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INTRODUCTION

Consider a sequence of independent random variables $X_1$, $X_2$, ..., $X_k$, ... with mean 0 and variance $\sigma_k^2$. Let $F_n(x)$ denote the distribution function of the sum

$$S_n = \frac{X_1 + \cdots + X_n}{s_n},$$

where $s_n^2 = \sigma_1^2 + \cdots + \sigma_n^2$. The classical forms of the central limit theorem state that, under certain conditions, the distribution function $F_n(x)$ approaches the Gaussian distribution

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}dt.$$

Berry [1] and Esseen [3] have studied the behavior of

$$M_n = \sup_{-\infty < x < \infty} |F_n(x) - \Phi(x)|$$

and in their main theorems have obtained bounds on $M_n$ which involve the moments of $X_k$ through the third.

A generalization of the above is to consider a system of random variables $(X_{nk})$, $k = 1, \ldots, k_n$, $n = 1, 2, \ldots$ such that
for each $n$, the random variables $X_{n1}, \ldots, X_{nk_n}$ are independent.

Let $F_n(x)$ denote the distribution function of the sum

$$S_n = X_{n1} + \ldots + X_{nk_n}.$$

From a well-known theorem of Khintchine (Theorem 1) it follows that if the random variables $X_{nk}$ are infinitesimal then the class of possible limit distributions of $F_n(x)$ coincides with the class of infinitely divisible distributions. Motivated by this theorem we consider

$$M_n = \sup_{-\infty < x < +\infty} |F_n(x) - F(x)|,$$

where $F(x)$ is an infinitely divisible distribution function.

Shapiro [5] has studied the behavior of $M_n$ in the case where both $X_{nk}$ and $F(x)$ have finite variances and has obtained bounds on $M_n$ which tend to zero as $n$ becomes infinite under necessary and sufficient conditions that $F_n(x)$ converges to $F(x)$.

The purpose of this study is to extend the results in [5] to include the case where neither $F(x)$ nor $X_{nk}$ need to have finite variances. Our main theorem (Theorem 8) gives a bound on $M_n$ under a mild assumption on $X_{nk}$ and a certain assumption on the derivative of the infinitely divisible distribution function $F(x)$. It is shown that if $F(x)$ satisfies
an additional condition, which is considerably weaker than that of having finite variance, then the bound tends to zero as \( n \) becomes infinite under the necessary and sufficient condition that \( F_n(x) \) converges to \( F(x) \) (c.f. Theorem 2). Finally we apply our general results to the distribution functions of normed sums

\[
S_n = \frac{X_1 + \ldots + X_n}{B_n},
\]

where \( X_1, \ldots, X_n, \ldots \) are independent identically distributed random variables and the constants \( B_n \) are suitably chosen.
CHAPTER I

Preliminaries

1.1 Random Variables, Distribution Functions and Characteristic Functions

Let \( \Omega \) be an abstract space, \( \mathcal{F} \) a Borel field of subsets of \( \Omega \), and \( P \) a probability measure defined on \( \mathcal{F} \). Any real-valued \( \mathcal{B} \)-measurable (Borel-measurable) function defined on \( \Omega \) is called a random variable. The distribution function \( F(x) \) of a random variable \( X \) is defined by

\[
(1.1.1) \quad F(x) = P(\{ \omega : X(\omega) \leq x \}) = P(X \leq x),
\]

where \( x \) is a real number, \(-\infty < x < +\infty\). We see that \( F(x) \) is non-decreasing, continuous from the right and \( F(-\infty) = 0 \), \( F(+\infty) = 1 \). It will be assumed in the remainder of this study that all random variables under consideration are defined on \( \Omega \).

The random variables \( X_1, X_2, \ldots, X_n \) are said to be independent if and only if

\[
(1.1.2) \quad P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) = \prod_{k=1}^{n} P(X_k \leq x_k)
\]
for all \( x_1, x_2, \ldots, x_n \). In the sequel we shall use the fact that if the random variables \( x_1, x_2, \ldots, x_n \) are independent and if \( f(x) \) is a \( \mathbb{F} \)-measurable function, then \( f(x_1), f(x_2), \ldots, f(x_n) \) are also independent random variables.

If \( g(x_1, \ldots, x_n) \) is any complex-valued \( \mathbb{F} \)-measurable function such that the integral

\[
\int_{\Omega} g(x_1(\omega), \ldots, x_n(\omega))d\mathbb{P}
\]

exists and is finite, we define the expectation of \( g(x_1, \ldots, x_n) \), denoted by \( \mathbb{E}[g(x_1, \ldots, x_n)] \), by

\[
(1.1.3) \quad \mathbb{E}[g(x_1, \ldots, x_n)] = \int_{\Omega} g(x_1(\omega), \ldots, x_n(\omega))d\mathbb{P}.
\]

The mean and the variance of a random variable \( X \), if exist, are defined to be

\[
\mathbb{E}[X] \quad \text{and} \quad \mathbb{E}[(X - \mathbb{E}[X])^2]
\]

respectively.

For any random variable \( X \) and for any real number \( t \), the expectation

\[
(1.1.4) \quad \phi(t) = \mathbb{E}[e^{itX}]
\]

always exists. We call \( \phi(t) \) the characteristic function of the
random variable $X$. If $F(x)$ is the distribution function of the random variable $X$, then its characteristic function $\phi(t)$ is also given by

$$\phi(t) = \int_{-\infty}^{+\infty} e^{itx} dF(x). \tag{1.1.5}$$

It follows immediately that every characteristic function satisfies

$$|\phi(t)| \leq 1. \tag{1.1.6}$$

If $X_1, \ldots, X_n$ are independent random variables with characteristic function $\phi_1(t), \ldots, \phi_n(t)$ respectively, and if we let

$$S = X_1 + \ldots + X_n, \tag{1.1.7}$$

then the characteristic function $\phi(t)$ of $S$ is given by

$$\phi(t) = \phi_1(t) \ldots \phi_n(t). \tag{1.1.8}$$

The correspondence between the distribution functions and the characteristic functions is one-to-one. Furthermore, if a sequence of distribution functions $\{F_n(x)\}$ converges to a distribution function $F(x)$ at every continuity point, then the
corresponding sequence of characteristic functions \( \{ \varphi_n(t) \} \) converges to the characteristic function \( \varphi(t) \) of \( F(x) \) and vice versa.

1.2 **Infinitely Divisible Distribution Functions**

A distribution function \( F(x) \) is said to be **infinitely divisible** if and only if, for every positive integer \( n \) there exist \( n \) independent identically distributed random variables \( X_{n1}, \ldots, X_{nn} \) such that the distribution function of the sum \( S_n = X_{n1} + \ldots + X_{nn} \) is \( F(x) \). The characteristic function \( \varphi(t) \) of any infinitely divisible distribution function can be represented by the formula of Levy and Khintchine (c.f. [4])

\[
\log \varphi(t) = igt + \int_{-\infty}^{+\infty} f(t,u)dG(u),
\]

where \( g \) is a constant, \( f(t,u) \) is given by

\[
f(t,u) = \begin{cases} 
\left( e^{itu} - 1 - \frac{itu}{1 + u^2} \right) \frac{1 + u^2}{u^2} & (u \neq 0), \\
-\frac{t^2}{2} & (u = 0),
\end{cases}
\]

and \( G(u) \) is a bounded non-decreasing function, which is continuous from the right and \( G(-\infty) = 0 \).
We can also represent \( \varphi(t) \) by Levy's formula

\[
(1.2.3) \quad \log \varphi(t) = \delta it - \frac{\sigma^2 t^2}{2} + \left( \int_{-\infty}^{0} \left( e^{itx} - 1 - \frac{itx}{1 + x^2} \right) d\mu(x) \right) + \left( \int_{0+}^{+\infty} \left( e^{itx} - 1 - \frac{itx}{1 + x^2} \right) d\nu(x) \right),
\]

where \( \delta, \sigma^2 \geq 0 \) are constants, \( \mu(x) \) and \( \nu(x) \) are non-decreasing functions defined on \((-\infty, 0)\) and \((0, +\infty)\) respectively, and are such that

\[
M(-\infty) = 0 = N(+\infty)
\]

and

\[
\int_{-\varepsilon}^{0-} x^2 d\mu(x) + \int_{0+}^{\varepsilon} x^2 d\nu(x) < +\infty
\]

for every finite \( \varepsilon > 0 \).

The connection between these two representations is given by the following:

\[
M(x) = \int_{-\infty}^{x} \frac{1 + u^2}{u^2} dG(u) \quad \text{for} \quad x < 0,
\]

\[
(1.2.4) \quad N(x) = -\int_{x}^{+\infty} \frac{1 + u^2}{u^2} dG(u) \quad \text{for} \quad x > 0,
\]

\[
\sigma^2 = G(0+) - G(0-).
\]
If an infinitely divisible distribution function has finite variance, we have another representation of its characteristic function, namely:

\[ \log \varphi(t) = \mu it + \int_{-\infty}^{+\infty} f_1(t, v) dK(v), \]

which is known as Kolmogorov's formula (c.f. [4]). Here the constant \( \mu \) is the mean of the distribution function, the function \( f_1(t, v) \) is given by

\[ f_1(t, v) = \begin{cases} \frac{e^{itv} - 1 - itv}{v^2} & (v \neq 0), \\ \frac{\mu^2}{2} & (v = 0), \end{cases} \]

where \( K(v) \) is a bounded non-decreasing function which is continuous from the right and \( K(-\infty) = 0 \).

It can be seen by two differentiations of both sides of (1.2.5) that \( K(\infty) = \sigma^2 \), the variance of the distribution function. The relationship of the formula of Levy and Khintchine (1.2.1) to Kolmogorov's formula (1.2.5) is given by

\[ \mu = \delta + \int_{-\infty}^{+\infty} u dG(u), \]

and
(1.2.8) \[ K(v) = \int_{-\infty}^{v} (1 + u^2) dG(u). \]

1.3 Some General Limit Theorems

The random variables \( X_{nk} \), \( k = 1, \ldots, k_n; \ n = 1, 2, \ldots \) are said to be infinitesimal if and only if,

\[ \lim_{n \to \infty} \max_{k} P(|X_{nk}| > \varepsilon) = 0 \]

for every \( \varepsilon > 0 \).

The following two theorems were referred to in the introduction.

Theorem 1 (Khintchine). In order that a distribution function \( F(x) \) be the limit distribution function of the distribution functions of the sums

\[(1.3.1) \quad S_n = X_{n1} + \ldots + X_{nk_n} + A_n, \]

where \( (X_{nk}) \), \( k = 1, \ldots, k_n; \ n = 1, 2, \ldots \) is a system of infinitesimal random variables, independent within each row and \( A_n \) are suitably chosen constants, it is necessary and sufficient that \( F(x) \) is an infinitely divisible distribution function.

Theorem 2. In order that the distribution functions of the sums (1.3.1) converge to a limit, it is necessary and
sufficient that there exist non-decreasing functions

\[ M(x) \ (M(-\infty) = 0) \text{ and } N(x) \ (N(+\infty) = 0), \]

defined in the intervals \((-\infty, 0)\) and \((0, +\infty)\) respectively, and a constant \(\sigma > 0\), such that

\[
\lim_{n \to \infty} \sum_{k=1}^{k_n} F_{nk}(x) = M(x) \quad (x < 0),
\]

(3.1.2)

\[
\lim_{n \to \infty} \sum_{k=1}^{k_n} [F_{nk}(x) - 1] = N(x) \quad (x > 0),
\]

at every continuity point of \(M(x)\) and \(N(x)\),

\[
\lim_{\xi \to 0} \lim_{n \to \infty} \sum_{k=1}^{k_n} \left\{ \int_{|x| < \xi} x^2 dF_{nk}(x) - \left( \int_{|x| < \xi} x dF_{nk}(x) \right)^2 \right\} = \sigma^2,
\]

(1.3.3)

\[
\lim_{\xi \to 0} \lim_{n \to \infty} \sum_{k=1}^{k_n} \left\{ \int_{|x| < \xi} x^2 dF_{nk}(x) - \left( \int_{|x| < \xi} x dF_{nk}(x) \right)^2 \right\} = \sigma^2,
\]

where \(F_{nk}(x)\) denotes the distribution function of \(X_{nk}\).

Proofs of these theorems can be found in \([4]\).
CHAPTER II

Statement of Some Earlier Results

In this chapter we state some of the earlier results concerning estimation of errors.

2.1 Esseen's Theorem

We shall make use of the following:

Theorem 3 (Esseen). Let $F(x)$ be a non-decreasing function such that $F'(x)$ exists everywhere and $|F'(x)| < B$ for some real number $B > 0$. Let $H(x)$ be a function of bounded variation on the whole real line. Let $F(-\infty) = 0 = H(-\infty)$ and $F(+\infty) = H(+\infty)$. Let $f(t)$ and $h(t)$ be the corresponding Fourier-Stieltjes transforms and for any $T > 0$ let

\[ (2.1.1) \quad \mathcal{E} = \int_{-T}^{T} \left| \frac{h(t) - f(t)}{t} \right| dt. \]

Then to every $p > 1$ there corresponds a finite positive real number $c(p)$, depending on $p$, such that

\[ (2.1.2) \quad |H(x) - F(x)| \leq \frac{p \mathcal{E}}{2\pi} + \frac{c(p)B}{T}. \]
2.2 Shapiro's Estimates

Let $F_n(x)$ denote the distribution function of the sum

$$S_n = X_{n1} + \ldots + X_{nk_n}$$

of infinitesimal random variables which are independent in each row. Let $F(x)$ be an infinitely divisible distribution function. Define

$$M_n = \sup_{-\infty < x < +\infty} |F_n(x) - F(x)|.$$

As mentioned in the introduction, in [5] Shapiro considers the case where both the infinitesimal random variables $X_{nk}$ and the distribution function $F(x)$ have finite variances. Bounds on $M_n$ are obtained in terms of the means and variances of the random variables $X_{nk}$ and $F(x)$. It also involves the function $K(v)$ which corresponds to $F(x)$ by the Kolmogorov's formula for the characteristic function of $F(x)$ (1.2.5). For later reference we state some of the results of [5] in detail.

Let $F_{nk}(x)$, $\mu_{nk}$ and $\sigma^2_{nk}$ denote the distribution function, the mean and the variance of $X_{nk}$. Let the infinitely divisible distribution function $F(x)$ have mean $\mu$ and variance $\sigma^2$, so that its characteristic function $\phi(t)$ has a representation (1.2.5)

$$\log \phi(t) = \mu it + \int_{-\infty}^{+\infty} f(t,v) dK(v).$$
Let $A > 0$ be such that $-A$ and $A$ are continuity points of $K(v)$ and let $0 < \delta \leq 2A$. Define

$$m = m(A, \delta) = \left\lfloor \frac{2A}{\delta} \right\rfloor + 1,$$

where bracket denotes the greatest integer function. Let

$$-A = x_0 < x_1 < \ldots < x_m = A$$

be such that $x_i$ $(i = 1, \ldots, m - 1)$ is a continuity point of $K(v)$ and

$$\max_{i=1, \ldots, m} |x_i - x_{i-1}| < \delta.$$

Let

$$K_n(v) = \sum_{k=1}^{k_n} \int_{-\infty}^{v} u^2 dF_{nk}(u + \mu_{nk})$$

and

$$E(n, t, m(A, \delta)) = \frac{5}{4} \delta |t|^3 \left( \sigma_n^2 + \sigma^2 \right)$$

$$+ \frac{t^2}{2} \sum_{i=0}^{m} \left| K_n(x_i) - K(x_i) \right|$$

$$+ 2|t| \frac{1}{A} \left[ K_n(+\infty) - K_n(A) + K(+\infty) - K(A) ight. + K_n(-A) + K(-A) \right],$$

where $\sigma_n^2$ and $\sigma^2$ are variances of $F_n(x)$ and $F(x)$ respectively.
Lemma 1 (Lemma 3 of [5]). Let $f_1(t,v)$ be as defined in (1.2.6), then we have

$$(2.2.8) \quad \left| \int_{-\infty}^{+\infty} f_1(t,v) \, d\left[K_n(v) - K(v)\right]\right| \leq E(n, t, m(A, \delta))$$

for any $A > 0$, $0 < \delta \leq 2A$ and any choice of $x_0, x_1, \ldots, x_m$ satisfying (2.2.4), (2.2.5).

Now let $\mu_n$ denote the mean of $S_n$. Using the above notation we define

$$(2.2.9) \quad g(n, m(A, \delta)) = \left[ \frac{1}{3} \sigma_n^2 \max_{\sigma_{nk}^2} \left( \frac{1}{3} \right) + \left[ \frac{5}{6} \delta \left( \sigma_n^2 + \sigma^2 \right) \right] \right]^{\frac{1}{2}}$$

$$+ \left[ \frac{1}{2} \sum_{i=0}^{n} \left( K_n(x_i) - K(x_i) \right) \right]^{\frac{1}{2}}$$

$$+ \left[ \frac{1}{4} \left\{ K_n(+\infty) - K_n(A) + K(+\infty) - K(A) \right. \right.$$

$$\left. + K_n(-A) + K(-A) \right\} + 2\left| \mu_n - \mu \right| \right]^{\frac{1}{2}}.$$

Now we are able to state the general result of [5].

Theorem 4. Let $F(x)$ be any infinitely divisible distribution function with mean $\mu$ and variance $\sigma^2$ and with corresponding $K(v)$ given by Kolmogorov's formula. Let $(X_{nk})$ be a system of random variables, independent within
each row with mean $\mu_{nk}$ and variance $\sigma_{nk}^2$. Let $F(x)$ be the distribution of

$$S_n = X_{n1} + \ldots + X_{nk}\,$$

and suppose that $F'(x)$ exists and $|F'(x)| < B$ for all $x$. Assume that $\sigma_{nk}^2 \leq 1$, $k = 1, \ldots, k_n$. Then it follows that for any $p > 1$

$$(2.2.10) \quad K_n = \sup_{-\infty < x < +\infty} |F_n(x) - F(x)| \leq k(p,B)g(n,m(A,\delta)),\,$$

where $k(p,B)$ is a constant depending only on $B$ and on $p > 1$.

By choosing $\delta > 0$ such that $\pm \frac{1}{\sqrt{\delta}}$ are continuity points of $K(v)$ we may then replace $A$ in (2.2.9) by $\frac{1}{\sqrt{\delta}}$. Hence (2.2.10) becomes

$$(2.2.11) \quad M_n = \sup_{-\infty < x < +\infty} |F_n(x) - F(x)| \leq k(p,B)g(n,\delta),\,$$

where

$$(2.2.12) \quad g(n,\delta) = g(n,m(\frac{1}{\sqrt{\delta}},\delta)).\,$$

The behavior of $g(n,\delta)$ is given in the following theorem.
Theorem 5. Using the notation of Theorem 4, if we assume that

a) \( X_{nk} = \mu_{nk} \) are infinitesimal,

b) \( \lim_{n \to \infty} F_n(x) = F(x) \) at every continuity point of \( F(x) \),

c) \( \lim_{n \to \infty} \sigma_n^2 = \sigma^2 \),

then there exists a sequence \( \{ \delta_n \} \) of real numbers such that

d) \( 0 < \delta_n < 1 \),

e) \( \lim_{n \to \infty} \delta_n = 0 \)

and \( f) \lim_{n \to \infty} g(n, \delta_n) = 0 \).

This theorem is only a part of Theorem 5 of [5]. Its proof can be found there. We remark that the sequence \( \{ \delta_n \} \) can be chosen so that the additional condition

g) \( \delta_{n+1} \leq \delta_n \)

is satisfied.

2.3 Truncated Random Variables

Let \((X_{nk}), k = 1, \ldots, k_n; n = 1, 2, \ldots \) be a system of infinitesimal random variables, independent within each row. For any \( a > 0 \) let \( X_{nk}^a \) be defined by
(2.3.1) \[ x_{nk}^a(\omega) = \begin{cases} \chi_{nk}(\omega) & \text{if } -a < \chi_{nk}(\omega) \leq a, \\ 0 & \text{otherwise}. \end{cases} \]

Let \( F_{nk}^a(x) \), \( \varphi_{nk}^a(t) \), \( \mu_{nk}^a(a) \), \( \sigma_{nk}^2(a) \) denote, respectively, the distribution function, the characteristic function, the mean, the variance of \( x_{nk}^a \). Let \( F_{nk}(x) \) denote the distribution function of the random variable \( X_{nk} \). Then we have

(2.3.2) \[ F_{nk}^a(x) = \begin{cases} 0 & \text{for } x \leq -a, \\ F_{nk}(x) - F_{nk}(-a) & \text{for } -a < x < 0, \\ F_{nk}(x) + 1 - F_{nk}(a) & \text{for } 0 \leq x \leq a, \\ 1 & \text{for } a < x. \end{cases} \]

Let

(2.3.3) \[ S_n = X_{n1} + \ldots + X_{nk}, \]

(2.3.4) \[ S_n^a = X_{n1}^a + \ldots + X_{nk}^a. \]

Let \( F_n(x) \), \( \varphi_n(t) \) denote the distribution function, the characteristic function of \( S_n \), let \( F_n^a(x) \), \( \varphi_n^a(t) \), \( \mu_n^a(a) \) and \( \sigma_n^2(a) \) denote the distribution function, the characteristic function, the mean and the variance of \( S_n^a \) respectively.
Let $F(x)$ be an infinitely divisible distribution function with corresponding function $G(u)$ given by the formula of Levy and Khintchine (1.2.1). For each $a > 0$ such that $-a$ and $a$ are continuity points of $G(u)$ we define

$$G^a(u) = \begin{cases} 
0 & \text{for } u \leq -a, \\
g(u) - G(-a) & \text{for } -a < u \leq a, \\
G(a) - G(-a) & \text{for } u > a, 
\end{cases}$$

(2.3.5)

$$\gamma^a = \gamma - \int_{|u| \geq a} \frac{1}{u} \, dG(u).$$

(2.3.6)

Since $G^a(u)$ is also bounded, non-decreasing, continuous from the right and $G^a(-\infty) = 0$, hence $\gamma^a$ and $G^a(u)$ determine a unique infinitely divisible distribution function, which will be denoted by $F^a(x)$. The following results are obtained in [6].

**Theorem 6.** If $F_n(x)$ converges to $F(x)$, then for any $a > 0$ such that $-a$ and $a$ are continuity points of $G(u)$, $F_n^a(x)$ converges to $F^a(x)$. In particular, if $G(u)$ is non-increasing outside of the interval $[-a,a]$ then $F_n^a(x)$ converges to $F(x)$.

**Theorem 7.** If $F_n^a(x)$ converges to $F^a(x)$, then

$$\lim_{n \to \infty} \int_{-\infty}^{+\infty} x^k dF_n^a(x) = \int_{-\infty}^{+\infty} x^k dF^a(x)$$

for every positive integer $k$. 
CHAPTER III

The Main Result

3.1 Notations

Let F(x) be an infinitely divisible distribution

function with characteristic function $\phi(t)$. Hence according to

the formula of Levy and Khintchine (1.2.1), a constant $\gamma$ and

a function $G(u)$ are uniquely determined by $\phi(t)$. Let $\gamma^a$,

$G^a(u)$, $\phi^a(t)$ and $F^a(x)$ be as defined in § 2.3. According to

the theorem of [7], the mean and variance of $F^a(x)$ exist for

each $a > 0$. Let $\mu(a)$ and $\sigma^2(a)$ denote the mean and variance

of $F^a(x)$, respectively. By evaluating the derivative of $\phi^a(t)$

at 0 we find

\begin{equation}
(3.1.1) \quad \mu(a) = \gamma^a + \int_{-\infty}^{+\infty} uG^a(u) \, du.
\end{equation}

Since $F^a(x)$ is an infinitely divisible distribution with finite

variance, its characteristic function can be represented by

Kolmogorov's formula (1.2.5). We have

\begin{equation}
(3.1.2) \quad \log \phi^a(t) = \mu(a)it + \int_{-\infty}^{+\infty} f_1(t,v) dK^a(v).
\end{equation}
According to (1.2.8) we have

\[(3.1.3) \quad K^a(v) = \int_{-\infty}^{v} (1 + u^2) dG^a(u).\]

Let \((X_{nk}), k = 1, \ldots, k_n; n = 1, 2, \ldots\) be a system of random variables which are independent within each row. Let \(X_{nk}^a, F_{nk}^a(x), \varphi_{nk}^a(t), \mu_{nk}^a(a), \sigma_{nk}^2(a), s_n, s_n^a, F_n(x), \varphi_n(t), F_n^a(x), \varphi_n^a(t), \mu_n^a(a), \sigma_n^2(a)\) be defined as in §2.3. Define

\[(3.1.4) \quad K_n^a(v) = \sum_{k=1}^{k_n} \int_{-\infty}^{v} x^2 dF_{nk}^a(x + \mu_{nk}^a(a)).\]

For any \(A > 0\) such that \(-A\) and \(A\) are continuity points of \(G(u)\), and hence also continuity points of \(K^a(v)\), let \(0 < \delta \leq 2A\) and define \(m = m(A, \delta), x_i (i = 0, 1, \ldots, m)\) as in (2.2.3)-(2.2.5). Let

\[(3.1.5) \quad E^a(n, t, m(A, \delta)) = \frac{5}{4} \delta |t|^3 \left( \sigma_n^2(a) + \sigma^2(a) \right) + \frac{t^2}{2} \sum_{i=0}^{m} |K_n^a(x_i) - K^a(x_i)| + \frac{2|t|}{A} \left( K_n^a(+\infty) - K_n^a(A) + K^a(+\infty) - K^a(A) + K_n^a(-A) + K^a(-A) \right).\]
It follows from Lemma 1 that

\[(3.1.6) \quad \left| \int_{-\infty}^{\infty} f_1(t,v) d\left[K_n^a(v) - K^a(v)\right] \right| \leq E^a(n,t,m(A,\delta)),\]

where \(f_1(t,v)\) is the function given by (1.2.6).

We define

\[(3.1.7) \quad g^a(n,m(A,\delta)) = \left\{ \frac{1}{3} \sigma_n^2(a) \max \sigma_{nk}^2(a) \right\}^{\frac{1}{2}} \]

\[+ \left\{ \frac{5}{6} \delta \left( \sigma_n^2(a) + \sigma_{\infty}^2(a) \right) \right\}^{\frac{1}{2}} \]

\[+ \left\{ \frac{1}{2} \sum_{i=0}^{m} \left| K_n^a(x_i) - K^a(x_i) \right| \right\}^{\frac{1}{2}} \]

\[+ \frac{4}{A} \left[ K_n^a(+\infty) - K_n^a(A) + K^a(+\infty) - K^a(A) \right. \]

\[\left. + K_n^a(-A) + K^a(-A) \right] + 2 \left| \lambda_n(a) - \lambda(a) \right|^{\frac{1}{2}}.

If we assume that \(\frac{dF^a(x)}{dx}\) exists and \(\left| \frac{dF^a(x)}{dx} \right| \leq B(a)\) for every \(x\), then it would follow from Theorem 4 that

\[(3.1.8) \quad \sup_{-\infty < x < r} |F_n^a(x) - F^a(x)| \leq k(p,B(a))g^a(n,m(A,\delta)).\]

As will appear later, this assumption on the derivative of \(F^a(x)\) would have to be made for infinitely many \(a\)'s. To avoid this
situation we shall not use Theorem 4 in obtaining bounds on $M_n^{(2.2.2)}$. Our bounds will involve a quantity we now define:

$$g^a(n, m(A, \delta), r) = g^a(n, m(A, \delta)) + \frac{1}{\Lambda(1+\Lambda)} \left\{ \frac{\int |u|^r dG(u)}{r} \right\},$$

where $r$ is a positive real number.

### 3.2 Preliminary Lemmas

The following lemmas will be used to obtain the main theorem in the next section.

**Lemma 2.** Let $z_1$ and $z_2$ be two complex numbers such that $|z_1| \leq 1$ and $|z_2| \leq 1$; then

$$|z_1 - z_2| \leq |\log z_1 - \log z_2|.$$

**Proof:** We use the mean value theorem for function of complex variable (c.f. [2] p.115) which states that if $f(z)$ is analytic, then

$$f(z_1) - f(z_2) = \mu (z_1 - z_2)$$

for some $\mu$ in the convex closure of the image of the line joining $z_1$ and $z_2$ given by the derivative of $f(z)$. Applying this theorem to

$$f(z) = e^z.$$
We see that if \( F_1 \) and \( F_2 \) have negative or zero real parts, then \( |\alpha| \leq 1 \). Therefore, for such \( F_1 \) and \( F_2 \)

\[
|e^{F_1} - e^{F_2}| \leq |F_1 - F_2|.
\]

But this implies that

\[
|z_1 - z_2| \leq |\log z_1 - \log z_2|
\]

if \( |z_1| \leq 1 \) and \( |z_2| \leq 1 \). Q.E.D.

**Lemma 3.** Let \( F_n(x) \) and \( F_n^a(x) \) be defined as in §2.3, then

(3.2.2) \[
|F_n(x) - F_n^a(x)| \leq \sum_{k=1}^{\infty} \left\{ F_{nk}(-a) + 1 - F_{nk}(a) \right\}.
\]

**Proof:** \( F_n(x) = P(S_n \leq x) \)

\[
= P(S_n \leq x \text{ and } X_{nk} \in (-a,a) \text{ for all } k) + P(S_n \leq x \text{ and } X_{nk} \not\in (-a,a) \text{ for some } k)
\]

\[
\leq P(S_n^a \leq x) + P(X_{nk} \not\in (-a,a) \text{ for some } k)
\]

\[
\leq F_n^a(x) + \sum_{k=1}^{\infty} P(X_{nk} \not\in (-a,a))
\]

\[
= F_n^a(x) + \sum_{k=1}^{\infty} \left\{ F_{nk}(-a) + 1 - F_{nk}(a) \right\}.
\]

Therefore, \( F_n(x) - F_n^a(x) \leq \sum_{k=1}^{\infty} \left\{ F_{nk}(-a) + 1 - F_{nk}(a) \right\}. \)
On the other hand,

\[ F_n^a(x) = P(S_n^a \leq x) \]

\[ = P(S_n^a \leq x \text{ and } X_{nk} \in (-a, a] \text{ for all } k) \]

\[ + P(S_n^a \leq x \text{ and } X_{nk} \notin (-a, a] \text{ for some } k) \]

\[ \leq P(S_n \leq x) + P(X_{nk} \notin (-a, a] \text{ for some } k) \]

\[ \leq F_n(x) + \sum_{k=1}^{n} P(X_{nk} \notin (-a, a]) \]

\[ = F_n(x) + \sum_{k=1}^{n} \{F_{nk}(-a) + 1 - F_{nk}(a)\}. \]

Therefore, \( F_n^a(x) - F_n(x) \leq \sum_{k=1}^{n} \{F_{nk}(-a) + 1 - F_{nk}(a)\}. \)

Hence we have

\[ |F_n(x) - F_n^a(x)| \leq \sum_{k=1}^{n} \{F_{nk}(-a) + 1 - F_{nk}(a)\}. \]

Q.E.D.

**Lemma 4.** Let \( G(u) \) be the function corresponding to an infinitely divisible distribution function \( F(x) \) through the formula of Levy and Khintchine (1.2.1). Then for any real numbers \( a, r \) such that \( a > 1, 0 < r < 1 \), we have

\[ (3.2.3) \quad |\varphi^a(t) - \varphi(t)| \leq 4|t|^r \int_{|u| > a} |u|^r \, dG(u), \]

where \( \varphi(t) \) is the characteristic function of \( F(x) \) and \( \varphi^a(t) \) is...
the characteristic function determined by \( \varphi^a \) and \( G^a(u) \) (c.f. (2.3.5) and (2.3.6)).

**Proof:** By using Lemma 2 and (2.3.5), (2.3.6) we have

\[
|\varphi^a(t) - \varphi(t)| \leq |\log \varphi^a(t) - \log \varphi(t)|
\]

\[
= \left| -it \int_{|u| > a} \frac{1}{u} dG(u) - \int f(t,u) dG(u) \right|
\]

\[
= \left| \int_{|u| > a} (e^{itu} - 1) \frac{1 + u^2}{u^2} dG(u) \right|
\]

\[
\leq \left( \int_{|u| > a} (e^{itu} - 1) \frac{1 + u^2}{u^2} dG(u) \right).
\]

But for \( a > 1 \) we have

\[
\frac{1 + u^2}{u^2} \leq 2
\]

for \( |u| > a \). Hence, for \( |u| > a \geq 1 \),

\[
|\left( e^{itu} - 1 \right) \frac{1 + u^2}{u^2}| \leq 2 \left| e^{itu} - 1 \right|
\]

\[
= 4 \left| \sin \frac{tu}{2} \right|.
\]

For \( \left| \frac{tu}{2} \right| \geq 1 \) we have

\[
4 \left| \sin \frac{tu}{2} \right| \leq 4 \leq 4 \left| \frac{tu}{2} \right|^r \leq 4 |tu|^r.
\]

For \( \left| \frac{tu}{2} \right| \leq 1 \) and \( 0 < r \leq 1 \) we have

\[
4 \left| \sin \frac{tu}{2} \right| \leq 4 \left| \frac{tu}{2} \right| \leq 4 \left| \frac{tu}{2} \right|^r \leq 4 |tu|^r.
\]
Therefore,

\[ |(e^{itu} - 1) \frac{1 + u^2}{u^2} | \leq 4 |tu|^r \]

for \(|u| > a > 1, 0 < r \leq 1\). Hence we have

\[ |\varphi^n(t) - \varphi(t)| \leq \int_{|u| > a} 4 |tu|^r dG(u) \]

\[ = 4 |t| \int_{|u| > a} |u|^r dG(u). \quad Q.E.D. \]

3.3 The General Result

Using the notation of § 3.1 we now have the main theorem.

**Theorem 8.** Let \(F(x)\) be an infinitely divisible distribution function with corresponding \(G(u)\) given by the formula of Levy and Khintchine (1.2.1). Let \((X_{nk})\), \(k = 1, \ldots, k_n; n = 1, 2, \ldots\) be a system of random variables, independent within each row. Let \(F_n(x)\) be the distribution function of the sum

\[ S_n = X_{n1} + \ldots + X_{nk_n} \]

and suppose that \(F'(x)\) exists and \(|F'(x)| < B\) for all \(x\). Assume ¹ also that \(\sigma_{nk}(a) \leq 1\) for all \(n, k\). Then it follows that, for

¹ The assumption \(\sigma_{nk}(a) \leq 1\) is really quite weak as will be seen later (c.f. the footnote to the proof of Theorem 9).
each \( p > 1, 0 < r \leq 1, a \geq 1 \),

\[(3.3.1) \quad H_n = \sup_{-\varphi < x < \varphi} |F_n(x) - F(x)| \leq k(p,B)g^a(n,m(A,\delta),r) \]

\[+ \sum_{k=1}^{k_n} \left\{ F_{nk}(-a) + 1 - F_{nk}(a) \right\}, \]

where \( k(p,B) \) is a constant depending only on \( B \) and \( p > 1 \), and where \( g^a(n,m(A,\delta),r) \) is given by (3.1.9).

**Proof:** We write

\[(3.3.2) \quad |F_n(x) - F(x)| \leq |F_n(x) - F_n^a(x)| + |F_n^a(x) - F(x)|. \]

Applying (3.1.6) of Lemma 1 with \( K_n(v) = K_n^a(v) \), \( K(v) = K^a(v) \), we have

\[(3.3.3) \quad \left| \int_{-\infty}^{+\infty} f_1(t,v)d[K_n^a(v) - K^a(v)] \right| \leq E^a(n,t,m(A,\delta)), \]

where \( E^a(n,t,m(A,\delta)) \) is as given in (3.1.5).

Applying Lemma 2, we have

\[(3.3.4) \quad |\varphi_n^a(t) - \varphi^a(t)| \leq |\log \varphi_n^a(t) - \log \varphi^a(t)|. \]

Hence, by Lemma 4, we have

\[(3.3.5) \quad |\varphi_n^a(t) - \varphi(t)| \leq |\varphi_n^a(t) - \varphi^a(t)| + |\varphi^a(t) - \varphi(t)| \]

\[\leq |\log \varphi_n^a(t) - \log \varphi^a(t)| + \left( \int_{|u| > a} \cdot dG(u) \right) |t|^r. \]
Define

(3.3.6) \( \psi_n^a(t) = \mu_n(a)it + \int_{-\infty}^{+\infty} f_1(t,v)dk_n(v), \)

we have

(3.3.7) \[ |\log \psi_n^a(t) - \log \psi^a(t)| \leq |\log \psi_n^a(t) - \psi_n^a(t)| \]

\[ + |\psi_n^a(t) - \log \psi^a(t)|. \]

Let

(3.3.8) \( \tilde{F}_{nk}^a(x) = F_{nk}^a(x + \mu_{nk}(a)) \)

and let \( \tilde{\psi}_{nk}^a(t) \) denote the corresponding characteristic function. Let

(3.3.9) \( \beta_{nk}^a(t) = \tilde{\psi}_{nk}^a(t) - 1. \)

We have

(3.3.10) \( \tilde{\psi}_{nk}^a(t) = 1 + \frac{1}{2} \theta \sigma_{nk}(a)t^2, \quad |\theta| \leq 1. \)

---

2 This part of the proof through equation (3.3.24) is analogous to that found in [5].
Therefore,

\[(3.3.11) \quad |\beta_{nk}(t)| = \frac{1}{2} |\theta| \sigma_{nk}(a) t^2 \leq \frac{1}{2} \sigma_{nk}(a) t^2.\]

Let

\[(3.3.12) \quad T_n = \frac{1}{g^a(n, m(A, \delta), r)}\]

and assume that \(|t| \leq T_n\). Then we have

\[(3.3.13) \quad |\beta_{nk}(t)| \leq \frac{1}{2} \sigma_{nk}(a) T_n^2\]

\[= \frac{\sigma_{nk}(a) T_n^2}{2 g^a(n, m(A, \delta), r)^2}\]

\[\leq \frac{\sigma_{nk}(a)}{2 \left[\frac{1}{3} \sigma_{nk}(a) - \sigma_{nk}(a)\right]^{2/5}}\]

\[\leq \frac{3}{2} \left(\sigma_{nk}(a)\right)^{2/5}\]

\[\leq \frac{4}{5}.\]

Hence,

\[(3.3.14) \quad \log \phi_{nk}(t) = \log \left(1 + \beta_{nk}(t)\right)\]
\[
\left( \alpha^{-1} \right)^{2} = \frac{2}{3} + t \left( \frac{2}{3} \right)
\]

\[
\left( \alpha^{-1} \right)^{2} = \frac{2}{3} + \left( \frac{2}{3} \right)
\]

\[
\left( \alpha^{-1} \right)^{2} = \frac{2}{3} + \left( \frac{2}{3} \right)
\]

\[
\left( \alpha^{-1} \right)^{2} = \frac{2}{3} + \left( \frac{2}{3} \right)
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\left( \alpha^{-1} \right)^{2} = \frac{2}{3} + \left( \frac{2}{3} \right)
\]

\[
\left( \alpha^{-1} \right)^{2} = \frac{2}{3} + \left( \frac{2}{3} \right)
\]

\[
\left( \alpha^{-1} \right)^{2} = \frac{2}{3} + \left( \frac{2}{3} \right)
\]

So that

\[
\left( \alpha^{-1} \right)^{2} = \frac{2}{3} + \left( \frac{2}{3} \right)
\]

\[
\left( \alpha^{-1} \right)^{2} = \frac{2}{3} + \left( \frac{2}{3} \right)
\]
\[
\sum_{k=1}^{k_n} \mu_{nk}(a) \imath t + \sum_{k=1}^{k_n} \int_{-\infty}^{+\infty} f_1(t, v) v^2 \mu_{nk}(v + \mu_{nk}(a)) \, dv \\
= \sum_{k=1}^{k_n} \{ \mu_{nk}(a) \imath t + \int_{-\infty}^{+\infty} (e^{\imath tv} - 1 - \imath tv) \mu_{nk}(v + \mu_{nk}(a)) \, dv \} \\
= \sum_{k=1}^{k_n} \{ \mu_{nk}(a) \imath t + \int_{-\infty}^{+\infty} (e^{\imath tv} - 1) \mu_{nk}(v + \mu_{nk}(a)) \, dv \} \\
= \sum_{k=1}^{k_n} \{ \mu_{nk}(a) \imath t + \varphi_{nk}(t) \} ,
\]

and

\[(3.3.17) \quad \log \varphi_n^a(t) = \sum_{k=1}^{k_n} \log \varphi_{nk}(t) \]

\[= \sum_{k=1}^{k_n} \{ \mu_{nk}(a) \imath t + \log \varphi_{nk}(t) \} .
\]

Therefore we have

\[(3.3.18) \quad |\log \varphi_n^a(t) - \psi_n^a(t)| \leq \sum_{k=1}^{k_n} |\log \varphi_{nk}(t) - \beta_{nk}(t)| \]

\[\leq \frac{5}{2} \sum_{k=1}^{k_n} |\beta_{nk}(t)|^2 .\]
Using (3.3.11) we have
\[
\left| \log \Phi_n(t) - \psi_n(t) \right| \leq 2 \sum_{k=1}^{\infty} \sigma_{nk}^2(a).
\]
Since
\[
(3.3.20) \quad \beta_{nk}(t) = \int_0^t \left( e^{itv} - 1 \right) dF_n(v + \Lambda_{nk}(a))
\]
(3.3.21) \quad \sum_{k=1}^{\infty} \beta_{nk}(t) = \int_0^t \sum_{k=1}^{\infty} \left( e^{itv} - 1 \right) dF_n(v + \Lambda_{nk}(a))
\]

hence
\[
(3.3.22) \quad \psi_n(t) - \mu_n(t) = \int_0^t \left( e^{itv} - 1 \right) dF_n(v).
\]
Therefore we have
\[
(3.3.22) \quad \psi_n(t) = \mu_n(t) + \int_0^t \left( e^{itv} - 1 \right) dF_n(v).
\]
Hence,

(3.3.23) \[ | \psi_n^a(t) - \log \psi^a(t) | \leq | \mu_n(a) - \mu(a) | \cdot |t| \\
+ \int_{-\infty}^{\infty} f_n(t, v) d[\kappa_n^a(v) - \kappa^a(v)] \\
\leq | \mu_n(a) - \mu(a) | \cdot |t| \\
+ E^a(n, t, m(A, \delta)). \]

Hence, applying (3.3.19) and (3.3.23), we have

(3.3.24) \[ | \log \psi_n^a(t) - \log \psi^a(t) | \leq | \log \psi_n^a(t) - \psi_n^a(t) | \\
+ | \psi_n^a(t) - \log \psi^a(t) | \\
\leq \frac{5}{8} t^4 \max \sigma_{nk}(a) \sigma_n^2(a) \\
+ | \mu_n(a) - \mu(a) | \cdot |t| \\
+ E^a(n, t, m(A, \delta)). \]

From (3.3.5) and (3.3.24) we have
(3.3.25) \[ |\varphi_n(t) - \varphi(t)| \leq \frac{5}{8} t^4 \max \sigma^2_{nk}(a) \sigma^2_n(a) \]

\[ + |\mu_n(a) - \mu(a)||t| \]

\[ + E^a(n,t,m(A,\delta)) \]

\[ + 4 \left( \int_{|u|>a} |u|^r dG(u) \right) |t|^r \]

\[ \equiv h^a(t,n,m(A,\delta),r). \]

Therefore,

\[ (3.3.26) \int_{-T_n}^{T_n} \left| \frac{\varphi_n(t) - \varphi(t)}{t} \right| dt \leq 2 \int_0^{T_n} h^a(t,n,m(A,\delta),r) dt \]

\[ = 2 \int_0^{T_n} \left\{ \frac{5}{8} t^3 \max \sigma^2_{nk}(a) \sigma^2_n(a) + |\mu_n(a) - \mu(a)| \right. \]

\[ + \frac{5}{4} \delta t^2 (\sigma^2_n(a) + \sigma^2(a)) + \frac{5}{2} \sum_{i=0}^n |K^a_n(x_i) - K^a(x_i)| \]

\[ + \frac{2}{A} \left( K^a_n(\pm \infty) - K^a_n(A) + K^a(\pm \infty) - K^a(A) \right) \]

\[ + K^a_n(-A) + K^a(-A) \right\} dt \]

\[ + 4 \left( \int_{|u|>a} |u|^r dG(u) \right) t^{r-1} dt \]
\[ = \frac{5}{16} \max \sigma_{nk}^2(a) \sigma_n^2(a) T_n^2 + 2 | \mu_n(a) - \mu(a) | T_n\]
\[ + \frac{5}{6} \delta(\sigma_n^2(a) + \sigma^2(a)) T_n^3 + \frac{1}{2} \sum_{i=0}^{m} | K_n^a(x_i) - K^a(x_i) | T_n^2 \]
\[ + \frac{4}{A} (K_n^a(+\infty) - K_n^a(A) + K_n^a(-\infty) - K^a(A) + K_n^a(-A) + K^a(-A)) T_n \]
\[ + \frac{8}{r} \left( \int_{|u| > a} r dG(u) \right) T_n^r . \]

From (3.3.12) we have

\[ T_n \leq \frac{1}{\left( \frac{1}{3} \sigma_n^2(a) \max \sigma_{nk}^2(a) \right)^{\frac{1}{3}}}, \quad T_n \leq \frac{1}{\left( \frac{5}{6} \delta(\sigma_n^2(a) + \sigma^2(a)) \right)^{\frac{1}{3}}} \]
\[ T_n \leq \frac{1}{\left( \frac{1}{2} \sum_{i=0}^{m} | K_n^a(x_i) - K^a(x_i) | \right)^{\frac{1}{3}}} \]
\[ T_n \leq \frac{1}{\left( \frac{4}{A} (K_n^a(+\infty) - K_n^a(A) + K_n^a(-\infty) - K^a(A) + K_n^a(-A) + K^a(-A)) + 2 | \mu_n(a) - \mu(a) | \right)^{\frac{1}{4}}} \]

and

\[ T_n \leq \frac{1}{\left( \frac{8}{r} \int_{|u| > a} r dG(u) \right)^{\frac{1}{4+a}}} . \]
Hence

\[ \frac{5}{16} \max \sigma_n^2(a) \sigma_n^2(a) T_n^4 \leq \left( \frac{1}{3} \sigma_n^2(a) \max \sigma_n^2(a) \right)^{\frac{1}{5}} \]

\[ \frac{5}{6} \delta \left( \sigma_n^2(a) + \sigma^2(a) \right) T_n^3 \leq \left( \frac{5}{6} \delta \left( \sigma_n^2(a) + \sigma^2(a) \right) \right)^{\frac{1}{3}} \]

\[ \frac{1}{2} \sum_{i=0}^{m} |K_n^a(x_i) - K^a(x_i)| T_n^2 \leq \left( \frac{1}{2} \sum_{i=0}^{m} |K_n^a(x_i) - K^a(x_i)| \right)^{\frac{1}{3}} \]

\[ \left[ \frac{4}{A} \left( K_n^a(A) + K_n^a(\infty) - K_n^a(A) + K_n^a(-\infty) - K_n^a(A) + K_n^a(A) + K_n^a(-A) + K_n^a(-A) \right) \right]^{\frac{1}{2}} \]

\[ 2 \left| \mu_n(a) - \mu(a) \right| T_n \leq \left[ \frac{4}{A} \left( K_n^a(A) + K_n^a(+\infty) - K_n^a(A) + K_n^a(+\infty) \right. \right. \]

\[ - K_n^a(A) + K_n^a(-A) + K_n^a(-A) + 2 \left| \mu_n(a) - \mu(a) \right| \right]^{\frac{1}{2}} \]

\[ \frac{8}{r} \int_{|u| > a} |u|^r dG(u) T_n^r \leq \left( \frac{8}{r} \int_{|u| > a} |u|^r dG(u) \right)^{\frac{1}{r}} \]

Therefore,

\[ (3.3.27) \int_{T_n}^{T_n} \frac{\varphi_n^a(t) - \varphi(t)}{t} |dt \leq g^a(n, m(A, \delta), r), \]

where \( g^a(n, m(A, \delta), r) \) is the quantity defined in (3.1.9).

Applying Theorem 3, we have
(3.3.28) $M_n^a = \sup_{-\infty < x < +\infty} |F_n^a(x) - F(x)|$

\[ \leq \frac{p}{2\pi} g^a(n, m(A, \delta), r) + c(p) \frac{B}{T_n} \]

\[ = \left[ \frac{p}{2\pi} + c(p)B \right] g^a(n, m(A, \delta), r) \]

\[ = k(p,B)g^a(n, m(A, \delta), r). \]

Applying Lemma 3, (3.3.2) and (3.3.28) we have

(3.3.29) $M_n = \sup_{-\infty < x < +\infty} |F_n(x) - F(x)|$

\[ \leq \sup_{-\infty < x < +\infty} |F_n(x) - F_n^a(x)| + \sup_{-\infty < x < +\infty} |F_n^a(x) - F(x)| \]

\[ \leq \sum_{k=1}^{n} \left\{ F_{nk}(-a) + 1 - F_{nk}(a) \right\} \]

\[ + k(p,B)g^a(n, m(A, \delta), r). \quad Q.E.D. \]

3.4 Behavior of the Estimate

We shall now examine, under suitable conditions, the behavior of

(3.4.1) $D(n, A, \delta, a, r) = k(p,B)g^a(n, m(A, \delta), r)$
which is the bound on

\[ \sup_{-\infty < x < \infty} |F_n(x) - F(x)| \]

given in Theorem 8. These conditions are motivated by Theorem 2.

The following lemmas will be needed.

**Lemma 5.** Let \( \{Q_n(a)\} \), \( n = 0, 1, 2, \ldots \) be defined for \( a > 0 \) and be such that

i) for each \( n \), \( Q_n(a) \geq Q_n(b) \geq 0 \) for \( a < b \),

ii) \( \lim_{a \to \infty} Q_n(a) = 0 \),

iii) \( \lim_{n \to \infty} Q_n(a) = Q_0(a) \) at every continuity point of \( Q_0(a) \),

then for any sequence \( \{a_n\} \) such that each \( a_n \) is a continuity point of \( Q_0(a) \), \( a_n \leq a_{n+1} \) for all \( n \) and

\[ \lim_{n \to \infty} a_n = +\infty \]

we have

\[ \lim_{n \to \infty} Q_n(a_n) = 0. \]
Proof: Let $\varepsilon > 0$ be given. Choose a positive integer $K$ such that

$$Q_0(a_K) < \frac{\varepsilon}{2}. \quad (3.4.3)$$

Choose $n_K$ such that

$$Q_n(a_{n_K}) < Q_0(a_K) + \frac{\varepsilon}{2} < \varepsilon \quad (3.4.4)$$

for all $n > n_K$.

Now, for any $n > n_K$ we have $n > K$, hence $a_n > a_K$. Therefore, using (i) and (3.4.4), we have

$$Q_n(a_n) \leq Q_n(a_{n_K}) < \varepsilon. \quad (3.4.5)$$

Q.E.D.

Lemma 6. Let $\{X_{nk}\}$, $F_n(x)$ and $F(x)$ be as in Theorem 8. Let $M(x)$ and $N(x)$ correspond to $F(x)$ by Levy's formula (1.2.3). If

i) the random variables $X_{nk}$ are infinitesimal,

ii) $\lim_{n \to \infty} F_n(x) = F(x)$ at all continuity points of $F(x)$,

then for any sequence $\{a_n\}$ such that $-a_n$ and $a_n$ are continuity points of $M(x)$ and $N(x)$, respectively, $a_n \leq a_{n+1}$ and

$$\lim_{n \to \infty} a_n = +\infty,$$

we have
(3.4.6) \[ \lim_{n \to \infty} \sum_{k=1}^{n} \left\{ F_{nk}(-a_n) + 1 - F_{nk}(a_n) \right\} = 0. \]

**Proof:** Let

(3.4.7) \[ Q_n(a) = \sum_{k=1}^{n} \left\{ F_{nk}(-a) + 1 - F_{nk}(a) \right\}, \]

(3.4.8) \[ Q_0(a) = M(-a) - N(a). \]

According to Theorem 2 (1.3.2), we have

(3.4.9) \[ \lim_{n \to \infty} Q_n(a) = Q_0(a) \]

for every \( a \) such that \(-a\) and \( a \) are continuity points of \( Q_0(a) \).

Hence by Lemma 5 we have

(3.4.10) \[ \lim_{n \to \infty} Q_n(a_n) = 0. \] Q.E.D.

**Lemma 7.** Let \( g(n, a, \delta) \) be non-negative and be such that to each \( a \) there corresponds a sequence \( \{\delta_n(a)\} \) of positive real numbers such that

i) \( \delta_n(a) \geq \delta_{n+1}(a) \) for every \( n \),

ii) \( \lim_{n \to \infty} g(n, a, \delta_n(a)) = 0. \)
Then there exist sequences \( \{a_n\} \) and \( \{\delta_n\} \) such that

\[
(3.4.11) \quad \lim_{n \to \infty} g(n, a_n, \delta_n) = 0.
\]

In addition, \( \{a_n\} \) can be chosen so that

\[
(3.4.12) \quad a_n \leq a_{n+1}
\]

for every \( n \), and

\[
(3.4.13) \quad \lim_{n \to \infty} a_n = +\infty.
\]

**Proof:** Let \( \{\varepsilon_k\} \) be a sequence of positive real numbers such that \( \varepsilon_{k+1} \leq \varepsilon_k \) and

\[
\lim_{k \to \infty} \varepsilon_k = 0.
\]

Let \( \{\bar{a}_n\} \) be a sequence such that (3.4.12) and (3.4.13) hold. By the hypothesis, for each fixed \( k \) there exists a sequence \( \{\delta_n(\bar{a}_k)\} \) such that

\[
(3.4.14) \quad \lim_{n \to \infty} g(n, \bar{a}_k, \delta_n(\bar{a}_k)) = 0.
\]

Choose \( n_1 \) such that

\[
(3.4.15) \quad g(n, \bar{a}_1, \delta_n(\bar{a}_1)) < \varepsilon_1
\]

for all \( n > n_1 \).
For each $k > 1$ let $n_k$ be a positive integer such that $n_k > n_{k-1}$ and

\[(3.4.16) \quad g(n, \bar{a}_k, \delta_n(\bar{a}_k)) < \xi_k \]

for all $n > n^*$.

Define

\[(3.4.17) \quad \delta_n = \begin{cases} 
\delta_1(\bar{a}_1) & \text{for } n \leq n_1, \\
\delta_n(\bar{a}_k) & \text{for } n_k < n \leq n_{k+1}, \ k = 1, 2, \ldots 
\end{cases} \]

\[(3.4.18) \quad a_n = \begin{cases} 
\bar{a}_1 & \text{for } n \leq n_1, \\
\bar{a}_k & \text{for } n_k < n \leq n_{k+1}, \ k = 1, 2, \ldots 
\end{cases} \]

We claim that \{a_n\} and \{\delta_n\} are the desired sequences. To verify (3.4.11), let $\xi > 0$ be given. Choose $K$ such that $\xi_k < \xi$

for all $k > K$. Let $N = n_{K+1}$. If $n > N$, then

$$n_k < n \leq n_{k+1}$$

for some $k > K$. Therefore,

$$\delta_n = \delta_n(\bar{a}_k) \quad \text{and} \quad a_n = \bar{a}_k.$$
Hence,

\[(3.4.19) \quad g(n, a_n, \delta_n) = g(n, \tilde{a}_k, \delta_n(\tilde{a}_k)) \leq e_k < \varepsilon.\]

Since \(\{a_n\}\) satisfies (3.4.12) and (3.4.13), hence the sequence \(\{a_n\}\) as defined by (3.4.18) also satisfies (3.4.12) and (3.4.13).

Q.E.D.

Now we show that, under suitable conditions, we can choose \(A, \delta\) and \(a\) so that \(D(n, A, \delta, a, r)\) approaches zero as \(n\) becomes infinite. Let \(0 < \delta < 1\) be such that

\[-\frac{1}{\sqrt{\delta}} \quad \text{and} \quad \frac{1}{\sqrt{\delta}}\]

are continuity points of \(G(u)\). Using the notation of (3.1.7) and (3.1.9), we define

\[(3.4.20) \quad g(n, a, \delta) = g^a(n, m(\frac{1}{\sqrt{\delta}}, \delta)),\]

\[(3.4.21) \quad g(n, a, \delta, r) = g^a(n, m(\frac{1}{\sqrt{\delta}}, \delta), r) \cdot \frac{1}{1+\lambda}\]

\[= g(n, a, \delta) + \left\{ \frac{8 \int \int_{R} dG(u)}{w > a} \right\}.\]

Finally we define

\[(3.4.22) \quad Q_n(a) = \sum_{k=1}^{k_n} \left\{ F_{n_k}(-a) + 1 - F_{n_k}(a) \right\}.\]
Then (3.4.1) becomes

\[(3.4.23) \quad D(n, \Delta, \delta, \alpha, r) = k(p, B)g(n, \alpha, \delta, r) + Q_n(\alpha).\]

This leads to the main result of this section.

**Theorem 9.** Let \( (X_{nk}) \), \( F_n(x) \) and \( F(x) \) satisfy the assumptions of Theorem 8. Assume further that the random variables \( (X_{nk}) \) are infinitesimal, \( F_n(x) \) converges to \( F(x) \) at every continuity point of \( F(x) \), and that for some \( r \) such that \( 0 < r \leq 1 \)

\[(3.4.24) \quad \int_{-\infty}^{+\infty} |u| F dG(u) < +\infty,\]

where \( G(u) \) is as in Theorem 8. Then there exist sequences \( \{a_n\} \) and \( \{\delta_n\} \) such that

\[(3.4.25) \quad M_n = \sup_{-\infty < x < +\infty} |F_n(x) - F(x)| \leq D(n, \frac{1}{\sqrt{\delta_n}}, \delta_n, a_n, r)\]

and

\[(3.4.26) \quad \lim_{n \to \infty} D(n, \frac{1}{\sqrt{\delta_n}}, \delta_n, a_n, r) = 0.\]

**Proof:** Since \( X_{nk} \) are infinitesimal, \( X_{nk}^a \) as well as \( X_{nk}^a - \mu_{nk}(a) \) are also infinitesimal. By Theorems 6 and 7 we have
\[
\lim_{n \to \infty} F_n^a(x) = F^a(x)
\]

and

\[
\lim_{n \to \infty} \sigma_n^2(a) = \sigma^2(a).
\]

Hence, by Theorem 5, there exists a sequence \( \{ \delta_n(a) \} \) such that

\[
\delta_n(a) \geq \delta_{n+1}(a)
\]

and

\[
\lim_{n \to \infty} g(n, a, \delta_n(a)) = 0.
\]

By Lemma 7, we can find sequences \( \{ a_n \} \) and \( \{ \delta_n \} \) such that

\[
\lim_{n \to \infty} g(n, a_n, \delta_n) = 0,^3
\]

\[
a_n \leq a_{n+1}
\]

and

\[
\lim_{n \to \infty} a_n = +\infty.
\]

Clearly, from the proof of Lemma 7, \( a_n \) can be chosen so that both \(-a_n\) and \( a_n \) are continuity points of \( G(u) \).

---

^3 Now, it follows in particular from Lemma 4 of [5] that

\[
\lim_{n \to \infty} \max_k \sigma_{nk}^2(a) = 0
\]

which justifies the footnote to Theorem 8.
Therefore, by Lemma 6, we have

$$\lim_{n \to \infty} \sum_{k=1}^{k_n} \left\{ F_{n_k}(-a_n) + 1 - F_{n_k}(a_n) \right\} = 0.$$ 

Since

$$\int_{-\infty}^{+\infty} |u|^r dG(u) < +\infty$$

it follows that

$$\lim_{n \to \infty} \int_{\{u| > a_n\}} |u|^r dG(u) = 0.$$ 

Therefore from (3.4.20)-(3.4.23) we see that (3.4.26) holds.

As soon as \(a_n \geq 1\) we have

$$\sup_{-\infty < x < +\infty} |F_n(x) - F(x)| \leq D(n, \frac{1}{\sqrt{\delta_n}}, \delta_n, a_n, r)$$

from Theorem 8.
4.1 Stable Distribution Functions

A distribution function $F(x)$ is said to be stable if and only if, for every $a_1 > 0$, $b_1$, $a_2 > 0$, $b_2$ there exist constants $a > 0$ and $b$ such that

$$F(a_1x + b_1) * F(a_2x + b_2) = F(ax + b),$$

where $*$ denotes the convolution operation. It is well known that every stable distribution function is an infinitely divisible distribution function. An important property of stable distributions is in the following theorem (c.f. [4] p. 162).

Theorem 10. In order that the distribution function $F(x)$ be a limit distribution function for the sums

$$S_n = \frac{X_1 + \ldots + X_n}{B_n} - A_n,$$

where $(X_n)$, $n = 1, 2, \ldots$ are independent identically distributed random variables, $A_n$ and $B_n$ are suitably chosen constants, it is
necessary and sufficient that \( F(x) \) be stable.


**Theorem 11.** In order that a distribution function \( F(x) \) be stable, it is necessary and sufficient that the logarithm of its characteristic function \( \varphi(t) \) be represented by the formula

\[
\log \varphi(t) = i \delta t + c |t|^\gamma \left\{ 1 + i \beta \frac{t}{|t|} \omega(t, \alpha) \right\},
\]

where \( \alpha, \beta, \gamma, c \) are constants (\( \delta \) is any real number, \(-1 \leq \beta \leq 1, 0 < \alpha \leq 2, c \geq 0 \) and

\[
\omega(t, \alpha) = \begin{cases} 
\tan \frac{\pi \alpha}{2} & \text{if } \alpha \neq 1, \\
\frac{2}{\pi} \log |t| & \text{if } \alpha = 1.
\end{cases}
\]

The functions \( M(x) \) and \( N(x) \) and the constant \( \sigma^2 \) in Levy's formula (1.2.3) are either

1) \( \alpha = 2: \) \( \sigma^2 > 0, \) \( M(x) \equiv 0, \) \( N(x) \equiv 0, \)

or 2) \( 0 < \alpha < 2: \) \( \sigma^2 = 0, \) \( M(x) = c_1 |x|^\alpha, \) \( N(x) = -c_2 |x|^\alpha, \)

where \( c_1 > 0, c_2 > 0, c_1 + c_2 > 0. \)

In case (1) we have either normal or degenerate distribution function. Any non-degenerate stable distribution is called a proper stable distribution. By § 36 of [4], it
follows that all proper stable distribution functions have bounded derivatives of all orders at every point. In fact, using the notation just introduced for \( 0 < \alpha < 2 \), the bound on \( F'(x) \) is given by

\[
|F'(x)| \leq \frac{1}{\pi x} \Gamma\left(\frac{1}{\alpha}\right) c^{-1/\alpha} = \frac{1}{\pi}.
\]

Thus for proper stable distribution functions the assumption in Theorem 8 on the derivative of \( F(x) \) is always satisfied.

**Lemma 8.** If \( F(x) \) is a stable distribution function with corresponding function \( G(u) \) given by the formula of Levy and Khintchine (1.2.1), then there exists a real number \( r \) such that \( 0 < r \leq 1 \) and

\[
(4.1.6) \quad \int_{-\infty}^{+\infty} |u|^r dG(u) < +\infty.
\]

**Proof:** From (1.2.4) we have

\[
\int_{-\infty}^{+\infty} |u|^r dG(u) = \int_{-\infty}^{0} |x|^r \frac{x^2}{1 + x^2} dM(x) + \int_{0}^{+\infty} x^r \frac{x^2}{1 + x^2} dN(x).
\]

If \( M(x) \) and \( N(x) \) are as given in case (1) of Theorem 11, then (4.1.6) clearly holds. If (2) is the case, then we have

\[
\int_{-\infty}^{+\infty} |u|^r dG(u) = c_1 \int_{-\infty}^{0} |x|^r \frac{x^2}{1 + x^2} \frac{dx}{|x|^{1+\delta}} + c_2 \int_{0}^{+\infty} x^r \frac{x^2}{1 + x^2} \frac{dx}{x^{1+\delta}}.
\]
This integral is finite if we choose \( r \) so that

\[
0 < r \leq 1 \quad \text{and} \quad r < \alpha.
\]

This choice of \( r \) is always possible.

Q.E.D.

This lemma removes one of the hypotheses in Theorem 9.

The next two lemmas will be used to simplify the general estimate given in Theorem 8 to the stable case (see Theorem 12).

**Lemma 9.** Let \( F(x) \) be a stable distribution function with representation given by (4.1.3) with \( 0 < \alpha < 2 \). Then

\[
\mu(a) = \gamma + (c_1 - c_2) \left\{ \frac{u}{(1 + u^2)u^\alpha} \int_a^\infty \right. - \left. \frac{u^{2-\alpha}}{1 + u^2} \right\},
\]

(4.1.7)

\[
\sigma^2(a) = (c_1 + c_2) \frac{a^{2-\alpha}}{2 - \alpha}
\]

(4.1.8)

and
(4.1.9) \( K^a(v) = \begin{cases} 
0, & v < -a, \\
c_1 \left( \frac{a^{2-d}}{2-a} - \frac{v^{2-d}}{2-a} \right), & -a \leq v < 0, \\
c_1 \left( \frac{a^{2-d}}{2-a} \right) + c_2 \left( \frac{v^{2-d}}{2-a} \right), & 0 \leq v < a \\
(c_1 + c_2) \left( \frac{a^{2-d}}{2-a} \right), & a \leq v. 
\end{cases} \)

**Proof:** From (1.2.4) we have

(4.1.10) \( dG(u) = \begin{cases} 
\frac{u^2}{1 + u^2} \, dX(u) = c_1 \frac{u^{1-d}}{1 + u^2} \, du, \\
\frac{u^2}{1 + u^2} \, dN(u) = c_2 \frac{u^{1-d}}{1 + u^2} \, du.
\end{cases} \)

Therefore,

(4.1.11) \( \gamma^a = \gamma - \int_{\{u| > a} \frac{1}{u} \, dG(u) \)

\[ = \gamma - \int_{-\infty}^{-a} \frac{1}{u} \, c_1 \frac{u^{1-d}}{1 + u^2} \, du - \int_{a}^{+\infty} \frac{1}{u} \, c_2 \frac{u^{1-d}}{1 + u^2} \, du \]

\[ = \gamma + c_1 \int_{-\infty}^{-a} \frac{du}{(1 + u^2)|u|^\alpha} - c_2 \int_{a}^{+\infty} \frac{du}{(1 + u^2)u^\alpha} \]
Hence, from (3.1.1), we have

\[ \mu(a) = \gamma a + \int_{-\infty}^{+\infty} u \text{d}G^a(u) \]

\[ = \gamma a + \int_{-a}^{a} u \text{d}G(u) \]

\[ = \gamma a + \int_{-a}^{0} u c_1 \frac{u^{1-\alpha}}{1 + u^2} \text{d}u + \int_{0}^{a} u c_2 \frac{u^{1-\alpha}}{1 + u^2} \text{d}u \]

\[ = \gamma a - c_1 \int_{-a}^{0} \frac{u^{2-\alpha}}{1 + u^2} \text{d}u + c_2 \int_{0}^{a} \frac{u^{2-\alpha}}{1 + u^2} \text{d}u \]

\[ = \gamma a - (c_1 - c_2) \int_{0}^{a} \frac{u^{2-\alpha}}{1 + u^2} \text{d}u \]

\[ = \gamma + (c_1 - c_2) \left\{ \left[ +\infty \frac{\text{d}u}{(1 + u^2)^{\alpha}} \right]_{a}^{+\infty} - \left[ \frac{u^{2-\alpha}}{1 + u^2} \text{d}u \right]_{0}^{a} \right\} \]

This proves (4.1.7).

For \(-a < v < 0\), we have

\[ X^a(v) = \int_{-\infty}^{v} (1 + u^2) \text{d}G^a(u) \]
For $0 < v < a$, we have

\[
K^a(v) = \int_{-\infty}^{v} (1 + u^2) dG^a(u) = \int_{-a}^{v} c_1 |u|^{1-\alpha} du = c_1 \left( \frac{a^{2-\alpha}}{2-\alpha} - \frac{|v|^{2-\alpha}}{2-\alpha} \right).
\]

But $G(0^+) - G(0^-) = \sigma^2$, which is zero (Theorem 11). From this (4.1.9) follows. Formula (4.1.8) follows from (4.1.9) and the fact that $\sigma^2(a) = K^a(+\infty)$. Q.E.D.
Lemma 10. Let $F(x)$ be a stable distribution function of Theorem 11 with $0 < \alpha < 2$. If $0 < r < \alpha$, then

\[(4.1.12) \quad \int_{|u| > a} |u|^r dG(u) \leq \frac{c_1 + c_2}{(\alpha - r)a^{\alpha - r}},\]

where $G(u)$ is the corresponding function of $F(x)$ given by the formula of Levy and Khintchine (1.2.1).

Proof: Using (4.1.10) we have

\[
\begin{align*}
\int_{|u| > a} |u|^r dG(u) &= \int_{-\infty}^{-a} c_1 \frac{|u|^{1+r-\alpha}}{1 + u^2} du + \int_{a}^{+\infty} c_2 \frac{u^{1+r-\alpha}}{1 + u^2} du \\
&= (c_1 + c_2) \int_{a}^{+\infty} \frac{u^{1+r-\alpha}}{1 + u^2} du \\
&\leq (c_1 + c_2) \int_{a}^{+\infty} \frac{u^{1+r-\alpha}}{u^2} du \\
&= \frac{c_1 + c_2}{(\alpha - r)a^{\alpha - r}}.
\end{align*}
\]

Q.E.D.

4.2 Error Estimate for Normed Sums

Let $(x_n)$, $n = 1, 2, \ldots$ be a sequence of independent random variables with a common distribution function $F(x)$. For each $n$, let
where \( B_n \) are suitably chosen constants. Let \( F_n(x) \) denote the distribution function of \( S_n \). To apply our general result, define

\[
X_{nk} = \frac{X_k}{B_n}.
\]

The results expressed in (4.2.2)-(4.2.9) follow easily.

\[
(4.2.2) \quad \begin{cases}
\frac{X_k}{B_n} & \text{for } -aB_n < X_k \leq aB_n, \\
0 & \text{otherwise}.
\end{cases}
\]

\[
(4.2.3) \quad F_{nk}(x) = \overline{F(xB_n)}.
\]

\[
(4.2.4) \quad \begin{cases}
0 & \text{for } x \leq -a, \\
\overline{F(xB_n)} - \overline{F(-aB_n)} & \text{for } -a < x < 0, \\
\overline{F(xB_n)} + 1 - \overline{F(aB_n)} & \text{for } 0 \leq x < a, \\
1 & \text{for } a \leq x.
\end{cases}
\]

\[
(4.2.5) \quad \mu_{nk}(a) = \int_{-a}^{a} x \overline{F(xB_n)} = \frac{1}{B_n} \int_{-aB_n}^{aB_n} x \overline{F(x)}.
\]

\[
(4.2.6) \quad \mu_n(a) = \frac{aB_n}{B_n} \int_{-aB_n}^{aB_n} x \overline{F(x)}.
\]
\[ (4.2.7) \quad \sigma_{nk}^2(a) = \frac{1}{B_n^2} \left\{ \int_{-aB_n}^{aB_n} x^2 d\overline{F}(x) - \left( \int_{-aB_n}^{aB_n} xd\overline{F}(x) \right)^2 \right\}. \]

\[ (4.2.8) \quad \sigma_{nk}^2(a) = \frac{n}{B_n^2} \left\{ \int_{-aB_n}^{aB_n} x^2 d\overline{F}(x) - \left( \int_{-aB_n}^{aB_n} xd\overline{F}(x) \right)^2 \right\}. \]

\[ (4.2.9) \quad K_n^a(v) = \sum_{k=1}^{k_n} \int_{-\infty}^{v} u^2 dF_{nk}(u + \mu_{nk}(a)) \]

\[ = n \int_{-\infty}^{v+ \mu_{nk}(a)} (u - \mu_{nk}(a))^2 dF_{nk}(u) \]

\[ \begin{cases} 0 & v + \mu_{nk}(a) \leq -a, \\ -a \int_{-a}^{v+ \mu_{nk}(a)} (u - \mu_{nk}(a))^2 dF(uB_n) & -a < v + \mu_{nk}(a) < 0, \\ -a \int_{-a}^{v+ \mu_{nk}(a)} (u - \mu_{nk}(a))^2 dF(uB_n) + [\mu_{nk}(a)]^2 \left[ 1 - \overline{F}(aB_n) + \overline{F}(-aB_n) \right] & 0 \leq v + \mu_{nk}(a) < a, \\ -a \int_{-a}^{a} (u - \mu_{nk}(a))^2 dF(uB_n) + [\mu_{nk}(a)]^2 \left[ 1 - \overline{F}(aB_n) + \overline{F}(-aB_n) \right] & a \leq v + \mu_{nk}(a). \end{cases} \]
Finally we have

\begin{equation}
\sum_{k=1}^{k_n} \left\{ F_{n,k}(-a) + 1 - F_{n,k}(a) \right\} = n \left\{ F_{n,k}(-a) + 1 - F_{n,k}(a) \right\} = \left\{ F(-aB_n) + 1 - F(aB_n) \right\}.
\end{equation}

The following is an immediate consequence of Theorem 8, Lemmas 9 and 10, and (4.2.1)-(4.2.10).

**Theorem 12.** Let \( \{X_n\}, n = 1, 2, \ldots \) be a sequence of independent random variables with a common distribution function \( F(x) \). For each \( n \), let

\[ S_n = \frac{X_1 + \ldots + X_n}{B_n}, \]

where \( B_n \) are constants. Let \( F_n(x) \) denote the distribution function of \( S_n \). Let \( F(x) \) be a stable distribution function with exponent \( \alpha \), as given by Theorem 11, with \( 0 < \alpha < 2 \). Let \( B \) be as given in (4.1.5). Assume that for each \( n \) we have

\begin{equation}
\frac{1}{B_n^2} \left\{ \int_{-aB_n}^{aB_n} x^2 dF(x) - \left( \int_{-aB_n}^{aB_n} xdF(x) \right)^2 \right\} \leq 1.
\end{equation}

\[ ^4 \text{We note that this is } \sigma_{nk}^2(a) \leq 1 \text{ so that by footnote to Theorem 9 this is a weak assumption.} \]
Then, for each $p > 0$, $0 < r \leq 1$, $\alpha > 1$, we have

\begin{equation}
M_n = \sup_{-a < x < a} |F_n(x) - F(x)|
\end{equation}

\leq k(p, B)g^a(n, m(a, \delta), r) + n \left\{ \bar{F}(-aB_n) + 1 - \bar{F}(aB_n) \right\},

where $k(p, B)$ is a constant and

\begin{equation}
g^a(n, m(a, \delta), r) = \left\{ \frac{n}{3B_n^4} \left[ \frac{\int_{-aB_n}^{aB_n} x^2 dF(x)}{B_n^2} - \left( \frac{\int_{-aB_n}^{aB_n} xdF(x)}{aB_n} \right)^2 \right] \right\}^{\frac{1}{3}}
\end{equation}

\begin{equation}
+ \left\{ \frac{5}{6} \delta \left[ \frac{n}{B_n^2} \left\{ \frac{\int_{-aB_n}^{aB_n} x^2 dF(x)}{B_n^2} - \left( \frac{\int_{-aB_n}^{aB_n} xdF(x)}{-aB_n} \right)^2 \right\} + \frac{(c_1 + c_2)a^{2-\alpha}}{2 - \alpha} \right\}^{\frac{1}{4}}
\end{equation}

\begin{equation}
+ \left\{ \frac{1}{2} \sum_{i=0}^{m} |K^a_n(x_i) - K^a(x_i)| \right\}^{\frac{1}{3}}
\end{equation}

\begin{equation}
+ \left\{ \frac{4}{A} \left[ \frac{n}{B_n^2} \left( \frac{\int_{-aB_n}^{aB_n} x^2 dF(x)}{B_n^2} - \left( \frac{\int_{-aB_n}^{aB_n} xdF(x)}{-aB_n} \right)^2 \right) - K^a_n(A) \right] \right\}^{\frac{1}{3}}
\end{equation}

\begin{equation}
+ \frac{(c_1 + c_2)a^{2-\alpha}}{2 - \alpha} - K^a(A) + K^a(-A) + K^a(-A) \right\}^{\frac{1}{3}}
\end{equation}
where $K_n^a(v)$ and $K^a(v)$ are given by (4.2.9) and (4.1.9), and $A$, $J$, $m(A, J)$ are given in §3.1.

4.3 An Example

We consider a sequence of independent random variables

$(X_n), \ n = 1, 2, ...$ with a common distribution function $F(x)$. Let $F(x)$ have density function

\begin{equation}
\bar{f}(x) = \frac{1}{\pi} \frac{1 - \cos x}{x^2}.
\end{equation}

We consider the normed sums

\begin{equation}
S_n = \frac{X_1 + \ldots + X_n}{n}.
\end{equation}

Again, let $F_n(x)$ denote the distribution function of $S_n$. By employing the technique of characteristic functions we find that the characteristic functions of $F(x)$ and $F_n(x)$ are

\begin{equation}
\phi(t) = \begin{cases} 
1 - |t| & \text{for } |t| \leq 1, \\
0 & \text{for } |t| > 1
\end{cases}
\end{equation}
and

\[
\Phi_n(t) = \begin{cases} 
1 - \frac{|t|}{n} & \text{for } |t| \leq n, \\
0 & \text{for } |t| > n,
\end{cases}
\]

respectively. We see that \( \Phi_n(t) \) converges to

\[
(4.3.5) \quad \Phi(t) = e^{-|t|}
\]

which is the characteristic function of the well known Cauchy's distribution function

\[
(4.3.6) \quad F(x) = \frac{1}{\pi} \left( \frac{x}{\pi/2} + \arctan x \right).
\]

To apply the result of Theorem 12 we put

\[
(4.3.7) \quad x_{nk} = \frac{x_k}{n}.
\]

Hence we have

\[
(4.3.8) \quad \mu_{nk}(x) = 0,
\]

\[
(4.3.9) \quad \mu_n(x) = 0,
\]

\[
(4.3.10) \quad \sigma_{nk}^2(x) = \frac{1}{n^2} \int_{-n}^{n} x^2 \left( \frac{1}{\pi} \frac{1 - \cos x}{x^2} \right) dx
\]

\[
= \frac{2}{\pi n^2} (nx - \cos nx),
\]

\[
(4.3.11) \quad \sigma_n^2(x) = \frac{2}{\pi n} (nx - \cos nx).
\]
For \(-a < v < a\) we have

\[
K_n^a(v) = \frac{n}{n^2} \int_{-na}^{nv} u^2 \frac{1}{\pi} \frac{1 - \cos u}{u^2} \, du
\]

\[
= \frac{1}{n\pi} \left[ (1 - \cos u)du \right]_{-na}^{nv}
\]

\[
= \frac{1}{n\pi} \left[ n(v + a) - (\cos nv - \cos na) \right].
\]

Hence

\[
(4.3.12) \quad K_n^a(v) = \begin{cases} 
0 & \text{for } v < -a, \\
\frac{1}{n\pi} \left[ n(v + a) - (\cos nv - \cos na) \right] & \text{for } -a \leq v < a, \\
\frac{2}{n\pi} \left[ na - \cos na \right] & \text{for } a \leq v.
\end{cases}
\]

From (4.3.5) we have

\[
(4.3.13) \quad \log \mathcal{V}(t) = -|t|.
\]

Comparing (4.3.13) with (4.1.3) we find that the constants in the representation of \(\mathcal{V}(t)\) given by Theorem 11 are

\[
(4.3.14) \quad \gamma = 0, \quad c = 1, \quad \alpha = 1, \quad \beta = 0.
\]

Hence

\[
(4.3.15) \quad \sigma^2 = 0.
\]
According to the proof of Theorem 11 as given in [4], for the case \( \alpha = 1 \) we find that

\[
c = (c_1 + c_2) \frac{\pi}{2}, \quad \beta = \frac{c_2 - c_1}{c_1 + c_2}.
\]

Hence

\[
(4.3.16) \quad c_1 = c_2 = \frac{1}{\pi}.
\]

Therefore

\[
(4.3.17) \quad dM(x) = \frac{dx}{\pi x^2}, \quad dN(x) = \frac{dx}{\pi x^2},
\]

hence we have

\[
(4.3.18) \quad dG(u) = \frac{1}{\pi} \frac{du}{1 + u^2}.
\]

Applying Lemma 9 we have

\[
(4.3.19) \quad \mu(a) = 0,
\]

\[
(4.3.20) \quad \sigma^2(a) = \frac{2a}{\pi},
\]

\[
(4.3.21) \quad K^a(v) = \left\{ \begin{array}{ll}
0 & \text{for } v < -a, \\
\frac{a + v}{\pi} & \text{for } -a \leq v < a, \\
\frac{2a}{\pi} & \text{for } a \leq v.
\end{array} \right.
\]

For simplicity we let
(4.3.22) \[ a = A = \frac{1}{\sqrt{3}}. \]

Then we have

(4.3.23) \[ \left\{ \frac{1}{3} \sigma_n^2(a) \max \sigma_{nk}^2(a) \right\}^{\frac{1}{3}} \leq \left\{ \frac{8}{3 \pi^2} \frac{1}{n^2 \sqrt{8}} \right\}^{\frac{1}{3}} \]

(4.3.24) \[ \left\{ \frac{5}{6} \delta \left( \sigma_n^2(a) + \sigma^2(a) \right) \right\}^{\frac{1}{4}} \leq \left\{ \frac{5}{16} \sqrt{\delta} \right\}^{\frac{1}{4}} \]

(4.3.25) \[ \left\{ \frac{1}{2} \sum_{i=0}^{n} |K_n^a(x_i) - K^a(x_i)| \right\}^{\frac{1}{3}} = \left\{ \frac{1}{2} \sum_{i=0}^{m} \left| \cos nx_i - \cos na \right| \right\}^{\frac{1}{3}} \leq \left\{ \frac{1}{2} (m + 1) \frac{2}{n \pi} \right\}^{\frac{1}{3}} \leq \left\{ \frac{1}{2} \frac{4}{\delta \sqrt{\delta}} \frac{2}{n \pi} \right\}^{\frac{1}{3}} \]

\[ = \left\{ \frac{4}{n \pi \delta \sqrt{\delta}} \right\}^{\frac{1}{3}} \]

(4.3.26) \[ \left\{ \frac{4}{A} \left( K_n^a(\omega) - K_n^a(A) + K^a(\omega) - K^a(A) + K_n^a(-A) + K^a(-A) \right) \right\}^{\frac{1}{2}} + 2 \left| K_n^a(a) - K^a(a) \right| \leq 0. \]

Hence,
Applying Lemma 10 (4.1.12), (4.3.16) and (4.3.27) to (3.1.9) we have

(4.3.28) \[ g^a(n,m(A,\delta),r) \leq \left\{ \frac{8}{3\pi^2} \frac{1}{n^2 \sqrt{\delta}} \right\}^{\frac{1}{4}} + \left\{ \frac{2}{\pi} \sqrt{\delta} \right\}^{\frac{1}{4}} + \left\{ \frac{4}{\pi n \delta \sqrt{\delta}} \right\}^{\frac{1}{4}} \]

Applying (4.2.10) we have

(4.3.29) \[ \sum_{k=1}^{k_n} \left\{ F_{nk}(-a) + 1 - F_{nk}(a) \right\} = n \left\{ \tilde{F}(-an) + 1 - \tilde{F}(an) \right\} \]

\[ = n \frac{2}{a} \int_{an}^{+a} \frac{1 - \cos x}{x^2} \, dx \]

\[ \leq \frac{4}{\pi a} \]

\[ = \frac{4}{\pi} \sqrt{\delta} . \]

Hence, from Theorem 12 we have
(4.3.30) \( M_n = \sup_{-\infty < x < +\infty} |F_n(x) - F(x)| \)

\[
\leq k(p, \beta) \left[ \left\{ \frac{8}{3n^2 \sqrt{\beta}} \right\} + \left\{ \frac{5}{\pi} \sqrt{\delta} \right\} \right]
\]

\[
+ \left\{ \frac{4}{(\pi n \delta \sqrt{3})} \right\}^{\frac{1}{3}} + \left\{ \frac{16}{r(1 - r)} \right\} \left\{ \sqrt{\delta} \right\} \right]
\]

\[+ \frac{4}{n} \sqrt{\delta}.\]

Taking

(4.3.31) \( \delta = \frac{1}{8/15}, \quad r = \frac{3}{5} \)

we find that

(4.3.32) \( M_n = \sup_{-\infty < x < +\infty} |F_n(x) - F(x)| \leq \frac{c}{n^{1/15}}, \)

where \( c \) is a constant.

We remark that (4.3.32) gives an alternative proof of the fact that \( F_n(x) \) converges to \( F(x) \).
BIBLIOGRAPHY


