able to run one unit per second of time, while the tortoise can run only half as fast. It is then in order to look at the total handicap which Achilles must overcome if he is to stand a chance of winning the race. It will take Achilles one second to cover the first one unit of handicap, during which time the tortoise moves one-half unit further on. In making up this second handicap in one-half second the tortoise moves on another one-fourth unit, etc. The total handicap which Achilles must overcome is then given by the unending summation: \[ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots. \] Zeno's trouble can be located at this point, i.e., he could not conceive of a single number as being the sum of such an unending problem of addition.

Let the sum of such an addition be represented by \( S_n \), where \( n \) stands for the number of terms added together. Then for example:

\[
\begin{align*}
S_2 &= 1 + \frac{1}{2} = \frac{3}{2} = 1.5 \\
S_3 &= 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4} = 1.75 \\
S_4 &= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8} = 1.875 \\
&\hspace{1cm} \vdots \\
S_{25} &= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{24}} = \frac{33554431}{33554432} \approx 2
\end{align*}
\]

Note: \( \approx \) is read approximately equal to.
To write the above type of sum in general terms, note that each term of the sum can be obtained from the preceding term by multiplying that term by the same common ratio, designate that ratio by $r$, and the first term of the sum by $a$. The sum of $n$ such terms may then be written:

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$$

$$S_n = a(1 + r + r^2 + r^3 + \ldots + r^{n-1})$$

Note

$S_2 = a(1 + r)$
$S_3 = a(1 + r + r^2)$
$S_4 = a(1 + r + r^2 + r^3)$

By applying the distributive law we note:

$$(1-r)(1+r) = 1-r^2, \quad 1+r = (1-r^2)/(1-r)$$

hence

$S_2 = a(1-r^2)/(1-r)$

$$(1-r)(1+r+r^2) = 1-r^3, \quad 1+r+r^2 = (1-r^3)/(1-r)$$

hence

$S_3 = a(1-r^3)/(1-r)$

$$(1-r)(1+r+r^2+r^3) = 1-r^4, \quad 1+r+r^2+r^3 = (1-r^4)/(1-r)$$

hence

$S_4 = a(1-r^4)/(1-r)$

Continuing this process we infer that:

$$S_n = a(1-r^n)/(1-r) = a(1/(1-r) - r^n(1-r))$$
We have guessed at a formula for $S_n$ by looking at several specific cases; this formula can be proved as follows:

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

**multiply by $r$**

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$$

**subtract**

$$S_n - rS_n = a - ar^n$$

$$S_n(1-r) = a - ar^n$$

$$S_n = (a-ar^n)/(1-r) = a(1-r^n)/(1-r)$$

In any given sum of such nature the first term $a$ and the common ratio $r$ are certainly fixed numbers. The only symbol left namely $n$, the number of terms, is allowed to vary, and in the case of Achilles' race becomes large without bound. In order to answer the paradox of Zeno we must examine such sums. Consider first the sum:

$$S_n = 1 + 1/2 + 1/4 + 1/8 + \cdots + 1/2^n$$

$$S_n = 1( 1/(1-1/2) - (1/2)^n/ (1-1/2) )$$

$$= (2 - 2(1/2)^n)$$

$$= 2( 1 - (1/2)^n )$$

(i)

What observations can be made?

Note first that $(1/2)^n$ is always a positive number, though perhaps quite small, for any positive integer $n$, so that the
quantity indicated as (i) is always less than one. Note further that as \( n \) increases in size the quantity \( 1 - (1/2)^n \) gets closer and closer to one. In other words we may make the difference between 1 and \( 1 - (1/2)^n \) as small as we please by taking \( n \) large enough. For instance, what value of \( n \) will insure that \( 1 - (1/2)^n \) is within \( 1/1,000,000 \) of one?

Before looking at this problem we will introduce the following notation:

Considering real numbers, \( a < b \) means that there exists a positive number \( k \) such that \( b - a = k \).

Geometrically \( a < b \) means \( a \) is to the left of \( b \) on the number line. The notation \( a < b \) introduces an element of doubt. All that we really know is that \( a \) is not larger than \( b \). Statements of the form \( a < b, b > a, a < b, \) and \( b > a \) are termed inequalities.

Properties of inequalities:

(i) if \( a < b \) then \( a + c < b + c \) for any constant \( c \).

(ii) if \( a < b \) then \( a \cdot c < b \cdot c \) whenever \( c > 0 \).

(iii) if \( a < b \) then \( a \cdot c > b \cdot c \) whenever \( c < 0 \).

(iv) if \( 0 < a < b \) then \( 1/a > 1/b \).
Now to return to the problem indicated above, that is,
for what $n$ is $1 - (1 - (1/2)^n) < 1/1,000,000$?

$$\frac{1}{2^n} < \frac{1}{1,000,000}$$

implies

$$2^n > 1,000,000$$

if $n \geq 20$

and

$$1 - (1 - (1/2)^n) < 1/1,000,000$$

Note

$2^{18} = 262,144$

$2^{19} = 524,288$

$2^{20} = 1,048,576$

We must add twenty terms before $1 - (1 - (1/2)^n)$ is within $1/1,000,000$ of one. Note further that for any integer larger than 20, that number of terms when added together gives a sum even closer to one. We thus note two facts concerning this particular sum:

1) the sum gets larger and larger as more terms are added but

2) the sum gets as close to two as we please by adding enough terms, but does not exceed two.

The question is, what one number best represents the sum of an unlimited number of terms such as $1 + 1/2 + 1/4 + \cdots$.

It does not make sense to add a particular number of terms together and proclaim their sum to be the best representative of the entire sum, for any sum thereafter would be larger than the original sum. The number two itself would
seem to be the best choice to represent the entire sum. The number two, to be sure, is never a calculated sum of this series of numbers no matter how many terms are added, but on the other hand two is never exceeded no matter how many terms are added. In fact we may get as close to two as we please simply by adding enough terms, and any further addition of terms merely makes our sum even closer to two. It would appear then that two is somehow a limit to our finite additions, and we do in fact define two as the best number to represent the entire sum of numbers.

We might look once again at the formula previously developed to see the reasonableness of the above definition.

\[ S_n = a \left( \frac{1}{1-r} - \frac{r^n}{1-r} \right) \]

In a specific situation \( n \) is the only number allowed to change, and indeed \( n \) takes on larger and larger integral values. But in the formula \( n \) affects only the term \( r^n \), so that it is this term we must examine as \( n \) gets larger. In the race between Achilles and the tortoise the value of \( r \) is \( 1/2 \), i.e., the tortoise runs one-half as fast as Achilles, so that we must examine \((1/2)^n\) as \( n \) becomes larger.

\[
(1/2)^2 = 1/4, \quad (1/2)^3 = 1/8, \quad (1/2)^4 = 1/16, \quad (1/2)^5 = 1/32, \\
(1/2)^6 = 1/64, \quad \text{etc.}
\]
Note that as \( n \) gets larger the value of \((1/2)^n\) gets smaller and in fact gets so small in value as to approach zero for an ever increasing \( n \). As \( r^n \) approaches zero, certainly \( r^n/(1-r) \) also approaches zero, so that the formula for \( S_n \) reduces to \( S_n = a(1/(1-r)) \) in the case in which an unlimited number of terms are to be added.

Returning to Zeno's race, Achilles must make up the total handicap indicated by the unlimited sum \( 1 + 1/2 + 1/4 + 1/8 + \cdots \). \( S_n = 1(1/(1-1/2)) = 2 \). As soon as Achilles accomplishes a run of two units he will have caught the tortoise.

Note here that the value of \( r \) is less than one. If the value of \( r \) had been two for example, i.e., the tortoise runs twice as fast as Achilles, then Achilles would have to make up in handicaps the sum of \( 1 + 2 + 4 + 8 + \cdots \), and certainly this sum grows large without bound. In such a race the tortoise would win in a breeze.

The existence then of a sum of such numbers depends upon the value of \( r \). The existence of such sums has been investigated by mathematicians, and in abeyance to their preference for deductive mathematics they have constructed a mathematical science to deal with unending sums of numbers.
In mathematics we are often concerned with quantitative relationships existing between two sets of numbers. To each number in the first set there corresponds a unique number in the second set. The two sets of numbers and the correspondence between these two sets comprise what is termed a function. The first set is called the domain set, the second set is termed the range set. With each number \( x \) in the domain set there corresponds a number \( y \) in the range set. Different functions are denoted by different letters. A particular function might be designated by \( f \), and the range number corresponding to a particular domain element \( x \) is denoted by \( f(x) \).

For example consider the function given by \( f(x) = x^2 + 3 \). Thus \( f(2) = 7 \), \( f(-2) = 7 \), \( f(-0) = 3 \), etc.

Consider a real valued function of the positive integers, i.e., a function whose domain set consists of the positive integers. Such a function is called an infinite sequence.

Examples: 1, 1/2, 1/3, 1/4, \( \ldots \), 1/n, \( \ldots \)

1, 1, 1, 1, 1, \( \ldots \), 1, \( \ldots \)

In every infinite sequence there is a first term, a second term, a third term, etc., an \( n \)th term, etc. The notation
for an infinite sequence is as follows: \( a_1, a_2, a_3, a_4, \ldots, a_n, \ldots \), where \( a_n \) represents the \( n \)th functional value or more simply the sequence may be represented by \( \{a_n\} \). In the sequence 1, 2, 3, 4, 5, 5, 5, 5, 5, 5, 5, \ldots, 5, \ldots, we note the continual occurrence of the term 5, i.e., \( \{a_n\} \) possesses the property P such that \( a_n = 5 \). Four terms fail to possess property P while infinitely many terms possess this property.

Given a sequence \( a_n \) and a property P, almost every term possesses P if only a finite number of terms fail to have this property. Putting this in more precise language; almost every term possesses property P if there exists a subscript \( v \) such that \( \{a_n\} \) possesses P for every \( n > v \).

\[ a_1, a_2, a_3, \ldots, a_v, a_{v+1}, a_{v+2}, \ldots \]

Possess property P

Recall our analysis of the race between Achilles and the tortoise; we looked at the term \( (1/2)^n \), and noticed that as \( n \) becomes large \( (1/2)^n \) becomes increasingly small. In fact we concluded that as \( n \) becomes large \( (1/2)^n \) approaches zero. Using this type of reasoning we will define a certain type of sequence called the null sequence.
Definition: $a_n$ is a null sequence (approaches or converges to zero) if for every $\epsilon > 0$, almost every term has an absolute value less than $\epsilon$.

Pictorially: assign any interval centered at zero, property $P$: $|a_n| < \epsilon$.

If for a small $\epsilon$, an infinite number of terms of the sequence group themselves about zero, then the sequence is null. You might reply, what if an even smaller $\epsilon$ is picked? If it is possible to find a $v$ corresponding to this new $\epsilon$ such that for $n > v$, $|a_n| < \epsilon$ then the sequence is null. If, as the game proceeds, i.e., picking $\epsilon$ finding $v$, picking a smaller $\epsilon$ finding its corresponding $v$, etc., it is always true that the corresponding $v$ can be found then the sequence is null.

It might be well here to look at a specific sequence and determine whether it is null? Consider $1, 1/2, 1/3, 1/4, \cdots, 1/n, \cdots$, is this a null sequence? Pick an $\epsilon > 0$; question—how many terms are there such that $|1/n| \geq \epsilon$? If it can be shown that the number of such terms is finite, then surely there must be an infinite number of terms such that $|1/n| < \epsilon$, and thus $\{1/n\}$ would be null.
Note that $1/n$ is always positive, and thus absolute value signs are unnecessary. $1/n > \varepsilon$ implies that $n \leq 1/\varepsilon$

now if $n < 1/\varepsilon$ then $1/n > \varepsilon$, certainly we can find a subscript $v > 1/\varepsilon$ such that for every $n > v$, $1/n < \varepsilon$. Clearly there are only a finite number of subscripts less than $v$ and hence an infinite number of them greater than $v$. Thus the sequence $\{1/n\}$ is null.

What about the sequence $\{a_n\}$ such that $a_n = q^n$, where $0 < q < 1$; is it null or otherwise? In other words for $\varepsilon > 0$, how many terms satisfy the property $q^n < \varepsilon$? If the answer is almost all of them then the sequence is null. We will attack the problem by examining $q^n > \varepsilon$, if this inequality is satisfied by only a finite number of terms then surely $q^n < \varepsilon$ is satisfied by an infinite number of terms.

$0 < q < 1$ implies that $1/q > 1$

hence we may write $1/q = 1 + r$ where $r$ is a positive number.

$1/q^n = (1 + r)^n = 1 + nr + \cdots$ all positive terms

thus $1/q^n > 1 + nr > nr$

$1/q^n > nr$ implies that $q^n < 1/nr$
If for an \( n \), \( q^n > \varepsilon \) then \( 1/nr > \varepsilon \) hence \( 1/n > \varepsilon r \), which implies \( n < 1/\varepsilon r \)

How many \( n \)'s are there such that each is less than a fixed number? only a finite number of them.

Hence \( q^n > \varepsilon \) holds only for a finite number of exponents \( n \), and thus \( q^n < \varepsilon \) holds for almost every term.

If \( 0 < q < 1 \) then \( \{q^n\} \) is a null sequence.

Notation: If \( \{a_n\} \) is a null sequence then it converges to zero; \( a_n \to 0 \).

Property: If \( a_n \to 0 \), then \( ca_n \to 0 \), where \( c \) is any constant

Case 1: \( c = 0 \), obviously null

Case 2: \( c \neq 0 \)

assign \( \varepsilon \)

consider \( \varepsilon/|c| \), legitimate since \( c \neq 0 \)

\(|a_n| < \varepsilon/|c| \) this is not necessarily true for all \( n \) but it is true after several terms say \( \nu \) of them.

this implies that \( |c| \cdot |a_n| < \varepsilon \)

this implies that \( |ca_n| < \varepsilon \) for \( n > \nu \)
Definition: a sequence \( \{a_n\} \) converges to the number 'a' \( \left( a_n \to a \right) \) provided that the sequence \( \{a - a_n\} \) is null.

The sequence \( \{a - a_n\} \) written in its longer form is:

\[
a - a_1, a - a_2, a - a_3, \ldots, a - a_n, \ldots
\]

Notation: \( \lim a_n = a \)

What does it mean to say that \( \{a - a_n\} \) is a null sequence? \( \{a - a_n\} \to 0 \), provided that for every \( \varepsilon > 0 \) there exists an integer \( v \) such that \( |a - a_n| < \varepsilon \) for \( n > v \)

Pictorially: \( -\varepsilon < a - a_n < \varepsilon \) implies that \( a - \varepsilon < a_n < a + \varepsilon \) for \( n > v \)

Almost every term is within \( \varepsilon \) distance of a.

Definition: a sequence \( \{a_n\} \) is termed convergent if there exists a real number 'a' such that \( a_n \to a \).

Example: where \( 0 < q < 1 \), let \( \{a_n\} \) be given as follows:

\[
a_1 = 1, \ a_2 = 1 + q, \ a_3 = 1 + q + q^2, \ldots, \ a_n = 1 + q + q^2 + \cdots + q^n, \ldots
\]

If \( q \) is given the value of one-half, \( a_n \) is seen to give precisely the total handicap which Achilles had to make up in order to win his race with the tortoise.
\[ a_n = 1 - \frac{q^n}{1-q} = \frac{1}{1-q} - \frac{q^n}{1-q} \]

\[ \frac{1}{1-q} - a_n = \frac{q^n}{1-q} = q^n(1/1-q) \]

Recall that \( q^n \) is a null sequence, and \( 1/1-q \) is a constant, hence \( q^n/1-q \) is a null sequence. Therefore \((1/1-q - a_n)\) is a null sequence which implies that \( a_n \to 1/1-q \), or \( \lim a_n = 1/1-q \).

This is precisely the result which we had anticipated when examining the 'racing' paradox. Although at that time we guessed the result from looking at several specific examples, here we obtain the same formula as a result of deductive reasoning. As an exercise you might convince yourselves of a fact only hinted at before, and that is that any periodic decimal is a rational number.
CALCULUS

Isaac Newton and Gottfried Leibniz brought calculus to its fruition during the seventeenth century. The need for calculus arose as a tool to help in the fashioning of universal laws. In particular mathematicians of the time were united in their struggle to understand the concept of instantaneous rate of change. Instantaneous velocity must be understood if one is to deal with bodies moving at varying speeds, and this is precisely the situation in which mathematicians found themselves during the seventeenth century. Kepler's second law for example states that a planet moves, not at a constant speed as the Greeks and other pre-renaissance scientists had believed, but in a continually varying speed. Similarly, according to Galileo, bodies rising or falling near the surface of the earth travel at continually varying speeds. Pendulum and projectile motion which were studied at the time also involve varying speeds.

In attempting to understand what these seventeenth-century scientists were up against, consider the distance-time formula \( s = t^2 \), where \( s \) stands for distance and \( t \) for time. This formula suggests a functional relation existing
between the two variables distance and time. The domain set here consists of all the non-negative values of time which may be substituted for $t$ in the formula. The corresponding values of the distance variables $s$ may be found by squaring the various $t$ values. For example at time $t = 0$ the particle hasn't moved at all, but at time $t = 1$ the particle has moved one unit. To help visualize the particle's movement construct a short table for several $t$ values.

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s$</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>16</td>
<td>25</td>
<td>36</td>
</tr>
</tbody>
</table>

What can be said concerning the movement of the particle? Note that in the first unit of time the particle has moved through one unit of distance, while in the second unit of time the particle has covered three units of distance. It is obvious that the particle is speeding up. There are other questions which could be asked, e.g., what is the average speed of the particle during the six units of time? Recalling the basic formula, rate equals distance divided by time, it is a simple matter to divide 36 distance units by 6 time units and reply that the particle is moving 6 units of distance.
per each unit of time. It is, however, a much different question to ask how fast the particle is moving at a specific instant of time, say at time equals three. The uninitiated might be tempted to reason: in three units of time it has traveled 9 distance units, hence it is traveling at nine divided by three equals three units per unit of time. This is a curious conclusion considering that the particle has moved through five units in the time unit preceding three, and through seven units during the succeeding time unit. One ought to conclude that the speed traveled at time equals three is some value between five and seven. What should be done if a better estimate is desired? The obvious answer is to calculate average speeds for shorter and shorter intervals of time around t = 3.

\[
\begin{align*}
  t & \quad 2 \ldots 2 \ 1/2 \ldots 2 \ 3/4 \ldots 3 \ldots \\
  & \quad 3 \ 1/4 \ldots 3 \ 1/2 \ldots 4 \\
  s & \quad 4 \ldots 6 \ 1/4 \ldots 7 \ 9/16 \ldots 9 \ldots 10 \ 9/16 \\
  & \quad \ldots 12 \ 1/4 \ldots 16 \\
\end{align*}
\]

As a convenient notation let \( V_3 \) stand for the particle's velocity at time \( t = 3 \), then

- first approximation \( 5 < V_3 < 7 \)
- second approximation \( 5 \ 1/2 < V_3 < 6 \ 1/2 \)
- third approximation \( 5 \ 3/4 < V_3 < 6 \ 1/4 \)

--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------
If this sort of calculation is continued, it seems intuitively evident that the upper and lower approximations are converging upon each other and indeed to the common limit of 6. And so once again we meet the idea of limit, although here in a slightly different manner. In discussing the sequential limit we allowed the value of the variable subscript 'n' to become large without bound, but it took on only integral values. In the context of speed at an instant we allow the value of time to vary, this time approaching a fixed instant of time by assuming all possible intermediate values.

Reviewing what has been done in a symbolic form, we have the formula for motion:

\[ s = t^2 \]

then \[ V_3 = 6 = 2.3 \]

If we were to perform the same calculations for other instants of time we would find:

\[ V_1 = 2 = 2.1 \]
\[ V_2 = 4 = 2.2 \]
\[ V_4 = 8 = 2.4 \]
\[ V_5 = 10 = 2.5 \]
There appears a very definite pattern here in that if one wishes the speed at any instant of time \( t \) he merely multiplies that value of \( t \) by 2 to obtain:

\[
V_t = 2 \cdot t
\]

As an exercise do the same for the motion formulas:

\[
s = t^3, \ s = 2 \cdot t^2, \text{ and } s = 2 \cdot t^3.
\]

Instantaneous velocity then would seem to involve the concept of limit. When calculating the velocity of a particle moving in a straight line it is the usual procedure to divide the distance traveled in a particular time interval by the time interval. If we consider the particle's motion as given by the functional relation \( s = f(t) \), where \( s \) is the distance covered in time \( t \), then it is sufficient to set up the ratio \( f(t_2) - f(t_1)/(t_2 - t_1) \) in order to obtain the velocity between times \( t_1 \) and \( t_2 \). However, if the motion is such that its velocity is not constant then this ratio yields only the average velocity between times \( t_1 \) and \( t_2 \). If this ratio remains constant no matter what values of time are substituted into it then the motion is said to be uniform. Of the two types of velocity, instantaneous and average, when the term velocity is used it will mean instantaneous.
If the motion is not uniform, and we desire its velocity say at time $t_1$, then it seems intuitively evident that we must take the limit of this ratio as the time interval becomes small, i.e., as $t_2$ approaches $t_1$ (symbolically this is written $t_2 + t_1$). Thus far in our discussion on limits we considered only the case of a sequential limit, where $n$ got very large but did so only through integral values. In taking the limit of the average velocity ratio, $t_2$ will approach $t_1$ by taking on all real values of time within the interval. Note that $t_2$ can never equal $t_1$ for in that case average velocity would not be defined.

We know what is meant by a discrete limit, but what is meant by limit by a continuous approach? Consider the function $f(x) = x^2 - 1 / x - 1$, this function is defined for all values of $x$ except $x = 1$, for in that case we would be doing the impossible, dividing by zero. If $x \neq 1$, then $f(x)$ may be reduced to the function $f(x) = x + 1$. Graphically this means the function represents a straight line, except that it does not pass through the line $x = 1$. 
The questionable point on the graph is the point $(1, 2)$. It is certainly not a point on the graph of the original function, but what about the value of the function as $x$ approaches $1$. It is geometrically evident that $f(x)$ approaches $2$ as $x$ approaches $1$ from either side. To picture this phenomena algebraically, we will look at a formula expressing the difference between $f(x)$ and $2$.

\[ f(x) - 2 = x + 1 = 2 = x - 1 \]

Clearly the difference between $f(x)$ and $2$ may be made as small as we please by taking values of $x$ closer and closer to the value $1$, but we may not make $x$ equal to $1$. For example if $x$ equal $.9$ or $1.1$, then the difference between $f(x)$ and $2$ is $.1$, and if $x$ equals $.99$ or $1.01$ then the difference is $.01$. More generally if $e$ is any small positive number then the difference between $f(x)$ and $2$ will be less than $e$ provided that $x$ is chosen so that its distance from $1$ is less than some number $d$. And now analogously to the sequential limit we may write a definition of what is meant by the limit of a function with respect to a continuous approach.
The number $L$ is said to be the limit of a function $f(x)$ at some value of $x$ say $x_1$, if for every positive number $\varepsilon$ we may find a positive number $\delta$ such that $|f(x) - L| < \varepsilon$ for all $x$ different from $x_1$ satisfying $0 < |x - x_1| < \delta$. If these conditions are satisfied it is customary to write:

$$\lim_{x \to x_1} f(x) = L$$

Taking a lead from our discussion of instantaneous velocity and the definition of limit, we are in a position to define a new function. Recall that a function consists of a domain set and a correspondence which assigns one and only one range number to each domain number. For a point $x$ in the domain of $f$, construct the ratio $f(x_1) - f(x)/(x_1 - x)$, where $x_1$ represents a value in the domain near to but not identical with $x$. Consider the limit of this ratio as $x_1$ approaches $x$; if this limit exists, it is said to be the value of the derived function at $x$. The derived function is given the notation $f'(x)$, i.e.,

$$f'(x) = \lim_{x_1 \to x} \frac{f(x_1) - f(x)}{(x_1 - x)}$$

and is called the derivative.

Several examples of derived functions:

I. let $f(x) = c$ where $c$ is any constant-

$$f(x_1) - f(x)/(x_1 - x) = c - c/(x_1 - x) = 0$$
so that \( f'(x) = \lim_{x_1 \to x} 0 = 0 \)

II. let \( f(x) = x \)

\[
f(x_1) - f(x) / (x_1 - x) = x_1 - x / (x_1 - x) = 1
\]

so that \( f'(x) = \lim_{x_1 \to x} 1 = 1 \)

III. let \( f(x) = x^2 \)

\[
f(x_1) - f(x) / (x_1 - x) = x_1^2 - x^2 / (x_1 - x) = x_1 + x
\]

so that \( f'(x) = \lim_{x_1 \to x} (x + x) = 2x \)

Interpretations of the derived function:

I. The derivative is defined in terms of a quite general relation existing between two variables \( x \) and \( f(x) \). If we think of \( x \) as the time variable \( t \), and the corresponding variable as distance \( s \), then \( y = f(x) \) becomes \( s = f(t) \), a distance-time relation. The basic ratio becomes \( f(t_1) - f(t) / (t_1 - t) \), which stands for the distance traveled in a time interval divided by the time interval. Taking the limit of this ratio as \( t_1 \) approaches \( t \) we obtain \( f'(t) \). But this is precisely what we did experimentally to obtain velocity. Thus it would seem proper to define velocity as the derived function \( f'(t) \).
Note that the ratio \( f(x_1) - f(x) / (x_1 - x) \) can be read as the rate of change of the function value with respect to a change in \( x \). It tells us how fast the function changes per change in \( x \). Velocity can then be thought of as the instantaneous rate of change of distance with respect to time. But velocity itself is in general a function of time, and thus it is in order to think of the rate of change of velocity with respect to time, i.e., to think of quantity which measures the variability of velocity. This measure is appropriately enough a derived function, but this time the derived function of velocity, in other words a second derived function. This second derived function is denoted by \( f''(t) \), and in the case of the distance-time function is called acceleration.

As an example consider a stone thrown vertically up, its distance above ground level given as the function of time:

\[
s = 96 + 60t - 16t^2
\]

Questions which might be asked include:

What is the stone's velocity at any instant of time?
What is the stone's maximum height?

At what velocity was the stone released?

At what velocity does the stone strike the ground?

What is the acceleration of the stone's travel?

\[ V(\text{velocity}) = f'(t) = \lim_{t_1 \to t} \frac{(96+60t_1-16t_1^2)-(96+60t-16t^2)}{t_1-t} \]

\[ V = f'(t) = 60 - 32t \]

To find the stone's velocity at any instant, for example, when \( t \) equals one second,

\[ V_1 = f'(1) = 60 - 32 \cdot 1 = 32 \text{ ft/sec} \]

When the stone reaches maximum height its velocity is zero.

\[ V = 0 = 60 - 32t \]

\[ t = 60/32 = 15/8 \text{ sec.} \]

The stone will hit the ground when \( s = 0 \)

\[ 96 + 60t - 16t^2 = 0 \]

Solving this equation we find that

\[ t = \frac{15 + \sqrt{609}}{8} \text{ sec.} \]
II. A geometric interpretation:

Consider a function $y = f(x)$, and two points on its graph.

The expression $f(x_i)$ represents the vertical distance from the $x$-axis to the point $P_i$. And $f(x)$ represents the corresponding distance to the point $P$. Thus $f(x_i) - f(x)$ represents the vertical rise from $P$ to $P_i$, while $x_i - x$ represents the corresponding horizontal distance.

The ratio $f(x_i) - f(x)/(x_i - x)$ then represents the rate of change of height with respect to horizontal displacement of the secant line $PP_i$. Now if $x_i$ is chosen closer to $x$, the secant line will in effect revolve about the point $P$, but as the two points $P$ and $P_i$ approach each other the rotating secant line approaches a limiting
position. This limiting position is defined as the tangent line to the curve at the point \((x, f(x))\), and the rate of change in height of the tangent line is defined as the limit of the rate of change of the secant line. We will assume that the direction of a curve at any point is given by the direction of the tangent line at that point. If it is desired to determine maximum or minimum points on the curve (points either higher or lower than any points in the immediate neighborhood of the points in question) one must look for points of horizontal tangency. For at these points the curve is neither rising nor falling, i.e., the rate of change of the tangent line is zero. For those sections of the graph where \(f'(x)\) is greater than zero and considering \(x\) as moving from left to right the curve is ascending, if \(f'(x)\) is less than zero the curve is descending. For those points for which \(f'(x)\) is zero and in addition \(f'(x)\) is greater than zero to the left of the point but less than zero to the right, the point in question is a maximum point. A similar statement can be made for minimum points.

Consider the problem of constructing an open box from a square piece of metal eighteen inches on a side, by cutting equal squares from each corner of the metal and turning up the sides.
\[ V = (18 - 2x)^2 \cdot x = 324x - 72x^2 + 4x^3 \]

\[ f'(x) = 324 - 144x + 12x^2 \]

Setting \( f'(x) \) equal to zero;

\[ 324 - 144x + 12x^2 = 0 \]

\[ x = 9 \text{ or } x = 3 \]

Now if \( x = 9 \), we don't have any box at all, the only other possibility is for \( x \) to equal 3. Testing \( f'(x) \) to the left and right of \( x = 3 \), we obtain;

\[ f'(2) = 324 - 144 \cdot 2 + 12 \cdot 2^2 = 84 > 0 \]

\[ f'(4) = 324 - 144 \cdot 4 + 12 \cdot 4^2 = -160 < 0 \]

The box with dimensions of 12 by 12 by 3 is of maximum volume.

In this criterion for determining whether a point is maximum or minimum, the derivative runs through a succession of values ranging from positive to negative in the case of a maximum, keeping in mind that \( x \) moves from left to right. In other words the rate of change of
f'(x) is decreasing as x moves through a maximum point. But the rate of change of f'(x) is precisely given by the expression f''(x). It is in order then to look at f''(x). If f''(x) is negative then the point is a maximum. A similar statement can be made for minimum points.

Returning to our example:

\[ V = f(x) = 324x - 72x^2 + 4x^3 \]
\[ f'(x) = 324 - 144x + 12x^2 \]
\[ f'(3) = 0 \]
\[ f''(x) = -144 + 24x \]
\[ f''(3) = -144 + 72 = -72 < 0 \]

Hence the point (3,f(3)) is a maximum.

The second major problem of calculus has its roots in the ancient problem of determining areas of irregular shapes. We spoke earlier of the problem of finding the areas of such regular shapes as rectangles and triangles. It is a simple matter to place within such shapes a certain number or fraction thereof of units of area, usually square units, and thus quote the desired area as so many square units. It is an entirely different matter to determine the area of a figure surrounded by curved lines. Archimedes in the third century B.C. dealt with this problem by placing within the region an approximating region of a polygonal
boundary whose area could be determined as the sum of a number of triangles; he then chose another approximating polygonal region containing the first as his second approximation. He continued in this way eventually obtaining the desired area as the limit of a sequence of approximating polygonal areas. Each individual region whose area was desired was treated in its own special way: the calculus was destined to replace these specialized methods by one general and powerful method.

Consider a positive, continuous function on the interval from $x = a$ to $x = b$. Note: a function is continuous on an interval if its entire graph on that interval may be drawn without lifting the pencil.

Problem: to determine the area of the region bounded from below by the $x$-axis, from above by the curve $f(x)$, and from the left by the line $x = a$ and from the right by the line $x = b$. 

\[
\begin{array}{c}
\text{\hspace{1cm} } \\
\end{array}
\]
We will proceed by replacing the shaded region by a number of rectangles whose total area will approximate the desired area. To accomplish this divide the interval from $a$ to $b$ into a number of subintervals. At each point of division erect a perpendicular line intersecting the curve. At some point either interior to or at an endpoint of each subinterval erect a perpendicular to the curve which will serve as an altitude to the rectangle erected on that particular subinterval. We might for example choose to use the original interval from $a$ to $b$ as the base of one approximating rectangle whose altitude is erected say at $x_1^\#$.

The shaded area is then approximated by the rectangular area $S_1 = f(x_1^\#)(b-a)$.

We might have divided the interval $ab$ into two subintervals by the division point $x_1$.

![Diagram]

Then in each subinterval choose points say $x_1^\#, x_2^\#$, and erect appropriate altitudes. The sum of the areas of these
two rectangles then represent a second approximation to
the desired area.

\[ S_2 = f(x_1^#)(x_1 - a) + f(x_2^#)(b - x_1) \]

We could continue this process of dividing the interval
from \( a \) to \( b \) into an ever increasing number of subintervals
in such a way that the width of the widest rectangle tends
to zero. It seems intuitively obvious that the sequence
of such approximating sums \( S_1, S_2, S_3, \cdots, S_n, \cdots \) would
tend to the limit \( A \) (the desired area) as \( n \) becomes large
without bound. It can be shown that the number \( A \) in no
way depends upon the choice of subdivision points nor
does it depend upon the points at which the altitudes are
erected.

With these ideas in mind we proceed to a definition
of the integral. Consider a continuous function
\( f(x) \) on the interval from \( x = a \) to \( x = b \). Divide this
interval into \( n \) subintervals; division points designated
by; \( x_0 = a, x_1, x_2, x_3, \cdots, x_{n-1}, x_n = b \). In each sub-
interval pick a point, either interior to or at an endpoint
of the subinterval, and evaluate the function at that point,
then form the product of the subinterval's length and the
value of the function at the chosen point. Do this for
each subinterval, and finally form the sum of all such
products.
Thus: $S_n = f(x_1^#)(s_1-a) + f(x_2^#)(x_2-x_1) + f(x_3^#)(x_3-x_2) + \cdots + f(x_n^#)(b-x_{n-1})$

As a convenient notation we will use the symbol $\Sigma$ to indicate that a sum is to be taken. To describe the terms of the desired sum, think of a typical term as $f(x_i^#)(x_i-x_{i-1})$, where $i$ is understood to be a variable assuming the values 1, 2, 3, 4, 5, ..., $n$. $S$ may then be written:

$$S_n = \sum_{i=1}^{n} f(x_i^#)(x_i - x_{i-1})$$

We might make one further innovation to this symbolism by letting $\Delta x_i$ stand for the length of the interval $x_i-x_{i-1}$, so that:

$$S_n = \sum_{i=1}^{n} f(x_i^#)(\Delta x_i)$$

Now choose a larger and larger number of subdivision points in such a way that the longest subinterval tends to zero as $n$ becomes large without bound. If the limit of this summation exists, and it will if $f$ is a continuous function, then that limit is said to be the integral of $f(x)$ on the interval from $x = a$ to $x = b$, and is denoted as follows:

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^#)(\Delta x_i) = \int_{a}^{b} f(x) \, dx$$
If one thinks that the area of the region bounded by the curve, the x-axis, the lines x = a, and x = b exists without question, and that this area may be calculated by the process of summing rectangles, then the above mentioned limit needs no proof as to its existence. However as an analytical definition (no mention of area) the existence of such limit does need to be proved and is proved in rigorous texts of calculus.

Several examples of integration:

I. If \( f(x) = c \), where \( c \) is any constant then \( \int_{a}^{b} c \, dx = ? \)

\[
S_n = \sum_{i=1}^{n} f(x_i) \Delta x_i , \quad \text{but } f(x) = c \text{ for any value of } x,
\]

hence \( S_n = c \sum_{i=1}^{n} \Delta x_i \)

\[
= c(x_n - x_0 + x_2 - x_1 + x_3 - x_2 + \cdots + x_{n-1} - x_{n-2} + x_n - x_{n-1})
\]

\[
= c(x_n - x_0)
\]

\[
\int_{a}^{b} c \, dx = c(x_n - x_0) = c(b - a)
\]

Does this result agree with the area interpretation of the integral?

\[
\text{area} = \text{height} \cdot \text{base} = c(b-a)
\]
II. If \( f(x) = x \), then \( \int_{a}^{b} x \, dx = ? \)

Recall that we mentioned that it makes no difference to the value of the integral in what manner the interval \( ab \) is subdivided. In evaluating this particular integral the work is simplified if the interval \( ab \) is divided into \( n \) equal subintervals.

\[
S_n = \sum_{i=1}^{n} x_i \Delta x = \sum_{i=1}^{n} (a + i\Delta x) \Delta x
\]

\[
= (a + 1\Delta x + a + 2\Delta x + a + 3\Delta x + \cdots + n\Delta x) \Delta x
\]

\[
= n a \Delta x + \Delta x(1 + 2 + 3 + \cdots + n) \Delta x
\]

\[
S_n = n a \Delta x + (\Delta x)^2(1 + 2 + 3 + \cdots + n)
\]

call \( r \)

To find a simpler expression for \( r \) write:

\[
r = 1 + 2 + 3 + 4 + \cdots + n
\]

\[
r = n + (n-1) + (n-2) + \cdots + 1
\]

\[
2r = (1 + n) + (1 + n) + (1 + n) + \cdots + (1 + n)
\]

\[
2r = n(1 + n)
\]

\[
r = n(1 + n)/2
\]

thus \( S_n = n a \Delta x + (\Delta x)^2(\ n(1+n)/2 \ ) \)

note \( \Delta x = b-a/n \)

\[
S_n = n a(b-a/n) + (b-a/n)^2(\ n(1+n)/2 \ )
\]

\[
S_n = a(b-a) + (b-a)^2(n+1)/2n
\]

\[
S_n = a(b-a) + n(b-a)^2 + (b-2)^2/2n
\]

\[
S_n = a(b-a) + (b-a)^2/2 + (b-a)^2/2n
\]
\[ S_n = a(b-a) + \frac{(b-a)^2}{2} + \frac{(b-a)^2}{(a - \frac{1}{n})} \]

now \( \frac{1}{n} \) is null hence \( \frac{(b-a)^2}{(2 \cdot \frac{1}{n})} \) is null, and

\[ \lim_{n \to \infty} S_n = \int_a^b x \, dx = a(b-a) + \frac{(b-a)^2}{2} = \frac{b^2 - a^2}{2} \]

Does this result agree with the area interpretation?

The shaded figure is a trapezoid;

\[ A = \frac{1}{2} (b-a)(b+a) = \frac{b^2 - a^2}{2} \]

We might go on in this manner to obtain \( \int_a^b x^2 \, dx = \frac{b^3}{3} - \frac{a^3}{3} \), and finally \( \int_a^b x^r \, dx = \frac{b^{r+1}}{r+1} - \frac{a^{r+1}}{r+1} \).

Applications of the integral:

I. Our discussion of the integral was prompted by looking at the problem of finding the area under the graph of \( f(x) \), considered as positive and continuous. In the definition of the integral, however, nothing was said about the function being positive, in fact the function may be both positive and negative or even everywhere negative on the interval from \( x = a \) to \( x = b \). The heart of the
integral is the process of taking the limit of a certain sequence of sums. The terms of the sums are of the form $f(x_1^i)\Delta x_1^i$, where $i$ ranges from 1 to $n$. The limiting process occurs as $n$ becomes large without bound in such a way that the largest $\Delta x$ approaches zero. Thus if the function oscillates, being at times above and below the $x$-axis, the integral $\int_a^b f(x)dx$ represents the algebraic sum of the areas bounded by the graph of $f(x)$, the $x$-axis, and the lines $x = a$ and $x = b$.

II. Noting that the integral involves the limit of a sum of small quantities, it is possible to define many geometric and physical concepts as integrals. Consider for example the function $f(x) = x$ on the interval from $x = 0$ to $x = 3$. Rotate the region bounded by the graph of $f(x)$, the $x$-axis, and the line $x = 3$, about the $x$-axis. The volume which has thus been swept out is a cone.

Problem: find its volume
Divide the interval from 0 to 3 into \( n \) subintervals, and on each erect a perpendicular to the graph of \( f(x) \), forming \( n \) rectangles. Now as the shaded region is revolved about the \( x \)-axis so too are the \( n \) rectangles, each rectangle forming a circular disk. The volume of a typical disk may be represented by \( \pi (f(x_i^#))^2 \Delta x_i \). The sum of \( n \) such disks approximates the desired volume.

\[
S_n = \sum_{i=1}^{n} \pi (f(x_i^#))^2 \Delta x_i
\]

The constant \( \pi \) is uneffected by the summation, hence \( S_n \) may be written:

\[
S_n = \pi \sum_{i=1}^{n} (f(x_i^#))^2 \Delta x_i
\]

If \( n \) is allowed to become large without bound, the limit of \( S_n \) strongly suggests the integral \( \int_{a}^{b} (f(x))^2 \, dx \). It would thus seem natural to define the volume of such a solid of revolution by the integral:

\[
V = \pi \int_{a}^{b} (f(x))^2 \, dx
\]

In the present case \( V = \pi \int_{0}^{3} x^2 \, dx = \pi \left(3^3/3 - 0^3/3 \right) = \pi 27/3 = 9\pi \).

As a check recall the volume formula for a right circular cone: \( V = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi 3^2 \cdot 3 = 9\pi \).
III. As a third example we might look at a problem totally unrelated to areas and volumes. Physicists tell us that if a force on a body is constant then the work required to move that body through a certain distance is given by the product of that applied force and the distance through which the body is moved. How much work is performed, however, if the force is not constant?

Consider a body whose mass is thought of as concentrated at the point \( x \). The force that is applied to this body is not constant but is given as a function of the body's position \( x \). The problem arises when we attempt to calculate the expended work in moving the body from position \( a \) to position \( b \). As in the case of velocity, the force is not constant throughout the entire interval, but if the interval is divided into smaller subintervals the force tends to approximate a constant over these smaller subintervals. Divide the interval into \( n \) subintervals, pick a point in each and evaluate the force function at that point. Form the sum

\[
S_n = \sum_{i=1}^{n} f(x_i^*) \Delta x_i,
\]

note once again the similarity to the integral. It would thus seem to be in order to define the work done by a variable force \( f(x) \) moving a body through the distance from \( x = a \) to \( x = b \) by the integral,

\[
W = \int_{a}^{b} f(x) \, dx.
\]
1. Discuss the conditions under which two quadrilaterals are congruent.

2. What is a valid argument?

3. What is a true conclusion?

4. Are the conclusions obtained in deductive reasoning always true?

5. Of what use is intuition in mathematics?

6. Make up examples in which different postulates imply the same conclusion.

7. Make up five examples of invalid arguments.

8. Find the rational number equivalent to $0.9232323\cdots$

9. Find any maximum or minimum points of the function $f(x) = 4x^3 + 9x^2 - 12x + 5$.

10. A bin with square base and no top is to be constructed from 432 square feet of lumber. Find the maximum capacity of the bin.

11. Given $s = 100t - 10t^2$
    
    A. Find expressions for velocity and acceleration at any time $t$. 

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11. Continued

B. Find the velocity of t =5, t =10, t =15.

C. If a ball rolls up a plane according to above equation, how far up does it roll?

12. Find the area bounded by the graph of

\[ Y = \frac{1}{2} x^3 + 3x^2 + \frac{1}{2} x + 2 \] and the lines \( x = 1, \)
\( x = 4, \) and \( Y = 0. \)

13. If \( f(x) \) is a given function define \( f'(x). \)

If \( f(x) = \frac{1}{x} \) obtain \( f'(x) \) by definition.

14. Consider the sequence \( \{a_n\} \) where \( a_n = \frac{n}{1+n^2} \)

The first five terms are:

A number which you think is the limit of \( \{a_n\} \) is:

What does it mean to say that the above number is a limit?
FINAL EXAM

(1) What is meant by a "mathematical science"? Be sure to state all essential ingredients.

(2) What is meant by an argument being valid?

Argument:  Given -- all intelligent students can pass Math 416. Smith can pass Math 416.

Conclusion--Smith is an intelligent student.

Valid  Proof:
Not Valid

(3) Consider the function f(x). Define f'(x).

State two applications of f'(x).

a.

b.

(4) A particle is moving vertically (positive direction up): Its position s, measured from some origin, at time t is given by s = t^2 - 11t + 10.

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(4) continued

a. Find a formula giving velocity at any time t.

b. Find the velocity for:
   \[ t = 0 \]
   \[ t = 1 \]
   \[ t = 2 \]
   \[ t = 3 \]
   \[ t = 4 \]

c. When is the particle moving upward?

When is the particle moving downward?

d. When is the particle at rest?

(5) What is meant by

\[ \int_{a}^{b} f(x) \, dx \]

State one application.

(6) Consider the positive integers as the only numbers known. Let '0' be an operation defined as follows:

i) a and b are any two positive integers then:

\[ a \circ b = a + (2b) \]
I. Are the positive integers closed under this operation?

II. Is the commutative property satisfied?

III. How about associativity?

(7) The group axioms are given as follows:
If a, b, and c are any three elements of the group, and if '°' is the given operation,

I. \( a ° b \) is also a group element

II. \( (a ° b) ° c = a ° (b ° c) \)

III. There is an element i of the group such that for any element \( a \), \( a ° i = a \)

IV. For any a there exists an \( a' \) such that \( a ° a' = i \)

Give two examples of a group.

Theorem: If a, b, c are group elements
If \( a ° c = b \)
Then \( c = a' ° b \)

Proof
(8) If a farmer digs his potatoes today, he will have 600 bushels worth $1 a bushel. Every week he waits, the price drops ten cents a bushel, and the crop increases one hundred bushels.

If \( x \) represents the number of weeks waited, represent the total gross income \( R \) as a function of \( x \).

How many weeks should he wait to make the most money?
BIBLIOGRAPHY


