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UNDER SINUSOIDAL LOADS

DISSERTATION

Presented in Partial Fulfillment of the Requirements
for the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Mohammed Tayib Akrawi, License, B.C.E., M.Sc.

* * * * * * * *

The Ohio State University
1964

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NOTATIONS

t  Time

p(t)  Intensity of load at any time

P_0  Maximum intensity of load

\omega  Circular frequency

\Delta  Dilatation

e  Base of natural logarithms, Component of strain

i  \sqrt{-1}, a subscript used as a summation variable

j  A subscript used as a summation variable

J(t)  Creep compliance

G(t)  Relaxation modulus

a  Radius of circular area

r, z  Coordinates of cylindrical system

P, Q, P'
and Q'
  Linear operators

\sigma_{ij}  Stress tensor

\epsilon_{ij}  Strain tensor

S_{ij}  Deviator of stress tensor

e_{ij}  Deviator of strain tensor

G  Shear modulus

K  Bulk modulus, an elliptic integral in Appendix A

\mu  Poisson's ratio

\sigma_r  Radial horizontal stress

\sigma_z  Vertical stress
NOTATIONS (Contd.)

\( \tau_{rz} \)  
Radial-vertical Shear stress

\( U \)  
Horizontal displacement

\( w \)  
Vertical displacement

\( f_i, F_i, g_i \)

\( h_i, Q_i, q_i \)

\( G_i, \) and \( H_i \)  
Functions of geometry \((r, z, \) and \( a)\)

\( s \)  
An algebraic parameter independent of time

\( \sigma^* \)  
Radial horizontal stress of the associated elastic solution

\( U^* \)  
The horizontal displacement of the associated elastic solution

\( w^* \)  
The vertical displacement of the associated elastic solution

\( A \)  
The elastic element of a mechanical model, an identifying letter for an Appendix, and a dimensionless ratio in Appendix A

\( B \)  
The viscous element of a mechanical model, an identifying letter for an Appendix

\( C \)  
A constant = 3K, an identifying letter for an Appendix

\( D \)

\( \gamma \)  
Retardation or relaxation time

\( X_i, Y_i, \) and \( Z_i \)  
Functions appearing in the equations of the vertical displacement of the Maxwell, Kelvin, and standard linear solid models, respectively

\( \phi_i \)  
Functions appearing in the equations of the radial stress, the horizontal, and the vertical displacements of the Burgers model

\( \lambda, \phi, \tau, \)  
\( \psi, \theta, \)  

and \( \nu \)  
Angles
NOTATIONS (Contd.)

$\alpha$  A constant in the equations of the Burgers model

$\gamma$  A constant in the equations of the Burgers model, an angle in Appendix A

$V$  Velocity

$V_1-V_{10}$  Angles in equations 38-43

$x$  Distance in Figure 44

$R$  Radial distance in Appendix A

$E', K'$, $E(k, \phi)$, and $F(k, \phi)$  Elliptic integrals defined in Appendix A

$J$  Combination of elliptic integrals defined in Appendix A

$b$  A dimensionless ratio in Appendix A
CHAPTER 1

INTRODUCTION

There are two procedures of major importance to a design engineer. The first is the prediction of the behavior of a material under loading and unloading. The second is the determination of the state of stresses and displacements in the material caused by prescribed loads and for given boundary conditions. The stresses and displacements which control the design of any structure are functions of parameters which reflect the properties of the materials used. Hence the second procedure is dependent on the first. The tasks involved in the first are experimental, while the second is mostly analytic, the results of which maybe checked by experiments. This dissertation falls in the second category.

During the past century the theory of elasticity has been used for the prediction of stresses and displacements in many engineering problems and a great deal of literature is available on this subject. However, the observed behavior of many engineering materials, such as portland cement concrete, foundation soils, and bituminous mixtures, do not agree with the behavior assumed by the theory of elasticity. This shortcoming is due to the fact that in elasticity, the material properties $E$ and $\mu$, (modulus of elasticity and Poisson's ratio, respectively) are assumed to be constants, whereas in reality a material's properties often are not constants, but are time-dependent. Materials for which the stress and strain tensors are related by differential or integral operators with respect to time are
called viscoelastic. For linear viscoelastic materials the stress and strain tensors are related by linear differential or integral operators which are functions of time only.\(^1\)

Over the years numerous investigators from different branches of science have studied viscoelastic stress analysis and have solved many problems of practical importance. On the experimental level, many investigators concerned themselves with finding the rheologic properties of different materials.

A problem which has received a great deal of attention is that of a semi-infinite body under different types of loading. Problems dealing with the semi-infinite homogeneous body are of practical importance in the field of foundation engineering and as a first step toward the solution of layered systems such as in the case of highway and airport pavements.

This work deals with the stress analysis of a semi-infinite viscoelastic foundation under loads which are fixed in space but which vary with time according to the relations: \(p(t) = p_0 \sin (\omega t)\), and \(p(t) = p_0 \sin^2 (\omega t)\). These types of loading may occur in many engineering problems. An example is shown for using some of the results of this

\(^1\) For the definition of a linear viscoelastic material given above see references 12, 17, and 26 listed at the end of this dissertation.
work as an approximate solution for the problem of a circular load of constant magnitude moving over the surface of a viscoelastic foundation. It should also be mentioned that many types of complicated loadings may be expressed in terms of a sinusoidal loads by using the Fourier series for loads, which are periodic functions of time, or the Fourier integral for loads which are non-periodic functions of time. In this work, solutions for the viscoelastic medium are compared with those for the elastic medium under the same types of loadings. Using the Laplace transform and making use of the solutions of elasticity, expressions for stresses and displacements are obtained for several types of linear viscoelastic materials.
CHAPTER 2
SURVEY OF LITERATURE

In the following paragraphs, the literature related to this work is surveyed, starting with that concerning the semi-infinite elastic body, since this solution will be used later in this work.

Semi-infinite elastic body

In 1831, Lame and Clapeyron(1) gave the first solution for the problem of a normal load on a semi-infinite elastic body. Using Fourier's theorem, they expanded a function of two variables as a quadruple integral. In 1885, Boussinesq(2) introduced the use of the direct, inverse, and the logarithmic potential functions into the field of elasticity. He solved the fundamental problem of a point load action on the surface of a semi-infinite elastic body. Lamb(3) solved an extension of the problem, for a symmetrical distribution normal load on the surface, by expanding the load in a Fourier-Bessel series using Bessel's Function of the zero order.2

In 1916, Terazawa(4) generalized the special problems and presented a complete solution for the stresses and displacements at any point in the semi-infinite elastic body, subjected to any given normal pressure. Terazawa expanded the load function on the boundary in a Fourier-Bessel double integral. Using vector notations and cylindrical coordinates, he

Note: Numbers in brackets refer to the references listed at the end of this work.

2. The above paragraph is based on reference 4.
solved the equation of equilibrium of the semi-infinite elastic body by assuming a dilatation $\Delta$ to be 
$$\Delta = e^{-kz} \left( \sin m \theta \right)$$
where $k$ is a positive constant and $m$ is an integer, positive, negative, or zero, so that the solution may be unique around the origin. From the equation, $\text{div. grad. } \Delta = 0$, he found the dilatation to be 
$$\Delta = C_m e^{-kz} J_m(kr) \left( \cos m \theta \right)$$
where $C_m$ is a constant of integration. Then he found the components of displacements and the stresses corresponding to each solution of $(\cos m\theta$ and $\sin m\theta)$.

For the case of a uniform circular load on the boundary, Terazawa ended up with infinite integrals which are the Laplace transform of products of Bessel functions of the zero and first order. He evaluated those infinite integrals by following methods proposed by Nagaoka(4). The solutions are in terms of elliptic integrals. It is interesting to mention that in Terazawa's work, concerning the evaluation of the infinite integrals, no reference is made to the work done by German mathematicians, such as Gegenbauer, Weber, Sonine, and Schafheitlin(5,6), which were done prior to Nagaoka.

In 1929, Love(7) published the general solution of the problem of a semi-infinite body by using "the potential method." Love recognized

3. The dilatation $\Delta$ is defined for "small" strains as: 
$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$
where $u$, $v$, and $w$ are the displacements in the $x$, $y$, and $z$ directions, respectively.
Terazawa's work as being earlier. Results of both works checked.

The writer found the general solution⁴ of the semi-infinite elastic body under a uniform circular load by using Burmister's stress function⁵, equations of elasticity \[ \text{(9), Eqns. 179, p.343} \], and by expanding the load in a Fourier-Bessel double integral.⁶ Infinite integrals identical with those obtained by Terazawa were obtained.

The Waterways Experiment Station(10) has presented Love's work in a tabular form using a more familiar notation for the elastic constants. The expressions for the stress and displacement at any point in the material involve a combination of elliptic integrals. A numerical result for the stress or displacement at a given point, which is difficult to obtain from the equations, is presented by Ahlvin and Ulery(11). Eight tables of numerical values are presented for functions of \( r \) and \( z \) in terms of the radius of the loaded area. From these functions one can easily determine the stress, strain, or displacement at any point.

Since in this work use will be made of the solution of the elastic body the general expressions for the stresses and displacements are shown in Appendix A.

---

⁴. Not published.

⁵. The idea of using a stress function to solve two dimensional problems was first introduced by G. B. Airy, Brit., Assoc., Advancement Sci. Rept., 1862.
Semi-infinite viscoelastic body

Lee(12) has proposed a general method for the solution of linear viscoelastic problems when the loading is quasi-static, so that inertia forces due to deformation are negligible. Lee's paper is the extension of the work done by Alfrey(13) and Tsien(14) who used two differential operators instead of four used by Lee.

In a general problem dealing with a linear viscoelastic material under loading, the stress-strain relations, the equilibrium equations, the compatibility equations, and the boundary conditions are all time-dependent. By applying the Laplace transform\(^6\) to the boundary conditions and the equations mentioned above, Lee removes the time-dependence of the problem and transforms the partial differential equations in the time variable into equations which are functions of an algebraic parameter (say \(s\)). The equations in the \(s\) domain represent a stress analysis problem (called the associated elastic problem) of an elastic body of the same shape as the viscoelastic body with elastic constants (\(E\) and \(\mu\)), a function of the parameter \(s\). Once the viscoelastic problem is transformed into an associated elastic problem, the solution of the latter problem belongs to the field of elasticity, and if this solution is known, then the solution of the associated elastic problem will be known. Taking the inverse Laplace

---

6. The Laplace transform of a function \(f(t)\) is defined as:

\[
\mathcal{L}\left[ f(t) \right] = \int_0^\infty f(t) \ e^{-st} \, dt = F(s),
\]

where \(s\) is any parameter. Thus \(f(t)\) is transformed from the \(t\) domain to the \(s\) domain.
transform of the solution of the associated elastic problem, the solution of the original viscoelastic problem (in the time domain) is obtained.

The Laplace transform method is convenient for the solution of problems of constant geometry and whose load function and geometry functions separate out, and where the elastic constants appear either as multiplying factors or as rational fractions. The method has several limitations: (1) The functions involved must have a Laplace transform. This puts a restriction on the type of problems that can be solved. Take for example a load-time function of the type \( p(t) = e^{-t^2} \). The method does not work, because \( e^{-t^2} \) does not have a Laplace transform. (2) Knowing the solution of the problem in the field of elasticity is a prerequisite. (3) The inverse Laplace transform may be very difficult to obtain in some cases.

As an example for the application of the method discussed in the foregoing paragraphs, Lee found \( \mathcal{F}_r(r, z, t) \) due to a concentrated load applied on the surface of a semi-infinite linear viscoelastic medium. In the example, he assumed the linear viscoelastic material

7. The inverse Laplace transform of a function \( G(s) \), is defined as:
\[
L^{-1} G(s) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} G(s)e^{st} \, ds = g(t) \quad \text{where } i = \sqrt{-1} \text{ and } c \text{ is chosen to the right of any singularity of } G(s).
\]
In this work, \( G(s) \) may be the radial stress, the horizontal or the vertical displacement of the associated elastic solution, and \( g(t) \) may be the radial stress or the displacement in the time domain.
to behave in shear as a delayed elastic Voigt material, the mechanical model of which consists of a spring connected in parallel to a dashpot, and assumed the material to be perfectly elastic under hydrostatic pressure. In the same work, another example is shown for the treatment of the problem of a concentrated load moving over the surface of a semi-infinite viscoelastic body. 8

Lee, in another paper(15) suggests the use of integral operator stress-strain relations for stress analysis problems of general linear viscoelastic materials and for problems whose boundary conditions are such that they cannot be solved by the Laplace transform method. The integral operator relates stress to strain through either the creep compliance or the relaxation modulus, depending on which has been determined experimentally, as shown below:

\[
\varepsilon(t) = \int_0^t J(t-\tau) \frac{d\sigma(\tau)}{d\tau} d\tau \quad \text{or} \quad \sigma(t) = \int_0^t G(t-\tau) \frac{d\varepsilon(\tau)}{d\tau} d\tau
\]

where \(\sigma\) is a component of stress, \(\varepsilon\) is the related strain, \(t\) is the time, \(J(t)\) is the creep compliance, and \(G(t)\) is the relaxation modulus.

8. The writer tried to solve the problem of a uniform circular load moving over the surface of a semi-infinite linear viscoelastic body by the method Lee used for the concentrated load. The problem is not easy to solve. For the uniform load there are four expressions for the stress or displacement, depending on the geometric location of the point (see Appendix A). One also has to show that the combinations of the elliptic integrals involved have a Laplace transform.
The topic of flexible plates resting on viscoelastic foundations has attracted the attention of many investigators, due to the practical importance of the solutions for foundations and highway engineering problems. Using the Laplace transform method and the plate theory, Hoskin and Lee (17) have analysed the problem of a flexible plate subject to a uniform circular load of constant magnitude, resting on a linear viscoelastic foundation. The authors have solved the problem for two types of foundations, namely, a Maxwell material, the mechanical model of which consists of a spring connected in series to a dashpot, and a standard linear solid material, the mechanical model of which consists of a Kelvin unit connected in series to a spring.

The plate theory analysis is approximate, for the foundation reaction is assumed to be normal and the shear in the foundation is neglected. If the foundation is considered to be a continuum, the analysis becomes more complicated (18).

Reissner (19) has analyzed the deflection of plates on viscoelastic foundations. He has developed a theory of subgrade action which is simpler than the continuum theory, but is an improvement over the common Winkler (or heavy liquid) foundation analysis, in that his method supplies some of the shear coupling which is absent in the Winkler foundation assumption.
Hoskin and Radok (20) have analyzed elastic plates on viscoelastic subgrades and have applied their results to the design of pavements. However this type of analysis is only approximate, for the accuracy of assuming the pavement layer as a plate will depend on the thickness of the pavement. Also the pavement structure is not one layer made of one material, but rather it consists of several layers of different materials. Furthermore, bituminous mixtures used in highway pavements are found to be viscoelastic too (see reference 32 for example).

Pister (21) has solved the problem of a viscoelastic plate on a viscoelastic foundation for an axisymmetric loading. By adopting the plate theory of Reissner (19), the Boussinesq solution for surface loading of a half-space, and the assumption of a frictionless interface, expressions are obtained for the plate deflection, the plate bending moment, and the interface pressure, for an elastic plate on an elastic subgrade. From the expressions mentioned above, and using the Laplace transform method, corresponding expressions are found for a linear viscoelastic plate resting on a linear viscoelastic foundation. As an example of the application of the method, the problem of a linear viscoelastic plate under a uniform circular load resting on a linear viscoelastic foundation is solved. Both the plate and the foundation are assumed to be incompressible and are assumed to behave as Maxwell materials. Expressions for the
plate deflection and the interface pressure are found, from which numerical results are obtained for the maximum plate deflection and the maximum interface pressure by using a computer and Fortran programming for a number of cases.

The topic of an elastic beam on a viscoelastic foundation, or a viscoelastic beam on an elastic foundation, has enjoyed attention too.

Freudenthal and Lorch(22) have solved the problem of an infinite elastic beam on a linear viscoelastic foundation. Pister and Westman(23) have presented a qualitative, simplified analysis of the problem of a viscoelastic pavement over an elastic subgrade: The system is subjected to a concentrated load, moving over the surface of the pavement. They have used the differential equation governing the deflection of an elastic beam on a Winkler foundation, and a simple mechanical model (a Maxwell element connected in parallel with a spring), to represent the modulus $E$ of the bituminous material. The authors have presented graphs for the variation with velocity of the deflection under the load and the curvature of the surface of the pavement. Comparison is made with corresponding graphs assuming the beam to be elastic. In this work, the authors have also found the error involved in assuming the modulus $E$ to be the same in tension and compression instead of a bilinear analysis.
From the foregoing survey of the solutions of viscoelastic problems, it is found that the authors have intended to show qualitatively the basic ideas involved in viscoelasticity. Simplifying assumptions regarding the nature of the viscoelastic material and the theories used, are common to all works.
CHAPTER 3

OBJECTIVE AND ANALYSIS

Objective

This work is aimed at developing general expressions for the stresses and displacements in a semi-infinite viscoelastic body, due to two types of uniform circular loads whose intensity vary with time. The loads are suddenly applied at the surface of an initially undisturbed material and are maintained thereafter. The problem will be analyzed for two cases of sinusoidal loading, namely:

Case I: \( p(t) = p_0 \sin(\omega t) \)

Case II: \( p(t) = p_0 \sin^2(\omega t) \).

In the above expressions, \( p(t) \) is the intensity of the circular load at any time \( t \), \( p_0 \) is the maximum intensity of the load (amplitude of the sinusoidal load), and \( \omega \) is the circular frequency of the applied sinusoidal loads.

For both cases of loading shown above \( p(t) = 0 \) when \( t = 0 \).

Using the Laplace transform method suggested by Lee(12) and described in Chapter 2, the expressions for \( \sigma_r, U, \) and \( w \) (the radial stress, the horizontal and the vertical displacements, respectively) will be found in general terms for any linear viscoelastic material; then by selecting the operators involved, specific solutions will be obtained for several types of linear viscoelastic materials, as characterized by different mechanical models. Comparison of the results of the different models will be made at different times.
The stresses and displacements found by the theory of viscoelasticity will be compared with those found by the theory of elasticity.

A practical application of the solutions will be shown by finding the approximate solution for the problem of a uniform circular load, moving on the surface of a semi-infinite viscoelastic body.

Analysis

Introduction

Figure 1 shows a semi-infinite viscoelastic body under a circular load whose intensity varies with time. The material of the viscoelastic body is assumed to be homogeneous, isotropic, and linearly viscoelastic.

![Figure 1. Semi-infinite Viscoelastic Body Under a Varying Load.](image)

As in the case of the theory of elasticity, the homogeneity condition on the material requires that the viscoelastic material be uniformly distributed throughout the body. Isotropy requires that the material's properties (stress-strain relations) be independent of the direction within the material. Linearity of an elastic material requires the proportionality of stress and strain, i.e., material's obedience of Hooke's Law. In the case of a viscoelastic material, linearity
requires that the stress-strain tensors be related by linear differential or integral operators which are functions of time only.

For a viscoelastic material which exhibits a linear behavior, Boltzmann's principle of superposition holds (see p. 198, reference 26), and the elastic-viscoelastic analogy used in this work is valid(37). Whether a viscoelastic material is linear or non-linear, can only be determined experimentally. If a sample of a given viscoelastic material is loaded with a constant stress $\sigma$, and a creep curve $e(t)$ where $e$ denotes the resulting strain is observed, then if another sample of the same material is loaded with a constant stress $2\sigma$, and a creep curve $2e(t)$ is observed, the material is linear. Most Civil Engineering materials exhibit linearity at sufficiently low levels of stress compared to their strength (see p. 139, reference 32 and see reference 39).

For a homogeneous isotropic, linear viscoelastic material, the stress strain relations are given in the literature by the well-known equations:

$$P(t) S_{ij} = Q(t) e_{ij} \quad (1)$$

$$P'(t) \sigma_{ii} = Q'(t) \xi_{ii} \quad (2)$$

In equations 1 and 2, $P$, $Q$, $P'$ and $Q'$ are linear operators of the form:

$$a_o + a_1 \frac{\partial}{\partial t} + a_2 \frac{\partial^2}{\partial t^2} + a_3 \frac{\partial^3}{\partial t^3} + \cdots + a_n \frac{\partial^n}{\partial t^n}.$$
The coefficients $a$ and the summation variable $n$ are, in general, different for each operator. $\sigma_{ij}$ and $\epsilon_{ij}$ are the stress and strain tensors, respectively. The strains are considered to be infinitesimal. $S_{ij}$ and $e_{ij}$ are respectively their deviators, defined as

$$S_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij}$$

$$e_{ij} = \epsilon_{ij} - \frac{1}{3} \epsilon_{kk} \delta_{ij}$$

where,

$\delta_{ij}$ is the Kronecker delta, defined as

$$\delta_{ij} = 1, \ i = j; \quad \delta_{ij} = 0, \ i \neq j$$

Equation 1 represents the behavior of a linear viscoelastic material in shear, and equation 2 represents its behavior under average hydrostatic tension and the associated dilatation (volume change). An alternative way to represent the stress-strain relation of a viscoelastic material, is the use of the creep or relaxation function in the form of a hereditary integral:

$$\epsilon = \int_{-\infty}^{t} J(t - \tau) \frac{d\sigma(\tau)}{d\tau} d\tau$$

where $\epsilon$ is the component of strain, $\sigma$ is the related stress, $t$ is the time of loading, and $J(t)$ is the creep compliance. A convenient method to facilitate the formulation of equations 1 and 2 is the use of mechanical models, whose overall behavior approximates the overall behavior of the viscoelastic material. Standard textbooks on rheology and many of the references given at the end of this work contain detailed information about mechanical models. Alfrey and Doty(27) present a
method of finding the equivalence between operator equations and mechanical models.

In the case of an isotropic linear elastic material, corresponding to equations 1 and 2, the stress-strain relations involve two elastic constants, namely, the shear modulus, denoted by \( G \), and the bulk modulus denoted by \( K \). An alternative set of constants used in the theory of elasticity are \( E \) (modulus of elasticity) and \( \mu \) (Poisson's ratio). From theory of elasticity the relation between the four elastic constants mentioned above follows:

\[
E = \frac{9KG}{3K + G} \quad (3)
\]

\[
\mu = \frac{3K - 2G}{2(3K + G)} \quad (4)
\]

Equations 3 and 4 are equations 30 of (15).

Method of solution

Basically, there are two methods for the solution of viscoelastic problems. They are (1) the transform method, including both the Laplace transform, and the Fourier transform; (2) the use of hereditary functions. For details of the second method, the reader is referred to (15). For the problem of this dissertation, the Laplace transform method and the use of the solutions of elasticity, as pre-
sented by Lee(12), is more convenient. The method consists of two steps. The first step is to transform the viscoelastic problem into an associated elastic problem. The second step is to take the inverse Laplace transform of the solution of the associated elastic problem, thereby obtaining the solution of the original viscoelastic problem.

Concerning the first step, equations 1 and 2, after transformation become:

\[ P(s) S^*_{ij} = Q(s) e^*_{ij} \quad (5) \]
\[ P'(s) \sigma^*_{ii} = Q'(s) \varepsilon^*_{ii} \quad (6) \]

The operators \( P, Q, P', \) and \( Q' \), are polynomials in the parameter \( s \) which is independent of time. The relation between the elastic constants and the operators of equations 5 and 6, is

\[ Q(s)/P(s) = 2G \quad (7) \]
\[ Q'(s)/P'(s) = 3K \quad (8) \]

9. The Laplace transform method is more convenient for the following reasons: (1) solution of the elastic body is already known, (2) the problem involves a constant geometry and the load and geometry functions separate out, (3) in the solution of the elastic body, the elastic constants either appear as a multiplying factor (see eq.9), or as rational fraction multiplied by the load and geometry functions (see eqns. 12 and 13), and (4) for the sinusoidal loads considered in this work, the solution of the associated elastic problem consists of proper rational fractions, the inverse Laplace transform of which is easily obtained. The second method leads to integral equations and in the second method, one does not make use of the solution of the elastic body.
where G is the shear modulus and K is the bulk modulus. From equations 3 and 4 and equations 7 and 8, one can express the elastic constants E and \( \mu \) appearing in the solutions of the elastic body, in terms of their equivalent rate operators in the s domain.

Equations 9-13 are the general expressions for the stresses and displacements at any point in a semi-infinite elastic body, caused by a uniform circular load \( p(t) \), whose intensity \( p \) varies with time. The load is assumed to be quasi-static, so that inertia forces may be neglected.

\[
\sigma_r = \left[ p(t) \right] \cdot \mu \cdot i_1(r, z, a) + \left[ p(t) \right] \cdot F_1(r, z, a) \\
\sigma_z = \left[ p(t) \right] \cdot g_i(r, z, a) \\
\tau_{rz} = \left[ p(t) \right] \cdot h_i(r, z, a)
\]

\[
U = \left[ p(t) \right] \left[ \frac{1 + \mu}{E} \right] \left( 1 - 2\mu \right) Q_i(r, z, a) \\
+ \left[ p(t) \right] \left[ \frac{1 + \mu}{E} \right] q_i(r, z, a) \\
w = \left[ p(t) \right] \left[ \frac{1 - \mu^2}{E} \right] G_i(r, z, a) \\
- \left[ p(t) \right] \left[ \frac{1 + \mu}{E} \right] H_i(r, z, a)
\]

where \( \sigma_r \) is the radial horizontal stress, \( \sigma_z \) is the vertical stress, \( \tau_{rz} \) is the radial-vertical shear stress, \( U \) is the horizontal displacement, and \( w \) is the vertical displacement.

In the above equations, the subscript \( i \) has four values corresponding to the four values of \( r \). They are \( i = 1, r = 0; \ i = 2, r < a; \ i = 3, r = a; \ i = 4, r > a \), where \( a \) is the radius of the loaded area and \( r \) is the radial
offset from the center of the load. The functions $f_i$, $F_i$, $g_i$, ..., $H_i$ are functions of the geometry alone. They are independent of time. The expression for those functions corresponding to each $i$ is given in Appendix A.

Equation 10 represents the normal stress and equation 11 represents the vertical shearing stress in the material. These two equations do not have any elastic constants in them. This means that the normal and shearing stresses are independent of the type of material, or that it makes no difference whether the material is linearly elastic or linearly viscoelastic. The efforts of this work will therefore be concentrated on equations 10, 12, and 13. The combinations of elastic constants appearing in them are shown in Table 1, in terms of the operators which are functions of the parameter $s$.

The associated elastic problem

The radial stress and the displacements for the associated elastic problem are obtained by substituting in equations 9, 12, and 13 for $p(t)$ its Laplace transform in terms of the parameter $s$, and for the elastic constants, the equivalent operator form obtained from Table 1, as shown below.
Table 1. Elastic Constants in Terms of the Operators

<table>
<thead>
<tr>
<th>Elastic Constant</th>
<th>Operators–Functions of $s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>$(Q'/P' - Q/P)/(2Q'/P' + Q/P)$</td>
</tr>
<tr>
<td>$E$</td>
<td>$(3Q'/P') (Q/P)/(2Q'/P' + Q/P)$</td>
</tr>
<tr>
<td>$\frac{1 + \mu}{E}$</td>
<td>$P/Q$</td>
</tr>
<tr>
<td>$\frac{(1 + \mu)(1 - 2\mu)}{E}$</td>
<td>$3/(2Q'/P' + Q/P)$</td>
</tr>
<tr>
<td>$\frac{1 - \mu^2}{E}$</td>
<td>$(Q'/P' + 2Q/P)/(Q/P)$ $(2Q'/P' + Q/P)$</td>
</tr>
</tbody>
</table>
\( \boldsymbol{\sigma}^*(r,z,a,s) = \left[ p(s) \right] \left\{ \frac{Q'(s)/P'(s) - Q(s)/P(s)}{2Q'(s)/P'(s) + Q(s)/P(s)} \right\} f_1(r,z,a) + \left[ p(s) \right] F_1(r,z,a) \)  

(14)

\[ U^*(r,z,a,s) = \left[ p(s) \right] \left[ \frac{3}{2Q'(s)/P'(s) + Q(s)/P(s)} \right] Q_i(r,z,a) + \left[ p(s) \right] \left[ \frac{P(s)/Q(s)}{Q_i(r,z,a)} \right] q_i(r,z,a) \]  

(15)

\[ w^*(r,z,a,s) = \left[ p(s) \right] \left\{ \frac{Q'(s)/P'(s) + 2Q(s)/P(s)}{2Q'(s)/P'(s) + Q(s)/P(s)} \right\} G_i(r,z,a) - \left[ p(s) \right] \left[ \frac{P(s)/Q(s)}{H_i(r,z,a)} \right] \]  

(16)

Equations 14-16 hold true for any type of varying load as long as it is quasi-static. The equations are also true for any type of homogeneous, isotropic, linear viscoelastic material. These equations represent the solution of a semi-infinite body of the same shape as the elastic body, but with its elastic constants being functions of the rate operators: \( P(s), Q(s), P'(s), \) and \( Q'(s) \), where \( s \) is a parameter independent of time.

For a known load and viscoelastic material, equations 14-16 become a function of \( s \) multiplied by a function of geometry. The second step consists of taking the inverse Laplace transform of equations 14-16. This gives the solution of the original viscoelastic problem. The radial stress and the displacements will be functions of geometry and time of loading. The difficulty in obtaining the inverse transform of equations 14-16 depends on the form of the function of \( s \).
The loads and the linear viscoelastic materials for which solutions are obtained are shown below.

**Loads**

<table>
<thead>
<tr>
<th>Case</th>
<th>Load Function</th>
<th>Laplace Transform of Load</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case I:</td>
<td>$p(t) = p_0 \sin(\omega t)$</td>
<td>$\frac{p_0 \omega}{s^2 + \omega^2}$</td>
</tr>
<tr>
<td>Case II:</td>
<td>$p(t) = p_0 \sin^2(\omega t)$</td>
<td>$\frac{2p_0 \omega^2}{s(s^2 + 4\omega^2)}$</td>
</tr>
</tbody>
</table>

**Viscoelastic materials**

For an incompressible viscoelastic material ($\mu = 0.5$), and for a viscoelastic material for which $\mu = 0$, the equation of the radial stress (eqn. 14) becomes identical with the equation of the radial stress for an elastic material; with the corresponding values of $\mu$ as mentioned above. The equations of the horizontal and vertical displacements simplify and are dependent on $E$ only. Solutions will be obtained for these two types of materials later in this work. Consider now linear viscoelastic materials which are perfectly elastic under hydrostatic pressures, but are linearly viscoelastic in shear. This corresponds to saying that $Q'/P' = C$, where $C$ is a constant and that $Q/P$ is time dependent. $Q(t)$ and $P(t)$ are linear differential operators which can be represented by mechanical models. Differential equations of higher order correspond to more complicated models. Alfry and Doty(27) show how to derive one from the other.
In this work are analyzed linear viscoelastic materials whose behavior in shear is represented by (1) a Maxwell model, (2) a Kelvin model, (3) a standard linear solid model, and (4) a Burgers model. These models together with stress–strain relations are shown in Table 2. The dimension of the elastic element of the model is in FL$^{-2}$, and the dimension of the viscous element of the model is in FTC$^{-2}$.

Following the two steps discussed under "method of solution," $Q(s)/P(s)$ is substituted for each model in equations 14–16 with $Q'/P'=C$ for all models, and when the Laplace transform of the load function is substituted in the said equations, the associated elastic solution for that model is obtained. The inverse of the latter gives the solution of the viscoelastic material under consideration.

As an example, the associated elastic solution for a Maxwell body is shown below. The associated elastic solution for the other three models is shown in Appendix B.

Example: The associated elastic solution for a linear viscoelastic material, which behaves as a Maxwell body in shear, but is perfectly elastic under hydrostatic pressures is obtained as follows:

Substitute $Q(s)/P(s) = ABs/(Bs + A)$ in equations 14–16, and the proper transform of the load function, and obtain:
Table 2. Mechanical Models and the Corresponding Stress-Strain Relations in the Time and $s$ Domains for the Materials Analyzed in this Work

<table>
<thead>
<tr>
<th>Mechanical Model</th>
<th>$P(t) \sigma = Q(t) e$</th>
<th>$Q(s)/P(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maxwell</td>
<td>$(1/A D + 1/B) \sigma = (D) e$</td>
<td>$abs/(bs+a)$</td>
</tr>
<tr>
<td>Kelvin</td>
<td>$(1) \sigma = (A+BD) e$</td>
<td>$A + Bs$</td>
</tr>
<tr>
<td>Standard L. Solid</td>
<td>$(D + \frac{A_1 + A_2}{B}) \sigma = (A_1 D + \frac{A_1 A_2}{B}) e$</td>
<td>$\frac{C + A_1 s}{C_2 + s}$</td>
</tr>
<tr>
<td></td>
<td>$C_1 = \frac{A_1 A_2}{B}$</td>
<td>$C_2 = \frac{(A_1 + A_2)/B}{C_2 + s}$</td>
</tr>
<tr>
<td>Burgers</td>
<td>$\frac{B_2}{A_1} D^2 + \frac{(1 + A_2)}{A_1}$</td>
<td>$\frac{B_2 s^2 + A_2 s}{C_3 s^2 + C_4 s + C_5}$</td>
</tr>
<tr>
<td></td>
<td>$+ \frac{B_2}{B_1} D + \frac{A_2}{B_1}$</td>
<td>$C_3 = \frac{B_2}{A_1}$</td>
</tr>
<tr>
<td></td>
<td>$\sigma = \left[ \frac{B_2 D^2 + A_2 D}{B_1} \right] e$</td>
<td>$C_4 = 1 + \frac{A_2}{A_1} + \frac{B_2}{B_1}$</td>
</tr>
<tr>
<td></td>
<td>$D = \frac{\partial}{\partial t}$</td>
<td>$C_5 = \frac{A_2}{B_1}$</td>
</tr>
</tbody>
</table>
Case I: \( p(t) = p_0 \sin(\omega t); \quad p(s) = p_0 \frac{\omega}{s^2 + \omega^2} \).

\[
\sigma^*_r = \left[ \frac{p_0 \omega}{s^2 + \omega^2} \right] \left[ \frac{(CB - AB)s + CA}{(2CB + AB)s + 2CA} \right] f_1(r, z, a) + \left[ \frac{p_0 \omega}{s^2 + \omega^2} \right] \frac{f_1(r, z, a)}{3Q_1(r, z, a)} 
\]

\[
U^* = \left[ \frac{p_0 \omega}{s^2 + \omega^2} \right] \left[ \frac{A + Bs}{(2BC + AB)s + 2CA} \right] q_1(r, z, a) + \left[ \frac{p_0 \omega}{s^2 + \omega^2} \right] \left[ \frac{Bs + A}{ABs} \right] q_1(r, z, a) 
\]

\[
w^* = \left[ \frac{p_0 \omega}{s^2 + \omega^2} \right] \left[ \frac{(2AB + CB)s + CA}{(2CB + AB)s + 2CA} \right] \left[ \frac{Bs + A}{ABs} \right] G_1(r, z, a) - \left[ \frac{p_0 \omega}{s^2 + \omega^2} \right] \left[ \frac{Bs + A}{ABs} \right] H_1(r, z, a) 
\]

In equations 17–19, the functions of \( s \) are all rational proper fractions. Taking the inverse Laplace transform by partial fractions, the radial stress and the displacements in the original viscoelastic problem are obtained as shown below.

The radial stress:

\[
\sigma_r = \left[ \frac{k_0^2 + k_2^2}{4k_0^2 + \omega^2} \right]^{1/2} \sin(\omega t + \lambda_1) + \omega \left[ \frac{k_0 - 2k_0 k_1}{4k_0^2 + \omega^2} \right] e^{-2k_0 t} f_1(r, z, a) + \sin(\omega t) \frac{f_1(r, z, a)}{F_1(r, z, a)} 
\]

10. The term multiplied by \( q_1 \) in equation 18 is the same as the term multiplied by \( H_1 \) in equation 19, with an opposite sign. This is true for all models.
The horizontal displacement

\[
U(A/P_o) = \left[ k_o (A/C) \left( \frac{1}{4 k_o} + \frac{1}{4 k_o^2 + \omega^2} \right) \frac{1}{2} \sin (\omega t + \lambda_2) + k_o \omega (A/C) (1 - 2 k_o \tau) e^{-2k_o t} \right] \cdot Q_i(r, z, a)
+ \left[ \frac{1}{\tau_0} - \frac{1}{\tau} \left( 1 + \tau^2 \omega^2 \right)^{1/2} \cos (\omega t + \lambda_3) q_i \right]
\]

The vertical displacement

\[
w(A/P_o) = \left[ \frac{1}{2\tau_0} - \omega \left( \frac{1}{2\tau} \left( \frac{1}{4 k_o^2 + \omega^2} \right) \right) e^{-2k_o t} \right]
- \left[ \frac{1}{\tau} \left( 1 + \frac{1}{\tau_0} \left( \frac{1}{2\tau} \left( \frac{1}{4 k_o^2 + \omega^2} \right) \right) \cos (\omega t + \lambda_4) \right) \right] \cdot G_i(r, z, a)
- \left[ \frac{1}{\tau} \left( 1 + \frac{1}{\tau_0} \left( \frac{1}{2\tau} \left( \frac{1}{4 k_o^2 + \omega^2} \right) \right) \cos (\omega t + \lambda_3) \right) \right] \cdot H_i(r, z, a)
\]

The notation used in equations 17–20 is as follows:

\( A/C \) = one third of the ratio of the elastic element of the model to the bulk modulus of the viscoelastic material.

\( A/P_o \) = the ratio of the elastic element of the model to the maximum intensity of the applied load.

\( \tau \) = B/A = relaxation time of the Maxwell model in seconds.

\( k_0 \) = \( CA/B (2C+A) \) = \( 1/\tau [2(A/C)] \)

\( k_1 \) = \( (C-A)(2C + A) \) = \( (1-A/C)(2+A/C) \)

11. Equation 22 is written in the form shown for the purpose of obtaining numerical values later in this work, for the vertical displacement at any point in the material.
\[ k_2 = \frac{(2A + C)}{(A + 2C)} = \left[ \frac{2(A/C) + 1}{(A/C) + 2} \right] \]

\[ \lambda_1 = \tan^{-1} \frac{\omega k_1}{k_o} - \tan^{-1} \frac{\omega}{2k_o} \]

\[ \lambda_2 = \tan^{-1} \omega \tau - \tan^{-1} \frac{\omega}{2k_o} \]

\[ \lambda_3 = \tan^{-1} \omega \tau \]

\[ \lambda_4 = \tan^{-1} \frac{\omega k_2}{k_o} + \lambda_2 \]

Equations 20–22 consist of functions of time (in the square brackets) multiplied by functions of geometry. The functions of time are dimensionless. The functions of geometry \( f_1 \) and \( F_1 \) are dimensionless, but \( q_i, Q_i, G_i, \) and \( H_i \) have the dimension of length. The functions of geometry can be found easily for any point in the material, by the proper combination of the eight functions tabulated in reference (11).

It is noticed that the expression of the radial stress and the displacements consist of sinusoidal terms with a phase difference between the applied load and the material's response, and exponentially decaying terms (transients). For example the first square bracket of the right side of equation 20 represents a sinusoid oscillating about an axis displaced along an exponentially decaying curve.

In addition to the time of loading \( (t) \), the radial stress and the displacements are functions of parameters which arise from three different sources. The parameters \( p_0 \) and \( \omega \) arise from the load function \( p(t) \). The parameters \( \tau \) and \( A/C \) are properties of the viscoelastic material. The third type of parameters depend on the geometry or the location of the point at which the stress or displace-
ment is sought. A clear picture of the variation of the radial stress and
displacements with time will be obtained when the equations are plotted
from numerical results obtained later in this work. The effect of varying
each parameter will be studied.

Case II: \( p(t) = p_o \sin(\omega t); \quad p(s) = 2p_o \frac{\omega^2}{s(s^2 + 4\omega^2)}. \)

To obtain the radial stress and the displacements due to the second
type of loading, the new quantity \( 2p_o \omega^2/s (s^2 + 4\omega^2) \) is substituted in
equations 17-19 in the place of the quantity \( p_o \omega^2(s^2 + \omega^2) \), and the
associated elastic problem for the second type of loading is obtained
for the same material as for case one of loading. As far as trans­
formations are concerned, an extra \( s \) is introduced into the denominator
of the expressions of the associated elastic problem. Then taking the
inverse Laplace transform of the resulting expressions, as was done
for the first case of loading, the radial stress and the displacements for
the second case of loading are obtained as follows:

\[
\frac{\sigma_r}{p_o} = \left[ \frac{1}{4} - \frac{\omega^2}{4} \left( \frac{1 - 2k}{k^2 + \omega^2} \right) e^{-2k_0 t} - \frac{1}{4} \left\{ \frac{k^2 + 4k_1^2 \omega^2}{k^2 + \omega^2} \right\} \right]^{1/2} \\
\cos(2\omega t + \lambda_s) f_i(r, z, a) + \left[ \sin^2(\omega t) \right] F_i(r, z, a)
\]

(23)
\[
U(A/p_0) = \left[ \frac{1}{4} \left( \frac{A}{C} \right) + \frac{\omega^2}{4} \left( \frac{A}{C} \right) \right] \left\{ \frac{2k \gamma - 1}{k_0^2 + \omega^2} \right\} e^{-2k_0 t} \\
- \frac{k_0}{4} \left( \frac{A}{C} \right) \left\{ \frac{1 + 4 \gamma^2 \omega^2}{k_0^2 + \omega^2} \right\}^{1/2} \left[ \cos (2\omega t + \lambda_6) \right] 3Q_i(r, z, a) \\
+ \left[ \frac{1}{2} + \frac{t}{2T} - \frac{1}{4T \omega} \left\{ 1 + 4 \gamma^2 \omega^2 \right\}^{1/2} \right] \sin \left( 2\omega t + \lambda_7 \right)
\]

\[q_i(r, z, a)\] (24)

\[
w(A/p_0) = \left[ \frac{2T k_0 + 2k_2 - 1}{8T k_0} \right] + \frac{t}{4T \omega} \left[ \frac{\omega^2 (1 - 2k_2) (1 - 2k_0 \omega)}{8T k_0 (k_0^2 + \omega^2)} \right] e^{-2k_0 t} \\
- \frac{1}{8T \omega} \left\{ \frac{(k_0^2 + 4k_a^2 \omega^2) (1 + 4 \gamma^2 \omega^2)}{k_0^2 + \omega^2} \right\}^{1/2} \sin \left( 2\omega t + \lambda_6 \right) G_i (r, z, a) \\
- \left[ \frac{1}{2} + \frac{t}{2T} - \frac{1}{4T \omega} \left\{ 1 + 4 \gamma^2 \omega^2 \right\}^{1/2} \right] \sin \left( 2\omega t + \lambda_7 \right) H_i (r, z, a) \\
= \left[ X_8(t) \right] G_i (r, z, a) - \left[ X_6(t) \right] H_i (r, z, a) (25)
\]

where
\[
\lambda_5 = \tan^{-1} \left( \frac{2\omega k_1 k_0}{k_0} \right) - \tan^{-1} \left( \frac{\omega}{k_0} \right) \\
\lambda_6 = \tan^{-1} \left( \frac{2\omega \tau}{\tan^{-1} \left( \frac{\omega}{k_0} \right)} \right) \\
\lambda_7 = \tan^{-1} \left( \frac{2\omega \tau}{\tan^{-1} \left( \frac{2\omega k_2 k_0}{k_0} + \lambda_6 \right)} \right)
\]

other notations are as explained before.
2. Now consider the second type of linear viscoelastic material, namely, a material which behaves in shear as a delayed elastic material represented by a Kelvin model, and behaves perfectly elastic under hydrostatic pressures. The associated elastic solution for this material is shown in Appendix B. Taking the inverse Laplace transform of the associated elastic solution, the radial stress and displacements for both cases of loading are shown below.

Case I: \( p(t) = p \sin \omega t \).

\[
\frac{\sigma_r}{P_0} = \left[ \frac{\omega\left(m_0 + m_1\right)}{m_1^2 + \omega^2} e^{-m_1 t} + \left(m_0^2 + \omega^2\right)^{1/2} \right] \\
\sin(\omega t + \lambda_9) f_i(r, z, a) + \left[ \sin(\omega t) \right] F_i(r, z, a)
\]

\[
U(A/P_0) = \left[ \frac{\omega}{\tau(m_1^2 + \omega^2)} e^{-m_1 t} + \frac{(m_1^2 + \omega^2)^{1/2}}{\tau} \right] \\
\sin(\omega t + \lambda_{10}) 3 Q_i(r, z, a) + \left[ \frac{\tau \omega}{1 + \tau^2 \omega^2} e^{-t/\tau} + \frac{1 + \tau^2 \omega^2}{\tau^2} \right] \\
\sin(\omega t + \lambda_{11}) q_i(r, z, a)
\]

\[
w(A/P_0) = \frac{\tau \omega (Am_2 \tau^2 - 2)}{(m_1^2 - 1) (1 + \tau^2 \omega^2)} e^{-t/\tau} + \frac{\omega (Am_2 \tau^2 - 2m_1)}{(1 - m_1^2 \tau) (m_1^2 + \omega^2)} e^{-m_1 t} \\
+ \left\{ \frac{\tau^2 \omega^2}{(1 + \tau^2 \omega^2) (m_1^2 + \omega^2)} \right\}^{1/2} \sin(\omega t + \lambda_{12}) G_i(r, z, a) \\
- \left[ \frac{\tau \omega}{1 + \tau^2 \omega^2} e^{-t/\tau} + \frac{1 + \tau^2 \omega^2}{\tau^2} \right] \\
\sin(\omega t + \lambda_{11}) H_i(r, z, a) = \left[ Y_\tau(t) G_i(r, z, a) - \left[ Y_4(t) \right] H_i(r, z, a) \right]
\]
Case II: \( p(t) = p_0 \sin^2(\omega t) \).

\[
\frac{\sigma_r}{p_0} = \left[ \frac{m_0}{2m_1} - \frac{2\omega^2 (m_0 + m_1)}{m_1 (m_1^2 + 4\omega^2)} \right] e^{-m_1 t} - \frac{1}{2} \left\{ \frac{m_0^2 + 4\omega^2}{m_1^2 + 4\omega^2} \right\}^{1/2} \cos (2\omega t + \lambda_{13}) f_1(r,z,a) + \left[ \sin^2(\omega t) \right] F_1(r,z,a)
\]

\[
U(A/p_0) = \left[ \frac{1}{2m_1 \tau} - \frac{2 \omega^2}{m_1 \tau (m_1^2 + 4\omega^2)} \right] e^{-m_1 t}
\]

\[
- \frac{1}{2} \left[ \frac{1}{m_1^2 + 4\omega^2} \right]^{1/2} \cos (2\omega t + \lambda_{14}) Q_1 + \left[ \frac{1}{2} - \frac{2\tau^2 \omega^2}{1 + 4\tau^2 \omega^2} \right] e^{-t/\tau} - \frac{1}{2} \left[ \frac{1}{1 + 4\tau^2 \omega^2} \right]^{1/2} \cos (2\omega t + \lambda_{15}) q_1
\]

\[
w(A/p_0) = \left[ \frac{m_2 B}{2m_1} - \frac{\tau^2 \omega^2}{1 + 4\tau^2 \omega^2} \right] e^{-t/\tau} - \frac{2 \omega^2 (m_2 B - 2m_1)}{m_1 (1-m_1 \tau)(m_1^2 + 4\omega^2)} e^{-m_1 t}
\]

\[
- \frac{1}{2} \left\{ \frac{m_2^2 B^2 + 16 \omega^2}{(1+4\tau^2 \omega^2) (m_1^2 + 4\omega^2)} \right\}^{1/2} \cos (2\omega t + \lambda_{16}) G_1(r,z,a) - \left[ \frac{1}{2} - \frac{2\tau^2 \omega^2}{1 + 4\tau^2 \omega^2} \right] e^{-t/\tau} - \frac{1}{2} \left[ \frac{1}{1 + 4\tau^2 \omega^2} \right]^{1/2} \cos (2\omega t + \lambda_{15}) H_1
\]

\[
= \left[ Y_8(t) \right] G_1(r,z,a) - \left[ Y_6(t) \right] H_1(r,z,a)
\]

Notations used in the equations of the Kelvin model.

- \( A/C \) = one third of the ratio of the elastic element of the Kelvin model to the bulk modulus of the viscoelastic material.

- \( \tau \) = retardation time of the Kelvin model = \( B/A \), seconds.
\[ m_0 = \frac{(C - A)/B}{(A/C)/T} \]
\[ m_1 = \frac{(2C + A)/B}{(2 + (A/C))/T(A/C)} \]
\[ Bm_2 = \frac{(C + 2A)/B}{[1 + 2(A/C)]/T(A/C)} \]

\[ \lambda_9 - \lambda_{16} \] are angles in radians, defined as:
\[ \lambda_9 = \tan^{-1} \frac{\omega}{m_0} - \tan^{-1} \frac{\omega}{m_1} \]
\[ \lambda_{10} = \tan^{-1} \frac{\omega}{m_1} \]
\[ \lambda_{11} = \tan^{-1} \frac{\omega}{T} \]
\[ \lambda_{12} = \tan^{-1} \frac{2\omega}{Bm_2} - \tan^{-1} \frac{\omega}{m_1} - \lambda_{11} \]
\[ \lambda_{13} = \tan^{-1} \frac{2\omega}{m_2} - \tan^{-1} \frac{2\omega}{m_1} \]
\[ \lambda_{14} = -\tan^{-1} \frac{2\omega}{m_1} \]
\[ \lambda_{15} = -\tan^{-1} \frac{2\omega}{T} \]
\[ \lambda_{16} = \tan^{-1} \frac{4\omega}{Bm_2} - \tan^{-1} \frac{2\omega}{m_1} + \lambda_{15} \]

3. The radial stress and the displacements in a linear viscoelastic material which behaves as a standard linear solid in shear but is perfectly elastic under hydrostatic pressures are found below.

The mechanical model of a standard linear solid consists of a Kelvin unit connected in series to a spring, as shown in Table 2. The associated elastic solution for this type of material is shown in Appendix B, for Case I of loading. For Case II of loading, the associated elastic solution can be obtained by using the Laplace transform of the second type of
loading, as explained before. Taking the inverse Laplace transform of
the associated elastic solution, for each type of loading, the radial stress
and the displacements are found as shown below.

Case I: \( p(t) = p_0 \sin(\omega t) \)

\[
\frac{\sigma_r}{p_0} = \left[ \frac{(n_0^n - n_2^n)\omega}{n_2^n + \omega^2} \right] e^{-n_2t} + \left( \frac{n_0^n + n_2^n}{n_2^n + \omega^2} \right)^{1/2} \sin(\omega t + l_1) f_i(r, z, a) + \left[ \sin(\omega t) \right] F_i(r, z, a)
\]

(32)

\[
U(A_1/p_0) = \left[ \frac{A_1\omega(n_4^n - n_2^n)}{n_2^n + \omega^2} \right] e^{-n_2t} + A_1 \left( \frac{n_4^n + n_2^n}{n_2^n + \omega^2} \right)^{1/2} \sin(\omega t + l_2) 3Q_i
\]

+ \left[ \frac{\tau\omega(C_2^n - 1)}{1 + \tau^2 \omega^2} \right] e^{-t/\tau} + \tau \left( \frac{C_2^n + \omega^2}{1 + \tau^2 \omega^2} \right)^{1/2} \sin(\omega t + l_3) q_i
\]

(33)

\[
w(A_1/p_0) = \left[ \frac{\tau \omega(C_2^n - 1)}{(n_2^n - 1)(1 + \tau^2 \omega^2)} \right] e^{-t/\tau} + \frac{C_2^n + \omega^2}{(1 + \tau^2 \omega^2)} \left( \frac{n_4^n + n_2^n}{n_2^n + \omega^2} \right)^{1/2} \sin(\omega t + l_4) G_i(r, z, a)
\]

- \left[ \frac{\tau \omega(C_2^n - 1)}{1 + \tau^2 \omega^2} \right] e^{-t/\tau} + \tau \left( \frac{C_2^n + \omega^2}{1 + \tau^2 \omega^2} \right)^{1/2} \sin(\omega t + l_3) H_i
\]

= \left[ Z_7(t) \right] G_i(r, z, a) - \left[ Z_4(t) \right] H_i(r, z, a)
\]

(34)
Case II: \( p(t) = p_0 \sin^2 \omega t \).

The radial stress:

\[
\frac{\sigma_r}{p_0} = \left[ \frac{n_0}{2n_2} - 2W^2 \left( \frac{n_0 - n_1 n_2}{n_2(n_2^2 + 4W^2)} \right) e^{-n_2 t} \right.
- \frac{1}{2} \left\{ \frac{n_0^2 + 4n_1^2 (W^2)}{n_2^2 + 4(W^2)} \right\}^{1/2} \\
\left. \cos (2\omega t + l_5) \right\} f_1(r,z,a) + \left[ \sin^2 Wt \right] F_1(r,z,a)
\]

(35)

The horizontal displacement:

\[
U(A_1/p_0) = \left[ \frac{A_1 n_4}{2n_2} - 2A_1 W^2 \left\{ \frac{n_4 - n_2 n_3}{n_2(n_2^2 + 4W^2)} \right\} e^{-n_2 t} \right.
- \frac{A_1}{2} \left\{ \frac{n_4^2 + 4n_3^2 (W^2)}{n_2^2 + 4(W^2)} \right\}^{1/2} \\
\left. \cos (2\omega t + l_6) \right\} f_1(r,z,a) + \frac{C_2 \tau}{2} - \frac{2}{1 + 4\tau^2} \left( C_2 \tau - 1 \right) e^{-t/\tau} \\
\left. \frac{C_2^2 + 4(W^2)}{1 + 4\tau^2 (W^2)} \right\}^{1/2} \cos (2\omega t + l_7) \right\} q_1(r,z,a)
\]

(36)

The vertical displacement:

\[
w(A_1/p_0) = \left[ \frac{C_2 n_6 \tau}{2n_2} - \frac{2W^2 \tau^2 (C_2 \tau - 1)}{n_2 (\tau - 1)} \left( n_6 - n_5 \right) e^{-t/\tau} \right.
- \frac{2\tau W^2 (C_2 \tau - n_2) \left( n_6 - n_2 n_5 \right)}{n_2^2 (1 - n_2 \tau) \left( n_2^2 + 4\tau^2 W^2 \right)} e^{-n_2 t} \\
- \frac{\tau}{2} \left\{ \frac{(C_2^2 + 4W^2) \left( n_6^2 + 4n_5^2 \right)}{(1 + 4\tau^2 W^2) \left( n_2^2 + 4\tau^2 W^2 \right)} \right\} \left[ C_2 \tau - \frac{2\tau W^2 (C_2 \tau - 1)}{1 + 4\tau^2 W^2} \right] e^{-t/\tau} \\
\left. \cos (2\omega t + l_6) \right\} G_1(r,z,a) - \left[ \frac{C_2 \tau}{2} - \frac{2\tau W^2 (C_2 \tau - 1)}{1 + 4\tau^2 W^2} \right] e^{-t/\tau} \\
\left. \frac{C_2^2 + 4(W^2)}{1 + 4\tau^2 (W^2)} \right\} \left[ \cos (2\omega t + l_7) \right] \right\} H_i(r,z,a)
\]
\[ w \left( \frac{A_1}{p_o} \right) = \left[ Z_8(t) \right] G_4(r, z, a) - \left[ Z_6(t) \right] H_4(r, z, a) \] (37)

Notations used in equations 32-37.

\[ C_1 = \frac{A_1 A_2}{B} \quad ; \quad C_2 = \frac{(A_1 + A_2)}{B} \]

\[ \tau = \frac{B}{A_2}, \text{ seconds} \]

\[ n_0 = \frac{(C C_2 - C_1)}{(2C + A_1)} \]

\[ n_1 = \frac{(C - A_1)}{(2C + A_1)} \]

\[ n_2 = \frac{(2C C_2 + C_1)}{(2C + A_1)} \]

\[ n_3 = \frac{1}{(2C + A_1)} \]

\[ n_4 = \frac{C_2}{(2C + A_1)} \]

\[ n_5 = \frac{(C + 2A_1)}{(2C + A_1)} \]

\[ n_6 = \frac{(2C_1 + C C_2)}{(2C + A_1)} \]

\[ \alpha_1 - \alpha_8 \] are angles in radians defined as:

\[ \alpha_1 = \tan^{-1} \frac{\omega}{n_0} - \tan^{-1} \frac{\omega}{n_2} \]

\[ \alpha_2 = \tan^{-1} \frac{\omega}{n_4} - \tan^{-1} \frac{\omega}{n_2} \]

\[ \alpha_3 = \tan^{-1} \frac{\omega}{C_2} - \tan^{-1} \frac{\omega}{\tau} \]

\[ \alpha_4 = \alpha_3 + \tan^{-1} \frac{\omega}{n_5} \frac{n_5}{n_6} - \tan^{-1} \frac{\omega}{n_2} \]

\[ \alpha_5 = \tan^{-1} 2 \omega \frac{n_1}{n_0} - \tan^{-1} 2 \omega \frac{n_6}{n_2} \]

\[ \alpha_6 = \tan^{-1} 2 \omega \frac{n_3}{n_4} - \tan^{-1} 2 \omega \frac{n_6}{n_2} \]

\[ \alpha_7 = \tan^{-1} 2 \omega \frac{n_5}{n_6} - \tan^{-1} 2 \omega \frac{n_6}{n_2} \]

\[ \alpha_8 = \tan^{-1} 2 \omega \frac{n_5}{n_6} - \tan^{-1} 2 \omega \frac{n_6}{n_2} + \alpha_7 \]
4. The radial stress and the displacements in a linear viscoelastic material, which is perfectly elastic under normal pressures, but whose behavior in shear is represented by a Burgers model, are shown.

The Burgers model consists of a Kelvin unit connected in series to a Maxwell unit. The model and its stress-strain relations are shown in Table 2. As in the case of the previous models, the inverse Laplace transform is taken of the associated elastic solution for this model, shown in Appendix B, and the radial stress and the displacements for each type of loading, are obtained as shown below.

Case I: \( p(t) = p_0 \sin \omega t \).

\[
\frac{\sigma_r}{p_0} = \left\{ \begin{array}{l}
a_2 \left[ \frac{\left( a_o - \omega^2 \right)^2 + a_1^2 \omega^2}{(\alpha^2 - \gamma^2 - \omega^2)^2 + 4\alpha^2 \omega^2} \right]^{1/2} \sin (\omega t + \psi_1) \\
+ \frac{\omega a_2 \left[ (\gamma - \alpha) (\gamma - \alpha + a_1) + a_o \right]}{2 \gamma \left( \omega^2 + (\gamma - \alpha)^2 \right)} e^{(\gamma - \alpha)t} \\
+ \frac{\omega a_2 \left[ (\gamma + \alpha) (a_1 - \gamma - \alpha) - a_o \right]}{2 \gamma \left( \omega^2 + (\gamma + \alpha)^2 \right)} e^{-(\alpha + \gamma)t} \end{array} \right\}
\]

\[
f_1(r, z, a) + \left[ \sin \omega t \right] F_1(r, z, a)
\]

\[
e \left\{ \delta_1(t) \right\} f_1(r, z, a) + \left[ \delta_2(t) \right] F_1(r, z, a)
\]
\[
{U(A_1/p_o)} = \left\{ \begin{array}{l}
M \left[ \frac{(b_0 - \omega^2)^2 + b_1^2 \omega^2}{(\alpha^2 - \gamma^2 - \omega^2)^2 + 4 \alpha^2 \omega^2} \right]^{1/2} \\
\sin(\omega t + \psi_2) \\
+ \frac{\omega M}{2} \left[ \frac{(\gamma - \alpha)(\gamma - \alpha + b_1) + b_0}{\omega^2 + (\gamma - \alpha)^2} \right] e^{(\gamma - \alpha)t} \\
+ \frac{\omega M}{2} \left[ \frac{(\gamma + \alpha)(b_1 - \gamma - \alpha) - b_0}{\omega^2 + (\gamma + \alpha)^2} \right] e^{-(\alpha + \gamma)t} \\
\end{array} \right. \\
Q_i(r,z,a) \\
+ \left\{ \frac{b_o}{b_3\omega} - \frac{\omega}{b_3} \left( \frac{b_3^2 - b_1 b_0 + b_0}{b_3^2 + \omega^2} \right) e^{-b_3 t} \\
- \frac{1}{\omega} \left[ \frac{(b_0 - \omega^2)^2 + b_1^2 \omega^2}{b_3^2 + \omega^2} \right]^{1/2} \\
\cos(\omega t + \psi_3) \right\} q_i(r,z,a) \\
= \left\{ \phi_5(t) \right\} Q_i(r,z,a) + \left\{ \phi_6(t) \right\} q_i(r,z,a) \quad (39)
\]
\[ w \left( \frac{A_1}{p_0} \right) = \left\{ \begin{array}{l} \frac{N d_0 b_0}{b_3 \omega (\alpha^2 - \gamma^2)} - \frac{\omega N (b_3^2 - b_3 d_1 + d_3) (b_3^2 - b_1 b_3 + b_0)}{b_3 (b_3^2 + \omega^2) [(\alpha - b_3)^2 - \gamma^2]} e^{-b_3 t} \\
\quad - \frac{N}{\omega} \left[ \frac{\left\{ (d_o - \omega^2)^2 + d_1^2 \omega^2 \right\}}{b_3^2 + \omega^2} \right] \left[ \frac{(b_o - \omega^2)^2 + b_1^2 \omega^2}{(\alpha^2 - \omega^2 - \gamma^2)^2 + 4 \alpha^2 \omega^2} \right] \right. \\
\quad \left. \cos (\omega t + \psi_4) \right\} \\
\quad + \frac{N \omega [ (\gamma - \alpha)(\gamma - \alpha + b_1) + b_0 ]}{2 \gamma (\gamma - \alpha) [ (\gamma - \alpha)^2 + \omega^2 ]} \left[ (\gamma - \alpha) (\gamma - \alpha + d_1) + d_0 \right] e^{(\gamma - \alpha) t} \\
\quad + \frac{N \omega [ (\alpha + \gamma)(\alpha + \gamma - b_1) + b_0 ]}{2 \gamma (\alpha + \gamma) [ (\alpha + \gamma)^2 + \omega^2 ]} \left[ (\alpha + \gamma) (\alpha + \gamma - d_1) + d_0 \right] e^{-(\alpha + \gamma) t} \\
\quad G_i (r, z, a) \\
\quad - \left\{ \frac{b_o}{b_3 \omega} - \frac{\omega (b_3^2 - b_1 b_3 + b_0)}{b_3 (b_3^2 + \omega^2)} \right\} e^{-b_3 t} \\
\quad - \frac{1}{\omega} \left[ \frac{(b_o - \omega^2)^2 + b_1^2 \omega^2}{b_3^2 + \omega^2} \right]^{1/2} \right\} H_i (r, z, a) \\
\quad = \left\{ \phi_9(t) \right\} G_i (r, z, a) - \left\{ \phi_6(t) \right\} H_i (r, z, a) \quad (40) \]
Case II: \( p(t) = p_0 \sin^2 \omega t \)

\[
\sigma_r^2 \bigg/ p_0 = \left\{ \begin{array}{l}
a_0 \ a_2 \\
2 (\alpha^2 - \gamma^2)
\end{array} \right\} - \frac{a_2}{2} \left[ \frac{(a_0 - 4 \omega^2)^2 + 4 a_1^2 \omega^2}{(\alpha^2 - \gamma^2)^2 - 4 \omega^2 \gamma^2 + 16 \alpha^2 \omega^2} \right]^{1/2}
\]

\[
\cos (2 \omega t + \theta) \]

\[
+ \frac{a_2 \omega^2 \left[ (\gamma - \alpha) (\gamma - \alpha + a_1) + a_0 \right]}{\gamma (\gamma - \alpha) \left[ (\gamma - \alpha)^2 + 4 \omega^2 \right]} e^{(\gamma - \alpha)t} \]

\[
+ \frac{a_2 \omega^2 \left[ (\alpha + \gamma) (\alpha + \gamma - a_1) + a_0 \right]}{\gamma (\alpha + \gamma) \left[ (\alpha + \gamma)^2 + 4 \omega^2 \right]} e^{-(\alpha + \gamma)t} \}
\]

\[
f_i(r,z,a) + \left[ \sin^2 \omega t \right] F_i (r,z,a) = \varphi_3 f_i + \varphi_4 F_i \] (41)

\[
U (A_1/p_0) = \left\{ \begin{array}{l}
\frac{b_0 M}{2 (\alpha^2 - \gamma^2)} - \frac{M}{2} \left[ \frac{(b_o - 4 \omega^2)^2 + 4 b_1^2 \omega^2}{(\alpha^2 - \gamma^2)^2 - 4 \omega^2 \gamma^2 + 16 \alpha^2 \omega^2} \right]^{1/2}
\end{array} \right\}
\]

\[
\cos (2 \omega t + \theta) \]

\[
+ \left[ \frac{\omega^2 M \left[ (\gamma - \alpha) (\gamma - \alpha + b_1) + b_0 \right]}{\gamma (\gamma - \alpha) \left[ (\gamma - \alpha)^2 + 4 \omega^2 \right]} \right] e^{(\gamma - \alpha)t} \]

\[
+ \frac{\omega^2 M}{\gamma} \left[ \frac{(\alpha + \gamma) (\alpha + \gamma - b_1) + b_0 \right]}{(\alpha + \gamma) \left[ (\alpha + \gamma)^2 + 4 \omega^2 \right]} \right] e^{-(\alpha + \gamma)t} \}
\]

\[
Q_i(r,z,a) + \left\{ \begin{array}{l}
\frac{b_1 b_3 - b_0}{2 b_3^2} + \frac{b_0 t}{2 b_3} + \frac{2 \omega^2 (b_3^2 - b_1 b_3 + b_0)}{b_3^2 (b_3^2 + 4 \omega^2)} e^{-b_3 t} \\
- \frac{1}{4 \omega} \left[ \frac{(b_o - 4 \omega^2)^2 + 4 b_1^2 \omega^2}{b_3^2 + 4 \omega^2} \right]^{1/2}
\end{array} \right\} q_i(r,z,a) \]

\[
= \left\{ \begin{array}{l}
\varphi_7(t) \end{array} \right\} Q_i(r,z,a) + \left\{ \begin{array}{l}
\varphi_8(t) \end{array} \right\} q_i(r,z,a) \] (42)
\[
\begin{align*}
\mathcal{W}(A_1/P_0) &= \left\{ \frac{N b_3 (\alpha^2 - \gamma^2) (b_o d_1 + d_o b_1) - N b_o d_o (2 \alpha b_3 - \gamma^2 + \alpha^2)}{2 b_3 (\alpha^2 - \gamma^2)^2} \right. \\
&+ \frac{N d_o b_o}{2 b_3 (\alpha^2 - \gamma^2)} t + \frac{2 \omega^2 N (b_3^2 - b_1 b_3 + b_o) (b_3^2 - d_1 b_3 + d_o)}{b_3^2 (b_3^2 + 4 \omega^2)} e^{-b_3 t} \\
&\left. - \frac{N}{4 \omega} \left[ \left\{ \frac{(b_o - 4 \omega^2)^2 + 4 \omega^2 b_1^2}{(\alpha^2 - 4 \omega^2)^2 + 16 \alpha \omega^2} \right\} \right]^{1/2} \right. \\
&\left. + \frac{N \omega^2}{2} \left[ \frac{((\gamma - \alpha) (\gamma - \alpha + b_1) + b_o) \left\{ \frac{(\gamma - \alpha) (\gamma - \alpha + d_1) + d_o}{(\gamma - \alpha)^2 + 4 \omega^2} \right\} e^{(\gamma - \alpha) t}}{(\gamma - \alpha)^2} \right. \\
&\left. + \frac{N \omega^2}{4} \left[ \frac{(\gamma + \gamma) (\gamma + \gamma - b_1) + b_o) \left\{ \frac{(\gamma + \gamma) (\gamma + \gamma - d_1) + d_o}{(\gamma + \gamma)^2 + 4 \omega^2} \right\} e^{(\gamma + \gamma) t}}{(\gamma + \gamma)^2} \right. \\
&\left. - \frac{1}{2 b_3} + \frac{b_o}{2 b_3} \right. t + \frac{2 \omega^2 (b_3^2 - b_1 b_3 + b_o)}{b_3^2 (b_3^2 + 4 \omega^2)} e^{-b_3 t} \\
&\left. - \frac{1}{4 \omega} \left[ \frac{(b_o - 4 \omega^2)^2 + 4 b_1^2 \omega^2}{b_3^2 + 4 \omega^2} \right]^{1/2} \right. \\
&\left. \sin (2 \omega t + \theta) \right\} \right. \\
&\left. H_1(r, z, a) \right. \\
&\left. \right. \\
&= \left\{ g_{10}(t) \right\} G_1(r, z, a) - \left\{ g_8(t) \right\} H_1(r, z, a) \quad (43) \end{align*}
\]
Notations used in equations 38-43.

\[ C_3 = \frac{B_2}{A_1} ; \quad C_4 = 1 + \left( \frac{A_2}{A_1} \right) + \left( \frac{B_2}{B_1} \right) ; \quad C_5 = \frac{A_2}{B_1} \]

\[ \tau_1 = \frac{B_1}{A_1} ; \quad \tau_2 = \frac{B_2}{A_2} \]

\[ a_o = \frac{C_5}{C_3 - B_2} = \frac{1}{\tau_1 \tau_2 \left[ 1 - \left( \frac{A_1}{C} \right) \right]} \]

\[ a_1 = \frac{C_4 - A_2}{C_3 - B_2} = \frac{1 + A_2/A_1 \left[ 1 + \left( \frac{\tau_2}{\tau_1} \right) - \left( \frac{A_1}{C} \right) \right]}{(A_2/A_1) \tau_2 \left[ 1 - \left( \frac{A_1}{C} \right) \right]} \]

\[ a_2 = \frac{C_3 - B_2}{2C_3 + B_2} = \frac{1 - \left( \frac{A_1}{C} \right)}{2 + \left( \frac{A_1}{C} \right)} \]

\[ b_o = \frac{C_5}{C_3} = \frac{1}{\tau_1 \cdot \tau_2} \]

\[ b_1 = \frac{C_4}{C_3} = \frac{1}{\tau_2} \left( \frac{A_1}{A_2} + 1 \right) + \frac{1}{\tau_1} \]

\[ M = \frac{3 \ A_1/C}{2 + \left( \frac{A_1}{C} \right)} \]

\[ N = \frac{1 + (2 \ A_1/C)}{2 + \left( \frac{A_1}{C} \right)} \]

\[ b_3 = \frac{A_2}{B_2} = \frac{1}{\tau_2} \]

\[ d_0 = \frac{C_5}{C_3 + 2B_2} = \frac{1}{\tau_1 \cdot \tau_2 \left[ 1 + (2 \ A_1/C) \right]} \]

\[ d_1 = \frac{C_4 + 2A_2}{C_3 + 2B_2} = \frac{1 + \left( \frac{A_1}{A_2} \right) + \left( \frac{\tau_2}{\tau_1} \right) + 2 \ A_1/C}{\tau_2 \left[ 1 + (2 \ A_1/C) \right]} \]
\[
\alpha = \frac{2CC_4 + A_2}{4CC_3 + 2B_2} = \frac{2(A_1/A_2)^2 + 2 + 2(\tau_2/\tau_1)^2 + A_1/C}{2 \tau_2 \left[ \frac{2 + (A_1/C)}{2 + (A_1/C)} \right]}
\]

\[
\gamma^2 = \frac{2 + 2(A_1/A_2)^2 + 2(\tau_2/\tau_1)^2 + A_1/C}{2 \tau_2 \left[ \frac{2 + (A_1/C)}{2 + (A_1/C)} \right]} - \frac{2}{\tau_1 \tau_2 \left[ \frac{2 + (A_1/C)}{2 + (A_1/C)} \right]}
\]

\(\gamma^2\) is always positive for all values of \(A_1/A_2\), \(\tau_2/\tau_1\) and \(A_1/C\), hence \(\gamma\) is real.

Values of the angles appearing in eqns. 51-53.

\[
\psi_1 = V_1 - V_2 \ \text{where} \ V_1 = \tan^{-1} \frac{a_1 \omega}{a_0 - \omega^2}
\]

and \(V_2 = \tan^{-1} \frac{2 A \omega}{\alpha^2 - \gamma^2 - \omega^2}\)

\[
\psi_2 = V_3 - V_2 \ \text{where} \ V_3 = \tan^{-1} \frac{b_1 \omega}{b_0 - \omega^2}
\]

\[
\psi_3 = V_3 - V_4, \ \text{where} \ V_4 = \tan^{-1} \frac{\omega}{b_3}
\]

\[
\psi_4 = V_5 + V_3 - V_4 - V_2, \ \text{where} \ V_5 = \tan^{-1} \frac{d_1 \omega}{d_0 - \omega^2}
\]

\[
\theta_1 = V_6 - V_7, \ \text{where} \ V_6 = \tan^{-1} \frac{2a_1 \omega}{a_0 - 4 \omega^2}
\]

and \(V_7 = \tan^{-1} \frac{4 \alpha \omega}{\alpha^2 - \gamma^2 - 4 \omega^2}\)

\[
\theta_2 = V_8 - V_7, \ \text{where} \ V_8 = \tan^{-1} \frac{2 \omega b_1}{b_0 - 4 \omega^2}
\]

\[
\theta_3 = V_8 - V_9, \ \text{where} \ V_9 = \tan^{-1} \frac{2 \omega}{b_3}
\]

\[
\theta_4 = V_8 + V_{10} - V_9 - V_7, \ \text{where} \ V_{10} = \tan^{-1} \frac{2 \omega d_1}{d_0 - 4 \omega^2}
\]
For an incompressible viscoelastic material ($\mu = .5$), and for a viscoelastic material for which $\mu = 0$, one needs only consider the displacements. The radial stress is the same as that for an elastic material having the same Poisson's ratio.

An incompressible viscoelastic material has an infinitely large bulk modulus. This corresponds to setting $A/C = 0$, or $A_1/C = 0$ in the equations of the displacements which were derived in this chapter. Numerical results will be presented for the displacements in incompressible viscoelastic materials represented in shear by the four mechanical models considered in this work.

For a viscoelastic material for which $\mu = 0$, substitute $\mu = 0$ in the equations of displacements (eqns. 12 and 13) and obtain

$$U = p(t) \left[ \frac{1}{E} \right] \begin{bmatrix} Q_i + q_i \end{bmatrix}$$  \hspace{1cm} (44)

$$w = p(t) \left[ \frac{1}{E} \right] \begin{bmatrix} G_i - H_i \end{bmatrix}$$  \hspace{1cm} (45)

One needs only assume a modulus $E$ for the material. Assume for example that the linear viscoelastic material is of the second type, namely, a material which behaves in shear as a Kelvin body. The modulus $E$ for this type of material, in terms of the operators $P(s)$ and $Q(s)$, is

$$E = \frac{(2C + A) + Bs}{3C(A + Bs)}$$
The associated elastic solution for the viscoelastic material for which \( \mu = 0 \), becomes:

Case I: \( p(t) = p_0 \sin \omega t \).

\[
\frac{w^*}{p_0} = \frac{p_0 \omega^2}{s^2 + \omega^2} \left[ \frac{3C(A + Bs)}{(2C + A) + Bs} \right] \left[ G_i - H_i \right]
\]

Case II: \( p(t) = p_0 \sin^2 \omega t \)

\[
\frac{w^*}{p_0} = \frac{2p_0 \omega^2}{s(s^2 + 4\omega^2)} \left[ \frac{3C(A + Bs)}{(2C + A) + Bs} \right] \left[ G_i - H_i \right]
\]

The horizontal displacement is the same function of \( s \) shown above multiplied by the function of geometry \( (Q_i + q_i) \) of equation 44. Taking the inverse Laplace transform of the above equations, the displacements for this type of material \( (\mu = 0) \) are obtained as shown below.

\[
\frac{w_1}{p_0} = \frac{1}{\gamma} \left( \frac{1}{m_1^2 + \omega^2} \right)^{1/2} \sin (\omega t + \lambda_{17}) + \frac{\omega}{\gamma} \left( \frac{1 - m_1}{m_1^2 + \omega^2} \right) e^{-m_1 t} \left[ 3C \left[ G_i - H_i \right] \right] \tag{46}
\]

\[
\frac{w_2}{p_0} = \frac{1}{2\gamma m_1} - \frac{2\omega^2(1 - m)}{m_1(m_1^2 + 4\omega^2)} e^{-m_1 t} - \frac{1}{2\gamma} \left( \frac{1 + 4\omega^2}{m_1^2 + 4\omega^2} \right)^{1/2} \cos (2\omega t + \lambda_{18}) \left[ 3C \left[ G_i - H_i \right] \right] \tag{47}
\]

where \( \lambda_{18} = \lambda_{14} - \lambda_{15} \). The expressions for the \( \lambda \)'s and \( m_1 \) are as explained before.
CHAPTER 4
NUMERICAL RESULTS AND DISCUSSION
OF RESULTS

This chapter deals with the following subjects: the obtaining of the numerical results, presentation of the results, discussion of the results, and interpretation of the results.

Obtaining numerical results

A program was written for IBM 7094 to obtain numerical results for the radial stress and the displacements for the viscoelastic material represented in shear by a Burgers model, due to both types of loading. In order to obtain general results for any point in the semi-infinite body, equations 38-40 were written as the sum of products of functions of time and geometry. As a review, the equations are shown below.

\[
\begin{align*}
\sigma_1/r_0 &= \phi_1(t) \cdot f_1 + \phi_2(t) \cdot F_i \\
\sigma_2/r_0 &= \phi_3(t) \cdot f_i + \phi_4(t) \cdot F_i \\
U_1(A_1/p_0) &= \phi_5(t) \cdot Q_1 + \phi_6(t) \cdot q_1 \\
U_2(A_1/p_0) &= \phi_7(t) \cdot Q_1 + \phi_8(t) \cdot q_1 \\
w_1(A_1/p_0) &= \phi_9(t) \cdot G_1 - \phi_6(t) \cdot H_i \\
w_2(A_1/p_0) &= \phi_{10}(t) \cdot G_1 - \phi_8(t) \cdot H_i
\end{align*}
\]

The subscripts 1 and 2 (left side of above equations) refer to the two types of loading \( p(t) = p_0 \sin(\omega t) \) and \( p(t) = p_0 \sin^2(\omega t) \), respectively.

The functions \( \phi_2(t) \) and \( \phi_4(t) \) are the \( \sin(\omega t) \) and \( \sin^2(\omega t) \) functions, respectively. These are tabulated in standard mathematical tables.

The remaining functions are dependent on the variables \( \omega \), the...
value of the elements of the model, and on the bulk modulus \( K \) of the viscoelastic material. In order to make the results more useful, the functions \( \theta_1(t), \theta_3(t), \theta_5(t) \ldots, \theta_{10}(t) \) were expressed in terms of the quantities \( \omega, \tau_1 \) (the relaxation time of the Maxwell unit), \( \tau_2 \) (the retardation time of the Kelvin unit), \( A_1/C \), and \( A_2/A_1 \). Table 3 shows the values of each of these five variables, for which magnitudes of the functions \( \theta_i \) were found.

**Table 3. Values of the Parameters for the Burgers Model**

<table>
<thead>
<tr>
<th>( \omega ) (radians/second)</th>
<th>( A_1/C )</th>
<th>( \tau_1 ) (seconds)</th>
<th>( \tau_2 ) (seconds)</th>
<th>( A_2/A_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>10</td>
<td>0.5</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>20</td>
<td>0.9</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
</tbody>
</table>

Numerical values of \( \theta_i \) were found for the 243 possible combinations of Table 3. The range of loading time was taken from \( t = 0.05 \) seconds to \( t = 12.7 \) seconds. The increments of loading time were taken as 0.05 seconds up to \( t = 5 \) seconds and as 0.1 second from \( t = 5 \) seconds to \( t = 12.7 \) seconds. In this way, enough points were obtained which are essential at the higher frequencies.
For the Maxwell body, the Kelvin body, and the standard linear solid, numerical results are found for the vertical deflection. The equations of the vertical deflection for each model and due to the two types of loading, were expressed as follows:

Maxwell Body

\[
\begin{align*}
    w_1 \left( \frac{A}{P_0} \right) &= X_1(t) \cdot G_1 - X_4(t) \cdot H_1 \\
    w_2 \left( \frac{A}{P_0} \right) &= X_2(t) \cdot G_1 - X_6(t) \cdot H_1
\end{align*}
\]  

(eqn. 22)

(eqn. 25)

Kelvin Body

\[
\begin{align*}
    w_1 \left( \frac{A}{P_0} \right) &= Y_1(t) \cdot G_1 - Y_4(t) \cdot H_1 \\
    w_2 \left( \frac{A}{P_0} \right) &= Y_2(t) \cdot G_1 - Y_6(t) \cdot H_1
\end{align*}
\]  

(eqn. 28)

(eqn. 31)

Standard Linear Solid

\[
\begin{align*}
    w_1 \left( \frac{A_1}{P_0} \right) &= Z_1(t) \cdot G_1 - Z_4(t) \cdot H_1 \\
    w_2 \left( \frac{A_1}{P_0} \right) &= Z_2(t) \cdot G_1 - Z_6(t) \cdot H_1
\end{align*}
\]  

(eqn. 34)

(eqn. 37)

Numerical values were obtained for the functions \(X_1, Y_1,\) and \(Z_1\) \((1 = 4, 6, 7, \text{ and } 8)\) for the values of the parameters \(\lambda, \tau,\) and \(A/C\) shown in Table 3. In the case of the standard linear solid, the values of \(A_1/A_2\) considered were the same as those for the Burgers model of Table 3.

The special case of an incompressible viscoelastic material is included in the numerical results when the ratio \(A_1/C = 0\), or the ratio \(A/C = 0\) for the four models.

Presentation of the numerical results

A great amount of information in tabular form for the values of the functions \(\phi_1, X_1, Y_1,\) and \(Z_1\) versus time of loading was obtained. It is
not feasible to present those tables in this work; however, an extensive
analysis of the numerical results revealed that for many combinations
of the parameters involved, the variation of the particular function with
the time of loading did not change significantly. Hence a typical graph
suffices to represent several combinations. The graphs shown in this
chapter represent the major part of the results. The program for the
IBM 7094 is shown in Appendix C.

In the following are presented typical graphs of the functions
\( X_1(t), Y_1(t), Z_1(t), \) and \( \phi_1(t) \). Discussion follows the presentation of
each group of graphs.

The functions \( X_1 \)

Figure 2 through Figure 13 show the functions \( X_4, X_6, X_7, \) and
\( X_8 \) for frequencies of 1, 10, and 20 radians per seconds. From these
graphs (and knowing the functions of geometry \( G_1 \) and \( H_1 \)), one can find
the vertical deflection due to the two types of sinusoidal loading, in the
viscoelastic material represented in shear by a Maxwell model, for the
relaxation times and the ratio \( A/C \) shown on the graphs. From the
functions \( X_4 \) and \( X_7 \), one can find the vertical deflection due to the
\( \sin \omega t \) type of loading and from the functions \( X_6 \) and \( X_8 \), one can find
the vertical deflection due to the \( \sin^2 \omega t \) type of loading.

12. Numerical results are available in the Department of Civil Engineering,
The Ohio State University.
The maximum vertical deflection occurs at the point \( r = 0, z = 0 \). At this point, \( G_1 = 2a \), and \( H_1 = 0 \). Therefore for the maximum vertical deflection, one only needs \( X_7 \) and \( X_8 \), \( X_4 \) and \( X_6 \) are needed to obtain the vertical deflection at other points in the material.

Analysis of the numerical results showed that the functions \( X_4 \) and \( X_6 \) are independent of the ratio \( A/C \), whereas the functions \( X_7 \) and \( X_8 \) depend on \( A/C \); however, the variation of these two functions is not appreciable compared to their values at \( \tau = .01 \). The effect of varying the frequency \( \omega \) on the functions \( X_4 \) is shown in Figures 4, 7, and 13. These figures are self-explanatory.

The most important parameter affecting the variation of the functions \( X_4 \) is the relaxation time \( \tau \). As \( \tau \) increases, the absolute value of the functions \( X_4 \) decreases. The change in the absolute values of the functions is greater when \( \tau \) varies from 0.01 to 1 than when \( \tau \) varies from 1 to 100.

An interesting result is revealed in Figures 2 and 8; namely, when \( \tau = 0.01 \), although when a sinusoidal load is applied, no negative (upward) deflection occurs. This is possible because the accumulated downward deflection balances or is more than the upward vertical deflection caused by the negative part of the load cycle.

The graphs of the functions \( X_6 \) and \( X_8 \) are characterized by a continuous increase of the functions as the loading time increases.
This means an accumulation of the vertical deflection. The slope of the graphs of $X_6$ and $X_5$ are steeper at lower relaxation times; in fact, one can say that the slope is inversely proportional to the relaxation time. This is seen by comparing the slope of the graph of $X_6$ at $\tau = 0.01$ to the slope of $X_5$ at $\tau = 1$, and $\tau = 100$.

Results for the vertical deflection in a material represented in shear by a Maxwell model show a clear break from the results that would be obtained from the theory of elasticity. According to the theory of elasticity, applying a sinusoidal load to an elastic material, the material responds instantaneously. The vertical deflection reaches its peak value at the same time as the load does (at $t = 1.57$ seconds, for $\omega = 1$); furthermore, according to the theory of elasticity, the maximum value remains constant even after an infinite application of load cycles.

For a linear viscoelastic material, the story is different. From the graphs presented, two important aspects are revealed. They are (1) the lag or time difference between the peak value of the applied load and the peak values of the functions $X_1$, (2) the accumulation of the vertical deflection as time progresses.

The results found from the mathematical analysis are expected for real viscoelastic materials represented in shear by a Maxwell model. A small value of the relaxation time $\tau$ characterizes materials which deform and flow more easily than materials with a
high value of $\tau$. At $\tau = 100$, for example, the dashpot being rigid, the elastic element overshadows the behavior of the material and the variation of the vertical deflection with time is closer to that expected for an elastic material than the vertical deflection for an easily flowing material ($\tau = 0.01$). The absolute value at any time $t$, of the vertical deflection for more rigid materials (high value of $\tau$), is less than the corresponding value of the vertical deflection for a less rigid material (low value of $\tau$). This is also expected and is observed in practice.
Figure 2. $X_4$ versus Time ($\omega = 1 \text{ radian/second}$)
Figure 3. $X_4$ versus Time ($\omega = 1$ radian/second).
Figure 4. $X_4$ versus Time ($\omega = 10$ and 20 radians/second).
Figure 5. $X_6$ versus Time ($\omega = 1$ radian/second).
Figure 6. $X_6$ versus Time ($\omega = 1$ radian/second).
Figure 7. $X_6$ versus Time ($\omega = 10$ and 20 radians/second).
Figure 8. \( X_\tau \) versus Time (\( \omega = 1 \) radian/second).
Figure 9. $x_7$ versus Time ($\omega = 1$ radian/second).
Figure 10. $X_\gamma$ versus Time ( $\omega = 10$ and 20 radians/second )
Figure 11. $X_8$ versus Time ($\omega = 1$ radian/second).
Figure 12. $X_8$ versus Time ($\omega = 1$ radian/second).
Figure 13. $X_8$ versus Time ($\omega = 10$ and 20 radians/second).
The functions $Y_i$

Figure 14 through Figure 21, show the functions $Y_4$, $Y_6$, $Y_7$, and $Y_8$ for the values of the parameters $\omega$, $\tau$, and $A/C$ shown on the figures. The values of the functions $Y_i$ at $\tau = 100$, become very small, therefore curves for this case are not plotted.

Analysis of the numerical results revealed that the functions $Y_4$ and $Y_6$ are independent of the ratio $A/C$, but the functions $Y_7$ and $Y_8$ depend on the ratio $A/C$. The latter two functions vary with $A/C$ more than $X_7$ and $X_8$ do.

The graphs shown indicate that the functions $Y_i$ are in general elastic or delayed elastic. The vertical deflection is not accumulative as was seen in the case of the material represented by the Maxwell model. All functions $Y_i$ change more when $\tau$ varies from 0.01 to 1 than when $\tau$ varies from 1 to 100. The maximum vertical deflection in an incompressible viscoelastic material, is less than that for a material with $A/C = .9$ (see Figure 20). The Maxwell body deflects more than the Kelvin body, for the same $\tau$ and ratio $A/C$, but for both models, higher deflections result at the lower values of the relaxation or retardation times,
Figure 14. $Y_4$ versus Time ($\omega = 1$ radian/second).
Figure 15. $Y_4$ versus Time ($\omega = 1$ radian/second).
Figure 16. $Y_6$ versus Time ($\omega = 1$ radian/second).
Figure 17. $Y_6$ versus Time ($\omega = 1$ radian/second).
Figure 18. $Y_6$ versus Time ($\omega = 10$ and 20 radians/second).
Figure 19. $Y_7$ versus Time ($\omega = 1$ radian/second).
Figure 20. $Y_8$ versus Time ($\omega = 1\ \text{radian/second}$).
Figure 21. $Y_8$ versus Time ($\omega = 10$ and 20 radians/second).
The functions $Z_1$

Figure 22 through Figure 28 show the functions $Z_4$, $Z_6$, $Z_7$, and $Z_8$ versus the time of loading for $\omega = 1$ radian per second. A sample curve for the function $Z_8$ versus the time of loading for the frequencies $\omega = 10$ and $\omega = 20$ radians per seconds, is shown in Figure 29. It is noticed that curves 1, 3, and 4 for the function $Z_4$ are similar in shape to the corresponding curves for $Z_7$. Similarity is also observed between the curves of the functions $Z_6$ and $Z_8$. Only curve 4 for $Z_8$ is shown. Corresponding to curves 1 and 3, one uses the relation between $Z_6$ and $Z_8$; namely, at any time $t$, $Z_6 = 2 Z_8$. This relation is only valid for $A_1/C = 0$, and all values for $\omega$ and $\tau$. The relation is seen to hold true by comparing Figures 25 and 28.

Tables 4 and 5 show the combinations of parameters for which curves are plotted and labeled accordingly.

Numerical results showed that the functions $Z_4$ and $Z_6$ do not depend on the ratio $A/C$, but the functions $Z_7$ and $Z_8$ do. Results also showed that the most important parameters in the case of the standard linear solid, are: the ratio $A_1/A_2$ and the retardation time of the Kelvin unit ($\tau$). Actually it is the combination of these two quantities which is important. Results for two extreme cases are shown. The first extreme case is when one has a large ratio $A_1/A_2$ (say 100), combined with a small retardation time $\tau$. In this case, the deformation of the
Kelvin unit is dominant and it takes a long time for the deformation to approach the elastic case. This case is also characterized by high absolute values of the deformation. To follow the above analyses, one compares curves 1, 3, and 4 of Figures 22 and 23 on which the peak point values are shown. One notices that curve 1 is closer to being elastic than curves 3 and 4. One also notices the high absolute values of curve 4 due to $A_1/A_2$ being high. Another characteristic of this case is a retarded deformation remaining after the first cycle of loading. This is seen in curve 3 of Figure 24 and curve 4 of Figure 25.

The second extreme is that when $A_1/A_2$ is small and $\tau$ is large. In this case the deformation of the spring $A_1$ dominates the total deformation, and the behavior of the material is close to being elastic. After the first cycle of loading, there is a very small amount of retarded deformation. This extreme case is shown in curve 1 of all the figures. Materials represented by the first case lag in response behind the materials represented by the second.

Other cases fall between the two extremes described above.
Table 4. Combinations of Parameters for the Functions $Z_4$ and $Z_6$ Represented by the Curves Labeled in Figures 22 and 24 for $\omega = 1$ radian/second

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Table 5. Combinations of Parameters for the Functions $Z_7$ and $Z_8$ Represented by the Curves Labeled in Figures 26 and 28 for $\omega = 1$ radian/second

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</table>
Figure 22. $Z_4$ versus Time ($\omega = 1$ radian/second).
Figure 23. $Z_4$ versus Time ($\omega = 1$ radian/second).
Figure 24. \( Z_6 \) versus Time \((\omega = 1 \text{ radian/second})\).
Figure 25. $Z_6$ versus Time ($\omega = 1$ radian/second).
Figure 26. $Z_7$ versus Time ($\omega = 1$ radian/second).
Figure 27. $Z_q$ versus Time ($\omega = 1$ radian/second).
Figure 28. $Z_8$ versus Time ($\omega = 1$ radian/second).
Figure 29. $Z_g$ versus Time ($\omega = 10$ and 20 radians/second and $A_1/C = 0$.)
The functions $\phi_i$

Figures 30 and 31 show the variation of the functions $\phi_1$ and $\phi_3$, respectively, with the time of loading. In each figure are plotted several curves labeled 1, 2, ... etc. The combination of the parameters of the model which each curve represents is shown in Table 6. Curve 1a is not shown because of its extreme closeness to curve 1.

Once the elements of the Burgers model are known, the radial stress at any time $t$, and at any point in the material, can be found for both types of loading. Curve 1 represents the functions $\phi_1$ and $\phi_3$ for an elastic material with $\mu = .5$. It also represents an incompressible viscoelastic material ($A_1/C = 0$) for all $\omega$, $\tau_1$, $\tau_2$, and $A_2/A_1$.

The curves of Figures 30 and 31 closely approximate the sinusoidal $\sin t$ and $\sin^2 t$, respectively. The reason is that the material is assumed to be elastic under normal pressures, i.e. a constant bulk modulus is assumed. The detail variation of each curve will not be discussed, but instead the important features, such as the maximum absolute value of the functions, the lag, and the permanent stress left in the material after the first cycle of loading and its variation with the time of loading, will be pointed out.

From Table 6 one can find the effect of each parameter $A_1/C$, $\tau_1$, $\tau_2$, and $A_2/A_1$ by varying that parameter and keeping the others constant. As an example, let it be required to find the
absolute values of $\theta_1$ and $\theta_3$. In Figures 30 and 31, curves 1 are compared to curves 5. It is found that a low value of $\tau_1$ gives a high maximum value for the functions $\theta_1$ and $\theta_3$. On the other hand a high value of $\tau_1$ gives small absolute values for maximum $\theta_1$ and $\theta_3$. This is seen by comparing the case $\tau_1 = 0.01$, $\tau_2 = 100$, and $A_2/A_1 = 100$ to the case $\tau_1 = 100$, $\tau_2 = 100$, and $A_2/A_1 = 100$, in Table 6.

The maximum radial stress in the semi-infinite viscoelastic body occurs at the center of the uniform load ($r = 0, z = 0$). At this point, $f_1 = 1$ and $F_1 = 0.5$. Hence for $\omega = 1$ radian/second, the expressions for the maximum radial stress become

\begin{align*}
\sigma_{r1}/p_0 &= \theta_1 + 0.5 \sin t \quad \text{(eqn. 44)} \\
\sigma_{r2}/p_0 &= \theta_3 + 0.5 \sin^2 t \quad \text{(eqn. 45)}
\end{align*}

Equations 44 and 45 are plotted in Figures 32 and 33, respectively, for curve 3 of Figures 30 and 31. It is noticed that for both types of loading, the maximum radial stress lags 0.13 seconds behind the maximum load intensity. The amount of permanent stress left in the material at different times is shown in the figures.

The variation of the functions $\theta_5$ to $\theta_{10}$ with the time of loading is shown in Figures 34 through 43. The combinations of parameters represented by the different curves is shown in Tables 7 and 8.

Results showed that for an incompressible viscoelastic material $\theta_5 = \theta_7 = 0$, for all values of $\omega$, $\tau_1$, $\tau_2$, and $A_2/A_1$. This
result is verified by considering equation 12, since when $A_1/C = 0$, 
\[ \mu = 0.5, \text{ and } 1 - 2\mu = 0. \] 
Hence $\phi_5 = \phi_7 = 0$.

In Table 8 the ratio $A_1/C$ is not shown, since the numerical results showed that $\phi_6$ and $\phi_8$ are independent of $A_1/C$ and $\phi_9$ and $\phi_{10}$ are either independent of $A_1/C$ or that their variation with $A_1/C$ was not appreciable.

Other curves were plotted for the functions $\phi_1$, $\phi_3$, $\phi_8$, and $\phi_{10}$, for the frequencies $\omega = 10$ and $\omega = 20$ radians per second. The curves obtained were similar in shape to the curves of Figures 4 and 7 of the Maxwell body. At the two frequencies mentioned above, the maxima and minima values of the functions occur at different times, but the slope of the curves (rate of change of the function with time) is the same. These curves are not shown in this work, but instead are filed with the numerical results.

The horizontal displacement can be found from Figures 34 to 39 for the different combinations of parameters and for both types of loading. These curves show that the horizontal displacement accumulates as the cycles of loading are repeated, the rate of rise and the lag being dependent on the combinations of the parameters of the model. It is interesting to notice curve 2 of Figure 35. This curve has no negative values of $\phi_6$. This case is similar to the case of Figure 8 of the Maxwell body. The explanation offered
there applies here too.

The vertical deflection can be found from the curves of \( \phi_6 \) and \( \phi_9 \) for the Sin type of loading or from \( \phi_8 \) and \( \phi_{10} \) for the Sin\(^2\) type of loading.

The curves of \( \phi_6 \) and \( \phi_8 \) are numbered to match the curves of \( \phi_9 \) and \( \phi_{10} \) respectively. A great similarity is noticed between the curves of \( \phi_8 \) and \( \phi_{10} \) of the Burgers model and the curves of \( \chi_6 \) and \( \chi_8 \) of the Maxwell model.

Results indicate that the vertical deflection for both types of loading is accumulative. The rate of increase of the deflection depends on the combination of the parameters of the model. The functions \( \phi_8 \) and \( \phi_{10} \) are characterized by their slope which increases with the decrease of \( \tau_1 \). Curves 9 of Figures 39 and 43 have a very flat slope.

Some numerical values at the peak points are shown for two different times. Although the slope is very flat, given enough time, the increase in the function will accumulate substantially.

Many other remarks can be made about the graphs presented in this chapter; however, the features of each curve are self-evident.

When the graphs are combined with Tables 6, 7, and 8, they are very useful in determining the radial stress and the displacements, once the elements of the Burgers model are known. The values of the elements of the model have to be determined experimentally.

Solutions for a variety of combinations of the parameters of the model are shown, since even for a given viscoelastic material,
such as bituminous mixtures, the value of the elements of the model are functions of the temperature of the mix, the asphalt content, and other factors. In the case of a given foundation soil, the elements of the model will vary with the variation of moisture content and density of the soil.

Significance of results from an engineering point of view

The results presented in the foregoing sections are significant from an engineering point of view. Results showed that the displacements are either accumulative (Maxwell and Burgers models) or that there were retarded deformations left in the material after the first cycle of loading (Kelvin and standard linear solid). This is in agreement with the observed behavior of engineering materials for which there is no explanation in the theory of elasticity. Permanent deformations in highway pavements may accumulate to a level at which failure occurs. This situation can not be predicted by the theory of elasticity. Also the maximum vertical deflection is used in the structural design of flexible pavements, but results showed that the maximum vertical deflection is a function of the time of loading, or in other words, a function of the number of load applications; therefore a viscoelastic type of analysis is required for the design of pavements.

It is suggested that a mathematical analysis similar to the one undertaken in this work be followed in the stress analysis of engineering structures subject to sinusoidal vibrations. Inertia forces are important in the latter case and must be included in the analysis.
Table 6. Combinations of Parameters for the Functions $\phi_1$ and $\phi_3$ Represented by the Curves Labeled in Figures 30 and 31 for $\omega = 1$ radian/second

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Table 7. Combinations of Parameters for the Functions $\phi_5$ and $\phi_7$, Represented by the Curves Labeled in Figures 34 and 37 for $\omega = 1$ radian/second

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Table 8. Combinations of Parameters for the Functions $\phi_2$, $\phi_3$, $\phi_9$ and $\phi_{10}$ Represented by the Curves Labeled in Figure 35, 36 and Figure 38 through Figure 43 for $\omega = 1$ radian/second

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Figure 30. $\phi_1$ versus Time ($\omega = 1$ radian/second).
Figure 31. $\theta_3$ versus Time ($\omega = 1$ radian/second).
Figure 32. Maximum $\sigma_r$ versus Time for Curve No. 3 of Figure 30.
Figure 33. Maximum $\frac{\sigma_{r_2}}{P_0}$ versus Time for Curve No. 3 of Figure 31.
Figure 34. $\phi_5$ versus Time ($\omega = 1$ radian/second).
Figure 35. $\phi_6$ versus Time ($\omega = 1$ radian/second).
Figure 36. $\theta_6$ versus Time ($\omega = 1$ radian/second).
Figure 37. $\psi_7$ versus Time ($\omega = 1$ radian/second).
Figure 38. $\phi_8$ versus Time ($\omega = 1$ radian/second).
Figure 39. $\theta$ versus Time ($\omega = 1$ radian/second).
Figure 40. $\theta_g$ versus Time ($\omega = 1$ radian/second).

See Table 8.
Figure 41. \( \beta_9 \) versus Time \((\omega=1 \text{ radian/second}) \).
Figure 42. $\phi_{10}$ versus Time ($\omega = 1$ radian/second).
Figure 43. $\theta_{10}$ versus Time ($\omega = 1$ radian/second).
CHAPTER 5
MOVING LOADS

The problem of a uniform load moving on the surface of a semi-infinite viscoelastic body has not been solved yet. In this chapter an approximate method is suggested for the solution of the problem by making use of some of the results found in this work.

The solution of the problem of a moving load on an elastic body is simple. At any instant of time the moving load is at a known position on the surface, and the stresses and the displacements at any point in the material are the same as those caused by a static load placed at the same position as the moving load. A sequence of the values for the stresses and displacements corresponding to a number of positions of the static load represent the solution of the problem of the moving load. The above procedure gives correct results because the properties of the elastic medium are not time dependent. This means that the history of loading has no influence on the results, but in the case of a moving load on a linear viscoelastic medium, the properties of the material are time dependent and the history of loading is of major importance. Therefore, the procedure used for the elastic medium cannot be used. It is believed that the approximate method suggested below will give fair results, however the degree of approximation can be found only when the exact solution is known.

Figure 44 is shown to illustrate the approximate method. In the figure, the curves labeled 1, show the maximum vertical deflection
Figure 44. Comparison between $\frac{W}{2(1-\mu^2)} \left( \frac{p_0}{E} \right)_a$ and the Geometric Function $\sin^2 \left( \frac{\pi}{6a}x \right)$. Uniform Circular Load Moving on a Semi-infinite Elastic Body, and the Geometric Function $\sin^2 \left( \frac{\pi}{6a}x \right)$. Distance in radii.
and the radial stress in an elastic medium caused by a moving load. The curves labeled 2, show the geometric function $\sin^2 \left( \frac{\pi}{6a} x \right)$, where $x$ is the distance measured as shown in the figure and $a$ is the radius of the loaded area. The distance $6a$ is half the period of the $\sin^2$ function. One finds that curves 1 are close to curves 2. This means that a static load whose intensity varies with time according to the relation $p(t) = p_o \sin^2 \omega t$ will approximately produce the same radial stress and deflection in the material as will the moving uniform circular load having a constant intensity $p_o$.

Using the above analogy for a linear viscoelastic material the results found in this work for the radial stress and the vertical deflection due to the $\sin^2$ type of loading approximately represent the radial stress and the vertical deflection caused by a uniform moving load of constant intensity. To make use of the results the frequency of the sinusoidal load must be correlated with the velocity of the travelling load, as shown below.

$$\omega = \left( \frac{\pi}{6a} \right) V$$

(eqnn. 46)

In a practical problem the quantities $a$ and $V$ of equation 46 are known. In addition the properties of the viscoelastic material are known. As an example let it be required to find the radial stress and the maximum vertical deflection caused by a uniform circular load of intensity $p_o$, travelling over a semi-infinite viscoelastic medium which is represented in shear by a Burgers model. Let the following information regarding
the load and the material be given: \( a = 1 \text{ ft.}, V = 1.3 \text{ MPH}, A_1/C = 0, \)

\( \tau_1 = 1, \quad \tau_2 = 100, \) and \( A_2/A_1 = 1. \) The solution of the problem
is as follows: the frequency of the \( \sin^2 \) type of loading corresponding

to the velocity of the load is found from equation 46 to be 1 radian/second.

To find approximately the radial stress at any point in the viscoelastic
material, curve 3 of Figure 31 is used for the range of time, \( t = 0 \), to
\( t = 3.14 \) seconds. A graph similar to the first cycle of Figure 33 will
result. The vertical scale (absolute values) will depend on the depth \( z \)
where the radial stress is sought.

The expression of the maximum vertical deflection for a material
represented by a Burger model is:

\( W_{\text{max}} (A_1/P_0) = \phi_{10}(t). \quad 2a = 2 \phi_{10}(t). \) At any instant of time during the period \( t = 0 \), to \( t = 3.14 \)
seconds, the approximate value of the maximum vertical deflection can
be found from curve 6 of Figure 43. The first cycle of the curve is
plotted to a larger scale in Figure 45 and is compared to the maximum
vertical deflection caused by the same load moving on an elastic material.

The absolute values of the two curves are not of much significance
since the vertical scale represents two different functions, but it is
interesting to examine the shapes of the curves. For an elastic
material, the maximum vertical deflection at any point on the surface
such as \( P_1 \) increases as the load approaches the point, reaching a
maximum value when the center of the load comes over the point,
then the deflection decreases as the load moves away. The resulting
curve is symmetrical around a vertical line passing through the point t = 1.57 seconds. If the point $P_1$ is on the surface of a linear viscoelastic medium and the load approaches it, the resulting curve is not symmetrical. The maximum vertical deflection at $P_1$ occurs after the load has departed, and left $P_1$. For this example, the maximum vertical deflection at $P_1$ occurs when the travelling load has reached a point $P_2$ which is 0.92 ft. away from $P_1$. The area under the viscoelastic curve between the limits of time, $t = 0$ to $t = 1.57$ seconds, is less than the area under the curve between the limits of time, $t = 1.57$ seconds, to $t = 3.14$ seconds. This indicates an accumulation of the vertical deflection due to the memory of the material.

Results obtained in this work are for the continuous application of load cycles. In terms of moving loads it will represent a rather severe case, since in practice, vehicles do not follow each other immediately. It might be possible to find a correlation between the number of continuous cycles of loading and the number of alternating cycles (on and off), which will produce the same vertical deflection.
Figure 45. Comparison between the Maximum Vertical Deflection in a Viscoelastic and Elastic Material Caused by a Uniform Circular Moving Load.
The objectives of the study were realized. Using the Laplace transform method and solutions of elasticity, general expressions for the stresses and displacements in a semi-infinite medium of any linear viscoelastic material and under any type of quasi-static circular load, were found. Specific solutions were found for the radial stress and the horizontal and vertical deflection for several types of linear viscoelastic materials and for two types of sinusoidal loads. A program was written for an IBM computer and numerical results were found for the vertical stress and the horizontal and vertical displacements for a linear viscoelastic material represented in shear by a Burgers model for a great variety of parameters reflecting the properties of the material. Numerical results were also found for the vertical deflection in three types of linear viscoelastic materials represented in shear by a Maxwell model, a Kelvin model, and a standard linear solid model, due to two types of sinusoidal loading.

An approximate method is suggested for the solution of the problem of a uniform circular load of constant intensity moving over the surface of a semi-infinite linear viscoelastic body.

Following are the conclusions drawn from the study.

1. The Laplace transform method was found to be very convenient for the solution of the stress analysis problem of this work involving a constant geometry. The equations of the radial stress and the dis-
placements for the associated elastic solution are proper rational fractions whose inverse Laplace transform is obtained without much difficulty.

2. The equations, especially that of the vertical deflection, become more complex as the mechanical model representing the material's behavior in shear become more complicated.

3. It was found that the radial stress in linear viscoelastic materials which behave elastic under normal pressures is close to the radial stress in elastic materials.

4. It was found that for linear viscoelastic materials represented in shear by a Maxwell model or by a Burgers model, the deformations increase continuously as the cycles of loading were repeated. The deformations in the materials represented by a Kelvin model, or by a standard linear solid model, were less but approached the elastic case.

5. When applying a Sin(\omega t) type of loading to a linear viscoelastic material represented in shear by either a Maxwell model or by a Burgers model, the presence of the negative (upward) deflection in the material depends on the relaxation time of the Maxwell model or of the Maxwell unit in the Burgers model.

6. When sinusoidal loads are applied to linear viscoelastic materials, the radial stress and the displacements at any point in the material are functions of parameters reflecting the properties of the material. For many combinations of the values of the elements within the same model, the same radial stress and displacements are obtained.
7. Function arising from the geometry of the material are independent of time and are the same for elastic and viscoelastic materials.

8. The approximate method suggested for the solution of the problem of uniform circular loads moving on linear viscoelastic foundations is a useful application of the results. The approximate method can serve as a simple check on the results of the exact solution. The degree of approximation can only be found after the exact solution is known.

9. Results pointed to the advisability of applying the theory of viscoelasticity instead of the theory of elasticity in solving Civil Engineering problems dealing with stress analysis in materials which exhibit rheologic properties.
APPENDIX A

Elastic Body—Stresses and Displacements
Expressions for the Stresses and Displacements in a Semi-infinite Elastic Body Under a Uniform Circular Load, \( p(t) \)

The subscript \( i \) appearing in the expressions of stresses and displacements refers to the four cases of geometry shown below.

\[ \begin{align*}
   i = 1, & \quad r = 0 \\
   i = 2, & \quad r < a \\
   i = 3, & \quad r = a \\
   i = 4, & \quad r > a
\end{align*} \]
Stresses and displacements -- elastic body

\[ \sigma_r = \left[ p(t) \right] \cdot \mu \cdot f_i(r,z,a) + \left[ p(t) \right] F_i(r,z,a) \]

\[ \sigma_z = \left[ p(t) \right] g_i(r,z,a) \]

\[ \tau_{rz} = \left[ p(t) \right] h_i(r,z,a) \]

\[ U = \left[ p(t) \right] \left[ \frac{1 + \mu}{E} \right] Q_i(r,z,a) \]

\[ + \left[ p(t) \right] \left[ \frac{1 + \mu}{E} \right] q_i(r,z,a) \]

\[ w = \left[ p(t) \right] \left[ \frac{1 - \mu^2}{E} \right] G_i(r,z,a) - \left[ p(t) \right] \left[ \frac{1 + \mu}{E} \right] H_i(r,z,a) \]

Values of the functions \( f_i, F_i, \ldots, H_i \) are given on the following pages.

These functions are found from reference 10, by substituting in Column C of Table 1, p. 82851 b, the values of the derivatives found from Table 2, p. 82851 C, and rearranging terms.
Functions $f_i$ and $F_i$ appearing in the expression of $\sigma_r$

\[ f_1 = \frac{z - R}{R} \]
\[ f_2 = \frac{1}{\pi} \left[ \pi - \frac{a^2 + r^2}{r^2} J - \frac{z R_2}{r^2} \frac{E'}{\frac{R_2}{r^2}} + \frac{z (2a^2 + z^2)}{R_2 r^2} k' \right. \]
\[ \left. - 2 \left( \frac{\pi - J - \frac{z}{R_2}}{K'} \right) \right] \]
\[ f_3 = \frac{1}{\pi} \left\{ \frac{2z}{R_2} K' - \pi + \frac{z}{R_2 a^2} \left[ (2 a^2 + z^2) K' - R_2^2 E' \right] \right\} \]
\[ f_4 = \frac{1}{\pi} \left[ \frac{\pi a^2}{r^2} + \frac{a^2 + r^2}{r^2} J - \frac{z R_2}{r^2} \frac{E'}{\frac{R_2}{r^2}} \right. \]
\[ \left. + \frac{z (2 a^2 + z^2)}{R_2 r^2} K' - 2 \left( \frac{J - \frac{z}{R_2}}{R_2} \right) \right] \]

\[ F_1 = \frac{z a^2}{2 R^2} - \frac{1}{2} \left( 1 - \frac{z}{R} \right) \]
\[ F_2 = -\frac{1}{2 \pi} \left\{ \pi - \frac{a^2 + r^2}{r^2} J - \frac{z R_2}{r^2} \frac{E'}{\frac{R_2}{r^2}} + \frac{z (2a^2 + z^2)}{R_2 r^2} K' \right. \]
\[ \left. + \frac{z}{r^2 R_2} \left[ 2 (a^2 + z^2) K' - R_2^2 (1 + \text{Ab}) E' \right] \right\} \]
\[ F_3 = \frac{1}{2 \pi} \left\{ \frac{z}{R_2 a^2} \left[ (2a^2 + z^2) K' - R_2^2 E' \right] + z \left[ \frac{2 (a^2 + z^2)}{R_2} \frac{K'}{a^2} \right. \right. \]
\[ \left. \left. - 6 a^2 + 2 z^2 \right] E' \right\} \]
\[ F_4 = \frac{1}{2 \pi} \left\{ -\frac{\pi a^2}{r^2} + \frac{a^2 + r^2}{r^2} J - \frac{z R_2}{r^2} \frac{E'}{\frac{R_2}{r^2}} + \frac{z (2a^2 + z^2)}{R_2 r^2} K' \right. \]
\[ \left. + \frac{z}{r^2 R_2} \left[ 2 (a^2 + z^2) K' - R_2^2 (1 + \text{Ab}) E' \right] \right\} \]
Functions $g_i$ appearing in the expression of $\mathcal{J}_z$

\[
g_1 = \frac{z}{R} \left( 1 - \frac{a^2}{R^2} \right) - 1
\]

\[
g_2 = -\frac{1}{2\pi} \left\{ 2 \left( \pi - J - \frac{z}{R_2} K' \right) + z \left[ \frac{2K'}{R_2} - \frac{R_2}{r^2} (1 - Ab) E' \right] \right\}
\]

\[
g_3 = \frac{1}{2\pi} \left[ \frac{2z}{R_2} K' - \pi - z \left( \frac{2}{R_2} K' - \frac{2}{R_2} E' \right) \right]
\]

\[
g_4 = -\frac{1}{2\pi} \left\{ 2 \left( J - \frac{z}{R_2} K' \right) + z \left[ \frac{2K'}{R_2} - \frac{R_2}{r^2} (1 - Ab) E' \right] \right\}
\]

Functions $h_i$ appearing in the expression of $\mathcal{T}_{rz}$

\[
h_1 = 0
\]

\[
h_2 = -\frac{z^2}{2\pi r R_2} \left[ (1 + \frac{1}{k^2}) E' - 2K' \right]
\]

\[
h_3 = -\frac{1}{\pi a R_2} \left[ (2a^2 + z^2) E' - z^2 K' \right]
\]

\[
h_4 = h_2
\]
Functions $Q_i$ and $q_i$ appearing in the expression of $\mathcal{U}$

$$Q_1 = 0$$

$$Q_2 = \frac{r}{2\pi} \left[ \pi + \frac{a^2 - r^2}{r^2} J + \frac{z}{r^2 R_2} \right]$$

$$\left\{ R_2^2 E' - (2 R_1 R_2 b - z^2) K' \right\}$$

$$Q_3 = \frac{a}{2\pi} \left[ \pi + \frac{z R_2}{a^2} E' - \frac{z R_2}{a^2} K' \right]$$

$$Q_4 = \frac{r}{2\pi} \left[ \frac{\pi a^2}{r^2} - \left( \frac{a^2 - r^2}{r^2} \right) J + \frac{z}{r^2 R_2} \right]$$

$$\left\{ R_2^2 E' (2 R_1 R_2 b - z^2) K' \right\}$$

$$q_1 = 0$$

$$q_2 = \frac{-z}{\pi r} (R_1 b K' - R_2 E')$$

$$q_3 = \frac{-z}{\pi a R_2} \left[ (2 a^2 + z^2) K' - R_2^2 E' \right]$$

$$q_4 = q_2$$
Functions $G_1$ and $H_1$ appearing in the expression of $w$

\[
G_1 = 2 \left( R - z \right)
\]

\[
G_2 = - \frac{2}{\pi} \left[ z \left( \pi - J \right) - \frac{1}{R_2} \left\{ R_2^2 E' + (a^2 + r^2) K' \right\} \right]
\]

\[
G_3 = \frac{2 R_2}{\pi} E' - z
\]

\[
G_4 = - \frac{2}{\pi} \left[ z J - \frac{1}{R_2} \left\{ R_2^2 E' + (a^2 - r^2) K' \right\} \right]
\]

\[
H_1 = z \left( \frac{z}{R} - 1 \right)
\]

\[
H_2 = - \frac{z}{\pi} \left[ \pi - J - \frac{z}{R_2} K' \right]
\]

\[
H_3 = \frac{z}{2\pi} \left[ \frac{2 z}{R_2} K' - \pi \right]
\]

\[
H_4 = - \frac{z}{\pi} \left[ J - \frac{z}{R_2} K' \right]
\]
Notations for Appendix A

\[ R = \sqrt{z^2 + a^2} \]
\[ R_1 = \sqrt{z^2 + (a - r)^2} \]
\[ R_2 = \sqrt{z^2 + (a + r)^2} \]
\[ k = \frac{R_1}{R_2} \]
\[ k' = \frac{1 - k^2}{R_1} \]

\[ E = \frac{\pi}{2} \int_{0}^{\pi} \sqrt{1 - k^2 \sin^2 \theta} \, d\theta \]
\[ K = \frac{\pi}{2} \int_{0}^{\pi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \]
\[ E' = \frac{\pi}{2} \int_{0}^{\pi} \sqrt{1 - k'^2 \sin^2 \theta} \, d\theta \]
\[ K' = \frac{\pi}{2} \int_{0}^{\pi} \frac{d\theta}{\sqrt{1 - k'^2 \sin^2 \theta}} \]
\[ E(k, \phi) = \phi \int_{0}^{\pi} \sqrt{1 - k^2 \sin^2 \theta} \, d\theta \]
\[ F(k, \phi) = \phi \int_{0}^{\pi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \]
\[ J = K' E(k, \phi) - (K' - G') F(k, \phi) \]

\[ \sin \phi = \frac{z}{R_1} \]
\[ \sin \nu = \frac{2 a z}{R_1 R_2} \]
\[ \cos \nu = \frac{z^2 - a^2 + r^2}{R_1 R_2} \]
\[ \cos \nu = \frac{R_1 R_2}{2 r^2} (1 - Ac) \]
\[ A = \frac{2 + a^2 - r^2}{R_1 R_2} \]
\[ A = \cos (\theta - \gamma) \]

\[ A = \frac{az \cos v + (a^2 - r^2)}{az} \sin v \]

\[ b = \frac{z^2 + a^2 + r^2}{R_1 R_2} \]

\[ b = \frac{z \cos v + a \sin v}{z} \]
APPENDIX B

Associated Elastic Solutions
2. The foundation behaves in shear as a delayed elastic material represented by a Kelvin model.

The associated elastic solution for the Kelvin body shown below is obtained by substituting in equations 14-16, for \( \frac{Q(s)}{p(s)} = A + Bs \) with \( \frac{Q'(s)}{p'(s)} = C \).

\[
\begin{align*}
\sigma^*(r,z,a,s) & = \left[ \frac{P_0 \omega}{s^2 + \omega^2} \right] \left[ \frac{(C - A)}{(2C + A + Bs)} \right] \cdot f_1(r,z,a) \\
& + \left[ \frac{P_0 \omega}{s^2 + \omega^2} \right] F_1(r,z,a)
\end{align*}
\]

\[
U^*(r,z,a,s) = \left[ \frac{P_0 \omega}{s^2 + \omega^2} \right] \left[ \frac{1}{2C + A + Bs} \right] Q_1(r,z,a)
\]

\[
+ \left[ \frac{P_0 \omega}{s^2 + \omega^2} \right] \left[ \frac{1}{A + Bs} \right] q_1(r,z,a)
\]

\[
w^*(r,z,a,s) = \left[ \frac{P_0 \omega}{s^2 + \omega^2} \right] \left[ \frac{(C + 2A) + 2 Bs}{(A + Bs)(2C + A + Bs)} \right] G_1(r,z,a)
\]

\[
- \left[ \frac{P_0 \omega}{s^2 + \omega^2} \right] \left[ \frac{1}{A + Bs} \right] H_1(r,z,a)
\]

3. The foundation material behaves in shear as a standard linear solid.

In equations 14-16, one substitutes for \( \frac{Q(s)}{p(s)} \) the quantity \( \frac{(C_1 + A_1 s)}{s^2 + s} \) and for \( \frac{Q'}{p'} \) by \( C \), then one obtains the associated elastic solution for the standard linear solid as shown below.

\[
\begin{align*}
\sigma^*(r,z,a,s) & = \left[ \frac{P_0 \omega}{s^2 + \omega^2} \right] \left[ \frac{(CC_2 - C_1) + (C - A_1) s}{(2CC_2 + C_1) + (2C + A_1) s} \right] \cdot f_1(r,z,a) \\
& + \left[ \frac{P_0 \omega}{s^2 + \omega^2} \right] F_1(r,z,a)
\end{align*}
\]
U*(r, z, a, s) = \left[ \frac{P_0 \omega}{s^2 + \omega^2} \right] \left[ \frac{C_2 + s}{2CC_2 + C_1} + \frac{(2C + A_1)s}{s} \right] 3Q_1(r, z, a)

w^*(r, z, a, s) = \left[ \frac{P_0 \omega}{s^2 + \omega^2} \right] \left[ \frac{C_2 + s}{C_1 + A_1 s} \right] q_1(r, z, a)

G_1(r, z, a) = \left[ \frac{P_0 \omega}{s^2 + \omega^2} \right] \left[ \frac{C_2 + s}{C_1 + A_1 s} \right] H_1(r, z, a)

4. The associated elastic solution for the Burgers model is shown below.

\sigma^*_r = \left[ \frac{P_0 \omega}{s^2 + \omega^2} \right] \left[ \frac{(CC_3 - B_2)s^2 + (CC_4 - A_2)s + CC_5}{(2CC_3 + B_2)s^2 + (2CC_4 + A_2)s + 2CC_5} \right] f_i(r, z, a)

U^* = \left[ \frac{P_0 \omega}{s^2 + \omega^2} \right] \left[ \frac{C_3 s^2 + C_4 s + C_5}{s(B_2 s + A_2)} \right] q_1(r, z, a)

w^* = \left[ \frac{P_0 \omega}{s^2 + \omega^2} \right] \left[ \frac{(CC_3^2 + CC_4 s + C_5^2)}{s(B_2^2 s + A_2)} \frac{(CC_3 + B_2)s^2 + (CC_4 + 2A_2 s + CC_5)}{s(B_2 s + A_2)} \cdot \frac{(CC_3^2 + 2B_2 s^2 + (CC_4 + 2A_2 s + CC_5)}{s(B_2^2 s + A_2)} \right] H_1(r, z, a)
APPENDIX C

IBM Programs
1. FLOATING (K1,K2,L12,M1(14),N1(14),LL3,L4,LL7,LL9,LL12,LL15,LL16,LL21)
2. FLOATING (MN2,MN5,MN6)
3. DIMENSION (XM(11),XTA(11),XAC(11),XALA(11),I=0,1,LL,3)
4. START
5. READ INPUT,INP,(LXM(1),XTA(1),XAC(1),XALA(1),I=0,1,LL,3)
6. F INP (4FE,2) -
7. DO THROUGH (L11,J=0,1,LL,3)
8. DO THROUGH (L1,J=0,1,LL,3)
9. W=KM(1)-
10. TA=XTA(J)-
11. AC=XAC(KK)-
12. W2=W*W-
13. TA2=TA+TA-
14. K=1./[TA+K2*AC]-
15. K1=K1.-AC2/2.*AC2-
16. K2=K2.*AC/.1/(AC+2.1)-
17. L11=1./[IN+TA]+SQRT(1.+TA2+W2)-
18. L12=1./[IN+TA]+SQRT(1.+W2+TA2)-
19. L1=1./[1./2.+TA+W]-
20. L14=SQRT(1.+K*W2)-
21. L15=M/2.+TA+1.-/2.*K+TA)-
22. L16=SQRT(K+K2*K2+W2)-
23. L17=2.*TA+K2+K2-1./10.*TA+K1)-
27. LL2=LL3-LL-
LL4=ATAN(1.7274/1+LL2-)
LL7=ATAN((2.+4w+T)-
LL8=ATAN((2.+4w+E/1+LL7-ATAN(1.1w/K1-)
M(11)=w=TA/(1.+w+2+T)-
M(12)=1.4(w+T)+Sqrt((1.+w+T)-TA=)
M(13)=2.+w+2+T=TA/((1.+w)+2+T-)
M(14)=2.52+T/((1.+w)+2+T=TA=)
M(15)=1.5=TA+AC+AC/TA=TA=+2+AC+AC+12.+AC1.P.2)-
M(16)=Sqrt((1.+w)/2.4)+w+2+TA=TA=AC+AC-)
M(17)=Sqrt((1.+w+2+T=TA)=((1.+w+AC1.P.2+(2.+AC).P.2)))-
M(18)=(1.+AC2.3/((2.+AC=-)
M(19)=1.+w+2+TA=15+AC+AC/112.+AC1.P.2+4.+w+2+TA=TA+AC+AC3)=-
M(20)=Sqrt((1.+w+AC1.P.2+(4.+w+TA=AC3.P.2=-
M(21)=Sqrt((1.+w+2+T=TA)=((2.+AC1.P.2+(2.+w+TA=AC1.P.2)))-
M(22)=(1.+AC1.P.2+TA=AC-)
M(23)=(2.+AC1.P.2+TA=AC-)
M(24)=LL1=LL3-1
LL12=ATAN((2.+w+H2)-ATAN.(11+M1)-LL1=1
P=Provided (AC.60)+(LL12=LL11-
LL15=ATAN(2.+w+TA)=
LL16=ATAN(14.+w+H2)-ATAN((2.+w+11)-LL15=-
P=Provided (AC.60)+(LL16=LL15-
White output +FM11,FM,TA,AC-)
F FM11 (/20x.+Y4.F7.+ONZ T=FM,2.2+H
2MY0,9X,2MY7,4L,2MY8/-)-
F FM1 (/20x.+Y4.F7.+ONZ T=FM,2.2+H
2MY0,9X,2MY7,4L,2MY8/-)
F FM1 (/20x.+Y4.F7.+ONZ T=FM,2.2+H
2MY0,9X,2MY7,4L,2MY8/-)
F FM1 (/20x.+Y4.F7.+ONZ T=FM,2.2+H
2MY0,9X,2MY7,4L,2MY8/-)
J3H

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58

L2

K0.2+T/(2.*TA)-L(L2)*SIN((2.*W+T)*L7)-

59

EX=CAPE.-1.*X=TA)-

60

PRIVIEU-1.*X=T.L.-10.1;EX=0.-

61

X7=L4)+L(S)+EX=L4)*P.2-L(1)*L(2)*COS(12.*W+T)*LL4)/L(4)-

62

X9=L7)+T(4.+TA)+L(5)*L(8)*EX=L(7)*L(9)*SIN(12.*W+T)*LL8)-

63

EX=CAPE.-1.*TA)-

64

EX=CAPE.-1.*M+T)-

65

PRIVIEU-1.*T./TA.L.-10.1;EX=0.-

66

PRIVIEU-1.*M+T.L.-10.1;EX=0.-

67

Y4=M((1)*EX=M(2)*SIN(12.*W+T)*LL11)-

68

Y5=M(3)+EX=M(4)*COS(12.*W+T)*LL15)-

69

Y7=M(6)+EX=M(7)*COS(12.*W+T)*LL12)/M(7)-

70

Y8=M(8)+M(9)*EX=M(9)*EX=M(10)*COS(12.*W+T)*LL16)/M(11)-

71

L2

WRITE OUTPUT xF03,T(4*,X0,X7,X8;Y4,Y6,Y7,Y8)-

72

F F H 2

(F0.2+T/A.L.-FA;4,F4.4,F11.5)-

73

PRIVIEU-1.T.X.+0.;TRANSF G-4.NEXT-4.)

74

TR=5.-

75

DTX=1.-

76

STA=1.72-

77

TRANSF G (L.)-4.

78 NEXT

DO THROUGH (L.),JJ=0.1;JJ.L.3-

79

A1=x(1.+JJ)-

80

C=(A1*1.+1./1.-

81

NN2=(2.*C.AC/TA)/12.+AC)-

82

NN5=(1.+2.*AC/12.+AC)-

83

NNU=(2.+AC/TA)*1/12.+AC)-

84

P3=ATAN.(W/C)-ATAN.(-W+TA)-

85

P4=P3*ATAN.(4W5+W/W)-ATAN.(-W/W2)-

86

P7=ATAN.(-2.*W/C)-ATAN.(-2.*W*TA)-
116  L6   WRITE OUTPUT (FM, (12, 24, 16, 27, 28))
117  FM  IF (C2, 4F11, G)
118     PROVIDE (TV, 0, 0), TRANSFER (L1)
119     TV=TV-
120     DIV=DIV-
121     STV=STV-72-
122     TRANSFER TO (L7)-
123     L1 CONTINUE -
124     LR CALL SUBROUTINE (1=ENDJOB, 1)-
125     END PROGRAM (START)-
SIOUENT LANGUAGE STATEMENTS

1 DIMENSION (KE351,PHI(12),NM(3),KAZA(33) -

2  FLOATING (F3)

3 START

4 READ INPUT, IMP, T(AMM1), XAC(I), XTI(1,2), XZA(I), XAZA(I,1,J)

5 F IMP

6 DO THROUGH SUM, I=0,1,2,3

7 DO THROUGH SUM, J=0,1,2,3

8 DO THROUGH SUM, K=0,1,2,3

9 DO THROUGH SUM, N=0,1,2,3

10 N=K+1

11 AC=XAC(J)

12 T1=XTI(K)

13 T2=XTI(L)

14 AZA=XAZA(N)

15 A1A2=1/ZA

16 A=1/T1*(1+AC1)

17 A=1/(A1A2+1+T2)*T1*(1+AC1)

18 AZ=1−AC1/2+AC1

19 B=1/(T1+T2)

20 D1=1/T2*(A1A2+1.1.1)/T1

21 B=1/T2

22 D1=1/T1*(1.1.2.2−AC1)

23 D1=1/ALPHA2+T2/1.1.2.2−AC1/(T2+1.1.2.2−AC1)

24 ALPHAM=ALPHA2+2.2+D2/1.1+AC1/(2.2+D2+2.2−AC1)

25 ALPHMSQ=ALPHAM2+2/(T1+T2+1.2.2)

26 GAMMA=SQR(T)(GAMMSQ)

27 M2=W.P.2
29  vice.1+2.1+alpha.w/(alphau-gamsq-x21)-
30  provided (alphau-gamsq-w2.l.o.1,v2+v2+3.1416-
31  provided (alphau-gamsq-w2.l.o.1,v2+1.570796-
32  v3=atan.1+i1+(i8-w21)-
33  provided (i8-w2.l.o.1,v3+v3+3.1416-
34  provided (i8-w2.e.o.1,v3+1.570796-
35  v4=atan.1+i3)-
36  v5=atan.1+i1+(i1-w21)-
37  provided (i1-w2.l.o.1,v5+v5+3.1416-
38  provided (i1-w2.e.o.1,v5+1.570796-
39  v7=atan.1+i4+(alphau-gamsq-4.*x21)-
40  provided (alphau-gamsq-4.*w2.l.o.1,v7+v7+3.1416-
41  v8=atan.1+i2+(i8-w21)-
42  provided (i8-w2.l.o.1,v8+v8+3.1416-
43  v9=atan.1+i3)-
44  v10=atan.1+i1-(i1-w21)-
45  provided (i1-w2.l.o.1,v10+v10+3.1416-
46  ps1=atan.1+i4+(i4-w21)-v2-
47  provided (i4-w2.l.o.1,ps1+ps1+3.1416-
48  provided (i4-w2.e.o.1,ps1+1.570796-v2-
49  ps12=v3+v2-
50  ps13=v3-v4-
51  ps14=v3+v3-v4-v2-
52  theta1=atan.1+i1+(i4-w21)-v7-
53  provided (i4-w2.l.o.1,theta1+theta1+3.1416-
54  theta2=v8-v7-
55  theta3=v8-v9-
56  theta4=v8+v10-v9-v7-
57  g6=alphau-gamsq-
139
REFERENCES


