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BROOKS, James Keith, 1938–
A TRANSFORMATION THEORY FOR BANACH SPACES.
The Ohio State University, Ph.D., 1964
Mathematics

University Microfilms, Inc., Ann Arbor, Michigan
A TRANSFORMATION THEORY

FOR BANACH SPACES

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

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INTRODUCTION

In 1961 Reichelderfer published a paper entitled "A Transformation Theory for Measure Space" [4] (the number in [ ] refers to the Bibliography). The present work consists of contributions to this abstract transformation theory. Reichelderfer's results [4] are developed in a setting which we shall now briefly describe. Let $T$ be a function (transformation) whose domain is a non-empty set $S$ and whose range is $S'$. Let $\mathcal{M}$ be a $\sigma$-field of subsets of $S$ and $\mu$ a measure defined on $\mathcal{M}$. Similarly, let $(S', \mathcal{M}', \mu')$ be a measure space. Suppose that $\mathcal{D}$ is a certain collection of subsets $D$ of $S$ and that for each set $D$ and each point $s'$ in $S'$ a non-negative extended real number $W'(s', D)$ is assigned, subject to the following conditions:

i) For each $D$ in $\mathcal{D}$, $W'$ vanishes outside $TD$.

ii) For each $s'$ in $S'$, $W'$ is under additive on $\mathcal{D}$.

iii) For each $s'$ in $S'$, $W'$ is inner continuous on $\mathcal{D}$.

iv) For each $D$ in $\mathcal{D}$, $W'$ is measurable $\mu'$.

Then for each $D$ in $\mathcal{D}$ we may consider the non-negative extended real number $\int_{S'} W'(\cdot, D) d\mu'$ as a weight $WD$ attached to $D$. 

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One of the main results of that paper is the establishment, under certain standard hypotheses, of necessary and sufficient conditions in order that there exists a non-negative extended real valued function $f$ defined on $S$ which is integrable $\mu$ such that

$$\int_D f d\mu = \int_S W'(\cdot, D) d\mu', \quad D \in \mathcal{D}.$$

When these conditions are satisfied it is shown that a transformation formula always holds in the following sense:

Let $H'$ be a real valued function defined on $S'$ which is measurable $\mu'$. Then for each $D$ in $\mathcal{D}$, $H'W'(\cdot, D)$ is measurable $\mu'$ and $H'\cdot T f$ is measurable $\mu$ on $D$. Moreover $H'W'(\cdot, D)$ is integrable $\mu'$ if and only if $H'\cdot T f$ is integrable $\mu$ on $D$.

When these functions are integrable then the transformation formula

$$ (*) \quad \int_D H'\cdot T f d\mu = \int_S H'W'(\cdot, D) d\mu'$$

holds.

The present work has the above paper as its foundation. There are two main objectives in this paper. The first is to extend the ranges of $H'$ and the measures $\mu$ and $\mu'$ so that the transformation formula (*) holds under suitable hypotheses. For this purpose we shall employ the concept of a general bilinear vector integral in which the integrand and measure take values in Banach spaces ($B$-spaces) $\mathcal{D}$ and $\mathcal{X}$ respectively.
for which there is a suitable bilinear functional defined on
\( \mathcal{A} \times \mathcal{Y} \) with values in a third B-space. An introduction to this
general theory is presented (Chapter I). The second objective
of this paper is to extend the range of \( W' \) to a B-space so
that the transformation formula (*) holds under suitable
hypotheses. To achieve these objectives, the hypotheses HB
are introduced in section 1.24 (Chapter I). In a special case
these hypotheses reduce essentially (cf. 1.25) to Reichelderfer's
hypotheses [4].

First the transformation theory is extended to the case
when \( H' \) is a B-valued function and the measures \( \mu \) and \( \mu' \) are
complex valued (Chapter II). The Proof of Theorem 2.10
(Chapter II) uses techniques which enable us (Chapter V) to
extend the range of \( W' \) to a B-space. Then the strongest
result (Chapter V) concerning the existence of the trans-
formation formula (*) occurs when \( W' \) is complex valued. A
possible approach to the extension of the range of \( W' \) to the
complex numbers is the decomposition of \( W' \) into real and
imaginary parts, which are again decomposed into positive and
negative parts. However, many difficulties occur with this
approach. The methods used (Chapter V) avoid such an approach
and the extension is made without any type of decomposition.
Chapter III deals with the case when \( \mu \) and \( \mu' \) take their values
in the B-space \( \mathcal{Y} \) while \( H' \) is complex valued. We then employ
the conjugate space \( \mathcal{Y}^* \) of \( \mathcal{Y} \) (Chapter III), so that the preceding
results (Chapter II) can be used to establish results for vector measures. Chapter IV deals with the case when $H'$ has the $B$-space $\mathcal{J}$ as its range and $\mu$ and $\mu'$ have the $B$-space $\mathcal{K}$ as their range.

Results are established in this paper which are independent of the transformation theory. The multiplication required (Chapter I) on the product $\mathcal{J} \times \mathcal{K}$ of two $B$-spaces is of the type studied in Bartle's paper [1] with certain natural conditions that permit us to show that the class of null sets is independent of the choice of two possible semi-variations for the measures (Theorem 1.8). This in turn permits us to establish a formula relating the general bilinear vector integral and an integral with a scalar integrand and a vector valued measure (Theorem 1.23). When $\mu$ is an $\mathcal{K}$-valued vector measure and $\mathcal{K}^*$ is the separable conjugate space of $\mathcal{K}$, Theorem 3.6 (Chapter III) states that if a set is $x^* \mu$-null for every $x^*$ in $\mathcal{K}^*$, then the set is $\mu$-null. Theorem 3.13 (Chapter III) gives a necessary and sufficient condition for a measurable $\mu$ function to be integrable $\mu$ in terms of the measures $x^* \mu$, where $x^*$ belongs to the closed unit sphere in $\mathcal{K}^*$. 
CHAPTER I

PRELIMINARIES

1.1 The following conventions will hold throughout this paper. \( \mathbb{R} \) will denote the real number system \((+\infty \text{ and } -\infty \text{ are excluded})\) and \( \mathbb{R}^+ \) will denote the set of non-negative real numbers \((+\infty \text{ is excluded})\). The extended real number system is the set obtained by adding \(+\infty\) and \(-\infty\) to \( \mathbb{R} \). \( \mathbb{C} \) will denote the complex number field. Each of the symbols \( \mathbb{X} \), \( \mathbb{Y} \), and \( \mathbb{Z} \) will always denote a Banach space (B-space) over the complex number field, and the norm of an element in a B-space will be denoted by \( |.| \). A partition of a set \( A \) is a disjoint family of subsets of \( A \) whose union is \( A \). Thus the empty set may appear in a partition. In this paper we shall often work with non-empty spaces denoted by \( S \) and \( S' \). If \( E \subseteq S \) then \( H(\cdot,E) \) denotes the characteristic function of \( E \) as a subset of \( S \); it is defined by

\[
H(s,E) = \begin{cases} 
1, & s \in E, \\
0, & s \in S - E 
\end{cases}
\]

Similarly, if \( E' \subseteq S' \), then \( H'(\cdot,E') \) denotes the characteristic function of \( E' \) as a subset of \( S' \).
1.2 Many concepts of measure and integration are used in this paper. We shall give a brief introduction to the theory of the general bilinear vector integral. This will be taken mainly from Bartle's paper [1]. Then in a special case this will reduce to the integration theory [3] in which the integrand is vector valued and the measure complex valued. In another special case the theory will reduce to the integration theory [3] in which the integrand is complex valued and the measure vector valued. The definitions and results involved with the former theory will be regarded as standard material and these concepts can be found in Chapter 3 in Linear Operators [3].

The assumption in 1.3 will hold throughout the development of the bilinear integral and whenever there is a process of multiplication in the sense defined below.

1.3 Assume $\mathbb{J}$, $\mathbb{K}$, and $\mathbb{B}$ are $\mathbb{B}$-spaces over the complex numbers such that there is a bilinear mapping $\rho$ defined on $\mathbb{J} \times \mathbb{K}$ into $\mathbb{B}$ with the following properties:

i) $|\rho(y,x)| \leq K|y||x|$ for some fixed positive number $K$ for all $y$ in $\mathbb{J}$ and $x$ in $\mathbb{K}$.

ii) Fix $x$ in $\mathbb{K}$. If $\rho(y,x) = 0$ for all $y$ in $\mathbb{J}$ then $x = 0$.

iii) If $\mathbb{J}$ or $\mathbb{K}$ is the field of complex numbers, then $\rho$ will be the usual scalar product.

For brevity we will denote $\rho(y,x)$ by $yx$. The hypothesis
in iii) justifies this notation.

1.4 Definitions. Assume $\mathcal{M}$ is a $\sigma$-field of subsets of a non-empty set $S$. Let $\mu$ be an $\mathcal{I}$-valued set function defined on $\mathcal{M}$ such that if $\{M_i\}$ is a sequence of pairwise disjoint sets in $\mathcal{M}$, then $\mu(\bigcup M_i) = \sum \mu(M_i)$. Then $\mu$ is said to be a vector measure or an $\mathcal{I}$-measure when we desire to indicate the range explicitly. The triple $(S,\mathcal{M},\mu)$ is called a vector measure space or an $\mathcal{I}$-vector measure space. If $\mathcal{I}$ is $\mathbb{R}$ or $\mathcal{C}$, $\mu$ is called a measure and $(S,\mathcal{M},\mu)$ is called a measure space. If the range of $\mu$ is $\mathbb{R}^+$, $\mu$ is called a non-negative measure and $(S,\mathcal{M},\mu)$ is called a non-negative measure space.

1.5 Definition. Let $(S,\mathcal{M},\mu)$ be a vector measure space. Define a set function $\nu$ on $\mathcal{M}$ as follows: For $M$ in $\mathcal{M}$

$$\nu(M,\mu) = \sup \sum_{i=1}^{n} |\mu(M_i)|,$$

where the supremum is taken over all partitions of $M$ into a finite number of disjoint sets $M_i$ in $\mathcal{M}$. $\nu$ is called the total variation of $\mu$.

Remark. It follows that $\nu$ is a monotone non-negative extended real valued countably additive set function defined on $\mathcal{M}$. If the range of $\mu$ is $\mathbb{R}$ or $\mathcal{C}$, then $\nu$ is bounded and $\nu(M,\mu) \leq 4 \sup |\mu(E)|$, $E \in \mathcal{M}$, $E \subseteq M$ [3, lemma 3.1.5, p.97]. Thus in this case $(S,\mathcal{M},\nu)$ is a non-negative measure space.
If \( \mu \) is an arbitrary \( \mathcal{I} \)-valued measure \( \nu \) may not be finite valued. Because of this, additional concepts are needed before an integration theory can be developed. These are presented in sections 1.6-1.12.

**1.6 Definitions.** Let \((S, \mathcal{M}, \mu)\) be an \( \mathcal{I} \)-vector measure space. The semi-variation of \( \mu \) with respect to \( \mathcal{D} \) is the extended real-valued non-negative set function \( |||\mu||| \) defined on \( \mathcal{M} \) as follows: Let \( M \in \mathcal{M} \),

\[
|||\mu||| (M) = \sup \left\{ \sum_{i=1}^{n} y_{i} \mu(M_{i}) \right\},
\]

where the supremum is taken over all partitions of \( M \) into a finite number of disjoint sets \( M_{i} \) in \( \mathcal{M} \) and all finite collections of elements \( y_{i} \) in \( \mathcal{D} \) with \( |y_{i}| \leq 1 \). For convenience we write \( |||M||| \) instead of \( |||\mu|||(M) \) when the vector measure \( \mu \) is understood. We can extend the definition of \( |||\mu||| \) to arbitrary subsets of \( S \) as follows: If \( A \) is a subset of \( S \), then

\[
|||A||| = \inf \{ |||M||| : M \in \mathcal{M}, A \subseteq M \}.
\]

This extension is a monotone sub-additive set function defined on the collection of all subsets of \( S \). We say that a subset \( A \) of \( S \) is a \( \mu \)-null set if \( |||A||| = 0 \). A proposition regarding points of \( S \) holds almost everywhere \( \mu \) (a.e.\( \mu \)) if it holds on \( S - A \) where \( A \) is a \( \mu \)-null set.
1.7 Definition. Let \( (S, \mathcal{M}, \mu) \) be an \( \mathbb{F} \)-vector measure space. The semi-variation of \( \mu \) with respect to \( \mathcal{G} \) is denoted by \( ||\mu||_\mathcal{G} \).

This then coincides with the usual semi-variation [3, definition 4.10.3]. It can be shown [3, lemma 4.10.4, p. 320] that in this case for each \( M \) in \( \mathcal{M} \),

\[
||M|| = ||\mu||_\mathcal{G}(M) \leq 4 \sup \{\mu(F) \mid F \in \mathcal{M}, F \subseteq M\}.
\]

Thus in this case the semi-variation is bounded. It is readily verified that if \( \mu \) is a complex valued measure, then the semi-variation of \( \mu \) with respect to \( \mathcal{G} \) reduces to the total variation of \( \mu \).

We shall now establish a result which shows a connection between the semi-variation of \( \mu \) with respect to \( \mathcal{G} \) and the semi-variation of \( \mu \) with respect to \( \mathcal{G} \).

1.8 Theorem. Let \( (S, \mathcal{M}, \mu) \) be an \( \mathbb{F} \)-vector measure space and let \( M \) belong to \( \mathcal{M} \). Then \( |||M||| = 0 \) if and only if \( ||M|| = 0 \).

Proof. Recall that

\[
(1) \quad |||M||| = \sup \left\{ \sum_{i=1}^{n} y_i \mu(M_i) \right\},
\]

where the supremum is extended over all partitions of \( M \) into a finite number of disjoint sets \( M_i \) in \( \mathcal{M} \) and all finite collections of elements \( y_i \) in \( \mathcal{G} \) with \( |y_i| \leq 1 \).

\[
(2) \quad ||M|| = \sup \left\{ \sum_{i=1}^{n} \alpha_i \mu(M_i) \right\},
\]
where the supremum is extended over all partitions of $M$ into a finite number of disjoint sets $M_i$ in $M$ and all finite collections of complex numbers $a_i$ with $|a_i| \leq 1$. Now assume $||M|| = 0$. Let $\sum a_i \mu(M_i)$ be a sum of the type appearing in (2). Let $y$ belong to $\mathcal{D}$, $y \neq 0$. Since $|\sum \frac{y}{|y|} a_i \mu(M_i)| = 0$ we have

\begin{equation}
(3) \quad y(\sum a_i \mu(M_i)) = 0.
\end{equation}

Since (3) holds for every $y$ in $\mathcal{D}$, by 1.3 ii), we have

$\sum a_i \mu(M_i) = 0$. Thus $||M|| = 0$.

Conversely, suppose $||M|| = 0$. Let $\sum y_i \mu(M_i)$ be a sum of the type appearing in (1). Since

$|\mu(M_i)| \leq ||M_i|| \leq ||M|| = 0$

for each $i$, we have $\sum y_i \mu(M_i) = 0$. Thus $||M|| = 0$.

1.9 Assume $(S,M,\mu)$ is an $\mathbb{F}$-vector measure space. Let $\mathbb{F}^*$ be the conjugate space of $\mathbb{F}$. Then for each $x^* \in \mathbb{F}^*$, $x^* \mu$ is a complex valued measure and $(S,M,x^* \mu)$ is a measure space.

It easily follows that $\nu(M,x^* \mu) \leq |x^*| ||\mu||(M), M \in M$.

1.10 Definition. Let $(S,M,\mu)$ be an $\mathbb{F}$-vector measure space. We say that the vector measure $\mu$ has the $*$-property relative to $\mathcal{D}$ if there exists a non-negative real valued
measure \( \lambda \) defined on \( \mathbb{R} \) such that \( \lambda(M) \to 0 \) if and only if 
\[ ||M|| \to 0, \text{ and } |||S||| < \infty. \]
We call \( \lambda \) a control measure for \( \mu \).

1.11 It can be shown [3, lemma 4.10.5, p.321] that if 
\((S,\mathbb{M},\mu)\) is an \( \mathbb{X} \)-vector measure space, then \( \mu \) always has the 
\( * \)-property relative to \( \mathbb{E} \). Furthermore there exists a control 
measure \( \lambda \) defined on \( \mathbb{R} \) such that \( \lambda(M) \leq ||M||, M \in \mathbb{M} \). In case 
\( \mathbb{X} = \mathbb{E} \) then the total variation of \( \mu \) is a control measure for \( \mu \).

1.12 In this paper it will always be assumed that \( \mu \) has 
the \( * \)-property relative to \( \mathbb{J} \) when \( \mathbb{J} \) is an arbitrary \( B \)-space.
Thus we shall always have \( |||S||| < \infty \). Bartle states [1] that 
it is not difficult to show from a theorem of Saks [5], that 
if \( \mu \) has a control measure \( \lambda \) then \( |||S||| < \infty \).

Throughout the following sections we shall assume that 
\((S,\mathbb{M},\mu)\) is an \( \mathbb{X} \)-vector measure space.

1.13 Definitions. Let \( f \) be a \( \mathbb{J} \)-valued function defined 
on \( S \) such that \( f \) has a representation of the form 
\[ f = \sum_{i=1}^{n} y_i \mathbb{H}(\cdot,M_i), \]
where \( y_i \in \mathbb{J} \) and \( M_i \in \mathbb{M} \) for each \( i \). Then \( f \) is called a simple
measurable \( \mu \) function. A sequence of \( \mathbb{J} \)-valued functions \( f_n \)
defined on \( S \) is said to converge in measure \( \mu \) to a \( \mathbb{J} \)-valued
function $f$ defined on $S$ if $\lim \{s \in S : |f_n(s) - f(s)| \geq \varepsilon \} ||| = 0$ for every positive number $\varepsilon$. The sequence is said to be almost uniformly convergent $\mu$ to $f$ if for every positive number $\varepsilon$ there is a subset $A_\varepsilon$ of $S$ such that $|||A_\varepsilon||| < \varepsilon$ and the convergence of the sequence $\{f_n\}$ to $f$ is uniform on $S - A_\varepsilon$. We say a $\mathcal{B}$-valued function defined on $S$ is measurable $\mu$ if it is the limit in measure $\mu$ of a sequence of simple measurable $\mu$ functions.

1.1.4 Definitions. Let $f$ be a $\mathcal{B}$-valued simple measurable $\mu$ function defined on $S$ and let $M \in \mathcal{M}$. Then $f$ can be represented in the form $\sum_{i=1}^{n} y_i H(\cdot, M_i)$, where $y_i \in \mathcal{B}$ and $M_i \in \mathcal{M}$ for each $i$. We define the integral of $f$ over $M$ with respect to $\mu$ to be

$$\int_M f d\mu = \sum_{i=1}^{n} y_i \mu(M \cap M_i), M \in \mathcal{M}.$$ 

In view of this definition we shall call a simple measurable $\mu$ function a simple integrable $\mu$ function.

Remark. Let $f$ be a $\mathcal{B}$-valued simple integrable $\mu$ function defined on $S$. Then it follows that the value of the integral of $f$ with respect to $\mu$ is independent of the choice of representation of $f$ as given above. Also the integral of $f$ with
with respect to \( \mu \) is a countably additive set function defined on \( \mathbb{W} \) with range \( \mathcal{S} \). For each fixed set \( M \) in \( \mathbb{W} \) the integral on \( M \) is a linear mapping defined on the linear space of \( \mathcal{D} \)-valued simple integrable \( \mu \) functions defined on \( S \), and has values in \( \mathcal{S} \).

1.15 Lemma. (a) If a sequence of \( \mathcal{D} \)-valued functions \( f_n \) defined on \( S \) converges in measure \( \mu \) to a \( \mathcal{D} \)-valued function \( f \) defined on \( S \), then some subsequence of \( \{f_n\} \) converges almost uniformly \( \mu \) to \( f \).

(b) A sequence of \( \mathcal{D} \)-valued measurable \( \mu \) functions \( f_n \) defined on \( S \) converges a.e.\( \mu \) to a \( \mathcal{D} \)-valued function \( f \) defined on \( S \) if and only if it converges almost uniformly \( \mu \) to \( f \).

Proof. [1, lemma 4.3, p.346].

1.16 Remark. It can be shown [1, lemma 2.2, p.340] that almost uniform \( \mu \) convergence implies convergence in measure \( \mu \). Using this fact and the above lemma we conclude that a function is measurable \( \mu \) if and only if it is the a.e.\( \mu \) limit of a sequence of simple integrable \( \mu \) functions. Furthermore if a sequence of \( \mathcal{D} \)-valued measurable \( \mu \) functions \( f_n \) defined on \( S \) converges a.e.\( \mu \) to a \( \mathcal{D} \)-valued function \( f \) defined on \( S \), then \( f \) is measurable \( \mu \).
1.17 **Definition.** A $\mathfrak{g}$-valued function $f$ defined on $S$ is said to be integrable $\mu$ if there exists a sequence of $\mathfrak{g}$-valued simple integrable $\mu$ functions $f_n$ defined on $S$ satisfying the conditions:

i) $\lim_{n} f_n = f$ a.e. $\mu$

ii) The sequence $\{\lambda_n\}$ of integrals

$$\lambda_n(M) = \int_M f_n d\mu, \quad M \in \mathcal{M}$$

converges in the norm of $\mathfrak{g}$ for each $M$ in $\mathcal{M}$. We then define $\int_M f d\mu = \lim_{n} \lambda_n(M)$, $M \in \mathcal{M}$. $\int_M f d\mu$ is called the integral of $f$ on $M$ with respect to $\mu$. The sequence $\{f_n\}$ is called a determining sequence for $f$.

1.18 Clearly if $f$ is integrable $\mu$ then $f$ is measurable $\mu$. It can be shown that the integral $\int_M f d\mu$ in 1.17 is independent of the choice of the determining sequence for $f$. A $\mathfrak{g}$-valued function $f$ defined on $S$ is said to be integrable $\mu$ on a set $M$ in $\mathcal{M}$ if $\mathbb{H}(\cdot, M)f$ is integrable $\mu$. The following properties can be proved [1]. If $M$ is in $\mathcal{M}$, the set of $\mathfrak{g}$-valued functions integrable $\mu$ over $M$ is a linear space over the complex numbers and the integral over $M$ is a linear mapping of this space into $\mathfrak{g}$. If $f$ is integrable $\mu$ over $S$, the integral of $f$ is a countably additive set function defined on $\mathcal{M}$ with values in $\mathfrak{g}$. 
Also \( \lim_{||M|| \to 0} \int_M f \, d\mu = 0. \)

1.19 Theorem. If \( \{f_n\} \) is a sequence of \( \mathcal{Y} \)-valued integrable \( \mu \) functions which are defined on \( S \) and if \( f \) is a \( \mathcal{Y} \)-valued function defined on \( S \) such that:

i) \( \lim_{n \to \infty} f_n = f \) \( \text{a.e.} \mu. \)

ii) Given a positive number \( \varepsilon \) there is a positive number \( \delta \) such that if \( M \) is in \( \mathcal{M} \) and \( ||M|| < \delta \), then \( |\int_M f_n \, d\mu| < \varepsilon \), for each \( n \), that is the integrals \( \int_M f_n \, d\mu \) are uniformly absolutely continuous; then we may conclude that \( f \) is integrable \( \mu \) on \( S \) and \( \int_M f \, d\mu = \lim_{n \to \infty} \int_M f_n \, d\mu \), uniformly for \( M \) in \( \mathcal{M} \).

Proof. [1, Theorem 4.10, p. 347].

1.20 If \( X = \mathbb{C} \) and \( \mathcal{Y} \) is an arbitrary \( B \)-space, then the concepts and results in sections 1.4–1.19 coincide with the standard integration theory [3]. If \( \mathcal{Y} = \mathbb{C} \) and \( X \) is an arbitrary \( B \)-space, then the concepts and results in sections 1.4–1.19 coincide with the integration theory [3] in which complex valued integrands and vector measures are considered. It can be shown that if either \( \mathcal{Y} \) or \( X \) is \( \mathbb{C} \), the integrability of a function \( f \) relative to \( \mu \) implies the integrability of the function \( |f| \) relative to \( \mu \).
1.21 Theorem. Let \((S, \mathcal{M}, \mu)\) be an \(\mathbb{F}\)-vector measure space. If \(\{f_n\}\) is a sequence of complex valued integrable \(\mu\) functions defined on \(S\) and \(f\) is a complex valued function defined on \(S\) such that \(\lim f_n = f\) a.e.\(\mu\) and if \(g\) is a real valued integrable \(\mu\) function defined on \(S\) such that \(|f_n| \leq g\) a.e.\(\mu\) for each \(n\), then \(f\) is integrable \(\mu\) and \(\int_M f_n d\mu = \lim \int_M f_n d\mu, M \in \mathcal{M}\).

Proof. [3, theorem 4.10.10, p. 328].

1.22 Lemma. Let \((S, \mathcal{M}, \mu)\) be an \(\mathbb{F}\)-vector measure space. Let \(U\) be a bounded linear operator from \(\mathbb{F}\) into a \(B\)-space \(\mathcal{B}\). Then \(U\mu\) is a vector measure defined on \(\mathcal{M}\) with values in \(\mathcal{B}\), and for any complex valued function \(f\) defined on \(S\) which is integrable \(\mu\), \(f\) is integrable \(U\mu\) and \(U(\int_M f d\mu) = \int_M f dU\mu, M \in \mathcal{M}\).

Proof. [3, theorem 4.10.8, p. 323].

The next theorem gives us a connection between the bilinear integral and the integral with a complex valued integrand and vector measure.

1.23 Theorem. Assume \((S, \mathcal{M}, \mu)\) is an \(\mathbb{F}\)-vector measure space. Let \(f\) be a complex valued function defined on \(S\) integrable \(\mu\). Let \(y\) belong to \(\mathcal{B}\). Then \(yf\) is integrable \(\mu\) and \(\int_M yf d\mu = y\int_M f d\mu, M \in \mathcal{M}\).
Proof. Since $f$ is integrable $\mu$ there exists by definition 1.17 a determining sequence of complex valued simple integrable $\mu$ functions $f_n$ defined on $S$; hence $\lim f_n = f$ a.e.$\mu$ and the integrals $\int_M f_n \, d\mu$ converge in the norm of $\mathcal{X}$ for each $M$ in $\mathcal{M}$.

Thus $y f_n$ is simple integrable $\mu$ for each $n$ and using theorem 1.8 it follows that $\lim y f_n = y f$ a.e.$\mu$. Also since the multiplication involved is bilinear we have

$$(1) \quad \int_M y f_n \, d\mu = y \int_M f_n \, d\mu, \quad M \in \mathcal{M}, \quad n \geq 1.$$ 

Now $|\int_M y f_n \, d\mu - \int_M y f_m \, d\mu| = |y \int_M f_n \, d\mu - \int_M f_m \, d\mu| \leq K |y| |\int_M f_n \, d\mu - \int_M f_m \, d\mu|, \quad M \in \mathcal{M}, \quad m, \quad n \geq 1.$

Thus the sequence of integrals $\int_M y f_n \, d\mu$ converges in the norm of $\mathcal{X}$ for each $M$ in $\mathcal{M}$ since the sequence of integrals $\int_M f_n \, d\mu$ converges in the norm of $\mathcal{X}$ for each $M$. Therefore $\{y f_n\}$ is a determining sequence for $y f$. Using (1) we have

$$\int_M y f \, d\mu = \lim \int_M y f_n \, d\mu = \lim y \int_M f_n \, d\mu = y \int_M f \, d\mu, \quad M \in \mathcal{M}.$$ 

1.24 In this section the standard hypotheses HB under which the transformation theory is to be developed are listed. The semi-variationds used in this section are with respect to $\mathcal{G}$. 
(S, M, \mu) is an \mathbb{R}-vector measure space.

(S', M', \mu') is an \mathbb{R}-vector measure space.

T is a single valued function (transformation) from S onto S'.

\mathcal{D} is a collection of subsets D of S having the following properties: The empty set and the space S belong to \mathcal{D}. \mathcal{D} is a subset of \mathcal{M} and T \mathcal{D}, that is \{TD: D \in \mathcal{D}\}, is a subset of \mathcal{M}'. If 1^D and 2^D belong to \mathcal{D}, then there is a countable number of pairwise disjoint sets D_i in \mathcal{D} such that 1^D \cap 2^D = UD_i. If M is an element of \mathcal{M} and \zeta is a positive number, then there exists a countable number of pairwise disjoint sets D_i in \mathcal{D} such that M \subset UD_i and ||UD_i - M|| < \zeta.

**Definition 1.** An element D_0 in \mathcal{D} is said to be of type \gamma if it is an element of a countable partition of S into sets D_i, i \geq 0, Y, Z, where the D_i \in \mathcal{D}, i \geq 0, such that ||Y|| = 0 and ||TZ|| = 0.

If D is an element in \mathcal{D} there is a countable sequence of sets D_j in \mathcal{D} such that each D_j is of type \gamma, D_j \subseteq D_{j+1} for every j and UD_j = D.

\mathcal{S}' is a \sigma-field of subsets \mathcal{S}' of S' having the following properties. \mathcal{S}' is a subset of \mathcal{M}' and T^{-1}\mathcal{S}' is a
subset of $\mathcal{M}$. For each element $M'$ in $\mathcal{M}$ there are sets $B'_1$ and $B'_2$ in $\mathcal{S}'$ such that $B'_1 \subseteq M' \subseteq B'_2$ and $||B'_2 - B'_1|| = 0$.

**Definition 2B.** Denote by $\mathcal{O}'$ all the subsets $O'$ of $S'$ for each of which there is a countable family of pairwise disjoint sets $D_i$ in $\mathcal{D}$ such that $T^{-1}O' = \bigcup D_i$.

**Remark.** If $M'$ is an element of $\mathcal{M}$ and $\zeta'$ is a positive number, then there exists a set $O'$ in $\mathcal{O}'$ such that $M' \subseteq O'$ and $||O' - M'|| < \zeta'$.

**Definition 3B.** An element $O'_i$ in $\mathcal{O}'$ is said to be of type $\gamma'$ if it is an element of a countable partition of $S'$ into sets $O'_i$, $i \geq 0$, $\gamma'$, $Z'$, where the $O'_i \in \mathcal{O}'$ for $i \geq 0$ and $||T^{-1}Y'|| = 0$ and $||Z'|| = 0$.

**Definition 4B.** An element $M'$ in $\mathcal{M}$ is said to be of type $\zeta'$ if there is a countable sequence of sets $O'_j$ in $\mathcal{O}'$ such that each $O'_j$ is of type $\gamma'$, $O'_j \subseteq O'_{j+1}$ for every $j$, and $\cup O'_j = M'$.

**Remark.** Let $M'$ be a set in $\mathcal{M}$ which is of type $\zeta'$. Then there exists a sequence of functions $H'(\cdot, O'_j)$ defined on $S'$ where $O'_j$ is a set in $\mathcal{O}'$ of type $\gamma'$ such that

$H'(\cdot, O'_j) \leq H'(\cdot, O'_{j+1})$ for each $j$ and $\lim H'(\cdot, O'_j) = H'(\cdot, M')$. 
HB8. If $M'$ is an element of $\mathfrak{M}'$, there is a countable sequence of sets $M'_j$ in $\mathfrak{M}'$ and two sets $U'$ and $V'$ in $\mathfrak{M}'$ such that each $M'_j$ is of type $\mathfrak{z}'$, $M'_j \supseteq M'_{j+1}$ for every $j$, $0 = ||U'|| = ||V'||$ and $\bigcup_j M'_j \cup U' = M' \cup V'$.

Remark. Let $M'$ be a set in $\mathfrak{M}'$. Then there exists a sequence of functions $H'(*,M'_j)$ defined on $S'$ where $M'_j$ is a set in $\mathfrak{M}'$ of type $\mathfrak{z}'$ such that $H'(*,M'_j) \geq H'(*,M'_{j+1})$ for each $j$ and $\lim_{j \to \infty} H'(*,M'_j) = H'(*,M')$ a.e.$\mu'$.

HB9. $W'$ is a non-negative real valued function defined on $S' \times \mathfrak{D}$ and satisfying the following conditions:

i) If $D$ is in $\mathfrak{D}$, then $W'(*,D) = 0$ a.e.$\mu'$ on $S' - TD$.

ii) If $D$ is a set in $\mathfrak{D}$ for which there is a countable number of pairwise disjoint sets $D_i$ in $\mathfrak{D}$ and two subsets $E$ and $F$ satisfying the relations:

\[ UD_i \subseteq D, D - UD_i = E \cup F, ||E|| = 0, \text{ and } ||TF|| = 0, \]

then $W'(*,D) = \sum W'(*,D_i)$ a.e.$\mu'$.

iii) $W'$ is a.e.$\mu'$ inner continuous on $\mathfrak{D}$, that is, if a set $D$ in $\mathfrak{D}$ is the union of a countable number of sets $D_j$...
in $\mathcal{D}$ such that $D_j \subseteq D_{j+1}$ for every $j$, then $\lim W'(\cdot, D_j) = W'(\cdot, D)$ a.e.$\mu'$.

iv) For each $D$ in $\mathcal{D}$, $W'(\cdot, D)$ is measurable $\mu'$.

A function $W'$ having these properties is termed a weight function for the transformation $T$.

1.25 Suppose Reichelderfer's hypotheses $H_1$, $H_2$, and $H_9$ [4] are modified as follows: The completeness condition in $H_1$ and $H_2$ is deleted. $H_9$ is replaced by the statement: $W'$ is a non-negative extended real valued function defined on $S' \times \mathcal{D}$ and satisfying the following conditions:

i) If $D \in \mathcal{D}$, then $W'(\cdot, D) = 0$ a.e.$\mu'$ on $S' - TD$.

ii) For each $D$ in $\mathcal{D}$ such that $D$ contains a countable number of pairwise disjoint sets $D_i$ in $\mathcal{D}$, $\sum W'(\cdot, D_i) \leq W'(\cdot, D)$ a.e.$\mu'$.

iii) If a set $D$ in $\mathcal{D}$ is the union of a countable number of sets $D_j$ in $\mathcal{D}$ such that $D_j \subseteq D_{j+1}$ for every $j$, then $\lim W'(\cdot, D_j) = W'(\cdot, D)$ a.e.$\mu'$.

iv) For each $D$ in $\mathcal{D}$, $W'(\cdot, D)$ is measurable $\mu'$.

In a conversation Reichelderfer stated that if his hypotheses [4] are modified as indicated above, then with minor modifications in the proofs in that paper, the same
results found in his paper [4] which lead to the transformation formula can be established. In view of this remark if the system \((S, \mathcal{M}, \mu), (S', \mathcal{M}', \mu')\), \(T, \mathcal{D}\), and \(W') satisfy HB1 - HB8 where \(\mu\) and \(\mu'\) are non-negative real valued measures, then this system satisfies HL - H8 with the completeness condition deleted in HL and H2. It can be shown [2, lemma 2.6.2] that if \(W\) satisfies HB9, then \(W'\) is underadditive a.e.\(\mu'\), that is whenever a set \(D\) in \(\mathcal{D}\) contains a countable number of pairwise disjoint sets \(D_i\) in \(\mathcal{D}\) we have \[\sum W'(*, D_i) \leq W'(*, D)\) a.e.\(\mu'\).

In view of this, if \(W\) satisfies HB9, then \(W'\) is a weight function in the modified sense of H9 [4].

1.26 Definition. Let \(D\) belong to \(\mathcal{D}\) and let \(O'\) be a set in \(\mathcal{D}'\) which is of type \(\gamma'\). Then by definition 3B there are pairwise disjoint sets \(O' = O'_0, O'_i, i > 0, Y', Z'\) in \(\mathcal{M}'\) whose union is \(S'\) such that \(||T^{-1} Y|| = 0\) and \(||Z'|| = 0\).

For each \(i \geq 0\) there is a countable number of pairwise disjoint sets \(D_{ij}\) in \(\mathcal{D}\) such that \(T^{-1} O'_i = \bigcup_j D_{ij}\). By HB4 it follows that for each pair of integers \(i \geq 0, j \geq 1\) there is a countable number of pairwise disjoint sets \(D_{ijk}\) in \(\mathcal{D}\) such that \(D \cap D_{ijk} = \bigcup_k D_{ijk}\). Put \(E = D \cap T^{-1} Y'\) and \(F = D \cap T^{-1} Z'\). It follows that the sets \(D_{ijk}, i \geq 0, j \geq 1, k \geq 1\) are a countable number of pairwise disjoint sets in \(\mathcal{D}\) which together with \(E\) and \(F\)
satisfy the relations

\[ U_{ijk} D_{ijk} \subseteq D, \quad D - U_{ijk} D_{ijk} = E \cup F; \quad ||E|| = 0 \text{ and } ||F|| = 0. \]

We call the sequence of sets \( \{D_{ijk}\}, \ E, \ F \) an \( O' \) induced partition of \( D \).

**Remark.** If \( W' \) satisfies HB9 and if the sequence \( \{D_{ijk}\}, \ E, \ F \) is an \( O' \) induced partition of a set \( D \) in \( D \), then by condition ii) of HB9 we have

\[ \sum_{ijk} W'(\ast,D_{ijk}) = W'(\ast,D) \text{ a.e.} \mu'. \]

**1.27 Remark.** Assume that \((S,\mathbb{M},\mu), \ (S',\mathbb{M}',\mu'), \ T, \ D, \ \mathcal{B}'\) and \( W' \) is a system satisfying HB1 - HB9. Let us call this system the first system. Let \( \lambda \) and \( \lambda' \) be vector measures (having the same range but not necessarily the range of \( \mu \) and \( \mu' \)) defined on \( \mathbb{M} \) and \( \mathbb{M}' \) respectively. Assume that \( \lambda \) and \( \lambda' \) are continuous relative to \( \mu \) and \( \mu' \) respectively. Now consider the second system \((S,\mathbb{M},\lambda), \ (S',\mathbb{M}',\lambda'), \ T, \ D, \ \mathcal{B}'\) and \( W' \). It easily follows that an element \( D \) in \( D \) which is of type \( \gamma \) in the first system is a set of type \( \gamma \) in the second system. Also an element \( O' \) in \( D' \) which is of type \( \gamma' \) in the first system is a set of type \( \gamma' \) in the second system. The converse of each of these two statements is not necessarily true. Using these facts and the fact that \( \lambda \) and \( \lambda' \) are continuous relative to \( \mu \) and \( \mu' \) respectively, we see that \((S,\mathbb{M},\lambda), \ (S',\mathbb{M}',\lambda'), \ T, \ D, \ \mathcal{B}'\) and \( W' \) satisfy HB1 - HB9.
Also if \( \{D_{i\ell k}\} \), \( E, F \) is an \( O' \) induced partition of a set \( D \) in \( \mathcal{D} \) where \( O' \) is a set of type \( \gamma' \) in the first system, then

\[
\sum_{i\ell k} W'('D_{i\ell k}) = W'('D) \quad \text{a.e.}\lambda'.
\]

The following lemma is an extension of a result found in the proof of theorem 3.2.22 in Linear Operators [3].

1.28 Lemma. Assume that \((S,\mathcal{M},\mu)\) is an \( F \)-vector measure space and that \( f \) is a \( \mathcal{D} \)-valued function defined on \( S \) which is measurable \( \mu \). Then there exists a sequence of simple integrable \( \mu \) functions \( f_n \) which are \( \mathcal{D} \)-valued and are defined on \( S \) such that the \( f_n \) converge in measure \( \mu \) to \( f \) and for each \( n \), \(|f_n| \leq 2|f| \) on \( S \).

Proof. Since \( f \) is measurable \( \mu \), by the definition of a measurable function there exists a sequence of simple integrable \( \mathcal{D} \)-valued functions \( g_n \) defined on \( S \) which converges in measure \( \mu \) to \( f \). There is a subsequence \( \{k_n\} \) such that for every \( n \) there exists an \( A_n \) in \( \mathcal{M} \) and an \( \varepsilon_n \) satisfying the following conditions:

\[
\lim \varepsilon_n = 0, \lim |||A_n||| = 0, \quad |g_{k_n}(s) - f(s)| < \varepsilon_n, \quad s \notin A_n.
\]

Let \( B_n = \{s \in S : s \notin A_n \text{ and } |g_{k_n}(s)| > 2\varepsilon_n \} \). Thus \( B_n \in \mathcal{M} \) for every \( n \). Define \( f_n \) by the equation \( f_n = H('B_n)g_{k_n} \). Thus \( f_n \) is a simple integrable \( \mu \) function defined on \( S \) for each \( n \). If \( s \in B_n \), then \(|f_n(s) - f(s)| < \varepsilon_n \). If \( s \notin A_n \) and \(|g_{k_n}(s)| < \varepsilon_n \)
\[ |f_n(s) - f(s)| < 3 \varepsilon_n, \text{ for } s \notin A_n. \]

Thus, for \( s \in A_n \),

\[ |f_n(s) - f(s)| < 3 \varepsilon_n. \]

Therefore, \( f_n \) converges to \( f \) in measure \( \mu \). Now if \( s \in A_n \) or if \( |g_k(s)| < 2 \varepsilon_n \), then

\[ f_n(s) = 0 \text{ and } |f_n(s)| \leq 2|f(s)|. \]

If \( s \notin A_n \) and \( |g_k(s)| > 2 \varepsilon_n \), then

\[ |f(s)| \geq |g_k(s)| - |g_k(s) - f(s)| > |g_k(s)| - \varepsilon_n > 1/2 |g_k(s)| = 1/2 |f_n(s)|. \]

Hence \( |f_n| \leq 2|f| \) on \( S \) for each \( n \).

**Remark 1.** In view of lemma 1.15 we could also have the condition that \( \lim f_n = f \) a.e.\( \mu \) in the conclusion of the above lemma.

**Remark 2.** Assume \((S,\mathcal{M},\mu)\) is a vector measure space. Let \( f \) and \( g \) be complex valued functions defined on \( S \) such that \( f \) is measurable \( \mu \) and \( g \) is integrable \( \mu \). Suppose \( |f| \leq |g| \) a.e.\( \mu \).

Then using lemma 1.28, the fact that \( |g| \) is integrable \( \mu \), and theorem 1.21, one can easily show that \( f \) is integrable \( \mu \).

The following hypothesis is to hold for sections 1.29-1.32.

Let \((S,\mathcal{M},\mu), (S',\mathcal{M}',\mu')\), \( T, \mathcal{D} \) and \( \mathcal{B} \) satisfy HB1-HB8, where \( \mu \) and \( \mu' \) are \( \mathcal{X} \)-measures. Assume \( \mathcal{G} \) and \( \mathcal{B} \) are \( \mathcal{B} \)-spaces over the complex numbers and either \( \mathcal{G} \) or \( \mathcal{G} \) is the complex number field.

Fix \( D \) in \( \mathcal{D} \). Let \( f \) be a \( \mathcal{G} \)-valued function defined on \( S \) such that

if \( M' \in \mathcal{M}' \) and \( |||M'|||| = 0 \) then \( f = 0 \) a.e.\( \mu \) on \( T^{-1}M' \). In view of theorem 1.8 the semi-variation can be chosen with respect
to $\mathbb{J}$ or $\mathbb{S}$. Let $W'(\cdot, D)$ be a function defined on $S'$ with values in $\mathbb{J}$ such that $W'(\cdot, D) = 0$ a.e.$\mu'$ on $S'$ - TD. Lemmas 1.29-1.31 are proved in the same way Reichelderfer [4] proves lemmas 1.8-1.10 respectively. The proofs will be omitted.

1.29 Lemma. Assume that $H_1$ and $H_2$ are real valued functions defined on $S'$ such that $H_1 \leq H_2$ a.e.$\mu'$ on TD. Then

$$H_1^{|W'(\cdot, D)|} \leq H_2^{|W'(\cdot, D)|} \text{ a.e.$\mu'$},$$

$$H_1^{|Tf|} \leq H_2^{|Tf|} \text{ a.e.$\mu$ on D}.$$ 

1.30 Lemma. Assume $H_j^j, j \geq 0$ is a sequence of functions defined on $S'$ with range $\mathbb{S}$ such that $\lim H_j^j = H_0^j$ a.e.$\mu'$ on TD. Then

$$\lim H_j^{|W'(\cdot, D)|} = H_0^{|W'(\cdot, D)|} \text{ a.e.$\mu'$},$$

$$\lim H_j^{|Tf|} = H_0^{|Tf|} \text{ a.e.$\mu$ on D}.$$ 

1.31 Lemma. Assume $H_1^j$ and $H_2^j$ are functions defined on $S'$ with range $\mathbb{S}$ such that $H_1^j = H_2^j$ a.e.$\mu'$ on TD. Then

$$H_1^{|W'(\cdot, D)|} = H_2^{|W'(\cdot, D)|} \text{ a.e.$\mu'$},$$

$$H_1^{|Tf|} = H_2^{|Tf|} \text{ a.e.$\mu$ on D}.$$ 

The following theorem is similar to theorem 2.13 in Reichelderfer's paper [4] but the classical proof cannot be used because the functions are $B$-valued.
1.32 Theorem. Assume that $f$ and $W'(*,D)$ are measurable $\mu$ and $\mu'$ respectively. Also let $H'$ be a measurable $\mu'$ function defined on $S'$ with range $S$. Then $H'W'(*,D)$ is measurable $\mu'$ and $H'\circ Tf$ is measurable $\mu$ on $D$.

Proof. Since $H'$ is measurable $\mu'$ there exists a sequence of $S$-valued simple integrable $\mu'$ functions $H'_k$ defined on $S'$ such that the sequence converges a.e. $\mu'$ to $H'$. Now each $H'_k$ can be represented in the form $\sum_{i=1}^{n_k} Z_{i,k} H'(*,M_{i,k}'),$ where $M_{i,k}' \subseteq M_{i,k}$ and $Z_{i,k} \in S$ for all $i$ and $k$. Using HB6 for each $i \geq 1, k \geq 1$ there exists a $B_{i,k}'$ in $S'$ such that $B_{i,k}' \subseteq M_{i,k}'$ and $|M_{i,k}' - B_{i,k}'| = 0$. Let $H_k^*$ be the function

$$\sum_{i=1}^{n_k} Z_{i,k} H'(*,B_{i,k}')$$

for every $k$. It then follows that $\lim H_k^* = H'$ a.e. $\mu'$. Again by HB6, $T^{-1}B_{i,k}'$ belongs to $M$. Thus $H_k^*T = \sum_{i=1}^{n_k} Z_{i,k} H(*,T^{-1}B_{i,k}')$ is simple integrable $\mu$. Since $\lim H_k^* = H'$ a.e. $\mu'$ we can apply lemma 1.30 to obtain $\lim H_k^*Tf = H'\circ Tf$ a.e. $\mu$ on $D$. $H_k^*Tf$ is measurable $\mu$ for each $k$ since $f$ is measurable $\mu$. Since the a.e. $\mu$ limit of a sequence of measurable $\mu$ functions
is measurable \( \mu \) it follows that \( H^*T_f \) is measurable \( \mu \) on \( D \). Also since \( H' \) and \( W'(\cdot, D) \) are measurable \( \mu' \) it follows that \( H'W'(\cdot, D) \) is measurable \( \mu' \). Thus the conclusion follows.

**Lemma.** Assume \((S, \mathcal{M}, \mu)\) is an \( \mathbb{F} \)-vector measure space. Let \( x^* \) belong to the conjugate space \( \mathbb{F}^* \). Assume that \( f \) is a complex valued function defined on \( S \) measurable \( \mu \). Then \( f \) is measurable \( x^*\mu \).

**Proof.** Since \( f \) is a measurable \( \mu \) function it is the a.e.\( \mu \) limit of a sequence of simple integrable \( \mu \) complex valued functions \( f_n \). Obviously then \( f_n \) is simple integrable \( x^*\mu \) for each \( n \) and \( \lim f_n = f \) a.e.\( x^*\mu \). Thus \( f \) is measurable \( x^*\mu \).

**Definition.** Assume \( \mathbb{X} \), \( \mathcal{Y} \), and \( \mathbb{B} \) are \( \mathcal{B} \)-spaces over the complex numbers and that the \( \mathbb{F} \)-vector measure spaces \((S, \mathcal{M}, \mu)\) and \((S', \mathcal{M}', \mu')\) together with \( T, \mathcal{D}, \mathbb{B}', \) and \( W' \) satisfy HBL–HB9. Let \( W'(\cdot, D) \) be integrable \( \mu' \) for each \( D \in \mathcal{D} \). Assume \( f \) is a complex valued function defined on \( S \) which is integrable \( \mu \) such that

\[
\int_D f \, d\mu = \int_S W'(\cdot, D) \, d\mu', \quad D \in \mathcal{D}.
\]

Fix \( D \) in \( \mathcal{D} \). Assume there is a bilinear multiplication defined between \( \mathcal{Y} \) and \( \mathbb{X} \) into \( \mathbb{B} \) of the type defined in 1.3. Under these conditions let \( \mathcal{S}'(D, \mathcal{Y}, \mathbb{X}) \) be the class of all \( \mathcal{Y} \)-valued measurable \( \mu' \) functions \( H' \) defined on \( S' \) so that
i) $H'W'(*,D)$ is integrable $\mu'$,

ii) $H'\circ Tf$ is integrable $\mu$ on $D$, and

iii) $\int_D H'\circ Tf \, d\mu = \int_S H'W'(*,D) \, d\mu'$.

1.35 Remark. Obviously $\S'(D,\mathfrak{g},\mathfrak{k})$ is a linear space over the complex numbers, that is if $H_1'$ and $H_2'$ are in $\S'$ and $\alpha_1$ and $\alpha_2$ are complex numbers, then $\alpha_1 H_1' + \alpha_2 H_2'$ is in $\S'$. When $\mathbb{R}^+$ is the range of $f$ and $\mu$ and $\mu'$, and when $\mathbb{R}$ is the range of $H'$ then $\S'(D,\mathfrak{g},\mathfrak{k})$ coincides with Reichelderfer's class $\S'(D)$ [4]. In this paper some of the ranges may be $\mathbb{R}^+$. Even though $\mathbb{R}^+$ is not a B-space we shall permit $\mathbb{R}^+$ to be substituted for $\mathfrak{g}$ or $\mathfrak{k}$.

1.36 Theorem. Let $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{m}$ be B-spaces over the complex numbers and either $\mathfrak{g}$ or $\mathfrak{k}$ is the field of complex numbers. Let $(S,\mathcal{M},\mu)$ and $(S',\mathcal{M}',\mu')$ be two measure spaces and let $D$ belong to $\mathcal{M}$. Let $T$ be a transformation from $S$ into $S'$. Let $f$ and $W'(*,D)$ be functions defined on $S$ and $S'$ respectively and have $\mathfrak{g}$ as their range. Let $f$ be measurable $\mu$ and let $W'(*,D)$ be measurable $\mu'$. Assume that $H'$ is a measurable $\mu'$ function defined on $S'$ with range $\mathfrak{g}$ and there exists a sequence of measurable $\mu'$ functions $H_j'$ defined on $S'$ with range $\mathfrak{g}$ such that $H_j'\circ Tf$ is measurable $\mu$ on $D$ for each $j$. Assume

(1) $\lim_{j} H_j'W'(*,D) = H'W'(*,D)$ a.e.$\mu'$,

(2) $\lim_{j} H_j'\circ Tf = H'\circ Tf$ a.e.$\mu$ on $D$. 
Suppose that there exist functions \( g \) and \( h \) defined on \( S \) and \( S' \) respectively, and having \( \mathfrak{B} \) as their range such that \( g \) is integrable \( \mu \) on \( D \) and \( h \) is integrable \( \mu' \). Assume that the following relations are satisfied for every \( j \):

\[
(3) \quad |H_j^i \mathfrak{W}'(\cdot, D)| \leq |g| \text{ a.e.}\mu'
\]

\[
(4) \quad |H_j^i \cdot T_f| \leq |g| \text{ a.e.}\mu \text{ on } D.
\]

Thus for each \( j \), \( H_j^i \mathfrak{W}'(\cdot, D) \) is integrable \( \mu' \) and \( H_j^i \cdot T_f \) is integrable \( \mu \) on \( D \). Furthermore for each \( j \) assume

\[
(5) \quad \int_D H_j^i \cdot T_f d\mu = \int_S H_j^i \mathfrak{W}'(\cdot, D) d\mu'.
\]

Then \( \int_D H' \cdot T_f d\mu = \int_S H' \mathfrak{W}'(\cdot, D) d\mu' \).

\textbf{Proof.} In view of (1), (2), (3) and (4) we can use the Lebesgue dominated convergence theorem and obtain

\[
\lim \int_D H_j^i \cdot T_f d\mu = \int_D H' \cdot T_f d\mu,
\]

\[
\lim \int_S H_j^i \mathfrak{W}'(\cdot, D) d\mu' = \int_S H' \mathfrak{W}'(\cdot, D) d\mu'.
\]

Using (5) we then have \( \int_D H' \cdot T_f d\mu = \int_S H' \mathfrak{W}'(\cdot, D) d\mu' \).

\textbf{Lemma 1.37.} Assume the system \((S, M, \mu), (S', M', \mu'), T, \mathcal{D}, \) and \( S' \) satisfies HB1-HB8 and \((S, M, \mu) \) and \((S', M', \mu') \) are measure spaces. Let \( \mathcal{X} \) be a \( B \)-space over the complex numbers. Assume \( f \) is a function defined on \( S \) with range \( \mathcal{X} \) that is integrable \( \mu \) such that if \( M' \in \mathcal{M}' \) and \( \nu(M', \mu') = 0 \) then \( f = 0 \) a.e.\( \mu \) on \( T^{-1}M' \).
Let $W'$ be a function defined on $S' \times \mathcal{D}$ with range $\mathcal{E}$ such that if $D_i, \mathcal{D}_i, i \geq 1$, is a sequence of sets in $\mathcal{D}$ where the $D_i$ are pairwise disjoint and $\bigcup \mathcal{D}_i \subseteq D_i$, $\bigcup \mathcal{D}_i = E \cup F$, $|E| = 0$, $|TF| = 0$, then

$$(1) \quad W'(*,D) = \sum \sum W'(*,D_{i_j}) \text{ a.e.} \mu'. $$

Furthermore let $W'(*,D)$ be integrable $\mu'$ for each $D$ in $\mathcal{D}$ such that any series of the type appearing in (1) is termwise integrable. Assume that if $D$ is in $\mathcal{D}$ then $W'(*,D) = 0$ a.e. $\mu'$ on $S' - TD$. Also let $\int_D f d\mu = \int_S W'(*,D) d\mu'$, $D \in \mathcal{D}$. Now let $0'$ be a set in $\mathcal{D}'$ which is of type $\gamma'$ such that $H'(*,0') \text{ TF}$ is measurable $\mu$ on $D$ for each $D$ in $\mathcal{D}$. Then $\int_D H'(*,0') \text{ TF} d\mu = \int_S H'(*,0') W'(*,D) d\mu'$.

**Proof.** We shall use the notation of definition 1.26. Let 

$\{D_{i_jk}\}, E, F$ be an $O'$ induced partition of $D$. Now since

$W'(*,D) = \sum \sum W'(*,D_{i_jk}) \text{ a.e.} \mu', \quad H'(*,0') W'(*,D) = \sum \sum H'(*,0') W'(*,D_{i_jk}) \text{ a.e.} \mu'. \quad \text{Since we have termwise integrability,}$

$$(1) \quad \int_S H'(*,0') W'(*,D) d\mu' = \sum \sum \int_S H'(*,0') W'(*,D_{i_jk}) d\mu' = \sum \sum \int_{TD_{i_jk}} H'(*,0') W'(*,D_{i_jk}) d\mu'. $$
Now for each $i \geq 0$, $TD_{ijk} \subseteq S^i - O'$. If $i > 0$ then $TD_{ijk} \subseteq S^i - O'$ and $H'(\cdot, O') W'(\cdot, D_{ijk}) = 0$ on $TD_{ijk}$, so that

$$\int_{TD_{ijk}} H'(\cdot, O') W'(\cdot, D_{ijk}) d\mu^i = 0,$$

$$i > 0, j > 0, k > 0.$$

On the other hand $TD_{Ojk} \subseteq O'$ and $H'(\cdot, O') W'(\cdot, D_{Ojk}) = W'(\cdot, D_{Ojk})$ on $TD_{Ojk}$, so that

$$\int_{TD_{Ojk}} H'(\cdot, O') W'(\cdot, D_{Ojk}) d\mu^i = \int_{TD_{Ojk}} W'(\cdot, D_{Ojk}) d\mu^i = \int_{D_{Ojk}} f d\mu, j > 0, k > 0.$$

Combining these relations we obtain

$$\int_S H'(\cdot, O') W'(\cdot, D) d\mu = \sum_{jk} \int_{D_{Ojk}} f d\mu.$$

Since $f = 0$ a.e. on $E \cup F$, $\int_{E \cup F} H'(\cdot, O') \cdot Tf d\mu = 0$. Thus

$$\int_D H'(\cdot, O') \cdot Tf d\mu = \sum_{ijk} \int_{D_{ijk}} H'(\cdot, O') \cdot Tf d\mu.$$ If $i > 0$ then

$TD_{ijk} \subseteq S^i - O'$ and $H'(\cdot, O') \cdot Tf = 0$ on $D_{ijk}$ so that

$$\int_D H'(\cdot, O') \cdot Tf d\mu = 0, i > 0, j > 0, k > 0.$$ On the other hand $TD_{Ojk} \subseteq O'$ and $H'(\cdot, O') \cdot Tf = f$ on $D_{Ojk}$ so that
Combining these relations we obtain

\[
\int_D H'(\cdot, O') \cdot T d\mu = \int_{Ojk} f d\mu, \quad j > 0, \; k > 0.
\]

\[
\int_D H'(\cdot, O') \cdot T d\mu = \sum_{jk} \int_{Ojk} f d\mu = \int_S H'(\cdot, O') W'(\cdot, D) d\mu'.
\]
CHAPTER II
THE TRANSFORMATION THEORY
FOR COMPLEX MEASURES

Let \((S, \mathcal{M}, \mu)\) and \((S', \mathcal{M}', \mu')\) be measure spaces. In this chapter the transformation formula will be extended to the case when \(\mu\) and \(\mu'\) are real valued or complex valued measures.

Throughout this chapter we shall assume that these measure spaces together with \(T, \mathcal{D}, \mathcal{B},\) and \(W'\) satisfy \(\text{HB1-HB9}\.\) Then the system \((S, \mathcal{M}, \nu(*, \mu), (S', \mathcal{M}', \nu(*, \mu'))\), \(T, \mathcal{D}, \mathcal{B},\) and \(W'\) satisfies Reichelderfer's hypotheses \(\text{H1-H9} [4]\) modified in the sense described in 1.25 so that all the results leading to the transformation formula in that paper hold in this system (cf. 1.27).

2.1 Definition. The transformation \(T\) is said to be of bounded variation with respect to \(W'\) — briefly \(\text{BVW}'\) — if \(W'(*, S)\) is integrable \(\mu'\).

Remark. \(W'(*, S)\) integrable \(\mu'\) on \(S'\) implies the integrability of \(W'(*, D)\) relative to \(\mu'\) for each \(D\) in \(\mathcal{D}\). Also \(W'(*, S)\) is integrable \(\mu'\) if and only if \(W'(*, S)\) is integrable \(\nu(*, \mu')\). We note that \(T\) is \(\text{BVW}'\) if and only if \(T\) is \(\text{BVW}\) in the sense of Reichelderfer's definition \([4, \text{definition 2.2}]\) where \(W_D = \int_S W'(*, D) d\nu(*, \mu'), \ D \in \mathcal{D}\).
2.2 Definition. Let \( f \) be a complex valued function defined on \( S \) and measurable \( \mu \) such that for some positive number \( \alpha \),

\[
\frac{1}{\alpha} |f| \text{ is a lbfW in the sense of Reichelderfer's definition [4, definition 2.5] where } (S, \mathcal{M}, \nu(\cdot, \mu)) \text{ and } (S', \mathcal{M}', \nu(\cdot, \mu')) \text{ are the measure spaces under consideration. Thus}
\]

\[
\frac{1}{\alpha} \int_D |f| \, d\nu(\cdot, \mu) \leq \int_D W'(\cdot, D) \, d\nu(\cdot, \mu'), \quad D \in \mathcal{D}.
\]

Such an \( f \) is called a lower bound function for the weight function \( W' \) — briefly, a lbfW'.

Remark. If \( \mu \) and \( \mu' \) are non-negative measures and \( f \) is non-negative this definition coincides with Reichelderfer's definition [4, definition 2.5] when \( \alpha = 1 \).

2.3 Lemma. Assume that \( T \) is BVW' and \( f \) is a lbfW'. Let \( M' \) be a set in \( \mathcal{M}' \) such that \( \nu(M', \mu') = 0 \). Then \( f = 0 \) a.e.\( \mu \) on \( T^{-1}M' \).

Proof. From lemma 2.6 in Reichelderfer's paper [1] it follows that \( \frac{1}{\alpha} |f| = 0 \) a.e.\( \nu(\cdot, \mu) \) on \( T^{-1}M' \), which implies \( f = 0 \) a.e.\( \mu \) on \( T^{-1}M' \).

In view of lemma 2.3 and the properties of the functions involved lemmas 2.4, 2.5, 2.6 and theorem 2.7 follow from lemmas 1.29, 1.30, 1.31 and theorem 1.32 respectively. The proofs are omitted.
2.4 Lemma. Assume \( T \) is BVW' and \( f \) is a lbfW'. Fix \( D \) in \( \mathcal{D} \).
Suppose that \( H_1^1 \) and \( H_1^2 \) are real valued functions defined on \( S' \)
and \( H_1^1 \leq H_1^2 \) a.e.\( \mu' \) on \( TD \). Then \( H_1^W'(*)D \leq H_2^W'(*)D \) a.e.\( \mu' \)
and \( H_1^*T|f| \leq H_2^*T|f| \) a.e.\( \mu \) on \( D \).

2.5 Lemma. Assume \( T \) is BVW' and \( f \) is a lbfW'. Fix \( D \) in \( \mathcal{D} \).
Suppose that \( H_j^1 \), \( j \geq 0 \), is a sequence of B-valued functions
defined on \( S' \) such that \( \lim H_j^1 = H_0^1 \) a.e.\( \mu' \) on \( TD \). Then

\[
\lim H_j^W'(*D) = H_0^W'(*D) \quad \text{a.e.}\mu';
\]

\[
\lim H_j^*Tf = H_0^*Tf \quad \text{a.e.}\mu \text{ on } D.
\]

2.6 Lemma. Assume \( T \) is BVW' and \( f \) is a lbfW'. Fix \( D \) in \( \mathcal{D} \).
Suppose that \( H_1^1 \) and \( H_1^2 \) are B-valued functions defined on \( S' \) and
\( H_1^1 = H_1^2 \) a.e.\( \mu' \) on \( TD \). Then \( H_1^W'(*)D = H_2^W'(*)D \) a.e.\( \mu' \) and
\( H_1^*Tf = H_2^*Tf \) a.e.\( \mu \) on \( D \).

2.7 Theorem. Assume \( T \) is BVW' and \( f \) is a lbfW'. Suppose
\( H' \) is a B-valued function defined on \( S' \) and is measurable \( \mu' \).
Then for each \( D \) in \( \mathcal{D} \) \( H'^W'(*)D \) is measurable \( \mu' \) and \( H'^*Tf \) is
measurable \( \mu \) on \( D \).

2.8 Definition. Assume \( T \) is BVW' and that there exists a
complex valued function \( f \) defined on \( S \) which is integrable \( \mu \)
such that
\[ \int_{D} f d\mu = \int_{S} W'(\ast, D)d\mu', \quad D \in \mathcal{D}. \]

Then the transformation $T$ is said to be absolutely continuous with respect to $W'$—briefly $ACW'$, and $f$ is said to be a gauge for $W'$ relative to $\mu$ and $\mu'$—briefly a $g(W',\mu,\mu')$.

**Remark.** It easily follows that if $\mu$ and $\mu'$ are non-negative real valued measures and $T$ is $ACW'$ and $f$ is a $g(W',\mu,\mu')$ then $f \geq 0$ a.e.$\mu$. Thus without loss of generality we could assume in this special case that $f \geq 0$ on $S$. In this case $T$ is $ACW'$ if and only if $T$ is $ACW$ in the sense of Reichelderfer's definition [4, definition 3.4].

### 2.9 Lemma

Assume $T$ is $ACW'$ and $f_1$ and $f_2$ are gauges for $W'$ relative to $\mu$ and $\mu'$. Then $f_1 = f_2$ a.e.$\mu$.

**Proof.** Since $f_1$ and $f_2$ are gauges for $W'$ relative to $\mu$ and $\mu'$, \[ \int_{D} f_1 d\mu = \int_{D} f_2 d\mu', \quad D \in \mathcal{D}. \] Let $h = f_1 - f_2$. Thus \[ \int_{D} h d\mu = 0, \quad D \in \mathcal{D}. \] If it can be shown that $\int_{M} h d\mu = 0$, $M \in \mathcal{M}$, then $h = 0$ a.e.$\mu$ and the conclusion follows. Let $M$ be a set in $\mathcal{M}$ and let $\varepsilon$ be a positive number. Since $h$ is integrable $\mu$, there exists a positive number $\delta$ such that \[ \int_{M_{\varepsilon}} |h| d\nu(\ast, \mu) < \varepsilon \] whenever $M_{\varepsilon} \in \mathcal{M}$ and $\nu(M_{\varepsilon}, \mu) < \delta$. By HB4 there is a countable number of pairwise disjoint sets $D_i$ in $\mathcal{D}$ such that $M \subseteq UD_i$ and $\nu(UD_i - M, \mu) < \delta$. Thus
\[
\int_M h d\mu = \left| \int_{U_1} h d\mu - \int_{U_1 - M} h d\mu \right| = \left| \sum_{D_i} \int_{D_i} h d\mu - \int_{U_1 - M} h d\mu \right|
\]

= \left| \int_{U_1 - M} h d\mu \right| \leq \int_{U_1 - M} |h| d\nu(\cdot, \mu) < \varepsilon.

Since \( \varepsilon \) is an arbitrary positive number, \( \int_M h d\mu = 0 \).

**Remark.** If \( T \) is ACW' and \( f \) is a \( g. (W', \mu, \mu') \) then \( f \) is unique in the sense that any other \( g. (W', \mu, \mu') \) must equal \( f \) almost everywhere relative to \( \mu \).

**2.10 Theorem.** Assume \( T \) is ACW' and \( f \) is a \( g. (W', \mu, \mu') \). Then \( f \) is a \( lbfW' \) and \( \alpha \) may be chosen so that \( \alpha \leq \frac{4}{\varepsilon} \).

**Proof.** Define \( \lambda(M) = \int_M f d\mu, M \in M \). Then \( \lambda \) is a countably additive complex valued set function defined on \( M \). We note that \( \nu(M, \lambda) = \int_M |f| d\nu(\cdot, \mu), M \in M \). Fix \( D \) in \( \mathcal{D} \). Let \( \varepsilon \) be a positive number. Since \( f \) is integrable \( \mu \) there exists a positive number \( \delta \) such that \( \int_M |f| d\nu(\cdot, \mu) < \frac{\varepsilon}{\delta} \) whenever \( M \in M \) and \( \nu(M, \mu) < \delta \).

Since \( \nu(D, \lambda) \leq 4 \sup |\lambda(M)|, M \in M, M \subseteq D \), there exists a set \( M \) in \( M \) such that \( M \subseteq D \) and

\[
(1) \quad \nu(D, \lambda) < 4|\lambda(M)| + \frac{\varepsilon}{2}.
\]

By \( HB^4 \) there exists a countable number of pairwise disjoint sets \( D_i^* \) from \( \mathcal{D} \) such that \( M \subseteq \bigcup D_i^* \) and \( \nu(\bigcup D_i^* - M, \mu) < \delta \). In view of \( HB^4 \) \( D \cap (\bigcup D_i^*) \) can be expressed as the union of a countable
number of pairwise disjoint sets $D_i$ in $D$. Thus

$$UD_i \subseteq D, M \subseteq UD_i, \nu(UD_i - M, \mu) \leq \nu(UD_i^* - M, \mu) < \delta.$$ 

Now $|\lambda(UD_i) - \lambda(M)| = \int_{UD_i - M} |f|d\nu(\cdot, \mu) < \delta.$

Thus

$$|\lambda(M)| \leq |\lambda(UD_i)| + \frac{\delta}{8}.$$ 

Using (1) and (2) we have

(3) $\nu(D, \lambda) \leq 4 \{ |\lambda(UD_i)| + \frac{\delta}{8} \} + \frac{\delta}{2} \leq$

$$\leq 4 \sum |\lambda D_i| + \varepsilon = 4 \sum |\int_{D_i} f d\mu| + \varepsilon =$$

$$= 4 \sum |\int_{S_i} W^*(\cdot, D_i) d\mu'*| + \varepsilon \leq$$

$$\leq 4 \int_{S_i} W^*(\cdot, D_i) d\nu(\cdot, \mu') + \varepsilon =$$

$$= 4 \int_{S_i} W^*(\cdot, D_i) d\nu(\cdot, \mu') + \varepsilon \leq$$

$$\leq 4 \int_{S_i} W^*(\cdot, D) d\nu(\cdot, \mu') + \varepsilon.$$ 

Thus

$$\int_D |f|d\nu(\cdot, \mu) \leq 4 \int_{S_i} W^*(\cdot, D) d\nu(\cdot, \mu') + \varepsilon.$$ 

Since $\varepsilon$ was an arbitrary positive number we have

$$\int_D |f|d\nu(\cdot, \mu) \leq 4 \int_{S_i} W^*(\cdot, D) d\nu(\cdot, \mu').$$

Therefore $f$ is a $\ast fW'$ and $\alpha$ may be chosen so that $\alpha \leq 4.$

2.11 Lemma. Assume $T$ is $ACW'$ and $f$ is a $g.(W', \mu, \mu').$

Fix $D$ in $\mathcal{D}$. Let $H'$ be a real valued function defined on $S'$
for which there exists a sequence of non-negative functions $H_j$ in $S'(D,\mathcal{C},\mathcal{C})$, such that $H_j^i \leq H_{j+1}^i$ a.e.$\mu$ on $T_D$ for each $j$ and $\lim H_j^i = H^i$ a.e.$\mu$. Also assume that $H^iW'('D)$ is integrable $\mu'$ on $S'$ and $H^iTF$ is integrable $\mu$ on $D$. Then $H'$ belongs to $S'(D,\mathcal{C},\mathcal{C})$.

**Proof.** In view of theorem 2.10 and lemmas 2.4 and 2.5 we have $|H_j^iW'(\cdot,D)| \leq |H^iW'(\cdot,D)|$ a.e.$\mu$ for every $j$, $|H_j^iTF| \leq |H^iTF|$ a.e.$\mu$ on $D$ for every $j$, $\lim H_j^iW'(\cdot,D) = H^iW'(\cdot,D)$ a.e.$\mu'$, $\lim H_j^iTF = H^iTF$ a.e.$\mu$ on $D$.

Thus from the properties of the functions involved the hypotheses of theorem 1.36 is satisfied. Thus $H'$ is in $S'(D,\mathcal{C},\mathcal{C})$.

### 2.12 Lemma.** Assume $T$ is ACW and $D \in \mathcal{D}$. Let $H'$ be a real valued function defined on $S'$ for which there exists a sequence $\{H_j^i\}$ of non-negative functions in $S'(D,\mathcal{C},\mathcal{C})$, such that $H_{j+1}^i \leq H_j^i$ a.e.$\mu$ on $T_D$ for each $j$ and $\lim H_j^i = H^i$ a.e.$\mu$. Then $H'$ belongs to $S'(D,\mathcal{C},\mathcal{C})$.

**Proof.** Let $f$ be a $g.(W',\mu,\mu')$. In view of theorem 2.10 and lemmas 2.4 and 2.5 we have $|H_j^iW'(\cdot,D)| \leq |H^iW'(\cdot,D)|$ a.e.$\mu$ for each $j$; $|H_j^iTF| \leq |H^iTF|$ a.e.$\mu$ on $D$ for each $j$,.
From the properties of the functions involved we deduce that $H'W'(\cdot, D)$ is integrable $\mu'$ and $H'\circ Tf$ is integrable $\mu$ on $D$ and the hypotheses of theorem 1.36 is satisfied. Therefore $H'$ is in $\mathcal{Q}'(D, \mathbb{C}, \mathbb{C})$.

2.13 Lemma. Assume $T$ is $ACW'$ and $f$ is a $g.(W', \mu, \mu')$. Let $M'$ be a set in $\mathbb{M}'$. Then for every $D$ in $\mathcal{D}$, $H'(*, M')W'(\cdot, D)$ is integrable $\mu'$ and $H'(\cdot, M')\circ Tf$ is integrable $\mu$ on $D$.

Proof. Since $H'(\cdot, M')$ is a bounded measurable $\mu'$ function and $W'(\cdot, D)$ is integrable $\mu'$ it follows that $H'(\cdot, M')W'(\cdot, D)$ is integrable $\mu'$. By theorems 2.10 and 2.17 $H'(\cdot, M')\circ Tf$ is measurable $\mu$ on $D$. Since $|H'(\cdot, M')\circ Tf| \leq |f|$ it follows that $H'(\cdot, M')\circ Tf$ is integrable $\mu$ on $D$.

2.14 Lemma. Assume $T$ is $ACW'$ and $O'$ is a set in $\mathcal{O}'$ which is of type $\gamma'$. Then for every $D$ in $\mathcal{D}$ the function $H'(\cdot, O')$ is in $\mathcal{Q}'(D, \mathbb{C}, \mathbb{C})$.

Proof. In view of lemma 2.3 and theorems 2.7 and 2.10, and HB9, the hypothesis of lemma 1.37 is satisfied. Thus $H'(\cdot, O')$ belongs to $\mathcal{Q}'(D, \mathbb{C}, \mathbb{C})$ since the integrability requirements are satisfied by lemma 2.13.

2.15 Lemma. Assume $T$ is $ACW'$ and $M'$ belongs to $\mathbb{M}'$ and is of type $\nu'$.
Then for every $D$ in $\mathcal{D}$ the function $H'(\cdot, M')$ is in $S'(D, \mathcal{C}, \mathcal{C})$.

**Proof.** Let $f$ be a $g(W', \mu, \mu')$. By lemma 2.13

$H'(\cdot, M')W'(\cdot, D)$ is integrable $\mu'$ and $H'(\cdot, M')Tf$ is integrable $\mu$ on $D$. By the remark following definition 4B there is a sequence of functions $H'(\cdot, O^j)$ where $O^j$ is a set in $\mathcal{D}'$ of type $\gamma'$ for every $j$ such that $H'(\cdot, O^j) \leq H'(\cdot, O_{j+1}^j)$ for every $j$ and

\[ \lim_{j \to \infty} H'(\cdot, O^j) = H'(\cdot, M'). \]

By lemmas 2.14 and 2.11, $H'(\cdot, M')$ is in $S'(D, \mathcal{C}, \mathcal{C})$.

**2.16 Lemma.** Assume $T$ is ACW' and $M'$ is any set in $\mathcal{M}'$.

Then for every $D$ in $\mathcal{D}$ the function $H'(\cdot, M')$ is in $S'(D, \mathcal{C}, \mathcal{C})$.

**Proof.** Let $f$ be a $g(W', \mu, \mu')$. By lemma 2.13

$H'(\cdot, M')W'(\cdot, D)$ is integrable $\mu'$ and $H'(\cdot, M')Tf$ is integrable $\mu$ on $D$. By the remark following HB8 there exists a sequence of functions $H'(\cdot, M'_j)$ where $M'_j$ is a set of type $\nu'$ for each $j$.

Also $H'(\cdot, M'_j) \geq H'(\cdot, M'_{j+1})$ for each $j$ and $\lim_{j \to \infty} H'(\cdot, M'_j) = H'(\cdot, M')$ a.e.$\mu'$. By lemmas 2.15 and 2.12 $H'(\cdot, M')$ is in $S'(D, \mathcal{C}, \mathcal{C})$.

**2.17 Lemma.** Assume $T$ is ACW'. Let $M'$ be a set in $\mathcal{M}'$ and let $x$ belong to $\mathcal{X}$. Then for every $D$ in $\mathcal{D}$ the function $xH'(\cdot, M')$ is in $S'(D, \mathcal{C}, \mathcal{C})$.

**Proof.** $H'(\cdot, M')$ belongs to $S'(D, \mathcal{C}, \mathcal{C})$ by lemma 2.16.
Thus

(1) \[ \int_S H'(\cdot, M') W'(\cdot, D) d\mu' = \int_D H'(\cdot, M') \circ T d\mu. \]

We obtain by multiplication of both sides of (1) by \( x \),

\[ x \int_S H'(\cdot, M') W'(\cdot, D) d\mu' = x \int_D H'(\cdot, M') \circ T d\mu, \]

or

\[ \int_S x H'(\cdot, M') W'(\cdot, D) d\mu' = \int_D x H'(\cdot, M') \circ T d\mu. \]

Therefore \( H'(\cdot, M') \) belongs to \( S'(D, \mathbb{F}, \mathbb{G}) \).

Remark. By the remark in 1.35 and by lemma 2.17, \( S'(D, \mathbb{F}, \mathbb{G}) \) contains all \( \mathbb{F} \)-valued simple measurable \( \mu' \) functions.

2.18 Theorem. Assume \( T \) is ACW' and let \( f \) be a \( g.(W', \mu, \mu') \). Fix \( D \) in \( D \). Suppose that \( H' \) is an \( \mathbb{F} \)-valued function defined on \( S' \) and is measurable \( \mu' \). Also assume that \( H'W'(\cdot, D) \) is integrable \( \mu' \) and \( H' \circ T \) is integrable \( \mu \) on \( D \). Then \( H' \) is in \( S'(D, \mathbb{F}, \mathbb{G}) \).

Proof. Since \( H' \) is measurable \( \mu' \) we can use lemma 1.28 and the remark following it to obtain a sequence of simple measurable \( \mu' \) functions \( H'_j \) defined on \( S' \) such that \( \lim_j H'_j = H' \) a.e.\( \mu' \) and \( |H'_j| \leq 2|H'| \) for each \( j \). Using lemma 2.5 and theorem 2.10 we can obtain

\[ |H'_j W'(\cdot, D)| \leq 2|H'W'(\cdot, D)| \text{ for each } j, \]

\[ |H'_j \circ T f| \leq 2|H' \circ T f| \text{ on } D \text{ for each } j, \]

\[ \lim_j H'_j W'(\cdot, D) = H'W'(\cdot, D) \text{ a.e.} \mu', \text{ and} \]

\[ \lim_j H'_j \circ T f = H' \circ T f \text{ a.e.} \mu \text{ on } D. \]

Thus from the above remark in 2.17 and theorem 2.7 and the
properties of the functions involved the hypotheses of theorem 1.36 is satisfied and consequently \( H' \) belongs to \( \mathcal{S}'(D, x, \mathcal{C}) \).

Remark. In sections 2.19 - 2.23 we shall assume \( T \) is ACW', \( f \) is a \( g(W', \mu', \mu') \), and \( D \) is a fixed set in \( \mathcal{D} \).

2.19 Definition. Let \( \mathcal{S}'(D) \) denote the set of non-negative measurable \( \mu' \) functions \( H' \) defined on \( S' \) such that

\[
\int_D H' \circ T |f| \, \text{d}v(\cdot, \mu) \leq 4 \int_S H'W'(\cdot, D) \, \text{d}v(\cdot, \mu'),
\]

where the integrals may or may not be finite.

2.20 Lemma. Let \( O' \) be a set in \( \mathcal{O}' \) which is of type \( \gamma' \). Then \( H'(\cdot, O') \) is in \( \mathcal{S}'(D) \).

Proof. The proof is similar to the proof of lemma 1.37 where \( \mu \) and \( \mu' \) are replaced by \( \nu(\cdot, \mu) \) and \( \nu(\cdot, \mu') \) respectively and \( |f| \) takes the place of \( f \). Certain equalities in that proof are replaced by inequalities. Reasoning in the proof of lemma 1.37 leading to (4) yields

\[
\int_S H'(\cdot, O') W'(\cdot, D) \, \text{d}v(\cdot, \mu') = \sum \int_{T_{D_{0jk}}} W'(\cdot, D_{0jk}) \, \text{d}v(\cdot, \mu').
\]

But since \( f \) is a \( \text{lbf} W' \) by theorem 2.10 we have

\[
\int_{D_{0jk}} |f| \, \text{d}v(\cdot, \mu) \leq 4 \int_{T_{D_{0jk}}} W'(\cdot, D_{0jk}) \, \text{d}v(\cdot, \mu').
\]

Thus

\[
\sum \int_{D_{0jk}} |f| \, \text{d}v(\cdot, \mu) \leq 4 \int_S H'(\cdot, O') W'(\cdot, D) \, \text{d}v(\cdot, \mu').
\]
By an argument used in the above mentioned proof we obtain
\[ \int_D H'(\cdot, O_j) \cdot T|f| \, dv(\cdot, \mu) = \sum \int_D |f| \, dv(\cdot, \mu). \]
Thus \[ \int_D H'(\cdot, O_j) \cdot T|f| \, dv(\cdot, \mu) \leq 4 \int_D H'(\cdot, O_j) W'(\cdot, D) \, dv(\cdot, \mu') \]
and \( H'(\cdot, O_j) \) is in \( C'(D) \).

2.21 Lemma. Let \( M' \) be a set in \( M' \) of type \( \gamma' \). Then \( H'(\cdot, M') \) is in \( C'(D) \).

Proof. By the remark following definition \ref{definition} there is a sequence of functions \( H'(\cdot, O_j) \) defined on \( S' \) where \( O_j \) is a set in \( \mathcal{S}' \) of type \( \gamma' \) for every \( j \). Also \( H'(\cdot, O_j) \leq H'(\cdot, O_{j+1}) \) for every \( j \) and \( \lim_{j \to \infty} H'(\cdot, O_j) = H'(\cdot, M') \). Using lemmas \ref{lemma2.4}, \ref{lemma2.5}, theorem \ref{theorem2.7}, lemma \ref{lemma2.20}, and the Lebesgue monotone theorem we have
\[ \int_D H'(\cdot, M') \cdot T|f| \, dv(\cdot, \mu) = \lim \int_D H'(\cdot, O_j) \cdot T|f| \, dv(\cdot, \mu) \leq \]
\[ \leq 4 \lim_{j \to \infty} \int_S H'(\cdot, O_j) W'(\cdot, D) \, dv(\cdot, \mu') = \]
\[ = 4 \int_S H'(\cdot, M') W'(\cdot, D) \, dv(\cdot, \mu'). \]
Thus \( H'(\cdot, M') \) is in \( C'(D) \).

2.22 Lemma. Let \( M' \) be a set in \( M' \). Then \( H'(\cdot, M') \) is in \( C'(D) \).

Proof. By the remark following \ref{remark} there is a sequence of functions \( H'(\cdot, M_j) \) defined on \( S' \) where \( M_j \) is a set of type \( \gamma' \).
for each $j$ and $H'(\cdot, M_{j+1}) \geq H'(\cdot, M_j)$.

Also $\lim H'(\cdot, M_j) = H'(\cdot, M')$ a.e. $\mu$. Using lemmas 2.4, 2.5, theorem 2.7, lemma 2.22, and the Lebesgue dominated convergence theorem we have

$$\int_D H'(\cdot, M_j) f \, dv(\cdot, \mu) = \lim_{j \to \infty} \int_D H'(\cdot, M_j) f \, dv(\cdot, \mu) \leq \lim_{j \to \infty} \int_D H'(\cdot, M'_{j+1}) f \, dv(\cdot, \mu) =$$

$$= \int_D H'(\cdot, M') W'(\cdot, D) \, dv(\cdot, \mu').$$

Thus $H'(\cdot, M')$ is in $\mathcal{C}'(D)$.

**Remark.** If $H_1'$ and $H_2'$ are in $\mathcal{C}'(D)$ and $\alpha_1$ and $\alpha_2$ are two non-negative real numbers it follows that $\alpha_1 H_1' + \alpha_2 H_2'$ is in $\mathcal{C}'(D)$. Thus by lemma 2.22 and the above statement $\mathcal{C}'(D)$ contains all simple measurable $\mu'$ non-negative functions defined on $S'$.

**2.23 Lemma.** Let $H'$ be a non-negative function defined on $S'$ which is measurable $\mu'$. Then $H'$ belongs to $\mathcal{C}'(D)$.

**Proof.** Since $H'$ is measurable $\mu'$ and non-negative there exists a sequence of non-negative simple measurable $\mu'$ functions $H_j'$ defined on $S'$ such that $H_j' \leq H_{j+1}'$ for every $j$ and $\lim H_j' = H'$. By the above remark $H_j'$ belongs to $\mathcal{C}'(D)$ for every $j$. Using theorem 2.7 and the Lebesgue monotone theorem we have
Thus $H'$ is in $\mathcal{C}'(D)$.

**2.24 Theorem.** Assume $T$ is $ACW'$ and $f$ is a.a.$(W',\mu,\mu')$. Fix $D$ in $\mathcal{D}$. Let $H'$ be an $\mathbb{I}$-valued function defined on $S'$ and measurable $\mu'$. Also let $H'W'(\cdot,D)$ be integrable $\mu'$. Then $H'\circ Tf$ is integrable $\mu$ on $D$.

**Proof.** By theorem 2.27 $H'\circ Tf$ is measurable $\mu$ on $D$. It suffices to prove that $|H'\circ Tf|$ is integrable $\nu(\cdot,\mu)$ on $D$. Now since $|H'|$ is a non-negative function defined on $S'$ which is measurable $\mu'$, by lemma 2.23 $|H'|$ is in $\mathcal{C}'(D)$. Using this and the integrability of $|H'|W'(\cdot,D)$ relative to $\nu(\cdot,\mu')$ we have

$$\int_D |H'\circ Tf| \, d\nu(\cdot,\mu) \leq 4 \int_D |H'|W'(\cdot,D) \, d\nu(\cdot,\mu') < \infty.$$ 

Thus $|H'\circ Tf|$ is integrable $\nu(\cdot,\mu)$ on $D$.

**2.25 Theorem.** Assume $T$ is $ACW'$ and $f$ is a.a.$(W',\mu,\mu')$. Fix $D$ in $\mathcal{D}$. Suppose $H'$ is an $\mathbb{I}$-valued function defined on $S'$ and is measurable $\mu'$ and $H'W'(\cdot,D)$ is integrable $\mu'$. Then $H'$ is in $\mathcal{C}'(D,\mathbb{I},\mathcal{C})$.

**Proof.** By theorem 2.24 $H'\circ Tf$ is integrable $\mu$ on $D$. Then by theorem 2.18 $H'$ is in $\mathcal{C}'(D,\mathbb{I},\mathcal{C})$.

**2.26** In the remaining sections of this chapter it will
be assumed that \( \mu \) and \( \mu' \) are non-negative real valued measures, 
\( T \) is \( \text{ACW}' \), and \( f \) is a non-negative real valued \( g.(W',\mu,\mu') \).
Thus \( f \) is a \( \text{glbfW} \) in the sense of Reichelderfer's definition 
\([4, \text{definition 3.4}]\). Thus the system \((S,W,\mu), (S',W',\mu')\), \( T, D, \)
\( S' \), and \( W' \) satisfies Reichelderfer's \( E1-H9 \) \([1]\) modified in the 
sense described in 1.25 so that the results leading to the 
transformation formula in his paper hold.

2.27 Lemma. Assume that \( H' \) is an \( \mathbb{Z} \)-valued function de­

defined on \( S' \) and is measurable \( \mu' \). Fix \( D \) in \( D \). Then \( H'W'((\cdot,D) \)
is integrable \( \mu' \) if and only if \( H'\cdot Tf \) is integrable \( \mu \) on \( D \).

Proof. Assume \( H'W'((\cdot,D) \) is integrable \( \mu' \). By theorem
2.24 \( H'\cdot Tf \) is integrable \( \mu \) on \( D \). Assume now that \( H'\cdot Tf \) is 
integrable \( \mu \) on \( D \). Now \( |H'| \) is a non-negative real valued 
function which is measurable \( \mu' \) and \( |H'|\cdot Tf \) is integrable \( \mu \) 
on \( D \). Therefore we have by a theorem of Reichelderfer \([4, 
\text{theorem 4.10}]\) that \( |H'| \) is in \( \mathcal{Q}'(D,D',\mathbb{R}^+) \); this then implies 
that \( |H'|W'((\cdot,D) \) is integrable \( \mu' \). Using theorem 2.7 we have
\( H'W'((\cdot,D) \) is integrable \( \mu' \) on \( S' \).

2.28 Theorem. Assume \( T \) is \( \text{ACW}' \) and \( f \) is a \( g.(W',\mu,\mu') \),
where \( \mu \) and \( \mu' \) are non-negative measures. Fix \( D \) in \( D \). Let
\( H' \) be an \( \mathbb{Z} \)-valued function defined on \( S' \) and measurable \( \mu' \) for
which either \( H'W'((\cdot,D) \) is integrable \( \mu' \) or \( H'\cdot Tf \) is integrable 
\( \mu \) on \( D \). Then \( H' \) is in \( \mathcal{Q}'(D,D',\mathbb{R}^+) \).

Proof. The result follows from lemma 2.27 and theorem 2.18.
CHAPTER III

THE TRANSFORMATION THEORY

FOR VECTOR MEASURES

In this chapter the transformation theory will be extended to the case when \((S,\mathcal{M},\mu)\) and \((S',\mathcal{M}',\mu')\) are vector measure spaces and \(\mu\) and \(\mu'\) have the B-space \(\mathbb{Z}\) as their range. It will be assumed in this chapter that the conjugate space \(\mathbb{Z}^*\) of \(\mathbb{Z}\) is separable. Also throughout this chapter we shall assume that \((S,\mathcal{M},\mu)\), \((S',\mathcal{M}',\mu')\), \(T, \mathcal{D}, \mathcal{B}'\) and \(\mathcal{W}'\) satisfy H1-89.

3.1 Remark. Let \(x^* \in \mathbb{Z}^*\). Since \(\nu(x^*,x^*\mu) \leq |x^*| ||\mu||(\cdot)\) on \(\mathbb{M}\), \(x^*\mu\) is continuous relative to \(\mu\). Likewise \(x^*\mu'\) is continuous relative to \(\mu'\). In view of remark 1.27, the system \((S,\mathcal{M},x^*\mu)\), \((S',\mathcal{M}',x^*\mu')\), \(T, \mathcal{D}, \mathcal{B}'\) and \(\mathcal{W}'\) satisfies H1-89 and all the results established in Chapter II hold for this system.

This fact will be used throughout this chapter.

3.2 Definition. The transformation \(T\) is said to be of bounded variation with respect to \(\mathcal{W}'\) — briefly \(BV\mathcal{W}'\), if \(\mathcal{W}'(\cdot, S)\) is integrable \(\mu'\).

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Remark. Using remark 2 in 1.28 we see that if $T$ is $\text{BVW}'$ then $W'(\cdot, D)$ is integrable $\mu'$ for every $D \in \mathcal{D}$. This definition is consistent with definition 2.1 when $\mathcal{I} = \mathcal{C}$. By lemma 1.22 it follows that if $T$ is $\text{BVW}'$ then $T$ is $\text{BVW}'$ in the sense of definition 2.1 when $x^\circ \mu$ and $x^\circ \mu'$ are the measures considered, where $x^\circ \in \mathcal{I}^\circ$.

3.3 Definition. Assume $T$ is $\text{BVW}'$ and that there exists a complex valued function $f$ defined on $\mathcal{S}$ which is integrable $\mu$ such that
\[ \int_D f \, d\mu = \int_D W'(\cdot, D) \, d\mu', \quad D \in \mathcal{D}. \]
Then the transformation $T$ is said to be absolutely continuous with respect to $W'$—briefly $\text{ACW}'$, and $f$ is said to be a gauge for $W'$ relative to $\mu$ and $\mu'$—briefly $g_f(W', \mu, \mu')$.

Remark. This definition is consistent with definition 2.8 when $\mathcal{I} = \mathcal{C}$. By lemma 1.22 if $f$ and $W'(\cdot, D)$, $D \in \mathcal{D}$, are integrable $\mu$ and $\mu'$ respectively then they are integrable $x^\circ \mu$ and $x^\circ \mu'$ respectively for every $x^\circ \in \mathcal{I}^\circ$.

3.4 Lemma. Assume $T$ is $\text{ACW}'$ and $f$ is a $g_f(W', \mu, \mu')$. Let $x^\circ$ belong to $\mathcal{I}^\circ$. Then $f$ is a $g_f(W', x^\circ \mu, x^\circ \mu')$ in the sense of definition 2.8.
Proof. Since $f$ is a $g^* (W', \mu_1, \mu')$ we have

$$\int_D f \, d\mu = \int_{g^*} W' (\cdot, D) \, d\mu', \quad D \in \mathcal{D}.$$  

Then

$$x^* \int_D f \, d\mu = x^* \int_{g^*} W' (\cdot, D) \, d\mu', \quad D \in \mathcal{D}.$$  

By lemma 1.22

$$\int_D f \, d(x^* \mu) = \int_{g^*} W' (\cdot, D) \, d(x^* \mu'), \quad D \in \mathcal{D}.$$  

Thus $f$ is a $g^* (W', x^* \mu, x^* \mu')$.

3.5 Lemma. Assume $T$ is ACW and $f$ is a $g^* (W', \mu_1, \mu')$. Let $x^*$ belong to $E^*$. Let $M'$ be a set in $M'$ such that $||M'|| = 0$.

Then $f = 0$ a.e. $x^* \mu$ on $T^{-1} M'$.

Proof. By lemma 3.4 $f$ is a $g^* (W', x^* \mu, x^* \mu')$. Also if $||M'|| = 0$ then $\nu(M', x^* \mu') = 0$. Then by theorem 2.10 and lemma 2.3 the result follows.

3.6 Theorem. Assume $(S, M, \mu)$ is an $E$-vector measure space and that the conjugate space $E^*$ of $E$ is separable. Let $E$ be a subset of $S$ such that $E$ is $x^* \mu$ null for every $x^*$ in $E^*$. Then $E$ is $\mu$-null.

Proof. Let $S^o = \{x^* \in E^* : |x^*| \leq 1\}$. For each $x^*$ in $S^o$ there exists a set $S_{x^*}$ such that
Let the sequence \( \{x_n^s\} \) be dense in \( S^s \). Then for every \( n \) there exists an \( S_n \) with the properties listed in (1). Let \( S = \cap S_n \).

Thus \( S \subseteq S_n \) for every \( n \), \( S \subseteq \mathbb{R} \), and \( E \subseteq S \). The theorem will be proved if \( |S| = 0 \). Recall that

\[
(2) \quad |S| = \sup \left\{ \sum_{i=1}^{k} |\alpha_i\mu(M_i)| \right\},
\]

where the supremum is extended over all partitions of \( M \) into a finite number of disjoint sets \( M_i \) in \( \mathbb{R} \) and all finite collections of complex numbers \( \alpha_i \) with \( |\alpha_i| \leq 1 \). Let \( \sum_{i=1}^{k} \alpha_i\mu(M_i) \) be a sum of the type appearing in (2). We assert that \( \sum_{i=1}^{k} \alpha_i\mu(M_i) = 0 \).

Let \( \varepsilon \) be a positive number and let \( x^s \) belong to \( S^s \). Then there exists a subsequence \( \{x_n^s\}_1^k \) of \( \{x_n^s\} \) such that \( \lim_{n \to \infty} x_n^s = x^s \).

Since \( \lim_{n \to \infty} x_n^s = x^s \), for every \( i, 1 \leq i \leq k \), there exists an \( N_i \) such that

\[
|x_{n_i}^s\mu(M_i) - x^s\mu(M_i)| < \frac{\varepsilon}{k}, \quad j \geq N_i.
\]

Let \( N = \max \{n_{N_i} \}_{i=1}^{k} \). Thus
\[
\left| x^* \sum_{i=1}^{k} \alpha_i \mu(M_i) \right| = \left| \sum_{i=1}^{k} \alpha_i x^* \mu(M_i) \right| \leq \sum_{i=1}^{k} \left| x^* \mu(M_i) \right| \leq \\
\leq \sum_{i=1}^{k} \left| x^*_N \mu(M_i) \right| + \varepsilon \leq v(S, x^*_N) + \varepsilon.
\]

Since \( S \subseteq S_N \) and \( v(S_N, x^*_N) = 0 \) we have \( v(S, x^*_N) = 0 \). Thus

\[
|x^* \sum_{i=1}^{k} \alpha_i \mu(M_i)| < \varepsilon. \quad \text{Since} \ \varepsilon \ \text{is an arbitrary positive number,}
\]

\[
(3) \quad |x^* \sum_{i=1}^{k} \alpha_i \mu(M_i)| = 0.
\]

We conclude that \( \sum_{i=1}^{k} \alpha_i \mu(M_i) = 0 \) since (3) holds for every \( x^* \) in \( S^* \). Thus \( ||S|| = 0 \).

3.7 Lemma. Assume \( T \) is ACW and \( f \) is a \( g(W', \mu, \mu') \). Let \( M' \) be a set in \( W' \) such that \( ||M'|| = 0 \). Then \( f = 0 \) a.e. on \( T^{-1}M' \).

Proof. Let \( x^* \) belong to \( S^* \). Then by lemma 3.5 \( f = 0 \) a.e. on \( T^{-1}M' \). Then by theorem 3.6 \( f = 0 \) a.e. on \( T^{-1}M' \).

3.8 Theorem. Assume \( T \) is ACW' and \( f \) is a \( g(W', \mu, \mu') \). Fix \( D \) in \( D \). Let \( H' \) be a complex valued function defined on \( S' \) which is measurable \( \mu' \). Then \( H'W'(\cdot, D) \) is measurable \( \mu' \) and \( H'\circ Tf \) is measurable \( \mu \) on \( D \).
Proof. Using lemma 3.7 and the properties of the functions involved, the hypothesis of theorem 1.32 is satisfied. Consequently the result follows.

3.9 Lemma. Assume $T$ is $ACW'$ and $f_1$ and $f_2$ are gauges for $W'$ relative to $\mu$ and $\mu'$. Then $f_1 = f_2$ a.e. on $S$.

Proof. By lemma 3.4 $f_1$ and $f_2$ are gauges for $W'$ relative to $x^*\mu$ and $x^*\mu'$ for every $x^*$ in $\mathbb{X}^*$. Thus by lemma 2.9 $f_1 = f_2$ a.e. $x^*\mu$, $x^* \in \mathbb{X}^*$. Using theorem 3.6 we have $f_1 = f_2$ a.e. $\mu$.

Remark. If $T$ is $ACW'$ and $f$ is a $g_*(W',\mu,\mu')$ then $f$ is unique in the sense that any other $g_*(W',\mu,\mu')$ must equal $f$ almost everywhere on $S$ relative to $\mu$.

3.10 Theorem. Assume $T$ is $ACW'$ and $f$ is a $g_*(W',\mu,\mu')$. Fix $D$ in $\mathcal{D}$. Let $H'$ be a complex valued function defined on $S'$ which is measurable $\mu'$ such that $H'W'(\cdot,D)$ is integrable $\mu'$ and $H'^*Tf$ is integrable $\mu$ on $D$. Then $H'$ belongs to $\mathcal{S}'(D,\mathbb{C},\mathbb{X})$.

Proof. Let $x^*$ belong to $\mathbb{X}^*$. Since $H'W'(\cdot,D)$ is integrable $x^*\mu'$ and $H'^*Tf$ is integrable $x^*\mu$ on $D$, using lemma 1.33, $H'$ belongs to $\mathcal{S}'(D,\mathbb{C},\mathbb{X})$ by theorem 2.18. Consequently

$$\int_D H'^*Tf \, d(x^*\mu) = \int_{S'} H'W'(\cdot,D) \, d(x^*\mu').$$

Thus using lemma 1.22
Since (1) holds for every $x^*$ in $\mathbb{Z}^*$ we have
\[ \int_D H^* T f \, d\mu = \int_S H' W'(*) \, d\mu'. \]

Therefore the result follows.

3.11 Definition. Let $S$ be a non-empty set and $\mathcal{M}$ a $\sigma$-field of subsets of $S$. Let $Q$ be a non-empty set of complex valued measures defined on $\mathcal{M}$. Let $h$ be a complex valued function defined on $S$ such that $h$ is integrable $\mu$ for each $\mu$ in $Q$. Then $h$ is said to be integrably uniformly absolutely continuous (i.u.a.c.) with respect to $Q$ if for every positive number $\varepsilon$ there exists a positive number $\delta$ such that

\[ \int_M |h| \, dv(*) \mu < \varepsilon, \quad \mu \in Q, \]

whenever $M \in \mathcal{M}$ and $v(M,\mu) < \delta$ for every $\mu$ in $Q$.

3.12 Lemma. Assume $T$ is ACM$^*$ and $f$ is a $g_*(W',\mu,\mu')$. Fix $D$ in $\mathcal{D}$ and let $x^*$ belong to $\mathbb{Z}^*$. Assume $H'$ is a complex valued function defined on $S'$ measurable $\mu'$ such that $H' W'(*)$ is integrable $\mu'$. Then $H^* T f$ is integrable $x^* \mu$ on $D$.

Proof. Using lemmas 1.33, 1.22 and 3.4 we see that $H'$ is measurable $x^* \mu'$, $H' W'(*)$ is integrable $x^* \mu'$, and $f$ is a
Thus the hypotheses of theorem 2.24 is satisfied. Consequently $H' \cdot T_\delta$ is integrable $x^* \mu$ on $D$.

**3.13 Theorem.** Assume $(S, \mathcal{M}, \mu)$ is an $\mathcal{F}$-vector measure space and $S^* = \{x^* \in \mathcal{X}^* : |x^*| \leq 1\}$ where $\mathcal{X}^*$ is the conjugate space of $\mathcal{X}$. Let $h$ be a complex valued function defined on $S$ which is measurable $\mu$. Then $h$ is integrable $\mu$ if and only if $h$ is i.u.a.c. with respect to $\{x^* \mu : x^* \in S^*\}$.

**Proof.** Assume $h$ is integrable $\mu$. Then

$$|x^* \sum_M h \mu| \leq |\sum_M h \mu|, \quad x^* \in S^*, \quad M \in \mathbb{M}.$$  

Using lemma 1.22 we have

(1) $$|\sum_M h d(x^* \mu)| \leq |\sum_M h \mu|, \quad x^* \in S^*, \quad M \in \mathbb{M}.$$  

Let $\varepsilon$ be a positive number. Since $h$ is integrable $\mu$ there exists a positive number $\delta_\perp$ such that

(2) $$|\sum_M h \mu| < \frac{\varepsilon}{4},$$

whenever $M \in \mathbb{M}$ and $||M|| \leq \delta_\perp$. It can be shown using theorem 4.9.2 and corollary 4.9.3 in *Linear Operators* [3] that there exists a control measure $\lambda$ for $\mu$ such that

(3) $$\lambda(M) \leq \sup_{x^* \in S^*} |x^* \mu(M)|, \quad M \in \mathbb{M}.$$  

Since $\lambda$ is a control measure for $\mu$ there exists a positive number $\delta$ such that $||M|| \leq \delta_\perp$ whenever $M \in \mathbb{M}$ and $\lambda(M) \leq \delta.$
Using (3) we have

\[(4) \lambda(M) \leq \sup_{x^* \in S^*} |x^* \mu(M)| \leq \sup_{x^* \in S^*} \nu(M, x^* \mu), \quad M \in \mathbb{M}.\]

Thus using (1), (2), and (4) we have

\[\int_M |h| d\nu(x^*, x^* \mu) \leq \varepsilon,\]

whenever \( M \in \mathbb{M} \) and \( \nu(M, x^* \mu) \leq \delta \) for all \( x^* \) in \( S^* \).

Therefore \( h \) is i.u.a.c. with respect to \( \{x^* \mu : x^* \in S^*\} \).

Conversely suppose \( h \) is i.u.a.c. with respect to \( \{x^* \mu : x^* \in S^*\} \). Since \( h \) is measurable \( \mu \), by lemma 1.28 and remark following it there exists a sequence of simple integrable \( \mu \) complex valued functions \( h_n \) defined on \( S \) such that \( |h_n| \leq 2|h| \) on \( S \) and \( \lim n h_n = h \) a.e. \( \mu \). Let \( \varepsilon \) be a positive number. Using lemma 1.22 it follows that

\[(5) \quad |x|^n \int \overline{h_n} |\overline{d\mu} = |\int \overline{h_n} |\overline{d(x^* \mu)}| \leq \int \overline{h_n} |\overline{d\nu(x^*, x^* \mu)}| \leq 2 \int \overline{h}|\overline{d\nu(x^*, x^* \mu)}, \quad M \in \mathbb{M}, \quad n \geq 1, \quad x^* \in S^*.\]

Since \( h \) is i.u.a.c. with respect to \( \{x^* \mu : x^* \in S^*\} \), there exists a positive number \( \delta \) such that

\[(6) \quad \int |\overline{h}| \overline{d\nu(x^*, x^* \mu)} < \frac{\varepsilon}{2},\]

whenever \( M \in \mathbb{M} \) and \( \nu(M, x^* \mu) \leq \delta \) for every \( x^* \) in \( S^* \). But if \( M \in \mathbb{M} \) and \( ||M|| \leq \delta \) then

\[(7) \quad \nu(M, x^* \mu) \leq |x^*||M|| \leq ||M|| < \delta, \quad x^* \in S^*.\]

Thus using (5), (6), and (7) we have...
whenever $M \in \mathbb{R}$, $x^* \in X^*$, and $|M| < \delta$. Since (8) holds for every $x^*$ in $S^*$ we have

$$| \int_{M} h_n \, d\mu | \leq \varepsilon, \quad n \geq 1,$$

whenever $M \in \mathbb{R}$ and $|M| < \delta$. By theorem 1.19 it follows that $h$ is integrable $\mu$.

**3.14 Theorem.** Assume $T$ is ACW' and $f$ is a $g.(W',\mu,\mu')$.

Let $H'$ be a complex valued function defined on $S'$ which is measurable $\mu'$ such that $H'W'(\cdot,D)$ is integrable $\mu'$ where $D$ is a fixed set in $\mathcal{D}$. Let $S^*$ be the closed unit sphere in $\mathbb{X}^*$. By lemma 3.12 $H'^{o}Tf$ is integrable $x^*\mu$ on $D$ for each $x^*$ in $S^*$.

Suppose that $H'^{o}Tf$ is i.u.a.c. with respect to $(x^*\mu : x^* \in S^*)$ on $D$. Then $H'$ belongs to $\mathcal{S}(D,\mathcal{S},\mathcal{X})$.

**Proof.** $H'^{o}Tf$ is integrable $\mu$ on $D$ by theorems 3.8 and 3.13. Thus $H'$ belongs to $\mathcal{S}(D,\mathcal{S},\mathcal{X})$ by theorem 3.10.
CHAPTER IV
THE TRANSFORMATION THEORY FOR THE BILINEAR INTEGRAL

Assume throughout this chapter that \( \mathcal{J} \), \( \mathcal{I} \), and \( \mathcal{B} \) are \( B \)-spaces over the complex numbers. Let the conjugate space \( \mathcal{I}^* \) of \( \mathcal{I} \) be separable. Let \( \rho \) be a bilinear map from \( \mathcal{J} \times \mathcal{I} \) into \( \mathcal{B} \) such that properties i), ii), and iii) in 1.3 are satisfied. The \( \mathcal{I} \)-vector measure spaces \((\mathcal{S}, \mathcal{M}, \mu)\) and \((\mathcal{S}', \mathcal{M}', \mu')\) together with \( \mathcal{T}, \mathcal{D}, \mathcal{B}', \) and \( \mathcal{W}' \) satisfy HB1 - HB9. Also assume that \( \mu \) and \( \mu' \) have the \( * \)-property relative to \( \mathcal{J} \). We will show that the transformation theory can be extended to the case when \( \mathcal{H}' \) has \( \mathcal{J} \) as its range and \( \mathcal{H}' \) satisfies suitable hypotheses.

4.1 Theorem. Assume \( \mathcal{T} \) is \( ACW' \) and \( f \) is a \( g.(W', \mu, \mu') \).
Let \( M' \) belong to \( \mathcal{M}' \) such that \( ||M'|| = 0 \) where the semi-variation is with respect to \( \mathcal{J} \). Then \( f = 0 \) a.e.\( \mu \) on \( T^{-1}M' \).

Proof. We have by theorem 1.8 that \( ||M'|| = 0 \). In view of lemma 3.7 and theorem 1.8 again, it follows that \( f = 0 \) a.e.\( \mu \) on \( T^{-1}M' \).

4.2 Lemma. Assume \( \mathcal{T} \) is \( ACW' \) and \( f \) is a \( g.(W', \mu, \mu') \).
Let \( \mathcal{H}' \) be a function defined on \( \mathcal{S}' \) with range \( \mathcal{J} \) such that \( \mathcal{H}' \) is measurable \( \mu' \). Fix \( D \) in \( \mathcal{D} \). Then \( \mathcal{H}'W'(\cdot, D) \) is measurable.
Proof. In view of theorem 4.1 and theorem 1.32 the result follows.

4.3 Lemma. Assume T is ACW'. Let M' be a set in W' and let y belong to J. Then for every D in S, the function yH'(*,M') is in S'(D,J,Z).

Proof. Let f be a g*(W',μ',μ'). By lemma 4.2 and remark 2 in 1.28 we have H'(*,M') W'(*,D) is integrable μ' and H'(*,M') Tf is integrable μ on D. Thus by theorem 3.10 H'(*,M') belongs to S'(D,C,Z). Hence

\[ \int_D H'(*,M') Tfdμ = \int_S H'(*,M') W'(*,D) dμ'. \]

Using theorem 1.23 we have

\[ \int_D yH'(*,M') Tfdμ = \int_S yH'(*,M') W'(*,D) dμ'. \]

Consequently yH'(*,M') belongs to S'(D,J,Z).

4.4 Remark. By the remark in 1.35 and lemma 4.3, S'(D,J,Z) contains all J-valued simple integrable μ' functions defined on S'.

4.5 Theorem. Assume T is ACW' and f is a g*(W',μ,μ'). Fix D in S and let H' be a J-valued function defined on S' measurable μ' such that H'W'(*,D) is integrable μ' and H' Tf is integrable μ on D. Now there exists a sequence of J-valued

μ' and H' Tf is measurable μ on D.
functions $H'_n$ defined on $S'$ such that $H'_n$ is simple integrable for each $n$, $\lim H'_n = H'$ a.e. $\mu'$, and hence by lemma 1.30
$\lim H'_n W'(\cdot, D) = H' W' (\cdot, D)$ a.e. $\mu'$ and $\lim H'_n T f = H' T f$ a.e. $\mu'$ on $D$. Suppose in addition we have the conditions:

(1) $\lim \int_D H'_n T f d\mu = \int_D H' T f d\mu$ and

(2) $\lim \int_{S'} H'_n W'(\cdot, D) d\mu' = \int_{S'} H' W'(\cdot, D) d\mu'$.

Then $H'$ belongs to $S'(D, E, \mathcal{F})$.

Proof. Using the remark in 4.4 we have

$$\int_D H' T f d\mu = \lim \int_D H'_n T f d\mu = \lim \int_{S'} H'_n W'(\cdot, D) d\mu' =$$

$$= \int_{S'} H' W'(\cdot, D) d\mu'.$$

Thus $H'$ belongs to $S'(D, E, \mathcal{F})$.

Remark. If the integrals $\int H'_n T f d\mu$, $n \geq 1$, and the integrals $\int H'_n W'(\cdot, D) d\mu'$, $n \geq 1$ are uniformly absolutely continuous, then by theorem 1.19 conditions (1) and (2) hold.
CHAPTER V

BANACH VALUED WEIGHT FUNCTIONS

Assume the complex measure spaces \((S, \mathbb{M}, \mu)\) and \((S', \mathbb{M}', \mu')\) with \(T, \mathcal{D}, \) and \(\mathcal{S'}\) satisfy HB1 - HB8. In this chapter we shall extend the definition of a weight function for the transformation \(T\) so that it may take its values in a \(B\)-space \(\mathcal{I}\). \(\mathcal{O}'(\mathcal{D}, \mathcal{S}, \mathcal{C})\) will now denote the extension of \(\mathcal{O}'\) in 1.34 to the case in which \(W'\) and \(f\) have \(\mathcal{I}\) as their range.

5.1 HB9*. \(W'\) is a function defined on \(S' \times \mathcal{D}\) with the \(B\)-space \(\mathcal{I}\) as its range and satisfying the following conditions:

i) If \(D\) is in \(\mathcal{D}\) then \(W'(\cdot, D) = 0\) a.e.\(\mu'\) on \(S' - T D\).

ii) If \(D\) is a set in \(\mathcal{D}\) for which there is a countable number of pairwise disjoint sets \(D_1\) in \(\mathcal{D}\) and two subsets \(E\) and \(F\) satisfying the relations:

\[
U D_1 \subseteq D, \ D - UD_1 = E \cup F,
\]

\[
||E|| = 0 \text{ and } ||TF|| = 0,
\]

then \(W'(\cdot, D) = \sum W'(\cdot, D_1) \text{ a.e.}\mu'.\)

iii) \(W'\) is a.e.\(\mu'\) inner continuous on \(\mathcal{D}\) - that is, if a set
D in $\mathcal{D}$ is the union of a countable number of sets $D_j$ in $\mathcal{D}$ such that $D_j \subseteq D_{j+1}$ for every $j$, then $\lim W'(\cdot, D_j) = W'(\cdot, D)$ a.e. $\mu'$.

iv) For each $D$ in $\mathcal{D}$, $W'(\cdot, D)$ is measurable $\mu'$.

A function $W'$ having these properties is termed a weight function for the transformation $T$.

**Remark.** If the range of $W'$ is the non-negative real number system then HB9* coincides with HB9.

5.2 **HB10.** Let $W'$ satisfy HB9*. There exists a countable subset $\mathcal{D}^\#$ of $\mathcal{D}$ and a set $X'$, $X' \subseteq S'$, where $X'$ is $\mu'$-null, such that for every $D$ in $\mathcal{D}$ there exists a sequence $D_i^\# \subseteq \mathcal{D}$ depending on $D$ such that $D_i^\# \subseteq D$ for every $i$ and $\lim W'(\cdot, D_i^\#) = W'(\cdot, D)$ on $S' - X'$.

Throughout this chapter HB9* and HB10 are always assumed.

5.3 **Definition.** $v'$ is a non-negative extended real valued function defined on $S' \times \mathcal{D}$ as follows: $v'(s', D) =$

$$= \sup_{i=1}^{n} \left| W'(s', D_i) \right|,$$

where the supremum is taken over all pairwise disjoint collections of sets $\{D_i\}_{i=1}^{n}$ in $\mathcal{D}$ such that $D_i \subseteq D$ for each $i$.

The function $v'(s', \cdot)$ is simply the total variation of $W'(s', \cdot)$ on $\mathcal{D}$ for each $s'$ in $S'$. We shall call $v'$ the
variation of $W'$.

**Remark.** In view of the remark in 1.25, if the range of $W'$ is the non-negative real number system then $\nu'(*,D) = W'(*,D)$ a.e.$\mu'$, $D \in \mathcal{D}$.

5.4 **Lemma.** For each $D$ in $\mathcal{D}$, $\nu'(*,D)$ is measurable $\mu'$.

**Proof.** Fix $D$ in $\mathcal{D}$. Let $\mathcal{D}_0$ be the set of all sets in $\mathcal{D}^\#$ which are contained in $D$. Obviously $\mathcal{D}_0$ is not empty and is countable. Let $s'$ belong to $S'$. Now

\begin{equation}
\nu'(s',D) \geq \sum_{i=1}^{n} |W'(s',D_i^\#)|,
\end{equation}

where $\{D_i^\#\}_{i=1}^{n}$ is contained in $\mathcal{D}_0$ and the $D_i^\#$ are pairwise disjoint.

Now let $s' \in S' - X'$ where $X'$ is the set mentioned in HB10. Let $a$ be a real number such that $a < \nu'(s',D)$. Then using the definition of $\nu'$ there exists a collection of pairwise disjoint sets $\{D_i\}_{i=1}^{n}$ in $\mathcal{D}$ such that $D_i \subseteq D$ for each $1 \leq i \leq n$ and $a < \sum_{i=1}^{n} |W'(s',D_i^\#)|$. By HB10 there exists a collection of sets $\{D_i^\#\}_{i=1}^{n}$ in $\mathcal{D}_0^\#$ such that $D_i^\# \subseteq D_i$, for each $1 \leq i \leq n$ and $a < \sum_{i=1}^{n} |W'(s',D_i^\#)|$. Note that the $D_i^\#$ are pairwise disjoint.

Thus using (1) we have

\begin{equation}
\nu(s',D) = \sup \sum_{i=1}^{n} |W'(s',D_i^\#)| \text{ a.e.}\mu', \text{ where the}
\end{equation}
The supremum is taken over all pairwise disjoint collections of sets \( \{D_i\}_{i=1}^n \) where the \( D_i \) belong to \( \mathcal{D}_\# \) for every \( l \leq i \leq n \). We note since \( W'(*,D) \) is measurable \( \mu' \) for every \( D \) in \( \mathcal{D} \), every function of the form

\[
(3) \sum_{i=1}^n |W'(*,D_i^\#)|, \text{ where the } D_i^\# \text{ belong to } \mathcal{D}_\# \text{ and are}
\]

pairwise disjoint, is measurable \( \mu' \). There is only a countable number of functions of the type appearing in (3) since \( \mathcal{D}_\# \) is countable. Using this fact and (2) we conclude that \( v'(*,D) \) is measurable \( \mu' \).

The properties of \( v' \) set forth in the lemma below may be verified easily by direct arguments.

5.5 Lemma. The following statements are valid.

i) If \( D \in \mathcal{D} \) then \( v'(*,D) = 0 \) a.e. \( \mu' \) on \( S' - T \cdot D \).

ii) Let \( s' \) belong to \( S' \). If \( D \) contains a countable number of pairwise disjoint sets \( D_i \) where \( D, D_i \) are in \( \mathcal{D} \) for each \( i \), then \( \Sigma v'(s',D_i) \leq v'(s',D) \).

5.6 Definition. The transformation \( T \) is said to be of bounded variation with respect to \( W' \) — briefly \( BVW' \) — if \( v'(*,S) \) is integrable \( \mu' \).

Remark. We note that if the range of \( W' \) is \( \mathbb{R}^+ \) then \( T \) is \( BVW' \) in the sense of definition 2.1 if and only if \( T \) is
BVW' in the sense of definition 5.6. Whenever T is BVW'
there is not any loss of generality in assuming that v' is
finite valued since v' would only take the value +∞ on a
µ'-null set which has no effect when v' enters into an integral.
We note that if T is BVW' we can use lemmas 5.4 and 5.5 to
conclude that v'(*)D) is integrable µ' for each D in D. Also
if T is BVW' then W'(*)D) is integrable µ' for each D in D.

5.7 Definition. Assume T is BVW' and that there exists
an x-valued function f defined on S which is integrable µ such
that
\[ \int_D f dµ = \int_S W'(*,D) dµ', D \in D. \]
Then the transformation T is said to be absolutely continuous
with respect to W'-briefly ACW' — and f is said to be a gauge
for W' relative to µ and µ' — briefly a g.(W',µ,µ').

Remark. Suppose T is ACW' in the sense of the above
definition and f is a g.(W',µ,µ'). Then if the range of W' is
R+ and f is complex valued then T is ACW' and f is a g.(W',µ,µ')
in the sense of definition 2.8.

5.8 Lemma. Assume T is ACW' and f is a g.(W',µ,µ').
Then f is unique in the sense that any other g.(W',µ,µ') is
equal to f a.e. µ.

Proof. The proof follows in a manner similar to the proof
in lemma 2.9.
5.9 Theorem. Assume $T$ is ACW' and $f$ is a $g(W',\mu,\mu')$.
Let $\mathbb{E}$ be the complex number field. Then

$$\int_D |f| dv(\cdot,\mu) \leq 4 \sum_S \nu'(\cdot,D) dv(\cdot,\mu'), \quad D \in \mathbb{D}.$$ 

Proof. Let $D$ belong to $\mathbb{D}$. The proof is exactly the same as the proof of theorem 2.10 except (3) in that proof now becomes

$$\nu(D,\lambda) \leq 4 \sum S, |W'(\cdot,D_i)| dv(\cdot,\mu') + \varepsilon \leq$$

$$\leq 4 \sum S, \nu'(\cdot,D_i) dv(\cdot,\mu') + \varepsilon \leq 4 \int \nu'(\cdot,D) dv(\cdot,\mu') + \varepsilon.$$ 

As before it follows that

$$\int_D |f| dv(\cdot,\mu) \leq 4 \sum_S \nu'(\cdot,D) dv(\cdot,\mu').$$

5.10 Lemma. Assume $T$ is BVW'. Let $M'$ be a set in $\mathbb{M}$ such that $\nu(M',\mu') = 0$. Then for every positive number $\varepsilon$ there is a countable number of pairwise disjoint sets $D_i$ in $\mathbb{D}$ such that

$$T^{-1}M' \subseteq \bigcup D_i \quad \text{and} \quad \sum S, \nu'(\cdot,D_i) dv(\cdot,\mu') < \varepsilon.$$ 

Proof. Let $\varepsilon$ be a positive number. Since $\nu'(\cdot, S)$ is integrable $\mu'$ there exists a positive number $\delta$ such that

$$\int_{M'} \nu'(\cdot,S) dv(\cdot,\mu') < \varepsilon,$$

whenever $M' \in \mathbb{M}$ and $\nu(M',\mu') < \delta$. Since $\nu(M',\mu') = 0$, by HB7 there is a set $0'$ in $\mathbb{D}$ such that $M' \subseteq 0'$ and $\nu(0',\mu') < \delta$.

There exists a countable number of pairwise disjoint sets $D_i$ in
Theorem. Assume $T$ is $ACW'$ and $f$ is a $g.(W',\mu,\mu')$.

Let $M' \in \mathcal{M}$ such that $\nu(M',\mu') = 0$. Then $f = 0$ a.e. on $T^{-1}M'$.

Proof. By lemma 5.10, for every positive integer $n$ there exists a countable number of pairwise disjoint sets $D_i^n$ in $\mathcal{D}$ such that $T^{-1}M' \subseteq \bigcup_i D_i^n$ and $\sum_{TD_i} \nu'(\cdot,D_i^n)\nu(\cdot,\mu') < \frac{1}{4n}$.

Let $X_n = \bigcup_i D_i^n$. Then $X_n \in \mathcal{M}$. Let $\lambda(M) = \int_M fd\mu$, $M \in \mathcal{M}$. Then $\lambda$ is an $\mathcal{F}$-measure defined on $\mathcal{M}$. We assert that $||\lambda||(X_n) \leq \frac{1}{n}$.

Let $\delta$ be a positive number. Since $f$ is integrable $\mu$ there exists a positive number $\delta$ such that $|\int_M fd\mu| < \frac{\epsilon}{\delta}$ when $M \in \mathcal{M}$ and $\nu(M,\mu) < \delta$. Recall (1.7) that $||\lambda||(X_n) \leq 4 \sup |\lambda(M)|$, $M \subseteq X_n$, $M \in \mathcal{M}$. Therefore there exists a set $M_o$ in $\mathcal{M}$ such that $M_o \subseteq X_n$ and $||\lambda||(X_n) \leq 4|\lambda(M_o)| + \frac{\epsilon}{2}$. Now by HB4 there exists a sequence of pairwise disjoint sets $D_j$ in $\mathcal{D}$ such that $M_o \subseteq \bigcup D_j$ and $\nu(\bigcup D_j - M_o,\mu) < \delta$. Again using HB4 for every $i$ and $j$ there
exists a sequence of pairwise disjoint sets $D_{ijk}$ from $\mathcal{D}$ such that $D_i^P \cap D_j = U_k D_{ijk}$. Thus $(U_i D_i^P) \cap (U_j D_j) = U_{ijk} D_{ijk}$ and the $D_{ijk}$ are pairwise disjoint for all $i$, $j$, and $k$. Also

$$M. \subseteq U_{ijk} D_{ijk} \text{ and } \nu(U_{ijk} D_{ijk} - M, \mu) < \delta. \text{ Thus } |\lambda(M_\mu)| \leq$$

$$\leq |\lambda(U_{ijk} D_{ijk})| + \frac{\epsilon}{\delta}. \text{ Therefore } |||\lambda|||_2 \leq 4|\lambda(U_{ijk} D_{ijk})| + \epsilon \leq$$

$$\leq 4 \Sigma_{ijk} |\lambda(D_{ijk})| + \epsilon = 4 \Sigma_{ijk} \int S \nu^*(\cdot, D_{ijk}) d\mu | + \epsilon \leq$$

$$\leq 4 \Sigma_{ijk} \int S \nu^*(\cdot, D_{ijk}) d\nu(\cdot, \mu) | + \epsilon \leq$$

$$\leq 4 \Sigma_{ijk} \int S \nu^*(\cdot, D_{ijk}) d\nu(\cdot, \mu) | + \epsilon \leq \frac{1}{n} + \epsilon. \text{ Since } \epsilon \text{ is an arbitrary positive number, } |||\lambda|||_2(X) \leq \frac{1}{n}. \text{ Now let } X = \cap_n X_n. \text{ Then }$$

$$X \in \mathfrak{M}, |||\lambda|||_2(X) = 0, \text{ and } T^{-M} \subseteq X. \text{ Using the fact that if }$$

$$M \in \mathfrak{M} \text{ and } M \subseteq X \text{ then } |\lambda(M)| \leq |||\lambda|||_2(M) \leq |||\lambda|||_2(X) = 0, \text{ we have }$$

$$\int_M f d\mu = 0, M \in \mathfrak{M}, M \subseteq X. \text{ Thus } f = 0 \text{ a.e. } \mu \text{ on } X \text{ which implies }$$

$$f = 0 \text{ a.e. } \mu \text{ on } T^{-M}.$$

In view of theorem 5.11 and the properties of the functions involved, lemmas 5.12, 5.13 and theorem 5.14 will follow easily from lemmas 1.29, 1.30 and theorem 1.32 respectively.

5.12 Lemma. Let $T$ be $ACW'$ and let $f$ be a $g.(W', \mu, \mu')$.

Fix $D$ in $\mathcal{D}$. Assume $H_1'$ and $H_2'$ are real valued functions defined on $S'$ such that $H_1' \leq H_2'$ a.e.$\mu'$ on $TD$. Then $H_1'|W'(\cdot, D)| \leq$

$$\leq H_2'|W'(\cdot, D)| \text{ a.e.$\mu'$ and } H_1'|T|f| \leq H_2'|T|f| \text{ a.e.$\mu$ on } D.$
5.13 Lemma. Assume $T$ is $ACW'$ and $f$ is a $g.(W',\mu,\mu')$. Fix $D$ in $\mathcal{D}$. Let $H_j^i$, $j \geq 0$, be a sequence of complex valued functions defined on $S'$ such that $\lim H_j^i = H'_i$ a.e.$\mu'$ on $TD$. Then

$$\lim H_j^i W'(\cdot,D) = H'_i W'(\cdot,D) \text{ a.e.}\mu'$$

and

$$\lim H_j^i \circ T f = H'_i \circ T f \text{ a.e.}\mu \text{ on } D.$$  

5.14 Theorem. Assume $T$ is $ACW'$ and $f$ is a $g.(W',\mu,\mu')$. Let $D$ belong to $\mathcal{D}$ and let $H'$ be a complex valued function defined on $S'$ measurable $\mu'$. Then $H' W'(\cdot,D)$ is measurable $\mu'$ and $H' \circ T f$ is measurable $\mu$ on $D$.

5.15 Lemma. Assume $T$ is $ACW'$ and $O'$ is a set in $\mathcal{O}'$ which is of type $\gamma'$. Then for every $D$ in $\mathcal{D}$ the function $H'(\cdot,O')$ is in $\mathcal{S}'(D,\mathcal{C},\mathcal{C})$.

Proof. Let $f$ be a $g.(W',\mu,\mu')$. Using theorem 5.14 we have $H'(\cdot,O') W'(\cdot,D)$ is integrable $\mu'$ and $H'(\cdot,O') \circ T f$ is integrable $\mu$. We shall show that the hypotheses of lemma 1.37 is satisfied and then we are done. To show the termwise integration condition holds let $g_n = \sum_{i=1}^{N} W'(\cdot,D_i)$. Now since $\sum W'(\cdot,D_i) = W'(\cdot,D)$ a.e.$\mu'$, we have $\lim g_n = W'(\cdot,D)$ a.e.$\mu'$. Also $|g_n| \leq v'(\cdot,S)$ for each $n$. Thus the termwise integration condition follows from the Lebesgue dominated convergence theorem. Using theorems 5.11, 5.14 and HB9* the rest of the hypotheses of lemma 1.37 is satisfied. Thus $H'(\cdot,O')$
belongs to $\mathcal{S}'(D,\mathbb{C},\mathbb{C})$.

Lemmas 5.16, 5.17 and theorem 5.18 are proved in a fashion similar to the proofs of lemmas 2.15, 2.16 and theorem 2.18 respectively. The proofs are omitted.

5.16 Lemma. Assume $T$ is $\mathcal{A} \cup W'$ and $M'$ belongs to $\mathbb{W}'$ and is of type $\gamma'$. Then for every $D$ in $\mathcal{D}$, $H'($, $M')$ is in $\mathcal{S}'(D,\mathbb{C},\mathbb{C})$.

5.17 Lemma. Assume $T$ is $\mathcal{A} \cup W'$ and $M'$ is any set in $\mathbb{W}'$. Then for every $D$ in $\mathcal{D}$ the function $H'($, $M')$ is in $\mathcal{S}'(D,\mathbb{C},\mathbb{C})$.

5.18 Theorem. Assume $T$ is $\mathcal{A} \cup W'$ and $f$ is a $g.(W',\mu',\mu')$. Fix $D$ in $\mathcal{D}$. Suppose that $H'$ is a complex valued function defined on $S'$ measurable $\mu'$. Also assume $H'W'($, $D)$ is integrable $\mu'$ and $H'\cdot f$ is integrable $\mu$ on $D$. Then $H'$ belongs to $\mathcal{S}'(D,\mathbb{C},\mathbb{C})$.

5.19 Lemma. Assume $T$ is $\mathcal{A} \cup W'$ and $f$ is a $g.(W',\mu',\mu')$. Let $\mathfrak{F}$ be the complex number field and let $O' \in \mathcal{D}'$ and be of type $\gamma'$. Fix $D$ in $\mathcal{D}$. Then

$$\int_{D} H'($, $O') \cdot T|f|dv($, $\mu) \leq 4 \int_{S} H'($, $O') \cdot v'($, $D)dv($, $\mu')$$

Proof. The reasoning follows somewhat like that used in the proof in lemma 1.37 where $\mu$ and $\mu'$ are replaced by $\nu($, $\mu)$ and $\nu($, $\mu')$ respectively and $|f|$ and $\nu'$ take the place of $f$ and $W'$ respectively. Certain equalities in that proof are replaced by inequalities. We shall use the same notation as in the proof of lemma 1.37. Due to the underadditivity of $\nu'$ we have
Thus from similar reasoning as in the proof of lemma 1.37

\[ \sum_{i,j,k} \nu'\left(\cdot, D_{ijk}\right) \leq \nu'\left(\cdot, D\right). \]

From theorem 5.9 we have

\[ \int_{D} |f| \text{d}v\left(\cdot, \mu\right) \leq 4 \int_{S} \nu'\left(\cdot, D_{Ojk}\right) \text{d}v\left(\cdot, \mu'\right). \]

Using theorem 5.11 we obtain

\[ \int_{D} H'\left(\cdot, O'\right) T |f| \text{d}v\left(\cdot, \mu\right) = \sum_{j,k} \int_{D} |f| \text{d}v\left(\cdot, \mu\right). \]

Thus

\[ \int_{D} H'\left(\cdot, O'\right) T |f| \text{d}v\left(\cdot, \mu\right) \leq 4 \int_{S} H'\left(\cdot, O'\right) \nu'\left(\cdot, D\right) \text{d}v\left(\cdot, \mu'\right). \]

By a sequence of lemmas similar to lemmas 2.21, 2.22 and 2.23 we can establish the following theorem. The proof is omitted due to the close parallel of the arguments used in Chapter II for the corresponding result.

\textbf{5.20 Theorem.} Assume T is ACW' and f is a g.\left(W', \mu, \mu'\right).

Let \mathbb{C} be the complex number field. Fix D in \mathbb{C}. Assume H' is a complex valued function defined on S' measurable \mu'. Also let H'v'\left(\cdot, D\right) be integrable \mu'. Then H'W'\left(\cdot, D\right) is integrable \mu' and H'\cdot Tf is integrable \mu on D.

\textbf{5.21 Theorem.} Assume T is ACW' and f is a g.\left(W', \mu, \mu'\right).
Let \( \mathbb{C} \) be the complex number field. Fix \( D \) in \( \mathbb{D} \). Assume \( H' \) is a complex valued function defined on \( S' \) measurable \( \mu' \). Let \( H'\nu'(\cdot,D) \) be integrable \( \mu' \). Then \( H' \) belongs to \( S'(D,\mathbb{C},\mathbb{C}) \).

**Proof.** The result follows using theorems 5.20 and 5.18.
SUMMARY

In this paper we study a mathematical system composed of two measure spaces \((\mathcal{S}, \mathcal{M}, \mu)\) and \((\mathcal{S}', \mathcal{M}', \mu')\), a transformation \(T\) from \(\mathcal{S}\) onto \(\mathcal{S}'\), and sets \(\mathcal{D}\) and \(\mathcal{D}'\) of subsets of \(\mathcal{S}\) and \(\mathcal{S}'\) respectively, satisfying HB1-HB8 in 1.24. We assume that \(W'\) is a weight function for \(T\) satisfying HB9 which is integrable \(\mu'\) for each \(D\) in \(\mathcal{D}\). We also assume there exists a function \(f[\cdot](W', \mu, \mu')\) defined on \(\mathcal{S}\) integrable \(\mu\) such that \(\int_D f d\mu = \int_{\mathcal{S}'} W'(\cdot, D) d\mu'\) for each \(D\) in \(\mathcal{D}\), that is \(T\) is absolutely continuous with respect to the weight function \(W'(ACW')\). We seek answers to the following question. Assume that the measures \(\mu\) and \(\mu'\) and the functions \(W'\) and \(f\) take values in appropriate Banach spaces. Let \(H'\) be a function defined on \(\mathcal{S}'\) with values in a Banach space. Given \(D\) in \(\mathcal{D}\) what conditions are sufficient to insure that \(H'W'(\cdot, D)\) is integrable \(\mu\), \(H'Tf\) is integrable \(\mu\) on \(D\), and the transformation formula \(\int_D H'Tf d\mu = \int_{\mathcal{S}'} H'W'(\cdot, D) d\mu'\) holds? We denote the class of functions \(H'\) which satisfy the above conditions by \(\mathcal{S}'\).

Assume the measures \(\mu\) and \(\mu'\) are complex valued. If the \(\mathbb{C}\)-valued function \(H'\) defined on \(\mathcal{S}'\) is measurable \(\mu'\) and \(H'W'(\cdot, D)\) is integrable \(\mu'\), then \(H'\) is in \(\mathcal{S}'(D, \mathbb{C}, \mathfrak{S})\) (theorem 2.25). An unsolved problem is whether the integrability of \(H'Tf\) on \(D\) re-
Relative to \( \mu \) implies the integrability of \( H'W'(\cdot, D) \) relative to \( \mu' \).

Assume \((S, M, \mu)\) is an \( \mathbb{Z} \)-vector measure space and that \( \mathbb{Z}^* \) is separable. Then if a subset of \( S \) is \( x^*\mu \)-null for every \( x^* \) in \( \mathbb{Z}^* \), then it is \( \mu \)-null (theorem 3.6). This is one of the main results which enables us to use the conjugate space of \( \mathbb{Z} \) in extending the transformation theory to vector valued measures.

The following is a result used to establish the necessary integrability conditions in this case. Assume \((S, M, \mu)\) is an \( \mathbb{Z} \)-vector measure space. Let \( h \) be a complex valued function defined on \( S \) which is measurable \( \mu \). Then \( h \) is integrable \( \mu \) if and only if \( h \) is integrably uniformly absolutely continuous (i.u.a.c.) with respect to \( \{x^*\mu : x^* \in S^*\} \), where \( S^* \) is the closed unit sphere in \( \mathbb{Z}^* \). We now state the main result (theorem 3.14) found in Chapter III. Let \( H' \) be a complex valued function defined on \( S' \) which is measurable \( \mu' \) such that \( H'W'(\cdot, D) \) is integrable \( \mu' \).

Suppose that \( H'Tf \) is i.u.a.c. with respect to \( \{x^*\mu : x^* \in S^*\} \). Then \( H' \) belongs to \( S'(D, \mathcal{E}, \mathcal{I}) \).

Since the class of null sets is independent of the choice of two possible semi-variations for the measures (theorem 1.8) we have the following result (theorem 1.23). Let \((S, M, \mu)\) be an \( \mathbb{Z} \)-vector measure space and assume \( h \) is a complex valued function defined on \( S \) which is integrable \( \mu \). Let \( y \) belong to \( \mathcal{Y} \). Then \( yh \) is integrable \( \mu \) and \( \int_M yh d\mu = y \int_M h d\mu \), \( M \in \mathcal{M} \).
The main result (theorem 4.5) concerning the transformation theory in the bilinear case is the following: Assume \( H' \) is a \( \mathbb{R} \)-valued function defined on \( S' \) measurable \( \mu' \) such that \( H'W'(*)D \) is integrable \( \mu' \) and \( H'\circ Tf \) is integrable \( \mu \) on \( D \). There exists a sequence of \( \mathbb{R} \)-valued functions \( H_n' \) defined on \( S' \) such that \( H_n' \) is simple integrable \( \mu' \) for each \( n \) and \( \lim H_n'W'(*)D = H'W'(*)D \) a.e.\( \mu' \) and \( \lim H_n'\circ Tf = H'\circ Tf \) a.e.\( \mu \) on \( D \). Suppose we have the conditions:

\[
\lim \int_D H_n'\circ Tf d\mu = \int_D H'\circ Tf d\mu \quad \text{and}
\]

\[
(*) \quad \lim \int_{S'} H_n'W'(*)D d\mu' = \int_{S'} H'W'(*)D d\mu'.
\]

Then \( H' \) belongs to \( \text{H'}(D,\mathbb{R}) \).

An unsolved problem is to find conditions to relate the integrability conditions in one space to the integrability conditions in the other space. Also due to a lack of strong convergence theorems in the bilinear integration theory, we have to postulate the above (*) interchange of limits. It would be very desirable to remove such a stringent condition.

In the last chapter we discuss the extension of the range of the weight function \( W' \) to a Banach space. An unsolved problem is to lessen the integrability requirements in the following result (theorem 5.18). Assume the system satisfies \( \text{HB9}' \) and \( \text{HB10} \) in addition to \( \text{HB1-8} \). Let \( \mu \) and \( \mu' \) be complex
measures. Suppose $H'$ is a complex valued function defined on $S'$ which is measurable $\mu'$. Also assume $H'W'(\cdot, D)$ is integrable $\mu'$ and $H'^*Tf$ is integrable $\mu$ on $D$. Then $H'$ belongs to $\mathcal{F}'(D, \mathcal{C}, \mathcal{C})$. If the range of $W'$ is $\mathcal{C}$, then the integrability of $H'v'(\cdot, D)$ relative to $\mu'$, where $v'$ is the variation of $W'$, implies the integrability of $H'^*Tf$ on $D$ relative to $\mu$ and the integrability of $H'W'(\cdot, D)$ relative to $\mu'$ (theorem 5.20).

Since the range of $W'$ has been extended, the investigation of a Jordan-type decomposition for $W'$ is possible. Another important investigation now possible is the establishment of necessary and sufficient conditions that $T$ be $ACW'$. 
BIBLIOGRAPHY


