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ABSTRACT AND APPLIED

DISSERTATION

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By

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Introduction

When one tries to differentiate a set function with respect to another set function, one usually looks at limits of the ratios of values that the pairs of set functions have on a collection of sets which "close down" in some sense on the point at which the derivative is sought. The totality of the collections or "sequences" of sets which one uses in the process of calculating these limits is referred to as a derivation basis. There is a considerable variety of derivation bases which have appeared in the literature and which have varying degrees of complexity.

Hayes and Pauc in [2] (the number in [ ] refers to the item with the same number in the list of references at the end of this paper) have abstracted some weakened forms of the classical Vitali properties which seem important when one talks about the differentiation of integrals, and they have shown that any two derivation bases satisfying these Vitali properties yield essentially the same value for the derivative of non-negative \( \mu \)-finite or Radon measures. A number of examples of bases are given which satisfy these weakened Vitali properties.

Monroe in [3] treats abstractly a derivation basis (called a net basis) which structurally is quite simple.
The relation of the derivative yielded by this latter type of basis to the derivative yielded by bases of the former type does not appear in any literature this author is familiar with. One of the reasons the results of [2] do not give information about the relation of the net derivatives to the Hayes-Pauc derivatives is that the former type basis is in one instance required to possess a property which is even weaker than a corresponding property discussed in [2].

In this work (see Ch. 4) simple criteria for essential equivalence of derivatives of set functions in terms of properties of the upper and lower derivatives are established. To wit: If \( [S, \mathcal{M}, \mu] \) is a non-negative sigma-finite measure space (see Chapter 1 for definitions) and if \( \sigma \) is a non-negative set function defined at least on \( \mathcal{M} \) and if \( \sigma^0(s) \), \( \sigma_0(s) \) (\( i = 1, 2 \)) are respectively the upper and lower derivatives of \( \sigma \) with respect to \( \mu \) at \( s \) under derivation bases \( B_i \) (\( i = 1, 2 \)) respectively and if the following two implicative statements hold whenever their hypotheses is satisfied, where \( A \) is a subset of \( S \), \( M_A \) is a \( \mu \)-measurable cover of \( A \), and \( \bar{\mu}(A) \) is the outer measure of \( A \) which is generated by \( \mu \),

1) \( \sigma^0(s) > a \) for a.e. \( \bar{\mu} \) \( s \) in \( A \) implies \( \sigma(M_A) \geq a \bar{\mu}(A) \),

2) \( \sigma_0(s) < a \) for a.e. \( \bar{\mu} \) \( s \) in \( A \) implies \( \sigma(M_A) \leq a \bar{\mu}(A) \),

then \( \sigma^0(s) = \sigma_0(s) = \sigma^0(s) = \sigma_0(s) \) almost everywhere \( \bar{\mu} \).
on that portion of $S$ which has at each point admissible sequences from both $B_1$ and $B_2$ which close down upon the point.

The criteria of the above result are then employed to show in Chapter 4 simultaneously that if $\sigma$ is an $\mathcal{M}$-measure absolutely continuous with respect to $\mu$ and if $B$ is a collection of sequences of $\mathcal{M}$-sets forming a derivation basis, then the $B$-derivative of $\sigma$ with respect to $\mu$ is essentially unaltered as long as one knows that either (1) $B$ possesses the weakened Vitali properties of Hayes and Pauc or (2) the classical Vitali properties or (3) the net regularity properties required by Monroe.

We show in Chapter 6 that, under certain measure space related topological properties, a net derivation basis may be constructed which satisfies the necessary regularity properties referred to above and observe that these conditions are always satisfiable in Euclidean n-space. We also show that if the setting is Euclidean n-space and $\sigma$ is a semi-regular measure whose domain of definition contains the Borel class of sets, then there is a net derivation basis which yields essentially the same value for the net derivative of $\sigma$ with respect to $\mu$ as the usual regular derivative of $\sigma$ with respect to $\mu$. It follows that for an important class of functions (the semi-regular measures) the value of the regular derivative in Euclidean n-space may be had almost everywhere by
looking at the limit of ratios on only one sequence of sets converging on the point under consideration.

We have in Chapter 7 given an elegant proof using some results of our own of a theorem concerning the abstract measure space differentiation of integrals when the derivative is taken relative to a net basis.

The setting in [6] is roughly this. T is a single valued mapping from a measure space S onto a measure space S', D is a class of measurable sets in S having certain properties. For each s' in S' and each D in D, W'(s', D) is a non-negative number termed the weight function for T which roughly describes the importance of s' in the image of D under T. The integral of W'(s', D) over S' is termed the weight WD attached to T. The transformation T is said to be absolutely continuous with respect to the weights W provided there exists a non-negative measurable integrable function f on S such that the integral of f on each D is equal to the weight WD.

We finally, in Chapter 8, construct in the setting of [6] a net derivation basis which, by a result of Chapter 7, enables us to differentiate in the setting of [6] any μ-finite integral when the integral is with respect to μ and when the integral exists finite or infinite on each measurable set. It is thus possible—when the transformation T from S onto S' is absolutely continuous with respect to the
weights $W$—to characterize the important greatest lower-bound function $f$ appearing in the transformation formulas of [6] as the net derivative of the outer measure $W^*$ generated by the weight function $W$ and a sequential covering class $\mathcal{D}$. Moreover it is noted that in the setting of Euclidean $n$-space the postulates of [6] may be satisfied in such a manner that, for measures $\sigma$ having certain properties, this net derivative of $\sigma$ with respect to $\mu$ is equal a.e. $\mu$ ($\mu =$ Lebesgue $n$-dimensional measure) to the regular derivative of $\sigma$ with respect to $\mu$. Also, we observe that if the theory of [6] is applied in the setting of [5] that the net derivative which yields the greatest lower-bound function $f$ is constant a.e. $\mu$ on the components of the inverse under $T$ of single points of $S'$. Thus, this net derivative enables us to gain a bit of knowledge about the structure of $f$. 
PART I

DERIVATIVES OF SET FUNCTIONS IN ABSTRACT MEASURE SPACES
1. The General Setting

Let $S$ be a non-empty set whose elements shall be designated by $s$. Let $\mathcal{M}$ be a \textit{sigma-field} (completely additive class of sets, sigma-algebra) of subsets of $S$, i.e., $\mathcal{M}$ contains (1) the empty set, (2) the complement of each of its elements and (3) the union of any countable sequence of its elements. Let $\mu$ be a non-negative, extended-real valued function defined on $\mathcal{M}$ which has the properties of a measure, i.e., (1) the $\mu$ of the union of a sequence of pairwise disjoint sets of $\mathcal{M}$ is equal to the sum of the $\mu$'s of each of the sets in the sequence, (2) $\mu$ of the empty set is the value zero. We also assume that $S$ is \textit{sigma-finite} with respect to $\mu$, i.e., there is a countable sequence of sets from $\mathcal{M}$ each of finite $\mu$-measure which together cover all of $S$. When the above conditions on the triple $[S, \mathcal{M}, \mu]$ are satisfied, we will refer to this triple as a sigma-finite, non-negative, extended-real valued measure space.

We shall use $\overline{\mu}$ to designate the usual \textit{outer measure} constructed from $\mu$, i.e., if $A$ is an arbitrary subset of $S$, then

$$\overline{\mu}(A) = \text{g.l.b. } \mu(M), \quad M \supseteq A, \ M \in \mathcal{M}.$$ 

A set $M_A$ in $\mathcal{M}$ will be called a $\mu$-measurable cover of a set $A \subseteq S$ provided $A \subseteq M_A$ and $\overline{\mu}(A) = \mu(M_A)$. It is well known that every subset of $S$ possesses a $\mu$-measurable cover. We note that if $M_A$ is a $\mu$-measurable cover of $A$.

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and if $M_o$ is in $\mathcal{M}$ with $M_o \supseteq A$, then $M_o \cap M_A$ is also a $\mu$-measurable cover of $A$.

A set $M_A$ in $\mathcal{M}$ will be called a $\mu$-cover of a set $A \subseteq \mathcal{S}$ provided $A \subseteq M_A$ and $\overline{\mu}(A \cap M) = \mu(M_A \cap M)$ for every $M$ in $\mathcal{M}$. If $M_A$ is a $\mu$-cover for $A$, then immediately from the definition $\mu(M_A) = \overline{\mu}(A)$; thus every $\mu$-cover is a $\mu$-measurable cover. The converse is not always true; however, if $A \subseteq M_A \in \mathcal{M}$ and $\mu(M_A) = \overline{\mu}(A) < \infty$, then $M_A$ is a $\mu$-cover of $A$. Thus $\mu$-measurable covers of finite measure are $\mu$-covers. The proof of this is as follows: let $M_A$ be a $\mu$-measurable cover of $A$; then, since $M_A$ is in $\mathcal{M}$ and is also in the class of sets measurable $\overline{\mu}$,

1) $\mu(M_A \cap M) + \mu(M_A - M) = \overline{\mu}(A \cap M) + \overline{\mu}(A - M)$.

Clearly

2) $\mu(M_A \cap M) \geq \overline{\mu}(A \cap M)$ and $\mu(M_A - M) \geq \overline{\mu}(A - M)$,

hence

3) $\mu(M_A \cap M) + \mu(M_A - M) \geq \overline{\mu}(A \cap M) + \overline{\mu}(A - M)$.

Because of the finiteness assumption we see that strict inequality in either of the inequalities 2) yields through 3) a contradiction of 1). Hence the conclusion.

A non-negative, extended-real valued set function $\sigma$ is an $\mathcal{M}$-measure provided its domain of definition contains $\mathcal{M}$; and when it is restricted to $\mathcal{M}$, it is a measure, e.g., $\overline{\sigma}$ is an $\mathcal{M}$-measure. An $\mathcal{M}$-measure, $\sigma$, is $\mu$-finite if it is finite valued for each $\mathcal{M}$-set of finite $\mu$-measure. A set
function $\sigma$ is a signed $\mathbb{M}$-measure if its domain of definition contains $\mathbb{M}$; and when its domain is restricted to $\mathbb{M}$, it must have the two properties required of a measure and take on at most one of the values $+\infty$, $-\infty$.

Let $\sigma$ be a set function defined on a family of subsets of $S$ and having its values in the extended reals (including both $+\infty$ and $-\infty$). Toward the end of defining in a quite general sense the derivative of $\sigma$ with respect to $\mu$, we introduce the following additional notation and terminology. Consider $s$ in $S$. We want to associate certain families of $\mathbb{M}$-sets with $s$. We shall adopt the notation $M(s)$ to denote a family of $\mathbb{M}$-sets associated with $s$. In order to compute certain limits of a set function defined on $M(s)$, we will have to introduce an indexing of the sets in $M(s)$. With this objective in mind, we recall that a non-empty set $\Delta$ is directed by a binary relation $\succeq$, defined on it if the binary relation is transitive and reflexive and if relative to the binary relation there exists upper bounds for pairs of elements in $\Delta$, i.e., if $\delta_1$ and $\delta_2$ are two elements in $\Delta$, then there exists an element $\delta_3$ in $\Delta$ such that $\delta_3 \succeq \delta_1$ and $\delta_3 \succeq \delta_2$. We shall refer to a set $\Delta$ directed by a binary relation, $\succeq$, as a directed set. A non-empty subset $\Delta_*$ of $\Delta$ is cofinal in $\Delta$ if for each $\delta$ in $\Delta$ there is a $\delta_*$ in $\Delta_*$ such that $\delta_* \succeq \delta$. 
It is immediate that $\Delta_\ast$ is a directed set under the inherited binary relation $\geq$.

A family $M(s)$ of $\mathbb{W}$-sets associated with an $s$ in $S$ will be called an admissible generalized sequence of sets associated with $s$ in $S$ provided

1. the family $M(s)$ is the functional image of a directed set $\Delta$, i.e., $M(s)$ is indexed with the elements of $\Delta$, thus $M(s) = \{M_\delta \mid \delta \in \Delta\}$,
2. $M$ in $M(s)$ implies $\mu(M)$ is finite. The set of images obtained when the domain of the function from $\Delta$ which indexes an admissible generalized sequence, $M(s)$, is restricted to a cofinal subset $\Delta_\ast$ of $\Delta$ is called a cofinal subsequence of $M(s)$ and will be denoted by $M_\ast(s)$. We note every cofinal subsequence of an admissible generalized sequence of sets is again an admissible generalized sequence of sets. We will speak of a generalized sequence of sets $M(s)$ if $M(s)$ is a family of $\mathbb{W}$-sets and require only that condition (1) in the definition of admissible generalized sequence be satisfied.

Let $E$ be a non-empty subset of $S$. A family, $B$, of admissible generalized sequences of sets is called a derivation basis associated with a subset $E$ of $S$ if

1. each $s$ in $E$ has an associated admissible generalized sequence of sets $M(s)$ in the family $B$,
2. every cofinal subsequence of each $M(s)$ in $B$ is again in $B$,
3. there is a
set $E_0 \subset E$ with $\mathcal{M}(E_0) = 0$ such that each $M(s)$ in $B$ which is associated with $s$ in $E - E_0$ consists only of sets of positive $\mu$-measure. From the definition of admissible generalized sequence, we also know that the sets in $M(s)$ for $s$ in $E$ are also of finite $\mu$-measure. The set $E$ in the above definition is called the domain of the derivation basis $B$. We shall designate by $B_*$ the totality of the $\mathfrak{R}$-sets which comprise the generalized sequences $M(s)$ in $B$. The sets in $B_*$ are called the constituents of the generalized sequences $M(s)$ in $B$.

Let $\sigma$ be an extended-real valued set function defined on a generalized sequence of sets $M(s)$ which is indexed with the members of a directed set $\Delta$. A cofinal subsequence $M_*(s)$ of $M(s)$ will be said to be mapped eventually above a real number $r$ by $\sigma$ if there is a $\delta_0$ in $\Delta_*$, where $\Delta_*$ is as required in the definition of cofinal subsequence, such that for $\delta_*$ in $\Delta_*$ with $\delta_* \geq \delta_0$ we have $\sigma(M_{\delta_*}) > r$. If this last inequality is reversed, then the cofinal subsequence will be said to be mapped eventually below $r$ by $\sigma$. If the strict inequality in the first of the above pair of definitions is replaced by $\geq$ (or $\leq$), then we say $M_*(s)$ is mapped eventually on or above (or on or below) $r$ by $\sigma$. In particular we note if in the above definitions $M_*(s) = M(s)$, then we say $M(s)$ is mapped eventually above (or "below," or "on or above," or "on or below") $r$ by $\sigma$. 
A number $L^0$ from the extended-real number system will be designated the **limit superior** of an extended-real valued function, $\sigma$, defined on a generalized sequence of sets $M(s)$ with directed indexing set $\Delta$ provided (1) that for each real number $r > L^0$, $M(s)$ is mapped eventually below $r$ by $\sigma$, and (2) that for each real number $r < L^0$ there is a cofinal subsequence of $M(s)$ which is mapped eventually above $r$ by $\sigma$.

A number $L_o$ from the extended-real number system will be designated as the **limit inferior** of an extended-real valued function $\sigma$ defined on a generalized sequence of sets $M(s)$ with directed indexing set $\Delta$ provided (1) that for every real number $r > L_o$ there is a cofinal subsequence of $M(s)$ which is mapped eventually below $r$ by $\sigma$, and (2) that for each real number $r < L_o$, $M(s)$ is mapped eventually above $r$ by $\sigma$.

We note that when $M(s)$ is an ordinary sequence of sets (i.e., when $\Delta$ is the natural numbers and the binary operation $\geq$ is the usual greater than or equal relation on the natural numbers) the above definition for limit superior and limit inferior of the function $\sigma$ on $M(s)$ reduces to the customary definition of these concepts.

The proofs of the following double statement are immediate. Let $M(s)$ be a generalized sequence of sets
with directed indexing set \( \Delta \). For any given real number \( r \) either \( M(s) \) is mapped eventually below (above) \( r \) by \( \sigma \) or there is a cofinal subsequence of \( M(s) \) which is mapped eventually on or above (on or below) \( r \) by \( \sigma \). Also, it is easily seen that if a cofinal subsequence of \( M(s) \) is mapped eventually on or above (on or below) a real number \( r_* \) by \( \sigma \), then for every \( r < r_* \) (\( r > r_* \)) there is a cofinal subsequence of \( M(s) \) which maps eventually above (below) \( r \) by \( \sigma \).

With the help of the statements noted in the preceding paragraph, we may show that, for a given generalized sequence of sets \( M(s) \) with directed indexing set \( \Delta \), there is a unique number \( L^0 \) in the extended-real number system which is the limit superior of an extended-real valued function \( \sigma \) defined on \( M(s) \).

The argument for uniqueness can be made as follows: suppose \( L_{1}^0 < L_{2}^0 \) are two distinct members of the extended-real number system each of which satisfies the definition of limit superior of \( \sigma \) on \( M(s) \). Let \( r^0 \) be a real number (finite) such that \( L_{1}^0 < r^0 < L_{2}^0 \), then we see if we choose \( r = r^0 \) in condition(1) of the definition of \( L_{1}^0 \) as the limit superior and if we choose \( r = r^0 \) in condition(2) of the definition of \( L_{2}^0 \) as the limit superior, a contradiction appears immediately, for \( M(s) \) cannot be mapped eventually
below \( r^0 \) by \( \sigma \) and simultaneously have a cofinal subsequence which maps eventually above \( r^0 \) by \( \sigma \).

The argument for existence can be made as follows: if for each real number \( r \) there is a cofinal subsequence of \( M(s) \) which is mapped eventually above \( r \) by \( \sigma \), then \( +\infty \) is the limit superior of \( \sigma \) on \( M(s) \) since condition (1) of the definition is vacuously fulfilled. If for each real number \( r \), \( M(s) \) is mapped eventually below \( r \) by \( \sigma \), then \( -\infty \) is the limit superior of \( \sigma \) on \( M(s) \) since condition (2) of the definition is vacuously fulfilled. If neither of these two cases involving infinity occurs, then, with the help of statements appearing two paragraphs above, we see that there must be real numbers \( r_0 < r^0 \) such that \( M(s) \) is mapped eventually below \( r_0 \) by \( \sigma \); and there is a cofinal subsequence of \( M(s) \) which is mapped eventually on or above \( r_0 \) by \( \sigma \). We now bisect the interval \([r_0, r^0]\) and find that precisely one of the "half" intervals thus formed has the same properties as the original interval. Designate this uniquely determined half interval by \([r_0, r^0]\) and perform the bisection-selection procedure on this interval. Continuing in this fashion, we construct a monotone decreasing sequence of closed intervals which determines a unique real number which we shall designate as \( L^0 \), for it is now easily seen that this number \( L^0 \) satisfies the definition of the limit superior of \( \sigma \) on \( M(s) \). This completes
the proof of the existence and uniqueness of the limit superior of \( \sigma \) on a generalized sequence of sets.

The existence of a unique number \( L_0 \) in the extended-real number system which is the limit inferior of an extended-real valued function \( \sigma \) defined on a generalized sequence of sets \( M(s) \) can be shown similarly.

It is also true that \( L_0 \leq L^0 \), for if otherwise, i.e., if \( L^0 < L_0 \), then there is a (finite) real number \( r_* \) such that \( L^0 < r_* < L_0 \) and using \( r = r_* \) in condition (1) in the definition \( L^0 \) and \( r = r_* \) in condition (2) in the definition of \( L_0 \) we immediately have the contradiction that \( M(s) \) is mapped eventually below \( r_* \) by \( \sigma \) and simultaneously \( M(s) \) is mapped eventually above \( r_* \) by \( \sigma \).

Let a derivation basis \( B \) with domain \( E \) having exceptional set \( E_o \) [see condition(3) in the definition of \( B \)] be given and let \( \sigma \) be an extended-real valued set function defined at least on \( B \) \( \notin \), then the upper and lower \( B \) derivative at \( s \) in \( E - E_o \) of \( \sigma \) with respect to \( \mu \) is defined as follows:

\[
D^* \sigma(s) = \sup \left[ \lim \sup \frac{\sigma(M)}{\mu(M)} \right]_{M(s) \text{ in } B} \quad M \text{ in } M(s)
\]

\[
D_* \sigma(s) = \inf \left[ \lim \inf \frac{\sigma(M)}{\mu(M)} \right]_{M(s) \text{ in } B} \quad M \text{ in } M(s)
\]
If $D^*\sigma(s) = D_\#\sigma(s)$, finite or infinite, then this common value is the $B$ derivative of $\sigma$ at $s$ and is denoted by $D\sigma(s)$.

1.1 Remark. We do not attempt to define $D^*\sigma(s)$ and $D_\#\sigma(s)$ for $s$ in $E_0$, the exceptional set which appears in condition (3) in the definition of $B$, since uncomputable ratios may appear in some of the differential quotients associated with points in $E_0$. However, in view of our remarks about the existence of the limit inferior and limit superior and in view of the requirements in the definition of $B$ relative to the set $E - E_0$, we see that $D^*\sigma(s)$ and $D_\#\sigma(s)$ exist, finite or infinite, a.e. $\mathbb{S}$ on $E$.

1.2 Remark. Should it happen that for a given point $s$ in $E - E_0$ all of the generalized sequences $M(s)$ in $B$ are cofinal subsequences of one master generalized sequence, then for this point we have from the definition of $L^0$ and $L_0$ that the prefixes "sup" and "inf" can be dropped in the definitions of $D^*\sigma(s)$ and $D_\#\sigma(s)$ provided this master sequence is used in calculating the "lim sup" and "lim inf".

1.3 Remark. One's first reaction is that the values for $D^*\sigma(s)$ and $D_\#\sigma(s)$ depend heavily on the choice of $B$. We shall see later that if any pair of derivation bases satisfy certain conditions then these bases yield essentially the same values for these upper and lower derivatives. Since the conditions alluded to above are usually
satisfied by the derivation bases occurring in applications, we have not indicated notationally any \( \mathbb{B} \)-dependence for the upper and lower derivatives.

1.4 Lemma. Let \( \sigma_1 \) and \( \sigma_2 \) be two extended-real valued set functions on a generalized sequence of sets \( M(s) \) with directed indexing set \( \Delta \) and having the additional property that they do not simultaneously have values at opposite infinities, then provided opposite infinities do not occur in the \( \limsup \) of \( \sigma_1 \) and the \( \limsup \) of \( \sigma_2 \),

1.4a) \[
\limsup_{M \in M(s)} [\sigma_1(M) + \sigma_2(M)] \leq \limsup_{M \in M(s)} \sigma_1(M) + \limsup_{M \in M(s)} \sigma_2(M).
\]

If opposite infinities do not occur in the \( \liminf \) \( \sigma_1 \) and the \( \liminf \) \( \sigma_2 \) and if \( \sigma_1 \) and \( \sigma_2 \) are not simultaneously at opposite infinities, then

1.4b) \[
\liminf_{M \in M(s)} \sigma_1(M) + \liminf_{M \in M(s)} \sigma_2(M) \leq \liminf_{M \in M(s)} [\sigma_1(M) + \sigma_2(M)].
\]

Proof. Let \( L^0 = \limsup [\sigma_1(M) + \sigma_2(M)] \), \( L^0 = \limsup \sigma_1(M) \), \( \gamma L^0 = \limsup \sigma_2(M) \), we show 1.4a) holds and remark that the proof of 1.4b) is similar.

Case \( \gamma L^0 \) \( \gamma L^0 \) are both finite. Let \( \varepsilon > 0 \) be given, then from the definition of \( \gamma L^0 \) we have that \( M(s) \) is mapped eventually below \( \gamma L^0 + \varepsilon/2 \) by \( \sigma_1 \). Similarly we have that
M(s) is mapped eventually below $2L^0 + \varepsilon/2$ by $\sigma_2$. Hence

it is now immediate from condition (1) of the definition of

it is immediate from condition (1) of the definition of

limit superior as applied to $\sigma_1 + \sigma_2$ that $12L^0 < 1L^0 + 2L^0 + \varepsilon$, and since $\varepsilon > 0$ was arbitrary, $1.4a)$ is immediate.

Case $L^0$ or $2L^0$ is $-\infty$ and opposite infinites do not occur. Suppose $1L^0$ is $-\infty$. Since opposite infinites do not occur, there is a finite number $r_2$ such that $M(s)$ is mapped eventually below $r_2$ by $\sigma_2$. From the definition of $1L^0$, we have for every negative real number $r$ that $M(s)$ is mapped eventually below $r$ by $\sigma_1$. Hence $M(s)$ is mapped eventually below $r + r_2$ by $\sigma_1 + \sigma_2$. Thus, since $r + r_2$ can be made less than any negative real number, we conclude that $12L^0$ is $-\infty$ and $1.4a)$ is established for this case.

Case $12L^0$ is $+\infty$. We observe that not both $1L^0$ and

$2L^0$ can be finite; for if so, by the first case above $12L^0$ would be finite or $-\infty$ which contradicts the assumptions of this case. We next note that neither $1L^0$ nor $2L^0$ are $-\infty$; for if so, we have by the second case above a contradiction to the assumption of this case. Hence at least one of $1L^0$ or $2L^0$ must be $+\infty$; and since opposite infinites do not occur, $1.4a)$ is established.

The case where either $1L^0$ or $2L^0$ is $+\infty$ and the case
where \( L^0 \) is \( \infty \) are immediate since opposite infinities are ruled out and hence need no amplification.

**1.5 Lemma.** Let a derivation basis \( B \) with domain \( E \) having exceptional set \( E_0 \) be given and let \( \sigma_1 \) and \( \sigma_2 \) be extended-real valued set functions defined at least on \( B_\ast \). Then, if at \( s_0 \) in \( E - E_0 \) opposite infinities do not occur in the pairs \( D^\ast \sigma_1(s_0), D^\ast \sigma_2(s_0) \) and if \( \sigma_1 \) and \( \sigma_2 \) are not simultaneously at opposite infinities on \( B_\ast \),

\[
1.5a) \quad D^\ast(\sigma_1 + \sigma_2)(s_0) \leq D^\ast \sigma_1(s_0) + D^\ast \sigma_2(s_0).
\]

If at \( s_0 \) in \( E - E_0 \) opposite infinities do not occur in the pairs \( D^\ast \sigma_1(s_0), D^\ast \sigma_2(s_0) \) and if \( \sigma_1 \) and \( \sigma_2 \) are not simultaneously at opposite infinities on \( B_\ast \),

\[
1.5b) \quad D^\ast \sigma_1(s_0) + D^\ast \sigma_2(s_0) \leq D^\ast(\sigma_1 + \sigma_2)(s_0).
\]

**Proof.** We prove 1.5a) and note that the proof of 1.5b) is similar.

With the help of 1.4a) we have

\[
\lim_{M \to M(s_0)} \sup_{M \in M(s_0)} \left[ \frac{\sigma_1 + \sigma_2}{\mu(M)} \right] \leq \sup_{M \in M(s_0)} \frac{\sigma_1}{\mu(M)}
\]

\[
+ \lim_{M \to M(s_0)} \sup_{M \in M(s_0)} \left[ \frac{\sigma_2}{\mu(M)} \right] \leq D^\ast \sigma_1(s_0) + D^\ast \sigma_2(s_0).
\]

Hence \( D^\ast(\sigma_1 + \sigma_2)(s_0) \leq D^\ast \sigma_1(s_0) + D^\ast \sigma_2(s_0) \).

**1.6 Theorem.** Let a derivation basis \( B \) with domain \( E \) having exceptional set \( E_0 \) be given, then if \( \sigma \) is an extended-real valued set function defined at least on \( B_\ast \) for
which there exists, finite or infinite, a \( B \) derivative of \( \sigma \) at \( s_o \),

1.6a) \( D\sigma(s_o) = -D[-\sigma](s_o) \).

If \( \sigma_1 \) and \( \sigma_2 \) are extended-real valued set functions which are defined at least on \( B \) and if at \( s_o \) in \( E - E_o \) \( D\sigma_1(s_o) \) and \( D\sigma_2(s_o) \) exist, finite or infinite, and are not at opposite infinities and if \( \sigma_1 \) and \( \sigma_2 \) are not simultaneously at opposite infinities on \( B \), then \( D(\sigma_1 + \sigma_2)(s_o) \) exists and

1.6b) \( D(\sigma_1 + \sigma_2)(s_o) = D\sigma_1(s_o) + D\sigma_2(s_o) \).

If \( \sigma_1 \) and \( \sigma_2 \) are extended-real valued set functions which are defined at least on \( B \) and if at \( s_o \) in \( E - E_o \) \( D\sigma_1(s_o) \) and \( D\sigma_2(s_o) \) exist, finite or infinite, and are not at like infinities and if \( \sigma_1 \) and \( \sigma_2 \) are not simultaneously at like infinities on \( B \), then \( D(\sigma_1 - \sigma_2)(s_o) \) exists and

1.6c) \( D(\sigma_1 - \sigma_2)(s_o) = D\sigma_1(s_o) - D\sigma_2(s_o) \).

Proof. The proof of 1.6a) is immediate from the definition of \( D\sigma(s) \), and we do not elaborate further here.

With the help of 1.5 and the fact that for \( s \) in \( E - E_o \), \( D^*_\sigma(s) \leq D^\*\sigma(s) \), we have for \( s_o \) in \( E - E_o \),

\[
\begin{align*}
D^*_\sigma_1(s_o) + D^*_\sigma_2(s_o) & \leq D^*(\sigma_1 + \sigma_2)(s_o) \\
& \leq D^*(\sigma_1 + \sigma_2)(s_o) \leq D^*_\sigma_1(s_o) + D^*_\sigma_2(s_o)
\end{align*}
\]
Since we are given $D_{*} \sigma_{1}(s_{0}) = D_{*} \sigma_{1}(s_{0})$ ($i = 1, 2$), 1.6b) is immediate.

One now easily has 1.6c) as an immediate consequence of 1.6a) and 1.6b).

1.7 Remark. If for some $s$ in $E - E_{o}$ we have $D_{*} \sigma(s) > \alpha$, where $\alpha$ is a constant, it follows from the definition of $D_{*}$ that there is a sequence $M(s)$ in $B$ for which
\[ \limsup_{M \in M(s)} \frac{\sigma(M)}{\mu(M)} > \alpha; \]
and consequently, we find from the definition of $\limsup$ that there is a cofinal subsequence of $M(s)$ which is mapped eventually above $\alpha$ by $\sigma/\mu$. This "entire tail end" of the cofinal subsequence which is mapped above $\alpha$ by $\sigma/\mu$ is again a cofinal subsequence of $M(s)$, hence by property (2) in the definition of $B$ this cofinal subsequence is again a sequence in $B$. Thus at each $s$ in $E - E_{o}$ for which $D_{*} \sigma(s) > \alpha$, we have a sequence $M(s)$ in $B$ such that for $M$ in $M(s)$, $\sigma(M) > \alpha \mu(M)$. Similarly at each point $s$ in $E - E_{o}$ for which $D_{*} \sigma(s) < \alpha$, we have a sequence $M(s)$ in $B$ such that for $M$ in $M(s)$, $\sigma(M) < \alpha \mu(M)$.

1.8 Remark. If $B$ is any derivation basis and $\sigma_{1}$ and $\sigma_{2}$ are any two set functions which agree on each element of $E_{*}$, then $D_{*} \sigma_{1}(s) = D_{*} \sigma_{2}(s)$ and $D_{*} \sigma_{1}(s) = D_{*} \sigma_{2}(s)$ everywhere on $E - E_{o}$, and if the derivative of one exists, finite or infinite, at $s$ in $E - E_{o}$, then $D\sigma_{1}(s) = D\sigma_{2}(s)$.
2. Some Classical Theorems

We list here some definitions and some classical theorems which hold in a sigma-finite, non-negative extended real valued measure space \( \{ S, \mathcal{M}, \mu \} \) and which we shall subsequently use.

A signed \( \mathcal{M} \)-measure \( \sigma_0 \) is singular with respect to \( \mu \) provided there is a set \( M_0 \) in \( \mathcal{M} \) with \( \mu(M_0) = 0 \) and \( \sigma_0 \) is zero on every measurable subset of \( S - M_0 \). A signed \( \mathcal{M} \)-measure \( \sigma_A \) is absolutely continuous with respect to \( \mu \) when for each positive \( \varepsilon \) there is a positive \( \delta(\varepsilon) \) such that when \( \mu \) has value less than \( \delta(\varepsilon) \), then the absolute value of the value of \( \sigma_A \) for the same set is less than \( \varepsilon \). This latter definition has an equivalent form when \( \sigma_A \) is \( \mu \)-finite. In this case \( \sigma_A \) is absolutely continuous with respect to \( \mu \) provided \( \sigma_A \) has the value zero when \( \mu \) has the value zero.

From the first definition of absolute continuity, we see, since \( |\sigma_A| = |-\sigma_A| \), that \( \sigma_A \) is absolutely continuous with respect to \( \mu \) if and only if \( -\sigma_A \) is absolutely continuous with respect to \( \mu \). Also, since \( |\sigma_1 + \sigma_2| \leq |\sigma_1| + |\sigma_2| \), we see that \( \sigma_1 \) and \( \sigma_2 \) each absolutely continuous with respect to \( \mu \) implies \( \sigma_1 + \sigma_2 \) is absolutely continuous with respect to \( \mu \).
2.1 In [1, Th. C, p. 134] we find under different wording but equivalent meaning the following:

**Theorem (Lebesgue decomposition).** If $S$ is sigma-finite with respect to a signed measure $\mu$, then there is a unique pair of sigma-finite signed measures $\sigma_0^+, \sigma_0^-$ such that $\sigma_\mu = \sigma_0^+ + \sigma_0^-$ with $\sigma_0^+$ being absolutely continuous with respect to $\mu$ and $\sigma_0^-$ being singular with respect to $\mu$.

In [1, exercise 3, p. 123] we find definitions of set functions which are equivalent to the first two definitions given below, and in [1, p. 122] we find the definition of the third set function appearing below. If $\sigma_\mu$ is a signed measure, then the functions defined for each $A$ in $\mathcal{M}$ by

$$
\sigma_\mu^+(A) = \text{l.u.b. } \sigma_\mu(M \cap A), \quad M \in \mathcal{M}
$$

$$
\sigma_\mu^-(A) = - \text{g.l.b. } \sigma_\mu(M \cap A), \quad M \in \mathcal{M}
$$

$$
|\sigma_\mu|(A) = \sigma_\mu^+(A) + \sigma_\mu^-(A)
$$

are the upper, lower and total variations of $\sigma_\mu$ on $A$.

2.2 In [1, Th. B, p. 123] we find the following:

**Theorem (Jordan decomposition).** The upper, lower and total variations of a signed measure $\sigma_\mu$ are non-negative measures and $\sigma_\mu(M) = \sigma_\mu^+(M) - \sigma_\mu^-(M)$ for every $M$ in $\mathcal{M}$. If $S$ is sigma-finite with respect to $\sigma_\mu$, then it is also with respect to $\sigma_\mu^+$ and $\sigma_\mu^-$, and at least one of the measures $\sigma_\mu^+$ and $\sigma_\mu^-$ is always finite.

In [1, Th. A, p. 125] we have that if $\sigma_\mu$ is a signed
\( \sigma \)-measure, then \( \sigma^* \) is absolutely continuous with respect to \( \mu \) if and only if \( \sigma^+ \) and \( \sigma^- \) are both absolutely continuous with respect to \( \mu \).

2.3 In [1, Th. B, p. 128] we find the following:

Theorem (Radon-Nikodym). If \( \sigma \) is an absolutely continuous with respect to \( \mu \), \( \mu \)-finite, signed \( \mathbb{W} \)-measure, then there is a finite valued measurable function \( f_0 \) on \( S \) such that \( \sigma(M) = \int_M f_0 \, d\mu \) for every \( M \) in \( \mathbb{W} \).

Note that \( f_0 \) is integrable on \( S \) if and only if \( \sigma(S) \) is finite.

2.4 In [1, Th. B, p. 112] the following appears:

Theorem (Lebesgue monotone convergence). If \( \{ f_n \} \) is a non-decreasing sequence of extended-real valued non-negative measurable functions and if \( \lim f_n(x) = f(x) \) a.e. \( \mu \) on \( S \), then \( \lim \int_M f_n \, d\mu = \int_M f \, d\mu \) for every \( M \) in \( \mathbb{W} \).

2.5 In [3, Th. 21.1, p. 155] the following appears:

Theorem. If \( f_* \) is a non-negative measurable function on \( S \), then there exists a non-decreasing sequence of non-negative simple functions \( \{ f_n \} \) for which \( \lim f_n(s) = f_*(s) \) everywhere in \( S \).
3. **Particular Settings**

We now examine three derivation bases and develop several properties for each of these. One of a particular pair of these properties is always mentioned sooner or later when properties of the derivative are developed. The other of these properties seems never to be mentioned in any writing known to me. The joint significance of these above mentioned two properties will become apparent in the theorems of the next section.

**Derivatives in the Hayes, Pauc Setting.** The first derivation basis we consider is discussed by Hayes and Pauc in [2]. The discussion is typical of those carried on about an abstract measure space. In particular, two weakened forms of the classical Vitali property are set forth and many of their consequences are explored. We will here repeat some of these consequences and add some to the list. The definitions and assumptions of Chapter 1 apply throughout this discussion.

3.1 In [2, p.224] a subcollection $V$ of $B_\mu$ is called a $B$-fine covering of a set $A \subseteq S$ provided for a.e. $s \in A$ there is an admissible generalized sequence $M(s)$ associated with $s$ which is in $B$ and whose constituents are in $V$. We remark that, in view of 1.7, if $A$ is a subset of a set at each point of which $D^*\sigma(s) > a$, a any real number, then the set of all $V$ in $B_\mu$ for which $\sigma(V) > a \mu(V)$
is a \( B \)-fine covering of \( A \). Also by 1.7, if \( A \) is a subset of a set at each point of which \( D \sigma(s) < a \), a any real number, then the set of all \( V \) in \( B \), for which \( \sigma(V) \leq a \mu(V) \) is a \( B \)-fine covering of \( A \).

3.2 Let \( V = \{ V_i \}_{i=1} \) be a countable collection of sets from \( \mathcal{W} \). Let the function \( \Phi_V(s) \) be defined for \( s \) in \( S \) to be the number of sets of \( V \) which contain \( s \), then 

\[
\Phi_V(s) - 1 \text{ gives the amount of excess covering of a point } s \text{ in } U V_1 \text{ by the sets of } V. \text{ For } s \text{ in } S, \Phi_V(s) = \sum C_{V_i}(s),
\]

where \( C_{V_i}(s) \) is the characteristic function of \( V_i \). Thus, since a constant function is always measurable, we see \( \Phi_V(s) - 1 \) is \( \mathcal{W} \)-measurable. Hence, we have for \( \sigma \) an \( \mathcal{W} \)-measure the function 

\[
\omega(V, \sigma) = \int_{U V_1} [\Phi_V(s) - 1] \, d \sigma,
\]

whose values are related to the amount of overlapping in the collection \( V \); hence this function is called the \( \sigma \)-overlap function. In particular, if \( \sigma(U V_1) \) is finite, we have

\[
\omega(V, \sigma) = \sum \sigma(V_i) - \sigma(U V_1).
\]

3.3 The following weakened form of the classical Vitali property appears in [2, p. 229].
Vitali $\sigma$-property (or Vitali property for $\sigma$). If $\sigma$ is an $\mathfrak{M}$-measure, then the derivation basis $B$ has the Vitali $\sigma$-property provided that for

a) any given $A \subseteq S$ with $\bar{\mu}(A) < +\infty$ and
b) any given $B$ - fine covering $V$ of $A$ and
c) any given $\mu$ - cover $M_A$ of $A$ and
d) any given $\varepsilon > 0$,

there must be a countable collection $V_\ast = \{V_i\}_{i=1} \subseteq V$ such that

1) $\bar{\mu}(A - UV_i) = 0$, (zero defect)
2) $\sigma(UV_i - M_A) < \varepsilon$, (measure less than $\varepsilon$)
3) $\omega(V_\ast, \sigma) < \varepsilon$, (overlap less than $\varepsilon$).

When $\sigma = \mu$, the above definition becomes:

Vitali $\mu$-property (or Vitali property for $\mu$). Since $\mu$ is an $\mathfrak{M}$-measure, the derivation basis $B$ has the Vitali $\mu$-property provided that for a), b), c), d), as above, there must be a countable collection $V_\ast \subseteq V$ such that

1) $\bar{\mu}(A - UV_i) = 0$, (zero defect)
2) $\mu(UV_i - M_A) < \varepsilon$, (measure less than $\varepsilon$)
3) $\omega(V_\ast, \mu) < \varepsilon$, (overlap less than $\varepsilon$).

We shall also consider the situation in which condition 2) of the Vitali $\sigma$-property is replaced by the weaker condition

$2') \sigma(UV_i) \leq \sigma(M_A) + \varepsilon$.

We shall refer to the thus modified Vitali $\sigma$-property as the Feeble Vitali $\sigma$-property.
3.4 The reader will note that the following lemma and its proof hold in any non-negative sigma-finite measure space \( \{S, \mathcal{M}, \mu\} \).

**Lemma.** If \( A \) is a subset of \( S \) with \( \mu(A) \) finite and if \( M_A \) is a \( \mu \)-measurable cover of \( A \) and if \( J \) is in \( \mathcal{M} \) with \( \mu(A - J) = 0 \), then \( \mu(M_A - J) = 0 \).

**Proof.** Since \( A = (A \cap M_A \cap J) \cup (A \cap M_A - J) \)
\[ \subseteq (M_A \cap J) \cup (A - J) \] and since \( \mu(A - J) = 0 \) and \( M_A \) is a \( \mu \)-cover of \( A \), we have \( \mu(M_A) = \mu(A) \leq \mu(M_A \cap J) \leq \mu(M_A) \), and thus \( \mu(M_A) = \mu(M_A \cap J) \). Hence, because \( \infty > \mu(M_A) \) = \( \mu(M_A - J) + \mu(M_A \cap J) \), we see that \( \mu(M_A - J) = 0 \).

5.5 Remark. Assuming that 2) in the Vitali \( \mu \)-property holds, we have the following inequality 3.5a).

If the corresponding property in the Vitali \( \sigma \)-property holds, we have 3.5b) by similar reasoning.

3.5a) \( \mu(UV_1) = \mu(UV_1 \cap M_A) + \mu(UV_1 - M_A) \)
\[ \leq \mu(M_A) + \varepsilon \),
3.5b) \( \sigma(UV_1) \leq \sigma(M_A) + \varepsilon \).

Note that 3.5b) is satisfied by definition for the Feeble Vitali \( \sigma \)-property.

Since \( \mu(M_A) = \mu(A) < \infty \), 3.5a) implies always that \( \mu(UV_1) < \infty \). Thus by the last remark of 3.2, 3) of the Vitali \( \mu \)-property may be replaced by \( \Sigma \mu(V_1) - \mu(UV_1) < \varepsilon \); and since \( \mu(UV_1) < \infty \), we have
3.5c) \( \Sigma \mu(V_1) < \mu(UV_1) + \varepsilon \).
If \( \sigma(UV_1) < \infty \), then we have similarly that 3) of the Feeble Vitali \( \sigma \)-property may be written in the form

\[ 3.5d) \quad \sigma(UV_1) \geq \sum \sigma(V_1) - \varepsilon. \]

It is important to note that 3.5d) holds even if \( \sigma(UV_1) = \infty \).

We next note that if 1) of the Vitali \( \mu \)-property holds and is used in 3.4 with \( UV_1 = J \), then \( \mu(M_A - UV_1) = 0 \).

Therefore, if \( \sigma \) is absolutely continuous with respect to \( \mu \), we have \( \sigma(M_A - UV_1) = 0 \). This latter together with \( \sigma(M_A) = \sigma(M_A \cap UV_1) + \sigma(M_A - UV_1) \) yields:

\[ 3.5e) \quad \sigma(M_A) \leq \sigma(UV_1). \]

Finally, if 1) of the Feeble Vitali \( \sigma \)-property holds, we have, since \( UV_1 \) is measurable \( \mathcal{H} \), that \( \mathcal{H}(A) = \mathcal{H}(A - UV_1) + \mathcal{H}(A \cap UV_1) - \mathcal{H}(A \cap UV_1) \). Whence,

\[ 3.5f) \quad \mathcal{H}(A) \leq \mathcal{H}(UV_1). \]

3.6 The following statement and its proof appear in [2, pp. 233-234] under the stronger hypothesis that \( \sigma \) be \( \mu \)-finite and that the Vitali \( \sigma \)-property holds.

**Lemma.** If \( \sigma \) is an \( \mathcal{B} \)-measure, \( B \) possesses the Feeble Vitali \( \sigma \)-property, \( A \subseteq S \) with \( \mathcal{H}(A) < \infty \), \( 0 < \alpha < \infty \), and there exists a \( B \)-fine covering \( V \) of \( A \) such that for \( V \) in \( V \)

\[ \sigma(V) \geq \alpha \mu(V), \]

then \( \sigma(M) \geq \mathcal{H}(A) \) for any \( M \) in \( \mathcal{B} \) with \( A \subseteq M \).

**Proof.** Let \( M \in \mathcal{B} \) with \( M \supseteq A \) and let \( M_A \) be a \( \mu \)-measurable cover of \( A \) such that \( M \supseteq M_A \); then for \( \varepsilon > 0 \)
\[ \sigma(M) \geq \sigma(M_A) \geq \sigma(U V_1) - \varepsilon \geq \sum \sigma(V_i) - 2\varepsilon \geq a \sum \mu(V_i) - 2\varepsilon \geq a \mu(U V_1) - 2\varepsilon \geq a \mu(A) - 2\varepsilon. \]

The justification of the second, third and sixth inequality in the sequence above is the Feeble Vitali \(\sigma\)-property and \(3.5b), 3.5d), 3.5f)\) respectively while the fourth is hypothesis. Since \(\varepsilon > 0\) was arbitrary, the conclusion follows.

**Corollary.** The above lemma holds when in the hypothesis we replace the Feeble Vitali \(\sigma\)-property with the Vitali \(\sigma\)-property.

This is so because the latter implies the former.

**3. 7 Theorem.** If \(\sigma\) is an \(\mathbb{R}\)-measure, \(B\) possesses the Feeble Vitali \(\sigma\)-property, \(0 < a < \infty\) and \(D^* \sigma(s) > a\) for \(s\) in \(A\), then \(\sigma(M) \geq a \mu(A)\) for \(M\) in \(\mathbb{R}\) with \(M \supseteq A\).

**Proof.** Since \(D^* \sigma(s) > a\) for \(s\) in \(A\), we have by 3.1 that the set of all \(V\) in \(B\), for which \(\sigma(V) > a \mu(V)\) is a \(B\)-fine covering of \(A\); hence, if \(\bar{\mu}(A)\) is finite, the conclusion follows from 3.6. If \(\bar{\mu}(A)\) is infinite, then we use the sigma-finite condition on \(S\) to write \(S\) as the union of a sequence of pairwise disjoint, measurable sets of finite \(\mu\)-measure, \(\{M_i\}_{i=1}^\infty\). Let \(A_1 = A \cap M_1\) and let \(M\) in \(\mathbb{R}\) with \(M \supseteq A\). Let \(i M_A\) be a \(\mu\)-measurable cover of \(A_1\) such that \(M \cap M_1 \supseteq i M_A \supseteq A_1\); then from the finite case we have

\[
\sigma(M) = \Sigma \sigma(M \cap M_1) \geq a \Sigma \bar{\mu}(A_1) = a \Sigma \mu(i M_A) = a \mu(U M_A) \geq a \bar{\mu}(A).
\]
Corollary. The above theorem holds if in the hypothesis we replace the Feeble Vitali $\sigma$-property with the Vitali $\sigma$-property.

3. 8 Lemma. If $\sigma$ is an $\mathcal{M}$-measure absolutely continuous with respect to $\mu$, $B$ possesses the Vitali $\mu$-property, $A \subseteq S$ with $\bar{\mu}(A)$ finite, $0 < a < \infty$ and there exists a $B$-fine covering $V$ of $A$ such that for $V$ in $V$

$$\sigma(V) \leq a \mu(V),$$

then $\sigma(M) \leq a \bar{\mu}(A)$ for every $M$ in $\mathcal{M}$ with $M \subseteq M_A$, $M_A$ a $\mu$-measurable cover of $A$.

Proof. Let $M_A$ be a $\mu$-measurable cover of $A$ and let $M$ be in $\mathcal{M}$ with $M \subseteq M_A$, then for $\varepsilon > 0$,

$$\sigma(M) \leq \sigma(M_A) \leq \sigma(UV_1) \leq \Sigma \sigma(V_1) \leq a \Sigma \mu(V_1)$$

$$\leq a (\mu(UV_1) + \varepsilon) \leq a(\mu(M_A) + 2\varepsilon) = a(\bar{\mu}(A) + 2\varepsilon).$$

The justification of the second, fifth and sixth inequality in the sequence above is the Vitali $\mu$-property and 3.5e), 3.5c), 3.5a) respectively while the justification of the fourth inequality is hypothesis.

3. 9 Theorem. If $\sigma$ is an $\mathcal{M}$-measure absolutely continuous with respect to $\mu$, $B$ possesses the Vitali $\mu$-property, $0 < a < \infty$ and $D_* \sigma(s) < a$ for $s$ in $A$, then for any $\mu$-measurable cover, $M_A$, of $A$ and any set $M$ in $\mathcal{M}$ with $M \subseteq M_A$ the inequality $\sigma(M) \leq a \bar{\mu}(A)$ is valid.

Proof. Since $D_* \sigma(s) < a$ for $s$ in $A$, we have by 3.1 that the set of all $V$ in $B_*$ for which $\sigma(V) < a \mu(V)$
is a $\mathcal{B}$-fine covering of $\mathcal{A}$; hence, if $\mathcal{U}(\mathcal{A})$ is finite, the conclusion follows from 3.8. If $\mathcal{U}(\mathcal{A})$ is infinite, then the conclusion is trivial.

3.10 Theorem. If $\sigma_0$ is an $\mathcal{M}$-measure singular with respect to $\mu$ and if $\mathcal{B}$ possesses the Vitali $\sigma_0$-property (or the Feeble Vitali $\sigma_0$-property), then the $\mathcal{B}$-derivative of $\sigma_0$ exists and is equal to zero almost everywhere $\mathcal{U}$ in $\mathcal{S}$.

Proof. Let $M_0$ in $\mathcal{M}$ with $\mu(M_0) = 0$ and with $\sigma_0$ being zero for every measurable subset of $\mathcal{S} - M_0$. Let $A_n = \{ s \mid D^* \sigma_0(s) > 1/n \} \cap (S - M_0)$. By 3.7, $0 = \sigma_0(S - M_0) \geq 1/n \mathcal{U}(A_n)$; hence, $\mathcal{U}(A_n) = 0$. Since $\{ s \mid D^* \sigma_0(s) > 0 \} \cap (S - M_0) = U A_n$ and since

$$\{ s \mid D^* \sigma_0(s) > 0 \} = [\{ s \mid D^* \sigma_0(s) > 0 \} \cap M_0]$$

$U [\{ s \mid D^* \sigma_0(s) > 0 \} \cap (S - M_0)]$, we see that $D^* \sigma_0(s) = 0$ a.e. $\mathcal{U}$ on $\mathcal{S}$. Consequently $D \sigma_0(s) = 0$ a.e. $\mathcal{U}$ on $\mathcal{S}$.

Regular Derivatives in Euclidean $n$-space. Throughout this discussion the definitions and assumptions of 1 apply.

Some proofs concerning regular derivatives in Euclidean $n$-space use the fact that the following Vitali property holds for certain measures. This property is called a strong Vitali property because its third requirement is stronger than the corresponding requirement in the Vitali property of the Hayes-Pauc approach just discussed.
Strong Vitali $\sigma$-property. If $\sigma$ is an $\mathcal{M}$-measure, then the derivation basis $B$ has the Strong Vitali $\sigma$-property if for

a) any given $A \subseteq S$ with $\mu(A) < \infty$ and

b) any given $B$-fine covering $V$ of $A$ and

c) any given $\mu$-cover $M_A$ of $A$ and

d) any given $\varepsilon > 0$,

there is a countable collection $V_* = \{V_1\}_{i=1} \subseteq V$ such that

1) $\mu(A - UV_1) = 0$, (zero defect)

2) $\sigma(UV_1 - M_A) < \varepsilon$, (overflow less than $\varepsilon$)

3) $\omega(V_*, \sigma) = 0$, (overlapping zero).

In particular when $\sigma = \mu$, the above property is explicitly:

Strong Vitali $\mu$-property. Since $\mu$ is an $\mathcal{M}$-measure, the derivation basis $B$ has the Strong Vitali $\mu$-property if for a), b), c), d) as above there is a countable collection $V_* = \{V_1\}_{i=1} \subseteq V$ such that

1) $\mu(A - UV_1) = 0$, (zero defect)

2) $\mu(UV_1 - M_A) < \varepsilon$, (overflow less than $\varepsilon$)

3) $\omega(V_*, \mu) = 0$, (overlapping zero).

3.11 Remark. For a given $B$ and any $\mathcal{M}$-measure $\sigma$ and for the $\mathcal{M}$-measure $\mu$, the Strong Vitali $\sigma$-property implies the Vitali $\sigma$-property and the Strong Vitali $\mu$-property implies the Vitali $\mu$-property. Thus, we immediately have the following results by 3.7 and 3.9.
3.12 Theorem. If $\sigma$ is an $\mathcal{M}$-measure, $\mathcal{B}$ possesses the Strong Vitali $\sigma$-property, $0 < a < \infty$ and $D^* \sigma(s) > a$ for $s$ in $A$, then $\sigma(M) \geq a \mathcal{H}(A)$ for $M$ in $\mathcal{M}$ with $M \supset A$.

3.13 Theorem. If $\sigma$ is an $\mathcal{M}$-measure absolutely continuous with respect to $\mu$, $\mathcal{B}$ possesses the Strong Vitali $\mu$-property, $0 < a < \infty$ and $D^* \sigma(s) < a$ for $s$ in $A$, then for any $\mu$-measurable cover $M_A$ of $A$ and any set $M$ in $\mathcal{M}$ with $M \subseteq M_A$, the inequality $\sigma(M) \leq a \mathcal{H}(A)$ is valid.

3.14 Remark. If $\sigma$ is an $\mathcal{M}$-measure which is absolutely continuous with respect to $\mu$ and if $V$ is a countable collection of $\mathcal{M}$-sets for which $\omega(V, \mu) = 0$, then the integrand in the definition of $\omega(V, \mu)$ is zero a.e. $\mu$ on $UV_1$, $V_1$ in $V$. Hence, this integrand is also zero a.e. $\sigma$ on $UV_1$, $V_1$ in $V$; thus $\omega(V, \sigma) = 0$. Consequently, it is easily seen that if a derivation basis $\mathcal{B}$ has the Strong Vitali $\mu$-property and if $\sigma$ is an $\mathcal{M}$-measure which is absolutely continuous with respect to $\mu$, then $\mathcal{B}$ has the Strong Vitali $\sigma$-property.

We review briefly some relevant definitions and terminology associated with the regular derivative in Euclidean $n$-space and indicate the sequence through which the Strong Vitali properties are known for a certain class of measures. The reader will be referred to standard references included in the list of references at the end of this paper for proofs or suggestions of proofs where these are omitted.
In what follows, with the exception of the discussion appearing in 3.15 and 3.16 where we permit more general spaces, the components of the triple \([S, \mathcal{M}, \mu]\) shall have the following meanings: The set \(S\) shall be Euclidean \(n\)-space \(\mathbb{R}^n\) with \(n > 1\); \(\mathcal{M}\) shall be the Borel sets of \(\mathbb{R}^n\), i.e., the smallest sigma-field containing the closed sets (or equivalently the open sets); \(\mu\) shall represent Lebesgue \(n\)-dimensional measure restricted to \(\mathcal{M}\).

We observe that if \(\mu\), as defined above, is extended to an outer measure on \(\mathbb{R}^n\) by the usual process of taking for each subset \(A\) of \(\mathbb{R}^n\) the greatest lower bound of \(\mu(M)\) for sets \(M\) in \(\mathcal{M}\) which cover \(A\) (Method I in [3]), then the outer measure \(\widetilde{\mu}\) thus constructed is equal to the usual Lebesgue outer measure on \(\mathbb{R}^n\). This is so since there is a Borel set which is for a given subset \(A\) of \(\mathbb{R}^n\) simultaneously a \(\mu\)-measurable cover of \(A\) when \(\mu\) is as defined above and also when \(\mu\) is Lebesgue measure on the Lebesgue measurable sets in \(\mathbb{R}^n\). The ordered \(n\)-tuples \(x = (x_1, x_2, \ldots, x_n)\) of real numbers are the points of \(\mathbb{R}^n\). A closed interval in \(\mathbb{R}^n\) is a set of the form

\[ \{x \mid a_i \leq x_i \leq b_i, \quad i = 1, 2, \ldots, n \} \]

where the \(a_i \leq b_i\) are fixed real numbers. A sequence (ordinary) of closed intervals (or more generally a sequence of sets) in \(\mathbb{R}^n\) is said to converge to a point \(x\) of \(\mathbb{R}^n\) if

1. \(x\) belongs to each of the intervals (sets) of the
sequence and (2) the limit of the diameters of the intervals (sets) in the sequence is zero. In [3, p.269] or [4, p.106] we find a sequence (ordinary) \( \{I_k\}_{k=1}^{\infty} \) of closed intervals (or more generally a sequence \( \{I_k\}_{k=1}^{\infty} \) of Borel sets) which converge to \( x \) in \( \mathbb{R}^n \) is said to be a regular sequence of intervals (or sets) with parameter of regularity \( \alpha \), where \( \alpha > 0 \) is a constant, provided there is for each \( k \) a non-degenerate cube \( J_k \), i.e., an interval \( J_k \) with equal edges none of which is zero length, with \( J_k \supseteq I_k \) for which \( \mu(I_k) / \mu(J_k) > \alpha \).

If for \( x \) in a set \( E \subseteq \mathbb{R}^n \) we define \( M(x) \) to be a regular sequence (ordinary) of closed intervals (i.e., the directed set \( \Delta \) used to index the family \( M(x) \) is the set of positive integers under the binary relation greater than or equal to) which converges to \( x \) and if for each \( x \) in \( E \) we place in \( B \) all of the regular sequences, \( M(x) \), of closed intervals which converge to \( x \), then \( B \) is a measure and the definitions for \( D^+ \sigma(x) \) and \( D^- \sigma(x) \) as given in Chapter 1 become the usual definitions for the regular upper and lower derivatives of \( \sigma \) at \( x \), e.g., see [3, p.269 - 270].

Sometimes a measure is termed a regular measure when for each set \( M \) in its domain, it has the property defined in 3.15 below, which implies an equivalent property concerning closed sets inside \( M \) (see [3, footnote 7*, p.111]). Sometimes the closed sets inside \( M \) are also required to be compact (see [1, p.224]). This latter fact
dictated the choice of terminology we use below. The reader is cautioned to distinguish between the terms regular measure and regular outer measure; the latter concept requires that every subset of the space possess a \( \mu \)-measurable cover (see [3, p.94]).

3.15 A measure \( \sigma \) whose domain shall be designated by \( \mathcal{M}_\sigma \) shall be called semi-regular provided (1) \( \mathcal{M}_\sigma \) contains the open sets of the topological space \( S \) and (2) for each \( M_\sigma \) in \( \mathcal{M}_\sigma \) and each \( \epsilon > 0 \) there is an open set \( O \supseteq M_\sigma \) such that \( \sigma(O - M_\sigma) < \epsilon \).

3.16 Remark. A metric (Caratheodory) outer measure (see [3, p.101]) is a non-negative set function, \( \sigma \), defined on the class of all subsets of a metric space \( S \) and satisfying: (1) If \( A \subseteq B \), then \( \sigma(A) \leq \sigma(B) \); (2) \( \sigma(\cup B_n) \leq \sum \sigma(B_n) \) where \( \{B_n\}_{n=1}^\infty \) is any sequence of subsets of \( S \); (3) \( \sigma(\emptyset) = 0 \); and (4) \( \sigma(A \cup B) = \sigma(A) + \sigma(B) \) if the sets \( A \) and \( B \) are separated by a positive distance. If the non-negative set function \( \sigma \) satisfies only conditions (1), (2), (3) above, then it is called simply an outer measure.

If \( \mathcal{M}_\sigma \) denotes the class of sets measurable with respect to an outer (or metric outer) measure \( \sigma \), then \( M_\sigma \) is in \( \mathcal{M}_\sigma \) if and only if \( \sigma(A) = \sigma(A \cap M_\sigma) + \sigma(A - M_\sigma) \) for every subset \( A \) of \( S \). There are procedures for constructing outer measures from a non-negative extended-real valued set
function \( \tau \) defined on a class \( \mathcal{C} \) of subsets of \( S \) having the properties that \( \emptyset \) is in \( \mathcal{C} \) and also that \( S \), and hence an arbitrary subset \( A \) of \( S \), may be covered by a countable number of sets from \( \mathcal{C} \). In particular, if \( \tau(\emptyset) = 0 \) and if for \( A \subseteq S \), \( \sigma(A) = \text{g.l.b.} \Sigma \tau(C_n), C_n \in \mathcal{C}, U C_n \supseteq A \), then \( \sigma \) is an outer measure. This procedure for defining an outer measure is called "Method I" in [3, p.90]. This procedure does not always yield a metric outer measure; however, under certain conditions (see [3, p.112]) a metric outer measure is produced. A variant of this Method I which is called "Method II" by Munroe (see [3, p.105]) always yields a metric outer measure.

It is shown [3, Th. 13.7.1, p.111] that if a metric outer measure, \( \sigma \), is constructed from \( \mathcal{C} \) and \( \tau \) by either Method I or Method II and if \( \mathcal{C} \) consists of open sets and if \( S \) is covered by a countable number of open sets each of finite \( \sigma \)-measure, then \( \sigma \) is semi-regular.

3.17 Remark. We call the readers' attention to one important class of measures on \( \mathbb{R}^n \) which is known to be semi-regular. This class of measures is designated as Lebesgue-Stieltjes measures. For the technique of construction and properties of Lebesgue-Stieltjes measures, we refer to [3, p.115-127]. In particular, we call attention to the remarks of [3, 2nd paragraph, p.117] and [3, 2nd paragraph, p.124] where Monroe notes first that
Lebesgue-Stieltjes outer measure in Euclidean space $\mathbb{R}^n$ satisfies the conditions stipulated in the preceding paragraph; hence, Lebesgue-Stieltjes measure is semi-regular and, secondly, the same is true for Lebesgue-Stieltjes outer measure in Euclidean space $\mathbb{R}^n$. It is noted [3, exercise g, p.120] that the usual Lebesgue outer measure on the real line is a Lebesgue-Stieltjes outer measure. The key to demonstrating this result appears in [3, Th. 11.4, p.92]. Also, it is noted [3, paragraph 2, p.121] that the usual Lebesgue outer measure in $\mathbb{R}^n$ is a Lebesgue-Stieltjes outer measure. Finally, we note that in [3, Th. 38.1, p.268] it is shown that the Lebesgue integral of a non-negative Lebesgue integrable function on $\mathbb{R}^n$ is a Lebesgue-Stieltjes measure.

Finally, we note that since Lebesgue-Stieltjes outer measure is constructed on $\mathbb{R}^n$ by Method I from a covering class consisting of all the open intervals of $\mathbb{R}^n$, each of these open intervals is a set which is measurable with respect to Lebesgue-Stieltjes measure. Thus, since in $\mathbb{R}^n$ there is a countable subcollection of open intervals (those with rational corner points) which is a base for the usual topology of $\mathbb{R}^n$, we have immediately that the open sets of $\mathbb{R}^n$ are in the class of sets measurable with respect to Lebesgue-Stieltjes measures in $\mathbb{R}^n$; hence, the Borel sets of $\mathbb{R}^n$ are measurable with respect to Lebesgue-Stieltjes
measures in \( \mathbb{R}^n \). Thus, we see the Lebesgue-Stieltjes measures in \( \mathbb{R}^n \) are \( \mathcal{M} \)-measures where \( \mathcal{M} \), as agreed in 3.14, is the class of Borel sets of \( \mathbb{R}^n \).

3.18 The following definition and theorem appear in [4, p.109]. A family \( I \) of \( \mathcal{M} \)-sets is said to cover a set \( A \) in the sense of Vitali if for every \( x \) in \( A \) there is a regular sequence of sets from \( I \) which converge to \( x \).

Theorem (Vitali). If a collection, \( I \), of closed sets covers a subset \( A \) of \( \mathbb{R}^n \) in the sense of Vitali, then there is a pairwise disjoint (finite or infinite) sequence, \( \{J_k\}_{k=1}^\infty \subseteq I \), such that \( \mu(A - UJ_k) = 0 \).

3.19 Remark. If the hypothesis of the theorem of Vitali, above, is altered to: If a collection, \( I \), of closed sets covers in the sense of Vitali all of \( A \) except for a subset of \( \mu \)-measure zero, then the conclusion remains valid. We observe that this modified hypothesis of the theorem of Vitali requires that for a.e. \( x \) in \( A \) there be a regular sequence of closed sets in \( I \) which converges to \( x \). Thus, if these sequences of closed sets are in a derivation basis \( B \) and if \( I \subseteq B \), this modified hypothesis of the theorem of Vitali is equivalent to requiring that the family \( I \) be a \( B \)-fine covering of \( A \).

3.20 Theorem. If \( \sigma \) is a semi-regular \( \mathcal{M} \)-measure (\( \mathcal{M} \) being the class of Borel sets of \( \mathbb{R}^n \)) and if \( B \) consists of regular sequences (ordinary) of closed sets associated
with the points of \( S = \mathbb{R}^n \), each of which converges to its associated point, then \( B \) possesses the Strong Vitali \( \sigma \)-property.

**Proof.** Let \( A \subseteq \mathbb{R}^n \) with \( \mu(A) < \infty \), let \( V \) be a \( \mathcal{B} \)-fine covering of \( A \), let \( M_A \in \mathcal{M} \) be a \( \mu \)-cover of \( A \) and let \( \varepsilon > 0 \) be given. By the semi-regularity of \( \sigma \), there is an open set \( G \supseteq M_A \) such that

\[
\sigma(G - M_A) < \varepsilon.
\]

Since \( V \) is a \( \mathcal{B} \)-fine covering of \( A \), there is for a.e. \( \bar{x} \) in \( A \) a sequence of closed sets \( M(x) \) in \( B \) whose constituents are in \( V \) and which, since it converges to \( x \), has its constituents eventually inside \( G \). The constituents of these subsequences inside \( G \) form a \( \mathcal{B} \)-fine covering of \( A \). Thus, by 3.18 and 3.19 there is a pairwise disjoint sequence \( \{ V_i \}_{i=1} \subseteq V \) such that \( \cup V_i \subseteq G \) and \( \sigma(A - \cup V_i) = 0 \). Thus, 1) and 3) of the Strong Vitali \( \sigma \)-property are satisfied; and since

\[
\cup V_i - M_A \subseteq G - M_A,
\]

we have that 2) of the Strong Vitali \( \sigma \)-property is satisfied.

**3.21 Remark.** If \( \mathcal{B} \) is the derivation basis mentioned in 3.14 which is customarily used for calculating the derivative in \( \mathbb{R}^n \) designated as the regular derivative, then \( \mathcal{B} \) is as required in 3.20. Also, we have noted in 3.17 that any Lebesgue-Stieltjes measure on \( \mathbb{R}^n \) is a semi-regular
Thus, by 3.20 we have that $B$ possesses the Strong Vitali $\sigma$-property where $\sigma$ is any Lebesgue-Stieltjes measure on $R^n$. In particular, $B$ possesses the Strong Vitali $\mu$ property where $\mu$ is Lebesgue measure on $R^n$ restricted to the Borel sets of $R^n$.

It is not generally true that the derivative of a singular measure is zero a.e. Consider the following example. Let a function $f(x)$ be defined on $R^1$ by

$$f(x) = 0 \text{ if } x \text{ is irrational},$$

$$f(x) = \infty \text{ if } x \text{ is rational},$$

then the set function $\sigma_0$ defined on each subset of $R^1$ by the formula

$$\sigma_0(A) = \sum f(x), \quad x \in A,$$

is, if we explicitly define $\sigma_0(\emptyset) = 0$, easily seen to be a metric outer measure whose class of measurable sets, $\mathcal{M}_0$, contains every subset of $R^1$. Thus, $\sigma_0$ is an $\mathcal{M}$-measure; and since outside the set of rationals $\sigma_0$ is zero, we see that $\sigma_0$ is singular with respect to $\mu$. It is easily checked that the regular derivative of $\sigma_0$ with respect to $\mu$ is identically $+\infty$ on $R^1$.

3.22 Theorem. If $\sigma_0$ is a semi-regular $\mathcal{M}$-measure on $R^n$ which is singular with respect to $\mu$, then for the regular derivative we have $D\sigma_0(x) = 0$ a.e. $\mu$ in $R^n$.

Proof. This is an immediate consequence of 3.20, 3.11 and 3.10.
Corollary. If $\sigma_o$ is a Lebesgue Stieltjes measure on $\mathbb{R}^n$ which is singular with respect to $\mu$, then for the regular derivative we have $D\sigma_o(x) = 0$ a.e. $\mu$ in $\mathbb{R}^n$.

Proof. Immediate from 3.17.

Derivatives with Respect to Nets. Some of the basic properties of the derivation basis we are about to consider were set forth by Saks [4, pp. 152-156] in the setting of a separable metric space. These ideas have been carried to an abstract measure space by Monroe [3, pp. 296-302]. We present here with a slight modification some of the important definitions and a few of the results appearing in [3, pp. 296-302]. We then add in this and succeeding sections some results of our own, e.g., 3.27, 3.29, 3.30, all of Chapter 4, all of Chapter 5, the results in Chapter 6 as applied to the construction of a monotone sequence of nets, in Chapter 7 we mention 7.6 especially and to some degree the rest of the section with the exception of 7.1 and 7.4, and all of Chapter 8.

The definitions and assumptions of Chapter 1 apply throughout this discussion.

3.23 A net $\mathcal{M}$ in $S$ is a countable class of non-empty pairwise disjoint measurable sets of finite $\mu$-measure whose union is $S$. Thus, a net is a partitioning of $S$ into a countable number of measurable sets of finite $\mu$-measure. A monotone sequence of nets $\{\mathcal{M}_i\}_{i=1}^\infty$ is a sequence of nets (partitions) in $S$ which has the property that for each
positive integer \( n \) every set of \( \mathcal{T}_{n+1} \) is a subset of some set of \( \mathcal{T}_n \). Thus, a sequence of nets is monotone provided succeeding nets of the sequence are refinements of their predecessors in the sequence. If a space \( S \) has a monotone sequence of nets introduced upon it, then with each \( s \) in \( S \) there is in each net \( \mathcal{T}_n \) a unique set which contains \( s \). Thus, if we designate by \( M_{ns} \) the unique set in \( \mathcal{T}_n \) which contains \( s \), we have a unique ordinary sequence \( M(s) = \{M_{ns}\}_{n=1}^{\infty} \) of measurable sets associated with each \( s \) in \( S \). This sequence has the property that for each \( n \), \( M_{ns} \supseteq M_{n+1} \) if \( s \). The natural choice for a derivation basis, \( B \), in this situation consists of the ordinary sequences \( M(s), s \) in \( S \), and all their subsequences. (Recall that an ordinary sequence is the functional image of the directed set \( \Delta \) of positive integers having the binary relation greater than or equal to.) We note that, because the totality of the sets occurring in a monotone sequence of nets is countable, it follows that the sequences \( M(s) \) which contain sets of \( \mu \)-measure zero are associated with points \( s \) in a subset \( E_0 \) of \( S \) which is of \( \mu \)-measure zero. Hence, the natural choice, as indicated above, for the derivation basis \( B \) generated by a monotone sequence of nets on \( S \) satisfies the requirements imposed on a derivation basis in Chapter 1 (see page 10). In view of remark 1.2, the definition for \( D \) and \( D^* \), as given in 1 may be modified by
omitting the "sup" and "inf" provided we use the single master sequence $M(s) = \{M_{ns}\}_{n=1}^{\infty}$ for calculating the "lim sup" and "lim inf" which remain. $B_\delta$ now consists of all of the sets $M_{ns}$, $(n = 1, 2, \ldots)$, $s \in S$; and we note that $B_\delta$ is a countable collection.

3.24 In [3, p.297] the following definition appears without reference to the special case involving the empty set.

Definition. (Regularity of a monotone sequence of nets with respect to $\mu$ on a set $M$ in $\mathbb{R}$). A monotone sequence of nets $\{M_i\}_{i=1}^{\infty}$ is regular with respect to $\mu$ on a non-empty set $M$ in $\mathbb{R}$ provided for each $\varepsilon > 0$ there is a sequence $\{K_i\}_{i=1}^{\infty}$ of sets from $B_\delta$ such that the following two conditions are satisfied:

1) $\mu(M - UK_i) = 0$
2) $\mu(UK_i) \leq \mu(M) + \varepsilon$.

We shall agree that every monotone sequence of nets is regular with respect to $\mu$ on the empty set. This agreement is in accord with the spirit of the above definition of regularity of a monotone sequence of nets on a non-empty set $M$ in $\mathbb{R}$ in that it essentially requires that there be a subcollection of $B_\delta$ which 1) covers almost all of $M$ and 2) $\mu$ of the union of the sets in this collection is less than or equal to $\mu(M) + \varepsilon$. When $M$ is the empty set, we could choose, in accord with the above mentioned
spirit, the subcollection of $B_\#$ to be the empty collection and agree that the union over an empty collection is the empty set. Thus, by making the agreement that every monotone sequence of nets is regular with respect to $\mu$ on the empty set, we avoid a lot of tedious, repetitive, special-case argument.

In [3, p.298] it is implied in the proof of lemma 4.3.2 that net regularity with respect to an $\mathbb{M}$-measure, $\sigma$, is to mean the above definition with $\mu$ replaced by $\sigma$ in both conditions. However, we modify this as indicated below.

**Definition (Regularity of a monotone sequence of nets with respect to a signed $\mathbb{M}$-measure $\sigma$ on a set $M$ in $\mathbb{M}$).** A monotone sequence of nets $\{\mathcal{M}_i\}_{i=1}^\infty$ is regular with respect to a signed $\mathbb{M}$-measure $\sigma$ on a non-empty set $M$ in $\mathbb{M}$ provided for each $\varepsilon > 0$ there is a sequence $\{K_i\}_{i=1}^\infty$ of sets from $B_\#$ such that the following two conditions are satisfied:

1) $\mu(M - UK_i) = 0$,

2) $\sigma(UK_i) \leq \sigma(M) + \varepsilon$.

We agree that every monotone sequence of nets is regular with respect to a signed $\mathbb{M}$-measure $\sigma$ on the empty set.

We note that this latter definition permits us to consider the first definition above as a special case simply by replacing $\sigma$ by $\mu$. We also note that the above regularity definition clearly depends not only on $\sigma$ but also on $\mu$; however, since in the situations we consider, $\mu$,
the basic measure of the triple \([S, \mathcal{M}, \mu]\), will be fixed, we do not explicitly call attention to it in the title of this regularity property.

3.25 The following theorem and proof appear in [3, Th. 43.1, p.297].

**Theorem.** If \(\{\mathcal{M}_i\}_{i=1}^n\) is a monotone sequence of nets in \(S\) and if \(K\) is a subcollection of sets from \(\mathcal{B}\), then there is a pairwise disjoint sequence \(\{K_i\}_{i=1}^n \subseteq K\) such that

\[
\bigcup_{i=1}^n K_i = \bigcup K \text{ in } K.
\]

**Proof.** Define inductively \(K_1 = \{K \in K \mid K \text{ is in } \mathcal{M}_1\}, \ldots, K_n = \{K \in K \mid K \text{ is in } \mathcal{M}_n\} \text{ and for which there does not exist a set } K_0 \text{ in } \bigcup_{i=1}^n K_i, i = 1, 2, \ldots, n - 1, \text{ such that } K \subseteq K_0\}, \ldots. \text{ For each positive integer } n, K_n \text{ is countable; therefore, } \bigcup_{n=1}^\infty K_n \text{ is countable. Clearly } \bigcup_{n=1}^\infty K_n \text{ is a pairwise disjoint collection. Thus, upon arranging the sets in } \bigcup K_n \text{ into a sequence } \{K_i\}_{i=1}^n \text{, the conclusion is established.}

3.26 This theorem below is a modification of one appearing in [3, Lemma 43.2, p.298].

**Theorem.** If \(\{\mathcal{M}_i\}_{i=1}^n\) is a monotone sequence of nets in \(S\) which is regular with respect to \(\mu\) on each \(M\) in \(\mathcal{M}\) and if \(\sigma\) is a signed \(\mathcal{M}\)-measure which is absolutely continuous with respect to \(\mu\), then \(\{\mathcal{M}_i\}_{i=1}^n\) is regular with respect to \(\sigma\) on each \(M\) in \(\mathcal{M}\).

**Proof.** We make the proof first for the case when
M is a subset of one of the sets $M_0$ of $\mathbb{R}_1$. Let $\varepsilon > 0$ be given and let $\delta > 0$ be the $\delta$ associated with $\varepsilon$ in the criteria for absolute continuity of $\sigma$ with respect to $\mu$ (see Chapter 2). Applying the regularity of $[\mathbb{R}_1]_{i=1}^N$ with respect to $\mu$ on $M$, we find, for the $\varepsilon$ of net regularity chosen to be $\varepsilon_0 = \frac{1}{2} \min \{\varepsilon, \delta\}$, there is a countable sequence $\{K_i\}_{i=1}^\infty \subseteq B_\mu$ such that

3.26a) $\mu(M - UK_i) = 0$ and $\mu(UK_i) \leq \mu(M) + \varepsilon_0$.

Observe that because of the monotone property of the sequence $[\mathbb{R}_1]_{i=1}^N$ all of the members of the sequence $[K_i]_{i=1}^\infty$ can be chosen to be inside $M_0$. From the following:

$$
\mu(UK_i - M) = \mu(UK_i - [M \cap U K_i]) = \mu(UK_i) - \mu(M \cap U K_i) - \mu(M - UK_i) = \mu(UK_i) - \mu(M) \leq \varepsilon_0
$$

we see because of 3.26a) that $\mu(UK_i - M) \leq \varepsilon_0 < \delta$. Thus, absolute continuity gives us $\sigma(UK_i - M) \leq \varepsilon$ and also $\sigma(M - UK_i) = 0$, whence

$$
\sigma(UK_i) = \sigma(UK_i - M) + \sigma(UK_i \cap M) + \sigma(M - UK_i) = \sigma(UK_i - M) + \sigma(M) \leq \varepsilon + \sigma(M),
$$

and regularity of $[\mathbb{R}_1]_{i=1}^N$ with respect to $\sigma$ on $M$ is established for this case.

Now let $M$ be arbitrary. Index the sets in $\mathbb{R}_1$ as $\{M_i\}_{i=1}^\infty$ and form the set $M_0 = M \cap M_i$. Let $\varepsilon > 0$ be given; then, since $[\mathbb{R}_1]_{i=1}^\infty$ is regular with respect to $\mu$ on each $M$ in $\mathbb{R}$, we have by the special case just proved that there
is a sequence \( \{K_{ij}\}_{j=1} \) with \( K_{ij} \subseteq M_1 \) for each \( j \) such that 
\[
\mu(M_1^O - U_j K_{ij}) = 0 \text{ and } \sigma(U_j K_{ij}) \leq \sigma(M_1^O) + \epsilon/2^j.
\]
Because of the disjointness involved, we have
\[
0 = \sum_i \mu(M_1^O - U_j K_{ij}) = \mu(U_i (M_1^O - U_j K_{ij}))
\]
\[
= \mu(U M_1^O - U_i U_j K_{ij}) = \mu(M - U_i U_j K_{ij})
\]
and
\[
\sigma(U_i U_j K_{ij}) = \sum_i \sigma(U_j K_{ij}) \leq \sum_i \sigma(M_1^O) + \epsilon
\]
\[
= \sigma(U M_1^O) + \epsilon = \sigma(M) + \epsilon.
\]
Since the collection \( \{K_{ij}\}_{i=1} \), \( j=1 \) is countable, we are done.

**3.27 Theorem.** If \( \sigma(\mu) \) is an \( \mathcal{M} \)-measure and if 
\( \{M_i\}_{i=1} \) is a monotone sequence of nets on \( S \) which is regular with respect to \( \sigma(\mu) \) on each \( M \) in \( \mathcal{M} \) and if \( B \) is the derivation basis associated with \( \{M_i\}_{i=1} \) as indicated in 3.23, then \( B \) possesses the Feeble Vitali \( \sigma \)-property (Strong Vitali \( \mu \)-property). In general, under the above hypothesis if \( \sigma \neq \mu \), then \( B \) will not possess property 2) of the Strong Vitali \( \sigma \)-property (or the Vitali \( \sigma \)-property); however, if \( \sigma \) is \( \mu \)-finite and absolutely continuous with respect to \( \mu \), then \( B \) possesses 2) of the Strong Vitali \( \sigma \)-property; and hence, in this case, \( B \) possesses the Strong Vitali \( \sigma \)-property.

**Proof.** Let \( A \) be a subset of \( S \) such that \( \mu(A) < \infty \), let \( M_A \) be a \( \mu \)-cover of \( A \), and let \( \epsilon > 0 \) be given. If \( V \)
is a $B$-fine covering of $A$, then for every $s$ in $A$ except possibly those $s$ in a set $A_0 \subseteq A$ with $\mathcal{N}(A_0) = 0$ there is associated a sequence of $V$-sets which is a subsequence $\{M_n(i)_s\}_{i=1}^{\infty}$ of $\{M_n(s)\}_{n=1}^{\infty}$, where $M_n(s)$ is the unique set of $V_n$ which contains $s$. If the sets of $\mathcal{N}$ are designated by $\{M_k\}_{k=1}^{\infty}$, we can apply the monotoneness and regularity of $\{\mathcal{N}_n\}_{n=1}^{\infty}$ with respect to $\sigma$ (or $\nu$, if we replace $\sigma$ by $\nu$ in 3.27c below) on $M_A \cap M_k$ to get sequences $\{J_{kj}\}_{j=1}^{\infty} \subseteq \mathbb{B}$ such that

3.27a) $U_j J_{kj} \subseteq M_k$,  
3.27b) $\mu(M_A \cap M_k - U_j J_{kj}) = 0$,  
3.27c) $\sigma(U_j J_{kj}) \leq \sigma(M_A \cap M_k) + \varepsilon/2^k$.

For each $s$ in $(A - A_0) \cap U_j J_{kj}$ we have, because of the monotone character of $\{\mathcal{N}_n\}_{n=1}^{\infty}$, that there is an $i_s(s)$ such that every member of $\{M_n(i)_s\}_{i=1}^{i_s(s)}$ is inside $U_j J_{kj}$. The totality of the sets in these "tail end" subsequences associated with $s$ in $(A - A_0) \cap U_j J_{kj}$ is countable, and from this collection, we may by 3.25 extract a pairwise disjoint subcollection taken notationally as $\{J_{km}\}_{m=1}^{\infty}$ which satisfies

3.27d) $(A - A_0) \cap U_j J_{kj} \subseteq \bigcup_{m=1}^{\infty} J_{km} \subseteq U_j J_{kj} \subseteq M_k$.

We now show that the sequence $\{J_{km}\}_{m=1}^{\infty}$, $k=1$ is the sub-collection of $V$ required in properties 1) and 3) of the
Feeble Vitali $\sigma$-property (or Strong Vitali $\mu$-property).

Since \( \{J_{*km}\}_{m=1} \), \( k=1 \) is pairwise disjoint, the $\sigma$-overlap (or $\mu$-overlap) is zero; hence, 3) of the Feeble Vitali $\sigma$-property (or Strong Vitali $\mu$-property) is immediate.

From

\[
A = (A - U_{j,k} J_{kj}) U (A \cap U_{j,k} J_{kj}) = (A - U_{j,k} J_{kj})
\]

\[
U[(A - A_o) \cap U_{j,k} J_{kj}] U (A_o \cap U_{j,k} J_{kj})
\]

we have

\[
A - U_{m,k} J_{*km} = [A - (U_{j,k} J_{kj} U U_{m,k} J_{*km})]
\]

\[
U[ [(A - A_o) \cap U_{j,k} J_{kj}] - U_{m,k} J_{*km}]
\]

\[
U [ (A_o \cap U_{j,k} J_{kj}) - U_{m,k} J_{*mk}] ;
\]

and since by 3.27d) the set in braces is empty, we have

\[
A - U_{m,k} J_{*km} \subseteq U_{k} [(M_{A} \cap M_{k}) - U_{j} J_{kj}] U \emptyset U A_o ,
\]

which with 3.27b) yields \( \overline{\mu}(A - U_{m,k} J_{*mk}) = 0 \). Thus 1) of the Feeble Vitali $\sigma$-property (or Strong Vitali $\mu$-property) is satisfied.

With the help of 3.27d) and 3.27c), it is easily seen that \( \{J_{*km}\}_{m=1} \), \( k=1 \) satisfies condition 2') of the Feeble Vitali $\sigma$-property; hence, \( B \) possesses the Feeble Vitali $\sigma$-property.

We now turn our attention to condition 2) of the Strong Vitali $\sigma$-property (or Strong Vitali $\mu$-property).

Inequality 3.27c) can be written as
3.27e) \[ \sigma(U_j J_{kj}) \leq \sigma(M_A \cap M_k) + \varepsilon/2^k = \sigma(M_A \cap M_k) \cap U_j J_{kj} \] + \sigma(M_A \cap M_k - U_j J_{kj}) + \varepsilon/2^k.

Whence, if

3.27f) \[ \sigma(M_A \cap M_k \cap U_j J_{kj}) < \infty, \]

we may write 3.27e) as

\[ \sigma(U_j J_{kj} - M_A \cap M_k \cap U_j J_{kj}) \leq \sigma(M_A \cap M_k - U_j J_{kj}) \]
\[ + \varepsilon/2^k; \]

hence, using 3.27a),

\[ \sigma(U_j J_{kj} - M_A) \leq \sigma(M_A \cap M_k - U_j J_{kj}) + \varepsilon/2^k. \]

Now summing both sides of the above inequality with respect to \( k \) yields, because of the disjointness of the sets involved,

\[ \sigma \left( \bigcup_k (U_j J_{kj} - M_A) \right) \leq \sigma \left( \bigcup_k (M_A \cap M_k - U_j J_{kj}) \right) + \varepsilon, \]

which in turn, because \( (U_j J_{kj}) \cap M_{k_0} = \emptyset \) unless \( k = k_0 \), reduces to

\[ \sigma(U_j, k J_{kj} - M_A) \leq \sigma(M_A - U_j, k J_{kj}) + \varepsilon. \]

Using this, together with the fact that 3.27d) implies

\[ U_j, k J_{kj} - M_A \supseteq U_m, k J_{mk} - M_A, \]

we have

3.27g) \[ \sigma(U_{m, k} J_{km} - M_A) \leq \sigma(M_A - U_j, k J_{kj}) + \varepsilon. \]

It is now apparent that \( \{ J_{km} \}_{m=1}^{k=1} \), a countable sub-collection of \( V \), will also satisfy 2) of the Strong Vitali \( \sigma \)-property if

3.27h) \[ \sigma(M_A - U_j, k J_{kj}) = 0; \]
or more generally, if the left side of 3.27h) can be made arbitrarily small. We observe that if \( \sigma = \mu \) in the discussion from 3.27e) on, then clearly the assumptions 3.27h) and 3.27f) are satisfied; and hence, \( B \) possesses the Strong Vitali \( \mu \)-property if \( \{ \mathfrak{M}_i \}_{i=1}^{\infty} \) is regular with respect to \( \mu \) on each \( M \) in \( \mathfrak{M} \). We also note if \( \sigma \neq \mu \) in the discussion from 3.27e) on that we can if \( \sigma \) is \( \mu \)-finite have 3.27f) satisfied, and if \( \sigma \) is absolutely continuous with respect to \( \mu \), we have 3.27h) satisfied. Thus, if \( \sigma \) satisfies these two conditions, \( B \) possesses the Strong Vitali \( \sigma \)-property.

We now turn our attention to an example which shows that the pair of conditions 1) \( \sigma \) is \( \mu \)-finite and 2) \( \sigma \) is absolutely continuous with respect to \( \mu \) cannot both be disposed of if the Strong Vitali \( \sigma \)-property is to be implied by net regularity with respect to \( \sigma \) on each \( M \) in \( \mathfrak{M} \).

Example: Let \( \mathfrak{M} \) be the class of Borel sets on \( \mathbb{R}^1 \) and let \( \mu \) be Lebesgue measure on \( \mathbb{R}^1 \) restricted to \( \mathfrak{M} \). Let \( \sigma \) be identically infinite on every subset of \( \mathbb{R}^1 \) except the empty set and let \( \sigma(\emptyset) = 0 \), then \( \sigma \) is an \( \mathfrak{M} \)-measure which is not \( \mu \)-finite and not absolutely continuous with respect to \( \mu \). For a monotone sequence of nets on \( \mathbb{R}^1 \) we define inductively the collections of half open intervals.

\[
\mathfrak{I}_n = \left\{ \left[ \frac{j}{2^n}, \frac{j+1}{2^n} \right] \mid j \text{ an integer} \right\}.
\]
Let $\mathcal{B}$ be the natural derivation basis induced by $\{\mathcal{W}_n\}_{n=1}^\infty$ (see 3.23); then, if $M$ is any non-empty set in $\mathcal{M}$, any covering of $M$ by sets from $\mathcal{B}$ will satisfy the definition of $\{\mathcal{W}_n\}_{n=1}^\infty$ being regular with respect to $\sigma$ on $M$, and because of our agreement about regularity on $\emptyset$, it is easily seen that $\{\mathcal{W}_n\}_{n=1}^\infty$ is regular with respect to $\sigma$ on each $M$ in $\mathcal{M}$. That the Strong Vitali $\sigma$-property (or the Vitali $\sigma$-property) is not possessed by $\mathcal{B}$ may be seen by considering the set $A = M_A = \{x \mid x \text{ is in } [0, 1] \text{ and } x \text{ is irrational}\}$. Clearly $\mathcal{W}(A) < \infty$. Let $\mathcal{V}$ be a $\mathcal{B}$-fine covering of $A$; then we have shown above that we may immediately conclude conditions 1) and 3) of the Strong Vitali $\sigma$-property (or the Vitali $\sigma$-property) are satisfied. Since any countable subcollection $\mathcal{V}_* = \{V_i\}_{i=1}^\infty$ of $\mathcal{V}$ which covers almost all, with respect to $\mu$, of $A$ must cover at least one rational point of $[0, 1]$, we see that

$$\sigma(UV_1 - M_A) = \infty \cdot \frac{1}{\varepsilon}$$

for any real $\varepsilon > 0$. Thus, the Strong Vitali $\sigma$-property (or Vitali $\sigma$-property) is not satisfied. This completes the proof of 3.27.

Remark. The example just considered above shows that the Feeble Vitali $\sigma$-property is weaker than the Vitali $\sigma$-property.

3.28 Theorem. If $\sigma$ is an $\mathcal{M}$-measure, $\mathcal{B}$ results from a monotone sequence of nets on $S$ which is regular.
with respect to \( \sigma \) on each \( M \) in \( \mathbb{M} \), \( 0 < a < \infty \) and \( D^* \sigma(s) > a \) for \( s \) in \( A \); then \( \sigma(M) \geq a \bar{\mathcal{H}}(A) \) for \( M \) in \( \mathbb{M} \) with \( M \supset A \).

**Proof.** By 3.27 \( \mathbb{B} \) possesses the Feeble Vitali \( \sigma \)-property; hence, this is an immediate consequence of the theorem in 3.7.

**3.29 Theorem.** If \( \sigma \) is an \( \mathbb{M} \)-measure absolutely continuous with respect to \( \mu \), if \( \mathbb{B} \) results from a monotone sequence of nets on \( S \) which is regular with respect to \( \mu \) on each \( M \) in \( \mathbb{M} \), if \( 0 < a < \infty \) and if \( A \subseteq S \) is such that \( s \) in \( A \) implies \( D^* \sigma(s) < a \), then for any \( \mu \)-measurable cover \( M_A \) of \( A \) and any set \( M \) in \( \mathbb{M} \) with \( M \subseteq M_A \), the inequality \( \sigma(M) < a \bar{\mathcal{H}}(A) \) is valid.

**Proof.** Immediate from 3.27 and 3.9.

**3.30 Theorem.** If \( \sigma_0 \) is an \( \mathbb{M} \)-measure singular with respect to \( \mu \) and \( \mathbb{B} \) results from a monotone sequence of nets on \( S \) which is regular with respect to \( \sigma_0 \) on each \( M \) in \( \mathbb{M} \), then the \( \mathbb{B} \)-derivative of \( \sigma_0 \) with respect to \( \mu \) exists and is equal to zero almost everywhere \( \bar{\mathcal{H}} \) in \( S \).

**Proof.** By 3.27 \( \sigma_0 \) possesses the Feeble Vitali \( \sigma_0 \)-property; hence, 3.10 applies and we are done.
4. A Theorem on Equivalence of Functions

The following theorem has a number of important and interesting consequences.

4.1 Theorem. Let \([S, \mathcal{M}, \mu]\) be a non-negative sigma-finite measure space. If \(\sigma\) is a non-negative set function defined at least on \(\mathcal{M}\) and if \(\tau^0(s)\) and \(\tau_0(s)\) are two non-negative extended real valued point functions defined on \(S\) and if the following two implicativest statements are satisfied where \(A\) represents a subset of \(S\) and \(M_A\) is a \(\mu\)-measurable cover of \(A\),

4.1 a) \(\tau^0(s) > a\) for \(s\) in \(A - A_0\) with \(\mathcal{I}(A_0) = 0\)
implies \(\sigma(M_A) \geq a \mathcal{I}(A)\),

4.1 b) \(\tau_0(s) < a\) for \(s\) in \(A - A_0\) with \(\mathcal{I}(A_0) = 0\)
implies \(\sigma(M_A) \leq a \mathcal{I}(A)\),

then \(\tau^0(s) \leq \tau_0(s)\) a.e. \(\mathcal{I}\) on \(S\).

Proof. Place \(A_{hk} = \{s| \mathcal{A}(s) > \frac{h + 1}{k} > \frac{h}{k} > \tau^0(s)\}\)
where \(h, k\) are positive integers. Let \(\{M_1\}\) be a pairwise disjoint sequence of \(\mathcal{M}\) - sets of finite \(\mu\)-measure which cover \(S\) and place \(i A_{hk} = M_1 \cap A_{hk}\). If \(i A_{hk} = \emptyset\), then
\(\mathcal{I}(i A_{hk}) = 0\). If \(i A_{hk} \neq \emptyset\), then by 4.1 a) and 4.1 b) we have, if \(i A_{hk}\) is a \(\mu\)-measurable cover of \(i A_{hk}\), that

\[
\frac{h}{k} \mathcal{I}(i A_{hk}) \geq \sigma(i A_{hk}) \geq \frac{h + 1}{k} \mathcal{I}(i A_{hk}).
\]

Thus, \(\mathcal{I}(i A_{hk}) = 0\), and since

\[
\{s|s\ is\ in\ S\ and\ \tau^0(s) > \tau_0(s)\} = U_i U_{h,k} i A_{hk},
\]
we have \(\tau^0 \leq \tau_0\) a.e. \(\mathcal{I}\) on \(S\).

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The proof of the next corollary is immediate.

4.2 Corollary. Assume the hypothesis of 4.1 with the addition that \( \tau^0(s) \geq \tau_0(s) \) a.e. \( \bar{\mu} \) on \( S \), then \( \tau^0(s) = \tau_0(s) \) (finite or infinite) a.e. \( \bar{\mu} \) on \( S \).

4.3 Theorem. Assume the hypothesis of theorem 4.1 for functions \( \tau^0_1, \tau^0_2 \). If the two inequalities \( \tau^0_1(s) \leq \tau^0_2(s) \) \((i = 1, 2)\) hold a.e. \( \bar{\mu} \) on \( S \), then

\[
\tau^0(s) = \tau^0_1(s) = \tau^0_2(s) = \tau^0_3(s) \quad \text{finite or infinite} \quad \text{a.e. } \bar{\mu} \text{ on } S.
\]

Proof. Since 4.1 a) holds for \( \tau^0_0 \) and 4.1 b) holds for \( \tau^0_2 \), we have by 4.1 that \( \tau^0_1(s) \leq \tau^0_2(s) \) a.e. \( \bar{\mu} \) on \( S \).

The preceding statement also holds with the subscripts 1 and 2 interchanged; hence, since \( \tau^0_1(s) \leq \tau^0_2(s) \) \((i = 1, 2)\) a.e. \( \bar{\mu} \) on \( S \), we have equality in the following:

\[
\tau^0(s) \leq \tau^0_1(s) \leq \tau^0_2(s) \leq \tau^0_3(s) \quad \text{a.e. } \bar{\mu} \text{ in } S.
\]

4.4 Theorem. If \( \sigma \) is an \( M \)-measure absolutely continuous with respect to \( \mu \) and if \( B_1 \) is a derivation basis associated with a monotone sequence of nets on \( S \) which is regular with respect to \( \mu \) on each \( M \) in \( \mathcal{M} \), \( B_2 \) is a derivation basis on \( S \) satisfying the Strong Vitali \( \mu \)-property, \( B_3 \) is a derivation basis on \( S \) satisfying both the Vitali \( \mu \)- and \( \sigma \)-properties; then (1) there exists a.e. \( \bar{\mu} \) on \( S \) a derivative (finite or infinite) of \( \sigma \) with respect to \( \mu \) for each of these derivation bases, and (2) these derivatives equal each other almost everywhere \( \bar{\mu} \) on \( S \) (i.e., when sets of \( \bar{\mu} \))
measure zero are neglected, the derivative is independent of the derivation basis).

**Proof.** Let $\tau^0(s), \tau^0(s)$ $(i = 1, 2, 3)$ be the upper and lower derivatives of $\sigma$ with respect to $\mu$ for each of the derivation bases $E_i$ $(i = 1, 2, 3)$ respectively; then for these functions, the hypothesis of 4.3 is satisfied because of theorems 3.7, 3.9; 3.14, 3.12, 3.13; 3.26, 3.28, 3.29 and the fact that the upper derivative always exceeds or equals the lower derivative. Hence, the conclusion is immediate.

**4.5 Remark.** If one has the task of introducing a derivation basis onto a non-negative sigma-finite measure space, the derivation basis with the simplest structure might be the easiest to construct. It would be desirable to know ahead of time that such a basis would give (essentially) the same values for the derivative as any other derivation basis with possibly a more complicated structure. Of the types of derivation bases discussed in section 3, the one resulting from a monotone sequence of nets is certainly the simplest structurally. In view of theorem 4.4 above, it is now clear that this basis does, at least for measures $\sigma$ which are absolutely continuous with respect to the basic measure $\mu$, give (essentially) the same value for the derivative as any of the other bases discussed. In the next section we shall exhibit a technique of constructing a monotone sequence of nets with the necessary regularity
properties which will, under certain predictable conditions, yield derivatives essentially the same as derivatives with respect to any of the bases discussed in this work, not only for measures, \( \sigma \), absolutely continuous with respect to \( \mu \), but for an even more general class of measures which are important in applications.

4.6 Remark. Because of 3.26 (net regularity inherited by absolutely continuous functions), 3.14 (Strong Vitali property inherited by absolutely continuous functions), 3.11 (the Strong Vitali properties imply the Vitali properties), and 3.27 (net regularity implies the Strong Vitali properties for functions \( \mu \)-finite and absolutely continuous), we have, as a result of Hayes and Pauc in [2, Th. 1.52, p. 234], a statement similar to that of theorem 4.4 under the additional hypothesis that \( \sigma \) be \( \mu \)-finite. We note that the result of Hayes and Pauc just referred to requires the prior development of a theory of the differentiation of integrals. Against this setting the somewhat stronger result of 4.4 with its simplicity and elegance of proof (together with the simplicity and elegance of the proof of its predecessor 4.1) can be better appreciated.

4.7 Remark. The best results indicated by Monroe in [3] for the relation of the net derivative to the regular derivative in \( \mathbb{R}^n \) appear in [3, exercises f and g, p. 302]
and involve knowing the diameters of the mesh of the net sets tend to zero. Theorem 4.4 sheds light on this problem of equivalence without this restriction.
PART II

AN IMPORTANT MONOTONE SEQUENCE

OF NETS
5. Construction of a Monotone Sequence of Nets

5.1 Suppose \([S, \mathcal{M}, \mu]\) is a non-negative sigma-finite measure space. Denote by \(\mathcal{P}_1 = \{N_j\}_{j=1}^\infty\) a pairwise disjoint sequence of non-empty \(\mathcal{M}\)-sets of finite \(\mu\)-measure which cover \(S\). Note that \(\mathcal{P}_1\) is a net in the sense of 3.23. Let \(\mathcal{D}_k = \{D_{*k}\}_{k=1}^\infty\) be a countable sequence of \(\mathcal{M}\)-sets and define inductively:

\[
\mathcal{P}_2 = [N_2]\ \text{there is } N_1 \text{ in } \mathcal{P}_1 \text{ such that either } \emptyset \neq N_2 \in D_{*1} \text{ or } \emptyset \neq N_2 = N_1 - D_{*1},
\]

\[
\mathcal{P}_3 = [N_3]\ \text{there is } N_2 \text{ in } \mathcal{P}_2 \text{ such that either } \emptyset \neq N_3 \in D_{*2} \text{ or } \emptyset \neq N_3 = N_2 - D_{*2},
\]

\[
\vdots
\]

\[
\mathcal{P}_n = [N_n]\ \text{there is } N_{n-1} \text{ in } \mathcal{P}_{n-1} \text{ such that either } \emptyset \neq N_n \in D_{*n-1} \text{ or } \emptyset \neq N_n = N_{n-1} - D_{*n-1},
\]

\[
\vdots
\]

5.2 Remark. The sequence of partitions constructed above is a monotone sequence of nets in the sense of 3.23. We emphasize that the derivation basis generated by a monotone sequence of nets satisfies the requirements relative to the exceptional set \(E_0\) appearing in the definition of a derivation basis (see 3.23).
5.3 Lemma. Each $D_k$ in $\mathcal{D}_k$ is the union of a countable pairwise disjoint sequence of sets $\{N_{k+1,m}\}_{m=1}^{\infty}$ in $\mathcal{M}_{k+1}$.

Proof. This is immediate from the construction of $\mathcal{M}_{k+1}$.

5.4 We now set forth two sets of restrictions on the collection $\mathcal{D}_k$ of 5.1 which we shall subsequently show are sufficient to guarantee that the monotone sequence of nets constructed in section 5.1 is in one instance regular on each $M$ in $\mathcal{M}$ with respect to $\mu$ and in the second instance regular on each $M$ in $\mathcal{M}$ with respect to certain $\mathcal{M}$-measures $\sigma$.

Let the sets of $\mathcal{D}_k$ be enumerated as $\{D_{k,i}\}_{i=1}^{\infty}$, and let the sets of $\mathcal{M}_j$ be enumerated as $\{N_{j,i}\}_{i=1}^{\infty}$. Our postulates are:

5.4 I For given $\epsilon > 0$ and given $M$ in $\mathcal{M}$ with $M \subseteq N_{j,i}$ for some $j$, there is a (countable) subcollection $\{D_{k,i}(i)\}_{i=1}^{\infty}$ of $\mathcal{D}_k$ for which

5.4 I a) $\mu(M - \cup D_{k,i}(i)) = 0$
5.4 I b) $\mu(\cup D_{k,i}(i)) < \mu(M) + \epsilon$.

5.4 II Let $S$ be a topological space and let $\sigma$ be a semi-regular $\mathcal{M}$-measure. For given $\epsilon > 0$ and given $M$ in $\mathcal{M}$ with $M \subseteq N_{j,i}$ for some $j$ and given set $\sigma$ open in the topology on $S$ with $M \subseteq \sigma$, there is a (countable) subcollection $\{D_{k,i}(i)\}_{i=1}^{\infty}$ of $\mathcal{D}_k$ for which

5.4 II a) $\mu(M - \cup D_{k,i}(i)) = 0$
5.4 II b) $\sigma(\cup D_{k,i}(i) - \sigma) < \epsilon$. 
5.5 Lemma. - If \( M \in \mathcal{M} \) is such that for some \( j, M \subseteq N_{1j} \) with \( N_{1j} \) in \( \mathcal{M}_1 \) and if 5.4 I) holds, then the monotone sequence of nets constructed in section 5.1 is regular on \( M \) with respect to \( M \).

Proof. By the assumptions of 5.4 I), there is for \( \varepsilon > 0 \) a subcollection \( \{D_{\pi_k(i)}\}_{i=1} \) of \( \mathcal{D}_\pi \) such that

\[
\mu(M - \bigcup_{i=1} D_{\pi_k(i)}) = 0
\]

and

\[
\mu(U_i D_{\pi_k(i)}) < \mu(M) + \varepsilon.
\]

From lemma 5.3 we have for each \( i \), \( D_{\pi_k(i)} = U_{m=1} N_{k(i)+1,m} \); hence, \( U_i D_{\pi_k(i)} = U_{m=1} N_{k(i)+1,m} \), and we are done.

5.6 Lemma. - If \( M \in \mathcal{M} \) is such that for some \( j, M \subseteq N_{1j} \) with \( N_{1j} \) in \( \mathcal{M}_1 \) and if 5.4 II) holds, then the monotone sequence of nets constructed in section 5.1 is regular with respect to \( \sigma \) on \( M \) where \( \sigma \) is a semi-regular \( \mathcal{M} \)-measure.

Proof. Since \( \sigma \) is semi-regular, there is for \( \varepsilon > 0 \) an open set \( \sigma \supseteq M \) such that \( \sigma(M - \sigma) < \varepsilon/2 \); and by 5.4 II) there is a subcollection \( \{D_{\pi_k(i)}\}_{i=1} \) of \( \mathcal{D}_\pi \) such that

\[
\mu(M - \bigcup_i D_{\pi_k(i)}) = 0,
\]

\[
\sigma(U_i D_{\pi_k(i)} - \sigma) < \varepsilon/2.
\]

From lemma 5.3 we have that each \( D_{\pi_k(i)} \) is the union of a countable pairwise disjoint sequence of net sets. Let us denote the totality of the net sets comprising the totality of the sets \( D_{\pi_k(i)} \) (\( i = 1, 2, \ldots \)) by \( \{N_m\}_{m=1} \). Thus,
Because of the monotone character of the sequence of nets, we observe that there is a subsequence of \( \{N_m\} \) which we denote by \( \{N_m(p)\} \), such that for each \( p \), \( N_m(p) \subseteq N_{1j} \), where \( N_{1j} \) is as designated in the hypothesis, and such that \( \mu(M - \bigcup_p N_m(p)) = 0 \). Since
\[
\bigcup_p N_m(p) - M \subseteq \bigcup_m N_m - M = \bigcup D_{*k(i)} - M
\]
\[
\subseteq \left( \bigcup D_{*k(i)} - \emptyset \right) \cup \left( \emptyset - M \right),
\]
we have
\[
\sigma(\bigcup_p N_m(p) - M) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon;
\]
and thus,
\[
\sigma(\bigcup_p N_m(p)) \leq \sigma(M) + \varepsilon.
\]
This establishes the lemma.

5.7 Theorem. If 5.4 I) holds, then the monotone sequence of nets constructed in 5.1 is regular with respect to \( \mu \) on each \( M \) in \( \mathbb{M} \).

Proof. Form \( M_j = M \cap N_{1j} \) where \( N_{1j} \) is in \( \mathbb{N}_1 \), then by lemma 5.5 there is a countable collection of net sets which we designate notationally as \( \{ jN_i \}_{i=1}^\infty \) such that
\[
\mu(M_j - U_{i=1}^j jN_i) = 0
\]
and
\[
\mu(U_{i=1}^j jN_i) \leq \mu(M_j) + \varepsilon/2^j
\]
hold for each \( j \). Because of the monotone character of the sequence of nets, we can require also that each \( jN_i \subseteq N_{1j} \).
Hence, the collection \( \{ j_{N_i} \}_{i=1}^{N} \) of net sets forming the set \( \bigcup_{j=1}^{j_{i-1}} \bigcup_{i=1}^{j_{i-1}} j_{N_i} \) clearly satisfies the net regularity conditions for \( M = \bigcup M_j \).

5.8 Theorem. If 5.4 II) holds, then the monotone sequence of nets constructed in 5.1 is regular with respect to \( \sigma \) on each \( M \) in \( \mathcal{M} \) where \( \sigma \) is a semi-regular \( \mathcal{M} \)-measure.

Proof. The proof is similar to that of 5.7 with lemma 5.6 replacing lemma 5.5. We omit the details.

5.9 Theorem. If \( \{ S, \mathcal{M}, \mu \} \) is a non-negative sigma-finite measure space in which the postulates 5.4 I) and 5.4 II) are satisfied, \( \sigma \) being a semi-regular \( \mathcal{M} \)-measure, then the monotone sequence of nets constructed in 5.1 is regular on each \( M \) in \( \mathcal{M} \) with respect to \( \mu \) and also regular on each \( M \) in \( \mathcal{M} \) with respect to \( \sigma \).

Proof. Immediate from 5.7 and 5.8.

5.10 Remark. The significance of the above theorem will become apparent after the results of the next section are available.
6. An Important Class of Spaces Permitting
the Construction of Section Five

6.1 In this section we note a situation in which the hypotheses of the preceding section are fulfilled. Thus, we know at least one important situation where a monotone sequence of nets can be constructed to which theorem 5.9 applies.

Reichelderfer in [5] develops some measure space properties in terms of certain topological properties required of some of the measurable sets. We extract here those results which are important for our purposes and list the assumptions and definitions under which they are derived. The reader is referred to [5] for details of proofs. The notations employed are those of [5] with the exception that we use $S$ as a generic notation for members of $\mathcal{E}$.

The following hypotheses, and definitions appear in [5, pp.182-184]. We use parenthesis around the statement titles when they come from [5] to distinguish them from the statement titles of the present work.

(h 1). $S$ is a connected, locally connected, locally compact, separable Hausdorff space.

(h 2). $S'$ is a Hausdorff space.

(h 3). $T$ is a continuous transformation from $S$ onto $S'$.

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(Definition 1). Let \( \mathcal{C} \) denote the set of all subsets \( S \) of \( S \) for each of which \( T_S \) consists of a single point in \( S' \) and \( S \) is a component of \( T^{-1}T_S \).

(h 4). Each set \( S \) in \( \mathcal{C} \) is compact in \( S \).

(Definition 2). Denote by \( \mathcal{M}_o \) the set of all subsets \( M_o \) of \( S \) each of which has the property that if \( S \) is any element in \( \mathcal{C} \) such that \( S \cap M_o \neq \emptyset \), then \( S \subseteq M_o \).

In [5, Remark 2, p.182] it is observed that \( \mathcal{M}_o \) is a sigma-field of subsets of \( S \).

(Definition 3). Let \( \mathcal{D} \) denote the set of all domains (connected open sets) \( D \) in \( S \) which also belong to \( \mathcal{M}_o \).

(h 5). \( \mathcal{M}_\star \) is a sigma-field of subsets \( M_\star \) of \( S \) such that \( \mathcal{D} \subseteq \mathcal{M}_\star \subseteq \mathcal{M}_o \).

(h 6). \( [S, \mathcal{M}_\star, \mu_\star] \) is a sigma-finite non-negative measure space.

(h 7). There exists a measure space \( [S, \mathcal{M}_\star, \mu_\star] \) such that \( \mathcal{M}_\star \supseteq \mathcal{M}_\star \cup \mathcal{B} \) where \( \mathcal{B} \) is the sigma-field of Borel sets \( B \) in \( S \) and \( \mu_\star \mid \mathcal{M}_\star = \mu_\star \), which has the following property. For each set \( M_\star \) in \( \mathcal{M}_\star \) and each positive real number \( \xi \), there exists a set \( \mathcal{O}_\star \) open in \( S \) (hence an element of \( \mathcal{M}_\star \)) such that \( M_\star \subseteq \mathcal{O}_\star \) and \( \mu_\star \mathcal{O}_\star \leq \mu_\star M_\star + \xi \).

(h 10). (Partial). There exists a countable open base for the topology of \( S' \).

6.2 Remark. In [5, h10, p.183] there is also the requirement that the boundaries of the base elements for
the topology on $S'$ have measure zero according to the measure on $S'$. This is not necessary for the results we are interested in, hence the omission above. For the same reason, we omit the requirement that the collection of base elements for the topology of $S'$ be enlarged by forming from it a collection which is closed under finite unions.

The following definition and results appear in [5, pp.188-191].

(Definition 3.1). Let $\mathcal{D}_*$ be the set of all subsets $D_*$ of $S$ for each of which there is a set $\mathcal{G}$ in the countable base of $h\,\mathcal{H}$ having $D_*$ as the component of the inverse under $T$ of $\mathcal{G}$.

(Remark 3.2). $\mathcal{D}_* \subseteq \mathcal{D}$ and $\mathcal{D}_*$ is countable.

Among other things it is shown [5, Lemma 3.3, p.189] that if $S$ is in $\mathcal{G}$ and $\mathcal{G}$ is open in $S$ such that $S \subseteq \mathcal{G}$, then there is a $D_*$ in $\mathcal{D}_*$ such that $S \subseteq D_* \subseteq \mathcal{G}$. With this result the following lemma is established which says in part:

(Lemma 3.4). Let $M_o$ be in the set $\mathcal{T}_o$ and let $\mathcal{G}$ be any open set in $S$ such that $M_o \subseteq \mathcal{G}$. Then there is a countable sequence of sets $\{D_{*j}\}$ in $\mathcal{D}_*$ such that $\ldots M_o \subseteq U D_{*j} \subseteq \ldots \subseteq \mathcal{G}$. The set $U D_{*j}$ is an open set in $S$ which is in $\mathcal{T}_o$.

We may now establish the following theorems:

6.3 Theorem. Let $M_*$ be in $\mathcal{T}_*$ and let $\xi > 0$ be given;
then there is a sequence \( \{D_{*j}\}_{j=1}^{\infty} \subseteq D_* \) which covers \( M_* \) and for which \( \mu_*(U_{j=1} D_{*j}) \leq \mu_* M_* + \varepsilon \). Thus, 5.4I holds if \( \mathcal{M} \) and \( \mu \) in its statement are replaced by \( \mathcal{M}_* \) and \( \mu_* \) respectively.

**Proof.** By (h 2) there exists a set \( \mathcal{O}_* \) open in \( S \) such that \( M_* \subseteq \mathcal{O}_* \) and \( \mu_* \mathcal{O}_* \leq \mu_* M_* + \varepsilon = \mu_* M_* + \varepsilon \).

By (Lemma 3.4) and (h 5) there is a sequence \( \{D_{*j}\}_{j=1}^{\infty} \subseteq D_* \) such that \( M_* \subseteq U D_{*j} \subseteq \mathcal{O}_* \). Hence, (using (h 2))

\[
\mu_*(U D_{*j}) \leq \mu_* \mathcal{O}_* \leq \mu_* M_* + \varepsilon.
\]

**6.4 Theorem.** Let \( \sigma \) and \( \mu \) be any \( \mathcal{M}_* \)-measures with \( \sigma \) semi-regular; then conditions 5.4II a) and 5.4II b) are satisfied for these measures with \( \mathcal{M}_* \) replacing \( \mathcal{M} \) in 5.4.

**Proof.** Let \( M_* \) in \( \mathcal{M}_* \) be given and let \( \mathcal{O} \) be an open set in \( S \) such that \( M_* \subseteq \mathcal{O} \); then by (lemma 3.4) there is a countable sequence of sets \( \{D_{*j}\} \) in \( D_* \) such that \( M_* \subseteq U D_{*j} \subseteq \mathcal{O} \); hence, \( M_* - U D_{*j} = \emptyset = U D_{*j} - \mathcal{O} \) and the conclusion is immediate.

**6.5 Remark.** As a result of theorem 6.3, theorem 6.4, and (h 6), which requires that \( S \) be sigma-finite with respect to \( \mu_* \), we see that the collection \( D_* \) of (Definition 3.1) can be used to construct, as indicated in chapter 5, a monotone sequence of nets on \( S \) which by theorem 5.9 will be regular with respect to \( \mu_* \) on every \( M_* \) in \( \mathcal{M}_* \) and also simultaneously regular with respect
to any semi-regular $\mathfrak{M}_\sigma$-measure $\sigma$ (hence $\sigma$ is an $\mathfrak{M}_\sigma$-measure) on each $M_\sigma$ in $\mathfrak{M}_\sigma$.

6.6 Remark. In [5] Reichelderfer has shown that the assumptions $(h)$ and definitions quoted above imply certain postulates or portions of postulates $(H)$ hold which in turn imply theorem 6.3. Specifically, the above mentioned postulates are the last condition of $(H4^*)$, $(H10)$ condition (i), $(H11)$, and $(H15)$ condition (i) (see [5, p. 184 and p. 186-187]). The proofs that the above mentioned postulates and portions of postulates hold under the assumptions $(h)$ and definitions quoted above appear in [5] in sections 3.11 (p. 190-191), 3.19, 3.21, 3.23 (p. 192-193). That the above mentioned four postulates imply theorem 6.3 is discussed later in this work (see Th. 8.1A and Th. 8.1B). We have chosen to demonstrate theorem 6.3 of this section by the more direct logical route, i.e., by showing that the conditions which imply the four postulates quoted above imply theorem 6.3 directly.

6.7 Remark. It is noted in [5, p. 194] that if $S$ is a non-empty domain in Euclidean $n$-space, $\mathfrak{M}_\mathbb{N}$ is the set of all subsets $M_\mathbb{N}$ of $S$ each of which is $n$-dimensionally Lebesgue measurable, $\mu_\mathbb{N} M_\mathbb{N}$ is the $n$-dimensional Lebesgue measure, $T$ is a continuous mapping whose domain of definition is $S$ and whose range $S'$ is a Borel subset
of Euclidean $n'$-space and if for each $s$ in $S$ the unique component of $T^{-1}Ts$ which contains $s$ is compact; then with $\mathcal{E}$, $\mathcal{M}_0$, $\mathcal{D}$ as defined above and $\mathcal{M}_s = \mathcal{M}_0 \cap \mathcal{M}_s$ and $\mu_* = \mu_1|\mathcal{M}_s$, the hypotheses (h) listed above in this chapter are all satisfied. Hence, in all such situations we have a monotone sequence of nets with the properties described in 6.5 above. A very special case arises when $n = n'$ and $T$ is the identity mapping.

6.8 Theorem. If $S = \mathbb{R}^n$ and $\mathcal{M}_s$ and $\mu_*$ are as required in 6.7 and if $T$ is the identity mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$, then $\mathcal{M}_s$ contains the Borel sets of $\mathbb{R}^n$; and hence, for any semi-regular $\mathcal{M}_s$ measure $\sigma$ for which $S$ is sigma-finite, we have the derivative of $\sigma$ with respect to the derivation basis generated by the monotone sequence of nets in 6.5 agrees a.e. $\bar{\mu}_*$ (hence a.e. $\mu_*$) with the regular derivative of $\sigma$ in $\mathbb{R}^n$.

Proof. Decompose $\sigma$ by means of 2.1 into measures $\sigma_A$ and $\sigma_o$ which are absolutely continuous and singular respectively with respect to $\mu_*$ where $\mu_*$ is as in 6.7. Observe $\sigma$ semi-regular implies both $\sigma_A$ and $\sigma_o$ are semi-regular. By 3.20, 6.5, and 4.4 the derivatives of $\sigma_A$ with respect to the two derivation bases under discussion agree a.e. $\bar{\mu}_*$ on $S$ and by 3.22 and 3.30, 6.5 we have that the derivative of $\sigma_o$ with respect to each of the derivation bases under discussion is zero a.e. $\bar{\mu}_*$ on $S$. Hence, the conclusion follows by 1.6.
6.9 Remark. The proof of 6.8 is essentially unaltered if the hypothesis that $T$ be the identity mapping is weakened to $T$ is a continuous mapping from a non-empty domain of Euclidean $n$-space onto a Borel subset of Euclidean $n'$-space such that $\mathbb{M}_0$ (see [def.2] above) contains the Borel sets in $S$. This remark is intended to draw attention to the fact that the collection $\mathbb{D}_\alpha$ of (definition 3.1) depends on $T$; and hence, the monotone sequence of nets of 6.5 depends on $T$. Thus, the statement above is calling attention to a certain freedom in the choice of $T$. 
PART III

DIFFERENTIATION OF INTEGRALS

AND AN APPLICATION
7. The Fundamental Theorem

In this section we shall see that a monotone sequence of nets with appropriate regularity conditions differentiates an important class of integrals and yields (essentially) the integrand as the value of the derivative. Throughout this section it is assumed \([S, \mathcal{M}, \mu]\) is a sigma-finite non-negative measure space.

7.1 Lemma. If \(\{\mathcal{T}_n\}_{n=1}^{\infty}\) is a monotone sequence of nets on \(S\), \(\mathcal{B}\) is the derivation basis generated by \(\{\mathcal{T}_n\}_{n=1}^{\infty}\) and if \(\sigma\) is an extended-real valued set function defined at least on \(\mathcal{B}\), then \(D^* \sigma(s)\) and \(D_* \sigma(s)\) are each measurable functions on the measurable set \(S - E_0\) where \(E_0\) is the exceptional subset of \(S\) appearing in the definition of the derivation basis generated by a monotone sequence of nets (see 3.23).

Proof. If for each \(s\) in \(S\) we let \(N_{ns}\) be the unique set in \(\mathcal{T}_n\) which contains \(s\) and place \(d_n(s) = \sigma(N_{ns})/\mu(N_{ns})\) when \(N_{ns} - E_0 \neq \emptyset\), then, as observed in 3.23,

\[
D^* \sigma(s) = \limsup_n d_n(s), \quad s \text{ in } S - E_0,
\]

\[
D_* \sigma(s) = \liminf_n d_n(s), \quad s \text{ in } S - E_0.
\]

Observe that on \(N_{ns_o}\) with \(s_o\) in \(S - E_0\), \(d_n(s)\) is constant; and thus, \(\{s \mid d_n(s) \text{ is defined and } d_n(s) > a\}\), where \(a\) is a real number, is the countable union of sets \(N_{ns}\) in \(\mathcal{T}_n\) and hence is a measurable set. Thus,
\[ \{ s \mid d_n(s) > a \} \cap (S - E_0) \text{ is a measurable set. Clearly now, } D^* \sigma(s) \text{ and } D_- \sigma(s) \text{ are measurable functions on } S - E_0. \]

**7.2 Lemma.** If \( \sigma \) is a \( \mu \)-finite \( \mathfrak{M} \)-measure absolutely continuous with respect to \( \mu \) and if \( \{ \mathfrak{M}_i \}_{i=1} \) is a monotone sequence of nets on \( S \) regular with respect to \( \mu \) on each \( M \) in \( \mathfrak{M} \), then there exists a \( \mathfrak{M} \) derivative of \( \sigma \) with respect to \( \mu \) which is finite a.e. \( \mathfrak{M} \) on \( S \) where \( \mathfrak{M} \) is the derivative basis generated by \( \{ \mathfrak{M}_i \} \).

**Proof.** We know by theorem 4.4 that the derivative exists finite or infinite a.e. \( \mathfrak{M} \) on \( S \). Hence, we need only show the derivative is finite a.e. \( \mathfrak{M} \) on \( S \). Since always \( 0 \leq D^* \sigma(s) \leq D_- \sigma(s) \), \( s \) in \( S - E_0 \), let

\[ A = \{ s \mid s \text{ in } S - E_0 \text{ and } D^* \sigma(s) = \infty \}, \]

and place \( A_j = A \cap M_j \), where the \( M_j \) are of finite \( \mu \)-measure and cover \( S \). Since \( s \) in \( A_j \) implies \( D^* \sigma(s) > a \) for every real \( a > 0 \), we have, by 3.26 and 3.28, that \( \sigma(A_j) \geq a \mathfrak{M}(A_j) \) holds for every \( a > 0 \). Here \( A_j \) is a \( \mu \)-measurable cover of \( A_j \). If \( \mathfrak{M}(A) > 0 \), then \( \mathfrak{M}(A_j) > 0 \) for some \( j \); and hence, \( \sigma(A_j) = \infty \). But this contradicts the \( \mu \)-finiteness of \( \sigma \); hence, \( \mathfrak{M}(A) = 0 \) and we are done.

**7.3 Remark.** The preceding lemma can be extended to a \( \mu \)-finite signed \( \mathfrak{M} \)-measure \( \sigma \) which is absolutely continuous with respect to \( \mu \) by applying 7.2 to each of the functions obtained by decomposing \( \sigma \) by 2.2 into its upper
and lower variations (note the remarks preceding 2.1 and 2.3) and applying 1.6.

7.4 Lemma. If $\{\sigma_n\}_{n=1}^\infty$ is a non-decreasing, i.e., $\sigma_n \leq \sigma_{n+1}$, sequence of $\mathbb{M}$-measures and if we define for $M$ in $\mathbb{M}$ that $\sigma_0(M) = \lim \sigma_n(M)$, then $\sigma_0$ is an $\mathbb{M}$-measure.

Proof. Clearly $\sigma_0 \geq 0$ and $\sigma_0(\emptyset) = 0$. Let $\{M_i\}$ be a sequence of pairwise disjoint $\mathbb{M}$-sets; then, since always $\sigma_n(M_i) \leq \sigma_0(M_i)$, we have

$$\sigma_0 \left( \bigcup M_i \right) = \lim \sigma_n \left( \bigcup M_i \right) = \lim \left[ \sum_1^n \sigma_n(M_i) \right] \leq \sum_1^n \sigma_0(M_i) .$$

We establish the reverse inequality by considering two cases.

Case $\sum \sigma_0(M_i) < \infty$. Given $\varepsilon > 0$, then if $n$ and $k$ are sufficiently large numbers we have,

$$\sum_1^n \sigma_0(M_i) - \varepsilon < \sum_1^k \sigma_0(M_i) < \sum_1^n \sigma_0(M_i) + \varepsilon$$

$$= \sigma_0 \left( \bigcup_1^n M_i \right) + \varepsilon < \sigma_0 \left( \bigcup_1^n M_i \right) + \varepsilon ;$$

hence, the reverse inequality holds for this case.

Case $\sum \sigma_0(M_i) = \infty$. Let $L$ any real number, then $L < \sum \sigma_0(M_i)$ and there is an $n$ such that $L < \sum_1^n \sigma_0(M_i)$.

Let $\{L_i\}_{i=1}^n$ be real numbers such that $L_i < \sigma_0(M_i)$ and $L < \sum_1^n L_i$, then there is an integer $N$ such that for $1 \leq i \leq n$, $L_i < \sigma_N(M_i)$. Hence,

$$L < \sum_1^n L_i < \sum_1^n \sigma_N(M_i) = \sigma_N \left( \bigcup_1^n M_i \right)$$

$$\leq \sigma_0 \left( \bigcup_1^n M_i \right) ;$$

and hence, $\sigma_0(\bigcup_1^n M_i) = \infty$. 
7.5 Lemma. If \( \{\sigma_n\}_{n=1}^\infty \) is a non-decreasing sequence of \( \mathcal{M} \)-measures absolutely continuous with respect to \( \mu \) and if we define for \( M \in \mathcal{T}_0 \) that \( \sigma_0(M) = \lim_{n \to \infty} \sigma_n(M) \), then, if \( \sigma_0 \) is \( \mu \)-finite, \( \sigma_0 \) is absolutely continuous with respect to \( \mu \); and if \( B \) is the derivation basis generated by a monotone sequence of nets on \( S \) regular with respect to \( \mu \) on each \( M \in \mathcal{T}_0 \) and if \( E_0 \subset S \) is the exceptional set occurring in the domain of \( B \), then the following \( B \) derivatives exist a.e. \( \tilde{\mu} \) on \( S \) as finite numbers and are related by:

\[
\lim D\sigma_n(s) = D\sigma_0(s) \text{ a.e. } \tilde{\mu} \text{ on } S.
\]

Proof. Let \( M \in \mathcal{T}_0 \) with \( \mu(M) = 0 \); then, by the absolute continuity of each \( \sigma_n \), we have \( \sigma_n(M) = 0 \) for every \( n \); and hence, \( \sigma_0(M) = 0 \). This, with the hypothesis that \( \sigma_0 \) is \( \mu \)-finite, implies \( \sigma_0 \) is absolutely continuous with respect to \( \mu \).

The second conclusion is reasoned as follows:

Lemma 7.4 and the conclusion just established permit us to apply 7.2 to guarantee the existence of \( D\sigma_0(s) \) (finite valued) a.e. \( \tilde{\mu} \) on \( S \). Also, the hypothesis and 7.2 guarantees for each \( n \) the existence of \( D\sigma_n(s) \) (finite valued) a.e. \( \tilde{\mu} \) on \( S \). Form for each \( i \tau_i = \sigma_0 - \sigma_i \) and note these functions are non-negative and, by 1.6, there exists \( D\tau_i \) (finite valued) a.e. \( \tilde{\mu} \) on \( S \). Since always \( \tau_i \geq \tau_{i+1} \) we have \( D^*\tau_i(s) \geq D^*\tau_{i+1}(s) \), \( s \in S - E_0 \). Observe that for \( s \) in \( A_m \) where
\[ A_m = \{ s \in S - E_0 \mid \lim_1 D^* \tau_i(s) > 1/m \}, \text{ } m \text{ a positive integer}, \]

we have \( D^* \tau_i(s) > 1/m \). Since \( \tau_i \) is absolutely continuous with respect to \( \mu \), we have by 3.26 and 3.28 that for \( A_m \) a \( \mu \)-measurable cover of \( A_m \)

\[ m \tau_i(\bar{A}_m) > \bar{\mu}(A_m) \]

holds for every \( i \). Thus,

\[ \bar{\mu}(A_m) \leq m \lim_1 \tau_i(\bar{A}_m) = m \lim_1 (\sigma_0 - \sigma_i)(\bar{A}_m) \]

Thus, since \( \{ s \in S - E_0 \mid \lim_1 D^* \tau_i(s) > 0 \} = \cup A_m \), we see

\[ \lim_1 D^* \tau_i(s) = 0 \text{ a.e. } \bar{\mu} \text{ on } S; \text{ and since } D \tau_i(s) \text{ exists a.e. } \bar{\mu} \text{ on } S, \text{ we have a.e. } \bar{\mu} \text{ on } S \]

and since \( D \sigma_o(s) \) is finite a.e. \( \bar{\mu} \) on \( S \), we are done.

7.6 Theorem. Let \( \{ \mathfrak{m}_i \}_{i=1} \) be a monotone sequence of nets on \( S \) regular with respect to \( \mu \) on each measurable set, and let \( B \) be the derivation basis generated by \( \{ \mathfrak{m}_i \}_{i=1} \).

If \( f(s) \) is an extended-real valued measurable function on \( S \) and if \( \sigma = \int f d \mu \) exists finite or infinite for every measurable set and is \( \mu \)-finite, then the \( B \) derivative \( D \sigma \) exists a.e. \( \bar{\mu} \) on \( S \) and \( D \sigma = f \) a.e. \( \bar{\mu} \) on \( S \).

If \( \sigma \) is a \( \mu \)-finite signed \( M \)-measure absolutely continuous with respect to \( \mu \), then \( D \sigma(s) \) exists a.e. \( \bar{\mu} \) on \( S \),
is $\mu$-integrable on each measurable set of finite $\mu$-measure and
\[ \sigma = \int D \sigma(s) \, d\mu. \]

**Proof.** Let $E_0$ be the exceptional set in $S$ for $\mathcal{B}$. Let $E$ be any measurable set and let $C_E$ be the characteristic function of $E$. We show first that $\int C_E = \int C_E \, d\mu$ has $D(\int C_E)(s) = C_E(s)$ a.e. $\mathcal{M}$ on $S$. Let $\mathcal{N}_{ns}$ be the unique set in $\mathcal{F}_n$ which contains $s$; then, for the functions $d_n(s)$ defined in 7.1, we have for $s$ in $S - E_0$
\[ \int_{\mathcal{N}_{ns}} C_E \, d\mu = \frac{\mu(\mathcal{N}_{ns} \cap E)}{\mu(\mathcal{N}_{ns})} \leq 1, \ s \text{ in } S - E_0. \]

Thus,
\[ 7.6a) \quad D^* \left[ \int C_E \right](s) = \limsup d_n(s) \leq 1 \ \text{a.e.} \ \mathcal{M} \text{ on } S. \]

For each positive integer $n$ define
\[ A_n = \{ s \mid s \in E - E_0 \text{ and } D^* \left[ \int C_E \right](s) < 1 - \frac{1}{n} \}. \]

By 7.1 $A_n$ is measurable. Let $\{M_i\}_{i=1}$ be a pairwise disjoint sequence of sets of finite $\mu$-measure which cover $S$ and form $A_{ni} = A_n \cap M_i$; then each $A_{ni}$ is measurable and of finite $\mu$-measure. Since $\int C_E$ is $\mu$-finite and is zero on sets of measure zero, it is absolutely continuous with respect to $\mu$; and hence, we can apply 3.29 to the $A_{ni}$ to get
\[ \int_{A_{ni}} C_E = \mu(E \cap A_{ni}) = \mu(A_{ni}) \leq (1 - 1/n) \mu(A_{ni}) . \]

This can only be true if $\mu(A_{ni}) = 0$ for each $n$ and each $i$. Since
we see

\[ 7.6b) \quad D_\ast \left[ \int C_E(s) \right] = 1 \text{ a.e. } \bar{\mu} \text{ on } E - E_0; \]

thus, together with 7.6a), we have the following derivative existing a.e. \( \bar{\mu} \) on \( E \) and

\[ 7.6c) \quad D\left[ \int C_E(s) \right] = 1 \text{ a.e. } \bar{\mu} \text{ on } E. \]

Since

\[ \int C_S = \int C_E + \int C_{-E}, \]

we have by 7.6c) that the derivatives of the two end expressions exist and equal 1 a.e. \( \bar{\mu} \) on \( S \) and a.e. \( \bar{\mu} \) on \(-E\) respectively. We apply 1.6 and conclude

\[ 7.6d) \quad D\left[ \int C_E(s) \right] = 0 \text{ a.e. } \bar{\mu} \text{ on } -E. \]

Summarizing 7.6c) and 7.6d), we have existing

\[ 7.6e) \quad D\left[ \int C_E(s) \right] = C_E(s) \text{ a.e. } \bar{\mu} \text{ on } S. \]

If \( f = \sum_{i=1}^{n} a_i C_{E_i} \) is a simple function (see [3, p.155]) then from 7.6e), 1.6 and \( D[a \sigma](s) = a D \sigma(s) \),

we have

\[ 7.6f) \quad D\left[ \int f(s) \right] = f \text{ a.e. } \bar{\mu} \text{ in } S. \]

Finally, let \( f \) be any extended-real valued measurable function. Decompose \( f \) into its positive and negative parts \( \Phi \) and \( \Phi^c \) where

\[ \Phi(s) = \begin{cases} f(s) & \text{if } f(s) \geq 0 \\ 0 & \text{otherwise} \end{cases} \]

\[ \Phi^c(s) = \begin{cases} -f(s) & \text{if } f(s) < 0 \\ 0 & \text{otherwise} \end{cases} \]

and note \( f = \Phi^c - \Phi \), and both \( \Phi \) and \( \Phi^c \) are non-negative and
measurable. By 2.5 there are sequences of non-decreasing, non-negative simple functions \( \{ \tilde{f}_n \} \) and \( \{ f_n \} \) such that

\[
7.6g) \lim \tilde{f}_n = \tilde{f} \quad \text{and} \quad \lim f_n = f.
\]

We note that the integral of a simple function is absolutely continuous with respect to \( \mu \). The requirement that \( \int f \, d\mu \) exist finite or infinite on each measurable set means that not both \( \int \tilde{f} \, d\mu \), \( \int f \, d\mu \) are infinite on the same measurable set; hence,

\[
\int f \, d\mu = \int \tilde{f} \, d\mu - \int f \, d\mu ;
\]

and since \( \int f \, d\mu \) is \( \mu \)-finite, we see each of \( \int \tilde{f} \, d\mu \) and \( \int f \, d\mu \) is \( \mu \)-finite. Finally by 2.4 and 7.6g), the remaining hypothesis of 7.5 is satisfied, and therefore by 7.5, \( \int \tilde{f} \, d\mu \) and \( \int f \, d\mu \) are each absolutely continuous with respect to \( \mu \), and the derivatives below exist finite a.e. \( \mu \) on \( S \). With the help of 7.6f) and 7.6g), we may write

\[
D \int \tilde{f} \, d\mu = \lim D \int \tilde{f}_n \, d\mu = \lim \tilde{f}_n = \tilde{f} \quad \text{a.e.} \quad \mu \quad \text{on} \quad S.
\]

7.6h)

\[
D \int f \, d\mu = \lim D \int f_n \, d\mu - \lim f_n = f \quad \text{a.e.} \quad \mu \quad \text{on} \quad S.
\]

Hence by 1.6, the left derivative below exists finite a.e. \( \mu \) on \( S \); and

\[
D \int f \, d\mu = D \int \tilde{f} \, d\mu - D \int f \, d\mu = \tilde{f} - f = f
\]

a.e. \( \mu \) on \( S \).

This proves the first part of the theorem.
The second conclusion of 7.6 now follows easily. By 2.3 there is a finite valued measurable function $f$ such that for $M$ in $\mathcal{M}$

$$\sigma(M) = \int_M f \, d\mu.$$  

By the first part of 7.6, we have $D \sigma(s) = f(s)$ a.e. $\mu$ on $S$; whence,

$$\sigma(M) = \int_M f \, d\mu = \int_M D \sigma(s) \, d\mu.$$

The following corollary to the preceding theorem 7.6 appears in [3, theorem 43.4, p.299].

7.7 Corollary. Let $\{\mathcal{M}_i\}_{i=1}^\infty$ be a monotone sequence of nets regular with respect to $\mu$ on each measurable set. If $f$ is $\mu$-integrable and $\sigma = \int f \, d\mu$, then

$$D \sigma = f \quad \text{a.e. } \mu \text{ on } S.$$

If $\sigma$ is an everywhere finite, signed $\mathcal{M}$-measure, absolutely continuous with respect to $\mu$, then $D \sigma$ exists a.e. $\mu$ on $S$, and

$$\sigma = \int D \sigma \, d\mu.$$

7.8 Corollary. Assume the conditions of 7.6 on a monotone sequence of nets on $S$. If $C_E$ is the characteristic function of any measurable set $E$, then

$$D \int C_E \, d\mu = C_E \quad \text{a.e. } \mu \text{ on } S.$$

7.9 Corollary. Assume the conditions of 7.6 on a
monotone sequence of nets on $S$. If $f$ is any simple function on $S$, then

$$D \int f \, d\mu = f \quad \text{a.e. } \mu \text{ on } S.$$ 

7.10 Remark. We have set 7.7, 7.8 and 7.9 down for emphasis since even though 7.7 is actually equivalent to 7.6 it cannot be obtained from the proof indicated in [3, pp.299-300] without considerable additional argument. Neither can 7.8 nor 7.9 be obtained from the proof in [3] mentioned above without considerable additional argument. The factor which has helped to make the proof given here elegant is our use of 3.29, a result of our own.
8. An Interesting Application of Derivative in the Abstract

Reichelderfer in [6] has assumed, among other things, that \([S, \mathcal{M}, \mu]\) and \([S', \mathcal{M}', \mu']\) are sigma-finite complete measure spaces, \(T\) is a single-valued mapping from \(S\) onto the set \(S'\), \(\mathcal{D}\) is a collection of subsets \(D\) of \(S\) such that for each \(D\) and each point \(s'\) in \(S'\) a non-negative extended-real number \(W'(s', D)\) is assigned so that the function \(W'(s', D)\) satisfies certain hypotheses, among which is the requirement that it be measurable \(\mathcal{M}'\) for each \(D\) in \(\mathcal{D}\). Then for each \(D\) in \(\mathcal{D}\) the non-negative extended-real number \(\int_{S'} W'(s', D) d\mu'\) is a weight \(W(D)\) attached to \(D\). Under certain standard hypotheses, necessary and sufficient conditions are derived in order that there exist a non-negative extended-real valued function \(f\) defined on \(S\) and integrable \(\mu\) such that

8.0a) \[\int_{D} f(s) d\mu = W(D) = \int_{S'} W'(s', D) d\mu', \quad D \text{ in } \mathcal{D}.\]

When these conditions are satisfied, it is shown that a transformation formula always holds in the following sense:

Let \(H'\) be a real valued measurable \(\mathcal{M}'\) function defined on \(S'\); then for each \(D\) in \(\mathcal{D}\), \(H'(s') W'(s', D)\) is measurable \(\mathcal{M}'\) and \(H'[T(s)]f(s)\) is measurable \(\mathcal{M}\). Moreover, \(H'(s') W'(s', D)\) is integrable \(\mu'\) over \(S'\) if and only if \(H'[T(s)]f(s)\) is integrable \(\mu\) over \(D\). When these functions
are integrable, their integrals have the same value—that is, one has the transformation formula

\[ \int_D H'[T(s)]f(s)d\mu = \int_S H'[s'] W'(s'. D) d\mu'. \]

In this section we will introduce a monotone sequence of nets onto S which will enable us to exhibit f as the derivative of a certain measure and which will shed some light on the structure of f.

Hypothesis H4 in [6, p.285] requires, among other things, that S can be expressed as the union of a countable number of sets \( *D_j \) in \( \mathcal{D} \) such that \( \mu(*D_j) \) is finite for every \( j \); and if \( M \) is in \( \mathcal{M} \) with \( M \subseteq *D_j \) for some \( j \), then for every positive real number \( \varepsilon \) there is a countable number of pairwise disjoint sets \( D_1 \) in \( \mathcal{D} \) such that \( M \subseteq U D_1 \) and \( \Sigma \mu(D_1) < \mu(M) + \varepsilon \).

The stronger hypothesis H4* requires in place of the statement above: If \( M \) is in \( \mathcal{M} \) and \( \xi \) is a positive real number, then there exists a countable number of pairwise disjoint sets \( D_1 \) in \( \mathcal{D} \) such that \( M \subseteq U D_1 \) and \( \Sigma \mu(D_1) \leq \mu(M) + \xi \). It is observed in [6] that this H4* implies H4 in view of the assumption S is sigma-finite with respect to \( \mu \). It is this H4* that is shown to hold in the setting [5].

We here introduce a postulate HM which is at the same time both stronger and weaker than the particular
part of H4 mentioned above. To be specific, in HM the set \( D \) from which certain covering collections are drawn is required to be countable. This is a stronger requirement than the corresponding requirement of H4 which places no restriction on the cardinality of the set \( D \) from which the covering collections are drawn. HM is weaker than H4 in that the covering collections referred to above are not required to consist of pairwise disjoint elements while this pairwise disjointness is required in H4.

**HM.** (1) Let \( S \) be expressed as the union of a countable number of sets \( *D_j \) in \( D \) such that \( \mu(*D_j) \) is finite for every \( j \). (2) Let \( D^* \) be a countable subfamily of \( D \) which has the property that if \( M \) is in \( M \) with \( M \subseteq *D_j \) for some \( j \), then for \( \epsilon > 0 \) there is a countable number of sets \( D_{*1} \) in \( D^* \) such that \( M \subseteq U D_{*1} \) and \( \mu(U D_{*1}) < \mu(M) + \epsilon \).

8.1 **Remark.** The assumption HM is compatible with the set of postulates appearing in [6]; and, in fact, it is true that some of the postulates added in later sections of [6] together with one of the basic postulates appearing in the first section of [6] (namely H4) actually imply HM. In particular:

Postulate H[104] [6, p.300] requires in part that there exists a countable subset \( D^* \) of \( D \) which contains the
empty set and has the following property: Given a countable sequence of sets $D_j$, $j \geq 0$, in $\mathcal{D}$ such that if $D_0 \supseteq U_{j \geq 1} D_j$, then for each integer $j \geq 0$ there exists a countable sequence of sets $D^*_j$ in $\mathcal{D}^*$ satisfying the following conditions: 1) $\ldots$, $\bigcup_k D^*_k = D_0$; $\ldots$. We note that for $D_0 = D \in \mathcal{D}$, we can choose for $j > 1$, $D_j = D = D_0$. Hence, it is immediate from the part of H10 just quoted that each set $D$ in $\mathcal{D}$ can be written as the union of a countable number of sets from $\mathcal{D}^*$. The proof of the following theorem is now immediate and we omit the details.

**Theorem 8.1A.** Postulates H4 and H10 i) appearing in [6] together imply postulate HM introduced in this chapter, the set $\mathcal{D}^*$ of H10 serving as the set $\mathcal{D}^*_M$ of HM.

It is shown in [6, Lemma 7.2, p.302] under the assumption of postulate H11 (see [6, p.302]) that a certain collection $\mathcal{G}$ of sets whose elements are generically designated by $S$, is a partition of $S$ -- that is, the sets in $\mathcal{G}$ are non-empty, pairwise disjoint, and their union is $S$. Also, if $S$ in $\mathcal{G}$ and $D$ in $\mathcal{D}$ are such that $S \cap D \neq \emptyset$, then $S \subseteq D$. Thus, one easily has that each $D$ in $\mathcal{D}$ is the union of a pairwise disjoint collection of $S$'s in $\mathcal{G}$. This result together with postulate H15 i) [6, p.309], which requires in part that there exist a countable subset $\mathcal{D}^{**}$ of $\mathcal{D}$ having the following properties: 1) For each $S$ in $\mathcal{G}$ and each $D$ in $\mathcal{D}$ such that $S \subseteq D$, there is a $D^{**}$ in $\mathcal{D}^{**}$ such
that \( S \subseteq D^{**} \subseteq D \cdots \), makes the proof of the following theorem immediate and again we omit the details.

**Theorem 8.1B.** Postulates \( H_4, H_{11} \) and \( H_{15i} \) of [6] together imply postulate \( HM \) of this chapter, the set \( D^{**} \) of \( H_{15} \) serving as the set \( D_* \) of \( HM \).

Since it might be of interest in some applications, we mention the result [6, Lemma 8.8, p.309] which says in part: If postulates \( H_{11}, H_{12} \) (see[6, p.302]), and \( H_{15i} \) hold, then \( S \) in \( S \) is such that \( S = \cap D^{**}, D^{**} \in D^{**} \), \( S \subseteq D^{**} \). Thus, we have the following theorem:

**Theorem 8.1C.** If \( D_* \) of \( HM \) is chosen to be the set \( D^{**} \) of \( H_{15i} \) and if \( H_{11} \) and \( H_{12} \) of [6] hold, then for \( S \) in \( S \), \( S = \cap D_* \), \( D_* \) in \( D_* \), \( S \subseteq D_* \), and \( TS \) is a single point in \( S' \).

**8.2 Remark.** We use the sequence \( \{D_j\}_{j=1} \) of \( HM \) to manufacture a pairwise disjoint sequence of sets \( _1N' \) of finite \( \mu \)-measure which cover \( S \) by the following two-step procedure:

**Step I.** Define inductively
\[
\begin{align*}
_1N' &= D_1 \\
_2N' &= D_2 - D_1 \\
&\vdots \\
nN' &= D_n - \bigcup_{j=1}^{n-1} D_j \\
&\vdots
\end{align*}
\]
Step II. From the collection \( \{ N_i \}_{i=1}^n \) delete all occurrences of the empty set and re-index the remaining sets as \( \{ N_i \}_{i=1}^n \). The collection \( \{ N_i \}_{i=1}^n \) is a net \( \mathcal{T}_1 \) on \( S \). We use this \( \mathcal{T}_1 \) and the collection \( \mathcal{D}_m \) of HM to construct, as in Chapter 5, a monotone sequence of nets on \( S \). Since condition (2) of HM implies 5.41, we have by 5.7 that the monotone sequence of nets thus constructed is regular with respect to \( \mu \) on each \( M \) in \( \mathcal{M} \).

8.3. We now repeat a few of the definitions and results of [6] which are pertinent to the goal of this section. The reader is referred to [6] for the remaining definitions and proofs should they be desired.

In [6, section 5.4, p.298] an everywhere finite outer measure \( W^* \) is defined on each \( A \subseteq S \) by the following when \( T \) is BVW (see [6, Def. 2.2, p.287]):

\[
W^*(A) = \inf \sum W(D_j), \quad D_j \text{ in } \mathcal{D}, \quad A \subseteq \bigcup D_j,
\]

where (see [6, Def. 2.1, p.287]) \( W(D_j) = \int_S W'(s', D_j) \, d\mu' \).

\( \mathcal{M}^* \) is used to denote the class of sets measurable with respect to \( W^* \). In [6, section 5.8, p.298] it is observed that if \( T \) is BVW and \( W \) is absolutely continuous with respect to \( \mu \), then \( \mathcal{D} \subseteq \mathcal{M}^* \). In [6, section 5.10, p.299] we find defined \( \mathcal{M}^{**} = \mathcal{M}^* \cap \mathcal{M} \supseteq \mathcal{D} \), and, in fact, it has been observed (unpublished) by James E. Duemmel that \( \mathcal{M}^{**} = \mathcal{M} \).

The proof of this observation is as follows: From H4 it is readily seen that if \( M \) in \( \mathcal{M} \) is a subset of \( \bigcup D_j \) for some
j, then there is a sequence of sets \( \{D_{im}\} \subseteq \mathcal{D} \) such that for each \( m \), \( M \subseteq \bigcup_{i} D_{im} \) and \( \sum_{i} \mu(D_{im}) < \mu(M) + 1/m \). Hence, \( M \subseteq \bigcap_{m} \bigcup_{i} D_{im} = M^* \) and \( \mu(M^* - M) = 0 \). Consequently, \( M^* - M \in \mathcal{M}^* \) (see [6, Lemma 5.7, p.298]). But \( M^* \in \mathcal{M}^* \) since \( \mathcal{D} \subseteq \mathcal{M}^* \) (see above). Thus, since \( \{S, \mathcal{M}, \mu\} \) is a complete measure space, we have \( M \in \mathcal{M}^* \). Since every set \( M \) in \( \mathcal{M} \) can be expressed as a countable union of sets \( M_j \) in \( \mathcal{M} \) such that \( M_j \subseteq \#D_j \), it is now clear that \( \mathcal{M} \subseteq \mathcal{M}^* \); and thus, \( \mathcal{M}^* = \mathcal{M} \).

We shall use this observation of Duemmel to write in simpler form some formulas from [6] which we quote below without specifically calling attention to the simplification each time. In [6, Theorem 5.10, p.299] it is shown, assuming the transformation \( T \) is BVW and assuming that \( W(\cdot) \) is over additive and absolutely continuous with respect to \( \mu \), that \( T \) is ACW, i.e., there exists (by the Radon - Nikodym theorem -- see 2.3) a non-negative, extended-real valued, \( \mathcal{M} \)-measurable function \( f \) defined on \( S \) and integrable \( \mu \) such that for \( M \) in \( \mathcal{M} \)

\[
W^*(M) = \int_{M} f(s) \, d\mu,
\]

and also that for \( D \) in \( \mathcal{D} \)

\[
W(D) = W^*(D) = \int_{D} f(s) \, d\mu.
\]

The reader should note that the last equation above is equation 8.0a). Since \( \mathcal{D} \supseteq \mathcal{D}_* \), we have by 8.2 and 7.6 the following theorem:
8.4 Theorem. Assume postulates H1 - H9 of [6, pp.284-287]. If the transformation T from S onto S' is ACW and if HM holds, then for the B derivative of W* with respect to μ where B is the derivation basis generated by the monotone sequence of nets of 8.2, we have that the Radon - Nikodym integrand associated with the measure W* (see 8.3 above) satisfies the following:
\[ f(s) = DW^*(s) \text{ a.e. } \mathcal{H} \text{ on } S. \]

Thus, under the assumption of theorem 8.4, we see that the function f of equation 8.0a) may be chosen as a certain net derivative.

8.4A Corollary. The conclusion of theorem 8.4 holds if the assumption that HM holds is replaced by the assumption H10i) of [6]. (See Theorem 8.1A.)

8.4B Corollary. The conclusion of theorem 8.4 holds if the assumption that HM holds is replaced by the assumptions H11 and H15i) of [6]. (See Theorem 8.1B.)

8.4C Corollary. Assume postulates H1 - H9 of [6, pp.284-287]. If the transformation T from S onto S' is ACW; WD = \( \int_S W'(s', D) \, d \mu' \), D in \( \mathcal{D} \), where W' is the weight function for T as in H9; and if W* is the outer measure generated by W and the sequential covering class \( \mathcal{D} \) by Method I of Munroe (see [3, pp. 90-91]); and if HM holds; then for the B derivative DW^*(s) of W* with respect to μ, ...
where \( B \) is the derivation basis generated by the monotone sequence of nets of 8.2, we have (recall for the net basis \( B, E_0 \in \mathcal{M} \))

\[ \int_D DW^*(s) \, d\mu = WD = \int_S W'(s'; D) \, d\mu', \quad D \in \mathcal{D}. \]

If \( H' \) is a real valued measurable \( \mathbb{R} \) function defined on \( S' \), then for each \( D \in \mathcal{D} \), \( H'(s') W'(s', D) \) is measurable \( \mathbb{R} \) and \( H'(T(s)) DW^*(s) \) is measurable \( \mathbb{R} \) on \( S \).

Moreover, \( H'(T(s)) W'(s', D) \) is integrable \( \mu' \) over \( S' \) if and only if \( H'(T(s)) DW^*(s) \) is integrable \( \mu \) over \( D \). When these functions are integrable, their integrals have the same value — that is, one has the transformation formula

\[ \int_D H'(T(s)) DW^*(s) \, d\mu = \int_S H'(s') W'(s', D) \, d\mu'. \]

Moreover, if \( f(s) \) is any extended-real valued function on \( S \) which is measurable \( \mathbb{R} \) and such that \( H'(T(s)) f(s) \) is measurable \( \mathbb{R} \) and integrable \( \mu \) on \( S \), then

\[ D \int H'(T(s)) f(s) \, d\mu = H'(T(s)) f(s) \text{ a.e. } \mathcal{G} \text{ on } S. \]

This corollary merely summarizes some of the results of [6] in the light of 8.4, 7.1 and 7.6.

**8.5 Remark.** Let \( B \) represent the derivation basis with exceptional set \( E_0 \) which is generated by the monotone sequence of nets discussed in 8.2. Under the hypothesis H11 of [6, p.302], every member of \( E_0 \) is in the collection \( \mathcal{E} \) of definition 7.6 in [6, p.303]; and hence, for each fixed \( s \) in \( S \), we have \( S \subseteq \cap_n N_{ns} \), where \( s \in S = \cap D, D \in \mathcal{D} \) with \( s \in D \); and \( N_{ns} \) is the unique set in \( \mathcal{T}_n \) which contains \( s \).
If the hypotheses of theorem 8.1C are satisfied, the statement regarding the relation of $S$ to $\cap_n N_{ns}$ can be strengthened to $S = \cap_n N_{ns}$, $s$ being fixed in $S$. This may be seen by noting $\cap_n N_{ns} \subseteq \cap D_s$, $D_s \in \mathfrak{D}_s$, $s \in D_s$, holds since for fixed $D_s$ containing $s$ we have by the construction of $\{M_i\}_{i=1}^\infty$ that for some $n$, $N_{ns} \subseteq D_s$. Thus, equality holds in the following:

$$S \subseteq \cap_n N_{ns} \subseteq \cap D_s = S, D_s \in \mathfrak{D}_s, s \in D_s.$$  

Assuming H12 of [6, p.302], TS is a single point.

Observing that for any two points $s_1$ and $s_2$ in $\cap N_{ns} = E_0$, the same sequence of net sets is used in calculating the derivatives $D^*(W)(s_1)$ and $D^*(W)(s_2)$ (where $W$ is as in 8.4), we conclude immediately that the net derivative of $W^*$ exists a.e. $\mathfrak{N}$ on $\cap_n N_{ns}$ and is constant a.e. $\mathfrak{N}$ on $\cap_n N_{ns}$. This with 8.4 yields the following interesting result:

8.6 Theorem. Assume postulates H1 - H9 of [6, pp.284-287] and H11 and H12 of [6, p.302]. If the transformation $T$ from $S$ onto $S'$ is ACW and if HM holds, then the function $f$ of equation 8.0a) is constant a.e. $\mathfrak{N}$ on the subsets $S$ of $S$, where $S = \cap D$, $S \subseteq D \in \mathfrak{D}$, and these sets $S$ map under $T$ into single points.

8.6A Corollary. The conclusion of theorem 8.6 holds if the assumption that HM holds is replaced by the assumption that H10i) of [6] holds. (See 8.1A.)
8.6B Corollary. The conclusion of theorem 8.6 holds if the assumption that HM holds is replaced by the assumption that H15i) of [6]. holds. (See 8.1B.)

8.7 Remark. The hypotheses of [6] are shown by Reichelderfer to hold in the topological setting of [5]. Hence the results of 8.4 and the results 8.6 hold when the theory of [6] is applied in the setting of [5]. In particular, in the setting of [5], theorem 8.6 tells us that the Radon-Nikodym integrand which appears in the transformation formulas in [5] is constant a.e. on each component of the inverse under T of single points in S'.

8.8 Remark. If Σ is any non-empty open set in R^n and if B is the usual derivation basis employed in calculating the regular derivative in R^n (see 3.14), then it is easily seen that for each x in Σ at which the regular derivative of a set function with respect to μ(M=Lebesgue measure restricted to the Borel sets of R^n) exists, the derivative may be computed by using the Σ-restricted derivation basis, B ∩ Σ, which consists only of those regular sequences of closed intervals, each of whose constituents are subsets of Σ. Also, the measure with respect to which the derivative is computed can be taken as μ_{Σ}, the restriction of μ to M ∩ Σ, i.e., the members of M which are subsets of Σ (note Σ is in M). One has immediately from definition that A ⊆ Σ implies μ_{Σ}(A) = μ(A). Also, every subset A of Σ
with $\mathbb{M}^\delta(A) < \infty$ has a $\mu_\sigma$-cover. The following statement can now be made: If $\sigma$ is a semi-regular $\mathbb{M} \cap \sigma$-measure, then $B \cap \sigma$ possesses the Strong Vitali $\sigma$-property. The proof of this is identical with the proof of 3.20 (pp.40-41) except now $\sigma$ replaces $R^n$, $B \cap \sigma$ replaces $B$, $\mu_\sigma$ replaces $\mu$, and the open set $G$ appearing in the proof is now such that $\sigma \supseteq G$. We can now state a result which is a modification of 3.22 and whose proof is identical to that of 3.22 except that the observation above replaces the reason 3.20.

**8.8A Theorem.** If $\sigma_0$ is a semi-regular $\mathbb{M} \cap \sigma$-measure on $\sigma$, a non-empty open subset of $R^n$, which is singular on $\sigma$ with respect to $\mu_\sigma$, then for the $B \cap \sigma$-derivative we have $D \sigma_0(x) = 0$ a.e. $\mu_\sigma$ in $\sigma$.

Now let $\sigma$ be any semi-regular $\mathbb{M} \cap \sigma$-measure such that $\sigma$ is sigma-finite with respect to $\sigma$ and use 2.1 to decompose $\sigma$ into measures $\sigma_A$ and $\sigma_0$ which are absolutely continuous and singular in $\sigma$ with respect to $\mu_\sigma$. Note that $\sigma_A$ and $\sigma_0$ are semi-regular since $\sigma$ is non-negative. By the remark in 8.8 that $B \cap \sigma$ possesses the Strong Vitali $\sigma$-property for any semi-regular $\mathbb{M} \cap \sigma$-measure $\sigma$, it is clear that $B \cap \sigma$ possesses the Strong Vitali $\mu_\sigma$-property.
If $B_N$ is a derivation basis generated by a monotone sequence of nets on $\mathcal{C}$ which is regular on each $M$ in $\mathcal{M} \cap \mathcal{C}$ with respect to each of $\sigma_0$ and $\mu_\sigma$, then by 3.30 and 8.8A the derivative of $\sigma_0$ with respect to $\mu_\sigma$ exists and is the same a.e. $\mu_\sigma$ on $\mathcal{C}$ for either of the derivation bases under discussion.

Also, by 4.4 the derivative of $\sigma_A$ with respect to $\mu_\sigma$ exists and is the same a.e. $\mu_\sigma$ on $\mathcal{C}$ for either of the derivation bases under discussion. Hence, by 1.6 we have the following result:

8.8B Theorem. If $\sigma$ is any semi-regular $\mathcal{M} \cap \mathcal{C}$ - measure on $\mathcal{C}$ such that $\mathcal{C}$ is sigma-finite with respect to $\sigma$, if $B \cap \mathcal{C}$ is the $\mathcal{C}$-restriction of the derivation basis $B$ used to calculate the regular derivative in $\mathbb{R}^n$, and if $B_N$ is the derivation basis generated by a monotone sequence of nets on $\mathcal{C}$ which is regular on each $M$ in $\mathcal{M} \cap \mathcal{C}$ with respect to both $\sigma_0$ and $\mu_\sigma$, $\sigma_0$ being the $\mu_\sigma$-singular on $\mathcal{C}$ component of the Lebesgue decomposition of $\sigma$ and $\mu_\sigma$ being the restriction of Lebesgue measure on $\mathbb{R}^n$ to the Borel sets in $\mathcal{C}$, then the $B \cap \mathcal{C}$ derivative (regular derivative) of $\sigma$ with respect to $\mu_\sigma$ is equal a.e. $\mu_\sigma$ on $\mathcal{C}$ to the $B_N$ (net) derivative of $\sigma$ with respect to $\mu_\sigma$.

8.9 Remark. In [6, section 9, pp.310-313] Reichelderfer has observed that if $T$ is a continuous transformation from a non-empty domain $S$ in Euclidean $n$-space onto a (Borel) set $S'$ in Euclidean $n'$ space, and if $\mu$ and $\mu'$ are
the Lebesgue $n$ and $n'$-dimensional measures of sets in $S$ and $S'$ respectively, while $\mathcal{M}$ and $\mathcal{M}'$ are the Lebesgue measurable subsets of $S$ and $S'$ respectively; and if $\mathcal{D}$ is the set of all domains (connected open sets) in $S$ while $\mathcal{D}^*$ is the set of all domains $D^*$ in $\mathcal{D}$, each of which is an $n$-dimensional polyhedron having rational vertices, then the postulates of [6] which do not directly concern the weight function $W'$ for $T$ (i.e., all except H9, H13, H14) are satisfied; hence, by theorem 8.1A or theorem 8.1B, also $\mathcal{HM}$ is satisfied with $\mathcal{D}_*$ of $\mathcal{HM}$ chosen to be $\mathcal{D}^*$ (above). Thus, if $W'$ is a weight function for $T$ in the sense defined in H9 (see [6, pp. 286-287]) and the weights generated by $W'$ are denoted by $W_D = \int_S W'(s', D) \, d\mu'$ for $D$ in $\mathcal{D}$; then, if $T$ is of bounded variation with respect to the weights $W$—that is, $W'(s', S)$, $s'$ in $S'$, is Lebesgue integrable—there exists a function $f$ defined and integrable on $S$ such that

$$\int_D f \, d\mu = WD \text{ for } D \text{ in } \mathcal{D} \text{ if and only if } W \text{ is over additive and absolutely continuous with respect to } \mu; \text{ and, by theorem 8.4, } f \text{ equals a.e.} \mu \text{ on } S \text{ the net derivative with respect to } \mu \text{ of the outer measure } W^* \text{ generated by } W \text{ from the sequential covering class } \mathcal{D} \text{. (The net derivative mentioned above refers to the derivative with respect to the derivation basis generated by the monotone sequence of nets of } 8.2 \text{ constructed by using the collection } \mathcal{D}^* \text{ above.)}$$
Moreover, since the collection \( \mathcal{D}^* \) (above) is easily seen to satisfy postulate 5.411 (p. 63) for any semi-regular \( \mathcal{M} \)-measure, \( \sigma \), we have by 5.8 that the monotone sequence of nets of 8.2 is regular with respect to any semi-regular \( \mathcal{M} \)-measure \( \sigma \) on each \( M \) in \( \mathcal{M} \). In particular, the monotone sequence of nets of 8.2 is regular with respect to \( \mu \) on each \( M \) in \( \mathcal{M} \); and if \( S \) is sigma-finite with respect to \( \sigma \), then the monotone sequence of nets of 8.2 is regular on each \( M \) in \( \mathcal{M} \) with respect to \( \sigma_0 \), the \( \mu \)-singular part of \( \sigma \) appearing in the Lebesgue decomposition of \( \sigma \). (Note that \( \sigma \) is also a Borel measure on \( S \); and hence, the monotone sequence of nets of 8.2 is regular with respect to \( \sigma \) on each Borel set.) Thus, if \( \sigma \) is any semi-regular \( \mathcal{M} \)-measure which satisfies the condition that \( S \) is sigma-finite with respect to \( \sigma \), we have by 8.8B that the net derivative of \( \sigma \) with respect to \( \mu \) in this case is equal a.e. \( \mu \) on \( S \) to the regular derivative of \( \sigma \) with respect to \( \mu \). By 3.17 (see also 8.3) if \( T \) is \( ACW \), then \( \mathcal{W}^* \) is a semi-regular Borel measure. Also, if \( H' \) is any non-negative \( \mathcal{M}' \)-measurable function defined on \( S' \), then, for \( f \) as above, \( H'(T(s)) f(s) \) is measurable \( \mathcal{M} \); and if \( H'(T(s)) f(s) \) is integrable \( \mu \) on \( S \), then by 3.17
\[
\int H'(T(s)) f(s) \, d\mu \text{ is a semi-regular Borel measure and}
\]
\[
D\int H'(T(s)) f(s) \, d\mu = H'(T(s)) f(s) \text{ a.e. } \mu \text{ on } S,
\]
the derivative with respect to \( \mu \) being essentially the same whether it is the net derivative (above) or the regular
derivative. The above statement is also true for an arbitrary \( \mathbb{R}' \)-measurable function \( H' \) defined on \( S' \). This may be seen by decomposing \( H' \) into its positive and negative parts and using the above statement together with 1.6.
REFERENCES


