FLUTTER ANALYSIS OF AN AIRFOIL
EXHIBITING CAMBER DEFORMATIONS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

Ralph Gibson Dale, Jr., B.Aero.E.

The Ohio State University
1964

Approved by

[Signature]
Adviser
Department of Aeronautical and Astronautical Engineering
ACKNOWLEDGMENTS

The author wishes to express his appreciation to his adviser, Dr. B. E. Gatewood, for the technical advice, criticism, and encouragement provided throughout the course of this work. The author is particularly indebted to Mr. C. J. Peirce and Mr. C. D. Stevens of Goodyear Aerospace Corporation for suggesting the problem considered in this dissertation and for the stimulating discussions on the areas of application. Also, the author wishes to thank the Goodyear Aerospace Corporation for the use of the experimental results for the airfoil model studied as a sample problem in this dissertation.

In addition, the author wishes to express his sincere thanks to his wife and family for providing patience and encouragement throughout this program of graduate study.
VITA

June 27, 1934  Born - Detroit, Michigan

1957       Bachelor of Aeronautical Engineering

1957-1958  Engineer, North American Aviation, Inc., Columbus, Ohio

1958-1964  Instructor and Research Associate, Department of Aeronautical and Astronautical Engineering, The Ohio State University, Columbus, Ohio

PUBLICATIONS


iii
FIELDS OF STUDY

Major Field: Aeronautical and Astronautical Engineering


Studies in Aeroelasticity. Professor B. E. Gatewood and Dr. W. R. Laidlaw.

Studies in Aerodynamics. Professor J. D. Lee.
## CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td><strong>Section</strong></td>
<td></td>
</tr>
<tr>
<td>I. DIFFERENTIAL EQUATIONS OF MOTION FOR FORCED OSCILLATION</td>
<td>18</td>
</tr>
<tr>
<td>II. DEVELOPMENT OF THE AERODYNAMIC PRESSURE DISTRIBUTION FOR A BODY EXHIBITING CAMBER OSCILLATIONS</td>
<td>32</td>
</tr>
<tr>
<td>III. DEVELOPMENT OF THE GENERALIZED AERODYNAMIC FORCES REQUIRED FOR A NORMAL MODE FLUTTER ANALYSIS</td>
<td>61</td>
</tr>
<tr>
<td>IV. INVESTIGATION OF THE RESPONSE CHARACTERISTICS OF A TWO-DIMENSIONAL AIRFOIL OF AIRMAT CONSTRUCTION</td>
<td>87</td>
</tr>
<tr>
<td>V. SUMMARY AND CONCLUSIONS</td>
<td>102</td>
</tr>
<tr>
<td>LIST OF SYMBOLS</td>
<td>105</td>
</tr>
<tr>
<td>APPENDIX A—Natural mode shapes and frequencies for a uniform free-free Timoshenko beam</td>
<td>127</td>
</tr>
<tr>
<td>APPENDIX B—Expansion of ( \cos^m \theta ) in terms of ( \cos(m\theta) )</td>
<td>134</td>
</tr>
<tr>
<td>APPENDIX C—Derangement of term in the series ( (C.1) )</td>
<td>139</td>
</tr>
<tr>
<td>APPENDIX D—Restrictions on the use of the Cauchy principal value</td>
<td>146</td>
</tr>
<tr>
<td>APPENDIX E—Summary of definite integrals</td>
<td>155</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>164</td>
</tr>
<tr>
<td>Table</td>
<td>Description</td>
</tr>
<tr>
<td>-------</td>
<td>-------------</td>
</tr>
<tr>
<td>1.</td>
<td>Aeroelastic Representations</td>
</tr>
<tr>
<td>2.</td>
<td>Sample Problem Data</td>
</tr>
<tr>
<td>3.</td>
<td>Bessel Coefficients for Numerical Example</td>
</tr>
<tr>
<td>4.</td>
<td>Numerical Values of the Functions $G_n^m(\alpha,\beta)$ for Numerical Example</td>
</tr>
<tr>
<td>5.</td>
<td>Numerical Values of the Function $G_1^1(\lambda_j, \lambda_p)$</td>
</tr>
<tr>
<td>6.</td>
<td>Numerical Values of the Function $G_2^1(\lambda_j, \lambda_p)$</td>
</tr>
<tr>
<td>7.</td>
<td>Numerical Values of the Function $G_1^2(\lambda_j, \mu_p)$</td>
</tr>
<tr>
<td>8.</td>
<td>Numerical Values of the Function $G_2^2(\lambda_j, \mu_p)$</td>
</tr>
<tr>
<td>9.</td>
<td>Numerical Values of the Function $G_1^3(\mu_j, \mu_p)$</td>
</tr>
<tr>
<td>10.</td>
<td>Numerical Values of the Function $G_2^3(\mu_j, \mu_p)$</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>1.</td>
<td>Theoretical Model</td>
</tr>
<tr>
<td>2.</td>
<td>Free Body Diagram of Beam Segment</td>
</tr>
<tr>
<td>3.</td>
<td>Rigid Body Displacements and Elastic Deformation Representation</td>
</tr>
<tr>
<td>4.</td>
<td>Coordinate Transformation between $x$ and $x'$ Coordinate System</td>
</tr>
<tr>
<td>5.</td>
<td>Change in the Natural Frequency of a Vibrating Free-Free Beam Due to Shear and Rotary Inertia</td>
</tr>
<tr>
<td>6.</td>
<td>Graphical Solution of Transcendental Equation of a Free-Free Timoshenko Beam</td>
</tr>
<tr>
<td>7.</td>
<td>Comparison of Mode Shapes and Natural Frequencies</td>
</tr>
<tr>
<td>8.</td>
<td>Theoretical Model for Sample Problem</td>
</tr>
<tr>
<td>9.</td>
<td>Airmat Structural Configuration</td>
</tr>
<tr>
<td>10.</td>
<td>Side View of Experimental Model</td>
</tr>
<tr>
<td>11.</td>
<td>Plane View of Experimental Model</td>
</tr>
<tr>
<td>12.</td>
<td>Comparison between Theoretical and Experimentally Determined Frequency Response Characteristics</td>
</tr>
</tbody>
</table>
INTRODUCTION

The ability to predict and prevent dynamic instabilities such as flutter is a significant problem in the design and development of present aircraft and proposed glide re-entry vehicles. The problem of lifting surface flutter has existed since the early days of aircraft, and has received much attention in recent years. In the early days of aircraft the governing design criterion was the ability of the craft to withstand the static loading conditions associated with maneuvering flight. Generally the aeroelastic structural requirements were automatically satisfied within the flight range of the vehicles. New developments in materials and structural design techniques have led to lifting surfaces that are very efficient in withstanding static loadings. Progress in this area, however, has always resulted in the reduction of the overall structural stiffness of the surface. Consequently, in many cases the structural rigidity requirements, and hence the aeroelastic requirements, must be used to design certain of the structural components rather than the structural strength requirements. Also in the search for new developments in light weight storable structures, concepts such as the Paraglider and "airmat" class of inflatable
structures have come under consideration. These structures
tend to exhibit a high degree of flexibility in the camber
line as well as in the spanwise direction. For these
structures the capability to predict dynamic instabilities
is required for situations where there exist the possibil-
ity of the camber line deformation modes coupling with the
classical wing bending-torsion modes.

The first paper in which the mechanism of flutter
was rather thoroughly discussed was published in 1935 by
Theodorsen.\textsuperscript{1} In his work, he developed the aerodynamic
forces (lift and moment) associated with an oscillating
airfoil or airfoil-aileron combination in a two-dimension-
al incompressible flow field. The airfoil or airfoil-
aileron segments were regarded as rigid elements leading
to a three degree of freedom flutter problem. Since
Theodorsen's paper many reports and books have been written
on the subject of aeroelastic instabilities. Conse-
quently, only the work most relevant to the present study
will be reviewed. No attempt will be made to review thor-
oughly all significant aspects of the references studied.
A comprehensive bibliography of earlier works up through
1955 is contained in reference 2.

The problems associated with the field of aeroelas-
ticity are a combination or coupling of those studied in

\textsuperscript{1} Superscripts are reference numbers in the Bibliography.
the separate areas of structures and aerodynamics. Problems such as flutter instability therefore require a consideration of three aspects that may be classified as

1. structural representation
2. aerodynamic representation
3. aeroelastic representation.

The third problem represents the joining of the first two areas to develop a meaningful analogue to the combined behavior of the elastic system moving in a fluid medium.

The table presented on the following page lists several combinations of aerodynamic and structural representations that have been used in predicting the occurrence of surface flutter and wing divergence. By no means is the table a complete compilation of all techniques presently used. This brief table is presented here to support the discussions given in the following subsections, pointing out certain areas of applicability of the theories relevant to the problem studied in this dissertation. Also, it serves to categorize this work with respect to the techniques presently in use.

**Structural representation**

The first predictions of lifting surface flutter were calculated using the first representation listed in the table. Through experience it was established that the flutter speed could be approximately determined through examination of a one-dimensional airfoil segment having
<table>
<thead>
<tr>
<th>No.</th>
<th>Structural Representation</th>
<th>Reference</th>
<th>Aerodynamic Representation</th>
<th>Reference</th>
<th>Aeroelastic Problem Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>One-dimensional rigid airfoil segment elastically supported, one and two degrees of freedom</td>
<td>2, chp.8, 9, 30, chp.3, 15</td>
<td>Two-dimensional unsteady flows, rigid camber line, harmonic pitch and vertical translation oscillation</td>
<td>2, chp.5 15, 16, 17</td>
<td>2, chp.8, 9 30, chp.3, 5</td>
</tr>
<tr>
<td>2</td>
<td>Two-dimensional wing, rigid chord, uncoupled spanwise bending and torsion degrees of freedom</td>
<td>2, chp.3 18, 19</td>
<td>Two-dimensional strip theory unsteady flows, rigid chord. Aerodynamic influence coefficients</td>
<td>1, 2, 15, 16, 20, 21, 29, 31</td>
<td>2, 30, 31</td>
</tr>
<tr>
<td>3</td>
<td>Two-dimensional wing, rigid chord, coupled spanwise bending-torsion degrees of freedom, influence coefficients and lumped mass representation</td>
<td>2, chp.3 18, 19</td>
<td>Two-dimensional strip theory, unsteady flow, with or without permitting parabolic camber oscillations. Three-dimensional aerodynamic influence coefficients</td>
<td>1, 2, 15, 16, 19, 21, 29, 31</td>
<td>2, 30, 31</td>
</tr>
<tr>
<td>4</td>
<td>Three-dimensional wing, composite elastic structure leading to an influence coefficient representation, lumped mass inertia representation</td>
<td>4, 5</td>
<td>Two-dimensional strip theory, with or without parabolic camber corrections. Three-dimensional aerodynamic influence coefficients</td>
<td>1, 2, 15, 16, 21, 29, 31</td>
<td>2, 31</td>
</tr>
<tr>
<td>5</td>
<td>One-dimensional elastic airfoil segment, elastically supported, 2 rigid and n elastic degrees of freedom a) classical beam representation, bending flexibility only b) Timoshenko beam, bending and shearing deformations, mass and rotary inertia included</td>
<td>10, 12, 13, 14</td>
<td>Two-dimensional strip theory, unsteady flow, arbitrary oscillations of the camber line</td>
<td>15, 16 This dissertation</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Two-dimensional wing, rigid chords, spanwise bending and shearing deformation permitted, with or without rotary inertia effects</td>
<td>10, 12, 13, 14</td>
<td>Two-dimensional strip theory</td>
<td>1, 2, 15</td>
<td>None</td>
</tr>
<tr>
<td>7</td>
<td>Two-dimensional wing, plate representation a) bending deformations only b) bending and shearing deformations, rotary inertia effects included</td>
<td>6, 7</td>
<td>Two-dimensional strip representation, permitting oscillation of the camber line. Three-dimensional aerodynamic influence coefficients</td>
<td>16 This dissertation</td>
<td>None</td>
</tr>
</tbody>
</table>

In the above table, the use of the term "aerodynamic influence coefficients" is to infer that methods such as the kernel function approach be used in the calculation of the aerodynamic forces. The aeroelastic problems considered in the listing are those of surface divergence and surface flutter.
the dimensions and displacements characteristic of a section located at the three quarter span position. The camber line of the airfoil strip was fixed, the rigid segment being supported elastically to permit rigid body rotation and vertical translation. Obviously this was a crude representation of the system, both from the structural and aerodynamic points of view. However, this simple model served to provide much insight into the mechanism of aeroelastic instabilities.

Techniques presently used employ the concept of an elastic axis in order to represent more closely the dynamic behavior of the lifting surface. Assuming the existence of an elastic axis and associated chordwise rigidity perpendicular to this axis, deformations may be represented by specification of the deformations of the axis and rotations about the axis. The representation of problems two and three listed in the table is then accomplished by superposition of the normal bending and torsional modes of oscillation referred to the elastic axis. This approach has been shown to be feasible for application with relatively high aspect ratio wings.

The problem of predicting the elastic behavior of low aspect ratio wings of classical monocoque or thick skin construction where camber deformations are present is treated theoretically by Levy and by Turner. Both of these methods are best suited to the influence coefficient
type of approach indicated by 4 in the table. For very thin wings, the surface may be regarded as a plate and classical plate theory\(^6,7\) may be used in developing a modal or influence coefficient representation. Mindlin\(^8\) has developed the plate equations to include the effect of rotary inertia and shearing stiffness. Leonard, Brook and McComb\(^9\) present the linear form of the dynamical equations of motion of free vibration for a rectangular plate of constant depth "air mat" construction. These equations are identical to those developed by Mindlin, except that the internal pressure plays the role of the modulus in the plate transverse shear stiffness.

In the light of the complications involved with the combined aerodynamic and structural representation for structures exhibiting large amounts of camber deformation, as indicated by problem 7 of the table, a strip theory approach analogous to representation 1 appears most logical. In this manner, streamwise strips may be handled in which rotary inertia and shearing stiffness play a paramount role. In such an approach the individual strip segment could be regarded as an elastic beam, supported elastically to permit rigid body degrees of freedom. This representation is listed as number 5 in the table and is the problem to be considered in this dissertation. In this representation a one-dimensional streamwise strip of the airfoil is displaced from the lifting surface, spanwise bending effects are neglected and no restraints are permitted along the
edges. In such an approach the individual strip may be regarded as an elastic beam with flow along the beam length. Shearing deformation and rotary inertia effects may be included when appropriate. Since this is the technique used in this work, consideration is given in the following paragraphs to available beam theories.

Timoshenko\textsuperscript{10} was the first to develop the governing equations for a beam including the effects of both shearing deformations and rotary inertia, even though their presence was discussed as early as 1858 by Bresse.\textsuperscript{11} Rogers\textsuperscript{12} presents different forms of the basic equations for the various subcases that could apply to specific problems. An extensive study was performed by Kraszewski\textsuperscript{13} to determine the effects of transverse shear and rotary inertia on the natural frequencies of oscillation for a uniform beam under several conditions of restraint. The general conclusion is that the effect of rotary inertia, which tends to reduce the frequency, becomes more and more important with an increase in the natural frequency. The effect of shearing stiffness, however, is greater than that of rotary inertia and has the same trend with increasing frequency.

In the forced equations of motion presented by Timoshenko and by Rogers the force and moment equilibrium equations were combined before solution. This eliminates the individual shear and bending slope terms producing a single differential equation in terms of the total displacement. Leonard\textsuperscript{14} analyzed the symmetrical natural
vibration of a uniform free-free Timoshenko beam with a concentrated mass in the center, keeping the two equilibrium equations separated and solving them simultaneously. The results are identical to those obtained by the single equation approach; however, the resulting forms for the deflection and bending rotation mode shapes are more amenable for use in a modal representation of the forced oscillation problem. In the same reference Leonard considers the problem of forced oscillation using a normal coordinate approach. He establishes the necessary equations for the case where the external applied force is time dependent and a function of the coordinate along the beam. An approach similar to this will be used in the structural representation of the work contained herein; however the forcing function used will be a function of the displacements as well as time and position.

Aerodynamic representation

For many years flutter analysis techniques were based on the application of aerodynamic strip theory as indicated in representations 1 through 4 in the table. This approach seems to be adequate for predicting flutter instabilities in the case of relatively high aspect ratio wings. The aerodynamics employed in strip theory analysis are almost exclusively based on Smilg and Wasserman's\textsuperscript{15} tabulation of two-dimensional quasi-steady aerodynamic coefficients. This approach is in essence a numerical
tabulation of the procedures developed in Theodorsen's paper and the work published by Kussner. The tabulation provides aerodynamic force and moment coefficients for oscillating wing-aileron-tab combinations, taking into account the effects of aerodynamic balance and gaps for both the aileron and tab. However, the airfoil segments are regarded as rigid elements having elastic restraints between components, each component being permitted pitch and vertical translation degrees of freedom.

Kussner's theory, in its general development given in reference 16, is not limited to the class of airfoil motions studied in reference 15; on the contrary it is valid for any arbitrary oscillatory deformation shape of the camber line. However, only the motions tabulated by Smilg and Wasserman were considered in detail. The general theory employs the following approach. The downwash for harmonic oscillation is expanded in a Fourier series representation based on the prescribed deflection pattern. The unknown pressure distribution is assumed to be represented by a Fourier sine series with the addition of a cotangent term as is customarily employed in steady airfoil theory. The coefficients in the pressure series are expressible in terms of the downwash coefficients and can be evaluated through applying the physical boundary conditions required on the bound vorticity associated with the two-dimensional flow field. Through this procedure he develops a large
number (54 in all) of highly specialized integral relations which he has tabulated for the rather restricted class of motions described in the preceding paragraph.

Spielberg\textsuperscript{19} extended the work of reference 15 to include the case of a parabolic camber mode of oscillation in order to investigate the effect of camber flexibility on the classical bending torsion flutter modes. The final form developed for the quasi-steady aerodynamic coefficients is completely compatible for use in conjunction with the normal strip theory approach as shown in representation number 3 in the listing. Performing a three-degree of freedom flutter analysis (i.e., translation, rotation and parabolic camber deformation) Spielberg demonstrated that the inclusion of camber oscillation tended to reduce the flutter speed in all cases investigated where rigid body rotation was involved. It is interesting to note that Spielberg makes the following statement concerning the general approach of Kussner's\textsuperscript{16} theory.

The above general approach to the non-stationary aerodynamics of a profile undergoing an arbitrary deformation of the mean chord line is presented here largely for academic interest. It is not considered to be of practical value in flutter analyses of actual wings and it is recommended, for the present at least, that theoretical investigations of the effect of camber changes on flutter be confined to the use of the relatively simple parabolic approximation for the camber deformation.
As discussed later, this dissertation shows that many of the problems associated with the above approach can be eliminated by the introduction of modal coordinates in the downwash expression and by employing the "exact" linearized pressure equation of two-dimensional incompressible flow theory.\textsuperscript{20}

Reissner\textsuperscript{20} developed an integral equation relating the pressure to the downwash distribution employing a real variable approach through representing the lifting surface and its wake by a vortex sheet. The vortex strength distribution was determined by forcing the physical boundary conditions on the fluid field. This is in contrast to the Theodorsen\textsuperscript{1} approach of working with the flow about a cylinder and employing conformal transformation techniques to develop the flow about the airfoil section. Although the application of the relations developed has been restricted to use with rigid airfoil segments, they are suitable for predicting the pressure on any thin surface executing arbitrary deformations, as long as the motion is harmonic in time.

Lately, extensive use has been made of the concept of aerodynamic influence coefficients.\textsuperscript{21} The aerodynamic influence coefficient expresses the pressure acting at one point of the surface resulting from a unit change in the downwash at another point. This concept is useful in considering aerodynamic surface not clearly characterized by
easily expressible bending-torsion modes of oscillation, such as is the case with low aspect ratio wings. Representations 3 and 4 of the table of aeroelastic representations indicate the position of this approach relative to strip theory techniques. While strip theory assumes that the aerodynamic forces acting at a certain wing station depend only on the motion of that segment, the adjacent sections having no influence, the use of aerodynamic influence coefficients serves to eliminate the necessity of such an assumption. The aerodynamic influence coefficient approach is still only an approximate method in the sense that a finite grid needs to be considered. This approach is somewhat limited, however, by the fact that its advantages can only be realized if the structure can be completely characterized by a set of structural influence coefficients. In many instances, as is the case with some inflatable structures, deflection influence coefficients have no practical application. In the case of the "airmat" class of structures shearing deformations make up approximately half the total displacement and concentrated forces cause local shear displacements non-characteristic of deformations under a distributed load. The superposition of discrete load patterns is therefore not realistic.

**Aeroelastic representation**

Many of the representations developed and used in aeroelastic calculations were originally considered in the
separate areas of structures and aerodynamics. Hence, certain inconsistencies appear in the theoretical model finally employed when the two fields are joined for aeroelastic analysis. Such inconsistencies are apparent upon examination of the table on aeroelastic representations. For example, problem 3 employs the elastic axis concept for the structural representation and two-dimensional strip theory for the aerodynamic terms. The inconsistency lies in that the structural model requires sections normal to the elastic axis be rigid in their own plane, while the streamwise airfoil segment is assumed rigid for the aerodynamic calculations. These two conditions are not necessarily compatible, as is the case for a swept wing. In an attempt to alleviate this ambiguity the airfoil representation has been permitted, in some instances, to assume a parabolic camber deformation shape. However, in most cases, the amount of deformation is fixed in the aerodynamic terms and is completely uncoupled from the structural terms. In the investigation of problem 5 contained in this dissertation, compatible coupling of the aerodynamic and structural terms is permitted in the camber deformation motion.

**Proposed aeroelastic representation**

The purpose of this paper is to consider the problem of flutter of one-dimensional airfoil section in a two-dimensional flow when large amounts of camber
deformation are present. This problem is represented as number 5 in Table 1. Particular emphasis is placed on the requirements for application to inflatable structures of the "airmat" type of construction. This class of structures is such that the transverse shear stiffness can be expected to be very low so that much of the deflection of the surface is shear deflection not involving deformation of the material in the cover. Also, since the mass per unit length is relatively small, with most of the weight concentrated in the surface structure, rotary inertia effects must be included in the dynamical equations. The airfoil section is therefore treated as an elastically supported Timoshenko beam in order to account for these effects. Rigid body rotation and vertical translation are permitted by the elastic support mechanism.

The flutter mechanism for an airmat structure is not completely unlike that of a flapping flag in that the motion is a self-induced sustained oscillation involving deformations of the camber line. While the flag derives its stiffness from tension placed on the material by the viscous and pressure drag forces, the inflatable structure derives its stiffness from internal pressure. In this dissertation, this type of flapping motion is referred to as camber flutter. Throughout the development of the mechanical and aerodynamic portions of the flutter equations a modal representation is used to describe the forced
displacement function. The equations as developed permit free motion of the one-dimensional surface in a two-dimensional flow field. The normal modes used for displacement representation consist of the rigid body translation and rotation modes with the elastic deformations being approximated through the superposition of a finite number of free-free vibration modes.

The aerodynamic pressure and generalized forces are evaluated using thin airfoil theory, replacing the surface by its mean camber line. The aerodynamic development begins with the basic integral equation relating the pressure on the surface to the surface distribution of downwash. The particular form of the equation to be used herein is that presented in reference 2. The downwash is related to the normal coordinates representation of the deformation shape through the linearized boundary condition requiring the flow to be tangent to the surface. This is the same procedure normally used in treating the classical bending-torsion flutter problem. However, the integrals resulting in the pressure and generalized force relations for the case of camber oscillations are significantly more difficult to treat than those associated with the rigid airfoil strip theory analysis. Accordingly a considerable amount of the work contained in this dissertation is directed to the evaluation of these integrals.
It should be pointed out that the theory as developed herein is not limited to use only in the analysis of inflatable structures, but may be employed in any situation where large amounts of camber deformation are present and a strip theory approach is applicable. Eliminating the time dependency in the equations also permits the theory to be used for static aeroelastic calculations.

Section I of this paper is devoted to the development of the dynamical equations of forced motion of a Timoshenko beam. In addition to the aerodynamic forces acting on the system, provisions have been made in the derivation to permit two concentrated loads of a completely general nature. This configuration was established to facilitate the adoption of possible restraint systems, such as the supporting mechanism associated with wind tunnel tests. Or, ultimately to permit elastic reactions needed when extending the theory to a form suitable for strip theory application with a finite inflatable lifting surface.

Sections II and III contain the development of the aerodynamic pressure and generalized forces associated with oscillations in the modal coordinates. The mode shapes are such that the problem becomes one of determining the forces resulting from deformations characterized by the circular and hyperbolic functions.

In Section IV a specific airfoil configuration of "airmat" construction is considered and its frequency response
characteristics evaluated. The results of the theory as developed in this dissertation are compared with some experimental results obtained by Mr. C. D. Stevens of Goodyear Aerospace Corporation during wind tunnel test of a rectangular airmat panel.

Section V contains general conclusions as to the usability and applicability of the technique developed in this work.
I. DIFFERENTIAL EQUATIONS OF MOTION
FOR FORCED OSCILLATION

The streamwise airfoil segments to be considered in this analysis will be represented by an equivalent Timoshenko beam as shown in Figure 1. Since in the aerodynamic representation to be presented in Section II only the motion of the mean camber line is considered, the geometry of the beam segment is important only to the extent that it affects the beam structural characteristics. Consider the beam segment to be of unit width in the z-direction and of length $\lambda$ in the x-direction. Assume that the cross-sectional properties of the beam, in the y-z plane, are constant over the length of the segment, and that the structural properties of the beam are such that the deflections due to bending and shearing deformations are of the same order of magnitude. The total transverse deflection due to the elastic deformations will then be the sum of the bending and shearing deformations. These assumptions seem rather restrictive for the classical type of airfoil cross-section; however, the representation is quite reasonable in regard to application to the present class of inflatable airmat configurations.
For convenience of mathematical representation, the rotations rather than the individual deformations will be carried through the equations of motion as the unknowns of the problem. That is, let $y_e(x,t)$ be the total elastic deflection of the beam mean camber line, and define $\alpha_e(x,t)$ and $\beta_e(x,t)$ to be the elastic rotations of the beam segment resulting from the bending strains and shearing strains respectively, hence

$$y'_e(x,t) = \alpha_e(x,t) + \beta_e(x,t) \tag{1.1}$$

The body is assumed to be subjected to a general transverse aerodynamic load of intensity $P(x,y,t)$, and a concentrated transverse load and moment $F_a(y,t)$ and $M_a(y,t)$ as shown in Figure 1. Both concentrated quantities are assumed to act at the same point on the beam, a distance aft of the leading edge. All forces are permitted to be general functions of time and beam displacements, as well as their respective time derivatives. The equations of motion may be developed from the equations of equilibrium when, in accordance with D'Alembert's principle, the inertia forces are included. Consider a free body diagram of a segment of the beam of length $dx$, as shown in Figure 2. The notation adopted in Figure 2 is as follows:

- $m, \nu$, mass per unit length and mass moment of inertia per unit length, respectively;
- $M, V$, moment and shearing forces acting on the cross-section of the beam segment;
\( \mathbf{F}, \mathbf{M}, \) fictitious applied distributed force and moment adopted as a mechanism to admit concentrated forces;

\( P(x,y,t), \) aerodynamic pressure acting on the system due to its motion in a uniform flow field. When the term "pressure" is used, it is to signify the local pressure difference between the lower and upper surfaces of the airfoil;

\( y, \) total displacement of the beam consisting of rigid body displacements, \( y_r, \) and elastic deformations \( y_e; \)

\( \alpha, \) total cross-sectional rotation consisting of rigid body rotation, \( \alpha_r, \) and elastic bending rotation \( \alpha_e; \)

\( \beta_e \) elastic rotation of the beam segment resulting from shearing deformation.

Equilibrium of the segment requires a balance of all applied and inertia forces. The equilibrium equations may therefore be written as

\[
\begin{align*}
    m \ddot{y} - \frac{\partial V}{\partial x} &= P(x,y,t) - \mathbf{F}(x,y,t) \\
    -V \ddot{x} + \frac{\partial \mathbf{M}}{\partial x} - V &= \mathbf{M}(x,y,t)
\end{align*}
\]

(1.2)

where a dot over a quantity indicates a differentiation with respect to time, and a prime (to be used subsequently) indicates differentiation with respect to the streamwise coordinate, \( x. \) The relation between bending moments and bending rotations, shearing forces and shearing rotations are given by elementary beam theory as
where $EI$ is the bending stiffness and $GAK$ is the equivalent effective stiffness in transverse shear. Under the assumptions that the beam cross-section properties are independent of the coordinate, $x$, the equilibrium equations (1.2) may be written in terms of the displacements and rotations as

$$m\ddot{y} - GAK(y'' - y_r'' - \alpha') = P(x, y, t) - \vec{F}(x, y, t)$$

(1.3)

$$-\nu \ddot{\alpha} + EI \ddot{\alpha} + GAK\left[(y_r - y_r') - (\alpha - \alpha_r)\right] = \vec{M}(x, y, t)$$

where

$$y(x, t) = y_r(x, t) + y_e(x, t)$$

$$\alpha(x, t) = \alpha_r(t) + \alpha_e(x, t)$$

In order to explicitly introduce the concentrated quantities into the equations of motion, define the fictitious distributed force and moment in the following manner. Let

$$\vec{F}(x, y, t) = F_a(y, t) \delta(x - e)$$

$$\vec{M}(x, y, t) = M_a(y, t) \delta(x - e)$$

where $F_a(y, t)$ and $M_a(y, t)$ are independent of $x$ and at most can be functions of the displacement and rotation occurring at the point $x = e$. In this representation $\delta(x - e)$ is the
Dirac delta function possessing the property that

\[ \int_{0}^{l} \delta(x-e) \, dx = 1 \]

Hence

\[ \int_{0}^{l} F(x,y,t) \, dx = F_{a}(y,t) \]

and

\[ \int_{0}^{l} M(x,y,t) \, dx = M_{a}(y,t) \]

Substitution into equation (1.3) gives

\[ m \ddot{y} - GAK(y'' - y_r - \alpha') = P(x,y,t) - F_{a}(y,t) \delta(x-e) \]

\[ -y \dddot{\varphi} + EI \dddot{\alpha} + GAK \left[ (y' - y_r') - (\alpha - \alpha_r) \right] = M_{a}(y,t) \delta(x-e) \]  

(1.4)

The force terms appearing on the right hand side of these equations include two general classes of force. The terms containing \( F_{a}(y,t) \) and \( M_{a}(y,t) \) are intended to be restraining or force input type of forces and may or may not be functions of the displacement at the point \( x = e \), depending on the particular situation being investigated. For example, if the surface is pinned at the point \( x = e \) these forces take on the values \( M_{a}(y,t) = 0 \) and

\[ F_{a}(y,t) = \int_{0}^{l} \left[ P(x,y,t) - m \ddot{y} + GAK(y'' - y_r'' - \alpha') \right] \, dx \]
while in the free flight configuration, $M_a = 0$ and $F_a = 0$. If, on the other hand, the forces are derived from elastic reactions they become functions of the displacement and take on the general form, $M_a(y,t) = -K_a \alpha(e,t)$ and $F_a(y,t) = K_y Y(e,t)$. This class of forces has been included in order to accommodate certain areas of practical interest. An example is the use of the theoretical development in conjunction with wind tunnel test of two-dimensional airfoils where the model is restrained by the tunnel balance system. The inclusion of these terms will also facilitate an expanded use of the theory for a strip theory analysis of finite aspect ratio wings when appropriate elastic axis techniques may be applied.

The distributed aerodynamic pressure term, $P(x,y,t)$, results from the motion of the body through the airstream, both in a direction parallel to $x$-axis in the direction of the free stream and in a direction normal to the flight path. Accordingly, these forces are a function of the body deformations as well as the time rate of deformation. As will be seen later, it is through these forces that the elastic and rigid body equations become coupled.

The two differential equations given in equation (1.4) completely define the motion of the beam segment once the boundary conditions and initial conditions are established. There are two basic approaches that may be applied in working with these equations. Timoshenko\textsuperscript{10}
and Bisplinghoff\textsuperscript{2} combine the two equations into a single equation in terms of one dependent variable. Although neither Timoshenko nor Bisplinghoff solves the resulting differential equation, this approach to a solution of the general problem is suggested. Leonard,\textsuperscript{14} on the other hand keeps the two equations separated, as in (1.4) and solves for the two dependent variables. In this latter approach the left hand sides of the two equations are coupled and consequently must be solved simultaneously. The right side remains uncoupled. In the approach suggested in reference 2, there is no direct coupling of force terms; however the right side contains space and time derivatives of the force terms. The question as to which approach is to be used then depends on the form desired for the force expressions. Since the general problem considered in this work involves the dynamic response when the distributed forces are aerodynamically derived, these forces depending on the motion, it is desirable that the force side be as simple as possible. Therefore, the two-variable approach will be taken and equation (1.4) is dealt with in a manner similar to that presented in reference 14.

In the development of a pair of solutions to equation (1.4), consider the following representation for the deflection variables $y$ and $\alpha_\theta$. 
Let
\[ y(x,t) = h(t) + (x-e) \alpha_r(t) + \sum_{i=r}^n \varphi_{y_i} f_i(t) \]
and
\[ \alpha_e(x,t) = \alpha(x,t) - \alpha_r(t) = \sum_{i=r}^n \varphi_{\alpha_i} f_i(t) \quad (1.5) \]

where \( y_r(t) = h(t) + (x-e)\alpha_r \) represents the rigid body displacements with \( h(t) \) denoting the vertical translation of the point of application of the concentrated loads. As used here, \( \alpha_r \) is positive when the slope \( \frac{dy}{dx} \) evaluated at \( x = e \) is positive. This is exactly opposite to the normal sign convention used in rigid chord airfoil theory, and care must be exercised when comparing the equations of Section II with equations found in the literature. The quantities \( \varphi_{y_i} \) and \( \varphi_{\alpha_i} \) \( (i = 1, 2, \ldots) \) are the natural displacement vibration mode and the bending slope mode respectively for a free-free Timoshenko beam. Expressions for these mode shapes and their associated natural frequencies are developed in Appendix A. The dependent variables \( h(t), \alpha_r(t) \) and \( f_i(t) \) \( (i = 1, 2, 3, \ldots) \) become the normal or generalized coordinates of the problem. A solution for these coordinates permits a complete description of the motion of the system in terms of the modal coordinates \((1.5)\).

Substitution of the assumed displacement functions into the left hand side of the governing differential
equations gives

\[
m \left\{ \ddot{h}(t) + (x-e) \dddot{\alpha}_r(t) + \sum_{i=1}^{n} \left[ \frac{\partial^2 y_i}{\partial x^2} \frac{\partial f_i}{\partial t} \right] \right\} - GAK \left\{ \sum_{i=1}^{n} \left[ \frac{\partial^2 y_i}{\partial x^2} \frac{\partial f_i}{\partial t} \right] \right\}
= P(x,y,t) - \frac{\vec{F}_a}{y} (y,t) \delta(x-e)
\]

(1.6)

and

\[
-\mathcal{V} \left\{ \dddot{\alpha}_r(t) + \sum_{i=1}^{n} \frac{\partial^2 y_i}{\partial x^2} \frac{\partial f_i}{\partial t} \right\} + EI \left\{ \sum_{i=1}^{n} \frac{\partial^2 y_i}{\partial x^2} \frac{\partial f_i}{\partial t} \right\} + GAK \left\{ \sum_{i=1}^{n} \frac{\partial y_i}{\partial x} \frac{\partial f_i}{\partial t} \right\} = \frac{\partial M_a}{\partial y} (y,t) \delta(x-e)
\]

(1.7)

The fact that the modal coordinates, \( y_i \) and \( \phi_{\alpha_i} \), are solutions to the free vibration equations provides for certain relations that permit a simplification of these equations. The free vibration equations developed in Appendix A give directly

\[
GAK (\phi''_y - \phi''_{\alpha_i}) = \omega_i^2 m \phi_1
\]

(1.8)

\[
EI \phi''_{\alpha_i} + GAK (\phi'_y - \phi'_{\alpha_i}) = -\omega_i^2 \mathcal{V} \phi_{\alpha_i}
\]

Substituting equations (1.8) into the equation of forced motion (1.6) gives

\[
m \dddot{h}(t) + m(x-e) \dddot{\alpha}_r(t) + m \sum_{i=1}^{n} \left( \dddot{f}_i(t) + \omega_i^2 f_i(t) \phi_y(x) \right) = P(x,y,t) - \frac{\vec{F}_a}{y} (y,t) \delta(x-e)
\]

and

\[
-\mathcal{V} \dddot{\alpha}_r(t) + \sum_{i=1}^{n} \left( \dddot{f}_i(t) + \omega_i^2 f_i(t) \phi_y(x) \right) \phi_{\alpha_i}(x) = \frac{\partial M_a}{\partial y} (y,t) \delta(x-e)
\]

(1.9)
Galerkin's method may be applied to (1.9) to get the best possible approximate solution for a finite number $n$ of modal functions. This is accomplished by integrating (1.9) over the beam to establish a set of two rigid body equations and $n$ elastic equations of motion in terms of the unknown generalized coordinates. There will result $n + 2$ differential equations that may be solved simultaneously for the normal coordinates. Direct integration of the equations contained in (1.9) and integration of the same pair of equations after multiplication by the quantity $(x - e)$ gives

$$
M \dddot{h}(t) + S_e \dddot{\xi}_r(t) + \sum_{i=1}^{n} \left\{ (f_i^2(t) + \omega_i^2 f_i(t)) \int_{0}^{l} m \varphi_{y_i} \, dx \right\} = \int_{0}^{l} P(x, y, t) dx - F_x(y(e))
$$

(1.10)

and

$$
S_e \dddot{h}(t) + I_e \dddot{\xi}_r(t) + \sum_{i=1}^{n} \left\{ (f_i^2 + \omega_i^2 f_i(t)) \int_{0}^{l} (x - e) m \varphi_{y_i} \, dx \right\} = \int_{0}^{l} P(x, y, t)(x - e) dx
$$

(1.11)

where
Next, multiply the first of equation (1.9) by \( \varphi_{y_1} \) and the second by \( \varphi_{x_j} \), and integrate the equations separately over the length of the beam. Subtracting the second from the first of the equations obtained in this manner gives

\[
\begin{align*}
\left( \dddot{\varphi} - e \dddot{a}_r \right) \left( \int_0^l \left( m \varphi_{y_j} + \varphi_{x_j} \right) \, dx \right) + \left( \dddot{\varphi} \right) \left( \int_0^l \left( m \varphi_{y_j} + \varphi_{x_j} \right) \, dx \right) = \sum_{i=1}^n \left( f_i - \omega_i \right) \left( \int_0^l \left( m \varphi_{y_i} \varphi_{x_i} + \varphi_{y_i} \varphi_{x_i} \right) \, dx \right) \\
= \int_0^l \varphi_{y_j} P(x,y,t) \, dx - \int_0^l \varphi_{y_j} f_0(y,t) \delta(x-e) \, dx - \int_0^l \varphi_{x_j} M_0(y,t) \delta(x-e) \, dx
\end{align*}
\]

(1.13)

The orthogonality conditions on the modal coordinates are developed in Appendix A and are given in equation (A.11) as

\[
\int_0^l \left( m \varphi_{y_i} \varphi_{y_j} + \varphi_{x_i} \varphi_{x_j} \right) \, dy = 0 \quad i \neq j
\]

(A.11)
Also, additional requirements are placed on the free vibration mode shapes in that the summation of the inertia forces and inertia moments must be identically zero. These conditions are given as

\[
\int_0^l m \varphi_{y_j} \, dx = 0 \quad \text{and} \quad \int_0^l (m x \varphi_{y_j} + \nu \varphi_{a_j}) \, dx = 0 \quad (A.12)
\]

Forcing these requirements on the first of equation (1.10) along with the equation obtained by subtracting the second of (1.10) from the first of (1.11) gives the rigid body equations of motion for the system,

\[
\mathbf{M} \ddot{\mathbf{h}}(t) + S_e \dddot{\mathbf{r}}(t) = \int_0^l \mathbf{P}(x,y,t) \, dx - \mathbf{F}_a(y_e)
\]

and

\[
S_e \dddot{\mathbf{h}}(t) + (\mathbf{M} + I_e) \dddot{\mathbf{r}}(t) = \int_0^l \mathbf{P}(x,y,t) (x-e) \, dx - \mathbf{M}_a(y_e)
\]

The elastic equations of motion reduce directly from equation (1.13) through application of the requirements set forth in equations (A.11) and (A.12). Hence

\[
M_j \dddot{f}_3(t) + \omega_j^2 M_j f_3(t) = \int_0^l \varphi_{y_j} P(x,y,t) - \int_0^l \varphi_{y_j} F_a(y,t) \delta(x-e) \, dx
\]

\[
- \int_0^l M_a(y,t) \varphi_{a_j} \delta(x-e) \, dx
\]

(1.15)
for \( j = 1, 2, 3, \ldots, n \) and where \( M_j \) is the generalized mass given by

\[
M_j = \int_0^l \left( m \varphi_{y_j}^2 + \nu \varphi_{\alpha_j}^2 \right) dx
\]

It should be noted that in the preceding development no allowance has been made for the existence of structural damping. Reference 2 points out that the damping phenomenon has a strong controlling effect on stability problems, while its influence is relatively minor when considering forced motion. However, structures exhibiting large amounts of shearing deformation tend to show strong damping characteristics. Since, in part, this dissertation is concerned with the flutter instability of inflatable airmat structures a provision for energy removal will be included. The effect of structural damping can be included in an approximate fashion, as is done in the classical bending-torsion flutter analysis, by assuming that the energy removed per cycle of oscillation is proportional to the square of the amplitude, independent of frequency, and in phase with the velocity. This amounts to replacing the term \( \omega_j^2 M_j f_j(t) \) appearing in equation (1.15) by \( (1 + ig_j) \omega_j^2 M_j f_j(t) \), where \( g_j \) is the damping coefficient associated with the \( j^{th} \) mode of vibration. Hence, equation (1.15) written to include structural damping in this approximate manner becomes
\[ M_j \ddot{f}_j(t) + \omega_j^2 (1+i g_i) M_j f_j(t) = \tilde{z}_j(y,t) \quad (j = 1, 2, \cdots n) \]  

(1.16)

where the generalized force has been denoted as

\[ \tilde{z}_j(y,t) = \int_0^l \Phi_{y_j} P(x,y,t) \, dx - \int_0^l \Phi_{y_j} f_a(y,t) \delta(x-c) \, dx \]

\[ - \int_0^l \Phi_{x_3} M_a(y,t) \delta(x-c) \, dx \]  

(1.17)

Equations (1.14) and (1.16) are the differential equations for the unknown normal coordinates whose solutions provide the values necessary for the complete description of the motion of the system by substitution into equations (1.5).

Through the modal representation of the displacement functions the mechanical side of the equations have been uncoupled, and although the force side is again coupled the representation is not unreasonable. Except for the additional terms involved in the definition of the generalized mass and force expressions, equations (1.14) and (1.16) appear to be identical in form to the equations for forced motion when shearing and rotary inertia effects are neglected.
II. DEVELOPMENT OF THE AERODYNAMIC PRESSURE DISTRIBUTION FOR A BODY EXHIBITING CAMBER OSCILLATIONS

The distributed force term, $P(x,y,t)$, appearing in equations (1.14) and (1.17) is the unsteady aerodynamic pressure force acting on the oscillating body. The body is assumed to move with a constant forward speed, $U$, with the oscillations of the mean camber line being confined to a direction normal to the free stream. The resulting pressure force is dependent on the instantaneous deformation shape of the camber line as well as the time rates of change of these deformations. The particular problem considered in this dissertation deals with sustained oscillation of the camber line where the motion is a simple harmonic function of time. This motion may be classified as a camber flutter situation and as such has many of the mathematical and physical characteristics normally associated with the classical bending-torsion flutter problem. Because of this similarity, the development of the pressure relations for the flexible airfoil may commence with relations originally formulated for the classical chordwise rigid airfoil flutter situation.
An "exact" integro-differential equation of linearized lifting surface theory has been developed by Reissner\textsuperscript{20} for the case of an airfoil oscillating harmonically in an incompressible flow field. The general equation developed relates the unknown bound vorticity, associated with the surface, to the assumed known distribution of vertical velocity of the wing surface. The pressure difference acting across the surface at any point is then related to the circulation through the application of the Biot-Savart law in conjunction with the unsteady Bernoulli equation. The restrictions of linearized lifting surface theory may be briefly summarized as follows:

i) The airfoil may be replaced by a surface generated by connecting the mean camber line of each section and the resulting surface executes motion of small amplitude and disturbs the flow only slightly.

ii) The boundary condition of tangential flow at the airfoil surface may be satisfied on the plane \( y = 0 \).

iii) The wake may be represented by a sheet of distributed vorticity extending downstream to infinity in the plane \( y = 0 \).

Since a strip theory approach is to be taken in this analysis, a two-dimensional incompressible flow is assumed. Reference 2 presents a pressure relation for these circumstances that serves as a convenient starting point for the
formulation of the pressure forces. The relation given is in terms of the pressure amplitude and may be written as

\[
\frac{P(x)}{\rho U} = \frac{2}{\pi} \sqrt{\frac{1-x^2}{1+x^2}} \int_{-1}^{1} \frac{P'((1+\xi^2)^{1/2})}{(1-\xi^2)^{1/2}} \, d\xi - \frac{2}{\pi} i \int_{-1}^{1} \Lambda_1(x^*,\xi^*) \, d\xi^* \\
+ \frac{2}{\pi} \left[ 1 - C(k) \right] \sqrt{\frac{1-x^2}{1+x^2}} \int_{-1}^{1} \frac{1+\xi^2}{1-\xi^2} \, d\xi^*
\]

(2.1)

where

\[
\Lambda_1(x^*,\xi^*) = \frac{i}{\pi} \ln \left[ \frac{1-x^2+1-\xi^2 \sqrt{1-x^22}}{1-x^22 \sqrt{1-\xi^22}} \right]
\]

and \( \mathcal{W}(\xi^*) \) - amplitude of vertical fluid velocity

\( k \) - the reduced frequency \((\omega b) / U\)

\( C(k) \) - Theodorsen function

\( b \) - semi-chord of the surface.

The first two integrals appearing in equation (2.1) exist only in the sense of their Cauchy principal value because of the form of the singularity at the point \( x^* = \xi^* \). The symbol \( (\int \) is used to denote that only the principal value is to be considered.

The coordinate system employed in the above relations has its origin at the midchord of the airfoil section with the x-axis extending downstream, and the y-axis vertically up. All coordinate distances have been non-dimensionalized on the half-chord length. This
coordinate system is related to that of Section I as shown in Figure 4. The transformation is given by the relations

\[ \chi' = \chi - b \quad \text{and} \quad \chi^* = \left( \frac{\chi}{b} - 1 \right) \] (2.2)

The linearized form of the boundary condition requiring the flow to be tangent to the camber line surface may be written in the \( x' \) coordinate system as

\[ \mathcal{W}(x', t) = \frac{\partial y(x', t)}{\partial t} + U \frac{\partial y(x', t)}{\partial x'} \]

For the case of simple harmonic motion this reduces to

\[ \mathcal{W}(x') = i \omega \mathcal{Y}(x') + U \frac{\partial \mathcal{Y}(x')}{\partial x'} \] (2.3)

since

\[ \mathcal{W}(x', t) = \overline{\mathcal{W}(x')} e^{i \omega t} \quad \text{and} \quad y(x', t) = \overline{\mathcal{Y}(x')} e^{i \omega t} \]

The camber line deformation function describing the motion of the airfoil segment relative to its equilibrium position is given in equation (1.5) as

\[ y(x, t) = h(t) + (x - c) \alpha_r(t) + \sum_{j=1}^{n} \phi_j f_j(t) \]

where \( x \) is measured from the leading edge. Assuming simple harmonic motion of frequency \( \omega \), the displacement
amplitude becomes

\[ \bar{y}(x) = \bar{h} + (x-e) \bar{y} + \sum_{j=1}^{n} \bar{\varphi}_{y_j}(x) \bar{f}_j \]  \hspace{1cm} (2.4)

The bar over the quantities denotes the respective amplitude of oscillation.

Applying the coordinate transformations of (2.2) to obtain expressions suitable for use in the pressure amplitude equations gives

\[ \bar{y}(x^*) = \bar{h} + (b^*x + b - e) \bar{y} + \sum_{j=1}^{n} \bar{\varphi}_{y_j}(x^*) \bar{f}_j \]  \hspace{1cm} (2.5)

According to Appendix A, the natural modes of oscillation employed in the representation of (2.5) are

\[ \varphi_{y_j}(x^*) = \hat{a}_j \cosh[\mu_j b(x^*+i)] + \hat{b}_j \sinh[\mu_j b(x^*+i)] \]

\[ + \hat{c}_j \cos[\lambda_j b(x^*+i)] + \hat{d}_j \sin[\lambda_j b(x^*+i)] \]

This may be reduced through trigonometric relations to the form

\[ \varphi_{y_j}(x^*) = a_j \sin \bar{\lambda}_j x^* + b_j \cos \bar{\lambda}_j x^* + c_j \sinh \bar{\mu}_j x^* + d_j \cosh \bar{\mu}_j x^* \]  \hspace{1cm} (2.6)

where the constants are functions of the space and time dependent frequencies, and can readily be shown to take the form
\[ a_j = c \left\{ \gamma_j \lambda_j \sin \overline{\lambda}_j + \frac{T_j}{\mu_j} \cos \overline{\lambda}_j \right\} \]

\[ b_j = c \left\{ -\gamma_j \lambda_j \cos \overline{\lambda}_j + \frac{T_j}{\mu_j} \sin \overline{\lambda}_j \right\} \]

\[ c_j = c \left\{ \lambda_j \sinh \overline{\mu}_j + \cosh \overline{\mu}_j \right\} \]

\[ d_j = c \left\{ \lambda_j \cosh \overline{\mu}_j + \sinh \overline{\mu}_j \right\} \]

where

\[ \gamma_j = \begin{pmatrix} m \omega_j^2 + \frac{K_j}{b^2} \overline{\mu}_j^2 \\ m \omega_j^2 - \frac{K_j}{b^2} \overline{\lambda}_j^2 \end{pmatrix} , \quad \lambda_j = \frac{\cos 2\overline{\lambda}_j - \cosh 2\overline{\mu}_j}{\sinh 2\overline{\mu}_j + \frac{\mu_j}{\lambda_j} \sin 2\overline{\lambda}_j} \]

and

\[ \overline{\lambda}_j = b \lambda_j , \quad \overline{\mu}_j = b \mu_j , \quad K_i = GAK \]

The constant, C, is arbitrary and may be used to normalize the mode shapes as desired. The constants defined in equation (2.7) may be determined once the structure is defined and its natural frequencies evaluated.

The boundary condition on the downwash, written in terms of the non-dimensional variable, \( x^* \), becomes (see Chapter 5 of reference 2 for the development of this relation)

\[ \overline{W}(x^*) = i \omega \overline{y}(x^*) + \frac{U}{b} \frac{\partial \overline{y}(x^*)}{\partial x^*} \]  

(2.8)
Substituting equations (2.5) and (2.6) into the above expression for the downwash amplitude, gives

\[ \overline{W}(x^*) = i \omega \overline{h} + \left\{ i \omega \left( b x^* + b - c \right) + \sum_{j=1}^{\infty} \alpha_j \right\} \overline{\alpha}_r 
+ \omega \sum_{j=1}^{\infty} \left\{ A_j(k) \sin \lambda_j x^* + B_j(k) \cos \lambda_j x^* 
+ C_j(k) \sinh \overline{\mu}_j x^* + D_j(k) \cosh \overline{\mu}_j x^* \right\} \overline{f}_j \]  

(2.9)

where the coefficients in the series are functions of the reduced frequency, \( k \), and are given as

\[ A_j(k) = i a_j - \frac{1}{k} b_j \lambda_j \]
\[ B_j(k) = i b_j + \frac{1}{k} a_j \lambda_j \]  

(2.10)

\[ C_j(k) = i c_j + \frac{1}{k} d_j \overline{\mu}_j \]
\[ D_j(k) = i d_j + \frac{1}{k} c_j \overline{\mu}_j \]

To simplify the writing of the pressure amplitude expression in terms of equation (2.9), let the elastic downwash amplitude mode be denoted as

\[ \Phi_j(k, x^*) = A_j(k) \sin \lambda_j x^* + B_j(k) \cos \lambda_j x^* + C_j(k) \sinh \overline{\mu}_j x^* + D_j(k) \cosh \overline{\mu}_j x^* \]
so that equation (2.9) becomes

\[ \overrightarrow{\nabla}(x^*) = i \omega \overrightarrow{h} + \left\{ i \omega (bx^* + b - e) + \mathcal{U} \right\} \overrightarrow{a}_r + \omega \sum_{j=1}^{n} \Phi_j(k, x^*) \overrightarrow{f}_j \]  

(2.11)

The amplitude of the distributed pressure acting on the oscillating surface may then be expressed in terms of the normal coordinates by substituting equation (2.11) into equation (2.1) to obtain

\[ \frac{\overline{F}(x^*)}{\overline{F}} = \frac{2}{\overline{F}} \left[ \left\{ \frac{1 - \chi^*}{1 + \chi^*} \right\} \left\{ \frac{1 + \xi^*}{1 - \xi^*} \right\} \right] \left\{ i \omega \overrightarrow{h} + \left[ i \omega (b \xi^* + b - e) \mathcal{U} \right] \overrightarrow{a}_r \right. 

+ \omega \sum_{j=1}^{n} \Phi_j(k, x^*) \overrightarrow{f}_j \right\} \overrightarrow{a}_r \left\{ \frac{1 - \chi^*}{1 + \chi^*} \right\} \left\{ \frac{1 + \xi^*}{1 - \xi^*} \right\} \frac{1}{\chi^* - \xi^*} \right\} \overrightarrow{a}_r \]  

(2.12)

The permissibility of interchanging the order of integration and summation within the integrals contained in the above relations does not present any doubt as long as the index of summation remains finite. However, if \( n \) goes to infinity the validity of such an interchange may be questioned, since the integrals exist only in the sense that their Cauchy principal values exist and are finite. A sufficient set of restrictions permitting the interchange for the case where \( n \) goes to infinity is developed in Appendix D. As used here the index in the series of modal downwashes always remains finite, and the above discussion
is provided here only for completeness.

Interchanging the order of integration and summation in equation (2.12) and rearranging its terms leads to the following relation for the pressure amplitude.

\[
\frac{\overline{P}(x^*)}{\rho U} = \text{Re} \left\{ \frac{2i\omega}{\pi} \left[ L-C(k) \right] \left[ \frac{1-x^*}{l-x^*} \int_{-l}^{l} \frac{1+\xi^*}{1-\xi^*} \, d\xi^* \right] + \frac{2i\omega}{\pi} \left[ \frac{1-x^*}{l-x^*} \frac{l}{\xi^*-l/k} - ik\Lambda_i \right] \right\} + \sum_{j=1}^{n} \text{Re} \left\{ \frac{2i\omega}{\pi} \left[ L-C(k) \right] \left[ \frac{1-x^*}{1+x^*} \int_{-l}^{l} \frac{1+\xi^*}{1-\xi^*} \, d\xi^* \right] \phi_j(k,\xi^*) \right\} + \frac{2i\omega}{\pi} \left[ \frac{1-x^*}{1+x^*} \frac{\phi(k,\xi^*)}{\xi^*-l/k} - i k \Lambda_i \phi_j(k,\xi^*) \right] \right\}
\]

(2.13)

To proceed with the analysis, the integrals contained in equation (2.13) must first be evaluated. These integrals are evaluated in the following subsection.

**Integral relations for the pressure equation**

The integrals appearing in equation (2.13) are characterized by the integrals of equation (2.1) for the following three classes of downwash amplitude.

1) linear distribution of downwash amplitude

2) downwash amplitude distribution represented by the circular functions
iii) downwash amplitude distribution represented by the hyperbolic functions.

The integrals requiring evaluation may be symbolized in the following manner:

\[
\int_1^l \left[ \frac{1 + \xi^*}{\sqrt{1 - \xi^*}} \right] \left\{ \frac{l^*}{\sinh \lambda \xi^*} \right\} d \xi^* \tag{2.14}
\]

\[
\int_1^l \left[ \frac{1 + \xi^*}{\sqrt{1 - \xi^*}} \frac{L}{x^* - \xi^*} \right] \left\{ \frac{l^*}{\sinh \lambda \xi^*} \right\} d \xi^* \tag{2.15}
\]

\[
\int \frac{l}{2} \ln \left[ \frac{1 - x^* \xi^* + \sqrt{1 - x^*^2}}{l - x^*^2 - \sqrt{1 - x^*^2}} \right] \left\{ \frac{l^*}{\sinh \lambda \xi^*} \right\} d \xi^* \tag{2.16}
\]

where, in the evaluation of the integrals, each of the six terms listed within the brackets must be considered separately for any particular integral. In order to avoid unnecessary repetition of the work involved in the formulation of the integrals, the details will be presented in each instance only for the case where the \( \sin \lambda \xi^* \) is the term appearing in the integrand. The integral values for the remaining cases are obtained in an analogous manner and the results are presented in the summary table of definite integrals contained in Appendix E.
The general approach to be used in the evaluation of equations (2.14), (2.15), and (2.16) employs the following sequence of operations:

1. Expand the downwash function in its respective power series representation.
2. Interchange the order of integration and summation when permissible.
3. Transform coordinates by setting $j' = \cos \theta$
4. Integrate termwise by using Glauert's integral relation.
5. Collect terms in the expanded series.

The power series expansions for the circular and hyperbolic functions are given as

\[ \sin Z = \sum_{j=0}^{\infty} \frac{(-1)^j Z^{2j}}{(2j+1)!}, \quad \sinh Z = \sum_{j=0}^{\infty} \frac{Z^{2j+1}}{(2j+1)!} \]
\[ \cos Z = \sum_{j=0}^{\infty} \frac{(-1)^j Z^{2j}}{(2j)!}, \quad \cosh Z = \sum_{j=0}^{\infty} \frac{Z^{2j}}{(2j)!} \]

The series relations in equation (2.17) may easily be shown to be uniformly convergent. Also, since the expansions for the hyperbolic functions are equivalent to the circular function expansions, providing the absolute value of each term is regarded in the expansion, the series may be said to be absolutely uniformly convergent.
In order to proceed with step 2 in the sequence of operations to be used, consider for a moment the principal value of the integral

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \phi(x, \xi) \sum_{n=1}^{\infty} f_n(\xi) d\xi \quad (2.18)$$

where the function $\phi(x, \xi)$ possesses a singularity characteristic of the non-bracketed portion of the integrand contained in equations (2.15) and (2.16). A sufficient set of conditions permitting the interchange in order of integration and summation in integral forms characterized by equation (2.18) is developed in Appendix D. The formulation of these conditions is summarized in the following statement: If the function $\phi(x, \xi)$ has either of the forms

$$\phi(x, \xi) = \left\{ \begin{array}{ll}
\frac{l+\xi}{1-\xi} & \frac{l}{x-\xi} \\
\end{array} \right.$$  

or

$$\phi(x, \xi) = \frac{1}{2} /n \left[ \frac{l-x^2 + \sqrt{l-x^2}}{1-x^2} \sqrt{l-x^2} \right]$$

with a singularity at the point $x = \xi$, a sufficient set of conditions to permit the identity

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \phi(x, \xi) \sum_{n=1}^{\infty} f_n(\xi) d\xi = \sum_{n=1}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi(x, \xi) f_n(\xi) d\xi$$
is that the series

$$F(\xi) = \sum_{n=1}^{\infty} f_n(\xi) \quad \text{and} \quad \bar{F}(\xi) = \sum_{n=1}^{\infty} |f_n(\xi)|$$

be uniformly convergent and that the integral

$$\int_{-1}^{1} \Phi(x,\xi) f_n(\xi) \, d\xi$$

eexist.

It can be seen that upon substitution of the relations of (2.17) into equations (2.15) and (2.16), integrals analogous to equation (2.18) will result. And the argument given above for the interchanging of the order of integration and summation is directly applicable. Note that no questions arise in dealing with equation (2.14) since the integral exists in the normal sense of Riemann integration, and uniform convergence alone provides the sufficient condition for the interchange of the order of integration and summation.

Without making the detailed substitution of equation (2.17) into equations (2.14) through (2.16) it can be seen that the resulting integrals can be generated through a consideration of the following three general integral forms

$$\mathcal{O}_{1}(n) = \int_{-1}^{1} \frac{\xi^{n}}{\sqrt{1 - \xi^{2}} - \xi^{2}} \, d\xi$$

(2.19)
Consider first the integral (2.19) and make a transformation of coordinates by setting

\[ \xi^* = \cos \theta \quad \text{and} \quad \chi^* = \cos \phi \]  

(2.22)

to obtain

\[ \mathcal{J}_1(n) = \int_{0}^{\pi} \frac{(1 + \cos \theta) \cos^n \theta}{\cos \phi - \cos \theta} \, d\theta \]

Writing \( 1 + \cos \theta = (1 + \cos \phi) - (\cos \phi - \cos \theta) \) reduces the integrand to the form

\[ \mathcal{J}_1(n) = -(1 + \cos \phi) \int_{0}^{\pi} \frac{\cos^n \theta}{\cos \phi - \cos \theta} \, d\theta - \int_{0}^{\pi} \cos^n \theta \, d\theta \]

(2.23)

A convenient expansion for the \( \cos^n \theta \), is developed in Appendix B, and is given as

\[ \cos^n \theta = \sum_{k=0}^{n-1} \left( \begin{array}{c} n-1 \\ k \end{array} \right) \frac{n!}{(n-k)!(k)!} \cos(n-2k)\theta \quad \text{for } n \text{ odd} \]

(2.24)
and

\[ \cos^2 \theta = \frac{1}{2^{n-1}} \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} \cos(n-2k)\theta = \frac{1}{2^n} \left( n! \right) \quad n \text{ even} \] (2.25)

Substituting the series (2.24) into (2.23), interchanging the order of integration and summation, in accordance with the requirements of Appendix D, gives for the case \( n \) is odd

\[ \mathcal{J}(n) = - \left[ \frac{1+\cos \Phi}{2^{n-1}} \right] \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} \left\{ \int_{0}^{\pi} \frac{\cos(n-2k)\theta}{\cos \theta - \cos \Phi} \, d\theta \right\} \]

Using the integral relation developed by Glauert,\(^{17}\) i.e.

\[ \int_{0}^{\pi} \frac{\cos n\theta}{\cos \theta - \cos \Phi} \, d\theta = \pi \frac{\sin n\Phi}{\sin \Phi} \] (2.26)

gives

\[ \mathcal{J}(n) = - \frac{\pi (1+\cos \Phi)}{2^{n-1}} \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} \left\{ \int_{0}^{\pi} \sin(n-2k)\Theta \, d\Theta \right\} \quad n \text{ odd} \] (2.27)
For the case where \( n \) is even, the same procedure produces

\[
q_1(n) = -\frac{\pi(n+\cos\varphi)}{2^{n-1}} \sum_{k=0}^{k=\frac{n-2}{2}} \frac{n!}{(n-k)!(k)!} \frac{\sin(n-2k)\varphi}{\sin\varphi} - \frac{\pi}{2^n} \frac{n!}{(\frac{n}{2})!(\frac{n}{2})!}
\]

\( n \text{- even} \) \hspace{1cm} (2.28)

The integral \( q_2(n) \) given in (2.20) may be evaluated by first expanding to the form

\[
q_2(n) = \int_{-1}^{1} \frac{\xi^n}{\sqrt{1-\xi^2}} d\xi + \int_{-1}^{1} \frac{\xi^{n+1}}{\sqrt{1-\xi^2}} d\xi \hspace{1cm} (2.29)
\]

Substitution of the transformed variable through \( \xi = \cos\theta \) gives the general integral to be evaluated as

\[
\int_{-1}^{1} \frac{\xi^n}{\sqrt{1-\xi^2}} d\xi = \int_{0}^{\pi} \cos^n\theta d\theta
\]

where \( n \) takes on even and odd integer values. Using the expansion given in equation (2.24) for \( n \text{- odd} \) produces

\[
\int_{0}^{\pi} \cos^n\theta d\theta = \frac{1}{2^{n-1}} \sum_{k=0}^{k=\frac{n-1}{2}} \frac{n!}{(n-k)!(k)!} \int_{0}^{\pi} \cos(n-2k)\theta d\theta = 0
\]
and for the case $n$ is even, substitution of equation (2.25) provides

$$
\int_0^{\pi} \cos^2 \theta \, d\theta = \int_0^{\frac{\pi}{2}} \left\{ \sum_{k=0}^{n-1} \frac{n!}{(n-k)! \, k!} \cos(n-2k)\theta + \frac{n!}{(n-2)! \, \left(\frac{n}{2}\right)!} \right\} \, d\theta
$$

$$
= \frac{1}{2^n} \frac{n!}{\left(\frac{n}{2}\right)! \, \left(\frac{n}{2}\right)!} \pi
$$

Using these results in equation (2.29) the integral $\mathcal{J}_2(n)$ may be written as

$$
\mathcal{J}_2(n) = \frac{1}{2^{n+1}} \frac{(n+1)!}{\left(\frac{n+1}{2}\right)! \, \left(\frac{n+1}{2}\right)!} \pi \quad n \text{ odd}
$$

(2.30)

$$
\mathcal{J}_2(n) = \frac{1}{2^n} \frac{n!}{\left(\frac{n}{2}\right)! \, \left(\frac{n}{2}\right)!} \pi \quad n \text{ even}
$$

The integral relation (2.21) requires slightly more manipulation in order to determine its value. Using the coordinate transformation (2.22) in (2.21) gives

$$
\mathcal{J}_3(n) = \int_0^{\pi} \frac{\cos^2 \theta}{2} \ln \left[ \frac{1 - \cos \theta \cos \Phi + \sin \Phi \sin \theta}{1 - \cos \theta \cos \Phi - \sin \Phi \sin \theta} \right] \sin \theta \, d\theta
$$
Employing the trigonometric identities

\[
\cos \phi \cos \theta \mp \sin \phi \sin \theta = \cos (\phi \mp \theta)
\]

and

\[
(1 - \cos \alpha) = 2 \left( \sin \frac{1}{2} \alpha \right)^2
\]

this relation is reduced to the more convenient form

\[
\int_0^\pi \frac{\cos \theta \sin \theta}{2} \left\{ \ln \left[ \sin \frac{1}{2} (\phi + \theta) \right]^2 - \ln \left[ \sin \frac{1}{2} (\phi - \theta) \right]^2 \right\} \, d\theta
\]

Integrating by parts, with

\[
u = \ln \left[ \sin \frac{1}{2} (\phi \pm \theta) \right]^2, \quad dv = \frac{\sin \theta \cos n \theta}{2} \, d\theta
\]

gives

\[
\int_0^\pi \frac{\sin \theta \cos n \theta}{2} \left. \cos \frac{1}{2} (\phi + \theta) \right|_0^\pi \ln \left[ \sin \frac{1}{2} (\phi + \theta) \right]^2
\]

\[-\frac{1}{2} \left. \ln \left[ \sin \frac{1}{2} (\phi - \theta) \right] \right|_0^\pi \frac{\sin \theta \cos n \theta}{2}
\]

\[-\int_0^\pi \frac{\left( \cos \frac{1}{2} (\phi + \theta) \right) \cos \frac{1}{2} (\phi + \theta) \, d\theta}{\sin \frac{1}{2} (\phi + \theta)}
\]

\[-\int_0^\pi \frac{\left( \cos \frac{1}{2} (\phi - \theta) \right) \cos \frac{1}{2} (\phi - \theta) \, d\theta}{\sin \frac{1}{2} (\phi - \theta)}
\]
Since
\[
\int 
\sin \theta \cos^{n+1} \theta \, d\theta = -\frac{1}{n+1} \int d(\cos \theta)^{n+1}
\]
\[
= -\frac{1}{n+1} (\cos \theta)^{n+1}
\]

the integral may be written as

\[
\mathcal{J}_3(n) = \left\{ -\frac{1}{n+1} \left(\frac{\cos \theta}{2}\right)^{n+1} \left[ \frac{\sin \frac{\phi}{2}(\phi+\theta)}{\sin \frac{\phi}{2}(\phi-\theta)} \right] \right\}^\pi_0
\]
\[
+ \frac{1}{n+1} \int_0^\pi \left(\frac{\cos \theta}{2}\right)^{n+1} \left[ \frac{\cos \frac{\phi}{2}(\phi+\theta)}{\sin \frac{\phi}{2}(\phi+\theta)} + \frac{\cos \frac{\phi}{2}(\phi-\theta)}{\sin \frac{\phi}{2}(\phi-\theta)} \right] d\theta
\]

The first term of this equation goes to zero at both the limits, \( \Theta = 0, \pi \), leaving only the integral term. The bracketed term of this equation may be simplified through the identity

\[
\frac{\cos \frac{\phi}{2}(\phi+\theta)}{\sin \frac{\phi}{2}(\phi+\theta)} + \frac{\cos \frac{\phi}{2}(\phi-\theta)}{\sin \frac{\phi}{2}(\phi-\theta)} = \frac{-2 \sin \phi}{\cos \phi - \cos \theta}
\]

Therefore

\[
\mathcal{J}_3(n) = -\frac{\sin \phi}{n+1} \int_0^\pi \left(\frac{\cos \theta}{2}\right)^{n+1} d\theta
\]

Again turning to the expansion of (2.24) for the case \( n \) is odd, the above integral may be expressed as

\[
\mathcal{J}_3(n) = -\frac{\sin \phi}{n+1} \left\{ -\frac{1}{2^n} \left[ \sum_{k=0}^{\frac{n+1}{2}} \frac{(n+1)!}{(n+1-k)!k!} \frac{\cos(n+2k-\theta)\theta}{\cos \phi - \cos \theta} \right] d\theta - \frac{1}{2^n} \left[ \frac{(n+1)!}{(n+1-k)!k!} \frac{\cos \phi - \cos \theta}{\cos \phi - \cos \theta} \right] \right\}
\]
In this integral, the order of integration may be reversed since the series can be shown to fulfill the requirements presented in the argument following equation (2.18) and developed in Appendix D. Accomplishing the interchange and applying Glauert's integral relation (2.26) the results for the case \( n \) is an odd integer are established directly

\[
\mathcal{I}_3(n) = \sum_{k=0}^{n} \frac{n!}{(n+1-k)!k!} \sin(n+1-2k) \frac{\pi}{2} \quad n-\text{odd} \quad (2.31)
\]

The integral \( \mathcal{I}_3(n) \), for the case \( n \) is even is obtained in an analogous manner and can be shown to reduce to

\[
\mathcal{I}_3(n) = \sum_{k=0}^{n} \frac{n!}{(n+1-k)!k!} \sin(n+1-2k) \frac{\pi}{2} \quad n-\text{even} \quad (2.32)
\]

In all of the results presented above, the independent variable is the transformed coordinate, \( \varphi \), which is related to the rectangular coordinate, \( x^* \), through

\[ \varphi = \cos^{-1} x^* \]

In most of the situations to follow, it is convenient to work in the transformed coordinate \( \varphi \), rather than reverting back to the linear variable, \( x^* \). For this reason the resulting values for the integrals \( \mathcal{I}_1, \mathcal{I}_2, \) and \( \mathcal{I}_3 \) contained in equations (2.27), (2.28), (2.29), (2.31) and (2.32) are left in their present form.
The integrals $\mathcal{J}_1(n)$, $\mathcal{J}_2(n)$ and $\mathcal{J}_3(n)$ permit direct evaluation of the first two subcases contained in equations (2.14), (2.15) and (2.16). Through setting $n = 0$ and $n = 1$ in the appropriate equations the following relations are easily generated.

From (2.30),

\[
\int_{-1}^{1} \frac{1 + \xi^2}{1 - \xi^2} \, d\xi^* = \pi, \quad \int_{-1}^{1} \frac{\xi^* d\xi^*}{1 - \xi^2} = \frac{\pi}{2} \tag{2.33}
\]

from (2.27) and (2.28),

\[
\int_{-1}^{1} \frac{1 + \xi^2}{1 - \xi^2} \frac{d\xi^*}{x^2 - \xi^2} = -\pi, \quad \int_{-1}^{1} \frac{\xi^* d\xi^*}{x^2 - \xi^2} = -\pi (1 + \cos \phi) \tag{2.34}
\]

and from (2.31) and (2.32)

\[
\int_{-1}^{1} (\lambda^2 \xi^2) d\xi^* = \pi \sin \phi, \quad \int_{-1}^{1} \xi^2 \lambda (\lambda^2 \xi^2) d\xi^* = \frac{\pi}{2} \sin 2 \phi \tag{2.35}
\]

The remaining four subcases of the integrals appearing in (2.14), (2.15) and (2.16) are now evaluated. Since the series expansions for the circular and hyperbolic
functions all have a similar format, details are presented only for the class of integrals containing the sine term in the integrand. The alternate forms may be evaluated in a similar manner. The complete set of these definite integrals is given in Appendix E, equations (E.10) through (E.21).

Consider first the integral (2.15) with the sine term in the integrand, that is

\[
R_i(\lambda, \chi^*) = \int \int \frac{\sin \lambda \xi^*}{\chi^* - \xi^*} d\xi^*
\]

Expand the \(\sin \lambda \xi^*\) in its power series representation (2.17) and interchange the order of integration and summation in agreement with Appendix D to get

\[
R_i(\lambda, \chi^*) = \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^{2n+1}}{(2n+1)!} \int \frac{1}{\chi^* - \xi^*} d\xi^*
\]

By employing the coordinate transformation of (2.22) and the results of (2.27) to evaluate the integral \([(2n-1) is an odd integer], the above series becomes

\[
R_i(\lambda, \chi^*) = -\sum_{n=0}^{\infty} \frac{(-1)^n \lambda^{2n+1}}{(2n+1)!} \frac{(1 + \cos \phi)}{2^{2n}} \sum_{k=0}^{\frac{n}{2}} \frac{(2n+1)!}{(2n+1-2k)! k!} \frac{\sin(2n+1-2k)\phi}{\sin \phi}
\]
A convenient derangement of the above double series is developed in Appendix C, with the results contained in equation (C.2). The form desired is given as

\[
\sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{(-1)^j j!}{2^{2j}(2j+1-id)} \sin(2j+1-2k) \varphi
\]

\[
\sum_{K=0}^{\infty} 2(-1)^K J_{2k+1}(\varphi) \sin(2K+1) \varphi
\]

(2.37)

where \( J_{2k+1}(\lambda) \) is the Bessel coefficient of the first kind with index \((2k+1)\). It is important for future purposes to note that the infinite sum on the right side of (2.36) converges uniformly for all values of the argument \( \lambda \). The truth of this statement is shown in Appendix C.

Substitution of (2.37) into the integral (2.36) gives the final result

\[
\mathcal{R}_i(\lambda, \varphi) = -2\pi \frac{1+\cos \varphi}{\sin \varphi} \sum_{K=0}^{\infty} (-1)^K J_{2K+1} \sin(2K+1) \varphi
\]

(2.38)

Equations (E.14) through (E.17) contain the results for the remaining subcases of the integral (2.15).
The integral
\[ R_2(\lambda) = \int_{-1}^{1} \sqrt{\frac{1+\xi^*}{1-\xi^*}} \sin \lambda \xi^* d\xi^* \]
is evaluated in a manner similar to that employed to obtain (2.38). Expanding the circular function in its power series representation, interchanging the order of integration, and applying the results of equation (2.30) leads directly to the equation
\[ R_2(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{\lambda^{2n+1}}{2^{2n+2} (2n+2)! (2n+1)!} \]

The resulting series may be recognized as the series expansion for the Bessel function as given by Watson for the case \( n = 1 \), since
\[ J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{z}{2})^{2k+n}}{k! (n+k)!} \]

Hence the final result for the integral \( R_2(\lambda) \) reduces directly to a constant multiplied by the Bessel coefficient of index one, \( J_1(\lambda) \).
\[ R_2(\lambda) = \pi \int J_1(\lambda) \] (2.39)
Similar results are obtained from equation (2.15) for the cosine and hyperbolic function cases. However, for the cases where the hyperbolic functions appear in the integrand, the integrated series representation is comparable to the modified Bessel functions of the first kind, \( I_n(\tau) \), of purely imaginary arguments. The evaluated results for these cases are given in Appendix E.

The third of the integral forms (2.16) may be evaluated in a manner identical to that used in the preceding two integrals \( R_1(\lambda, \varphi) \) and \( R_2(\lambda) \), through the use of the relations (2.31) and (2.32). Proceeding as before,

\[
R_3(\lambda, \varphi) = \int_{\lambda}^{\infty} \left[ \lambda (x^*)^5 \right] \frac{\sin \lambda x^*}{x^*} dx^*
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^{2n+1} \pi}{2^{2n+1} (2n+1)!} \sum_{k=0}^{\infty} \frac{(2n+1)!}{(2n+2-k)! k!} \sin (2n+2-k) \varphi
\]

where setting \( m = n + 1 \), gives

\[
R_3(\lambda, \varphi) = \sum_{m=1}^{\infty} \pi (-1)^{m-1} \lambda^{2m-1} \sum_{k=0}^{\infty} \frac{\sin (2m-2k) \varphi}{2^{2m-1-k} k! (2m-k)!}
\]

Equation (C.4), developed in the appendixes, presents the following derangement

\[
\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^j \lambda^{2j} \sin (2j-2k) \varphi}{2^{2j} (2j-k)! k!} = \sum_{k=0}^{\infty} 2(-1)^{R+k} J_R(\lambda) \sin (2k+2) \varphi
\]

\[
= \sum_{k=0}^{\infty} 2(-1)^{R+k} \frac{J_R(\lambda)}{2k+2} \sin (2k+2) \varphi
\]
Employing this derangement scheme in the preceding series gives

\[ P_3(\lambda, \varphi) = -\frac{2}{\lambda} \sum_{k=0}^{\infty} 2(-1)^{k+1} \tilde{J}^{(k)}(\lambda) \sin(2k+2)\varphi \] (2.40)

Note that the series appearing in equation (2.40) is very similar to that contained in equation (2.38) for \( R_1(\lambda, \varphi) \). These two relations were obtained for the case where the sine term was the term considered in both (2.14) and (2.15). As can be seen from the list of definite integrals contained in Appendix E, the series under question becomes identical if the sine appears in the integrand of (2.14) and the cosine in (2.16) or conversely.

**Pressure amplitude distribution**

Having developed the necessary integral relations for evaluation of the general set of integrals specified in equations (2.14), (2.15) and (2.16), it is now possible to evaluate the pressure amplitude expression (2.13). Therefore, employing the relations listed in the table of definite integrals contained in Appendix E, the pressure amplitude expression in terms of the transformed variable \( \varnothing \) is given as
where $J_n(\lambda)$ is the Bessel function of the first kind of index $n$ with real arguments, and $I_n(\mu)$ is the modified Bessel function of the first kind of purely imaginary argument. As should be expected, the modified Bessel functions occur in all terms directly associated with the
hyperbolic camber terms in a directly analogous manner as do the normal Bessel functions belong to the circular terms in the modal representation. However, it is impossible to make a general statement as to any specific set of Bessel coefficients belonging to the individual circular or hyperbolic function.

Equation (2.41) gives the pressure amplitude distribution acting on a two-dimensional harmonically oscillating airfoil segment exhibiting arbitrary camber deformations. All Bessel coefficients appearing in (2.41) may be evaluated once the structural dynamic characteristics of the airfoil segment have been defined. The pressure distribution is then determined through expressing the deformation shape in terms of the normal coordinates and specifying the frequency of oscillation.

Although developed for oscillating bodies, the equation is perfectly valid for steady state calculation. For arbitrary steady state deformations the pressure distribution may be determined from (2.41) by setting the frequency, $\omega$, and the reduced frequency, $k$, equal to zero. The Theodorsen function, $C(k)$, is equal to one for the zero frequency condition. In reducing equation (2.41) to the steady form, care must be taken not to ignore the terms containing $\frac{\omega}{k}$, for $\frac{\omega}{k} = \frac{V}{b}$ and the terms do not drop out. The steady form of equation (2.41) may be written as
\[
\frac{P(\Phi)}{P} = \alpha_\tau \left\{ -2U \sin \Phi \right\} - \frac{4U}{b} \sum_{j=1}^{\infty} \bar{f}_j \left\{ -b_j \bar{\lambda}_j \sum_{n=0}^{\infty} (-1)^n \bar{J}_{2n+1}(\bar{M}_n) \sin(2n+1)\Phi \right\}
\]

\[
+ q_j \bar{\lambda}_j \sum_{n=0}^{\infty} (-1)^{n+1} \bar{J}_{2n+2}(\bar{M}_n) \sin(2n+2)\Phi + d_j \bar{\mu}_j \sum_{n=0}^{\infty} \bar{I}_{2n+1}(\bar{\mu}_n) \sin(2n+1)\Phi
\]

\[+ \frac{1}{2} \left( a_j \bar{\lambda}_j \bar{J}_{\bar{\lambda}_j} + c_j \bar{\mu}_j \bar{I}_{\bar{\mu}_j} \right) \sin \Phi \frac{\sin \Phi}{1 + \cos \Phi} \right\}
\]

(2.42)

where the coefficients, \( a_j, b_j, c_j \) and \( d_j \) are defined by equation (2.7) and are functions of the basic structural parameters.

The pressure relation (2.41) is general enough that it can be employed in any static aeroelastic or flutter problem where a strip theory analysis is appropriate and a normal mode representation of the deformation shape is used. Although bulky in appearance, most of the terms can be evaluated directly once the mode shapes and natural frequencies of the particular structure under consideration have been determined.
III. DEVELOPMENT OF THE GENERALIZED AERODYNAMIC FORCES REQUIRED FOR A NORMAL MODE FLUTTER ANALYSIS

Classically speaking, generalized forces are defined as the work done by the external forces acting on a system in going through a unit virtual displacement of each of the generalized coordinates, one at a time, when all other coordinates are held fixed. Each coordinate therefore has associated with it a generalized force. In the problem under consideration, the generalized coordinates are the normal coordinates that specify the amount of any specific mode shape to be added in constructing the true deformation shape. The external forces are the aero­dynamic pressure forces resulting from the body displace­ments.

The general deformation function is given in equation (1.5) as

\[ y(x,t) = h(t) + (x - e) \alpha_r(t) + \sum_{i=1}^{n} \phi_i \beta_i(t) \]

where the generalized coordinates are the rigid displace­ments \( h(t) \) and \( \alpha_r(t) \), and the normal coordinates \( f_i(t) \) associated with the elastic deformations. Using the
coordinate as a subscript the generalized forces for the rigid body equations may be written as

$$\overline{\omega}_h = \int_0^l P(x,y) \, dx \quad \text{and} \quad \overline{\omega}_x = \int_0^l (x-e)P(x,y) \, dx$$

The forces associated with the elastic deformations may be written in a similar manner as

$$\overline{\omega}_p = \int_0^l \Phi_{yp} P(x,y) \, dx$$

with \( p = 1, 2, 3, \ldots n. \)

Transforming to the coordinate system employed in Section II, these relations become

$$\overline{\omega}_h = b \int_0^\pi P(y,\Phi) \sin \Phi \, d\Phi \quad (3.1)$$

$$\overline{\omega}_x = b \int_0^\pi P(y,\Phi)(\cos \Phi + 1 - \frac{e}{b}) \sin \Phi \, d\Phi \quad (3.2)$$

and

$$\overline{\omega}_p = b \int_0^\pi P(y,\Phi) \phi_{yp}(\Phi) \sin \Phi \, d\Phi \quad (3.3)$$

The generalized forces as defined above are in complete agreement with the force terms contained in the governing differential equations of forced motion, equations (1.14) and (1.17). Substituting equation (2.41) into equation (3.1) and integrating over the chord of the airfoil gives
the generalized force associated with rigid body translation of the body. This force is simply the total lift force per unit span acting on the segment resulting from oscillations of the body in the free stream. The integrals involved present no difficulty and may be evaluated immediately using Glauert's integral relation (2.26). Performing the necessary integrations gives

\[
\frac{\bar{z}}{\rho U b \pi} = \bar{h} \left\{ -2 i \omega \left( c(k) + \frac{i k}{2} \right) \right\} + \bar{a}_r \left\{ 2 i \omega b \left[ c(k) \left( -\frac{3}{2} + \frac{c}{b} + \frac{k}{K} \right) - \frac{i k}{2} \left( -\frac{c}{b} - \frac{i}{K} \right) \right] \right\}
\]

\[
\sum_{j=1}^{n} \tilde{f}_j \left\{ \omega c(k) \left[ A_j(k) \bar{J}_j(\tilde{\lambda}_j) + B_j(k) \bar{J}_o(\tilde{\lambda}_j) + C_j(k) \bar{I}_j(\tilde{\mu}_j) + D_j(k) \bar{I}_o(\tilde{\mu}_j) \right] \right\} + 2 i \omega k \left[ \frac{B_j(k)}{\lambda_j} \bar{J}_j(\tilde{\lambda}_j) + \frac{D_j(k)}{\tilde{\mu}_j} \bar{I}_j(\tilde{\mu}_j) \right]
\]  

Substitution of the pressure (2.41) into equation (3.2) and carrying out the indicated integration gives the aerodynamic moment per unit span acting about the point \( x = e \) due to the motion of the airfoil. The resulting generalized force is that associated with the rigid body pitching motion and can be written as
The generalized forces given in equations (3.4) and (3.5) reduce to the classical expressions associated with a rigid airfoil exhibiting translation and pitch oscillation when only the first two terms in each is considered. That is, the elastic coupling terms containing the $f_j$'s are absent in the rigid configuration.

The generalized forces associated with the elastic deformations may be obtained through substituting equations (2.6) and (2.41) into equation (3.3) and performing the necessary integration. In order to facilitate evaluation of required integrations the pressure equation (2.41) is organized in the following manner.
Let

\[ P(y, \varphi) = P_h + P_\alpha + \sum_{j=1}^{m} P_j^e(y, \varphi) \]  \hspace{1cm} (3.6)

where

- \( P_h(y, \varphi) \) is the pressure acting on the body due to simple harmonic translatory oscillation;
- \( P_\alpha(y, \varphi) \) is the pressure acting on the body due to simple harmonic pitching oscillation about the point \( x = e \);
- \( P_j^e(y, \varphi) \) is the pressure acting on the airfoil due to simple harmonic oscillation in the \( j^{th} \) elastic, free-free vibration mode.

Therefore, the generalized force associated with the \( p^{th} \) elastic degree of freedom may be written as

\[
\ddot{\omega}_p = b \int_0^\pi \varphi_P (\varphi) P_h(y, \varphi) \sin \varphi \, d\varphi + b \int_0^\pi \varphi_P (\varphi) P_\alpha(y, \varphi) \sin \varphi \, d\varphi \\
+ b \sum_{j=1}^{m} \int_0^\pi \varphi_P (\varphi) P_j^e(y, \varphi) \sin \varphi \, d\varphi
\]

or simply

\[
\ddot{\omega}_p = \ddot{\omega}_h + \ddot{\omega}_\alpha + \sum_{j=1}^{m} \ddot{\omega}_j
\]  \hspace{1cm} (3.7)

The integrals occurring in the rigid body coupling terms, \( \ddot{\omega}_h \) and \( \ddot{\omega}_\alpha \), fall into three particular integral forms that may be handled more easily when transformed.
back into the star coordinate system. Upon examining

equations (2.41) and (3.3) the integral forms requiring

some detailed consideration are

\[\int_0^\pi \frac{\phi_p(\phi)}{\sqrt{1 + \cos\phi}} \sin\phi \, d\phi = \int_{-1}^1 (1-x^*)^{1/2} \phi_p(x^*) \, dx^*\]

\[\int_0^\pi \phi_p(\phi) \sin\phi \, d\phi = \int_{-1}^1 x^* (1-x^2)^{1/2} \phi_p(x^*) \, dx^*\]

and

\[\int_0^\pi \phi_p(\phi) \sin 2\phi \sin\phi \, d\phi = 2 \int_{-1}^1 x^* (1-x^2)^{1/2} \phi_p(x^*) \, dx^*\]

where \(\phi_p(x^*)\) contains the hyperbolic and circular functions. These relations may readily be evaluated for the integrands involved through repeated application of an integral relation for the Bessel function. Watson\(^{25}\) gives the following convenient integral forms for the Bessel coefficients

\[J_{\nu}(\beta) = \frac{(\frac{1}{2} \beta)^{\nu}}{\Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} \int_{-1}^1 (1-t^2)^{\nu - \frac{1}{2}} e^{i\beta t} \, dt\]

\[I_{\nu}(\alpha) = \frac{(\frac{1}{2} \alpha)^{\nu}}{\Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} \int_{-1}^1 (1-t^2)^{\nu - \frac{1}{2}} e^{\alpha t} \, dt\]
valid for the real part of $(\nu + \frac{1}{2}) > 0$. In these relations $\Gamma(\nu + \frac{1}{2})$ refers to the usual gamma function.

When the circular and hyperbolic functions are expressed in their respective exponential forms, $\varphi_{p}(x^{*})$ contains terms of the form $e^{i\beta x^{*}}$ and $e^{\alpha x^{*}}$. Therefore, the particular forms of interest are where $\nu = 0$ and $\nu = 1$.

Since $\Gamma(\frac{1}{2}) = (\pi)^{\frac{1}{2}}$ and $\Gamma(\frac{3}{2}) = \frac{1}{2}(\pi)^{\frac{1}{2}}$, equations (3.9) and (3.10) simplify to

$$
\int_{-1}^{1} (1-x^{*2})^{-\frac{1}{2}} e^{i\beta x^{*}} dx^{*} = \frac{\pi}{\beta} J_{\nu}(\beta)
$$

(3.11)

$$
\int_{-1}^{1} (1-x^{*2})^{-\frac{1}{2}} e^{\alpha x^{*}} dx^{*} = \frac{\pi}{\alpha} I_{\nu}(\beta)
$$

and

$$
\int_{-1}^{1} (1-x^{*2})^{\frac{1}{2}} e^{i\beta x^{*}} dx^{*} = \frac{\pi}{\beta} J_{\nu}(\beta)
$$

(3.12)

$$
\int_{-1}^{1} (1-x^{*2})^{\frac{1}{2}} e^{\alpha x^{*}} dx^{*} = \frac{\pi}{\alpha} I_{\nu}(\beta)
$$

Applications of the Bessel function recurrence formula

$$
J_{\nu}'(\beta) = J_{\nu-\frac{1}{2}}(\beta) - \frac{\nu}{\beta} J_{\nu}(\beta)
$$

and

$$
I_{\nu}'(\alpha) = I_{\nu-\frac{1}{2}}(\alpha) - \frac{\nu}{\alpha} I_{\nu}(\alpha)
$$
permits the generation of additional relations needed in order to complete the required integrations. The resulting expressions may be written as

\[ \int_{L} x^*(1-x^*)^{-\frac{1}{2}} e^{i\beta x^*} \, dx^* = i \pi J_i(\beta) \]  
(3.13)

\[ \int_{L} x^*(1-x^*)^{-\frac{1}{2}} e^{i\alpha x^*} \, dx^* = \pi I_i(\alpha) \]  
and

\[ \int_{L} x^*(1-x^*)^{-\frac{1}{2}} e^{i\beta x^*} \, dx^* = \frac{\pi}{i\beta} \left[ J_o(\beta) - \frac{2}{\beta} J_i(\beta) \right] \]  
(3.14)

Having these relations, the integral forms involved in equation (3.8) may be evaluated on substitution of the mode shape as given in (2.6). As an illustrative case as to the procedure followed, consider only the sine term in the modal representation. The first of the integrals of (3.8) become

\[ \int_{L} (1-x^*)^{1/2} \sin \lambda_p x^* \, dx^* = \int_{L} (1-x^*)^{1/2} \left( \frac{e^{i\lambda_p x^*}}{2i} - \frac{e^{-i\lambda_p x^*}}{2i} \right) \, dx^* \]  
(3.15)

\[ = \frac{\pi}{2i} \left\{ J_o(-\lambda_p) - i J_i(\lambda_p) - J_o(-\lambda_p) + i J_i(-\lambda_p) \right\} \]

\[ = - \pi J_i(\lambda_p) \]
since \( J_0(a) = J_0(-a) \) and \( J_1(-a) = -J_1(a) \) for \( a \geq 0 \). Following the same approach, and using the results of equations (3.11) through (3.14) gives

\[
\int_{-1}^{1} \sqrt{1-x^2} \sin \lambda \rho x' d'x^* = \frac{\pi}{2l \lambda \rho} \left\{ J_i(\lambda \rho) + J_i(-\lambda \rho) \right\} = 0 \quad (3.16)
\]

and

\[
\int_{-1}^{1} x^* \sqrt{1-x^2} \sin \lambda \rho x' d'x^* = \frac{\pi}{l \lambda \rho} \left\{ J_o(\lambda \rho) - \frac{2}{\lambda \rho} J_i(\lambda \rho) + \frac{2}{\lambda \rho} J_i(-\lambda \rho) \right\} \quad (3.17)
\]

\[
= -\pi \left\{ J_o(\lambda \rho) - \frac{2}{\lambda \rho} J_i(\lambda \rho) \right\}
\]

Similar integrals result when the cosine and hyperbolic terms in the modal expression are substituted in (3.8). The complete class of definite integrals resulting from the above analysis are presented in Appendix E, equations (E.22) through (E.33).

The rigid body coupling terms appearing in the generalized force associated with the elastic degrees of freedom may now be established. Substituting equations (2.6) and (2.41) into (3.7) and employing the procedure developed above, the following relations may readily be determined.
\[
\frac{\dot{\eta}}{b_{\eta}^2 \pi} = \hat{h} \left\{ 2 i \omega c(\kappa) \left[ \alpha_p \frac{J_1(\bar{\lambda}_p)}{\bar{\lambda}_p} - b_p J_0(\bar{\lambda}_p) + C_p I_0(\bar{\mu}_p) - d_p \frac{I_1(\bar{\mu}_p)}{\bar{\mu}_p^2} \right] \right. \\
+ 2 \omega k \left( b_p \frac{J_1(\bar{\lambda}_p)}{\bar{\lambda}_p} + d_p \frac{I_1(\bar{\mu}_p)}{\bar{\mu}_p} \right) \right\} 
\]

and
\[
\frac{\dot{\alpha}}{b_{\eta}^2 \pi} = \hat{\alpha}_r \left\{ 2 i \omega c(\kappa) \left[ \frac{3}{2} - \frac{\varphi}{b} - \frac{i}{k} \right] \left[ -\alpha_p \frac{J_1(\bar{\lambda}_p)}{\bar{\lambda}_p} + b_p J_0(\bar{\lambda}_p) - C_p I_0(\bar{\mu}_p) + d_p \frac{I_1(\bar{\mu}_p)}{\bar{\mu}_p} \right] \right. \\
- 2 i \omega \left[ b_p \frac{J_1(\bar{\lambda}_p)}{\bar{\lambda}_p} + d_p \frac{I_1(\bar{\mu}_p)}{\bar{\mu}_p} \right] \left[ -\alpha_p \frac{J_1(\bar{\lambda}_p)}{\bar{\lambda}_p} + b_p J_0(\bar{\lambda}_p) - C_p I_0(\bar{\mu}_p) + d_p \frac{I_1(\bar{\mu}_p)}{\bar{\mu}_p} \right] \\
+ i k \left\{ -\alpha_p \left( \frac{I_1(\bar{\mu}_p)}{\bar{\lambda}_p^2} - \frac{1}{\bar{\lambda}_p} J_1(\bar{\lambda}_p) \right) + C_p \left( \frac{I_1(\bar{\mu}_p)}{\bar{\lambda}_p^2} - \frac{1}{\bar{\lambda}_p} I_1(\bar{\mu}_p) \right) \right\} \\
+ \left( 1 - \frac{\varphi}{b} - \frac{i}{k} \right) \left[ b_p \frac{J_1(\bar{\lambda}_p)}{\bar{\lambda}_p} + d_p \frac{I_1(\bar{\mu}_p)}{\bar{\mu}_p} \right] \left\{ \right\} 
\]

The generalized forces resulting from the work done by the pressure distribution due to elastic deformation in the \( j^{th} \) mode when the surface goes through a unit deformation in the \( p^{th} \) mode are denoted as \( \bar{\eta}_i \) in equation (3.7). These non-zero terms provide coupling between the elastic degrees of freedom. If it was not for these cross coupling terms, equations (1.16) could be solved independently for each of the respective normalized coordinates. An explanation of the flutter mechanism based on the frequency coalescence concept, as presented by Pines,\(^{26}\) requires the
presence of these cross coupling terms in the force side of the governing equations in order for sustained oscillations to occur. The argument for frequency coalescence states that the two or more degrees of freedom oscillate at the same frequency, energy being taken from one mode to drive another mode unstable. Consequently, if camber flutter is physically possible, these cross coupling terms should play a major role in the resulting motion.

The elastically coupled portion of the generalized force expression in the elastic equations of motion contains integrals similar to those encountered in the previous development of \( \tilde{\tilde{\zeta}} \), and \( \tilde{\zeta} \), after the order of integration and summation has been reversed. However, two additional integrals of some consequence appear, when equations (2.6) and (2.41) are substituted into (3.7), that are characterized by the form

\[
\int_0^{\pi} \sin(2n+2)\phi \sin\phi \; \varphi_p(\phi) d\phi
\]

and

\[
\int_0^{\pi} \sin(2n+1)\phi \sin\phi \; \varphi_p(\phi) d\phi
\]

Considering the form of \( \varphi_p \), as defined by (2.6), in conjunction with the coordinate transformation of equation (2.22), it can be seen that the above two integrals may be generated by repeated application of the single integral form
\[
\int_0^{\pi} \cos r \phi \, \phi_n(\phi) \, d\phi
\]
since
\[
\sin k \phi \sin \phi = \frac{l}{2} \left\{ \cos(k+1)\phi - \cos(k-1)\phi \right\} \quad (3.21)
\]

Therefore the general class of integrals that must be evaluated may be characterized as

\[
\int_0^{\pi} \cos r \phi \begin{pmatrix}
\sin(\bar{\lambda} \cos \phi) \\
\cos(\bar{\lambda} \cos \phi) \\
\sinh(\bar{\lambda} \cos \phi) \\
\cosh(\bar{\lambda} \cos \phi)
\end{pmatrix} \, d\phi \quad (3.22)
\]

The following observations in regard to the integrands of equation (3.22) are immediately evident.

\[\cos r \phi \sin(\bar{\lambda} \cos \phi)\]
antisymmetric w.r.t. \(\pi/2\) when \(r\) is even
symmetric w.r.t. \(\pi/2\) when \(r\) is odd

\[\cos r \phi \cos(\bar{\lambda} \cos \phi)\]
antisymmetric w.r.t. \(\pi/2\) when \(r\) is odd
symmetric w.r.t. \(\pi/2\) when \(r\) is even

\[\cos r \phi \sinh(\bar{\lambda} \cos \phi)\]
antisymmetric w.r.t. \(\pi/2\) when \(r\) is even
symmetric w.r.t. \(\pi/2\) when \(r\) is odd

\[\cos r \phi \cosh(\bar{\lambda} \cos \phi)\]
antisymmetric w.r.t. \(\pi/2\) when \(r\) is odd
symmetric w.r.t. \(\pi/2\) when \(r\) is even

In the above representation the abbreviation \(w.r.t.\) has been used in place of the phrase \(\text{with respect to}\).
Since the integral over a symmetric interval of an antisymmetric integrand is identically zero, equation (3.22) need be evaluated only for the cases where the integrand is symmetric. In the situations involved with symmetric integrands integration need be completed only over one half the interval, and the resulting value doubled to obtain the total value.

Watson \(^{25}\) presents certain relations directly applicable to the present circumstances. These relations stem from the integral representation of the Weber functions written in the form

\[
\mathcal{J}_\nu(z) + \mathcal{J}_{\nu'}(z) = \frac{4\cos \frac{1}{2} \nu \pi}{\pi} \int_0^\pi \cos \phi \cos(z \cos \phi) \, d\phi
\]

and

\[
\mathcal{J}_\nu(z) - \mathcal{J}_{\nu'}(z) = \frac{4\sin \frac{1}{2} \nu \pi}{\pi} \int_0^\pi \cos \phi \sin(z \cos \phi) \, d\phi
\]

where \(\mathcal{J}_{\nu'}(z)\) denotes the Weber function of index \(\nu'\). As demonstrated by Watson, when \(\nu'\) has integral values \(r\) the Weber functions reduce to Bessel functions of the first kind. Since

\[
\mathcal{J}_r(z) = (-1)^r \mathcal{J}_r(z)
\]

it may easily be shown that the first two subcases of (3.22) have the values
\[
\int_0^\pi \cos r\phi \sin(\lambda_p \cos \phi) \, d\phi = \begin{cases} 
0 & r \text{ even} \\
(-1)^{r/2} \pi \, J_r(\lambda_p) & r \text{ odd}
\end{cases}
\]

and
\[
\int_0^\pi \cos r\phi \cos(\lambda_p \cos \phi) \, d\phi = \begin{cases} 
(-1)^{r/2} \pi \, J_r(\lambda_p) & r \text{ even} \\
0 & r \text{ odd}
\end{cases}
\]

Now the Bessel functions of purely imaginary arguments (i.e., modified Bessel functions of the first kind) are related to the ordinary Bessel functions by the relation
\[
I_{\nu}(iz) = e^{-\frac{i\pi\nu}{2}} J_{\nu}(i z) \quad (3.25)
\]
Using the relationships between the circular and hyperbolic functions, equation (3.25) and the fact that \( I_{-r}(z) = I_r(z) \) for \( r \) even or odd, equations analogous to (3.24) for the cases where the integrand of (3.21) contain the hyperbolic terms may be established. Performing the necessary manipulations gives
\[
\int_0^\pi \cos r\phi \cosh(\lambda_p \cos \phi) \, d\phi = \begin{cases} 
\pi \, I_r(i \lambda_p) & r \text{ even} \\
0 & r \text{ odd}
\end{cases}
\]
\[
\int_0^\pi \cos r\phi \sinh(\lambda_p \cos \phi) \, d\phi = \begin{cases} 
0 & r \text{ even} \\
\pi \, I_{-r}(i \lambda_p) & r \text{ odd}
\end{cases}
\]

(3.26)
Substituting equation (3.21) into (3.20) and using the results of equations (3.24) and (3.25), the integrals of (3.20) may readily be evaluated. The resulting composite integrals may be written as

\[
\int_0^{\pi} \sin(2n+2)\phi \begin{bmatrix} \sin(J\cos\phi) \\ \cos(J\cos\phi) \\ \sinh(J\cos\phi) \\ \cosh(J\cos\phi) \end{bmatrix} \sin\phi d\phi = \begin{cases} \frac{(-1)^n}{\pi} \left[ J_{2n} + J_{2n+2} \right] & \text{if } J_{2n+3} = 0 \\ \frac{2}{\pi} \left[ I_{2n} + I_{2n+2} \right] & \text{if } J_{2n+3} = 0 \end{cases}
\]

and

\[
\int_0^{\pi} \sin(2n+1)\phi \begin{bmatrix} \sin(J\cos\phi) \\ \cos(J\cos\phi) \\ \sinh(J\cos\phi) \\ \cosh(J\cos\phi) \end{bmatrix} \sin\phi d\phi = \begin{cases} \frac{(-1)^n}{\pi} \left[ J_{2n} + J_{2n+1} \right] & \text{if } J_{2n+2} = 0 \\ \frac{2}{\pi} \left[ I_{2n} - I_{2n+1} \right] & \text{if } J_{2n+2} = 0 \end{cases}
\]

These integrals may be reduced still further by noting the recurrence relations

\[
J_{\nu+1}(\bar{z}) + J_{\nu-1}(\bar{z}) = \frac{2\nu}{\bar{z}} J_{\nu}(\bar{z})
\]

and

\[
I_{\nu+1}(\bar{z}) - I_{\nu-1}(\bar{z}) = \frac{2\nu}{\bar{z}} I_{\nu}(\bar{z})
\]

Hence, the right hand side of equation (3.27) may be reduced to read

\[
(-1)^n \frac{\pi}{2} \left[ J_{2n+1} + J_{2n+2} \right] = (-1)^n \frac{\pi(2n+2)}{\bar{z}} J_{2n+1}(\bar{z})
\]

\[
\frac{\pi}{2} \left[ I_{2n+1} - I_{2n+2} \right] = \frac{\pi(2n+2)}{\bar{z}} I_{2n+1}(\bar{z})
\]

and

\[
(-1)^n \frac{\pi}{2} \left[ J_{2n} + J_{2n+1} \right] = (-1)^n \frac{\pi(2n+1)}{\bar{z}} J_{2n+1}(\bar{z})
\]

\[
\frac{\pi}{2} \left[ I_{2n} - I_{2n+1} \right] = \frac{\pi(2n+1)}{\bar{z}} I_{2n+1}(\bar{z})
\]

(3.28)
It is now finally possible to evaluate in its entirety the elastic coupling portion of the generalized force associated with the elastic degrees of freedom. Substituting equations (2.6) and (2.41) into expanded form of in (3.7), using the results of equations (3.11) through (3.17) as applicable, and equation (3.28), the generalized force under consideration takes on the rather lengthy form given by
\[
\frac{\partial \bar{p}}{\partial \omega} = \int \frac{1}{2} \pi \omega \left[ 1 + C(\kappa) \right] \left[ A_j(\kappa) J_0(\bar{\lambda}_p) + B_j(\kappa) J_0(\bar{\lambda}_p) + C_j(\kappa) I_0(\bar{\mu}_p) + D_j(\kappa) I_0(\bar{\mu}_p) \right] \frac{d}{d\lambda_p} \left( \frac{(2n+1)}{\lambda_p} J_{2n+1}^2(\bar{\lambda}_p) \right)
\]

\[
- \gamma \omega \left[ A_j(\kappa) \sum_{n=0}^{\infty} \left( -1 \right)^n J_{2n+1}(\bar{\lambda}_p) \left( b_p(-1)^n \frac{(2n+1)}{\lambda_p} J_{2n+1}(\bar{\lambda}_p) + c_p \frac{(2n+2)}{\mu_p} I_{2n+1}(\bar{\mu}_p) \right) 
+ B_j(\kappa) \sum_{n=0}^{\infty} \left( -1 \right)^n J_{2n+1}(\bar{\lambda}_p) \left( a_p(-1)^n \frac{(2n+1)}{\lambda_p} J_{2n+1}(\bar{\lambda}_p) + c_p \frac{(2n+2)}{\mu_p} I_{2n+1}(\bar{\mu}_p) \right) 
+ C_j(\kappa) \sum_{n=0}^{\infty} I_{2n+1}(\bar{\mu}_p) \left( b_p(-1)^n \frac{(2n+1)}{\lambda_p} J_{2n+1}(\bar{\lambda}_p) + c_p \frac{(2n+2)}{\mu_p} I_{2n+1}(\bar{\mu}_p) \right) 
+ D_j(\kappa) \sum_{n=0}^{\infty} I_{2n+1}(\bar{\mu}_p) \left( a_p(-1)^n \frac{(2n+1)}{\lambda_p} J_{2n+1}(\bar{\lambda}_p) + c_p \frac{(2n+2)}{\mu_p} I_{2n+1}(\bar{\mu}_p) \right) 
+ \left( \frac{B_j(\kappa)}{2} J_0(\bar{\lambda}_p) + \frac{D_j(\kappa)}{2} I_0(\bar{\mu}_p) \right) \left[ -a_p J_0(\bar{\lambda}_p) + b_p I_0(\bar{\mu}_p) - c_p I_0(\bar{\mu}_p) + d_p I_0(\bar{\mu}_p) \right] \right]
\]

\[
- 4i \omega \kappa \mu \left[ A_j(\kappa) \sum_{n=0}^{\infty} \left( -1 \right)^n J_{2n+1}(\bar{\lambda}_p) a_p(-1)^n \frac{(2n+2)}{\lambda_p} J_{2n+1}(\bar{\lambda}_p) + c_p \frac{(2n+2)}{\mu_p} I_{2n+1}(\bar{\mu}_p) 
+ B_j(\kappa) \sum_{n=0}^{\infty} \left( -1 \right)^n J_{2n+1}(\bar{\lambda}_p) b_p(-1)^n \frac{(2n+1)}{\lambda_p} J_{2n+1}(\bar{\lambda}_p) + d_p \frac{(2n+1)}{\mu_p} I_{2n+1}(\bar{\mu}_p) 
+ C_j(\kappa) \sum_{n=0}^{\infty} I_{2n+1}(\bar{\mu}_p) a_p(-1)^n \frac{(2n+2)}{\lambda_p} J_{2n+1}(\bar{\lambda}_p) + c_p \frac{(2n+2)}{\mu_p} I_{2n+1}(\bar{\mu}_p) 
+ D_j(\kappa) \sum_{n=0}^{\infty} I_{2n+1}(\bar{\mu}_p) b_p(-1)^n \frac{(2n+1)}{\lambda_p} J_{2n+1}(\bar{\lambda}_p) + d_p \frac{(2n+1)}{\mu_p} I_{2n+1}(\bar{\mu}_p) \right] \right]
\]

(3.29)
In order to facilitate the writing of equation (3.29), as well as to formulate the equation for practical application, the following set of designations are given to the series contained in the expanded expression. Let

\[ G_1'(\alpha, \beta) = \sum_{r=0}^{\infty} (2r+1) \, J_{2r+1}(\alpha) \, J_{2r+1}(\beta) \]

\[ G_2'(\alpha, \beta) = \sum_{r=0}^{\infty} (2r+2) \, J_{2r+2}(\alpha) \, J_{2r+2}(\beta) \]

\[ G_1^2(\alpha, \beta) = \sum_{r=0}^{\infty} (-1)^r (2r+1) \, J_{2r+1}(\alpha) \, I_{2r+1}(\beta) \]

(3.30)

\[ G_2^2(\alpha, \beta) = \sum_{r=0}^{\infty} (-1)^r (2r+2) \, J_{2r+2}(\alpha) \, I_{2r+2}(\beta) \]

\[ G_1^3(\alpha, \beta) = \sum_{r=0}^{\infty} (2r+1) \, I_{2r+1}(\alpha) \, I_{2r+1}(\beta) \]

\[ G_2^3(\alpha, \beta) = \sum_{r=0}^{\infty} (2r+2) \, I_{2r+2}(\alpha) \, I_{2r+2}(\beta) \]

It is interesting to note that the first two series of (3.30), where the ordinary Bessel functions are involved
in combinations, appeared in some early works in applied mathematics. Kapteyn was able to develop integral equivalents to the two particular series in question. His results are briefly discussed by Watson. However, other than these integral relations, there does not appear to be any convenient representation of the above series. For the purpose of this dissertation, the series of (3.30) has been evaluated for discrete values of $\alpha$ and $\beta$. The resulting numerical values were obtained through the use of an IBM 7090 digital computer. For computational reasons the series was evaluated directly as given in (3.30) rather than adopting an equivalent integral form. The results of these calculations are presented in Tables 5 through 10. The ordinary and modified Bessel functions of the first kind were computed in two separate subroutines using a modification of the method given by Randels and Reeves. The ordinary and modified Bessel functions, order 0 to 22, were computed for each value of $\alpha$ and $\beta$ and put in storage. The required functions were then called as needed for the calculation of the terms in the six summations.

Substituting the series representation of (3.30) into (3.29) reduces the generalized force expression to its final form.
Substituting equations (3.18), (3.19) and (3.31) into equation (3.7), equations (3.4), (3.5) and (3.7) may be expressed in shorthand matrix notation as

\[
\frac{\tilde{B}_{3}}{2\pi f^{2}} = \tilde{b}_{3} \left\{ \left[ \frac{1}{1+i(\omega)} \right] \left[ A_{s}(k)J_{s}(\tilde{\lambda}_{s}) + B_{s}(k)J_{o}(\tilde{\lambda}_{o}) + C_{s}(k)I_{o}(\tilde{\lambda}_{o}) + D_{s}(k)I_{o}(\tilde{\lambda}_{o}) \right] + \left[ -\alpha_{p} J_{s}(\tilde{\lambda}_{s}) + b_{p} J_{o}(\tilde{\lambda}_{o}) - c_{p} I_{o}(\tilde{\lambda}_{o}) + d_{p} I_{o}(\tilde{\lambda}_{o}) \right] \right\}
\]

\[
= -2 \left[ A_{s}(k) \left( \frac{b_{p}}{\lambda_{p}} G_{s}^{q}(\tilde{\lambda}_{p}, \tilde{\lambda}_{p}) + \frac{d_{p}}{\mu_{p}} G_{s}^{u}(\tilde{\lambda}_{p}, \tilde{\lambda}_{p}) \right) - B_{s}(k) \left( \frac{a_{p}}{\lambda_{p}} G_{s}^{q}(\tilde{\lambda}_{s}, \tilde{\lambda}_{p}) + \frac{c_{p}}{\mu_{p}} G_{s}^{u}(\tilde{\lambda}_{s}, \tilde{\lambda}_{p}) \right) + C_{s}(k) \left( \frac{b_{p}}{\lambda_{p}} G_{s}^{q}(\tilde{\lambda}_{p}, \tilde{\lambda}_{p}) + \frac{d_{p}}{\mu_{p}} G_{s}^{u}(\tilde{\lambda}_{p}, \tilde{\lambda}_{p}) \right) + D_{s}(k) \left( \frac{a_{p}}{\lambda_{p}} G_{s}^{u}(\tilde{\lambda}_{s}, \tilde{\lambda}_{p}) + \frac{c_{p}}{\mu_{p}} G_{s}^{u}(\tilde{\lambda}_{s}, \tilde{\lambda}_{p}) \right) \right] + \left( \frac{B_{s}(k)}{\lambda_{s}} J_{s}(\tilde{\lambda}_{s}) + \frac{D_{s}(k)}{\lambda_{s}} I_{s}(\tilde{\lambda}_{s}) \right) \left( -a_{p} J_{s}(\tilde{\lambda}_{s}) + b_{p} J_{o}(\tilde{\lambda}_{o}) - c_{p} I_{o}(\tilde{\lambda}_{o}) + d_{p} I_{o}(\tilde{\lambda}_{o}) \right) \]

\[
= -2i k \left[ -A_{s}(k) \left( \frac{a_{p}}{\lambda_{p}} G_{s}^{q}(\tilde{\lambda}_{p}, \tilde{\lambda}_{p}) + \frac{c_{p}}{\mu_{p}} G_{s}^{u}(\tilde{\lambda}_{p}, \tilde{\lambda}_{p}) \right) + \frac{B_{s}(k)}{\lambda_{s}} \left( \frac{b_{p}}{\lambda_{p}} G_{s}^{q}(\tilde{\lambda}_{s}, \tilde{\lambda}_{p}) + \frac{d_{p}}{\mu_{p}} G_{s}^{u}(\tilde{\lambda}_{s}, \tilde{\lambda}_{p}) \right) \right] + \left( \frac{B_{s}(k)}{\lambda_{s}} J_{s}(\tilde{\lambda}_{s}) + \frac{D_{s}(k)}{\lambda_{s}} I_{s}(\tilde{\lambda}_{s}) \right) \left( -\alpha_{p} J_{s}(\tilde{\lambda}_{s}) + b_{p} J_{o}(\tilde{\lambda}_{o}) - c_{p} I_{o}(\tilde{\lambda}_{o}) + d_{p} I_{o}(\tilde{\lambda}_{o}) \right) \]

\[
+ C_{s}(k) \left( \frac{b_{p}}{\lambda_{p}} G_{s}^{u}(\tilde{\lambda}_{s}, \tilde{\lambda}_{p}) + \frac{d_{p}}{\mu_{p}} G_{s}^{u}(\tilde{\lambda}_{s}, \tilde{\lambda}_{p}) \right) + D_{s}(k) \left( \frac{a_{p}}{\lambda_{p}} G_{s}^{u}(\tilde{\lambda}_{s}, \tilde{\lambda}_{p}) + \frac{c_{p}}{\mu_{p}} G_{s}^{u}(\tilde{\lambda}_{s}, \tilde{\lambda}_{p}) \right) \right] \}
\]

(3.31)
\[
\frac{\alpha}{\pi \rho U \beta \omega} = \frac{h}{b} \left\{ (1 - \frac{\omega}{b}) + 2i \mathcal{C}(\mathcal{K}) \left( \frac{\omega}{b} - \frac{1}{2} \right) \right\} \\
+ \bar{\alpha}_r \left\{ 2i \mathcal{C}(\mathcal{K}) \left( \frac{\omega}{b} - \frac{1}{2} \right) \left( \frac{3}{2} - \frac{\omega}{b} - \frac{i}{K} \right) + iK (1 - \frac{\omega}{b}) \left( 1 - \frac{\omega}{b} - \frac{i}{K} \right) - \left( \frac{i}{2} + \frac{K}{8} \right) \right\} \\
+ \sum_{s=1}^{\tilde{f}_s} \left[ a_s b_s c_s d_s \right] \left[ \mathcal{Q} \right] \left\{ - 2 \left( \frac{1}{2} - \frac{\omega}{b} \right) \mathcal{C}(\mathcal{K}) \left[ \mathcal{V}_{q} \right] \right\} \\
- 2iK (1 - \frac{\omega}{b}) \left[ \mathcal{W}_s \right] + iK \left[ H_{s} \right] - \left[ K_{s} \right] \\
(3.33)
\]

\[
\frac{\bar{\alpha}_p}{\pi \rho U \beta \omega} = \frac{h}{b} \left\{ \left[ \begin{array}{c} 1 \ 1 \ 1 \ 1 \end{array} \right] \left\{ 2i \mathcal{C}(\mathcal{K}) \left[ \mathcal{V}_{p} \right] + 2iK \left[ \mathcal{W}_{s} \right] \right\} \right\} \left\{ \begin{array}{c} a_p \\ b_p \\ c_p \\ d_p \end{array} \right\} \\
+ \bar{\alpha}_r \left\{ \left[ \begin{array}{c} 1 \ 1 \ 1 \ 1 \end{array} \right] \left\{ \left[ 2i \mathcal{C}(\mathcal{K}) \left( \frac{3}{2} - \frac{\omega}{b} - \frac{i}{K} \right) - i \right] \left[ \mathcal{V}_{p} \right] + 2K \left[ \mathcal{N}_{p} \right] \right\} \right\} \left\{ \begin{array}{c} a_p \\ b_p \\ c_p \\ d_p \end{array} \right\} \\
+ \sum_{s=1}^{\tilde{f}_s} \left[ a_s b_s c_s d_s \right] \left[ \mathcal{Q} \right] \left\{ 2 \left[ 1 + \mathcal{C}(\mathcal{K}) \right] \left[ S_{p} \right] + 2 \left[ T_{p} \right] \right\} \\
+ 4iK \left[ R_{p} \right] \right\} \left\{ \begin{array}{c} a_p \\ b_p \\ c_p \\ d_p \end{array} \right\} \\
(3.34)
\]

where

\[
\left[ \mathcal{Q}' \right] = \left[ \begin{array}{cccc}
- i & \frac{\mathcal{K}}{K} & 0 & 0 \\
\frac{\mathcal{K}}{K} & i & 0 & 0 \\
0 & 0 & - i & \frac{\mathcal{K}}{K} \\
0 & 0 & \frac{\mathcal{K}}{K} & i \\
\end{array} \right] \\
(3.35)
\]
\[
\begin{bmatrix}
Q
\end{bmatrix} =
\begin{bmatrix}
i & \frac{J_i}{K} & 0 & 0 \\
-\frac{J_i}{K} & i & 0 & 0 \\
0 & 0 & i & \frac{K_i}{K} \\
0 & 0 & \frac{J_i}{K} & i
\end{bmatrix}
\]  \hspace{1cm} (3.36)

\[
\begin{bmatrix}
V_p
\end{bmatrix} =
\begin{bmatrix}
J_i(\varphi)
-\frac{J_i(\varphi)}{\beta}
I_i(\varphi)
I_o(\varphi)
\end{bmatrix}
\]  \hspace{1cm} (3.37)

\[
\begin{bmatrix}
W_p
\end{bmatrix} =
\begin{bmatrix}
0
\frac{J_i(\varphi)}{\beta}
0
\frac{I_i(\varphi)}{\beta}
\end{bmatrix}
\]  \hspace{1cm} (3.38)

\[
\begin{bmatrix}
H_i
\end{bmatrix} =
\begin{bmatrix}
\frac{J_i(\mu_i)}{\chi_i}
0
\frac{I_i(\mu_i)}{\chi_i}
0
\end{bmatrix}
\]  \hspace{1cm} (3.39)
\[ \mathbf{K}_j = \begin{bmatrix} J_x(\bar{\lambda}_j) \\ J_z(\bar{\lambda}_j) \\ I_2(\bar{\eta}_{\bar{2}}) \\ -I_2(\bar{\eta}_{\bar{2}}) \end{bmatrix} \] (3.40)

\[ \mathbf{N}_\rho = \begin{bmatrix} \frac{J_x(\bar{\lambda}_\rho)}{2} + \frac{J_z(\bar{\lambda}_\rho)}{\lambda_\rho} (\gamma - \frac{\phi_\rho}{b}) - \frac{J_z(\bar{\lambda}_\rho)}{\lambda_\rho} & \frac{I_2(\bar{\eta}_{\bar{2}})}{2} - \frac{I_2(\bar{\eta}_{\bar{2}})}{\lambda_\rho} (1 + \frac{\phi_\rho}{b}) \end{bmatrix} \] (3.41)

\[ \mathbf{S}_{\rho j} = \begin{bmatrix} -J_x(\bar{\lambda}_j) J_x(\bar{\lambda}_j) & J_y(\bar{\lambda}_j) J_x(\bar{\lambda}_j) & -I_2(\bar{\eta}_{\bar{2}}) J_x(\bar{\lambda}_j) & I_2(\bar{\eta}_{\bar{2}}) J_x(\bar{\lambda}_j) \\ -J_y(\bar{\lambda}_j) J_y(\bar{\lambda}_j) & J_y(\bar{\lambda}_j) J_y(\bar{\lambda}_j) & -I_2(\bar{\eta}_{\bar{2}}) J_y(\bar{\lambda}_j) & I_2(\bar{\eta}_{\bar{2}}) J_y(\bar{\lambda}_j) \\ -J_z(\bar{\lambda}_j) I_2(\bar{\eta}_{\bar{2}}) & J_z(\bar{\lambda}_j) I_2(\bar{\eta}_{\bar{2}}) & -I_2(\bar{\eta}_{\bar{2}}) I_2(\bar{\eta}_{\bar{2}}) & I_2(\bar{\eta}_{\bar{2}}) I_2(\bar{\eta}_{\bar{2}}) \\ -J_z(\bar{\lambda}_j) I_2(\bar{\eta}_{\bar{2}}) & J_z(\bar{\lambda}_j) I_2(\bar{\eta}_{\bar{2}}) & -I_2(\bar{\eta}_{\bar{2}}) I_2(\bar{\eta}_{\bar{2}}) & I_2(\bar{\eta}_{\bar{2}}) I_2(\bar{\eta}_{\bar{2}}) \end{bmatrix} \] (3.42)
The matrices defined by equations (3.37) through (3.44) are dependent only on the structural characteristics of the body and accordingly may be evaluated once the structural configuration is defined. The matrices defined by (3.35) and (3.36) as well as the multipliers of the diagonal matrices contained in (3.32), (3.33) and (3.34)
are functions of either the frequency of oscillation or the reduced frequency and consequently must be determined through the solution of the dynamical equations of forced motion.

The row matrix \( [a_j, b_j, c_j, d_j] \) contains the coefficients of the circular and hyperbolic functions appearing in the free-free vibration modes (2.6) and are associated with the pressure distribution resulting from the oscillation of the surface. The column matrix \( \begin{bmatrix} q_p \\ b_p \\ c_p \\ d_p \end{bmatrix} \) is again the coefficients in the modal representation; however, as they appear in the preceding expressions they are associated with the virtual displacement in the evaluation of the generalized force. Therefore the \( p \) subscript, as used here, indicates that equation (3.34) represents the generalized force belonging to the \( p^{th} \) equation of motion. These coefficients in the modal representation need not necessarily be evaluated theoretically as given in equation (2.7), but may be derived from experimentally determined modes as long as the results are expressible in terms of the circular and hyperbolic functions.

The generalized forces as given in (3.32), (3.33) and (3.34) are actually only the amplitude of the generalized forces associated with simple harmonic oscillation of the camber line and accordingly must be regarded as such when used with equations (1.14) and (1.16). That is to say,
only after assuming simple harmonic motion of the normal coordinates in (1.14) and (1.16) and dividing by $e^{i\omega t}$ is it appropriate to make the substitution for the generalized forces as they have been developed in this section.
IV. INVESTIGATION OF THE RESPONSE CHARACTERISTICS OF A TWO-DIMENSIONAL AIRFOIL OF AIRMAT CONSTRUCTION

In order to investigate the applicability of the theory developed in the preceding sections, a particular model was chosen which had already been tested for its aeroelastic characteristics. The model chosen was a two-dimensional inflatable structure of airmat construction. Figure 9 illustrates the general structural configuration of airmat considered. Figures 10 and 11 show the airfoil section and planform of the test model mounted in the wind tunnel. The model had a 60-inch chord, a spanwise dimension of 42 inches, and was two inches in depth. The mass and stiffness data for the model are presented in Table 2. The mass per unit length value used was established based on the total weight of the structure and its chord dimension. The rotary inertia term was evaluated assuming all mass to be concentrated in the fabric cover, discounting the distributing influence of the drop threads.

Since bending stiffness properties are not easily determined theoretically for inflatable structures, an effective bending stiffness parameter was determined experimentally. To calculate an effective EI value the
first natural frequencies of vibration were measured for the case of a cantilever support at the leading edge and for the case of a pinned leading edge with a free trailing edge. These frequency values along with the mass data were then substituted into the appropriate dynamical equations of motion for each case and a corresponding EI value determined. The two values so determined differed by nine per cent so that an average of the two was taken as the effective EI value. The classical beam equations, ignoring shearing deformations and rotary inertia effects, were used in this calculation. This simple approach seems justifiable since in both cases the first natural frequency was quite low and the influence of shearing stiffness and rotary inertia are minimum for this condition. An effective shearing stiffness value (GAK) was calculated using the approach customarily employed for inflatable structures\textsuperscript{14} where the stiffness is equal to the inflation pressure differential times the enclosed cross-sectional area of the beam. The pressure differential for the particular configuration investigated was 6 psi.

The bending and shearing stiffness values determined as described above were 504 lb.ft\textsuperscript{2} and 504 lb., respectively. By chance the stiffness parameters came out to be numerically equal and therefore it appears that in the calculations to follow, the effects of the two types of deformations may be of the same order of importance.
In the calculations of the mass, rotary inertia, and stiffness distributions, no allowance was made for the local influence of the edge or end surface material. The effects of these boundaries have the tendency to cancel each other since the edges add stiffness which would cause an increase in the natural frequencies while the ends and edges increase the mass concentration and tend to decrease the frequencies. The subsections to follow discuss various aspects of the sample calculations and the dynamic behavior of the airmat model.

Free-free vibration mode shapes and natural frequencies

The dynamical equations of motion for the free-free vibration of a uniform beam including shearing and rotary inertia effects are solved in Appendix A. The mode shapes are given by equation (A.7) and (A.8) and the transcendental equation for determining the natural frequencies is given in equation (A.6). The graphical solution of the resulting transcendental equation for the natural frequencies of the particular configuration investigated is presented in Figure 6. The frequencies determined for this case can be compared with those obtained using classical beam theory for a measure of the effect of shearing deformation and rotary inertia. Values for comparison are shown on the following page.
### Natural frequencies (rad/sec)

<table>
<thead>
<tr>
<th>Mode</th>
<th>Timoshenko beam</th>
<th>Classical beam</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>116.58</td>
<td>144.88</td>
</tr>
<tr>
<td>2</td>
<td>233.58</td>
<td>397.13</td>
</tr>
<tr>
<td>3</td>
<td>349.88</td>
<td>778.38</td>
</tr>
<tr>
<td>4</td>
<td>460.83</td>
<td>1286.70</td>
</tr>
<tr>
<td>5</td>
<td>568.00</td>
<td>1922.11</td>
</tr>
</tbody>
</table>

Kraszewski\textsuperscript{13} carried out an extensive investigation of the influence of shearing stiffness and rotary inertia on the natural frequencies of oscillation. To show the importance of these effects, some of the results given in reference 13 are reproduced in Figure 5 using the notation of this dissertation. It can be seen from this figure that the influence of rotary inertia and shearing stiffness results in the decrease in the natural frequencies; both effects increase with frequency, the shearing stiffness having the stronger influence. This trend is to be expected since inclusion of shearing terms in essence decreases the overall stiffness of the beam, and the addition of rotary inertia increases the dynamic loading.

The first five natural mode shapes are shown in Figure 7 for the free-free beam with and without shearing and rotary inertia effects. All modes have been normalized to have a unit deflection at the leading edge. The additional displacement for the case of the Timoshenko beam increases with increasing frequency of oscillation. The general deformation patterns do not differ significantly;
however, if a modal representation is used in a forced vibration analysis this small difference is noticeable in the resulting calculation of the generalized masses. For example, considering the model of this investigation and calculating the generalized masses using the different shapes gives

<table>
<thead>
<tr>
<th>Mode</th>
<th>Generalized mass (Timoshenko beam)</th>
<th>Generalized mass (Classical beam)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.02245</td>
<td>0.01792</td>
</tr>
<tr>
<td>2</td>
<td>0.03648</td>
<td>0.02277</td>
</tr>
<tr>
<td>3</td>
<td>0.04029</td>
<td>0.02321</td>
</tr>
<tr>
<td>4</td>
<td>0.04561</td>
<td>0.02346</td>
</tr>
<tr>
<td>5</td>
<td>0.04576</td>
<td>0.02361</td>
</tr>
</tbody>
</table>

Note that in the case of the Timoshenko beam, rotary inertia effects have been added in the generalized mass terms; however, this accounts for only about one per cent of the difference.

**Forced vibration in a vacuum**

The model to be considered here is that of a uniform beam pinned at the leading edge and free of support at the trailing edge. A harmonically varying moment is applied to the leading edge at the pin support as shown in Figure 8. The differential equations for forced oscillation of a free beam are developed in Section I of this dissertation using a free-free modal representation for the deformation shape. A theoretical model, suitable for analysis of the pin-free configuration may be developed from these equations through application of the appropriate boundary
conditions and force system. The adaptation is accomplished by setting the vertical translation at the leading edge, \( h(o) \), equal to zero and requiring the applied forces and inertia loading to be in equilibrium. In the notation of equations (1.14) and (1.15) the external force system is given as

\[
M_\alpha = M_o \sin nt
\]

\[
F_q = \frac{S_o M_o \sin nt}{(m + I_e)}
\]

where \( \Omega \) is the frequency of the applied force. The force \( F_a \) results from the pin constraints since the sum of the inertia loading and the applied forces must balance. Substituting (4.1) into the appropriate differential equations, (1.14) and (1.15), gives the equations of motion in terms of the normal coordinates in the form

\[
\ddot{\varphi}_r = \frac{-M_o \sin nt}{(m + I_e)}
\]

and

\[
\ddot{f}_j + (1 + ig_j) \omega_j^2 f_j = -\frac{M_s}{M_s} \left[ \Phi_j(o) + \frac{S_o}{(m + I_e)} \Phi_j(o) \right] \sin nt
\]

\( j = 1, 2, \ldots n \)

Equations (4.2) and (4.3) may be solved to obtain the dynamic response characteristics of the system. Using the numerical values listed in Table 2 and assuming a
damping coefficient of $g = 0.03$ these equations were solved to determine the frequency response of the angular rotation at the leading edge. In the solution of the governing differential equations, the initial conditions were taken as

\begin{align*}
\gamma(x,0) &= 0 \\
\dot{\gamma}(x,0) &= 0 \\
\alpha(x,0) &= 0 \\
\dot{\alpha}(x,0) &= 0
\end{align*}

Figure 12 presents a line plot (straight lines between calculated points) of the frequency response versus forcing frequency where the response is shown as the slope of the leading edge divided by the amplitude of the applied moment. In order to compare this curve with experimental results to be discussed later, the ordinate of Figure 12 has been taken as ten times the log of the ratio of the rotation to the applied moment, i.e.

\[
\frac{10 \log_{10} \left( \frac{\alpha(\omega)}{M_0} \right)}{\omega}
\]

**Forced oscillation in air, zero air speed**

The problem investigated here is exactly that of the preceding subsection except that the body is oscillating in free air. Due to the body motion normal to its rest position, momentum is transferred to the air in the immediate vicinity of the body. This produces a force acting on the body due to its own motion. These air forces, termed virtual mass terms in unsteady aerodynamics because the forces are proportional to the acceleration, may be developed
directly from equations (3.32), (3.33), and (3.34) by setting the airspeed equal to zero. In performing this operation, care must be taken to express the reduced frequency in terms of the velocity in order to avoid dividing by zero. Carrying out the necessary reduction gives for the virtual mass terms in the generalized force expressions

\[
\begin{align*}
\tilde{m} = \frac{\bar{h}}{b^2 \omega^2} \sum_{j=1}^{n} \int_{b_j}^{c_j} \int d \xi \int \frac{f_{d_j}}{b} \left[ a_j b_j c_j d_j \right] \left[ \begin{array}{c}
-1 \\
-1
\end{array} \right] \left[ \begin{array}{c}
\omega_j
\end{array} \right]
\end{align*}
\]

\[
\begin{align*}
\tilde{m}_{l} = \frac{\bar{h}}{b^2 \omega^2} \sum_{j=1}^{n} \int_{b_j}^{c_j} \int d \xi \int \frac{f_{d_j}}{b} \left[ a_j b_j c_j d_j \right] \left[ \begin{array}{c}
-1 \\
-1
\end{array} \right] \left[ \begin{array}{c}
\omega_j + \hat{H}_j
\end{array} \right]
\end{align*}
\]

\[
\begin{align*}
\tilde{m}_{s} = \frac{\bar{h}}{b^2 \omega^2} \sum_{j=1}^{n} \int_{b_j}^{c_j} \int d \xi \int \frac{f_{d_j}}{b} \left[ a_j b_j c_j d_j \right] \left[ \begin{array}{c}
-1 \\
-1
\end{array} \right] \left[ \begin{array}{c}
\omega_j
\end{array} \right]
\end{align*}
\]

\[
\begin{align*}
\tilde{m}_{e} = \frac{\bar{h}}{b^2 \omega^2} \sum_{j=1}^{n} \int_{b_j}^{c_j} \int d \xi \int \frac{f_{d_j}}{b} \left[ a_j b_j c_j d_j \right] \left[ \begin{array}{c}
-1 \\
-1
\end{array} \right] \left[ \begin{array}{c}
\omega_j + \hat{H}_j
\end{array} \right]
\end{align*}
\]

The governing differential equations of motion for the zero air speed case may be written as

\[
\ddot{\alpha} = \frac{1}{\eta + \mathcal{L}_e} \left\{ \varepsilon^{i \omega t} \tilde{m} \tilde{\alpha} - \mathcal{M}_a \right\}
\]

(4.5)
and

\[
M_j \ddot{f}_j + \omega^2_j M_j \left(1 + i \eta_j \right) f_j = e^{i \omega t} W \tilde{z}_j - \int_0^l M_j(x,t) \Phi_j \delta(x-a) \, dx
\]

\[
- \int_0^l \Phi_j \left[ e^{i \omega t} W \tilde{z}_n - \frac{S_e}{\eta + I_e} \left( e^{i \omega t} W \tilde{z}_d - M_d \right) \right] \delta(x-a) \, dx
\]

\[j = 1, 2, \ldots, n\]

To obtain a solution for the frequency response equation (4.4) is substituted into (4.5) and (4.6). The term, \(W\), appearing in equations (4.5) and (4.6) accounts for the span dimension of the theoretical model, since in the development of the aerodynamic expressions and equation (4.4) the loads are expressed in terms of a unit strip. Using the numerical values of the test model, the response has been calculated by using equations (4.4), (4.5), and (4.6) for discrete values of the forcing frequency. The results of these calculations are shown in Figure 12. In performing the calculations care was taken to choose forcing frequencies that would produce a maximum response; however, no particular attention was given to points in between these values. Thus the points appearing as minimum points in Figure 12 are not necessarily minimum response points.

From Figure 12 it can be seen that the virtual mass terms tend to lower the peak response frequencies and reduce the amplitude of motion for the unit force input.
This is to be expected since in effect the mass of the system has been increased.

**Frequency response in a uniform flow field**

The problem of calculating the frequency response for the situation where the model is placed in a uniform flow field may be treated in a manner analogous to the two preceding subcases investigated. However for this case the aerodynamic forces contribute both to the effective mass terms and the damping term, the aerodynamic damping either adding to or subtracting from the structural damping in the system. The governing differential equations for this case are the same as those appearing in equations (4.5) and (4.6) when the appropriate generalized aerodynamic force expressions are used. The appropriate aerodynamic force expressions for this situation are given in equations (3.32), (3.33), and (3.34), and must be used in their complete form.

The particular problem considered was one which had been studied through wind tunnel testing. For calculation purposes an airspeed of 100 miles per hour was chosen for comparison with test data. Equations (4.5) and (4.6) were solved by using the experimental model characteristics and by employing three modes in the deformation representation. Frequency response characteristics were calculated for the reduced frequency values $k = 0.6, 0.8, 1.0, 1.2, 1.5$, and $2.0$. The results of these calculations are shown in Figure 12 as points only, since no attempt was made to
determine the peak response frequency. The method used for calculating this is discussed in the following paragraph.

The information required for calculating the frequency response characteristics of a surface, using the analysis developed in this dissertation, consist of the mass and rotary inertia properties of the surface; the free-free vibration mode shapes and natural frequencies; and the eigenvalues $\lambda_j$ and $\mu_j$ defined by equation (A.5) of Appendix A. The generalized mass terms are calculated using the definition presented on page 30. The generalized aerodynamic forces are given by equations (3.32) through (3.34) and are complex functions of the eigenvalues (i.e. $\overline{\lambda}_j$'s and $\overline{\mu}_j$'s) and the reduced frequency. The matrices appearing in these force expressions are defined by equations (3.37) through (3.44) which may be evaluated once the eigenfunction has been established. To complete this manipulation the series defined in equation (3.30) must be evaluated for appropriate combinations of the $\overline{\lambda}$'s and $\overline{\mu}$'s. For the particular problem considered, this operation was performed through the use of an IBM computer program in the manner described on page 79. Numerical values for the required functions are presented in Table 3. To proceed with the analysis it is necessary to specify the frequency of the applied oscillatory moment. Since only the steady state response is of interest, this permits the evaluation of
the reduced frequency and the associated Theodorsen func-
tion. For convenience in determining numerical values of
the Theodorsen function the discrete set of forcing fre-
quencies used in the calculations were chosen such as to
permit the use of the tabulated results given in refer-
ence 1. The generalized forces are then calculated by sub-
stituting in equations (3.32) through (3.34) and carrying
out the indicated matrix operations. Substituting the
values determined in this manner into equations (1.14) and
(1.15), a system of four non-homogeneous equations is ob-
tained for the unknown normal coordinates. The coefficients
appearing in these equations are complex numbers and the
system of equations must be separated into real and imagi-
ary parts for solution. Values for the normal coordinates
are then obtained by solving the resulting eight equations.
The response of any point on the beam is then established
by substitution of the normal coordinates and mode deforma-
tion shapes in equation (1.5).

This process is repeated for each selected value of
the forcing frequency. In each case the detailed calcula-
tion of the generalized forces is a long and tedious task
for hand calculation. For this reason it is recommended
that in practical applications of this method a computer
program be developed for calculating these terms. Approp-
riate input for such a program might be the eigen values
and the desired values of reduced frequency.
**Experimental test program**

Goodyear Aerospace Corporation has conducted a series of tests on simplified airmat structures in its 43 by 61 inch subsonic wind tunnel. The models tested were essentially plate-like structures mounted on a leading edge pivot. Provisions were made for applying a controlled oscillating moment to the pivot and measuring the response as a function of frequency while operating the tunnel at constant dynamic pressure. The frequency response characteristics were determined at different tunnel speeds and inflation pressures for a family of rectangular panel surfaces. The experimental data shown in Figure 12 are based on a 42 by 60 inch panel inflated to 6 psi differential pressure and operated at a nominal 100 mph airspeed.

**Discussion of results**

Figure 12 shows frequency response characteristics for the two-dimensional model oscillating in different environments. As expected, the case of oscillation in a vacuum shows peak responses near the natural frequencies of the system. The effect of adding the virtual mass terms in the zero airspeed case is to reduce the frequencies at which maximum response occurs. Also for a given driving force the amplitude of motion can be expected to be less as indicated in the figure.

A comparison of the calculated and experimentally determined response characteristics for the 100 mph flow
situation shows general agreement, at least for the limited number of points considered. The points of maximum response are well defined and agreement between theory and test is within acceptable limits for engineering use. The major area of disagreement appears to be in that the calculated values do not predict accurately the response in the heavily damped regions of the curve. This conclusion may be incorrect, however, since only a limited number of points have been studied. Complete analysis of the problem would require many more values than are presented herein. A large number of calculations would be required in order to completely define the dynamic response characteristics of the system involved. Such a program could be carried out only by using a digital computer program. It is anticipated that a computer program will be initiated in the near future by the Goodyear Aerospace Corporation.

Based on the results presented in Figure 12 it is concluded that the theory developed appears at this time to be applicable for calculating the dynamic response of two-dimensional airfoil segments where chordwise deformations are involved.

**Flutter speed determination**

Flutter, in any form, is a dynamic instability involving aerodynamic loadings and as such the dynamical equations representing the phenomenon for any system will
always have the same general form. The dynamical equations developed in this work for structures exhibiting chordwise flexibility are identical in form to those classical bending-torsion flutter representations. The two systems differ only in the detail representation of the generalized force terms. For this reason any technique suitable for calculating the flutter speed of typical surface structures may be used in calculating the flutter speeds for the class of structures considered in this work. Methods such as those discussed in reference 15 are directly applicable to solution of the flutter determinant for this problem. Since in all situations the flutter problem is a double eigen value problem the flutter determinant must normally be solved by trial and error. All solution techniques of reference 15 require evaluation of the characteristic equation resulting from expanding the flutter determinant for arbitrarily chosen values of the reduced frequency. Therefore, the detailed calculation of the coefficients appearing in the flutter determinant is directly analogous to the procedure outlined in the discussion on frequency response characteristics. In fact the only difference between the flutter problem and the subcritical forced response problem is that the former results in a set of homogeneous equations and the latter a non-homogeneous set. Because of the large number of calculations required in performing a flutter analysis, no calculations were made for the model analyzed here.
V. SUMMARY AND CONCLUSIONS

The main objective of this study was to develop a method of analysis from which the critical flutter velocity and dynamic response characteristics of a two-dimensional airfoil segment exhibiting chordwise deformations could be determined. The airfoil strip was represented as an elastic uniform free-free beam and included the effects of shearing deformation and rotary inertia. The beam deformations were developed through a superposition of free-free vibration modes and Galerkin's method used to obtain an approximate solution to the governing differential equation in terms of normal coordinates.

The assumption of uniform geometric and structural properties along the chord places certain restrictions on the class of airfoil configurations for which the representation developed can be applied. The representation however appears to be quite adequate for describing the airmat type of configuration investigation in Section IV. If the structure involved cannot be approximated as a uniform beam, certain modifications could be made to the present analysis to include correcting terms. The aerodynamic terms are unaffected by the distribution of cross-sectional
properties in the sense that as long as the deformation shape can be approximated by a superposition of circular and hyperbolic functions, the generalized forces can be used as developed.

The effect of including rotary inertia and shearing flexibility is to decrease the natural frequencies of oscillation and to increase the magnitude of the generalized masses through the added increment of displacement in the natural modes of oscillation. As shown in the sample calculations, these changes can be significant and should not be ignored. The inclusion of these effects complicates the analysis only in that the calculation of the free vibration characteristics are slightly more involved.

A two-dimensional incompressible inviscid flow was assumed throughout the aerodynamic development. The aerodynamic pressure relations were determined through evaluation of the integral equation relating the surface pressure to the downwash field resulting from oscillation of the body. In most instances the integrals were evaluated in closed form in the sense that the results were expressible as products of Bessel coefficients. Closed form expressions were not realizable for six of the integral relations involved; however, a convenient series representation was obtained in these cases. The six integrals in question were reduced to infinite series of products of Bessel coefficients. To facilitate the evaluation of these terms a
suitable IBM computer program was developed. The fact that the results are expressible only in terms of Bessel functions does not complicate the situation since in all cases the argument of these functions is fixed by the free vibration characteristics of the structure. Accordingly most of the terms appearing in the generalized force expressions may be evaluated once the structure is defined.

The theory as developed is limited to the consideration of two-dimensional airfoil segments. Provisions have been made in the development of the equations which permit external concentrated forces, such as restraints, to be applied to the body. It appears that through formulating properly a system of constraints the present analysis could be extended for use in a strip theory approach for application to finite wings. To investigate completely the regions of applicability of the technique developed herein it is recommended that a computer program be developed which will permit rapid calculation of the generalized force terms for a complete range of reduced frequencies.

By eliminating the necessity of performing the large number of required calculations by hand the frequency response characteristics can be investigated for a more complete frequency range.

Finally in summary, the general analysis developed here appears to be a valid and practical means of predicting the aeroelastic characteristics of a chordwise flexible airfoil segment.
LIST OF SYMBOLS

\( A_j, B_j, C_j, D_j \) - amplitude coefficients in downwash representation defined by equation (2.10).

\( a_j, b_j, c_j, d_j \) - amplitude coefficients in natural vibration mode shapes.

\( \hat{a}_j, \hat{b}_j, \hat{c}_j, \hat{d}_j \) - transformed amplitude coefficients in natural vibration mode shapes.

\( b \) - semi-chord of airfoil section.

\( C(k) \) - Theodorsen function.

\( e \) - x-coordinate locating concentrated forces, measured positive aft from the leading edge.

\( F(y,t) \) - concentrated transverse load.

\( \overline{F}(x,y,t) \) - fictitious applied distributed force.

\( f_j \) - \( j^{th} \) normal coordinate.

\( G^\alpha_n(\alpha,\beta) \) - functions defined by equation (3.30).

\( \xi_j \) - structural damping coefficient of the \( j^{th} \) natural mode.

\( h(t) \) - vertical translation measured at the point \( x = e \).

\( \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3 \) - integrals defined by equations (2.19), (2.20), and (2.21).

\( I_e \) - total mass moment of inertia about the line \( x = e \).
\[ I_{\nu}(z) \] - modified Bessel coefficient of the first kind with index \( \nu \).
\[ J_{\nu}(z) \] - Bessel coefficient of the first kind with index \( \nu \).
K\(_{1}\) or GAK - equivalent effective shearing stiffness.
K\(_{2}\) or EI - bending stiffness.
\( \lambda_i \) - function defined by equation (2.7).
k - reduced frequency.
l - length of beam segment.
M - bending moment.
M - total mass of beam segment.
M\(_{\text{a}}\)(y,t) - applied concentrated moment.
\( \bar{M} \) - fictitious applied distributed moment.
\( \eta \) - structural property defined by equation (1.12).
\( M_i \) - generalized mass.
M\(_{\text{a}}\) - amplitude of applied moment.
m - mass per unit length.
P(x,y,t) - transverse aerodynamic load intensity.
S\(_{\text{e}}\) - static unbalance about the point \( x = e \).
\( \Delta_{\text{e}} \) - structural property defined by equation (1.12).
t - time.
U - free stream velocity.
V - shearing force.
\( \bar{W} \) - amplitude of vertical fluid velocity.
$w$ - width of beam segment.

$x^*$ - nondimensional coordinate measured in the x-direction with origin at the mid-chord point.

$x'$ - transformed coordinate defined by equation (2.2).

$y$ - total displacement.

$y_r$ - rigid body displacement.

$y_e$ - total elastic deflection

$x, y, z$ - Cartesian coordinates.

$\alpha$ - total cross-section rotation.

$\alpha_r$ - rigid body cross-section rotation.

$\alpha_e$ - elastic rotation resulting from bending strain.

$\beta_e$ - elastic rotation resulting from shearing strain.

$\delta(x-c)$ - Dirac delta function.

$\theta$ - transformed coordinate defined by equation (2.22).

$\lambda, \mu$ - function defined by equation (2.1).

$\lambda_j, \mu_j$ - spatial frequencies or eigenfunctions of free-free vibration modes.

$\lambda_j, \tilde{\mu}_j$ - structural parameters defined by equation (2.7).

$\nu$ - mass moment of inertia per unit length.

$\tilde{\omega}_j$ - generalized force.

$\xi^*$ - nondimensional coordinate measured in the x-direction with origin at the midchord point.
\( \rho \) - air density.

\( \varphi_{y_{i}} \) - \( i^{th} \) free-free natural displacement vibration mode.

\( \varphi_{a_{i}} \) - \( i^{th} \) free-free natural bending slope mode.

\( \Phi \) - transformed coordinate defined by equation (2.22).

\( \gamma_{j} \) - structural parameter defined by equation (2.7).

\( \Omega \) - frequency of applied harmonic forces.

\( \omega \) - frequency of oscillation.

\( \omega_{j} \) - \( j^{th} \) free-free vibration natural frequency.
TABLE 2. SAMPLE PROBLEM DATA

\[ M = \int_{0}^{l} m \, dx = ml = 0.097260 \text{ slugs} \]

\[ \eta = \int_{0}^{l} \nu \, dx = \nu' l = 0.000675 \text{ slugs ft}^2 \]

\[ S_\varepsilon = \int_{0}^{l} m(x-e) \, dx = (0.24315 - 0.09726e) \text{ slugs ft} \]

\[ A_\varepsilon = \int_{0}^{l} \nu(x-e) \, dx = (0.0016875 - 0.000675e) \text{ slugs ft}^3 \]

\[ I_\varepsilon = \int_{0}^{l} m(x-e)^2 \, dx = (0.8105 - 0.48630e + 0.09726e^2) \text{ slugs ft}^2 \]

For leading edge pinned

\[ l = 5 \text{ ft} \]

width = 42 in. = 3.5 ft

thickness = 2 in.

\[ M = 0.097260 \text{ slugs} \]

\[ \eta = 0.000675 \text{ slugs ft}^2 \]

\[ S_\varepsilon = 0.24315 \text{ slugs ft} \]

\[ A_\varepsilon = 0.0016875 \text{ slugs ft}^3 \]

\[ I_\varepsilon = 0.8105 \text{ slugs ft}^2 \]

\[ K_1 = \text{GAK} = 504 \text{ lb} \]

\[ K_2 = \text{EI} = 504 \text{ lb ft}^2 \]

\[ \nu (\text{rotary inertia}) = 0.000135 \text{ slugs ft}^2/\text{ft} \]

\[ m = 0.019452 \text{ slugs/ft} \]

generalized mass; \( M_1 = 0.02245, 0.036482, 0.04029, 0.045614, 0.044556 \)
**TABLE 3. BESSEL COEFFICIENTS FOR NUMERICAL EXAMPLE**

<table>
<thead>
<tr>
<th>( \lambda_i )</th>
<th>( j_0(\lambda_i) )</th>
<th>( j_1(\lambda_i) )</th>
<th>( j_2(\lambda_i) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5384</td>
<td>.0673</td>
<td>.1874</td>
<td>.4513</td>
</tr>
<tr>
<td>4.2344</td>
<td>.3716</td>
<td>.1503</td>
<td>.3006</td>
</tr>
<tr>
<td>5.9032</td>
<td>.1230</td>
<td>.2946</td>
<td>.2228</td>
</tr>
<tr>
<td>7.5428</td>
<td>.2603</td>
<td>.1457</td>
<td>.2170</td>
</tr>
<tr>
<td>9.1583</td>
<td>.1278</td>
<td>.2238</td>
<td>.1766</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \tilde{\mu}_1 )</th>
<th>( I_0(\tilde{\mu}_1) )</th>
<th>( I_1(\tilde{\mu}_1) )</th>
<th>( I_2(\tilde{\mu}_1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.7727</td>
<td>1.9540</td>
<td>1.2832</td>
<td>.5063</td>
</tr>
<tr>
<td>2.1412</td>
<td>2.5195</td>
<td>1.8132</td>
<td>.8259</td>
</tr>
<tr>
<td>2.2615</td>
<td>2.7503</td>
<td>2.0252</td>
<td>.9592</td>
</tr>
<tr>
<td>2.3045</td>
<td>2.8390</td>
<td>2.1065</td>
<td>1.0109</td>
</tr>
<tr>
<td>2.304</td>
<td>2.8379</td>
<td>2.1054</td>
<td>1.0103</td>
</tr>
</tbody>
</table>
TABLE 4. NUMERICAL VALUES OF THE FUNCTIONS $G_n(\alpha, \beta)$ FOR NUMERICAL EXAMPLE

### $G_1' (\overline{I}_i, \overline{I}_p)$

<table>
<thead>
<tr>
<th>$\overline{I}_p$</th>
<th>2.5384</th>
<th>4.2344</th>
<th>5.9032</th>
<th>7.5428</th>
<th>9.1583</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5384</td>
<td>.38992</td>
<td>.23512</td>
<td>.00928</td>
<td>.07514</td>
<td>.00303</td>
</tr>
<tr>
<td>4.2344</td>
<td>.72024</td>
<td>.53523</td>
<td>.09946</td>
<td>.24781</td>
<td></td>
</tr>
<tr>
<td>5.9032</td>
<td></td>
<td>.88781</td>
<td>.59369</td>
<td>.01646</td>
<td></td>
</tr>
<tr>
<td>7.5428</td>
<td>.38992</td>
<td></td>
<td>.23512</td>
<td>.00928</td>
<td></td>
</tr>
<tr>
<td>9.1583</td>
<td></td>
<td>.07514</td>
<td>.00303</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### $G_2' (\overline{I}_i, \overline{I}_p)$

<table>
<thead>
<tr>
<th>$\overline{I}_p$</th>
<th>2.5384</th>
<th>4.2344</th>
<th>5.9032</th>
<th>7.5428</th>
<th>9.1583</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5384</td>
<td>.43153</td>
<td>.37079</td>
<td>.06006</td>
<td>.18632</td>
<td>.07988</td>
</tr>
<tr>
<td>4.2344</td>
<td>.60197</td>
<td>.42307</td>
<td>.02703</td>
<td>.15523</td>
<td></td>
</tr>
<tr>
<td>5.9032</td>
<td></td>
<td>.99474</td>
<td>.69101</td>
<td>.10049</td>
<td></td>
</tr>
<tr>
<td>7.5428</td>
<td></td>
<td>1.12288</td>
<td>.78033</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9.1583</td>
<td></td>
<td>.38992</td>
<td>.23512</td>
<td>.00928</td>
<td></td>
</tr>
</tbody>
</table>

### $G_3' (\overline{\mu_i}, \overline{\mu}_p)$

<table>
<thead>
<tr>
<th>$\overline{\mu}_p$</th>
<th>1.7727</th>
<th>2.1412</th>
<th>2.2615</th>
<th>2.3045</th>
<th>2.3040</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.7727</td>
<td>1.70599</td>
<td>2.04103</td>
<td>2.73780</td>
<td>2.85175</td>
<td>2.85039</td>
</tr>
<tr>
<td>2.1412</td>
<td>.350777</td>
<td>3.93979</td>
<td>4.10583</td>
<td>4.10384</td>
<td></td>
</tr>
<tr>
<td>2.2615</td>
<td>4.62708</td>
<td>4.61444</td>
<td>4.61216</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.3045</td>
<td>4.80996</td>
<td>4.80761</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.3040</td>
<td></td>
<td>4.80527</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### $G_2 (\overline{\lambda_i}, \overline{\lambda}_p)$

<table>
<thead>
<tr>
<th>$\overline{\lambda}_j$</th>
<th>2.5384</th>
<th>4.2344</th>
<th>5.9032</th>
<th>7.5428</th>
<th>9.1583</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.7727</td>
<td>1.70599</td>
<td>2.04103</td>
<td>2.73780</td>
<td>2.85175</td>
<td>2.85039</td>
</tr>
<tr>
<td>2.1412</td>
<td>.350777</td>
<td>3.93979</td>
<td>4.10583</td>
<td>4.10384</td>
<td></td>
</tr>
<tr>
<td>2.2615</td>
<td>4.62708</td>
<td>4.61444</td>
<td>4.61216</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.3045</td>
<td>4.80996</td>
<td>4.80761</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.3040</td>
<td></td>
<td>4.80527</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### $G_3 (\overline{\mu_i}, \overline{\mu}_p)$

<table>
<thead>
<tr>
<th>$\overline{\mu}_p$</th>
<th>1.7727</th>
<th>2.1412</th>
<th>2.2615</th>
<th>2.3045</th>
<th>2.3040</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.7727</td>
<td>1.70599</td>
<td>2.04103</td>
<td>2.73780</td>
<td>2.85175</td>
<td>2.85039</td>
</tr>
<tr>
<td>2.1412</td>
<td>.350777</td>
<td>3.93979</td>
<td>4.10583</td>
<td>4.10384</td>
<td></td>
</tr>
<tr>
<td>2.2615</td>
<td>4.62708</td>
<td>4.61444</td>
<td>4.61216</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.3045</td>
<td>4.80996</td>
<td>4.80761</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.3040</td>
<td></td>
<td>4.80527</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
**TABLE 5. NUMERICAL VALUES OF THE FUNCTION \( G_1(\lambda_j, \lambda_p) \)**

<table>
<thead>
<tr>
<th>( \lambda_j )</th>
<th>1.0</th>
<th>1.6</th>
<th>2.2</th>
<th>2.8</th>
<th>3.4</th>
<th>4.0</th>
<th>4.6</th>
<th>5.2</th>
<th>5.8</th>
<th>6.4</th>
<th>7.0</th>
<th>7.6</th>
<th>8.2</th>
<th>8.8</th>
<th>9.4</th>
<th>10.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>1948</td>
<td>.2550</td>
<td>.2542</td>
<td>.1963</td>
<td>.1099</td>
<td>.0037</td>
<td>.0882</td>
<td>.1315</td>
<td>.1262</td>
<td>.0799</td>
<td>.0115</td>
<td>.0546</td>
<td>.0969</td>
<td>.1034</td>
<td>.7447</td>
<td>.0223</td>
</tr>
<tr>
<td>1.6</td>
<td>.3406</td>
<td>.3523</td>
<td>.2932</td>
<td>.1843</td>
<td>.0576</td>
<td>.0529</td>
<td>.1210</td>
<td>.1351</td>
<td>.1002</td>
<td>.0318</td>
<td>.0354</td>
<td>.0864</td>
<td>.1029</td>
<td>.0823</td>
<td>.0347</td>
<td></td>
</tr>
<tr>
<td>2.2</td>
<td>.3887</td>
<td>.3623</td>
<td>.2854</td>
<td>.1800</td>
<td>.0719</td>
<td>.0160</td>
<td>.0690</td>
<td>.0830</td>
<td>.0646</td>
<td>.0275</td>
<td>.0121</td>
<td>.0406</td>
<td>.0501</td>
<td>.0403</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.8</td>
<td>.3961</td>
<td>.3905</td>
<td>.3462</td>
<td>.2698</td>
<td>.1732</td>
<td>.0719</td>
<td>.0173</td>
<td>.0852</td>
<td>.1092</td>
<td>.1028</td>
<td>.0682</td>
<td>.0185</td>
<td>.0306</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.4</td>
<td>.4764</td>
<td>.5180</td>
<td>.4972</td>
<td>.4097</td>
<td>.2685</td>
<td>.1019</td>
<td>.0543</td>
<td>.1662</td>
<td>.2116</td>
<td>.1868</td>
<td>.1070</td>
<td>.0016</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.0</td>
<td>.6164</td>
<td>.6962</td>
<td>.6403</td>
<td>.4781</td>
<td>.2697</td>
<td>.0392</td>
<td>.1507</td>
<td>.2586</td>
<td>.2674</td>
<td>.1883</td>
<td>.0566</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.2</td>
<td>.8786</td>
<td>.8286</td>
<td>.6724</td>
<td>.4469</td>
<td>.2030</td>
<td>.0081</td>
<td>.1451</td>
<td>.2008</td>
<td>.1735</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.8</td>
<td>.8843</td>
<td>.8426</td>
<td>.7115</td>
<td>.5161</td>
<td>.2933</td>
<td>.0829</td>
<td>.0809</td>
<td>.1765</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.4</td>
<td>.9441</td>
<td>.9497</td>
<td>.8196</td>
<td>.6558</td>
<td>.4013</td>
<td>.1332</td>
<td>.0976</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7.0</td>
<td>1.1034</td>
<td>1.1276</td>
<td>1.0052</td>
<td>.7538</td>
<td>.4234</td>
<td>.0840</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7.6</td>
<td>1.2801</td>
<td>1.2635</td>
<td>1.0742</td>
<td>.7511</td>
<td>.3661</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8.2</td>
<td>1.3700</td>
<td>1.2982</td>
<td>1.0632</td>
<td>.7177</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8.8</td>
<td>1.3798</td>
<td>1.3022</td>
<td>1.0814</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9.4</td>
<td>1.4190</td>
<td>1.3852</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10.0</td>
<td>1.5594</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
**TABLE 6. NUMERICAL VALUES OF THE FUNCTION G^2(λ_1, λ_2)**

<table>
<thead>
<tr>
<th>λ_2</th>
<th>1.0</th>
<th>1.6</th>
<th>2.2</th>
<th>2.8</th>
<th>3.4</th>
<th>4.0</th>
<th>4.6</th>
<th>5.2</th>
<th>5.8</th>
<th>6.4</th>
<th>7.0</th>
<th>7.6</th>
<th>8.2</th>
<th>8.8</th>
<th>9.4</th>
<th>10.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>.0264</td>
<td>.0592</td>
<td>.0913</td>
<td>.1108</td>
<td>.1098</td>
<td>.0865</td>
<td>.0460</td>
<td>.0010</td>
<td>.0419</td>
<td>.0660</td>
<td>.0677</td>
<td>.0482</td>
<td>.0151</td>
<td>.0204</td>
<td>.0468</td>
<td>.0563</td>
</tr>
<tr>
<td>1.6</td>
<td>.1330</td>
<td>.2059</td>
<td>.2519</td>
<td>.2528</td>
<td>.2041</td>
<td>.1166</td>
<td>.0130</td>
<td>.0791</td>
<td>.1360</td>
<td>.1148</td>
<td>.1073</td>
<td>.0389</td>
<td>.0366</td>
<td>.0946</td>
<td>.1177</td>
<td></td>
</tr>
<tr>
<td>2.2</td>
<td>.3213</td>
<td>.3979</td>
<td>.4075</td>
<td>.3419</td>
<td>.2155</td>
<td>.0607</td>
<td>.0822</td>
<td>.1773</td>
<td>.2038</td>
<td>.1618</td>
<td>.0715</td>
<td>.0341</td>
<td>.1198</td>
<td>.1593</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.8</td>
<td>.5023</td>
<td>.5305</td>
<td>.4702</td>
<td>.3341</td>
<td>.1565</td>
<td>.0179</td>
<td>.1471</td>
<td>.2043</td>
<td>.1846</td>
<td>.1052</td>
<td>.0008</td>
<td>.0959</td>
<td>.1500</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.4</td>
<td>.5874</td>
<td>.5613</td>
<td>.4577</td>
<td>.3015</td>
<td>.1295</td>
<td>.0200</td>
<td>.1178</td>
<td>.1510</td>
<td>.1250</td>
<td>.0602</td>
<td>.0152</td>
<td>.0744</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.0</td>
<td>.5959</td>
<td>.5663</td>
<td>.4780</td>
<td>.3180</td>
<td>.2007</td>
<td>.0620</td>
<td>.0161</td>
<td>.1100</td>
<td>.1267</td>
<td>.1035</td>
<td>.0566</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.6</td>
<td>.6374</td>
<td>.6517</td>
<td>.5988</td>
<td>.4810</td>
<td>.3153</td>
<td>.1307</td>
<td>.0366</td>
<td>.1612</td>
<td>.2166</td>
<td>.1222</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.2</td>
<td>.7802</td>
<td>.8247</td>
<td>.7634</td>
<td>.5997</td>
<td>.3634</td>
<td>.1040</td>
<td>.1226</td>
<td>.2688</td>
<td>.1735</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.8</td>
<td>.9670</td>
<td>.9831</td>
<td>.8602</td>
<td>.6208</td>
<td>.3177</td>
<td>.0193</td>
<td>.2093</td>
<td>.3242</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.4</td>
<td>1.0855</td>
<td>1.0427</td>
<td>.8617</td>
<td>.5829</td>
<td>.2690</td>
<td>.0121</td>
<td>.2054</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7.0</td>
<td>1.1083</td>
<td>1.0427</td>
<td>.8601</td>
<td>.5990</td>
<td>.3124</td>
<td>.0840</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7.6</td>
<td>1.1279</td>
<td>1.1021</td>
<td>.9514</td>
<td>.7143</td>
<td>.3661</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8.2</td>
<td>1.2407</td>
<td>1.2512</td>
<td>1.1145</td>
<td>.7177</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8.8</td>
<td>1.4255</td>
<td>1.4245</td>
<td>1.2374</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9.4</td>
<td>1.5699</td>
<td>1.5120</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10.0</td>
<td>1.6128</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda_j^{ap}$</td>
<td>1.0</td>
<td>1.3</td>
<td>1.6</td>
<td>1.9</td>
<td>2.2</td>
<td>2.5</td>
<td>2.8</td>
<td>3.1</td>
<td>3.4</td>
<td>3.7</td>
<td>4.0</td>
<td>4.3</td>
<td>4.6</td>
<td>4.9</td>
<td>5.2</td>
<td></td>
</tr>
<tr>
<td>-----------------</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td></td>
</tr>
<tr>
<td>2.2</td>
<td>3.034</td>
<td>4.186</td>
<td>5.546</td>
<td>7.188</td>
<td>0.9201</td>
<td>1.1700</td>
<td>1.4829</td>
<td>1.8775</td>
<td>2.3777</td>
<td>3.0145</td>
<td>3.8282</td>
<td>4.8710</td>
<td>6.2108</td>
<td>7.9362</td>
<td>10.1629</td>
<td></td>
</tr>
<tr>
<td>2.8</td>
<td>2.135</td>
<td>2.853</td>
<td>3.632</td>
<td>4.488</td>
<td>0.5434</td>
<td>0.6183</td>
<td>0.7647</td>
<td>0.8934</td>
<td>1.0345</td>
<td>1.1871</td>
<td>1.3488</td>
<td>1.5147</td>
<td>1.6766</td>
<td>1.8209</td>
<td>1.9267</td>
<td></td>
</tr>
<tr>
<td>3.4</td>
<td>0.766</td>
<td>0.864</td>
<td>0.936</td>
<td>0.9626</td>
<td>0.0155</td>
<td>0.0686</td>
<td>0.2046</td>
<td>0.4131</td>
<td>0.7221</td>
<td>1.1702</td>
<td>1.6096</td>
<td>2.7116</td>
<td>3.9729</td>
<td>5.7246</td>
<td>8.1143</td>
<td></td>
</tr>
<tr>
<td>4.0</td>
<td>0.658</td>
<td>0.1176</td>
<td>0.1986</td>
<td>0.3107</td>
<td>0.4997</td>
<td>0.7568</td>
<td>1.1203</td>
<td>1.6280</td>
<td>2.3133</td>
<td>3.2987</td>
<td>4.6224</td>
<td>6.4260</td>
<td>8.8756</td>
<td>12.1937</td>
<td>16.6784</td>
<td></td>
</tr>
<tr>
<td>4.6</td>
<td>1.725</td>
<td>2.671</td>
<td>0.402</td>
<td>0.5868</td>
<td>0.8465</td>
<td>1.2053</td>
<td>1.6974</td>
<td>2.3681</td>
<td>3.2781</td>
<td>4.5078</td>
<td>6.1643</td>
<td>8.3903</td>
<td>11.3757</td>
<td>15.3733</td>
<td>20.7197</td>
<td></td>
</tr>
<tr>
<td>5.2</td>
<td>2.153</td>
<td>3.220</td>
<td>0.659</td>
<td>0.6610</td>
<td>0.9253</td>
<td>1.2619</td>
<td>1.7614</td>
<td>2.4034</td>
<td>3.2602</td>
<td>4.4004</td>
<td>5.9145</td>
<td>7.9214</td>
<td>10.5781</td>
<td>14.0911</td>
<td>18.7329</td>
<td></td>
</tr>
<tr>
<td>5.8</td>
<td>1.869</td>
<td>2.277</td>
<td>0.342</td>
<td>0.5304</td>
<td>0.7223</td>
<td>0.9738</td>
<td>1.3028</td>
<td>1.7297</td>
<td>2.2183</td>
<td>3.0019</td>
<td>3.9278</td>
<td>5.1202</td>
<td>6.6528</td>
<td>8.6197</td>
<td>11.1409</td>
<td></td>
</tr>
<tr>
<td>7.0</td>
<td>0.090</td>
<td>0.0236</td>
<td>0.0503</td>
<td>0.0956</td>
<td>0.1684</td>
<td>0.2819</td>
<td>0.4517</td>
<td>0.7129</td>
<td>1.0921</td>
<td>1.6163</td>
<td>2.2433</td>
<td>3.5822</td>
<td>5.1978</td>
<td>7.4757</td>
<td>10.6703</td>
<td></td>
</tr>
<tr>
<td>8.8</td>
<td>1.638</td>
<td>2.439</td>
<td>3.519</td>
<td>0.9989</td>
<td>0.6994</td>
<td>0.9724</td>
<td>1.3433</td>
<td>1.8454</td>
<td>2.5231</td>
<td>3.4350</td>
<td>4.6685</td>
<td>6.2961</td>
<td>8.4830</td>
<td>11.3975</td>
<td>15.2747</td>
<td></td>
</tr>
<tr>
<td>9.4</td>
<td>1.084</td>
<td>1.578</td>
<td>2.218</td>
<td>3.055</td>
<td>0.1154</td>
<td>0.5595</td>
<td>0.7475</td>
<td>0.9918</td>
<td>1.3076</td>
<td>1.7137</td>
<td>2.2330</td>
<td>2.8934</td>
<td>3.7282</td>
<td>4.7772</td>
<td>4.7578</td>
<td></td>
</tr>
<tr>
<td>10.0</td>
<td>0.0204</td>
<td>0.0245</td>
<td>0.0261</td>
<td>0.0228</td>
<td>0.0113</td>
<td>0.0138</td>
<td>0.0605</td>
<td>0.1409</td>
<td>0.2730</td>
<td>0.4829</td>
<td>0.8082</td>
<td>1.3033</td>
<td>2.0457</td>
<td>3.1154</td>
<td>4.7578</td>
<td></td>
</tr>
<tr>
<td>( \lambda^2 )</td>
<td>1.0</td>
<td>1.3</td>
<td>1.6</td>
<td>1.9</td>
<td>2.2</td>
<td>2.5</td>
<td>2.8</td>
<td>3.1</td>
<td>3.4</td>
<td>3.7</td>
<td>4.0</td>
<td>4.3</td>
<td>4.6</td>
<td>4.9</td>
<td>5.2</td>
<td></td>
</tr>
<tr>
<td>-------</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>0.0312</td>
<td>0.0557</td>
<td>0.0904</td>
<td>0.1382</td>
<td>0.2036</td>
<td>0.2920</td>
<td>0.4112</td>
<td>0.5715</td>
<td>0.7867</td>
<td>1.0753</td>
<td>1.4619</td>
<td>1.9794</td>
<td>2.6721</td>
<td>3.5986</td>
<td>4.8379</td>
<td></td>
</tr>
<tr>
<td>1.6</td>
<td>0.0696</td>
<td>0.1212</td>
<td>0.2013</td>
<td>0.3076</td>
<td>0.4523</td>
<td>0.6478</td>
<td>0.9108</td>
<td>1.2637</td>
<td>1.7366</td>
<td>2.3695</td>
<td>3.2160</td>
<td>4.3472</td>
<td>5.8590</td>
<td>7.8782</td>
<td>10.5749</td>
<td></td>
</tr>
<tr>
<td>2.2</td>
<td>0.1067</td>
<td>0.1902</td>
<td>0.3076</td>
<td>0.4689</td>
<td>0.6878</td>
<td>0.9823</td>
<td>1.3773</td>
<td>1.9055</td>
<td>2.6108</td>
<td>3.5518</td>
<td>4.8062</td>
<td>6.4782</td>
<td>8.7059</td>
<td>11.6737</td>
<td>15.6271</td>
<td></td>
</tr>
<tr>
<td>2.8</td>
<td>0.1285</td>
<td>0.2283</td>
<td>0.3681</td>
<td>0.5591</td>
<td>0.8165</td>
<td>1.1609</td>
<td>1.6199</td>
<td>2.2300</td>
<td>3.0398</td>
<td>4.1141</td>
<td>5.5366</td>
<td>7.4271</td>
<td>9.9309</td>
<td>13.2504</td>
<td>17.6523</td>
<td></td>
</tr>
<tr>
<td>3.4</td>
<td>0.1255</td>
<td>0.2218</td>
<td>0.3555</td>
<td>0.5362</td>
<td>0.7771</td>
<td>1.0956</td>
<td>1.5151</td>
<td>2.0662</td>
<td>2.7894</td>
<td>3.7378</td>
<td>4.9817</td>
<td>6.6132</td>
<td>8.7537</td>
<td>11.5633</td>
<td>15.2528</td>
<td></td>
</tr>
<tr>
<td>4.0</td>
<td>0.0958</td>
<td>0.1676</td>
<td>0.2652</td>
<td>0.3941</td>
<td>0.5613</td>
<td>0.7764</td>
<td>1.0512</td>
<td>1.4014</td>
<td>1.8466</td>
<td>2.4119</td>
<td>3.1296</td>
<td>4.0403</td>
<td>5.1961</td>
<td>6.6631</td>
<td>8.5257</td>
<td></td>
</tr>
<tr>
<td>4.6</td>
<td>0.0462</td>
<td>0.0780</td>
<td>0.1178</td>
<td>0.1651</td>
<td>0.2187</td>
<td>0.2766</td>
<td>0.3354</td>
<td>0.3899</td>
<td>0.4316</td>
<td>0.4749</td>
<td>0.5200</td>
<td>0.5653</td>
<td>0.6097</td>
<td>0.6634</td>
<td>0.8977</td>
<td></td>
</tr>
<tr>
<td>5.2</td>
<td>0.0102</td>
<td>0.0233</td>
<td>0.0476</td>
<td>0.0898</td>
<td>0.1522</td>
<td>0.2694</td>
<td>0.4387</td>
<td>0.6932</td>
<td>1.0685</td>
<td>1.6139</td>
<td>2.3968</td>
<td>3.5100</td>
<td>5.0801</td>
<td>7.2796</td>
<td>10.3341</td>
<td></td>
</tr>
<tr>
<td>5.8</td>
<td>0.0581</td>
<td>0.1087</td>
<td>0.1856</td>
<td>0.2999</td>
<td>0.4671</td>
<td>0.7082</td>
<td>1.0521</td>
<td>1.5394</td>
<td>2.2230</td>
<td>3.1765</td>
<td>4.4988</td>
<td>6.3240</td>
<td>8.8329</td>
<td>12.2697</td>
<td>16.9634</td>
<td></td>
</tr>
<tr>
<td>6.4</td>
<td>0.0847</td>
<td>0.1549</td>
<td>0.2586</td>
<td>0.4078</td>
<td>0.6197</td>
<td>0.9173</td>
<td>1.3320</td>
<td>1.9063</td>
<td>2.6971</td>
<td>3.7809</td>
<td>5.2604</td>
<td>7.2729</td>
<td>10.0023</td>
<td>13.6946</td>
<td>18.6784</td>
<td></td>
</tr>
<tr>
<td>7.0</td>
<td>0.0835</td>
<td>0.1511</td>
<td>0.2189</td>
<td>0.3869</td>
<td>0.5789</td>
<td>0.8433</td>
<td>1.2019</td>
<td>1.6965</td>
<td>2.3616</td>
<td>3.2578</td>
<td>4.4610</td>
<td>6.0718</td>
<td>8.2226</td>
<td>11.0879</td>
<td>14.8979</td>
<td></td>
</tr>
<tr>
<td>7.6</td>
<td>0.0569</td>
<td>0.1014</td>
<td>0.1642</td>
<td>0.2501</td>
<td>0.3658</td>
<td>0.5200</td>
<td>0.7232</td>
<td>0.9896</td>
<td>1.3363</td>
<td>1.7819</td>
<td>2.3625</td>
<td>3.1021</td>
<td>4.0146</td>
<td>5.2401</td>
<td>6.7469</td>
<td></td>
</tr>
<tr>
<td>8.2</td>
<td>0.0144</td>
<td>0.0237</td>
<td>0.0343</td>
<td>0.0449</td>
<td>0.0533</td>
<td>0.0558</td>
<td>0.0666</td>
<td>0.0746</td>
<td>0.0928</td>
<td>0.1297</td>
<td>0.1729</td>
<td>0.2353</td>
<td>0.3036</td>
<td>0.4245</td>
<td>0.5115</td>
<td></td>
</tr>
<tr>
<td>8.8</td>
<td>0.0297</td>
<td>0.0564</td>
<td>0.0980</td>
<td>0.1617</td>
<td>0.2574</td>
<td>0.3997</td>
<td>0.6087</td>
<td>0.9132</td>
<td>1.3530</td>
<td>1.9260</td>
<td>2.8833</td>
<td>4.1561</td>
<td>5.9559</td>
<td>8.1802</td>
<td>12.0101</td>
<td></td>
</tr>
<tr>
<td>9.4</td>
<td>0.0615</td>
<td>0.1134</td>
<td>0.1912</td>
<td>0.3050</td>
<td>0.4694</td>
<td>0.7019</td>
<td>1.0396</td>
<td>1.5124</td>
<td>2.1770</td>
<td>3.1062</td>
<td>4.2006</td>
<td>5.9966</td>
<td>8.6803</td>
<td>12.1046</td>
<td>16.8126</td>
<td></td>
</tr>
<tr>
<td>10.0</td>
<td>0.0715</td>
<td>0.1307</td>
<td>0.2176</td>
<td>0.3427</td>
<td>0.5203</td>
<td>0.7702</td>
<td>1.1197</td>
<td>1.6055</td>
<td>2.2778</td>
<td>3.2045</td>
<td>4.4771</td>
<td>6.2193</td>
<td>8.5973</td>
<td>11.8348</td>
<td>16.2321</td>
<td></td>
</tr>
</tbody>
</table>
## TABLE 9. NUMERICAL VALUES OF THE FUNCTION $G_1^2(\mu, \bar{\mu})$

<table>
<thead>
<tr>
<th>$\mu_1$</th>
<th>1.0</th>
<th>1.3</th>
<th>1.6</th>
<th>1.9</th>
<th>2.2</th>
<th>2.5</th>
<th>2.8</th>
<th>3.1</th>
<th>3.4</th>
<th>3.7</th>
<th>4.0</th>
<th>4.3</th>
<th>4.6</th>
<th>4.9</th>
<th>5.2</th>
<th>5.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.320</td>
<td>0.254</td>
<td>0.219</td>
<td>0.153</td>
<td>1.101</td>
<td>1.453</td>
<td>1.914</td>
<td>2.518</td>
<td>3.312</td>
<td>4.358</td>
<td>5.73</td>
<td>7.56</td>
<td>9.97</td>
<td>13.16</td>
<td>17.39</td>
<td>22.99</td>
</tr>
<tr>
<td>1.3</td>
<td>0.643</td>
<td>0.580</td>
<td>0.518</td>
<td>0.421</td>
<td>2.079</td>
<td>2.672</td>
<td>3.300</td>
<td>5.023</td>
<td>6.636</td>
<td>8.770</td>
<td>11.59</td>
<td>15.33</td>
<td>20.30</td>
<td>26.88</td>
<td>35.62</td>
<td>47.24</td>
</tr>
<tr>
<td>1.6</td>
<td>2.193</td>
<td>2.831</td>
<td>3.799</td>
<td>5.173</td>
<td>6.855</td>
<td>9.080</td>
<td>12.028</td>
<td>15.93</td>
<td>21.12</td>
<td>26.01</td>
<td>37.16</td>
<td>49.34</td>
<td>66.52</td>
<td>89.04</td>
<td>123.06</td>
<td></td>
</tr>
<tr>
<td>1.9</td>
<td>3.930</td>
<td>5.243</td>
<td>6.975</td>
<td>9.268</td>
<td>12.307</td>
<td>16.342</td>
<td>21.70</td>
<td>28.82</td>
<td>38.30</td>
<td>50.91</td>
<td>67.69</td>
<td>90.04</td>
<td>123.06</td>
<td>167.57</td>
<td>227.61</td>
<td></td>
</tr>
<tr>
<td>2.2</td>
<td>7.014</td>
<td>9.357</td>
<td>12.465</td>
<td>16.595</td>
<td>22.086</td>
<td>29.39</td>
<td>39.12</td>
<td>52.08</td>
<td>69.35</td>
<td>92.23</td>
<td>123.06</td>
<td>167.57</td>
<td>227.61</td>
<td>308.61</td>
<td>432.82</td>
<td></td>
</tr>
<tr>
<td>2.5</td>
<td>12.516</td>
<td>16.717</td>
<td>22.308</td>
<td>29.757</td>
<td>39.68</td>
<td>52.92</td>
<td>70.58</td>
<td>95.15</td>
<td>125.59</td>
<td>167.57</td>
<td>227.61</td>
<td>308.61</td>
<td>432.82</td>
<td>569.01</td>
<td>827.31</td>
<td></td>
</tr>
<tr>
<td>2.8</td>
<td>22.380</td>
<td>29.934</td>
<td>40.015</td>
<td>53.47</td>
<td>71.44</td>
<td>95.45</td>
<td>127.52</td>
<td>170.36</td>
<td>227.61</td>
<td>308.61</td>
<td>432.82</td>
<td>569.01</td>
<td>827.31</td>
<td>1167.82</td>
<td>1509.35</td>
<td></td>
</tr>
<tr>
<td>3.1</td>
<td>40.12</td>
<td>53.743</td>
<td>71.95</td>
<td>96.30</td>
<td>128.87</td>
<td>172.43</td>
<td>230.68</td>
<td>308.61</td>
<td>432.82</td>
<td>569.01</td>
<td>827.31</td>
<td>1167.82</td>
<td>1509.35</td>
<td>2037.53</td>
<td>2524.25</td>
<td></td>
</tr>
<tr>
<td>3.4</td>
<td>72.122</td>
<td>96.73</td>
<td>129.67</td>
<td>173.79</td>
<td>232.85</td>
<td>311.94</td>
<td>417.82</td>
<td>569.01</td>
<td>827.31</td>
<td>1167.82</td>
<td>1509.35</td>
<td>2037.53</td>
<td>2524.25</td>
<td>3189.82</td>
<td>4019.53</td>
<td></td>
</tr>
<tr>
<td>$\mu_1$</td>
<td>1.0</td>
<td>1.3</td>
<td>1.6</td>
<td>1.9</td>
<td>2.2</td>
<td>2.5</td>
<td>2.8</td>
<td>3.1</td>
<td>3.4</td>
<td>3.7</td>
<td>4.0</td>
<td>4.3</td>
<td>4.6</td>
<td>4.9</td>
<td>5.2</td>
<td>5.5</td>
</tr>
<tr>
<td>---------</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
<td>$g^{2}(\mu_1, \mu_1)$</td>
<td>0.036</td>
<td>0.066</td>
<td>0.107</td>
<td>0.164</td>
<td>0.242</td>
<td>0.348</td>
<td>0.491</td>
<td>0.683</td>
<td>0.943</td>
<td>1.291</td>
<td>1.759</td>
<td>2.38</td>
<td>3.22</td>
<td>4.35</td>
<td>5.86</td>
<td>7.88</td>
</tr>
<tr>
<td></td>
<td>0.118</td>
<td>0.191</td>
<td>0.294</td>
<td>0.433</td>
<td>0.623</td>
<td>0.880</td>
<td>1.227</td>
<td>1.693</td>
<td>2.320</td>
<td>3.162</td>
<td>4.29</td>
<td>5.80</td>
<td>7.83</td>
<td>10.56</td>
<td>14.21</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.311</td>
<td>0.478</td>
<td>0.706</td>
<td>1.016</td>
<td>1.436</td>
<td>2.002</td>
<td>2.764</td>
<td>3.791</td>
<td>5.170</td>
<td>7.02</td>
<td>9.50</td>
<td>12.84</td>
<td>17.31</td>
<td>23.30</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>6.696</td>
<td>9.369</td>
<td>12.984</td>
<td>17.866</td>
<td>24.451</td>
<td>33.32</td>
<td>45.26</td>
<td>61.34</td>
<td>82.95</td>
<td>112.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>13.122</td>
<td>18.203</td>
<td>25.071</td>
<td>34.343</td>
<td>46.84</td>
<td>63.70</td>
<td>86.39</td>
<td>116.92</td>
<td>158.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>25.276</td>
<td>34.816</td>
<td>47.779</td>
<td>65.23</td>
<td>88.76</td>
<td>120.51</td>
<td>163.24</td>
<td>220.76</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>48.086</td>
<td>65.995</td>
<td>90.19</td>
<td>122.85</td>
<td>166.89</td>
<td>226.26</td>
<td>306.23</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>90.656</td>
<td>124.00</td>
<td>169.05</td>
<td>229.86</td>
<td>311.87</td>
<td>422.43</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>169.76</td>
<td>231.64</td>
<td>315.21</td>
<td>428.02</td>
<td>580.18</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>316.32</td>
<td>430.79</td>
<td>585.41</td>
<td>794.11</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>587.14</td>
<td>798.46</td>
<td>1083.88</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1086.62</td>
<td>1476.66</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2006.63</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 1. Theoretical Model

Figure 2. Free Body Diagram of Beam Segment
Figure 3. Rigid Body Displacements and Elastic Deformation Representation

Figure 4. Coordinate Transformation between $x$ and $x'$ Coordinate System
Figure 5. Change in the Natural Frequency of a Vibrating Free-Free Beam Due to Shear and Rotary Inertia
Figure 6. Graphical Solution of Transcendental Equation of a Free-Free Timoshenko Beam
Timoshenko Beam Theory
in cycles per second
--- Classical Beam Theory
in cycles per second

Mode #1
\[ \omega = 4.26 \]
\[ \omega_0 = 4.45 \]

Mode #2
\[ \omega = 6.03 \]
\[ \omega_0 = 7.85 \]

Mode #3
\[ \omega = 7.38 \]
\[ \omega_0 = 11.00 \]

Mode #4
\[ \omega = 8.47 \]
\[ \omega_0 = 14.13 \]

Mode #5
\[ \omega = 9.42 \]
\[ \omega_0 = 12.30 \]

Figure 7. Comparison of Free-Free Mode Shapes with and without Rotary Inertia and Shear Deformations Included
Figure 8. Theoretical Model for Sample Problem

Figure 9. Airmat Structural Configuration
Figure 10. Side View of Experimental Model
Figure 11. Plane View of Experimental Model
Theoretical Frequency Response Curves

- Forced oscillation in a vacuum
- Forced oscillation in air (zero airspeed)
- Forced oscillation in air (airspeed $U = 100$ mph)

Experimental Frequency Response Curve

- Forced oscillation in air (airspeed $U = 100$ mph)

$10 \log_{10} \frac{\bar{\sigma}(\omega)}{\bar{f}}$

Figure 12. Dynamic Response Characteristics of an Inflatable Structure
APPENDIX A

NATURAL MODE SHAPES AND FREQUENCIES FOR A UNIFORM FREE-FREE TIMOSHENKO BEAM

The differential equations and boundary conditions governing the motion of a uniform free-free beam, when the effects of rotary inertia and shearing deformations are included, may be written as (see reference 2)

\[ m \ddot{y} - GAK(y'' - \alpha') = 0 \]  \hspace{1cm} (A.1)

\[ EI \alpha'' + GAK(y' - \alpha) - y'' \ddot{\alpha} = 0 \]

where

\[ \alpha' = \frac{M}{EI}, \quad \beta = \frac{V}{GAK} \quad \text{and} \quad y' = \alpha + \beta \]

The appropriate boundary conditions for the free-free beam are

\[ \alpha'(0,t) = 0 \quad \alpha'(l,t) = 0 \]

\[ y'(0,t) - \alpha(0,t) = 0 \quad y'(l,t) - \alpha(l,t) = 0 \]  \hspace{1cm} (A.2)

In the above equations \( \alpha \) and \( \beta \) represent the rotations due to bending and shearing strains respectively; \( m \) and \( v \) are the mass and rotary inertia per unit length of the beam.
EI and GAK are the effective bending and shearing stiffness constants.

The free oscillation solution to the above system of equations may be obtained using the displacement functions

\[ y(x,t) = \phi_y(x)e^{i\omega t} \quad \text{and} \quad \alpha(x,t) = \phi_\alpha(x)e^{i\omega t} \]

where \( \omega \) is the time dependent circular frequency of vibration. Because of the linearity of equation (A.1), a solution may be obtained by setting \( \phi_y = \lambda e^{\lambda x} \) and \( \phi_\alpha = B e^{\lambda x} \) where \( \lambda \) is an eigenvalue for the space deformation function, and \( A \) and \( B \) are arbitrary constants. Substituting these forms into the governing equations gives

\[ -\nu \omega^2 A - K_1 (\lambda^2 A - \lambda B) = 0 \]  
\[ K_2 \lambda^2 B + K_1 (\lambda A - B) + \nu \omega B = 0 \]  

where

\[ K_1 = GAK \quad \text{and} \quad K_2 = EI \]

The requirement for the existence of a non-trivial solution for \( A \) and \( B \) leads to the biquadratic equation

\[ \lambda^4 + \left( \frac{K_1 \nu \omega^2 + m K_2 \omega^2}{K_1 K_2} \right) \lambda^2 + \left( \frac{m \omega^4 - m K_1 \omega^2}{K_1 K_2} \right) = 0 \]

and to the following relation between \( A \) and \( B \)

\[ B = \frac{m \omega^2 + K_1 \lambda^2}{K_1 \lambda} A \]
Solving the characteristic equation for the roots gives

\[ \lambda = \pm \mu \quad \text{and} \quad \lambda = \pm i \nu \]

where

\[ \mu = \frac{\omega}{\sqrt{2}} \left\{ \frac{1}{K_1 K_2} \left[ (v K_1 + m K_2) + \frac{1}{K_1 K_2} \sqrt{(v K_1 - m K_2)^2 + 4 m K_1^2 K_2^2 \omega^2} \right] \right\}^{1/2} \]

\[ \lambda = \frac{\omega}{\sqrt{2}} \left\{ \frac{1}{K_1 K_2} \left[ (v K_1 + m K_2) + \frac{1}{K_1 K_2} \sqrt{(v K_1 - m K_2)^2 + 4 m K_1^2 K_2^2 \omega^2} \right] \right\}^{1/2} \]  

(A.4)

The general solution for the free simple harmonic vibration mode shapes then becomes

\[ \Phi_\nu(x) = C_1 \cosh \mu x + C_2 \sinh \mu x + C_3 \cos \nu x + C_4 \sin \nu x \]

(A.5)

\[ \Phi_\nu(x) = \frac{m \omega^2 + K_1 \mu^2}{K_1 \mu} (C_1 \cosh \mu x + C_2 \sinh \mu x) + \frac{m \omega^2 - K_1 \nu^2}{K_1 \nu} (C_3 \sin \nu x - C_4 \cos \nu x) \]

The constants \( C_1, C_2, C_3 \) and \( C_4 \) are arbitrary functions of the circular frequency, \( \omega \), and must be determined from the boundary conditions of the governing system. Substitution into the boundary conditions (A.2) leads to a set of four homogeneous equations for the determination of the constants. The existence of a non-trivial solution requires the vanishing of the coefficient determinant, which in turn leads to a discrete set of natural frequencies of
oscillation \( \omega_j \), and allowable spatial mode shapes \( \Phi_j \) and \( \Phi_{\text{ij}} \). The transcendental equation resulting from the applications of the boundary conditions may be written as

\[
\frac{(\frac{\lambda}{\mu})^2 - \gamma^2}{2 \gamma (\frac{\lambda}{\mu})} = \frac{1 - \cosh \mu \lambda \cos \lambda l}{\sinh \mu \lambda \sin \lambda l}
\]

(A.6)

where

\[
\gamma = \left( \frac{m \omega^2 + k \mu^2}{m \omega^2 - k \mu^2} \right)
\]

or

\[
\frac{1 - \zeta}{2 \zeta} = \frac{1 - \cosh \mu \lambda \cos \lambda l}{\sinh \mu \lambda \sin \lambda l} \quad \text{with} \quad \zeta = \gamma \left( \frac{\lambda}{\mu} \right)
\]

The solutions to this equation for the \( \omega_j \)'s permit the evaluation of the spacial frequencies \( \mu_i \) and \( \lambda_j \) along with the parameters \( \gamma_i \) and \( \zeta_i \). For \( \omega = \omega_j \) the set of homogeneous equations resulting from the application of the boundary conditions may be solved for the relative magnitudes of the amplitude coefficients. Substitution of these results into equation (A.5) gives for the free-free vibration mode shape the two relations
\[ \Phi_j(x) = C \left\{ \frac{\cos\lambda_j \cdot \cosh\mu_j x}{\sinh\mu_j x - \tau_j \sinh\lambda_j x} \right\} \left[ \cosh\mu_j x - \Psi_j \cos\lambda_j x \right] \]

\[ + \left[ \sinh\mu_j x + \frac{\lambda_j}{\mu_j} \sin\lambda_j x \right] \]  \hspace{2cm} (A.7)

and

\[ \Phi_{\alpha}(x) = C \left\{ \frac{m\omega_j^2 + K_i\mu_j^2}{K_i \mu_j} \left[ \cosh\mu_j x + \frac{\cos\lambda_j \cdot \cosh\mu_j x}{\sinh\mu_j x - \tau_j \sinh\lambda_j x} \right] \right\} \]

\[ - \frac{m\omega_j^2 - K_i\mu_j^2}{K_i \mu_j} \left[ \cos\lambda_j x + \frac{\cos\lambda_j \cdot \cosh\mu_j x}{\sinh\mu_j x - \tau_j \sinh\lambda_j x} \right] \]  \hspace{2cm} (A.8)

The orthogonality relation between the natural modes, equations (A.7) and (A.8), may be developed in a manner analogous to that presented in reference 14 for the case of a Timoshenko beam with a concentrated mass. Since \( \Phi_j \) and \( \Phi_{\alpha} \) are the natural modes corresponding to the natural frequency \( \omega_j \), the differential equation (A.1) must be satisfied identically for any of the infinite number of modes. For the \( i^{\text{th}} \) mode the equations become

\[ -\omega_i^2 m \Phi_i'' + K_i (\Phi_i'' - \Phi_i') = 0 \]  \hspace{2cm} (A.9)

and

\[ K_i \Phi_i'' + K_i (\Phi_i' - \Phi_i') + \omega_i^2 \nu \Phi_i = 0 \]
Multiplying the first of these equations by \( \Psi_{y_j} \), the second by \( \Psi_{y_j} \) where \( j \neq i \), adding the results, and integrating over the length of the beam gives

\[
\omega_i^2 \int_0^L \left( m \frac{\partial^2 \Psi_{y_j}}{\partial t^2} + \nu \frac{\partial \Psi_{\alpha_i}}{\partial t} \frac{\partial \Psi_{\alpha_j}}{\partial t} \right) \, dx = -K_i \int_0^L \left( \frac{\partial^4 \Psi_{\alpha_i}}{\partial t^4} - \frac{\partial^4 \Psi_{\alpha_i}}{\partial t^2} \right) \, dx
\]

Integrating by parts the last two terms appearing on the right hand side of this equation reduces the order of the derivatives involved. Performing the necessary integrations gives

\[
\omega_i^2 \int_0^L \left( m \frac{\partial \Psi_{y_j}}{\partial t} + \nu \frac{\partial \Psi_{\alpha_i}}{\partial t} \frac{\partial \Psi_{\alpha_j}}{\partial t} \right) \, dx = -K_i \int_0^L \left( \frac{\partial^3 \Psi_{\alpha_i}}{\partial t^3} - \frac{\partial^3 \Psi_{\alpha_i}}{\partial t^2} \right) \, dx
\]

\[
+ K_i \left\{ - \frac{\partial \Psi_{y_j}}{\partial t} \left. \left( \frac{\partial^3 \Psi_{y_i}}{\partial t^2} - \frac{\partial^3 \Psi_{y_i}}{\partial t^2} \right) \right|_0^L + \int_0^L \frac{\partial^2 \Psi_{y_i}}{\partial t^2} \left( \frac{\partial \Psi_{y_i}}{\partial t} - \frac{\partial \Psi_{\alpha_i}}{\partial t} \right) \, dx \right\}
\]

\[
+ K_i \left\{ - \frac{\partial \Psi_{\alpha_j}}{\partial t} \left. \frac{\partial \Psi_{\alpha_i}}{\partial t} \right|_0^L + \int_0^L \frac{\partial \Psi_{\alpha_i}}{\partial t} \frac{\partial \Psi_{\alpha_j}}{\partial t} \, dx \right\}
\]

(A.10)

Since \( i \) and \( j \) play the same role in the above equations, they may be interchanged to obtain an analogous expression for the \( j^{\text{th}} \) modal expression multiplied by the \( i^{\text{th}} \) shape. Performing the indicated operation and subtracting the results from equation (A.10) gives
The boundary conditions applicable to the free beam, equation (A.2), require that the right hand side of this equation be identically zero. Therefore the orthogonality condition satisfied by the natural vibration modes of a uniform free-free Timoshenko beam is given by

\[
(\omega_i^2 - \omega_s^2) \int_0^l \left( m \phi_{y_i} \phi_{y_j} + \nu \phi_{\alpha_i} \phi_{\alpha_j} \right) dx = -K_1 \left[ \phi_{\alpha_i} (\phi_{\alpha_j} - \phi_{\alpha_j}) \right] + K_2 \left[ \phi_{\alpha_j} (\phi_{\alpha_j} - \phi_{\alpha_j}) \right] + \delta_{ij}
\]

where \( i \neq j \).

Also it should be noted that in the absence of external forces the summation of inertia forces in the \( y \)-direction and the summation of inertia moments about the \( z \)-axis must be zero. The relations expressing these requirements may be obtained through integration of the modal equations of motion (A.9) and application of the free-free boundary conditions. Carrying out the necessary operations gives:

\[
\int_0^l \left( m \phi_{y_i} \phi_{y_j} + \nu \phi_{\alpha_i} \phi_{\alpha_j} \right) dx = 0 \quad \text{if} \quad i \neq j
\]

\[
\int_0^l m \phi_{y_i} \phi_{y_j} dx = 0
\]

\[
\int_0^l (\nu m \phi_{y_i} + \nu \phi_{\alpha_i}) dx = 0
\]
APPENDIX B

EXPANSION OF $\cos^n\theta$ IN TERMS OF $\cos(m\theta)$

In the development of the theoretical expressions contained in the main body of this dissertation frequent use is made of the expansion of the $\cos^n\theta$, where $n$ takes on integral values. The necessary equations will be established in this appendix. By generating the expansion of $\cos^n\theta$ for definite values of the power, $n$, it appears that the expansion for a general power will have one of the following forms, depending on whether $n$ is even or odd.

\[
\cos^n\theta = \frac{i}{2^{n-1}} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{(n-k)!k!} \cos((n-2k)\theta) \quad n\text{-odd}
\]

\[
\cos^n\theta = \frac{i}{2^{n-1}} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{(n-k)!k!} \cos((n-2k)\theta) - \frac{i}{2^n} \left( \frac{n!}{n/2!n/2!} \right) \quad n\text{-even}
\]

The validity of these relations will be shown by the use of a proof by induction. That is, assuming the equations to hold for any integer value, $n$, and showing the truth for $n = 1, 2, 3, 4$, then generation of the $(n+1)$ term by multiplying the $\cos^n\theta$ by $\cos\theta$ will constitute a proof.
Direct substitution of \( n = 1, 2, 3, 4 \) into the above expansions gives the following results known to be valid.

\[
\cos \theta = \cos \theta \\
\cos^2 \theta = \frac{1}{2} (\cos 2\theta + 1) \\
\cos^3 \theta = \frac{1}{4} (\cos 3\theta + 3 \cos \theta) \\
\cos^4 \theta = \frac{1}{8} (\cos 4\theta + 4 \cos 2\theta + 3)
\]

Assuming \( n \) to be odd we have

\[
\cos^n \theta = \frac{1}{2^n} \sum_{k=0}^{\frac{n-1}{2}} \frac{n!}{(n-k)!k!} \cos(n-2k)\theta
\]

Multiplying by the \( \cos \theta \) gives

\[
\cos^{n+1} \theta = \frac{1}{2^n} \sum_{k=0}^{\frac{n-1}{2}} \frac{n!}{(n-k)!k!} \cos(n-2k)\theta \cos \theta
\]

Now making the substitution

\[
\cos \alpha \cos \beta = \frac{1}{2} \cos (\alpha + \beta) + \frac{1}{2} \cos (\alpha - \beta)
\]

provides

\[
\cos^{n+1} \theta = \frac{1}{2^n} \sum_{k=0}^{\frac{n-1}{2}} \frac{n!}{(n-k)!k!} \left[ \cos(n+1-2k)\theta \cos(n-1-2k)\theta + \cos(n+1-2k)\theta \cos(n-1-2k)\theta \right]
\]

or

\[
\cos^{n+1} \theta = \frac{1}{2^n} \sum_{k=0}^{\frac{n-1}{2}} \frac{n!}{(n-k)!k!} \cos(n+1-2k)\theta + \frac{1}{2^n} \sum_{k=0}^{\frac{n-1}{2}} \frac{n!}{(n-k)!k!} \cos(n-1-2k)\theta
\]

In the second sum, let \( k = j - 1 \) thus giving
\[
\cos^{n+1} \theta = \frac{1}{2^n} \sum_{k=0}^{\frac{n+1}{2}} \frac{n!}{(n-k)! k!} \cos(n+1-2k)\theta + \frac{1}{2^n} \sum_{k=0}^{\frac{n+1}{2}} \frac{n!}{(n+1-k)! k!} \cos(n+1-2k)\theta
\]

Now in the first sum, add and subtract the next higher term in the running index \( k \) (i.e., \( k = \frac{n+1}{2} + 1 \)) to obtain

\[
\cos^{n+1} \theta = \frac{1}{2^n} \sum_{k=0}^{\frac{n+1}{2}} \frac{n!}{(n-k)! k!} \cos(n+1-2k)\theta - \frac{1}{2^n} \frac{n!}{(n-\frac{1}{2})(n-\frac{3}{2})!} \\
+ \frac{1}{2^n} \sum_{k=0}^{\frac{n+1}{2}} \frac{n!}{(n+1-k)! k!} \cos(n+1-2k)\theta
\]

But

\[
\frac{1}{(n-k)!} + \frac{k}{(n+1-k)!} = \frac{n+1}{(n+1-k)!}
\]

hence, combining the two summations leads to

\[
\cos^{n+1} \theta = \frac{1}{2^n} \sum_{k=0}^{\frac{n+1}{2}} \frac{(n+1)!}{(n+1-k)! k!} \cos(n+1-2k)\theta - \frac{1}{2^n} \frac{(n+1)!}{(n+1)(n+1)!} (\frac{1}{2})^{n+1}
\]

(B.2)

Note that this is the second of equations (B.1) which has been assumed to be valid for even integers. Since in the above development, \( n \) has been taken as an odd integer, \( (n+1) \) is an even integer. To continue the proof, multiply equation (B.2) by \( \cos \theta \) to generate \( \cos^{n+2} \theta \), where \( n \) is still taken to be odd.
This gives

\[ \cos^{n+2} \theta = \frac{1}{2^n} \sum_{k=0}^{\frac{n+1}{2}} \frac{(n+1)!}{(n+1-k)! k!} \cos(n+1-2k) \theta \cos \theta - \frac{1}{2^{n+1}} \frac{(n+1)!}{(\frac{n+1}{2})! (\frac{n+1}{2})!} \cos \theta \]

\[ = \frac{1}{2^{n+1}} \sum_{k=0}^{\frac{n+1}{2}} \frac{(n+1)!}{(n+1-k)! k!} \left[ \cos(n+2-2k) \theta + \cos(n-2k) \theta \right] - \frac{1}{2^{n+1}} \frac{(n+1)!}{(\frac{n+1}{2})! (\frac{n+1}{2})!} \cos \theta \]

In the second term within the summation let \( j = k + 1 \) to obtain

\[ \cos^{n+2} \theta = \frac{1}{2^n} \sum_{k=0}^{\frac{n+1}{2}} \frac{(n+1)!}{(n+1-k)! k!} \cos(n+2-2k) \theta + \frac{1}{2^n} \sum_{k=0}^{\frac{n+1}{2}} \frac{(n+1)!}{(n+2-k)! k!} \cos(n+2-2k) \theta \]

\[ - \frac{1}{2^{n+1}} \frac{(n+1)!}{(\frac{n+1}{2})! (\frac{n+1}{2})!} \cos \theta \]

Collecting terms under a single summation sign and using the identity

\[ \frac{1}{(n+1-k)!} + \frac{k}{(n+2-k)!} = \frac{n+2}{(n+2-k)!} \]

gives

\[ \cos^{n+2} \theta = \frac{1}{2^n} \sum_{k=0}^{\frac{n+1}{2}} \frac{(n+2)!}{(n+2-k)! k!} \cos(n+2-2k) \theta \]  \hspace{1cm} (B.3)
Since \( n \) is odd, \( (n+2) \) is also odd and equation (B.3) establishes the truth of the first of equations (B.1), as can be seen by replacing \( (n+2) \) by \( n \) in the above relation.

\[
\cos^n \theta = \sum_{k=0}^{\frac{n-1}{2}} \frac{n!}{(n-k)!k!} \cos^{(n-2k)} \theta
\]

The proof of (B.3) guarantees the truth of (B.2) and the proof by induction is complete.
APPENDIX C

DERANGEMENT OF TERMS IN THE SERIES

\[ \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^j}{k!} \frac{2^{j+k}}{(2j+1)(2j+1)!!} \sin(2j+1-2k) \phi \]

(C.1)

A finite sum has the same value, no matter how the terms of the sum are arranged. This, however, is not necessarily the case in an infinite sum containing both positive and negative terms. An example of this can be found in the convergent series

\[ S = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \cdots \]

which converges to a positive value greater than \( \frac{1}{2} \). Derangement of the series to the form

\[ t = \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \cdots \]

can be shown to converge to a value of one half the original sum \( s \).

Consequently, if a derangement of the above infinite sum (C.1) is to be employed, the equality of the sum of
this series with the alternate form must be established.
It is also desired that any resulting series converge uni­
formly in the variable \( \phi \) in order to permit term by term
integration when integration with respect to \( \phi \) is neces­
sary.

To facilitate the derangement let \( m = 2j + 1 \) and note
that \( m \) is always an odd integer. Hence

\[
\sum_{j=0}^{\infty} \sum_{k=0}^{2j+1} \frac{(-1)^j j!}{2^{2j}(2j+1-k)!} \frac{\sin(2j+1-2k) \phi}{k!} \sum_{m \geq 0}^{\infty} \sum_{k=0}^{m} \frac{(-1)^{m-k} (m-k)!}{2^{m-(m-k)!} k!} \sin(m-2k) \phi
\]

the right hand series is the sum of the finite sums

\[
S_1 = \lambda (\sin \phi) \\
S_3 = -\frac{\lambda^3}{2^3} \left( \frac{\sin 3 \phi}{3!} + \frac{\sin \phi}{2!} \right) \\
S_5 = \frac{\lambda^5}{2^5} \left( \frac{\sin 5 \phi}{5!} + \frac{\sin 3 \phi}{4!} + \frac{\sin \phi}{3! 2!} \right) \\
S_7 = -\frac{\lambda^7}{2^7} \left( \frac{\sin 7 \phi}{7!} + \frac{\sin 5 \phi}{6!} + \frac{\sin 3 \phi}{5! 2!} + \frac{\sin \phi}{4! 3!} \right)
\]

Summing the series by the diagonals produces the form

\[
\sin \phi \sum_{m=0}^{\infty} \lambda^m \frac{(-1)^{m-k} (m-k)!}{2^{m-1} (m-k)! (m-k+1)!} + \sin 3 \phi \sum_{m=0}^{\infty} \lambda^{m+2} \frac{(-1)^{m-k+1} (m-k+1)!}{2^{m+2} (m-k+1)! (m-k+2)!} + \cdots
\]

\[
= \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \sin j \phi \left\{ \lambda^j \frac{(-1)^{m-j} (m-j)!}{2^{m-j} (m-j)! (m-j+1)!} \right\}
\]
Now let \( m = 2r + 1 \) and \( j = 2k + 1 \) to reduce the series to the form

\[
\sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \sin(2k+1) \phi \left\{ \frac{2^r 2^k + 1}{(2^r 2^k + 1)^{r+k+1}} \right\} \tag{C.2}
\]

The summation over the index \( r \) can be directly related to the series expansion of the Bessel Function. By reference 25, note that this expansion may be written as

\[
2(-1)^r \frac{(-1)^r \left( \frac{1}{2} \right)^{2r+2k+1}}{[r+(2k+1)]! r!} = 2(-1)^r \mathcal{J}_{2k+1}(\lambda)
\]

where \( \mathcal{J}_{2k+1}(\lambda) \) is a Bessel Function of the first kind of index, \( 2k + 1 \). Therefore the series in question has been reduced to the form

\[
\sum_{k=0}^{\infty} 2(-1)^k \mathcal{J}_{2k+1}(\lambda) \sin(2k+1) \phi \tag{C.3}
\]

It is now necessary to show that the two series given by equations (C.1) and (C.3) converge to the same value before equality is established. The proof of the equality of the two series (C.1) and (C.2) is easier to show when proceeding in reverse fashion. That is, by showing that the series (C.2) exhibits the necessary properties to allow derangement to the series (C.1), rather than proceeding from (C.1) to (C.2).
To begin the proof of the equality of the two series it is necessary to show that the series (C.2) is absolutely convergent. This may be accomplished through a comparison test since, if the positive terms of a double series are less than those of another series which is known to converge, the former series converges. Since \(|\sin (2k+1)\phi| \leq 1\) the series

\[
\sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{\sin (2k+1)\phi}{2^{r+2k}(r+1+2k)!} \leq \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{1}{2^{r+2k}(r+1+2k)!} \tag{C.4}
\]

may be used for comparison.

Consider for a moment the series expansion of \(e^\lambda\) which is known to be uniformly and absolutely convergent for all values of \(\lambda\),

\[
e^\lambda = 1 + \frac{\lambda}{3!} + \frac{\lambda^2}{5!} + \frac{\lambda^3}{7!} + \frac{\lambda^4}{9!} + \frac{\lambda^5}{11!} + \cdots
\]

Since this series is absolutely convergent, no derangement of the terms can alter the value of the series, nor change convergence into divergence. It may readily be shown that the coefficients in the series expansion of \(e^\lambda\) given above can be expressed in the form

\[
e^\lambda = 2 \left\{ \frac{\lambda}{2} + \frac{\lambda^3}{2^3 \cdot 3!} \left( \frac{1}{3!} + \frac{1}{2!} \right) + \frac{\lambda^5}{2^5 \cdot 5!} \left( \frac{1}{5!} + \frac{1}{4!} + \frac{1}{3!} \right) + \frac{\lambda^7}{2^7 \cdot 7!} \left( \frac{1}{7!} + \frac{1}{6!} + \frac{1}{5!} + \frac{1}{4!} + \frac{1}{3!} \right) + \cdots \right\}
\]
Writing this expanded form of $e^\lambda$ as a double infinite series gives

$$e^\lambda = \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{\lambda^{2r+2k+1}}{2^{2r+2k}(r+1+2k)!r!}$$

This representation is identical to the series on the right hand side of (C.4) and consequently absolute convergence of the original series is assured. Since, if a double infinite series is absolutely convergent, no derangement of the terms can alter the value of the series, the series (C.2) may be summed in any desired manner. In particular, the series (C.2) may be rearranged to revert to the form (C.1). Therefore the identity

$$\sum_{j=0}^{\infty} \sum_{r=0}^{k} \frac{(-1)^j \lambda^{2j+1}}{2^{2j}(2j+1-k)!k!} \sin(2j+1-2k) \varphi = \sum_{k=0}^{\infty} 2(-1)^k \int_{2k+1}^{\lambda} \sin(2k+1) \varphi$$

is valid.

The same approach may be employed to establish the additional relation

$$\sum_{j=0}^{\infty} \sum_{r=0}^{k} \frac{(-1)^j \lambda^{2j}}{2^{2j-1}(2j-2-k)!k!} \sin(2j-2k) \varphi = \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \sin(2k+2) \varphi \left\{ \frac{\lambda^{2r+2k+2}}{2^{2r+2k+1}(r+k+1)!r!(r+2+2k)!} \right\}$$

$$= \sum_{k=0}^{\infty} 2(-1)^{k+1} \int_{2k+2}^{\lambda} \sin(2k+2) \varphi$$

(C.5)

Since the series (C.1) has been shown to be absolutely convergent, the preceding argument applies equally well to the
same series, but with all terms positive. Accordingly the equivalence of the following representations are easily verified:

\[
\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\lambda^{2j+1}}{2^{2j}(2j+1-k)! k!} \sin(2j+1-2k) \Phi = \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \sin(2k+1) \Phi \left\{ \frac{\lambda^{2r+2k+1}}{2^{2r+2k} r! (r+2k)!} \right\}
\]

\[
= \sum_{k=0}^{\infty} 2 \mathcal{I}_{2k+1} (\lambda) \sin(2k+1) \Phi
\]

(C.6)

and

\[
\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\lambda^{2j}}{2^{2j-1}(2j-k)! k!} \sin(2j-2k) \Phi = \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \sin(2k+2) \Phi \left\{ \frac{\lambda^{2r+2k+2}}{2^{2r+2k+1} r! (r+2k)!} \right\}
\]

\[
= \sum_{k=0}^{\infty} 2 \mathcal{I}_{2k+2} (\lambda) \sin(2k+2) \Phi
\]

(C.7)

where \( \mathcal{I}_n(\lambda) \) is the modified Bessel Function of the first kind of index \( n \) and is represented in series form as

\[
\mathcal{I}_n(\lambda) = \sum_{k=0}^{\infty} \frac{\lambda^{2k+n}}{2^{2k+n} k! (n+k)!}
\]

The proof that the series (C.3) and (C.5) converge uniformly for all values of the variable \( \Phi \), as described for integration purposes, follows directly from the inequality for the bound on the Bessel Function (see reference 25).
The proof for (C.1) proceeds as follows:

The magnitude of each term in the series is bounded by the maximum value attained by \( J_{2k+1} \), i.e.

\[
(-1)^k \frac{J_{2k+1}(\lambda) \sin(2k+1) \phi}{\max J_{2k+1}(\lambda)} \leq \left( \frac{J_{2k+1}(\lambda)}{\max} \right)_{\max}
\]

The results given by Cauchy and presented in Watson\(^{25}\) are

\[
\left| J_n(z) \right| \leq \frac{(\frac{1}{2}z)^n}{n!} e^{-\frac{x}{n+1}} \quad n > 0
\]

so

\[
\sum_{n=0}^{\infty} \left( \frac{J_{2n+1}(\lambda)}{\max} \right)_{\max} \leq \sum_{n=0}^{\infty} \frac{(\frac{1}{2} \lambda)^n}{n!} e^{-\frac{x}{n+1}}
\]

The series on the right converges for finite values of the parameter \( \lambda \), since by the ratio test

\[
\frac{\nu_{n+1}}{\nu_n} = \frac{\lambda}{2(n+1)} e^{-\frac{x}{n+1(n+2)}}
\]

which goes to the limit \( t = 0 \) as \( n \to \infty \) and the series \( \sum \nu_n \) converges since \( 0 < t < 1 \). Therefore the terms may be used as the positive constants in the Weierstrass M-test to show uniform convergence of the original series in the variable \( \phi \).

Since the inequality given in equation (C.8) applies whether \( z \) be real or complex, the series in equations (C.6) and (C.7) can be shown to be uniformly convergent in exactly the same manner as outlined above for \( z \) real.
RESTRICTIONS ON THE USE OF THE CAUCHY PRINCIPAL VALUE

In subsonic aerodynamics, frequent use is made of the Glauert integral relation which exists only in the sense of the Cauchy principal value. The governing integral equation of slender wing theory relating the downwash to the circulation distribution over the wing is given as

$$\omega_y = -\frac{1}{\mu} \int_{-\delta/2}^{\delta/2} \frac{d\Gamma(y)}{\delta_y - \gamma} \, dy$$

where $\Gamma(y)$ is the bound circulation, and is the unknown quantity sought. Glauert's approach for solving the above equation is to expand the unknown circulation in an infinite Fourier sine series and to perform the necessary integration term by term employing the principal value concept. The resulting equation is then used to determine the lift and moment acting on the wing. This procedure brings to question the permissibility of the interchange in the order of integration and summation when the integral exists only in the sense of the Cauchy principal value.

Since an inversion in the order of integration and summation is necessary in order to evaluate certain
relations in the unsteady aerodynamic theory presented in the main body of this dissertation, the restrictions on the above procedure will now be investigated. Only a sufficient set of conditions will be established; no attempt will be made to ascertain the necessary conditions.

The integral relation involved has the general form

$$\int \limits_{-l}^{l} \frac{\varphi(x,\xi)}{1-\xi} \frac{l}{\lambda-\xi} F(\xi) d\xi \quad |x|<l \quad (D.1)$$

which, because of the form of the singularity, exists only in the sense as defined by the Cauchy principal value. The symbol ( \( \int \) ) is used to signify that only the principal value is to be considered.

By definition, reference 24, the principal value of the integral appearing in equation (D.1) is

$$\int \limits_{-l}^{l} \frac{\varphi(x,\xi)}{1-\xi} \frac{l}{\lambda-\xi} F(\xi) d\xi = \lim_{\varepsilon \to 0} \left\{ \frac{x-\varepsilon}{\lambda-\varepsilon} \int \limits_{-l}^{l} \frac{1+\xi}{l-\xi} \frac{F(\xi)}{\lambda-\xi} d\xi \right\}$$

$$+ \int \limits_{x+\varepsilon}^{l} \frac{1+\xi}{l-\xi} \frac{F(\xi)}{\lambda-\xi} d\xi \quad (D.2)$$

Let the function \( F(\xi) \) be represented by the uniform convergent series of continuous functions, \( f_n(\xi) \),
so that

$$F(\xi) = \sum_{n=1}^{\infty} f_n(\xi) \tag{D.3}$$

The right hand side of equation (D.2) becomes

$$\lim_{\xi \to 0} \left\{ \int_{-1}^{x-\varepsilon} \frac{1}{\sqrt{1+\xi}} \sum_{n=1}^{\infty} f_n(\xi) d\xi + \int_{x+\varepsilon}^{1} \frac{1}{\sqrt{1-\xi}} \sum_{n=1}^{\infty} f_n(\xi) d\xi \right\} \tag{D.4}$$

It is now necessary to determine what additional restrictions, if any, need be placed on the series (D.3) in order to permit the interchange of the order of summation and integration when the integral exists only in the sense of its Cauchy principal value. That is, what conditions will suffice the writing of equation (D.4) as

$$\lim_{\xi \to 0} \left\{ \int_{-1}^{x-\varepsilon} \frac{1}{\sqrt{1+\xi}} \sum_{n=1}^{\infty} f_n(\xi) d\xi + \int_{x+\varepsilon}^{1} \frac{1}{\sqrt{1-\xi}} \sum_{n=1}^{\infty} f_n(\xi) d\xi \right\} \tag{D.5}$$

Since the function $\phi(x,\xi)$ defined by equation (D.1) is continuous over the subintervals in the above integration,
and the series expansion converges uniformly, the operation of summation and integration may be reversed. Hence, equation (D.4) takes the form

\[
\lim_{\varepsilon \to 0} \sum_{n=1}^{\infty} \left\{ \int_{-\varepsilon}^{1-\varepsilon} \frac{f_n(\xi)}{1-\xi} d\xi + \int_{x-\varepsilon}^{x+\varepsilon} \frac{f_n(\xi)}{x-\xi} d\xi \right\}
\]

(D.6)

Now, if the series in equation (D.6) can be shown to converge uniformly in \(\varepsilon\) for all values of \(x\) in the interval \(x < 1\), the desired results of equation (D.5) may be realized. Considering the functions \(g_n(x, \varepsilon)\) and \(h_n(x, \varepsilon)\) in this respect, noting that the concept of the existence of the principal value requires

\[-1 < (x - \varepsilon) < 1\quad \text{and}\quad -1 < (x + \varepsilon) < 1\]

the bounds on the functions \(g_n(x, \varepsilon)\) and \(h_n(x, \varepsilon)\) may be written as

\[
\left| g_n(x, \varepsilon) \right| < \left| \left( \frac{1 + \varepsilon}{1 - \varepsilon} \right)^{\xi=x-\varepsilon} \right| \int_{-\varepsilon}^{\varepsilon} \left| f_n(\xi) \right| d\xi
\]

\[
< \frac{\Gamma(2)}{\varepsilon^{\frac{1}{2}}} \int_{-\varepsilon}^{\varepsilon} \left| f_n(\xi) \right| d\xi
\]
\[ |h_n(x, \varepsilon)| < \left| \left( \frac{x + \varepsilon}{x - \varepsilon} \right) \right| \int_{x+\varepsilon}^{x-\varepsilon} |f_n(\xi)| \, d\xi \]

\[ < \frac{12}{\varepsilon^3} \int_{x+\varepsilon}^{x-\varepsilon} |f_n(\xi)| \, d\xi \]

Hence

\[ g_n(x, \varepsilon) + h_n(x, \varepsilon) \leq |g_n(x, \varepsilon) + h_n(x, \varepsilon)| \]

\[ \leq \frac{12}{\varepsilon^3} \left\{ \int_{x-\varepsilon}^{x+\varepsilon} |f_n(\xi)| \, d\xi + \int_{x+\varepsilon}^{x-\varepsilon} |f_n(\xi)| \, d\xi \right\} \quad \text{(D.7)} \]

The integrand in each of these integrals is continuous by the initial statement of the series expansion; therefore the above inequality is valid for any arbitrary x in the open interval \(-1 < x < 1\), with \(\varepsilon > 0\). Hence, the right hand side of equation (D.7) may be used as a limiting series which is completely independent of the choice of x. An appropriate test series may then be written containing the terms

\[ \mathcal{H}_n(\varepsilon) = \left\{ \int_{x-\varepsilon}^{x+\varepsilon} |f_n(\xi)| \, d\xi \right\} \]

having the property that

\[ g_n(x, \varepsilon) + h_n(x, \varepsilon) \leq \frac{12}{\varepsilon^3} \mathcal{H}_n(\varepsilon) \]
If the series

$$\sum_{n=1}^{\infty} \frac{12}{\xi^{2n}} H_n(\xi)$$ \hspace{1cm} (D.8)

converges, it may be used as the test series in the Weierstrass M-test to establish uniform convergence of the original series contained in equation (D.6). Convergence of the series (D.8) is guaranteed if the series

$$\sum_{n=1}^{\infty} |f_n(\xi)|$$ \hspace{1cm} (D.9)

converges uniformly in the interval $-1 < \xi < 1$. Since uniform convergence of the series obtained by summing over the integrated functions is sufficient to permit the interchange of the order of the summation and limiting processes, the above formulation establishes the requirement (D.9) necessary to permit the writing of equation (D.5).

The results developed may be summarized by the following statement: If the functions $f_n(\xi)$ in the expansion

$$F(\xi) = \sum_{n=1}^{\infty} f_n(\xi)$$

are continuous, and the series is uniformly convergent in $\xi$, the condition that the series
also be uniformly convergent is sufficient to permit the relation

\[
\sum_{n=1}^{\infty} \left| f_n(\xi) \right| = \sum_{n=1}^{\infty} \int_{-1}^{1} \left( \frac{1 + \xi}{1 - \xi} \right) \left( \sum_{n=1}^{\infty} f_n(\xi) \right) d\xi = \sum_{n=1}^{\infty} \int_{-1}^{1} \left( \frac{1 + \xi}{1 - \xi} \right) \frac{f_n(\xi)}{\chi - \xi} d\xi
\]

if the integrals exist in the sense of their Cauchy principal value.

In the unsteady pressure equation (3.1) the virtual mass term also contains a discontinuity of the form discussed above, and similarly, the integral exists only in the sense of its Cauchy principal value. It is easily shown that the same requirement of uniform convergence of the series of absolute values is a sufficient condition to permit the interchange of the order of integration and summation in this case, as before. Consider the integral

\[
\int_{-1}^{1} \mathcal{L}_i(x,\xi) F(\xi) d\xi \quad |x| < 1
\]

where

\[
\mathcal{L}_i = \frac{1}{2} \ln \left[ \frac{1 - \xi_x + \sqrt{1 - \xi^2}}{1 - \xi_x - \sqrt{1 - \xi^2}} \frac{1 - x^2}{1 - \xi^2} \right]
\]
and the function \( F(\xi) \) is given by the uniformly convergent series

\[
F(\xi) = \sum_{n=1}^{\infty} f_n(\xi)
\]

As before the discontinuity that presents difficulty exists at the point \( x = \xi \).

By definition

\[
\int_{-1}^{1} \mathcal{I}(x, \xi) F(\xi) \, d\xi = \lim_{\varepsilon \to 0} \left\{ \int_{-1}^{x-\varepsilon} \frac{1}{\ln \left[ \frac{1-x\xi + \sqrt{1-\xi^2}}{1-x\xi - \sqrt{1-\xi^2}} \frac{\sqrt{1-\xi^2}}{\sqrt{1-x^2}} \right]} \sum_{n=1}^{\infty} f_n(\xi) \, d\xi
\]

\[
+ \int_{x+\varepsilon}^{1} \frac{1}{\ln \left[ \frac{1-x\xi + \sqrt{1-\xi^2}}{1-x\xi - \sqrt{1-\xi^2}} \frac{\sqrt{1-\xi^2}}{\sqrt{1-x^2}} \right]} \sum_{n=1}^{\infty} f_n(\xi) \, d\xi
\]

\[
\equiv \lim_{\varepsilon \to 0} \sum_{n=1}^{\infty} \left( \overline{g}_n(x, \varepsilon) + \overline{h}(x, \varepsilon) \right)
\]

the functions \( \overline{g}_n(x, \varepsilon) \) and \( \overline{h}(x, \varepsilon) \) denoting the value of the \( n^{th} \) term of the series of integrals obtained after the order of integration and summation have been interchanged.

Examination of the terms shows that

\[
\left| \overline{g}_n(x, \varepsilon) \right| \leq \left( \frac{1}{2} \ln \left[ \frac{1-x^\xi + \sqrt{1-\xi^2}}{1-x^\xi - \sqrt{1-\xi^2}} \frac{\sqrt{1-\xi^2}}{\sqrt{1-x^2}} \right] \right) \int_{-1}^{x-\varepsilon} \left| f_n(\xi) \right| \, d\xi
\]

\[
\leq \frac{1}{2} \ln \left[ 1 + \frac{y}{\varepsilon^2} \right] \int_{-1}^{x-\varepsilon} \left| f_n(\xi) \right| \, d\xi
\]

(D.12)
independent of the value of $x$ in the range $-1 < x < 1$.

Therefore the same argument as presented previously may be used to establish a set of sufficient conditions. Hence it may be concluded that if the functions $f_n(\xi)$ in the series expansion

$$F(\xi) = \sum_{n=1}^{\infty} f_n(\xi)$$

are continuous, and the series converges uniformly, a sufficient condition for the relation

$$\int_{-\epsilon}^{\epsilon} \mathcal{L}_n(x, \xi) F(\xi) \, d\xi = \sum_{n=1}^{\infty} \int_{-\epsilon}^{\epsilon} \mathcal{L}_n(x, \xi) f_n(\xi) \, d\xi$$

is that the series $\sum_{n=1}^{\infty} |f_n(\xi)|$ converges uniformly in $\xi$ on the interval of integration.

The remaining integral in the pressure equation (3.1) presents no difficulty and is integrable in the normal sense as long as the functions $f_n(\xi)$ are continuous over the closed interval.
APPENDIX E

SUMMARY OF DEFINITE INTEGRALS

The following list of integrals is a compilation of pertinent integrals used in the development of the body of this dissertation. Examples of the technique used for generation of these relations may be found in the appropriate section where the integral is employed. In many cases the result of the integration is left in terms of the transform variables \( \varphi \) and \( \theta \), where \( \varphi = \cos^{-1} x \) and \( \theta = \cos^{-1} \xi \). This form is used to facilitate further operations on the integrals required in the body of this work. In the integrals to follow, the symbol \( \mathcal{J} \) is used to indicate that only the Cauchy principal value of the integral is considered.

\[
\int_{-1}^{1} \frac{1 + \xi^2}{1 - \xi^2} d\xi = \pi \quad (E.1) \quad \int_{-1}^{1} \frac{1 + \xi^2}{1 - \xi^2} \xi^n d\xi = \frac{\pi}{2} \quad (E.2)
\]

\[
\int_{-1}^{1} \frac{1 + \xi^2}{1 - \xi^2} \frac{d\xi}{\chi^2 - \xi^2} = -\pi \quad (E.3) \quad \int_{-1}^{1} \frac{1 + \xi^2}{1 - \xi^2} \frac{\xi^* d\xi^*}{\chi^* - \xi^*} = -\pi (1 + \chi^*) \quad (E.4)
\]

\[
\int_{-1}^{1} \Lambda_3(\chi, \xi^*) d\xi^* = \pi \sqrt{1 - \chi^2} \quad (E.5) \quad \int_{-1}^{1} \xi^* \Lambda_3(\chi, \xi^*) d\xi^* = \frac{\pi}{2} \chi^* \sqrt{1 - \chi^2} \quad (E.6)
\]

\[
\int_{-1}^{1} \frac{1 + \xi^2}{1 - \xi^2} \xi^n d\xi^* = \begin{cases} 
\frac{1}{2^n} \frac{(n+1)!}{(n+1)!} \pi & \text{n-odd} \\
\frac{1}{2^n} \left( \frac{n!}{\frac{n}{2}! \frac{n}{2}!} \right) \pi & \text{n-even}
\end{cases} \quad (E.7)
\]
\[
\int_{-1}^{1} \frac{dx^*}{1-x^*} \frac{\sin \lambda x^*}{\sin \phi} = J_\nu(\lambda) \pi \quad (E.10)
\]

\[
\int_{-1}^{1} \frac{dx^*}{1-x^*} \frac{\cos \lambda x^*}{\sin \phi} = J_\nu(\lambda) \pi \quad (E.11)
\]

\[
\int_{-1}^{1} \frac{dx^*}{1-x^*} \frac{\sinh \nu x^*}{\sin \phi} = I_{\nu}(\mu) \pi \quad (E.12)
\]

\[
\int_{-1}^{1} \frac{dx^*}{1-x^*} \frac{\cosh \nu x^*}{\sin \phi} = I_{\nu}(\mu) \pi \quad (E.13)
\]

\[
\int_{-1}^{1} \frac{dx^*}{1-x^*} \frac{\sin \lambda x^*}{x^*} = -2\pi (1+\cos \phi) \sum_{K=0}^{\infty} (-1)^K J_{2K+1}(\lambda) \sin (2K+1) \phi -\pi J_0(\lambda) \quad (E.14)
\]

\[
\int_{-1}^{1} \frac{dx^*}{1-x^*} \frac{\cosh \nu x^*}{x^*} = -2\pi (1+\cos \phi) \sum_{K=0}^{\infty} (-1)^K J_{2K+2}(\lambda) \sin (2K+2) \phi -\pi J_0(\lambda) \quad (E.15)
\]
\[
\int_{-1}^{1} \sqrt{\frac{1+\xi^2}{1-\xi^2}} \frac{\sinh \mu \xi^*}{\xi^* - \xi^*} d\xi^* = -2\pi (1+\cos \phi) \sum_{k=0}^{\infty} I_k(\mu) \sin(2k+1)\phi
\]
(E.16)

\[
\int_{-1}^{1} \sqrt{\frac{1+\xi^2}{1-\xi^2}} \frac{\cosh \mu \xi^*}{\xi^* - \xi^*} d\xi^* = -2\pi (1+\cos \phi) \sum_{k=0}^{\infty} I_k(\mu) \sin(2k+2)\phi - \pi I_1(\mu)
\]
(E.17)

\[
\int_{-1}^{1} \sinh \mu \xi^* \Lambda_1(x, \xi^*) d\xi^* = \frac{2\pi}{\lambda} \sum_{k=0}^{\infty} (-1)^k J_{2k+2}(\lambda) \sin(2k+2)\phi
\]
(E.18)

\[
\int_{-1}^{1} \cos \lambda \xi^* \Lambda_1(x, \xi^*) d\xi^* = \frac{2\pi}{\lambda} \sum_{k=0}^{\infty} (-1)^k J_{2k+1}(\lambda) \sin(2k+1)\phi
\]
(E.19)

\[
\int_{-1}^{1} \sinh \mu \xi^* \Lambda_1(x, \xi^*) d\xi^* = \frac{2\pi}{\lambda} \sum_{k=0}^{\infty} I_{2k+2}(\mu) \sin(2k+2)\phi
\]
(E.20)

\[
\int_{-1}^{1} \cosh \mu \xi^* \Lambda_1(x, \xi^*) d\xi^* = \frac{2\pi}{\lambda} \sum_{k=0}^{\infty} I_{2k+1}(\mu) \sin(2k+1)\phi
\]
(E.21)

where
\[
\Lambda_1(x, \xi^*) = \frac{1}{2} \ln \left[ \frac{1-x^2 + \sqrt{1-x^2}}{1-x^2 - \sqrt{1-x^2}} \right]
\]
(E.22)

\[
\int_{-1}^{1} \sqrt{\frac{1-x^*}{1+x^*}} \sin \lambda x^* dx^* = -\pi J_0(\lambda)
\]
(E.23)

\[
\int_{-1}^{1} \sqrt{\frac{1-x^*}{1+x^*}} \cos \lambda x^* dx^* = \pi J_0(\lambda)
\]
(E.24)

\[
\int_{-1}^{1} \sqrt{\frac{1-x^*}{1+x^*}} \sinh \mu x^* dx^* = -\pi I_0(\mu)
\]
(E.25)

\[
\int_{-1}^{1} \sqrt{\frac{1-x^*}{1+x^*}} \cosh \mu x^* dx^* = \pi I_0(\mu)
\]
(E.26)
\[
\int_{-1}^{1} \sqrt{1-x^2} \sin \lambda x^* dx^* = 0 \quad (E.26)
\]

\[
\int_{-1}^{1} \sqrt{1-x^2} \cos \lambda x^* dx^* = \frac{\pi}{\lambda} J_1(\lambda) \quad (E.27)
\]

\[
\int_{-1}^{1} \sqrt{1-x^2} \sinh \mu x^* dx^* = 0 \quad (E.28)
\]

\[
\int_{-1}^{1} \sqrt{1-x^2} \cosh \mu x^* dx^* = \frac{\pi}{\mu} I_1(\mu) \quad (E.29)
\]

\[
\int_{-1}^{1} x^* \sqrt{1-x^2} \sin \lambda x^* dx^* = -\pi \left[ J_0(\lambda) - \frac{2}{\lambda} J_1(\lambda) \right] \quad (E.30)
\]

\[
\int_{-1}^{1} x^* \sqrt{1-x^2} \cosh \mu x^* dx^* = 0 \quad (E.31)
\]

\[
\int_{-1}^{1} x^* \sqrt{1-x^2} \sinh \mu x^* dx^* = \pi \left[ I_0(\mu) - \frac{2}{\mu} I_1(\mu) \right] \quad (E.32)
\]

\[
\int_{-1}^{1} x^* \sqrt{1-x^2} \cosh \mu x^* dx^* = 0 \quad (E.33)
\]

\[
J_{\nu}(\Theta) = \left( \frac{i \Theta}{\sqrt{\nu + \frac{1}{2}}} \right)^{\nu + \frac{1}{2}} \int_{-1}^{1} (1-t^2)^{-\nu - \frac{1}{2}} e^{i \Theta t} dt \quad (E.34)
\]

\[
I_{\nu}(\alpha) = \left( \frac{i \alpha}{\sqrt{\nu + \frac{1}{2}}} \right)^{\nu + \frac{1}{2}} \int_{-1}^{1} (1-t^2)^{-\nu - \frac{1}{2}} e^{\alpha t} dt \quad (E.35)
\]
\[
\frac{2}{\pi} \int_{-1}^{1} \sin \lambda \, \chi^* \sqrt{1 - \chi^*} \int_{-1}^{1} \frac{1}{1 - \xi^*} \sinh \lambda \xi^* \, d\xi^* \, d\chi^* = 0 \tag{E.36}
\]

\[
\frac{2}{\pi} \int_{-1}^{1} \sin \lambda \, \chi^* \sqrt{1 - \chi^*} \int_{-1}^{1} \frac{1}{1 - \xi^*} \cos \lambda \xi^* \, d\xi^* \, d\chi^* = -2\pi \sum_{n=0}^{\infty} \frac{2(2n+2) \int J_n(\lambda \xi)^2 + 2\pi \int J_n(\lambda \xi)J_n(\lambda \xi)}{\lambda \xi} \tag{E.37}
\]

\[
\frac{2}{\pi} \int_{-1}^{1} \sin \lambda \, \chi^* \sqrt{1 - \chi^*} \int_{-1}^{1} \frac{1}{1 - \xi^*} \sinh \lambda \xi^* \, d\xi^* \, d\chi^* = 0 \tag{E.38}
\]

\[
\frac{2}{\pi} \int_{-1}^{1} \sin \lambda \, \chi^* \sqrt{1 - \chi^*} \int_{-1}^{1} \frac{1}{1 - \xi^*} \cosh \lambda \xi^* \, d\xi^* \, d\chi^* = -2\pi \sum_{n=0}^{\infty} \frac{2(2n+2) \int J_n(\lambda \xi)^2 + 2\pi \int J_n(\lambda \xi)J_n(\lambda \xi)}{\lambda \xi} \tag{E.39}
\]

\[
\frac{2}{\pi} \int_{-1}^{1} \cos \lambda \, \chi^* \sqrt{1 - \chi^*} \int_{-1}^{1} \frac{1}{1 - \xi^*} \sinh \lambda \xi^* \, d\xi^* \, d\chi^* = -2\pi \int J_n(\lambda \xi)J_n(\lambda \xi) \tag{E.40}
\]

\[
\frac{2}{\pi} \int_{-1}^{1} \cos \lambda \, \chi^* \sqrt{1 - \chi^*} \int_{-1}^{1} \frac{1}{1 - \xi^*} \cosh \lambda \xi^* \, d\xi^* \, d\chi^* = -2\pi \int J_n(\lambda \xi)J_n(\lambda \xi) \tag{E.41}
\]

\[
\frac{2}{\pi} \int_{-1}^{1} \cosh \lambda \, \chi^* \sqrt{1 - \chi^*} \int_{-1}^{1} \frac{1}{1 - \xi^*} \sinh \lambda \xi^* \, d\xi^* \, d\chi^* = -2\pi \int J_n(\lambda \xi)J_n(\lambda \xi) \tag{E.42}
\]

\[
\frac{2}{\pi} \int_{-1}^{1} \cosh \lambda \, \chi^* \sqrt{1 - \chi^*} \int_{-1}^{1} \frac{1}{1 - \xi^*} \cosh \lambda \xi^* \, d\xi^* \, d\chi^* = -2\pi \int J_n(\lambda \xi)J_n(\lambda \xi) \tag{E.43}
\]

\[
\frac{2}{\pi} \int_{-1}^{1} \sinh \lambda \, \chi^* \sqrt{1 - \chi^*} \int_{-1}^{1} \frac{1}{1 - \xi^*} \sinh \lambda \xi^* \, d\xi^* \, d\chi^* = 0 \tag{E.44}
\]

\[
\frac{2}{\pi} \int_{-1}^{1} \sinh \lambda \, \chi^* \sqrt{1 - \chi^*} \int_{-1}^{1} \frac{1}{1 - \xi^*} \cosh \lambda \xi^* \, d\xi^* \, d\chi^* = -2\pi \sum_{n=0}^{\infty} \frac{2(2n+2) \int J_n(\lambda \xi)^2 + 2\pi \int J_n(\lambda \xi)J_n(\lambda \xi)}{\lambda \xi} \tag{E.45}
\]
\[
\frac{2}{\pi} \int_{-i}^{i} \sinh \lambda \frac{1 - \chi^*}{1 + \chi^*} \left( \int_{-i}^{i} \frac{1 + \frac{\xi^*}{\chi^* - \frac{\xi^*}{\lambda^*}}}{1 + \frac{\xi^*}{\chi^* - \frac{\xi^*}{\lambda^*}}} d\xi^* d\chi^* = 0 \right) \quad (E.46)
\]

\[
\frac{2}{\pi} \int_{-i}^{i} \sinh \lambda \frac{1 - \chi^*}{1 + \chi^*} \left( \int_{-i}^{i} \frac{1 + \frac{\xi^*}{\chi^* - \frac{\xi^*}{\lambda^*}}}{1 - \frac{\xi^*}{\chi^* - \frac{\xi^*}{\lambda^*}}} \cosh \lambda \frac{\xi^*}{\chi^* - \frac{\xi^*}{\lambda^*}} d\xi^* d\chi^* = -2 \pi \sum_{k=0}^{\infty} \frac{\text{I}_k(\lambda)}{\lambda_k^{1/2} + \lambda_k^{1/2}} \text{I}\_k(\lambda) \text{I}\_k(\lambda) \right) \quad (E.47)
\]

\[
\frac{2}{\pi} \int_{-i}^{i} \cosh \lambda \frac{1 - \chi^*}{1 + \chi^*} \left( \int_{-i}^{i} \frac{1 + \frac{\xi^*}{\chi^* - \frac{\xi^*}{\lambda^*}}}{1 - \frac{\xi^*}{\chi^* - \frac{\xi^*}{\lambda^*}}} \sinh \lambda \frac{\xi^*}{\chi^* - \frac{\xi^*}{\lambda^*}} d\xi^* d\chi^* = -2 \pi \sum_{k=0}^{\infty} \frac{(-1)^k \text{I}_k(\lambda_k^1)}{\lambda_k^{1/2} + \lambda_k^{1/2}} \text{I}_k(\lambda_k^1) \text{I}_k(\lambda_k^1) \right) \quad (E.48)
\]

\[
\frac{2}{\pi} \int_{-i}^{i} \cosh \lambda \frac{1 - \chi^*}{1 + \chi^*} \left( \int_{-i}^{i} \frac{1 + \frac{\xi^*}{\chi^* - \frac{\xi^*}{\lambda^*}}}{1 - \frac{\xi^*}{\chi^* - \frac{\xi^*}{\lambda^*}}} \cosh \lambda \frac{\xi^*}{\chi^* - \frac{\xi^*}{\lambda^*}} d\xi^* d\chi^* = -2 \pi \text{J}_0(\lambda) \text{I}_0(\lambda) \right) \quad (E.49)
\]

\[
\frac{2}{\pi} \int_{-i}^{i} \cosh \lambda \frac{1 - \chi^*}{1 + \chi^*} \left( \int_{-i}^{i} \frac{1 + \frac{\xi^*}{\chi^* - \frac{\xi^*}{\lambda^*}}}{1 - \frac{\xi^*}{\chi^* - \frac{\xi^*}{\lambda^*}}} \sinh \lambda \frac{\xi^*}{\chi^* - \frac{\xi^*}{\lambda^*}} d\xi^* d\chi^* = -2 \pi \sum_{k=0}^{\infty} \frac{2(2k+1)}{(2k+2)} \text{I}_k(\lambda_k^1) \text{I}_k(\lambda_k^1) \right) \quad (E.50)
\]

\[
\frac{2}{\pi} \int_{-i}^{i} \cosh \lambda \frac{1 - \chi^*}{1 + \chi^*} \left( \int_{-i}^{i} \frac{1 + \frac{\xi^*}{\chi^* - \frac{\xi^*}{\lambda^*}}}{1 - \frac{\xi^*}{\chi^* - \frac{\xi^*}{\lambda^*}}} \cosh \lambda \frac{\xi^*}{\chi^* - \frac{\xi^*}{\lambda^*}} d\xi^* d\chi^* = -2 \pi \text{J}_0(\lambda) \text{I}_0(\lambda) \right) \quad (E.51)
\]

\[
\int_{-i}^{i} \sinh \lambda \frac{1 - \chi^*}{1 + \chi^*} \left( \int_{-i}^{i} \frac{1 + \frac{\xi^*}{\chi^* - \frac{\xi^*}{\lambda^*}}}{1 - \frac{\xi^*}{\chi^* - \frac{\xi^*}{\lambda^*}}} \sinh \lambda \frac{\xi^*}{\chi^* - \frac{\xi^*}{\lambda^*}} d\xi^* d\chi^* = - \pi^2 \text{J}_0(\lambda) \text{I}_0(\lambda) \right) \quad (E.52)
\]

\[
\int_{-i}^{i} \sinh \lambda \frac{1 - \chi^*}{1 + \chi^*} \left( \int_{-i}^{i} \frac{1 + \frac{\xi^*}{\chi^* - \frac{\xi^*}{\lambda^*}}}{1 - \frac{\xi^*}{\chi^* - \frac{\xi^*}{\lambda^*}}} \cosh \lambda \frac{\xi^*}{\chi^* - \frac{\xi^*}{\lambda^*}} d\xi^* d\chi^* = - \pi^2 \text{J}_0(\lambda) \text{J}_0(\lambda) \right) \quad (E.53)
\]

\[
\int_{-i}^{i} \sinh \lambda \frac{1 - \chi^*}{1 + \chi^*} \left( \int_{-i}^{i} \frac{1 + \frac{\xi^*}{\chi^* - \frac{\xi^*}{\lambda^*}}}{1 - \frac{\xi^*}{\chi^* - \frac{\xi^*}{\lambda^*}}} \sinh \lambda \frac{\xi^*}{\chi^* - \frac{\xi^*}{\lambda^*}} d\xi^* d\chi^* = - \pi^2 \text{I}_0(\lambda) \text{J}_0(\lambda) \right) \quad (E.54)
\]

\[
\int_{-i}^{i} \sinh \lambda \frac{1 - \chi^*}{1 + \chi^*} \left( \int_{-i}^{i} \frac{1 + \frac{\xi^*}{\chi^* - \frac{\xi^*}{\lambda^*}}}{1 - \frac{\xi^*}{\chi^* - \frac{\xi^*}{\lambda^*}}} \cosh \lambda \frac{\xi^*}{\chi^* - \frac{\xi^*}{\lambda^*}} d\xi^* d\chi^* = - \pi^2 \text{I}_0(\lambda) \text{I}_0(\lambda) \right) \quad (E.55)
\]
\begin{align}
\int_{-1}^{1} \cos \lambda x \frac{1-x^2}{1+x^2} \int_{-\xi}^{\xi} \sinh \lambda \xi \, d\xi \, dx = \pi^2 J_n(\lambda \xi) J_n(\lambda \xi) \\
\int_{-1}^{1} \cos \lambda x \frac{1-x^2}{1+x^2} \int_{-\xi}^{\xi} \cos \lambda \xi \, d\xi \, dx = \pi^2 J_n(\lambda \xi) I_n(\lambda \xi) \\
\int_{-1}^{1} \cos \lambda x \frac{1-x^2}{1+x^2} \int_{-\xi}^{\xi} \sinh \lambda \xi \, d\xi \, dx = \pi^2 J_n(\lambda \xi) I_n(\lambda \xi) \\
\int_{-1}^{1} \cos \lambda x \frac{1-x^2}{1+x^2} \int_{-\xi}^{\xi} \cosh \lambda \xi \, d\xi \, dx = \pi^2 J_n(\lambda \xi) I_n(\lambda \xi)
\end{align}

(E.56)  

(E.57)  

(E.58)  

(E.59)  

(E.60)  

(E.61)  

(E.62)  

(E.63)  

(E.64)  

(E.65)
\begin{align*}
\int_{-1}^{1} \frac{\cosh \lambda_3 x^*}{\sqrt{1 - x^*}} \int_{-\frac{1}{1 - x^*}}^{\frac{1}{1 - x^*}} \frac{\sinh \lambda_i \xi^* d\xi^*}{\sqrt{1 - \xi^*}} &= \pi^2 I_0(\lambda_i) I_0(\lambda_3) \quad (E.66) \\
\int_{-1}^{1} \frac{\cosh \lambda_3 x^*}{\sqrt{1 + x^*}} \int_{-\frac{1}{1 + x^*}}^{\frac{1}{1 + x^*}} \frac{\cosh \lambda_i \xi^* d\xi^*}{\sqrt{1 - \xi^*}} &= \pi^2 I_0(\lambda_i) I_0(\lambda_3) \quad (E.67) \\
\int_{-1}^{1} \frac{\sin \lambda_3 x^*}{\sqrt{1 - x^*}} \int_{-\frac{1}{1 - x^*}}^{\frac{1}{1 - x^*}} \frac{\sinh \lambda_i \xi^* d\xi^*}{\sqrt{1 - \xi^*}} &= \frac{2 \pi^2}{\lambda_i \lambda_3} \sum_{p=\pm 1}^{\infty} (2p + 1) J_{2p+1}(\lambda_i) J_{2p+1}(\lambda_3) \quad (E.68) \\
\int_{-1}^{1} \frac{\sin \lambda_3 x^*}{\sqrt{1 + x^*}} \int_{-\frac{1}{1 + x^*}}^{\frac{1}{1 + x^*}} \frac{\cos \lambda_i \xi^* d\xi^*}{\sqrt{1 - \xi^*}} &= 0 \quad (E.69) \\
\int_{-1}^{1} \frac{\sin \lambda_3 x^*}{\sqrt{1 - x^*}} \int_{-\frac{1}{1 - x^*}}^{\frac{1}{1 - x^*}} \frac{\sinh \lambda_i \xi^* d\xi^*}{\sqrt{1 - \xi^*}} &= \frac{2 \pi^2}{\lambda_i \lambda_3} \sum_{p=\pm 1}^{\infty} (-1)^p (2p + 1) I_{2p+1}(\lambda_i) J_{2p+1}(\lambda_3) \quad (E.70) \\
\int_{-1}^{1} \frac{\sin \lambda_3 x^*}{\sqrt{1 + x^*}} \int_{-\frac{1}{1 + x^*}}^{\frac{1}{1 + x^*}} \frac{\cosh \lambda_i \xi^* d\xi^*}{\sqrt{1 - \xi^*}} &= 0 \quad (E.71) \\
\int_{-1}^{1} \frac{\cos \lambda_3 x^*}{\sqrt{1 - x^*}} \int_{-\frac{1}{1 - x^*}}^{\frac{1}{1 - x^*}} \frac{\sin \lambda_i \xi^* d\xi^*}{\sqrt{1 - \xi^*}} &= 0 \quad (E.72) \\
\int_{-1}^{1} \frac{\cos \lambda_3 x^*}{\sqrt{1 + x^*}} \int_{-\frac{1}{1 + x^*}}^{\frac{1}{1 + x^*}} \frac{\cos \lambda_i \xi^* d\xi^*}{\sqrt{1 - \xi^*}} &= \frac{2 \pi^2}{\lambda_i \lambda_3} \sum_{p=\pm 1}^{\infty} (2p + 1) J_{2p+1}(\lambda_i) J_{2p+1}(\lambda_3) \quad (E.73) \\
\int_{-1}^{1} \frac{\cos \lambda_3 x^*}{\sqrt{1 - x^*}} \int_{-\frac{1}{1 - x^*}}^{\frac{1}{1 - x^*}} \frac{\sinh \lambda_i \xi^* d\xi^*}{\sqrt{1 - \xi^*}} &= 0 \quad (E.74) \\
\int_{-1}^{1} \frac{\cos \lambda_3 x^*}{\sqrt{1 + x^*}} \int_{-\frac{1}{1 + x^*}}^{\frac{1}{1 + x^*}} \frac{\cosh \lambda_i \xi^* d\xi^*}{\sqrt{1 - \xi^*}} &= \frac{2 \pi^2}{\lambda_i \lambda_3} \sum_{p=\pm 1}^{\infty} (-1)^p (2p + 1) I_{2p+1}(\lambda_i) J_{2p+1}(\lambda_3) \quad (E.75)
\end{align*}
\[ \int_{-1}^{1} \sinh \lambda_3 x^* e^{-\lambda_1 \xi \xi' d\xi d\xi'} = \frac{2\pi i}{\lambda_1 \lambda_3} \sum_{p=0}^{\infty} (-1)^p (2p+2) I_{2p+2}(\lambda_1) I_{2p+2}(\lambda_3) \] (E.76)

\[ \int_{-1}^{1} \sinh \lambda_3 x^* e^{-\lambda_1 \xi \xi' d\xi d\xi'} = 0 \] (E.77)

\[ \int_{-1}^{1} \sinh \lambda_3 x^* e^{-\lambda_1 \xi \xi' d\xi d\xi'} \cos \lambda_1 \xi \xi' d\xi d\xi' = \frac{2\pi i}{\lambda_1 \lambda_3} \sum_{p=0}^{\infty} (-1)^p (2p+2) I_{2p+2}(\lambda_1) I_{2p+2}(\lambda_3) \] (E.78)

\[ \int_{-1}^{1} \sinh \lambda_3 x^* e^{-\lambda_1 \xi \xi' d\xi d\xi'} = 0 \] (E.79)

\[ \int_{-1}^{1} \cosh \lambda_3 x^* e^{-\lambda_1 \xi \xi' d\xi d\xi'} = 0 \] (E.80)

\[ \int_{-1}^{1} \cosh \lambda_3 x^* e^{-\lambda_1 \xi \xi' d\xi d\xi'} \cos \lambda_1 \xi \xi' d\xi d\xi' = \frac{2\pi i}{\lambda_1 \lambda_3} \sum_{p=0}^{\infty} (-1)^p (2p+2) I_{2p+2}(\lambda_1) I_{2p+2}(\lambda_3) \] (E.81)

\[ \int_{-1}^{1} \cosh \lambda_3 x^* e^{-\lambda_1 \xi \xi' d\xi d\xi'} = 0 \] (E.82)

\[ \int_{-1}^{1} \cosh \lambda_3 x^* e^{-\lambda_1 \xi \xi' d\xi d\xi'} \sinh \lambda_1 \xi \xi' d\xi d\xi' = \frac{2\pi i}{\lambda_1 \lambda_3} \sum_{p=0}^{\infty} (-1)^p (2p+2) I_{2p+2}(\lambda_1) I_{2p+2}(\lambda_3) \] (E.83)
BIBLIOGRAPHY


