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VALUE PROBLEMS IN THE THEORY OF ELASTICITY

DISSERTATION
Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
Schoof of The Ohio State University

By

Chunchang Lo, B.Sc., M.Sc.

The Ohio State University
1964

Approved by

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Adviser
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CONTENTS

ACKNOWLEDGEMENTS ............................................................. ii
VITA ......................................................................................... iii
LIST OF TABLES ......................................................................... vii
LIST OF FIGURES ....................................................................... viii
NOTATION ................................................................................ xi
INTRODUCTION ........................................................................... 1

Chapter

I SERIES SOLUTION ............................................................... 5

(A) The Point Matching Method, Least Square
Approximation and Multiple Poles Scheme ........ 5
(B) Numerical Examples for Membrane Problems .... 9

II MATRIX INVERSION AND ROUND-OFF ERRORS ............... 20

(A) The Matrix Inversion ................................................. 20
(B) The Matrix Condition ............................................. 29
(C) Round-off Errors and the Upper Bound
of Errors ................................................................. 32

III SERIES CONVERGENCE AND ITS IMPROVEMENT ........... 37

IV SINGULAR INTEGRAL SOLUTION ..................................... 47

(A) Step Function Approximation, Numerical
Examples ................................................................. 50
(B) Polygonal Function Approximation,
Numerical Examples ................................................ 64

V COMBINATION OF SERIES SOLUTION AND SINGULAR
INTEGRAL SOLUTION, NUMERICAL EXAMPLE IN
HEAT TRANSFER ............................................................... 72
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>VI APPLICATION TO TORSION PROBLEMS</td>
<td>81</td>
</tr>
<tr>
<td>(A) Theory, Equation and Solutions of Torsion</td>
<td>81</td>
</tr>
<tr>
<td>With Multiply Connected Region</td>
<td></td>
</tr>
<tr>
<td>(B) Torsional Rigidity</td>
<td>90</td>
</tr>
<tr>
<td>(C) Numerical Examples</td>
<td>94</td>
</tr>
<tr>
<td>(D) Anisotropic Torsion</td>
<td>104</td>
</tr>
<tr>
<td>VII APPLICATION TO PLATE BENDING</td>
<td>112</td>
</tr>
<tr>
<td>(A) Theory and Equations, Thermal and Isothermal</td>
<td>112</td>
</tr>
<tr>
<td>Loading</td>
<td></td>
</tr>
<tr>
<td>(B) Singular Integral Solution</td>
<td>118</td>
</tr>
<tr>
<td>(C) Plates on Elastic Foundation</td>
<td>132</td>
</tr>
<tr>
<td>VIII APPLICATION TO PLANE ELASTICITY PROBLEMS WITH MIXED BOUNDARY CONDITIONS</td>
<td>137</td>
</tr>
<tr>
<td>(A) Theory and Equations, Thermal and Isothermal</td>
<td>137</td>
</tr>
<tr>
<td>Loading</td>
<td></td>
</tr>
<tr>
<td>(B) Series Solution And Singular Integral</td>
<td>146</td>
</tr>
<tr>
<td>Solution</td>
<td></td>
</tr>
<tr>
<td>(C) Numerical Example</td>
<td>165</td>
</tr>
<tr>
<td>APPENDIX A FREDHOLM INTEGRAL EQUATIONS AND GREEN'S FUNCTION</td>
<td>169</td>
</tr>
<tr>
<td>APPENDIX B EQUATIONS OF THE SINGULAR INTEGRAL SOLUTION FOR PLATE BENDING</td>
<td>178</td>
</tr>
<tr>
<td>BIBLIOGRAPHY.</td>
<td>186</td>
</tr>
</tbody>
</table>
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-1</td>
<td>Deflection at the center of the membrane</td>
<td>18</td>
</tr>
<tr>
<td>1-2</td>
<td>The largest ratio of the largest element to the smallest element in the last column of the coefficient matrix</td>
<td>19</td>
</tr>
<tr>
<td>2-1</td>
<td>Round-off errors for sample problems in Chapter I</td>
<td>35</td>
</tr>
<tr>
<td>4-1</td>
<td>Deflection, slopes and N-condition number of a rectangular membrane</td>
<td>69</td>
</tr>
<tr>
<td>6-1</td>
<td>Shear stress at point A of Fig. 1-1</td>
<td>95</td>
</tr>
<tr>
<td>6-2</td>
<td>Torsional rigidity of rectangular plate (Refer. Fig. 4-4)</td>
<td>96</td>
</tr>
<tr>
<td>8-1</td>
<td>Physical meaning of various terms in Eq. (8-37)</td>
<td>157</td>
</tr>
<tr>
<td>8-2</td>
<td>Terms of series Eq. (8-37) which give multiple values of stresses or displacement</td>
<td>159</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>1-1</td>
<td>A membrane with concave sides</td>
<td>10</td>
</tr>
<tr>
<td>1-2</td>
<td>Contour map for deflection, 3x3 and 7x3</td>
<td>12</td>
</tr>
<tr>
<td>1-3</td>
<td>Contour map for deflection, 7x7 and 15x7</td>
<td>13</td>
</tr>
<tr>
<td>1-4</td>
<td>Contour map for deflection, 15x15 and 31x15</td>
<td>14</td>
</tr>
<tr>
<td>1-5</td>
<td>Deviation curves along the boundary</td>
<td>15</td>
</tr>
<tr>
<td>1-6</td>
<td>Coordinates for three poles</td>
<td>17</td>
</tr>
<tr>
<td>3-1</td>
<td>Boundary values of the equilateral triangle and the triangle with concave sides</td>
<td>39</td>
</tr>
<tr>
<td>3-2</td>
<td>Coordinates for a concentrated loaded square slice</td>
<td>41</td>
</tr>
<tr>
<td>3-3</td>
<td>Stresses in a concentrated loaded slice</td>
<td>43</td>
</tr>
<tr>
<td>3-4</td>
<td>Coordinates for a concentrated load on semi-infinite plane</td>
<td>44</td>
</tr>
<tr>
<td>3-5</td>
<td>Stresses in a concentrated loaded square slice</td>
<td>46</td>
</tr>
<tr>
<td>4-1</td>
<td>A balloon subjected to the pressure of a thin wall tube</td>
<td>48</td>
</tr>
<tr>
<td>4-2</td>
<td>Step function approximation along the boundary</td>
<td>51</td>
</tr>
<tr>
<td>4-3</td>
<td>Coordinates for a segment on the boundary</td>
<td>53</td>
</tr>
<tr>
<td>4-4</td>
<td>Rectangular membrane</td>
<td>54</td>
</tr>
<tr>
<td>4-5</td>
<td>Density function along the boundary</td>
<td>55</td>
</tr>
<tr>
<td>4-6</td>
<td>Graphical representation of the elements of a matrix</td>
<td>57</td>
</tr>
</tbody>
</table>

viii
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-7</td>
<td>Graphical representation of the elements of a matrix</td>
<td>58</td>
</tr>
<tr>
<td>4-8</td>
<td>Deviation curves along the boundary</td>
<td>60</td>
</tr>
<tr>
<td>4-9</td>
<td>Density function along the boundary</td>
<td>61</td>
</tr>
<tr>
<td>4-10</td>
<td>Segments along the boundary</td>
<td>62</td>
</tr>
<tr>
<td>4-11</td>
<td>Integration of the density function along the boundary</td>
<td>63</td>
</tr>
<tr>
<td>4-12</td>
<td>Polygonal function along the boundary</td>
<td>65</td>
</tr>
<tr>
<td>4-13</td>
<td>Coordinates for a segment on the boundary</td>
<td>66</td>
</tr>
<tr>
<td>4-14</td>
<td>Deviation curve along the boundary</td>
<td>70</td>
</tr>
<tr>
<td>4-15</td>
<td>Density function ( p(B) ) along the boundary</td>
<td>71</td>
</tr>
<tr>
<td>5-1</td>
<td>Boundary conditions of a rectangular plate</td>
<td>73</td>
</tr>
<tr>
<td>5-2</td>
<td>Temperature in the rectangular plate</td>
<td>75</td>
</tr>
<tr>
<td>5-3</td>
<td>Temperature gradient ( \nabla \phi ) along two edges</td>
<td>75</td>
</tr>
<tr>
<td>5-4</td>
<td>Boundary points of a rectangular plate</td>
<td>76</td>
</tr>
<tr>
<td>5-5</td>
<td>Temperature in the rectangular plate</td>
<td>77</td>
</tr>
<tr>
<td>5-6</td>
<td>Temperature gradient ( \nabla \phi ) along two edges</td>
<td>77</td>
</tr>
<tr>
<td>5-7</td>
<td>The coordinates of one additional term of the integral solution</td>
<td>79</td>
</tr>
<tr>
<td>5-8</td>
<td>Temperature in the rectangular plate</td>
<td>80</td>
</tr>
<tr>
<td>5-9</td>
<td>Temperature gradient ( \nabla \phi ) along two edges</td>
<td>80</td>
</tr>
<tr>
<td>6-1</td>
<td>Coordinates of a multiply-connected region of a hollow square shaft</td>
<td>89</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
<td>------</td>
</tr>
<tr>
<td>6-2</td>
<td>Segments and matched points</td>
<td>97</td>
</tr>
<tr>
<td>6-3</td>
<td>Density function p(B) along the boundary</td>
<td>98</td>
</tr>
<tr>
<td>6-4</td>
<td>Lines of equal shear stress and $\xi$-contour</td>
<td>99</td>
</tr>
<tr>
<td>6-5</td>
<td>Segments and matched points of a square shaft with two rectangular holes</td>
<td>101</td>
</tr>
<tr>
<td>6-6</td>
<td>Density function p(B) along the boundary</td>
<td>102</td>
</tr>
<tr>
<td>6-7</td>
<td>Lines of equal shear stress and $\xi$-contour</td>
<td>103</td>
</tr>
<tr>
<td>6-8</td>
<td>Coordinates rotation</td>
<td>105</td>
</tr>
<tr>
<td>7-1</td>
<td>A small element of the plate</td>
<td>114</td>
</tr>
<tr>
<td>7-2</td>
<td>Coordinates for a concentrated moment</td>
<td>123</td>
</tr>
<tr>
<td>7-3</td>
<td>The influence of certain functions</td>
<td>125</td>
</tr>
<tr>
<td>7-4</td>
<td>The influence of the modified concentrated load</td>
<td>129</td>
</tr>
<tr>
<td>7-5</td>
<td>The influence of the modified concentrated moment</td>
<td>131</td>
</tr>
<tr>
<td>7-6</td>
<td>Coordinates for a concentrated moment</td>
<td>135</td>
</tr>
<tr>
<td>8-1</td>
<td>Stresses and displacements of a short uniformly loaded cantilever</td>
<td>167</td>
</tr>
<tr>
<td>8-2</td>
<td>Stresses and displacements of a short uniformly loaded cantilever</td>
<td>168</td>
</tr>
</tbody>
</table>
NOTATION

x, y, z  Rectangular coordinates.

r, \theta  Polar coordinates.

n, t  Normal and tangential coordinates.

u, v, w  Displacements in x, y and z directions respectively.

\Phi  Prandtl stress function for torsion, or Airy stress function for plane elasticity.

\Psi  Warping functions for torsion.

\alpha  Angle between the x-axis and the normal direction.

\phi  Angle between the radial direction and the normal direction.

\tau_x, \tau_y, \tau_z  Components of normal and shear stresses.

\epsilon_{xx}, \epsilon_{yy}, \epsilon_{xy}  Components of normal and shear strain.

\epsilon_{xx}, \epsilon_{yy}, \epsilon_{xy}  Components of normal and shear strain in tensor notation.

M_x, M_y, M_z  Components of bending and twisting moments.

M_t  Twisting moment for torsion.

Q_x, Q_y  Components of shear forces.

V_x, V_y  Components of free edge shear forces.

E  Young's Modulus.

G  Modulus of shear, \( G = \frac{E}{2(1+\nu)} \)
Poisson's ratio

\[ \nu \]

Coefficients of thermal expansion.

\[ \nabla^2 = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) = \left( \frac{\partial^2}{\partial x^2} + \frac{1}{\lambda} \frac{\partial}{\partial x} + \frac{1}{\nu^2} \frac{\partial}{\partial y} \right) \]

\[ \nabla^4 = \left( \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} \right) = \left( \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^3}{\partial x^3 \partial y} + \frac{\partial^4}{\partial y^4} \right) \]

\[ = \left( \frac{\partial^2}{\partial x^2} + \frac{1}{\lambda} \frac{\partial}{\partial x} + \frac{1}{\nu^2} \frac{\partial}{\partial y} \right)^2 \]
INTRODUCTION

The problems of mathematical physics have found wide application in the most diverse fields of engineering, but exact solutions usually can not be obtained except for a few rather simple cases. Therefore, in practice, these problems are treated approximately or experimentally. Among the approximate methods, the finite difference and variational methods which were developed at the beginning of this century are probably the most well-known. During the past decade, the basic approximate methods have undergone further development on questions of convergence and estimates of error and a number of new methods have appeared. One of them is known as the point matching method. The point matching method is one of the infinite series methods and can be considered as a special case of the interpolation method. It has the advantages that

(1) the method is simple, but yields reasonably accurate results for most problems

(2) it works the same way for problems with different boundary shapes.

During recent years, the digital computer has been widely used in
science and engineering, causing many laborious methods which are not convenient to operate on the computer to be abandoned. On the other hand, the point matching method has become very favorable because a single computer program for a given partial differential equation can be used for any problem governed by that equation no matter what the boundary shape is.

As to the origin of the point matching method, it has occurred to a number of people independently as early as J. C. Slater (1934) and J. Barta (1937). However, the name "point matching" is due to H. D. Conway (1960).

Since November, 1962, a research project has been in progress in the Department of Engineering Mechanics, The Ohio State University, for advancing the study of the point matching method and providing computer programs for use in industries and research institutes. That research project has been sponsored by Wright-Patterson Air Force Base and Prof. F. W. Niedenfuhr, Prof. A. W. Leissa, the writer and others have participated. The final report for the first year has been published as listed under reference (1). This dissertation is a part of the continuation of that research.

All methods which we shall discuss later will satisfy two conditions.

(a) It is possible to write a single computer program such that, without any modification, it will yield an approximate solution for any problem governed by a given differential equation no matter what the
boundary shape and boundary conditions are.

(b) As a property of the harmonic and the biharmonic functions, the maximum error of the approximation solution always occurs on the boundary. Since the errors on the boundary can be evaluated, the upper bound of the error of the approximate solution therefore can be obtained.

In this dissertation, we shall not only introduce methods for solving various boundary value problems, but also do two important additional things:

(a) Improve the convergence of solutions which represent problems having singularities or discontinuities on the boundary.

(b) Investigate and reduce the computing round-off error.

The point matching method, its least square approximation, and its multiple poles scheme will be introduced very briefly in Chapter I. Then in Chapter II, we shall discuss the round-off errors due to solving a set of linear simultaneous equations. This error almost represents the entire computing round-off error. The amount of error depends on the matrix condition which essentially governs the accuracy of the numerical solution. In Chapter III, we shall discuss the convergence of the series solution and its improvement. For a given problem of mathematical physics, one can set up a differential equation or if he likes, he can set up
an integral equation, and then solve it. Up to now, the literature is voluminous on analytical properties of both differential equations and integral equations, but the numerical treatment for two or three dimensional physical problems is almost limited to the former. In Chapter IV, we shall introduce the notion of how to set up a singular integral equation for a given problem and how to solve it numerically. This approach can eliminate some of the difficulties which might be encountered in the series solution. In Chapter V, we shall show the value of using the series solution and singular integral equation together, which not only yields a fast convergent solution, but also broadens the scope of the boundary value problems which we can handle. The rest of the chapters contain some of the details of the applications in torsion, plane elasticity and plate bending by methods which we mentioned above.
A). The Point Matching Method, Least Square Approximation
And Multiple Poles Scheme

For simplicity, we shall introduce the point matching method
and its modification by discussing the deflection of a uniformly
loaded membrane, the governing differential equation being
\[ w = - \frac{q}{T} \] 
\[ w = K \] on the boundary. Since \( K \) is an arbitrary constant for
a simply-connected region, we let it be zero, so we have the
boundary condition
\[ w = 0 \]
In Eq. (1-1), \( w \) is the deflection of the membrane, the constants
\( q \) and \( T \) are the lateral loading and the membrane tension, respect­
ively. The solution for this equation generally can be written as
\[ w = w_p + w_c \] 
where \( w_p \) and \( w_c \) are the particular and complementary solutions,
respectively. Physically, the contributions of \( w_p \) and \( w_c \) are
that the former defines the lateral loading while the later adjusts
the complete solution to satisfy a given boundary value on the
boundary. We can let the particular solution be

\[ \chi_p = -\frac{1}{4} r^2 \frac{d^2}{dr^2} \]

.................................(1-4)

and the complementary solution in polar coordinates be

\[ \chi_c = A_i \ln r + \sum_{n=1}^{\infty} \left( A_n r^n + B_n r^{-n} \right) \cos n\theta + \sum_{n=1}^{\infty} \left( A'_n r'^n + B'_n r'^{-n} \right) \sin n\theta \]

.................................(1-5)

where \( A_n, B_n, A'_n, \) and \( B'_n \) are constants to be determined. If we select a finite number of terms of the series of Eq. (1-5), say \( N \) terms, we can determine the constants by solving \( N \) linear simultaneous equations which are obtained by satisfying the boundary conditions at \( N \) discrete points. These equations can be written as

\[ \alpha_{ij} \chi_j = b_i, \quad i = 1, 2, \ldots, N; \quad j = 1, 2, \ldots, N. \]

.................................(1-6)

where \( \alpha_{ij} \) is the coefficient matrix, \( \chi_j \) are the unknown constants; i.e., \( A_n, B_n, A'_n, \) and \( B'_n \) in Eq. (1-5); and the \( b_i \) result from the combination of the particular solution and boundary values. This method is called the point matching method since the final solution satisfies the boundary condition only at \( N \) points. Boundary values between the matched points may be determined only after a particular problem has been solved and the boundary values between the matched points have been calculated. Furthermore, no mathematical analysis is known which predicts whether the deviation between the matched points will decrease as the number of matched points increases. In other words, whether or not the solution obtained by the point matching method is convergent is not known. In order to help establish the solution to be convergent in the
mean, the point matching method has been modified to be used in the least square sense.

If we have a polynomial \( \sum_{n=1}^{N} a_n U_n(s) \) with \( N \) terms to represent a given function \( f(s) \), we can determine the coefficients by requiring the sum of the square error along the boundary be minimum; i.e.

\[
\sum_{n=1}^{N} \int \left( \sum_{n=1}^{N} a_n U_n(s) - f(s) \right)^2 ds = c, \quad i = 1, 2 \ldots N
\]

or

\[
\sum_{n=1}^{N} a_n \int U_n(s) U_i(s) ds = \int f(s) U_i(s) ds, \quad i = 1, 2 \ldots N.
\] .................(1-7).

The integral in Eq. (1-7) can be obtained by using the rectangle approximation; i.e., replacing the integral by a set of equally spaced points, say \( M \) where \( M \geq N \). Then Eq. (1-7) can be written as

\[
\sum_{n=1}^{N} \int a_n S \sum_{j=1}^{M} \left[ U_n(s_j) U_i(s_j) \right] ds = \alpha S \sum_{j=1}^{M} f(s_j) U_i(s_j), \quad i = 1, 2 \ldots N
\]

or equivalently,

\[
\alpha_{k,i} = a_{k,i} \beta_i, \quad i = 1, 2 \ldots \tilde{N}; \quad j = 1, 2 \ldots \tilde{M};
\]

\[
\tilde{N} = N, \quad \tilde{M} = M
\]

where \( \alpha_{i,j} \) is the coefficient matrix which is obtained by satisfying the boundary condition at \( M \) points, and \( \alpha_{k,i} = \alpha_{i,k}^T \) is the transpose of \( \alpha_{i,j} \). One can see now that the least square approximation is merely point matching at \( M \) points with \( N \) unknowns to be determined. If \( M \) approaches \( N \), then the solutions obtained by point matching method and least square approximation become the same.
But for $M \leq N$ the operation of multiplying the coefficient matrix by its transpose will merely increase the round-off error and cause the condition of the matrix to worsen. On the other hand, if the ratio of $M$ to $N$ keeps increasing, the solution will converge to the solution which should be obtained from Eq. (1-7), but the round-off error increases rapidly as the ratio becomes too big. Therefore, there is an optimum ratio of $M$ to $N$ which yields the most accurate solution for a given number of unknowns. From the author's experience, this ratio is usually about 2.

In conclusion, we obtain an infinite series solution which is expanded about a fixed origin. Since there is no limitation as to where we can put the origin, we certainly can use several independent series each expanded about different origins or poles. The technique of using multiple poles has several advantages over using a series expanded about a single origin. The most important advantage is that it can treat a highly multiply-connected region. L. E. Hulbert (5) (1963) solved plane elasticity problems with many holes by this scheme. Because the solution we obtain is a numerical approximation, the terms retained in one series need not be the same as is retained in another series expanded respect to another origin. Some other advantages of the multiple poles scheme will be discussed in the next section.
B). **Numerical Examples for Membrane Problems**

A series in polar coordinates as Eq. (1-5), becomes a series of orthogonal functions at the boundary if the boundary is a circle. The greatest advantage of an orthogonal series is that one equation will determine one unknown constant of the series. If the series of Eq. (1-5) is applied to a nearly circular boundary, the generated matrices of Eq. (1-6) and Eq. (1-8) will be nearly orthogonal. A square matrix \( A \) is said to be orthogonal if \( A = (A^T)^{-1} \). For a rectangular orthogonal matrix, the inner product of two column vectors equals \( \delta_{ij} \), where \( \delta_{ij} = 0 \) if \( i \neq j \). For a nearly orthogonal matrix, the least square product matrix \( (A^TA) \) is nearly a diagonal matrix. If the boundary shape becomes far away from a circle like a very long rectangle, then the generated matrix may be ill conditioned, a phenomenon which we shall discuss in the next chapter. In order to have a severe test for the point matching method, we shall show a numerical example which has an unfavorable boundary for the series of Eq. (1-5).

**Example.**
A membrane has a boundary shape which is formed by three arcs with origins at the vertices of an equilateral triangle as shown in Fig. 1-1.
A MEMBRANE WITH CONCAVE SIDES

Fig. 1-1

This shape was demonstrated by Leissa and Brann(11) to be unfavorable for application of the point matching method. If we let the origin of the coordinates be at the center of the triangle, we have three axes of symmetry. Due to the symmetry and the requirement that $w$ is bounded as $r$ approaches zero, all the constants except $A_c$, $A_3$, $A_6$, $A_q$, ..., in the complementary solution series of Eq. (1-5) should be zero. Now, we start to form an approximate solution by retaining the first three terms $A_c$, $A_3$, $A_6$ of the series and determine them by satisfying the boundary conditions at three equally spaced points as shown in Fig. 1-1. Then we can draw the contour lines of $w$ from this approximate solution as shown in the upper half of Fig. 1-2. Dotted lines indicate the zero contour which represents the actual boundary shape for this solution. One can see that there is not any closed zero contour for this solution. However, the
other closed contour lines indicate the boundary shapes which this solution can also represent since the boundary value can be any constant. Similarly, we can solve the same problem by the least square approximation; i.e., to determine three unknowns by approximately satisfying the boundary at seven equally spaced points. The results are shown in the lower half of Fig. 1-2, so one can compare with the results from the point matching method. 3 X 3 and 7 X 3 in Fig. 1-2 means that the solution came from solving a three by three matrix and a seven by three matrix; i.e., three equations with three unknowns and seven equations with three unknowns. In order to see the convergence, we have also drawn the contour lines for solutions obtained from 7 X 7, 15 X 7, 15 X 15 and 31 X 15 matrices in Fig. 1-3 and Fig. 1-4. In all cases, the boundary points are equally spaced and the terms of the series in Eq. (1-5) are selected according to the sequence $A_0, A_3, A_7, \ldots$ until the total number of unknowns equals three, seven or fifteen. By comparing the contour lines obtained from different solutions as shown in Fig. 1-3 and Fig. 1-4, one can see that they are very much alike except that the zero contour lines have a slight deviation near the vertex of the triangle. To see the deviation along the desired boundary from A to B (Fig. 1-1), the deviation curves for each of the six cases discussed above were plotted and shown in Fig. 1-5. Among them, the solution obtained from 15 X 7 matrix gives the best accuracy, it has a maximum error 0.3% at the vertex of the triangle comparing
CONTOUR MAPS FOR DEFLECTION $w$, in $10^{-6}/T$

Fig. 1-3
CONTOUR MAPS FOR DEFLECTION \( W \), in \( 10^{-2} \text{t/T} \)

Fig. 1-4
\[ \theta (\text{at the center of the triangle}) = 0.8318 \, \text{g/T} \]

**DEVIATION CURVES ALONG THE BOUNDARY**

Fig. 1-3
with the deflection at the center. According to St. Venant's principle, the effect of this boundary residue will reduce rapidly at points away from the boundary. Therefore, one can estimate that the deflection at the center, \( w = 0.8318 \frac{q}{t} \) should have an accuracy within 0.1%. Looking at the deviation curve for the 31 \( \times \) 15 solution in Fig. 1-5, one may be surprised to observe that solution is less accurate than the solution obtained from a 15 \( \times \) 7 array. Mathematically, this should not have happened since a least square solution converges in the mean on the boundary. Again, looking at the deviation curve represented by the 15 \( \times \) 15 solution, it blows out near the vertex and it does not pass through some of the selected boundary points (e.g., 290°) in Fig. 1-5 as it is supposed to. We shall explain all of these deviations in the next chapter.

Now, let us solve the same problem by the multiple pole scheme. We shall use three poles, each of them at a vertex of the triangle, and each pole having an independent expanded series like Eq. (1-5). Because \( w \) need not be bounded at each pole, we can retain the negative order terms \( \left( n^m \right) \) of the expanded series. As to the positive terms, we can take them or not. In this example, we do not. If we arrange the coordinates for each pole as shown in Fig. 1-6, then the sine terms in the series can be dropped out since it has symmetry respect to each X axis. We let

\[
\begin{align*}
W_c &= W_{c1} + W_{c2} + W_{c3} = \sum_{n=1}^{3} W_{c5} \quad \ldots \ldots \ldots (1-9) \\
W_{c5} &= \sum_{n=1}^{N_5} A_n^{(s)} \frac{R_s^{-n}}{l_s} n_0 + A_n^{(s)} \ln R_s, S=1,2,3 \quad \ldots \ldots (1-10)
\end{align*}
\]
where $w_c$ is the complementary solution, $w_{c1}$, $w_{c2}$ and $w_{c3}$ are expanded series respect to three poles and $\theta_y$ and $\theta_z$ are the polar coordinates of a given point with respect to pole $S$. If $N$ in Eq. (1-10) is the same number for $S = 1$, 2, and 3, one can see from symmetry considerations that $A_n^{(1)} = A_n^{(2)} = A_n^{(3)}$, $n = 1, 2, 3, \ldots$. Therefore Eq. (1-9) and Eq. (1-10) can be written as

$$W_c = \sum_{n=1}^{N} A_n (\lambda_1^{-n} \cos n\theta_y + \lambda_2^{-n} \cos n\theta_z + \lambda_3^{-n} \cos n\theta_z) + A_0 (|n| \lambda_1 + |n| \lambda_2 + |n| \lambda_3) \quad \ldots \quad (1-11)$$

By using Eq. (1-9) and (1-11), we then obtain the solutions for $15 \times 15$, $15 \times 7$ and $7 \times 7$ arrays. For the first case, the matrix fails to invert by single-precision arithmetic. For the other two cases,
we have shown the deviation curve along the boundary in Fig. 1-5.

For comparison, the deflections of the membrane at the center for different cases are tabulated below.

**TABLE 1-1** DEFLECTION AT THE CENTER OF THE MEMBRANE

<table>
<thead>
<tr>
<th>Number of poles</th>
<th>1</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size of matrix</td>
<td>3x3</td>
<td>7x3</td>
</tr>
<tr>
<td>Size of matrix</td>
<td>3x3</td>
<td>7x3</td>
</tr>
<tr>
<td>$w$ (in $\frac{\mu}{T}$)</td>
<td>.8177</td>
<td>.8673</td>
</tr>
</tbody>
</table>

From the above table and Fig. 1-5, it seems that, for a simply connected region, the multiple pole scheme has no advantage over a single pole. Actually, it does have some advantage. If we use the positive degree terms of the series which are expanded about a single pole inside the region, the last term of the series will be $N_{\text{cs}N\theta}^N$ or $N_{\text{sn}N\theta}^N$. Then, the maximum ratio of the largest element to the smallest element in the last column of the coefficient matrix will be in the order of $(\frac{\lambda_{\text{max}}}{\lambda_{\text{min}}})^N$. When this ratio becomes too large, say $|C^N|$, the computer will fail to handle certain terms no matter how one scales this matrix. For this sample problem, we tabulate the largest ratios among the
elements in the last column of the coefficient matrix for each case as follows.

**TABLE 1-2**

<table>
<thead>
<tr>
<th>Number of poles</th>
<th>1</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size of matrix</td>
<td>3x3</td>
<td>7x7</td>
</tr>
<tr>
<td>Ratio</td>
<td>2.7x10^3</td>
<td>3.8x10^6</td>
</tr>
</tbody>
</table>

One can see now, if we use a single pole for this particular problem, we can at most take 17 or 18 unknown constants. But for multiple poles with retaining only the negative order terms of the series, there is no practical limitation about how many unknowns we can take as far as the computer underflow or overflow is concerned.

For a given boundary shape, the largest ratio depends only on the highest order N of the series. Therefore, the largest ratio for a square coefficient matrix NxN is the same as those for a rectangular matrix MxN. For the latter case, the coefficient matrix will be multiplied by its transpose before it can be inverted. If the largest ratio of the largest element to the smallest element of the coefficient matrix is very large, one can see that the small elements will lose their weight during the matrix multiplication. Therefore, the round off error will be large.
A). The Matrix Inversion

In the numerical treatment of boundary value problems, one must usually solve a set of linear simultaneous equations. If one fails to obtain an accurate solution for these linear equations, he would not be able to obtain an accurate solution for the given physical problem. Thus the matrix inversion is one of the factors which govern the accuracy of the solution. Another important factor is the convergence problem which we shall discuss in the next chapter.

Let us suppose that we are given a set of linear equations $AX = B$ to solve. Here $A$ represents a square matrix of the $n$th order and $X$ and $B$ vectors of the $n$th order. We may either treat this problem as it stands and attempt to find by some direct method such as iteration, or we may solve the more general problem of finding the inverse of the matrix $A$, and then allowing it to operate on $B$, giving the required solution of the equation as $X = A^{-1}B$. If we require the solution
to only one set of the equations, the direct approach has the advantage of involving less work. However, from the coefficients of the inverse, we can see at once how sensitive the solution is to small changes in the coefficients of \( A \) and of \( B \). We have, in fact,

\[
\frac{\partial x_i}{\partial b_j} = (A^{-1})_{ij}, \quad \frac{\partial x_j}{\partial a_{jk}} = - (A^{-1})_{kj}, \quad x_k
\]

This enables us to estimate the accuracy of the solution if we can judge the accuracy of the data. Therefore, with the advent of the high speed electronic computer, it has become standard practice to find the inverse.

There are many methods for solving linear equations and inverting matrices. Although new methods are sometimes merely rearrangements of old methods, such rearrangements may have considerable practical value. In 1951, L. Fox \( ^{30} \) gave an excellent compilation and comparison of the various methods of solving sets of linear equations. In the following, we shall introduce four methods which are believed to have some advantages over the others.

1) Gauss-Seidel iteration method

The scheme of iteration can be simply represented as

\[
\begin{align*}
\alpha_{11} x_1^{(n+1)} + \alpha_{12} x_2^{(n)} + \alpha_{13} x_3^{(n)} + \cdots + \alpha_{1n} x_n^{(n)} &= b_1 \\
\alpha_{21} x_1^{(n+1)} + \alpha_{22} x_2^{(n+1)} + \alpha_{23} x_3^{(n)} + \cdots + \alpha_{2n} x_n^{(n)} &= b_2 \\
& \vdots \\
\alpha_{n1} x_1^{(n+1)} + \alpha_{n2} x_2^{(n+1)} + \alpha_{n3} x_3^{(n+1)} + \cdots + \alpha_{nn} x_n^{(n)} &= b_n
\end{align*}
\]
where \( r = 0, 1, 2 \ldots \). We start by obtaining \( x_1^{(1)} = \frac{b_1}{\omega_{11}} \)
from the first equation, then substitute it into the second equation,
giving \( x_2^{(1)} = \frac{b_2 - Q_{21} x_1^{(1)}}{\omega_{22}} \). Repeating the same process, we can
obtain \( x_3^{(1)}, x_4^{(1)} \ldots \) until \( x_n^{(1)} \). Then using \( x_1^{(1)}, x_2^{(1)} \ldots x_n^{(1)} \),
we can obtain \( x_1^{(2)}, x_2^{(2)} \ldots x_n^{(2)} \) successively. Repeat this process
until \( x_j^{(m)} = x_j^{(m-1)} \frac{1}{\omega_{jj}} \), where \( \varepsilon \) is the error bound for the
solution. This method is always convergent if the matrix is positive
definite. Note that a real matrix obtained from least square
approximation \( A = A_1^T A_1 \) is positive definite. However, the
convergence of this method is slow unless the matrix has large
elements along the main diagonal. Furthermore, the iterative
method is a direct method; it does not give the inverse of the
matrix.

2) Gaussian elimination method.

This method consists of constructing an upper triangular
matrix by successive elimination. Let consider a set of equations
in \( n \) variables \( x_1, x_2 \ldots x_n \) which we denote by
\[
A^{(1)}X = B^{(1)}
\]
The upper suffix is used because the method of solution proceeds
by deriving a sequence of systems of equations each of order \( n \),
a term of the sequence denoted by
\[
A^{(n)}X = B^{(n)}
\]
Each of these systems is the equivalent of the original system.
The final system
\[
A^{(n)}X = B^{(n)}
\]
has for its matrix of coefficients, \( A^{(1)} \), an upper triangular matrix.

The details of this process are shown on the next page, and need no further explanation. Note that \( a_{11}^{(1)} = a_{11}^{(2)} = a_{i1}^{(3)} \), \( a_{12}^{(1)} = a_{12}^{(2)} = a_{13}^{(3)} \), \( a_{21}^{(1)} = a_{21}^{(2)} = a_{23}^{(3)} \), \( a_{22}^{(1)} = a_{22}^{(2)} = a_{23}^{(3)} \). Now, the solution can be obtained by back-substitution which starts from the last equation as shown in the example problem. Finally, the matrix \( A \) has been transformed into an I matrix while the final stage of vector \( B \) is the solution vector for \( X \). If we write the original equation as

\[
(A; B; I)
\]

Then, when \( A \) is transformed into an I matrix, the I matrix becomes the inverse of \( A \).

The operation of this method can be represented by matrix notation as

\[
\begin{pmatrix}
1 & 0 & 0 \\
\frac{-a_{11}^{(1)}}{a_{11}^{(2)}} & 1 & 0 \\
\frac{-a_{11}^{(2)}}{a_{11}^{(3)}} & \frac{-a_{12}^{(2)}}{a_{12}^{(3)}} & 1
\end{pmatrix}
\begin{pmatrix}
an_{11}^{(1)} \\
an_{11}^{(2)} \\
an_{11}^{(3)}
\end{pmatrix}
= \begin{pmatrix}
an_{11}^{(3)} \\
an_{12}^{(3)} \\
an_{13}^{(3)}
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & -\frac{a_{11}^{(3)}}{a_{11}^{(4)}} & -\frac{a_{12}^{(3)}}{a_{12}^{(4)}} \\
0 & 1 & -\frac{a_{21}^{(3)}}{a_{21}^{(4)}} \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
an_{11}^{(4)} \\
an_{12}^{(4)} \\
an_{13}^{(4)}
\end{pmatrix}
= \begin{pmatrix}
an_{11}^{(4)} \\
an_{12}^{(4)} \\
an_{13}^{(4)}
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
an_{21}^{(4)} \\
an_{22}^{(4)} \\
an_{23}^{(4)}
\end{pmatrix}
= \begin{pmatrix}
an_{21}^{(4)} \\
an_{22}^{(4)} \\
an_{23}^{(4)}
\end{pmatrix}
\]
\[
\begin{align*}
\begin{pmatrix}
    a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & b_1^{(1)} \\
    a_{21}^{(1)} & a_{22}^{(1)} & a_{23}^{(1)} & b_2^{(1)} \\
    a_{31}^{(1)} & a_{32}^{(1)} & a_{33}^{(1)} & b_3^{(1)}
\end{pmatrix} & \times (-a_{21}^{(1)}/a_{11}^{(1)})_3 \times (-a_{31}^{(1)}/a_{11}^{(1)})_2 \\
\begin{pmatrix}
    a_{11}^{(2)} & a_{12}^{(2)} & a_{13}^{(2)} & b_1^{(2)} \\
    a_{21}^{(2)} & a_{22}^{(2)} & a_{23}^{(2)} & b_2^{(2)} \\
    a_{31}^{(2)} & a_{32}^{(2)} & a_{33}^{(2)} & b_3^{(2)}
\end{pmatrix} & \times (-a_{21}^{(2)}/a_{22}^{(2)})_3 \\
\begin{pmatrix}
    a_{11}^{(3)} & a_{12}^{(3)} & a_{13}^{(3)} & b_1^{(3)} \\
    a_{21}^{(3)} & a_{22}^{(3)} & a_{23}^{(3)} & b_2^{(3)} \\
    a_{31}^{(3)} & a_{32}^{(3)} & a_{33}^{(3)} & b_3^{(3)}
\end{pmatrix} & \times (-a_{21}^{(3)}/a_{33}^{(3)})_3 \times (-a_{13}^{(3)}/a_{33}^{(3)})_2 \\
\begin{pmatrix}
    a_{11}^{(4)} & a_{12}^{(4)} & 0 & b_1^{(4)} \\
    a_{21}^{(4)} & a_{22}^{(4)} & 0 & b_2^{(4)} \\
    a_{31}^{(4)} & a_{32}^{(4)} & a_{33}^{(4)} & b_3^{(4)}
\end{pmatrix} & \times (-a_{12}^{(4)}/a_{22}^{(4)})_3 \\
\begin{pmatrix}
    a_{11}^{(5)} & 0 & 0 & b_1^{(5)} \\
    a_{21}^{(5)} & a_{22}^{(5)} & 0 & b_2^{(5)} \\
    a_{31}^{(5)} & 0 & a_{33}^{(5)} & b_3^{(5)}
\end{pmatrix} & \div (a_{11}^{(5)})_3 \div (a_{22}^{(5)})_2 \div (a_{33}^{(5)})_3 \\
\begin{pmatrix}
    1 & 0 & 0 & b_1^{(6)} \\
    0 & 1 & 0 & b_2^{(6)} \\
    0 & 0 & 1 & b_3^{(6)}
\end{pmatrix}
\end{align*}
\]
In order to minimize the round-off error, we should find the largest element (absolute value) of the matrix, say $a_{ij}$, as the first pivot: i.e., multiply the $i$th row by $(-a_{ij}/a_{jj})$ and add it to the $k$th row where $k = 1, 2, \ldots, n$ except $i$. Then, discard the $i$th row and $j$th column, and find the largest element in the remaining matrix as the second pivot, etc. In the sample problem, we used the elements along the main diagonal as the pivots. To select the largest elements as pivots is actually the most important advantage Gauss's method and Jordan's method have. This method has been adopted as a standard library subroutine in most computer centers in this country.

3) Jordan's elimination method.

This method differs from Gauss's method only that it reduces a given matrix $A$ directly to a diagonal matrix instead of a triangular matrix. The details of the process is shown in the next page. In order to obtain the inverse, we write an $I$ matrix beside the original equations as

$$(A \mid B \mid I)$$

Then, when matrix $A$ is transformed into a unit matrix, vector $B$ becomes the solution vector and the right side of $I$ matrix becomes the inverse of $A$. This method may be the simplest one of finding the inverse, and it has been used as the standard library subroutine in the OSU computer.
\[
\begin{align*}
\begin{pmatrix}
\alpha_{11}^{(1)} & \alpha_{12}^{(1)} & \alpha_{13}^{(1)} & b_1^{(1)} \\
\alpha_{11}^{(2)} & \alpha_{22}^{(2)} & \alpha_{23}^{(2)} & b_2^{(2)} \\
\alpha_{11}^{(3)} & \alpha_{22}^{(3)} & \alpha_{33}^{(3)} & b_3^{(3)} \\
\alpha_{11}^{(4)} & \alpha_{22}^{(4)} & \alpha_{33}^{(4)} & b_3^{(4)} \\
1 & 0 & 0 & b_1^{(5)} \\
0 & 1 & 0 & b_2^{(5)} \\
0 & 0 & 1 & b_3^{(5)}
\end{pmatrix}
\end{align*}
\]

\[x \left(-\frac{\alpha_{11}^{(1)}}{\alpha_{11}^{(2)}}\right); \quad x \left(-\frac{\alpha_{31}^{(3)}}{\alpha_{11}^{(2)}}\right)\]

\[x \left(-\frac{\alpha_{12}^{(2)}}{\alpha_{22}^{(2)}}\right); \quad x \left(-\frac{\alpha_{11}^{(2)}}{\alpha_{22}^{(2)}}\right)\]

\[x \left(-\frac{\alpha_{23}^{(3)}}{\alpha_{22}^{(2)}}\right); \quad x \left(-\frac{\alpha_{11}^{(3)}}{\alpha_{33}^{(3)}}\right)\]

\[x \left(-\frac{\alpha_{22}^{(4)}}{\alpha_{33}^{(4)}}\right); \quad x \left(-\frac{\alpha_{11}^{(4)}}{\alpha_{33}^{(4)}}\right)\]
h) Triangular decomposition method

A given square matrix $A$ can always be represented as the product of a lower unit triangular matrix and an upper triangular matrix as

$$A X = B$$

$$A = L U$$

or

$$\begin{pmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} \\
\alpha_{21} & \alpha_{22} & \alpha_{23} \\
\alpha_{31} & \alpha_{32} & \alpha_{33}
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 \\
\ell_{21} & 1 & 0 \\
\ell_{31} & \ell_{32} & 1
\end{pmatrix}
\begin{pmatrix}
u_{11} & u_{12} & u_{13} \\
0 & u_{22} & u_{23} \\
0 & 0 & u_{33}
\end{pmatrix}$$

where $\ell_{ij}$ and $u_{kl}$ are unknowns. Multiply the two triangular matrices and compare with the left side matrix, giving

$u_{11} = \alpha_{11}$

$u_{12} = \alpha_{12}$

$u_{13} = \alpha_{13}$

$\ell_{21} = (\alpha_{21}/u_{11})$

$u_{22} = (\alpha_{22} - \ell_{21} u_{12})$
If $A$ is a symmetrical matrix, we can let $L$ and $U$ be the transpose of each other, so we have

$$
\begin{pmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} \\
\alpha_{21} & \alpha_{22} & \alpha_{23} \\
\alpha_{31} & \alpha_{32} & \alpha_{33}
\end{pmatrix} =
\begin{pmatrix}
\alpha_{11} & 0 & \alpha_{13} \\
\alpha_{21} & \alpha_{22} & \alpha_{23} \\
\alpha_{31} & \alpha_{32} & \alpha_{33}
\end{pmatrix}
\begin{pmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} \\
\alpha_{21} & \alpha_{22} & \alpha_{23} \\
\alpha_{31} & \alpha_{32} & \alpha_{33}
\end{pmatrix}
\begin{pmatrix}
\alpha_{11} & 0 & \alpha_{13} \\
\alpha_{21} & \alpha_{22} & \alpha_{23} \\
\alpha_{31} & \alpha_{32} & \alpha_{33}
\end{pmatrix}
$$

As before, the unknown coefficients can be determined one by one:

i.e.,

$$
\begin{align*}
\alpha_{11} &= \sqrt{\alpha_{11}} \\
\alpha_{21} &= \frac{\alpha_{12}}{\sqrt{\alpha_{11}}} \\
\alpha_{31} &= \frac{\alpha_{13}}{\sqrt{\alpha_{11}}} \\
\alpha_{22} &= \sqrt{\alpha_{22} - \frac{\alpha_{12}^2}{\alpha_{11}}}
\end{align*}
$$

After $L$ and $U$ have been found, the solution vector and the inverse can easily be obtained by back-substitution. This method was first introduced by Cholesky (1924) for the symmetrical matrix, and Dwyer and Wangh (1945) and Banachiewicz (1948) modified it and applied to the unsymmetrical matrix. Fox, Huskey and Wilkinson (1948) showed that, for a symmetrical matrix of order 6, the errors of the solution obtained from this method are about $\frac{1}{70}$ as those obtained from Gauss's method or Jordan's method. Besides this, the method is quicker than the elimination methods.

To treat boundary value problems numerically, we would
like to use a least square approximation to satisfy the boundary conditions whenever it is possible. Therefore, as a conclusion, it is suggested that the triangular decomposition method be used for the solution of linear algebraic equations. If the round-off error is still large, the Gauss-Seidel iteration method can be followed to improve the solution.

B) The Matrix Condition

When solving sets of linear equations of order $n$ ($AX = B$), one notices that the errors which appear in the solutions may be large in one case and small in another even though the same method is used to obtain the solution. This variety depends on the distribution of the coefficients of matrix $A$. More strictly speaking, it depends on the stability of the matrix with which we deal. In geometry, if we want to find the intersection point of two straight lines, the intersection point becomes very unstable if these two lines become nearly parallel. This is the case of an unstable matrix of order two. In mathematics, the unstable matrix is called an ill-conditioned matrix. A matrix with a small determinant usually is called ill-conditioned. This statement contains a certain amount of truth. It is certainly the case that bad conditioning and small determinants tend to go together. However, a diagonal matrix with small coefficients certainly is not ill-conditioned even though the determinant may be small. More adequate measures of the condition of a matrix have been proposed.
by J. V. Neumann and H. H. Goldstine (9) (1947) and by A. M. Turing (11) (1948). According to their definition, an ill-conditioned matrix is characterized by large condition numbers, such as

1. **P - condition number**

\[ P = \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} \]

where \( \lambda_{\text{max}} \) and \( \lambda_{\text{min}} \) are the largest and smallest characteristic roots of the lambda-matrices derived from the matrix to be inverted.

2. **N - condition number**

\[ N = \frac{N(A) N(A^{-1})}{\eta} \]

where \( N(A) \) and \( N(A^{-1}) \) are the norms of matrix \( A \) and \( A^{-1} \) respectively; i.e.,

\[ N(A) = \left( \sum_{i,j} a_{ij}^2 \right)^{\frac{1}{2}} \]

and \( n \) is the order of \( A \).

In the next section, we shall see how large the \( N \) - condition number can be before it indicates a serious degree of ill-conditioning. It should also be noted that if we multiply each element of a matrix \( A \) by a constant \( k \), the \( N \) - condition number will not be changed. However, if we multiply \( k \) into a row or a column of \( A \), the \( N \) will be changed. Thus, the \( N \) - condition number can be varied by scaling operations.
In Section (A), we mentioned that, for a symmetrical matrix \(A\), the triangular decomposition method is more accurate than the elimination methods for the solution of \(AX = B\).

This is the case if we want to solve \(A_iX = B_i\), where \(A_i\) and \(B_i\) are \(m \times n\) and \(m \times l\) matrices, and \(m > n\). We can obtain the solution in least square approximation by multiplying the transpose of \(A\) into the equations; i.e., \((A_i^T A_i)X = (A_i^T B_i)\). It becomes \(AX = B\) if we consider \(A = A_i^T A_i\) and \(B = A_i^T B_i\). But if \(A_i\) is an unsymmetrical square matrix, certainly we can obtain a symmetrical matrix by multiplying by \(A_i^T\) and solve by the triangular decomposition method. However, in such a case, we shall not gain any advantage because the product matrix \((A_i^T A_i)\) is always more ill-conditioned than the original matrix \(A_i\). This was proved in (13) where it was shown that if \(\mu\) and \(\lambda\) represent the characteristic roots of matrices \(A_i\) and \((A_i^T A_i)\) respectively, then

\[
\lambda_{\min} \leq \frac{\mu_i}{\lambda_i} \leq \lambda_{\max}
\]

\[
1 \leq \left| \frac{\lambda_{\max}}{\lambda_{\min}} \right| \leq \frac{\lambda_{\max}^2}{\lambda_{\min}} \leq \frac{\lambda_{\max}}{\lambda_{\min}}
\]

Thus, the \(P\) condition number of \((A_i^T A_i)\) is always greater than that of \(A_i\) itself.
C) **Round Off Errors and the Upper Bound of the Errors**

Let us consider a set of linear equations and solve them on digital computer in the floating point mode. The solution we obtain will contain some errors. In order to eliminate or reduce these errors, it is important to divide them into two categories according to how they are introduced. The first category is connected with the errors accumulated during the process of solution. If we are given a set of equations

$$\Delta X = B$$

we may obtain an approximate solution $x$ with an error $\Delta x$ by any of the standard methods for linear equations. That is, substituting $(x+\Delta x)$ back into the original equations, we have

$$A(x + \Delta x) = B = b + \Delta b$$

Since we do not know $X$, we cannot find $\Delta x$. However, we can find $\Delta b$ as

$$\Delta b = b - A x$$

Therefore, $\Delta b$ is usually used as an indication for error $\Delta x$.

The average error of $B$ is usually represented as $N(\Delta b)/N(b)$, where

$$N(\Delta b) = \left( \sum_{i=1}^{n} \Delta b_i^2 \right)^{1/2}$$

and

$$N(b) = \left( \sum_{i=1}^{n} b_i^2 \right)^{1/2}.$$  

If we want to reduce this error, we can obtain an improved solution by either of the following two methods:

(a) Add $\Delta x$ to the first approximate solution $x$ where

$$\Delta x = A^{-1} \Delta r = A^{-1}(b - A x)$$

If $A^{-1}$ has already been obtained, the labor involved to obtain $\Delta x$ is very little. Note that $A^{-1}$
is an approximate inverse of $A$.

(b) Suppose that $A_i^{-1}$ is an approximate inverse of $A$. Then we can obtain from it a better inverse $A_2^{-1}$ by the formula

$$A_2^{-1} = A_i^{-1} (I - A A_i^{-1})^{-1}. $$

We write $E_1 = I - A A_i^{-1}$, $E_2 = I - A A_2^{-1}$, so that $E_1$ and $E_2$ give a measure of the incorrectness of the two inverses. We have $E_2 \ll E_1$, so that at each application of this process, the error is essentially squared. If $E_i$ is small, this process will lead a fast convergence for $A^{-1}$. On the other hand, if $E_i$ is very large, it becomes divergent.

The second category is connected with the effect of small errors in the formulation of the matrix $A$ itself. Suppose we want to solve a set of equations

$$A_{\text{exact}} X_{\text{exact}} = B_{\text{exact}}.$$  

But due to round-off or other reasons, we actually solve a set of equations

$$AX = B.$$  

If we do find the exact solution $X$ for the second set of equations, what is the difference between $X$ and $X_{\text{exact}}$ due to some small differences between matrix $A$ and $A_{\text{exact}},$ and $B$ and $B_{\text{exact}}$? The answer is: If the matrix $A$ is well-conditioned, the difference between $X$ and $X_{\text{exact}}$ is the same order as the difference between $A$ and $A_{\text{exact}}$ or $B$ and $B_{\text{exact}}$. However, if $A$ is ill-conditioned, the relative error of $X$ from $X_{\text{exact}}$ may be many orders of magnitude
larger than the error between $A$ and $A_{exact}$ or between $B$ and $B_{exact}$. Hence, one can consider that the matrix condition number is the error magnification index. As a matter of fact, the $N$-condition number is actually derived to measure the error magnification.

In the single precision arithmetic operation on an IBM 7094, we may have the coefficients of the matrix to 8 figure accuracy. In such a case, our experience is that if the $N$-condition number is over $10^{10}$, the solution vector may no longer be considered as an approximate solution even when we invert the matrix exactly. Should one deal with a matrix having an $N$-condition number over $10^{10}$, he should use double precision to generate the matrix as well as to solve it.

The errors for the sample problems in Chapter I are tabulated in Table 1-2, where $|\Delta p_i|/\epsilon_i$, $|\|max$ is a measure of the first kind of error and $N$-condition number is a measure of the second kind of error. The square matrices used in Table 2-1 were not premultiplied by their transpose, as the rectangular matrices were. In Chapter I, we showed that the solution obtained from 31X15 matrix was less accurate than the solution obtained from the 15X7 matrix, although we expected it the other way. Looking at Table 2-1, this question is answered since the round-off error (first kind) for the case of 31X15 is almost ten thousand times the error entering in the case of 15X7. Furthermore, the decreased accuracy of the multipole solution which we observed
in Table 1-1 can be now explained by the large value of the N - condition numbers found in Table 2-1.

Table 2-1 ROUND-OFF ERRORS FOR SAMPLE PROBLEMS IN CHAPTER I.

<table>
<thead>
<tr>
<th>number of poles</th>
<th>( A )</th>
<th>( B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 3 \times 3 )</td>
<td>3.0 \times 10^{-3}</td>
<td>( 6.8 \times 10^{-4} )</td>
</tr>
<tr>
<td>( 7 \times 7 )</td>
<td>2.5 \times 10^{-7}</td>
<td>( 4.9 \times 10^{-6} )</td>
</tr>
<tr>
<td>( 15 \times 15 )</td>
<td>5.1 \times 10^{-10}</td>
<td>( 9.4 \times 10^{-9} )</td>
</tr>
<tr>
<td>( 31 \times 15 )</td>
<td>1.2 \times 10^{-14}</td>
<td>( 4.0 \times 10^{-13} )</td>
</tr>
<tr>
<td>( 7 \times 7 )</td>
<td>7.7 \times 10^{-10}</td>
<td>( 1.3 \times 10^{-10} )</td>
</tr>
<tr>
<td>( 15 \times 15 )</td>
<td>1.8 \times 10^{-13}</td>
<td>( 1.8 \times 10^{-13} )</td>
</tr>
</tbody>
</table>

Finally, we shall introduce a method to measure the upper bound of the total error for the solution of the linear equations

\[
A_{\text{exact}} X_{\text{exact}} = B_{\text{exact}} \quad \ldots \ldots \ldots (2-1)
\]

Due to the second kind of error, the actual equations we treat are

\[
A X = B \quad \ldots \ldots \ldots (2-2)
\]

Solving Eq. (2-2), we obtain an approximation solution \( x \) (first kind error). Substituting it in Eq. (2-2), we have

\[
A x = B - \Delta b \quad \ldots \ldots \ldots (2-3)
\]

Rewrite Eq. (2-1) as

\[
\left[ A + (A_{\text{exact}} - A) \right] \left[ X + (X_{\text{exact}} - x) \right] = B + (B_{\text{exact}} - B) \]

or

\[
A x + (A_{\text{exact}} - A) x + A (X_{\text{exact}} - x) + (A_{\text{exact}} - A) (X_{\text{exact}} - x) = B + (B_{\text{exact}} - B) \quad \ldots \ldots \ldots (2-4)
\]
Substituting Eq. (2-3) in this, we obtain

\[ \left( X_{\text{exact}} - x \right) = A^{-1} \left[ \mathbf{a} + (b_{\text{exact}} - \mathbf{B}) - (A_{\text{exact}} - A) x - (A_{\text{exact}} - A) \left( x_{\text{exact}} - x \right) \right] \]

Now, let us define the norm of a matrix as

\[ N(A) = \left( \sum_{i,j} a_{ij}^2 \right)^{\frac{1}{2}} \]

The matrix norms have several basic relations such as the following:

(a) \( N(A_1 A_2) \leq N(A_1) N(A_2) \)

(b) \( N(A_1 + A_2) \leq N(A_1) + N(A_2) \)

(c) \( N(A^T) = |A| N(A) \)

Taking the norms of Eq. (2-4), we have

\[ N\left( X_{\text{exact}} - x \right) \leq N\left( A^{-1} \right) \left[ N(\mathbf{a}) + N\left( b_{\text{exact}} - \mathbf{B} \right) + N\left( A_{\text{exact}} - A \right) N\left( x_{\text{exact}} - x \right) \right] \]

Solving for \( N\left( X_{\text{exact}} - x \right) \) and divide both sides by \( N(x) \), we have

\[ \frac{N\left( X_{\text{exact}} - x \right)}{N(x)} \leq \frac{N\left( A^{-1} \right) \left[ N(\mathbf{a}) + N\left( b_{\text{exact}} - \mathbf{B} \right) + N\left( A_{\text{exact}} - A \right) N\left( x_{\text{exact}} - x \right) \right]}{N(x) \left[ 1 - N\left( A^{-1} \right) N\left( A_{\text{exact}} - A \right) \right]} \]

provided \( \left| N\left( A^{-1} \right) N\left( A_{\text{exact}} - A \right) \right| > 1 \). If we assume all the coefficients of matrix \( A \) and matrix \( B \) have the same relative error \( \epsilon \) (for single precision, \( \epsilon > 10^{-6} \)), then Eq. (2-5) can be written as

\[ \frac{N\left( X_{\text{exact}} - x \right)}{N(x)} \leq \frac{N\left( A^{-1} \right) \left[ N(\mathbf{a}) + \mathbf{N}(\mathbf{b}) \epsilon + \mathbf{N}(A) N(x) \epsilon \right]}{N(x) \left[ 1 - N(A) N(A^{-1}) \epsilon \right]} \]

Eq. (2-6) gives the upper bound of the error for the norm of the solution vector.
CHAPTER III

SERIES CONVERGENCE AND ITS IMPROVEMENT,
NUMERICAL EXAMPLES IN PLANE ELASTICITY

If we want to represent a given periodic function by a Fourier series expansion, we can write this series as

\[ f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos n\theta + b_n \sin n\theta \right) \quad \ldots \ldots \ldots \ldots \quad (3-1) \]

where \( a_n \) and \( b_n \) are the Fourier coefficients, given by

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, \mathrm{d}\theta \]
\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, \mathrm{d}\theta \quad \ldots \ldots \ldots \ldots \quad (3-2) \]

In many cases, the Fourier series converges slowly. However, the series will be rapidly convergent if the function \( f(\theta) \) itself and a certain number of its derivatives are continuous and periodic.

Let, for example, \( f(\theta) \), \( f'(\theta) \), \( f''(\theta) \) and \( f'''(\theta) \) be continuous and periodic, and \( f^{(4)}(\theta) \) integrable. Denote by \( a_n \) and \( b_n \) the Fourier coefficients of \( f(\theta) \); by \( a_n^{(1)} \), \( b_n^{(1)} \) those of \( f'(\theta) \); by \( a_n^{(2)} \), \( b_n^{(2)} \) those of \( f''(\theta) \); by \( a_n^{(3)} \), \( b_n^{(3)} \) those of \( f'''(\theta) \); and by \( a_n^{(4)} \), \( b_n^{(4)} \) those of \( f^{(4)}(\theta) \).

On integrating by parts the expressions for \( a_n \) and \( b_n \), we obtain

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(\theta) \sin n\theta \, \mathrm{d}\theta = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\sin n\theta}{n} f(\theta) \, \mathrm{d}\theta = \frac{1}{\pi n} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, \mathrm{d}\theta \]

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(\theta) \cos n\theta \, \mathrm{d}\theta = -\frac{1}{\pi n} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, \mathrm{d}\theta = -\frac{b_n^{(4)}}{n} \]

37
Similarly, \[ b_n = \frac{\alpha_n^{(4)}}{n^4} \]

Now employing the same formulas successively for \( f(t) \), \( f''(t) \), \( f'''(t) \), we find

\[
\begin{align*}
\alpha_n &= -\frac{b_n^{(0)}}{n^2} = -\frac{\alpha_n^{(2)}}{n^3} = \frac{\alpha_n^{(3)}}{n^4} = \frac{\alpha_n^{(4)}}{n^5} \\
b_n &= \frac{\alpha_n^{(1)}}{n^2} = -\frac{\alpha_n^{(2)}}{n^3} = -\frac{\alpha_n^{(3)}}{n^4} = \frac{\alpha_n^{(4)}}{n^5}
\end{align*}
\]

\[ \ldots \ldots \ldots (3-3) \]

The numbers \( \alpha_n^{(4)} \) and \( b_n^{(4)} \), as the Fourier coefficients of an integrable function, tend towards zero as \( n \to \infty \), and therefore the Fourier series for \( f(t) \) will in this case converge more rapidly than the series \( \sum \frac{1}{n^4} \). As a conclusion, the series which represents a periodic function will be rapidly convergent if the function itself and its derivatives are continuous. If the function is continuous, but its derivative is not, the convergence of the series will not be fast. The convergence becomes further slowed if the function itself is not a continuous function. Finally, the convergence becomes so slow that the solution has to be modified if the function to be represented is a singular function like that for a point load.

For two dimensional problems, infinite series like Eq. (1-5) can be obtained as the result of a certain transformation of the Fourier series of the boundary function; the coefficients of the series giving the solution are determined in terms of the Fourier coefficients of the boundary function. Therefore, if the value of the function (including the particular solution) along the boundary, which the complementary solution
has to represent, is a continuous function and so is its derivative; the series will be rapidly convergent. For the sample problem shown in Chapter I, the boundary value is shown in Fig. 3-1 in solid lines.

Because of symmetry, the boundary value is one sixth of the total boundary is used. For comparison, the membrane for the equilateral triangle is drawn and shown in the same figure. In both cases,
the boundary value is a continuous function but without continuous
derivatives at B. Still the curve for the equilateral triangle
is smoother than that for the "triangle" with concave sides. Therefore, the series for the former case should have a better convergence than that for the later. In the case of mixed boundary value problems, the boundary value itself may not be a continuous function (in most cases, the discontinuity is implicit). By mixed boundary value problem, we mean that the boundary condition is defined in terms of different orders of derivatives of the unknown function on different segments of the boundary. For instance, the plate bending problem studied in Ref. 3 and the third kind of elasticity problems have mixed boundary conditions. To show that discontinuous boundary values may occur, let us examine an example of conductive heat transfer in a plate. This problem is governed by \( \Delta T = 0 \), where \( T \) is the temperature. If part of the boundary is insulated \( \left( \frac{\partial T}{\partial n} = 0 \right) \), and the rest is subjected to a constant temperature \( T_c \), one can see that the normal temperature gradient is not zero along that part of the boundary which is subjected to the defined temperature \( T_0 \). Therefore, a discontinuity of \( \frac{\partial T}{\partial n} \) is expected at the point where the boundary condition is changed from \( \frac{\partial T}{\partial n} = 0 \) to \( T_c \). In this case, the series solution converges very slowly. The improvement of this situation shall be discussed in detail in Chapter V.

The worst case for the series convergence problem is when singularities appear on the boundary. In this case, a reasonably
accurate result may not be obtainable even if hundreds of unknowns are used. In order to improve the convergence, these singularities have to be removed before the series solution is applied. We shall show how to remove the singularities and how much the convergence can be improved by some examples in plane elasticity.

Consider a square plate whose principal axes coincide with the X and Y coordinates as shown in Fig. 3-2. We want to find the stress in the plate when it is acted upon by concentrated forces at C and D.

![Figure 3-2](image)

For plane elasticity, if we use Airy's stress function, the governing equation is

$$\nabla^4 \sigma = c$$

and the stresses are

$$\sigma_x = \frac{\partial^2 \sigma}{\partial y^2} ; \quad \sigma_y = \frac{\partial^2 \sigma}{\partial x^2} ; \quad \sigma_{xy} = -\frac{\partial^2 \sigma}{\partial x \partial y} \quad \text{...........(3-4)}$$

$$\tau_{tx} = \frac{\partial^2 \sigma}{\partial x \partial y} ; \quad \tau_{ny} = -\frac{\partial^2 \sigma}{\partial y^2} \quad \text{...........(3-5)}$$
and the solution for Eq. (3-4) can be represented in polar
cooridinated as

$$\xi = \xi_p + \xi_c$$ .............(3-6)

where

$$\xi_p = 0$$ .............(3-7)

and

$$\xi_c = A_0 \ln r + B_0 r^2 + C_0 r^2 \ln r + D_0 r^2 \theta + A_0 \theta$$

$$+ \frac{A_1}{r} \sin \theta + \left( B_1 r^3 + A_1 \ln r + B_1 \ln r \right) \cos \theta$$

$$- \frac{C_1}{r} \cos \theta + \left( D_1 r^3 + C_1 \ln r + D_1 \ln r \right) \sin \theta$$

$$+ \sum_{n=2}^{\infty} \left( A_n r^n + B_n r^{n+2} + C_n r^{n-2} + D_n r^{n+2} \right) \cos n\theta$$

$$+ \sum_{n=2}^{\infty} \left( A'_n r^n + B'_n r^{n+2} + C'_n r^{n-2} + D'_n r^{n+2} \right) \sin n\theta$$

The series solution Eq. (3-8) was given by J. H. Michell
(1899)(15). Because of symmetry and the requirement of finite
stresses at r=0, we retain only $A_2$, $A_4$, ... $A_{16}$, $B_0$, $B_2$, $B_4$, ...
$B_{16}$ to make a total of seventeen terms. To determine these
unknowns, 41 boundary equations were generated by approximately
satisfying 21 equally spaced points as shown in Fig. 3-2. The
boundary conditions are that the normal and the shear stresses
are zero except at C where the concentrated load P was replaced
by a line load $20 \frac{P}{a}$ distributed on a width of $\frac{a}{20}$. Solving this
41x17 set of equations in the least square sense, the stresses
distributed along the x and y axes, and the residues along the
boundary are shown in Fig. 3-3. Because of the huge boundary
error, the stresses obtained at this point within the plate have
very poor accuracy. This is caused by the presence of singularities, the concentrated boundary loads. Now, let us see how to remove these singularities and thereby improve the accuracy. For the concentrated load \( P \) applied at a point of a straight boundary of semi-infinite body, we have the solution obtained by Flamant (1892) from Boussinesq's three dimensional solution (see ref. (16), p. 86),

\[
\frac{n}{\kappa} = \frac{P}{k} \cosh \omega \quad \omega = \frac{b}{k}
\]

Where the origin of coordinates is now at the load \( P \), as shown in Fig. 3-4. Comparing Eq. (3-9) with Eq. (3-8), one can see that Eq. (3-9) is merely one term of Eq. (3-8), so it is a biharmonic function.

![Fig. 3-4](image)

Eq. (3-9) can be written in rectangular coordinates as

\[
F = \frac{P}{k} \cdot x \tan^{-1} \frac{y}{x}
\]

If we want to find the solution for two concentrated loads with respect to coordinates as shown in Fig. 3-2, it can be obtained
easily by a simple translation and superposition; i.e.,

\[ \bar{F} = \frac{P}{\pi} \lambda \left[ \tan^{-1} \frac{y + \xi \alpha}{\lambda} - \tan^{-1} \frac{y - \xi \alpha}{\lambda} \right] \ldots (3-11) \]

Obviously, Eq. (3-11) satisfies the boundary conditions only at points C and D (Fig. 3-2). Now, we use Eq. (3-11) as the particular solution for this problem though it is a biharmonic function itself. We then determine the same 17 unknown constants by satisfying the boundary conditions at the points shown in Fig. 3-2 except point C, at which the boundary condition has been satisfied already. After solving this 17 set of equations, the stresses along the x and y axes are evaluated and shown in Fig. 3-5. The maximum boundary residue was found to be 0.00734 \( \frac{P}{\lambda \alpha} \) near the corner of the square plate. Comparing with the maximum stresses at the center of the plate \( \bar{T} = 1.92465 \frac{P}{\lambda \alpha} (\bar{T}_{xx} = 0.63082 \frac{P}{\lambda \alpha}) \), the maximum deviation on the boundary is only 0.35%. By St. Venant's principle, the stresses at the center are believed to have an error of less than 0.1%. This problem was originally solved by J. N. Goodier(29)(1932) by the energy method. The stress curves presented in his paper matched with these very well. He claimed that his solution has an accuracy of within 5%.
STRESSES IN A CONCENTRATED LOAD SLICE

Fig. 3-5

Scale of Stresses

\( \sigma_x = \frac{1}{\pi} \)
CHAPTER IV

SINGULAR INTEGRAL SOLUTION

Let us introduce a new way of attacking boundary value problems by returning to the membrane problem, which as we saw before in Chapter I is governed by

\[ \frac{\partial^2 w}{\partial \xi^2} = \frac{\partial^2 w}{\partial \eta^2} \quad \ldots \ldots (4-1) \]

and \( w = 0 \) \( \ldots \ldots (4-2) \)
on the boundary. The solution for Eq. (4-1) is

\[ w = w_p + w_c \quad \ldots \ldots (4-3) \]

This time, we do not use a series like Eq. (1-5) to represent the complementary solution \( w_c \), but we still let the particular solution be

\[ w_p = -\frac{1}{4} \eta^2 \quad \ldots \ldots (4-4) \]

Geometrically, \( w_p \) represents a paraboloid of revolution. In order to visualize the shape, we can find a toy balloon which has a local paraboloidal surface as defined by Eq. (4-4). If we choose a given point on that balloon and draw a contour line (level curve) through that point, the resulting shape is then a complete solution for a membrane problem with a shape defined by that contour line. On an undeformed spherical balloon, all
contour lines are circles. But, if we have a very thin wall tube having a hollow rectangular section, we can press this tube upon the balloon and then the balloon will have a local deformation as shown in Fig. (4-1). The contact line between the balloon and the tube is on the same elevation; i.e., the resulting shape of the balloon surface represents the solution for a membrane problem with a rectangular boundary. This thin wall tube has exerted a line load upon the balloon, which causes a rectangular contour line.

Now, it seems that one may obtain the exact solution for a membrane problem with irregular boundary shape if he can find
the exact line load $p(B)$ applied along the boundary, where $p(B)$ indicates the line load intensity along the boundary B. Very naturally, he may represent this line load by an infinite series with unknown coefficients to be determined. However, this may lead to the problems of the slow convergence and matrix ill-conditioning as discussed in Chapter II and Chapter III. In order to avoid these difficulties, we introduce a moving coordinate system; i.e., instead of using a single series solution expanded with respect to a fixed origin, we use an infinite number of solutions expanded with respect to an infinity of origins, but each solution has only one term.

Let us now consider a concentrated load $P$ applied at the center of a circular membrane; then the deflection at an arbitrary point $A$ will be

$$\Delta W = k \frac{P}{R^2} \ln \frac{P}{r(B,A)} \frac{r(B,A)}{C}$$  \hspace{1cm} \ldots (4-5)$$

where $r(B,A)$ is the distance between the center point $B$ and an arbitrary point $A$, $C$ is the radius of the circle and $k$ is a constant to be determined. The solution represented by Eq. (4-5) is a singular solution for the membrane. Since the membrane must be in equilibrium everywhere, we cut out a circle around the applied load as a free body, yielding

$$P = -\frac{1}{2\pi} \int \frac{\Delta W}{\rho^2} \frac{P}{r(B,A)} \frac{r(B,A)}{C} \, d\rho$$

or

$$k = -\frac{1}{2\pi}$$  \hspace{1cm} \ldots (4-6)$$

So, Eq. (4-5) can be written as

$$\Delta W = -\frac{1}{2\pi} \frac{P}{r(B,A)} \frac{r(B,A)}{C} \ln \frac{P}{r(B,A)} \frac{r(B,A)}{C}$$  \hspace{1cm} \ldots (4-7)$$
If we let the boundary line load be \( p(B) \), then, the deflection at any point \( A \) in the region or on the boundary is

\[
W = -\frac{1}{2\pi} \int_{B} p(B) \ln \left[ \frac{2(r-Ax)}{R} \right] ds - \frac{1}{4} R^2 \frac{\dot{u}}{T} \tag{4-8}
\]

To determine the boundary load \( p(B) \), we let \( A \) be a boundary point and satisfy the boundary condition Eq. (4-2), we have

\[
W = -\frac{1}{2\pi} \int_{B} p(B) \ln \left[ \frac{2(r-Ax)}{R} \right] ds - \frac{1}{4} R^2 \frac{\dot{u}}{T} = 0 \tag{4-8a}
\]

In Eq. (4-8a), the unknown function \( p(B) \) is behind the integral sign, so it is called an integral equation. And because the term \( \ln \left[ \frac{2(r-Ax)}{R} \right] \) becomes infinite when \( A \) approaches \( B \). Therefore, Eq. (4-8a) is called a singular integral equation. After \( p(B) \) has been found, then Eq. (4-8) is an ordinary integral and is called the singular integral solution. Note that the singular solution, due to the infinitesimal load \( p(S)ds \), shown in Eq. (4-7) is the solution for a circular membrane with center at \( B \), not at any fixed center like \( X=0, Y=0 \). Therefore, the boundary load \( p(B) \) in Eq. (4-8) is not the line load which the thin wall tube has exerted upon the balloon as shown in Fig. 4-1. To avoid this misunderstanding, we shall call \( p(B) \) the density function along the boundary. In the following two sections, we shall introduce two approximate methods of determining \( p(B) \).

A) **Step Function Approximation, Numerical Example**

For a membrane with an irregular boundary shape, there is an unknown density function along the boundary which represents the complementary solution for that problem. This density function,
STEP FUNCTION ALONG THE BOUNDARY

Fig. 4-2

certainly, can be replaced approximately by a step function; i.e., we divide the boundary into N segments (they need not be equally spaced). Then the boundary of each segment is replaced by a straight edge and the line load $p(B)$ in each segment is assumed to be a constant $p_i$. This procedure is shown in Fig. 4-2 to specify the location of a line load in each segment, we let $B_i(x_i, y_i)$ be the center of the segment having a total length $S_i$ and we call the angle from the x-axis to the normal direction $\psi_i$ (see Fig. 4-3). Then, the deflection at point $A(x, y)$ due to the rectangle segment load $p_i$ will be
where $R_i$ = distance between $R_i$ and $A_i$; $\theta_i$ = angle between the $x$-axis and the vector $B_iA_i$; $SN = SN (\theta_i - \phi_i)$; $CN = CN (\theta_i - \phi_i)$ and $P_i = P_i S_i$. The complete solution for a given membrane problem can be written as

$$W = -\frac{1}{4} \int_0^T h \frac{\mathbf{P}_i \cdot \mathbf{F}_i}{\beta_0} d\tau + \sum_{i=1}^{N} W_i$$

One can see, in Eq. (4-10), that the $P_i^j$'s $(i=1,2...N)$ are unknown constants. They can be determined by solving the $N$ simultaneous equations which are obtained by satisfying the boundary condition Eq. (4-2) at $M$ discrete points as we did in Chapter I. If $M=N$, it is point matching; if $M>N$, it is least square approximation.
Let us now use Eq. (4-9) and Eq. (4-10) to solve a membrane problem with a rectangular section as shown in Fig. 4-4. 60 equally spaced points are chosen along the boundary to represent the centers of 60 segments, and 60 unknown $p_t$'s are determined by point-matching at the centers of these 60 segments. The constant $C$ in Eq. (4-9) is assigned as 2.0 (we shall explain why we pick $C=2.0$ in Chapter VII, Section B. We then represent the density function $p(B)$ along the boundary by a step function, as shown in Fig. 4-5, where the actual solution for the boundary pressure is
shown. From Eq. (4-10), the deflection at an arbitrary point can be obtained.

\[
Y
\]

\[
X
\]

RECTANGULAR MEMBRANE

**Fig. 4-4**

Solving the problem we find, at the center, \( w = 0.1139 \frac{q}{T} \), and the maximum slope at \((0, 0.5)\) is \( \frac{\partial w}{\partial y} = 0.1469 \frac{q}{T} \). Comparing with Timoshenko's (16) \( 0.165 \frac{q}{T} \), which is less than 1% error. The deviation curve along the boundary is as shown in Fig. 4-8.

One of the greatest advantages of the singular integral solution is that the concentrated load gives a large influence in the neighborhood of the concentrated load. This character causes the linear simultaneous equations to have a matrix with large elements along and near the main diagonal. This simply
DENSITY FUNCTION
ALONG THE BOUNDARY
Fig. 4-5
means that the matrix we have to invert is always well-conditioned. For instance, the rectangular membrane problem we just solved has a \( N \)-condition number as small as \( N=5.414 \) for a matrix as big as 60X60 (refer to Chapter II). In Fig. 4-6, the magnitude of the elements of this 60X60 matrix is shown graphically. As has been indicated previously, such a matrix usually yields very small round-off errors in numerical computing processes.

The above rectangular membrane actually has two axes of symmetry, but this symmetry property does not apply to the moving coordinates along the boundary. In other words, although the load at segment 1 equals the load at segment 20, still we have to evaluate the influence from segment 20 when we want to find the deflection at points in segment 1. However, we can save some computing time by merging the terms of the matrix; i.e., to obtain a set of simultaneous equations by satisfying the boundary conditions only in one fourth of the boundary and then merge those coefficients which have the same unknown loads. The 60X60 matrix then can be reduced to a 15X15 matrix. In Fig. 4-7, the magnitude of the elements of this 15X15 matrix is shown graphically also for comparison with Fig. 4-6 discussed previously. To check the residue along the boundary, the deviation curve is shown in Fig. 4-8. Now, we can solve the same problem by using least square theory. We determine the same 15 unknown \( P_i \) by approximately satisfying the boundary condition at 31 equally-spaced
GRAPHICAL REPRESENTATION OF THE ELEMENTS OF MATRIX A
Fig. 4-6
GRAPHICAL REPRESENTATION OF THE ELEMENTS OF MATRIX A

Fig. 4-7
points. Then, we can obtain the density function and the deviation curve as shown in Fig. 4-8 and Fig. 4-9 with label 31x15.

From Fig. 4-5, one can see that the density function curve $p(B)$ is flat near the middle of the sides, but has a steep slope near the corner B. Therefore, for an improved step function approximation, we should divide the boundary by having long segments near the middles of the sides but short segments near the corner. For illustration, we divide the boundary (one fourth) into 22 segments with different length as shown in the upper half of Fig. 4-10. The 22 unknowns are determined by approximately satisfying the boundary condition at 31 equally-spaced points and 64 unequally-spaced points which are shown in the lower half of Fig. 4-10. The solution for both cases are shown in Fig. 4-9 and their deviation curves are shown in Fig. 4-8.

From the four deviation curves in Fig. 4-8, one can see that the deviation is small (compared with $w = 0.1139$) except near the corner. Among them, the solution obtained from the $64 \times 22$ matrix appears much better than the rest. Note that these deviation curves intersect the zero line about two $N$ times, where $N$ is the number of unknowns. On the other hand, the series solution (Chapter I) usually has its deviation curve intersecting the zero line as many as the number of unknowns. Therefore, if the least square method is used, the optimum ratio of the number of boundary equations to the number of unknown constants is increased from 2 for the latter to 3 or 4 for the former. This
Fig. 4-9  Density Function $\rho(B)$ Along the Boundary

(A) 3 x 15

(B) 3 x 7.2

(C) 6.3 x 22
is especially important when we are interested in the derivative of $w$ instead of $w$ itself.

It should be mentioned here that the computer program for this singular integral solution is much more complicated than that for the series solution in the following respects:

(a) The deflection function Eq. (4-9) and its derivatives may become undefined at certain points; i.e., zero multiplied by infinity. Since the computer does not know how to apply the L'Hospital rule itself, the programmer has to insert a check statement and employ the L'Hospital rule if it is necessary.
(b) As we mentioned before that the geometrical symmetry property does not apply to the moving coordinates along the boundary. Therefore, the program should evaluate all the symmetrical boundary points (axial symmetry or polar symmetry) and merge all the coefficients who have the same unknowns.

(c) Because we have used a step function approximation for the density function (boundary line load), the slope in the tangential direction \( \left( \frac{\partial w}{\partial t} \right) \) becomes infinite at the points adjoining the density function rectangles like a, b, c, ... in Fig. 4-11. To avoid this error, the program used was so designed that, when the tangential slope at point c is evaluated, the influence from the density function will be integrated from \( +\xi \) to \( -\xi \) as shown in Fig. 4-11. One can see that if \( \xi \) is small, the exact density function in the range \( \xi > t > -\xi \) practically has a constant value, hence the contribution of the density function from \( 0 \) to \( \xi \) will be equal but has opposite sign to those from \( -\xi \) to \( 0 \).

\[ \int_{\xi}^{\xi} \text{FUNCTION ALONG THE BOUNDARY} \]

**Fig. 4-11**
B) Polygonal Function Approximation, Numerical Example

Instead of using many rectangles to represent the density function \( p(B) \) along the boundary, we can have a better approximation by employing many trapezoids; i.e., we divide the boundary into \( N \) segments, each segment being replaced by a straight line and the density function \( p(B) \) in each segment is assumed to vary linearly. We called this approximate representation a polygonal function. One can see that, unlike the step function, the polygonal function is a continuous function. Though the entire density function can be defined by \( N \) unknowns; i.e., \( p_1, \ldots, p_N \) which are the intensity of the density function at the points adjoining the segments, any trapezoid with a segment length \( S_i \) needs two unknowns \( p_i \) and \( p_{i+1} \) to define it as shown in Fig. 11-12. To avoid this difficulty, we let each unknown \( p_i \) define two triangles which lie in the two adjoining segments with length \( S_{i-1} \) and \( S_i \). Superimposing all these triangles along the boundary, they form a polygonal function. We let \( B_i(X_i, Y_i) \) be the point which has a density function value \( p_i \) adjoin two segments which have length \( S_{i-1} \) and \( S_i \), and normal angles \( \gamma_i \) and \( \beta_i \) from the x-axis respectively (Fig. 11-13). Then, the deflection at point \( A(Y, Y) \) due to two triangular density functions with bases \( S_{i-1} \) and \( S_i \) (like ACF in Fig. 11-12) will be
POLYGONAL FUNCTION ALONG THE BOUNDARY

Fig 1-12
COORDINATES FOR A SEGMENT ON THE BOUNDARY

Fig. 4-13
\[ \mathcal{W}_\lambda = -\frac{P_0}{2\pi T} \left[ \int_{-S_{\lambda-1}}^{0} (1 + \frac{t}{S_{\lambda-1}}) \ln \left( \frac{R(\lambda)}{C} \right) \, dt + \int_{0}^{S_{\lambda}} (1 - \frac{t}{S_{\lambda}}) \ln \left( \frac{R(\lambda)}{C} \right) \, dt \right] \]

\[ = \frac{P_0}{2\pi T} \left[ R_{\lambda} \cdot CN_{\lambda}^{(0)} \left( 1 + \frac{R_{\lambda} \cdot SN_{\lambda}^{(0)}}{S_{\lambda-1}} \right) \left( \tan^{-1} \frac{SN_{\lambda}^{(0)}}{CN_{\lambda}^{(0)}} - \tan^{-1} \frac{S_{\lambda-1} + R_{\lambda} \cdot SN_{\lambda}^{(0)}}{R_{\lambda} \cdot CN_{\lambda}^{(0)}} \right) \right. \]

\[ + \frac{R_{\lambda}}{S_{\lambda-1}} \ln R_{\lambda} \cdot \left( R_{\lambda} \cdot SN_{\lambda}^{(2)} + S_{\lambda-1} \cdot SN_{\lambda}^{(1)} - 0.5 \cdot R_{\lambda} \right) \]

\[ + \frac{1}{4} \frac{1}{S_{\lambda-1}} \left( R_{\lambda}^2 \cdot CN_{\lambda}^{(0)} - S_{\lambda-1}^2 - 2 \cdot R_{\lambda} \cdot S_{\lambda-1} \cdot SN_{\lambda}^{(1)} - R_{\lambda}^2 \cdot SN_{\lambda}^{(2)} \right) \ln \left( S_{\lambda-1}^2 + R_{\lambda}^2 \right) \]

\[ + 2 \cdot R_{\lambda} \cdot S_{\lambda-1} \cdot SN_{\lambda}^{(1)} + \frac{1}{2} \cdot S_{\lambda-1} \ln C + \frac{1}{2} \cdot R_{\lambda} \cdot SN_{\lambda}^{(0)} + \frac{3}{4} \cdot S_{\lambda-1} \]

\[ - R_{\lambda} \cdot CN_{\lambda}^{(2)} \left( 1 - \frac{R_{\lambda} \cdot SN_{\lambda}^{(2)}}{S_{\lambda}} \right) \left( \tan^{-1} \frac{S_{\lambda} - R_{\lambda} \cdot SN_{\lambda}^{(3)}}{R_{\lambda} \cdot CN_{\lambda}^{(2)}} + \tan^{-1} \frac{SN_{\lambda}^{(3)}}{CN_{\lambda}^{(2)}} \right) \]

\[ + \frac{R_{\lambda}}{S_{\lambda}} \ln R_{\lambda} \cdot \left( R_{\lambda} \cdot SN_{\lambda}^{(2)} - S_{\lambda} \cdot SN_{\lambda}^{(2)} - 0.5 \cdot R_{\lambda} \right) \]

\[ + \frac{1}{4} \frac{1}{S_{\lambda}} \left( R_{\lambda}^2 \cdot CN_{\lambda}^{(2)} - R_{\lambda}^2 \cdot SN_{\lambda}^{(2)} - S_{\lambda}^2 + 2 \cdot R_{\lambda} \cdot S_{\lambda} \cdot SN_{\lambda}^{(2)} \right) \ln \left( R_{\lambda}^2 + S_{\lambda}^2 \right) \]

\[ - 2 \cdot S_{\lambda} \cdot R_{\lambda} \cdot SN_{\lambda}^{(2)} + \frac{1}{2} \cdot S_{\lambda} \ln C - \frac{1}{2} \cdot R_{\lambda} \cdot SN_{\lambda}^{(2)} + \frac{3}{4} \cdot S_{\lambda} \right) \]

\[ = \frac{P_0}{2\pi T} \left\{ R_{\lambda} \cdot CN_{\lambda}^{(0)} \left( \tan^{-1} \frac{SN_{\lambda}^{(0)}}{CN_{\lambda}^{(0)}} - \tan^{-1} \frac{S_{\lambda-1} + R_{\lambda} \cdot SN_{\lambda}^{(0)}}{R_{\lambda} \cdot CN_{\lambda}^{(0)}} \right) \right. \]

\[ + \frac{R_{\lambda}}{S_{\lambda-1}} \ln R_{\lambda} \cdot \left( R_{\lambda} \cdot SN_{\lambda}^{(2)} - S_{\lambda} \cdot SN_{\lambda}^{(2)} - 0.5 \cdot R_{\lambda} \right) \]

\[ + \frac{1}{4} \frac{1}{S_{\lambda}} \left( R_{\lambda}^2 \cdot CN_{\lambda}^{(2)} - R_{\lambda}^2 \cdot SN_{\lambda}^{(2)} - S_{\lambda}^2 + 2 \cdot R_{\lambda} \cdot S_{\lambda} \cdot SN_{\lambda}^{(2)} \right) \ln \left( R_{\lambda}^2 + S_{\lambda}^2 \right) \]

\[ - 2 \cdot S_{\lambda} \cdot R_{\lambda} \cdot SN_{\lambda}^{(2)} + \frac{1}{2} \cdot S_{\lambda} \ln C - \frac{1}{2} \cdot R_{\lambda} \cdot SN_{\lambda}^{(2)} + \frac{3}{4} \cdot S_{\lambda} \ \ \ \ \ \ (4-11) \]

where \( R_{\lambda} \) = distance between \( B_{\lambda} \) and \( A_{\lambda} \); \( \theta_{\lambda} \) = angle between the \( x \)-axis and the vector \( B_{\lambda}A_{\lambda} \); \( SN_{\lambda}^{(1)} = \sin \left( \theta_{\lambda} - \psi_{\lambda-1} \right) \); \( CN_{\lambda}^{(1)} = \cos \left( \theta_{\lambda} - \psi_{\lambda-1} \right) \); \( SN_{\lambda}^{(2)} = \sin \left( \theta_{\lambda} - \psi_{\lambda} \right) \) and \( CN_{\lambda}^{(2)} = \cos \left( \theta_{\lambda} - \psi_{\lambda} \right) \).

The complete solution for a given membrane problem can then be written as

\[ \mathcal{W} = -\frac{1}{4} \cdot h^2 \cdot \frac{q}{T} + \sum_{\lambda=1}^{N} \omega_{\lambda} \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ (4-12) \]
The unknowns $p_1, p_2, \ldots, p_n$ can be determined like they were for the step function.

To compare with the step function approximation, we solve the same rectangular membrane problem by polygonal function approximation. One quarter of the boundary is divided into 16 segments having a length 0.1 except that two segments adjoining the corner point B have a length 0.05 (see Fig. 4-1h). Then the 16 unknowns $p_1, p_2, \ldots, p_{16}$ are determined by satisfying the boundary at 16 equally spaced points or approximately satisfying 64 points (Fig. 4-10). The solutions (density function $p(B)$) are shown in Fig. 4-15 and the deviation curves along the boundary are shown in Fig. 4-1h. Comparing Fig. 4-8 with Fig. 4-1h, it shows that, with 16 unknowns, the polygonal function approximation gives a solution having a better accuracy than those obtained from the step function with 22 unknowns.

For comparison, we tabulate in Table 4-1 the deflection at the center of the membrane, the slope at the middle of the long and short sides obtained from the singular integral equation with step function approximation and polygonal function approximation.
<table>
<thead>
<tr>
<th>Size of matrix</th>
<th>15x15</th>
<th>31x15</th>
<th>31x22</th>
<th>64x22</th>
<th>16x16</th>
<th>64x16</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda ) at center ( \left( \frac{4}{7} \right) )</td>
<td>.22782</td>
<td>.22780</td>
<td>.22777</td>
<td>.22775</td>
<td>.22734</td>
<td>.22738</td>
</tr>
<tr>
<td>( 2 \left( \frac{2w}{\beta x} \right) ) at ( \left( \frac{3}{7} \right) )</td>
<td>-.93726</td>
<td>-.93735</td>
<td>-.93721</td>
<td>-.93703</td>
<td>-.92785</td>
<td>-.92799</td>
</tr>
<tr>
<td>( 2 \left( \frac{2w}{\beta y} \right) ) at ( \left( \frac{3}{7} \right) )</td>
<td>-.73442</td>
<td>-.73097</td>
<td>-.73447</td>
<td>-.73445</td>
<td>-.73023</td>
<td>-.73204</td>
</tr>
<tr>
<td>( N )-condition number</td>
<td>7.76</td>
<td>4.71x10^3</td>
<td>6.63x10^7</td>
<td>8.60x10^2</td>
<td>1.08x10^4</td>
<td>1.14x10^3</td>
</tr>
</tbody>
</table>
Fig. 4-14 Deviation Curve Along the Boundary

deflection at the center $W_0 = 0.1137 \, \frac{\text{ft}}{}$
Fig. 4-15  DENSITY FUNCTION $\rho(B)$ ALONG THE BOUNDARY

(A) 16x16

(B) 64x16
CHAPTER V

COMBINATION OF SERIES SOLUTION AND SINGULAR INTEGRAL SOLUTION, NUMERICAL EXAMPLE IN HEAT TRANSFER

In Chapter III, we showed that the series solution converges extremely slow if singularities appear along the boundary. The convergence became fast after the singularities had been represented by a special term, which we called the particular solution in Chapter III. Actually, this particular solution is a special case of the singular integral solution; i.e., it assumed that the density function was concentrated at a point instead of distributed along a finite length of the boundary. In Chapter VIII, we shall say more about the special term in Eq. (3-9).

In Chapter III, we said that, after the singularities, the next factor to cause slow convergence for the series solution is the occurrence of certain discontinuities in the boundary value. Let us consider a rectangular plate which is insulated at both top and bottom faces, and has four edges either insulated or at a given temperature as shown in Fig. 5-1. Note that the edge AB is defined as an insulated edge while BC is defined by a given temperature.
The governing equation for steady-state conductive heat transfer with no sources or sinks is

$$\nabla^2 T = 0$$  

...(5-1)

**Boundary Conditions of a Rectangular Plate**

Fig. 5-1

where $T$ is the temperature. The insulated edge is represented by a zero normal temperature gradient:

$$\frac{\partial T}{\partial n} = 0$$  

...(5-2)
When the boundary condition is defined by a given temperature, then the temperature gradient $\frac{\partial T}{\partial n}$ along that boundary usually is not zero; i.e., the heat may either flow into the plate or flow out from the plate. At point B in Fig. 5-1, the boundary is defined by a zero normal temperature gradient at its left side, and a given temperature which may have a non-zero normal temperature gradient at its right side. Therefore, at point B, the boundary value $\frac{\partial T}{\partial n}$ most likely is discontinuous.

The solution of Eq. (5-1) is a harmonic function which can be represented by Eq. (1-5) by retaining only the positive order terms; i.e.,

$$T = A_0 + \sum_{n=1}^{N} \left( A_n \eta^n \cos n\theta + A'_n \eta^n \sin n\theta \right)$$

...(5-3)

We select the first 21 unknowns of Eq. (5-3) (N=10) to represent an approximate solution of the given problem and determine them by satisfying the boundary conditions in a least square sense at 45 equally-spaced points except that two points which are used to represent the boundary condition at point B. One point is immediately at the left side of B to represent the boundary condition $\frac{\partial T}{\partial n}=0$ and the other point is immediately at the right side of B to represent the boundary condition $T=6T_1$. The resulting temperature along several sections of the plate is shown in Fig. 5-2 and the temperature gradient in $y$ direction along edge ABC and edge GFE is shown in Fig. 5-3. One can see that this
Fig. 5-2 Temperature In the Rectangular Plate

Fig. 5-3 Temperature Gradient $\frac{\partial T}{\partial z}$ Along Two Edges
approximate solution has a very large residue error $\Delta T$ along the boundary BC. Fig. 5-3 shows that the gradient curve along ABC does not have any discontinuity as we predicted. As a conclusion, the series of Eq. 5-3 with 21 unknowns is far from being able to yield a solution for the given problem with reasonable accuracy.

It is well known that the hyperbolic-trigonometric series fits the heat transfer problem of a rectangular plate with zero temperature along two parallel edges better than the series in polar coordinates like Eq. (5-3) does. If we have the origin of the coordinates as shown in Fig. 5-2, the hyperbolic-trigonometric harmonic function is

$$T = \sum_{n=1}^{N} \left( A_n \sinh \frac{n\pi}{\alpha} y + B_n \cosh \frac{n\pi}{\alpha} y \right) \sin \frac{n\pi}{\alpha} \chi \quad \ldots \quad (5-4)$$

![Diagram of heat transfer problem with coordinates and temperature gradient](image)
Fig. 5-5 Temperature In the Rectangular Plate

Fig. 5-6 Temperature Gradient \( \frac{dT}{dy} \) Along Two Edges
Since Eq. (5-4) has satisfied the boundary condition along two parallel edges, we select only the first 12 unknowns ($N=6$) of Eq. (5-4) and determine them by satisfying the boundary condition in a least square sense at 23 points. The temperature in the plate and the boundary residues are shown in Fig. 5-5 and Fig. 5-6. Like the series solution in polar coordinates, the boundary residue error is too large. The reason for the poor convergence of Eq. (5-3) or Eq. (5-4) for the given problem is that the boundary value $\frac{\partial T}{\partial n}$ is discontinuous at boundary point B (see Fig. 5-1 or Fig. 5-4). If there is to be any chance of improving the convergence of these series solutions, one thing we have to do is to introduce a harmonic function which will give a discontinuous boundary value of $\frac{\partial T}{\partial n}$ at B along boundary ABC. It can be proved easily by examining the derivative of the singular integral solutions as represented by Eq. (6-17), Eq. (6-18), Eq. (6-20) and Eq. (6-21) that the singular integral solution which we introduced in Chapter IV can serve for this purpose if the density function has a discontinuity at B. Now, if we divide the boundary BC (Fig. 5-1) into several segments and use Eq. (4-11) to represent the singular integral solution, then add it to the series of Eq. (5-3), we should be able to obtain a final solution for the given problem with a reasonable accuracy. In order to show how much the singular integral solution will help the series convergence, we use only one term of the singular integral solution
of Eq. (4-11) by letting $i=1$, $a_0=0.3a$, $a_1=0$, $\gamma_0=90.0$ and $c=1.0$, and $p_0$ is an unknown to be determined. Add this special term into the series Eq. (5-3) which contains the same 21 unknowns, and the 22 total unknowns are determined by satisfying the same 15 boundary points as we did before. $p_0$ was found to be $7.599T_0$, and the final results of the temperature in the plate and $\partial T/\partial x$ along two edges are shown in Fig. 5-8 and Fig. 5-9 respectively. Comparing Fig. 5-2, Fig. 5-3 with Fig. 5-8 and Fig. 5-9, one can see that, with only one special term, the solution has been greatly improved.

Fig. 5-7
Fig. 5-8 Temperature In the Rectangular Plate

Fig. 5-9 Temperature Gradient $\frac{\partial T}{\partial y}$ Along Two Edges
CHAPTER VI

APPLICATION TO TORSION PROBLEMS

A) Theory, Equations and Solutions of Torsion with Multiply-Connected Regions

The theory of the problem of torsion of prismatical bars was developed by Saint-Venant in 1855. He assumed that

\[ u = -x, \quad v = \beta \frac{\partial \varphi}{\partial x}, \quad w = \beta \varphi (x, y) \]

where \( u, v, \) and \( w \) are displacement in \( x, y \) and \( z \) directions, \( \beta \) is the angle of rotation per unit length and \( \varphi (x, y) \) is called the warping function. Substituting Eq. (6-1) into the strain-displacement relations, gives

\[ 2 \varepsilon_y = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \beta \left( \frac{\partial \varphi}{\partial y} + \lambda \right) \]
\[ 2 \varepsilon_z = \frac{\partial u}{\partial z} + \frac{\partial v}{\partial z} = \beta \left( \frac{\partial \varphi}{\partial z} - \gamma \right) \]

where \( \varepsilon_y \) and \( \varepsilon_z \) are tensorial strains; \( 2 \varepsilon_y = \gamma_{y}, \quad 2 \varepsilon_z = \gamma_{z} \).

From the stress-strain relations, we obtain

\[ \sigma_y = \lambda \varepsilon_y \varepsilon_{y} = \lambda \varepsilon_y \left( \frac{\partial \varphi}{\partial y} + \lambda \right) \]
\[ \sigma_z = \lambda \varepsilon_z \varepsilon_{z} = \lambda \varepsilon_z \left( \frac{\partial \varphi}{\partial z} - \gamma \right) \]

where \( G \) is the modulus of elasticity in shear. Eliminate \( \varphi \) by differentiating the first with respect to \( x \), the second with
respect to \( y \), and subtracting from the first. We obtain
\[
\frac{\partial \tau_{y\lambda}}{\partial x} - \frac{\partial \tau_{x\lambda}}{\partial y} = 2 G/\beta \quad \ldots(6-4)
\]

Using the Prandtl stress function
\[
\begin{cases}
\tau_{x\lambda} = \frac{\partial F}{\partial y} \\
\tau_{y\lambda} = -\frac{\partial F}{\partial x}
\end{cases} \quad \ldots(6-5)
\]

and substituting it into Eq. (6-4), we find the governing equation is
\[
\nabla^2 F = -2 G/\beta \quad \ldots(6-6)
\]
The boundary of the prismatic bar is defined to be stress free; i.e.,
\[
\tau_{\nu t} = \tau_{x\lambda} \frac{dy}{ds} - \tau_{y\lambda} \frac{dx}{ds} = 0 \quad \ldots(6-7)
\]
Substituting Eq. (6-5) in Eq. (6-7), we obtain
\[
\frac{\partial F}{\partial s} = 0 \quad \ldots(6-8)
\]
This shows that the stress function must be constant along the boundary of the cross section. In the case of a simply connected boundary, this constant can be chosen arbitrarily. For simplicity, we take it equal to zero. For a multiply-connected boundary, we take the outer boundary \( F = 0 \) and on inner boundaries \( F = k_i \), where \( i = 1, 2, \ldots, H \), and \( H \) is the total number of holes; i.e.,

Simply connected boundary: \( F = 0 \) \quad \ldots(6-9)

Multiply connected boundary:
\[
\begin{cases}
\text{Outer boundary} & F_o = 0 \\
\text{inner boundary} & F_i = k_i, \ i = 1, 2 \ldots, H
\end{cases} \quad \ldots(6-10)
\]
We shall show how to evaluate \( k \) later. The solution of Eq. (6-6) can be written as

\[ \bar{\mathcal{X}} = \bar{\mathcal{X}}_p + \bar{\mathcal{X}}_c \]

...(6-11)

where the particular solution \( \bar{\mathcal{X}}_p \) usually is taken as

\[ \bar{\mathcal{X}}_p = -\frac{1}{2} G\beta r^2 \]

...(6-12)

and the complementary solution \( \bar{\mathcal{X}}_c \) can be the series of Eq. (1-5). Substitute Eq. (6-12) and Eq. (1-5) into Eq. (6-11) and Eq. (6-5), giving

\[ \bar{\mathcal{X}} = -\frac{1}{2} G\beta r^2 + A_0 + B_0 \ln r + \sum_{n=1}^{\infty} (A_n r^n + B_n r^{-n}) \cos n\theta \]

\[ + \sum_{n=1}^{\infty} (A'_n r^n + B'_n r^{-n}) \sin n\theta \]

...(6-13)

\[ \tau_{x_2} = \frac{\partial \bar{\mathcal{X}}}{\partial y} = -G\beta \gamma + B_0 r^{-1} \sin \theta \]

\[ - \sum_{n=1}^{\infty} \left[ A_n n^{-1} r^{n-1} \sin (n-1)\theta + B_n n^{-1} r^{-n-1} \sin (n+1)\theta \right] \]

\[ + \sum_{n=1}^{\infty} \left[ A'_n n^{-1} r^{n-1} \cos (n-1)\theta + B'_n n^{-1} r^{-n-1} \cos (n+1)\theta \right] \]

...(6-14)

\[ \tau_{y_3} = -\frac{\partial \bar{\mathcal{X}}}{\partial \gamma} = G\beta \gamma - B_0 r^{-1} \cos \theta \]

\[ - \sum_{n=1}^{\infty} \left[ A_n n^{-1} r^{n-1} \cos (n-1)\theta - B_n n^{-1} r^{-n-1} \cos (n+1)\theta \right] \]

\[ - \sum_{n=1}^{\infty} \left[ A'_n n^{-1} r^{n-1} \sin (n-1)\theta - B'_n n^{-1} r^{-n-1} \sin (n+1)\theta \right] \]

...(6-15)
The complementary solution \( \Phi_c \) can also be represented by a singular integral solution with step function approximation as Eq. (4-9). For torsion problems, there is no physical meaning like concentrated load for the term \( \log \frac{r}{c} \), hence we simply take \( k = 1 \) instead of \( k = -\frac{1}{2\pi T} \) as shown in Eq. (4-6). Substitute Eq. (6-12) and Eq. (4-9) in Eq. (6-11) and Eq. (6-5), giving

\[
\Phi = -\frac{1}{2} G/\beta h^2 - \sum_{i=1}^{N} P_i \left\{ \int_{-s_i/2}^{s_i/2} \ln \frac{r (B, A)}{c} \, dt \right\}
\]

\[
= -\frac{1}{2} G/\beta h^2 - \sum_{i=1}^{N} \frac{P_i}{S_i} \left( \frac{S_i^2}{4} - 0.5 R_i S_i N_i \right) \ln \left( R_i^2 - R_i S_i S_i N_i + \frac{S_i^2}{4} \right)
\]

\[
+ \left( \frac{S_i}{4} + 0.5 R_i S_i N_i \right) \ln \left( R_i^2 + R_i S_i S_i N_i + \frac{S_i^2}{4} \right) - S_i \left( 1 + \ln c \right)
\]

\[
+ R_i C N_i \left( \tan^{-1} \frac{S_i^2 - R_i S_i N_i}{R_i C N_i} + \tan^{-1} \frac{S_i^2 + R_i S_i N_i}{R_i C N_i} \right)
\] ....(7-16)

\[
\varpi_{x3} = \frac{\partial \Phi}{\partial y} = -G/\beta y - \sum_{i=1}^{N} P_i \left\{ \int_{-s_i/2}^{s_i/2} \frac{\sin \theta (B, A)}{r (B, A)} \, dt \right\}
\]

\[
= -G/\beta y - \sum_{i=1}^{N} \frac{P_i}{S_i} \left[ \sin \phi \left( \tan^{-1} \frac{S_i^2 - R_i S_i N_i}{R_i C N_i} + \tan^{-1} \frac{S_i^2 + R_i S_i N_i}{R_i C N_i} \right)
\]

\[
+ \frac{\cos \phi}{2} \left[ \ln \left( R_i^2 + R_i S_i S_i N_i + \frac{S_i^2}{4} \right) - \ln \left( R_i^2 - R_i S_i S_i N_i + \frac{S_i^2}{4} \right) \right]
\] ....(6-17)
\[ \tau_{ii}^{B} = -\frac{2\pi}{\alpha} = G \beta \chi + \sum_{i=1}^{N} p_i \left\{ \int_{-\frac{S_i}{2}}^{\frac{S_i}{2}} \cos \theta(B, A) \frac{d\theta}{R(B, A)} \right\} \]

\[
= G \beta \chi + \sum_{i=1}^{N} \frac{p_i}{S_i} \left\{ \cos \psi_i \left( \tan^{-1} \frac{\frac{S_i}{2} - R_i SN_i}{R_i CN_i} + \tan^{-1} \frac{\frac{S_i}{2} + R_i SN_i}{R_i CN_i} \right) - \frac{\sin \psi_i}{2} \left[ \ln \left( \frac{R_i^2 + R_i S_i \cdot SN_i + \frac{S_i^2}{4}}{R_i} \right) - \ln \left( \frac{R_i^2 - R_i S_i \cdot SN_i + \frac{S_i^2}{4}}{R_i} \right) \right] \right\}
\]

....(6-18)

where \( R_i \) = distance between \( B_i \) and \( A_i \); \( \theta \) = angle between the x-axis and the vector \( B_iA_i \); \( SN_i = \sin (\theta_i - \psi_i) \); \( CN_i = \cos (\theta_i - \psi_i) \), \( \theta(B, A) = \) angle between the x-axis and the vector \( BA \) and \( R_i = P_i \cdot S_i \) (refer to Fig. 4-3). Now, we let \( \Psi \) be represented by the singular integral solution with polygonal function approximation as Eq. (4-11). Again, we let \( k \) of Eq. (4-9) be -1 and substitute Eq. (6-12) and Eq. (4-11) in Eq. (6-11) and Eq. (6-5), giving

\[
\Psi = -\frac{1}{2} G \beta R^2 - \sum_{i=1}^{N} p_i \left\{ \int_{0}^{\frac{S_i}{2}} \ln \left( \frac{R_i}{R_i} \right) \frac{d\theta}{c} + \int_{-\frac{S_i}{2}}^{0} \ln \left( \frac{R_i}{R_i} \right) \frac{d\theta}{c} \right\}
\]

\[
= -\frac{1}{2} G \beta R^2 + \sum_{i=1}^{N} \frac{p_i}{S_i} \left\{ \frac{R_i}{CN_i^{(1)}} \left( 1 + \frac{R_i SN_i^{(1)}}{S_i^{(1)}} \right) \left( \tan^{-1} \frac{SN_i^{(1)}}{CN_i^{(1)}} - \tan^{-1} \frac{S_i^{(1)} + R_i SN_i^{(1)}}{R_i CN_i^{(1)}} \right) \right. \\
+ \frac{R_i}{S_i} \ln R_i \left( R_i^{(1)} - S_i^{(1)} \right) + S_i^{(1)} \cdot SN_i^{(1)} - 0.5 R_i \right) \\
+ \frac{1}{2} \left( R_i^2 CN_i^{(2)} - S_i^{(2)} - 2R_i S_i \cdot SN_i^{(2)} - R_i SN_i^{(2)} \right) \ln \left( \frac{S_i^{(2)} + R_i^2 + 2R_i S_i \cdot SN_i^{(2)}}{R_i^2} \right) \\
+ \frac{1}{2} S_i \ln R_i - \frac{1}{2} R_i \cdot SN_i^{(2)} + \frac{3}{4} S_i \\
- R_i \cdot SN_i^{(2)} \left( 1 - \frac{R_i SN_i^{(2)}}{S_i^{(2)}} \right) \left( \tan^{-1} \frac{S_i^{(2)} - R_i SN_i^{(2)}}{R_i CN_i^{(2)}} + \tan^{-1} \frac{SN_i^{(2)}}{CN_i^{(2)}} \right) \\
+ \frac{R_i}{S_i} \ln R_i \left( R_i S_i^{(2)} - S_i \cdot SN_i^{(2)} - 0.5 R_i \right)
\]
\[
\mathcal{L}_{k_3} = \frac{\partial F}{\partial \theta} = -G \beta x - \sum_{i=1}^{N} p_i \left\{ \int_{S_{i-1}}^{S_i} \frac{\sin \theta(B_A)}{\lambda(B_A)} (1 + \frac{x}{S_i}) \, dt + \int_{0}^{S_i} \frac{\sin \theta(B_A)}{\lambda(B_A)} (1 - \frac{x}{S_i}) \, dt \right\}
\]

\[
\mathcal{L}_{y_3} = -\frac{\partial F}{\partial \theta} = G \beta x + \sum_{i=1}^{N} p_i \left\{ \int_{S_{i-1}}^{S_i} \frac{\cos \theta(B_A)}{\lambda(B_A)} (1 + \frac{x}{S_i}) \, dt + \int_{0}^{S_i} \frac{\cos \theta(B_A)}{\lambda(B_A)} (1 - \frac{x}{S_i}) \, dt \right\}
\]

\[
\mathcal{L}_{z_3} = \frac{\partial F}{\partial \theta} = \frac{1}{4S^2} (R_z^2 \cdot C N_{x}^{(2)} - R_z^2 \cdot S N_{x}^{(2)} - S_{z}^2 + 2 R_z S_S N_{x}^{(2)} \ln (R_z^2 + S_{z}^2 - 2 R_z S_S N_{x}^{(2)}))
\]

\[
+ \frac{1}{S^2} (R_z^2 \cdot C N_{x}^{(2)} - R_z^2 \cdot S N_{x}^{(2)} - S_{z}^2 + 2 R_z S_S N_{x}^{(2)} \ln (R_z^2 + S_{z}^2 - 2 R_z S_S N_{x}^{(2)}))
\]

\[
+ \frac{1}{2} S_z \cdot \ln c - \frac{1}{2} R_z S_N^{(2)} + \frac{3}{2} S_z \}
\]

\[\cdots (6-19)\]

\[
\mathcal{L}_{z_3} = \frac{\partial F}{\partial \theta} = -G \beta y - \sum_{i=1}^{N} p_i \left\{ \int_{S_{i-1}}^{S_i} \frac{\sin \theta(B_A)}{\lambda(B_A)} (1 + \frac{x}{S_i}) \, dt + \int_{0}^{S_i} \frac{\sin \theta(B_A)}{\lambda(B_A)} (1 - \frac{x}{S_i}) \, dt \right\}
\]

\[
= -G \beta y + \sum_{i=1}^{N} p_i \left\{ \left[ \sin \gamma_i + \frac{R_i \sin \theta_i + C N_{x}^{(2)} \cos \gamma_i}{S_i} \right] \left( \tan^{-1} \frac{S_{i}^2 - 2 R_z S_S N_{x}^{(2)}}{R_z C N_{x}^{(2)}} \right) \right\}
\]

\[\cdots (6-21)\]
where \( R_i \) = distance between \( B_i \) and \( A_i \); \( \theta_i \) = angle between the 
\( x \)-axis and the vector \( \mathbf{B}_i \mathbf{A}_i \); \( SN_i^{(1)} = \sin (\theta_i - \psi_{i+1}) \); \( CN_i^{(1)} = \cos (\theta_i - \psi_{i+1}) \); 
\( SN_i^{(2)} = \sin (\theta_i - \psi_i) \) and \( CN_i^{(2)} = \cos (\theta_i - \psi_i) \); 
\( \theta(B,A) \) = angle between the \( x \)-axis and the vector \( \mathbf{B}A \). Note that, 
for \( i=1 \), \( S_{-1} = S_N \) and \( \psi_{-1} = \psi_N \). The geometrical relation 
of the segments is shown in Fig. 4-13.

For multiply connected regions, the inner boundaries 
are defined by a set of unknowns \( k_i \), \( i=1,2,...,H \) as shown in 
Eq. (6-10). From considering the equilibrium around a hole 
(inner boundary) of the membrane analogy, we can obtain an 
additional equation for each hole as (refer to reference (31), 
p. 326)

\[
\oint_{\partial S} \frac{\partial \phi}{\partial n} \, ds = -2 \pi \phi \, A_i 
\]

Substituting Eq. (6-11) in Eq. (6-22) and use the relation 
\[
\frac{1}{r} \cos \phi \, ds = d\theta 
\]

where \( r \) and \( \theta \) are polar coordinates, \( S \) is the length along the 
boundary and \( \phi \) is the angle between the radial line and the 
normal line (see Fig. 6-1), we obtain

\[
\oint_{\partial S} \frac{\partial \phi}{\partial n} \, ds = \oint_{\partial S} \frac{\partial \phi}{\partial \theta} \, ds + \oint_{\partial S} \frac{\partial \phi}{\partial r} \, ds \\
= \oint_{\partial S} (\cos \phi \, \frac{\partial \phi}{\partial n} + \sin \phi \, \frac{1}{r} \, \frac{\partial \phi}{\partial \theta}) \, ds - \oint_{\partial S} \frac{\partial \phi}{\partial S} \, ds
\]
\[
\int_{\gamma} \frac{\partial \phi}{\partial \eta} \, ds = -G \beta \int_{\gamma} \rho^2 \, d\theta - \int_{\gamma} d\psi
\]

\[
= -2G\beta A_{i} - \psi_{i} \bigg|_{\eta,0}^{\eta,2\pi}
\]

where \( \psi \) is the warping function as defined by Eq. (6-1) and it is the harmonic conjugate of \( \Phi_{c} \); i.e., they satisfy the Cauchy-Reimann relation

\[
\begin{align*}
\frac{\partial \psi}{\partial n} &= \frac{\partial \Phi_{c}}{\partial s} ; \quad \frac{\partial \psi}{\partial s} &= -\frac{\partial \Phi_{c}}{\partial n} \\
\frac{\partial \psi}{\partial \eta} &= \frac{\partial \Phi_{c}}{\partial \eta} ; \quad \frac{\partial \psi}{\partial \eta} &= -\frac{\partial \Phi_{c}}{\partial \eta}
\end{align*}
\]

\[
\text{(6-25)}
\]

For the series expression \( \Phi_{c} \) of Eq. (6-13), \( \psi \) is

\[
\psi = -B_{o} \theta - \sum_{n=1}^{\infty} (A_{n} r^{-n} - B_{n} r^{-n}) \sin n\theta + \sum_{n=1}^{\infty} (A'_{n} r^{-n} - B'_{n} r^{-n}) \cos n\theta
\]

\[
\text{(6-26)}
\]

Substitute Eq. (6-26) in Eq. (6-24) and compare with (6-22), giving

\[
-2G\beta A_{i} = -2G\beta A_{i} + 2\pi B_{o}
\]

or

\[
B_{o} = 0
\]

\[
\text{(6-27)}
\]

This means that, if only one series is used, the term with \( B_{o} \) should be excluded. For the case of multiple poles, \( B_{o} \) will be zero if there is only one pole in each hole. However, if there is more than one pole in a given hole, then one additional equation should be included for each hole; i.e.,
where \( M \) is the number of poles in a given hole. Note that if a pole is outside of a hole, the line integral of Eq. \((6-28)\) will vanish for all the terms of Eq. \((6-26)\). Similarly for the singular integral solution, we obtain an additional equation for each hole

\[
\sum_{i=1}^{M_{\text{h}}} \phi_{i}^{(j)} = 0 \quad \text{ \ldots (6-29)}
\]

where \( j \) indicates the hole's number and \( M_{\text{h}}^{(j)} \) is the number of segments around hole \( j \). Note that the density function is considered being infinitesimally outside of the inner or outer boundary as shown in Fig. 6-1.

![Coordinates of a Multiply Connected Region](image-url)
B) **Torsional Rigidity**

From the last section, we obtained the solution of Eq. (6-6) in terms of $G/\beta$. However, the constant $\beta$ usually is not known. Instead, the applied twisting moment $M_T$ is given. The relation between $G/\beta$ and $M_T$ is

$$M_T = \int \int (\tau_{y} \lambda - \tau_{x} \lambda_{y}) \, dx \, dy = - \int \int (\frac{\partial \lambda}{\partial x} \tau + \frac{\partial \lambda}{\partial y} \tau_{y}) \, dx \, dy$$

$$= - \int \int \left[ \frac{\partial \lambda}{\partial x} + \frac{\partial \lambda_{y}}{\partial y} \right] dxdy + 2 \int \int \Phi \, dx \, dy \quad \cdots (6-30)$$

or after transforming the first integral according to Green's lemma (reference (32), p. 498),

$$M_T = - \int \int (x \, dy - y \, dx) + 2 \int \int \Phi \, dx \, dy$$

$$= - k_c \int (x \, dy - y \, dx) - k_l \int (x \, dy - y \, dx) - \cdots$$

$$- k_h \int (x \, dy - y \, dx) + 2 \int \int \Phi \, dx \, dy$$

$$= - 2(\Phi \lambda_A - k_1 A_1 - k_2 A_2 - \cdots k_h A_h) + 2 \int \int \Phi \, dx \, dy$$

$$= 2(k_1 A_1 + k_2 A_2 + \cdots k_h A_h) + 2 \int \int \Phi \, dx \, dy \quad \cdots (6-31)$$
where $k_j$ and $A_j$ are the boundary value and the area of hole $j$, and $k_j$ is the boundary contour around hole $j$, and we note that the line integral is counter-clockwise along the outer boundary and clockwise along the inner boundaries as shown in Fig. 6-1, and $k_o$ is zero.

For the case of simply connected region,

$$M_t = 2 \int \int_R \frac{k}{E} d\chi d\Psi$$

$$= 2 \int \int_R (\overline{E} + \overline{C}) d\chi d\Psi = -G/\beta \int \int_R \kappa^2 d\chi d\Psi + 2 \int \int_R \overline{E} d\chi d\Psi$$

$$= -G/\beta J + 2 \int \int_R \overline{C} d\chi d\Psi \quad \ldots \ldots \ldots (6-32)$$

or

$$G/\beta = \frac{\overline{E}}{\overline{C}}, \int \int_R \overline{C} d\chi d\Psi = J$$

where $J$ is the polar moment inertia, and the complementary solution $\overline{E}$ is a function of the coordinates multiplied by $G/\beta$. If we let

$$C = \frac{G/\beta}{\int \int_R \overline{C} d\chi d\Psi} = J G/\beta$$

$$\ldots \ldots \ldots (6-34)$$

where $C$ is called the torsional rigidity, Eq. (6-33) can be written as

$$M_t = C G/\beta \quad \ldots \ldots \ldots (6-35)$$

Instead of Eq. (6-31), there is an alternative way to set up the relation between $M_t$ and $G/\beta$. Starting from...
Eq. (6-30) and using Eq. (6-12), we have

\[ M_t = - \int_\mathbb{R} \left( \kappa \frac{\partial \mathcal{F}}{\partial x} + \gamma \frac{\partial \mathcal{F}}{\partial y} \right) dx dy \]

\[ = - \int_\mathbb{R} \left( \kappa \frac{\partial \mathcal{F}}{\partial x} + \gamma \frac{\partial \mathcal{F}}{\partial y} \right) dx dy + G \beta \int_\mathbb{R} \left( \kappa \frac{\partial \mathcal{F}}{\partial x} + \gamma \frac{\partial \mathcal{F}}{\partial y} \right) dx dy \]

\[ = - \int_\mathbb{R} \left( \kappa \frac{\partial \mathcal{F}}{\partial x} + \gamma \frac{\partial \mathcal{F}}{\partial y} \right) dx dy + G \beta J \]

Using the relations of Eq. 6-25) and Green's lemma, we obtain

\[ M_t = G \beta J - \int_\mathbb{R} \left( \gamma \frac{\partial \mathcal{F}}{\partial x} - \kappa \frac{\partial \mathcal{F}}{\partial y} \right) dx dy \]

\[ = G \beta J - \int_\mathbb{R} (\kappa \mathcal{F} \frac{\partial \mathcal{F}}{\partial x} + \gamma \mathcal{F} \frac{\partial \mathcal{F}}{\partial y}) dx dy \]

\[ = G \beta J - \int_\mathbb{R} \mathcal{F} \left( \kappa \frac{\partial \mathcal{F}}{\partial x} + \gamma \frac{\partial \mathcal{F}}{\partial y} \right) ds \]

\[ = G \beta J + \int_\mathbb{R} \mathcal{F} \sin \phi \ ds \quad \ldots \ldots (6-36) \]

where \( \phi \) is the angle between the radial line and the normal line. Note that, unlike Eq. (6-31), there is no \( A_k Z_k \) in Eq. (6-36); i.e., it treats the multiply connected region the same way as it does the simply connected region. If the complementary solution \( \mathcal{F}_c \) is in a series like Eq. (6-13), \( \mathcal{F} \) is represented in Eq. (6-26), and then we have

\[ M_t = G \beta J + \int_\mathbb{R} \left[ \sum_{n=1}^{\infty} \left( -A_n r^{n+1} + B_n r^{-n+1} \right) \sin n\theta \right. \]

\[ + \sum_{n=1}^{\infty} \left( A_n' r^{n+1} - B_n' r^{-n+1} \right) \cos n\theta \] \( \left. \sin \phi \right] ds \quad \ldots \ldots (6-37) \]
However, if we let the complementary solution $\overline{\Phi}_c$ be represented by a singular integral solution, the numerical value of $\overline{\Phi}_c$ at any point $A$ is

$$\overline{\Phi}_c(A) = \int \mathcal{P}(B) \ln \frac{\mathcal{L}(B,A)}{C} \, ds$$

where $\mathcal{P}(B)$ is the value of the density function at boundary point $B$. The corresponding warping function is

$$\mathcal{M}(A) = -\int \mathcal{P}(B) \cdot \theta(B,A) \, ds \quad ....(6-38)$$

where $\theta(B,A)$ is the angle between the x-axis and the line $BA$ and $-\pi < \theta < \pi$. Substitute Eq. (6-38) in Eq. (6-36), yielding

$$M_i = G_{ij} \int \left[ \int \mathcal{P}(B) \theta(B,A) \, d\theta \right] \kappa \sin \phi \, ds \quad ....(6-39)$$

If we divide the boundary into $N$ segments and denote $\mathcal{M}_j$ at the center of segment $j$ by $\mathcal{M}_j$, Eq. (6-38) becomes (refer to Fig. 6-3)

$$\mathcal{M}_j = -\sum_{\lambda=1}^{N} \int_{-\frac{s_j}{2}}^{\frac{s_j}{2}} \theta(\theta, A) \, d\lambda \sum_{\lambda=1}^{N} \int_{-\frac{s_j}{2}}^{\frac{s_j}{2}} \frac{-1}{R_i} \tan \theta_i \cos \chi_i \sin \chi_i \, d\lambda \quad ....(6-40)$$

The symbols in Eq. (6-40) are the same as those in Eq. (4-9).

The line integral of Eq. (6-40) can be evaluated and Eq. (6-40) can be written as
\[ \left[ \frac{2}{2 \sin \psi_i} \left( \frac{z R_i \sin \theta_i + S_i \cos \psi_i}{z R_i \cos \theta_i + S_i \sin \psi_i} \right) + \sin 2\psi_i \right] - \frac{\cos \psi_i}{\sin \psi_i} \right] \tan \frac{2 R_i \sin \theta_i - S_i \cos \psi_i}{z R_i \cos \theta_i + S_i \sin \psi_i} \\
+ \frac{1}{4} \ln \left[ \frac{4 R_i^2 + S_i^2 - 4 R_i S_i \sin (\theta_i - \psi_i)}{4 R_i^2 + S_i^2 + 4 R_i S_i \sin (\theta_i - \psi_i)} \right] \left( \frac{2 R_i \cos \theta_i - S_i \sin \psi_i}{z R_i \cos \theta_i + S_i \sin \psi_i} \right) \right] \\
- \ln \left( \frac{\cos \psi_i}{\sin \psi_i} + \frac{2 R_i \sin \theta_i - S_i \cos \psi_i}{z R_i \cos \theta_i + S_i \sin \psi_i} \right) + \ln \left( \frac{\cos \psi_i}{\sin \psi_i} + \frac{2 R_i \sin \theta_i + S_i \cos \psi_i}{z R_i \cos \theta_i - S_i \sin \psi_i} \right) \right] \\
\ldots \ldots (6-41) \\

Assuming that \( \psi_\delta \) is a step function along the boundary, Eq. (6-36) becomes

\[ M_k = G \beta J + \sum_{\delta=1}^{N} \psi_\delta \int_{S_i} \rho \sin \phi \, ds \] \ldots \ldots (6-42) \\

From Eq. (6-41) and Eq. (6-42), \( G \beta \) can be found, and from Eq. (6-35), we obtain the torsional rigidity.

C) Numerical Examples

By using the series solution, numerical examples of torsion of a bar of regular polygonal cross section were
illustrated in Reference (2) and a cruciform section was demonstrated in Reference (1). In Chapter I, a cross section with three concave sides was also solved by a series solution. If we replace \( \frac{a}{r} \) of the membrane problem by \( 2G\beta \), the solution for the membrane becomes the solution of the torsion problem. From the results for the sample problems in Chapter I, we obtain the maximum shear stress at the center of the concave boundary as shown in Table 6.1.

<table>
<thead>
<tr>
<th>Number of poles</th>
<th>3x3</th>
<th>7x7</th>
<th>15x7</th>
<th>15x15</th>
<th>31x15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size of the matrix</td>
<td>3x3</td>
<td>7x7</td>
<td>15x7</td>
<td>15x15</td>
<td>31x15</td>
</tr>
<tr>
<td>( C_{\alpha A} ) in ( \mu )</td>
<td>2.403</td>
<td>2.407</td>
<td>2.407</td>
<td>2.407</td>
<td>2.403</td>
</tr>
</tbody>
</table>

In Chapter II, we solved a rectangular membrane by singular integral equation with either step function approximation or polygonal function approximation. If we let \( 2\frac{2^{\pi} \psi}{3} \), \( -2\frac{2^{\pi} \psi}{3} \) and \( \frac{q}{r} \) be replaced by \( \lambda_{\lambda', \phi} \), \( \lambda_{\phi, \phi} \) and \( \lambda_{\phi, \phi} \), Table 4-1 showed the shear stresses in the rectangular cross section. As for the torsional rigidity, the formula of Eq. (6-41) was not yet programmed for the digital computer by the time this dissertation has finished. However, Eq. (6-40) was approximately computed by replacing the integral
by $S_i\theta_i$ where $\theta_i$ is the average of the angles between the x-axes and the lines connecting the center points of eight subsegments of segment $S_i$ and point A. Then Eq. (6-40) is approximately represented by

$$N = -\sum_{i=1}^{N} P_i S_i \frac{\theta_i}{\lambda}$$

...(6-43)

Using Eq. (6-43) and Eq. (6-42), $G/\mu$ of the torsion of the rectangular cross section (Fig. 4-1) was found and shown in Table 6-2. The $G/\mu$ found by Timoshenko and Goodier (reference 16, p. 277) was also included in Table 6-2 for comparison.

Table 6-2. Torsional rigidity of rectangular shaft (refer to Fig. 4-1)

<table>
<thead>
<tr>
<th>Size of matrix</th>
<th>15x15</th>
<th>31x15</th>
<th>31x22</th>
<th>64x22</th>
<th>reference (16)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G/\mu (in \mu \epsilon)$</td>
<td>2.122</td>
<td>2.145</td>
<td>2.182</td>
<td>2.170</td>
<td>2.183</td>
</tr>
<tr>
<td>$C$</td>
<td>0.471</td>
<td>0.466</td>
<td>0.458</td>
<td>0.460</td>
<td>0.458</td>
</tr>
</tbody>
</table>

Now, we shall illustrate the method of solving torsion problems with multiply connected regions. First, consider a square hollow shaft with the outer boundary dimension 2a x 2a and the inner boundary dimension axa as shown in Fig. 6-2. This geometrical shape has four axes of symmetry. Hence we shall only consider one eighth of the boundary. Using the singular integral solution with step function approximation; i.e., Eq. (6-16), we divide the outer boundary FG into ten segments and inner
boundary JK into five segments, so there are 15 unknowns for the density function and one unknown \( k \), for the boundary value of the hole, Eq. (6-10). We determine these 16 unknowns by approximately satisfying the boundary condition at 32 equally-spaced points as shown in Fig. 6-2 and Eq. (6-29). The results of the density function is shown in Fig. 6-3, and the \( \Xi \) contour and the lines of equal shear stress are shown in Fig. 6-4. The dotted lines in Fig. 6-4 indicate the actual outer and inner

---

**SEGMENTS AND MATCHED POINTS OF A HOLLOW SQUARE SHAFT**

*Fig. 6-2*
Fig. 6-3  Density Function $\Psi(B)$

Outer Boundary

Inner Boundary
LINES OF EQUAL SHEAR STRESS \( \phi \) - CONTOUR

in \( G/\beta \) in \( G/\beta a^2 \)

Fig. 6.4
boundary of the problem which the approximate solution represents. One can see that these dotted lines are very close to the desired boundary except at the corner of the outer boundary. The torsional rigidity was found by Eq. (6-43), Eq. (6-42) and Eq. (6-35) to be $C=2.026G\alpha^4$, compared with $C=2.066G\alpha^4$ found by Synge and Cahill, which are reasonably close values.

Now, consider a square shaft with two rectangular holes as shown in Fig. 6-5. Again, we use a singular integral solution with the step function approximation (Eq. (6-16)). We have 32 unknowns for the density function and one unknown $k_1$ of Eq. (6-10) for the inner boundary value. These 33 unknowns are determined by 66 boundary condition equations and Eq. (6-16). The density function is shown in Fig. 6-6, and the $F$ contour and the lines of equal shear stress are shown in Fig. 6-7. The torsional rigidity was found to be $C=2.180G\alpha^4$. 
Lines of Equal Shear Stress

in $10^{-1} \frac{G\beta}{a}$

$\Phi$-Contour

in $10^{2} \frac{G\beta}{a^{2}}$

Fig. 6-7
D) Anisotropic Torsion

In the last three sections we showed that the isotropic torsion problem was solved by either a series solution or a singular integral solution. In the following, we shall show that these methods can be used to solve anisotropic torsion problems. Two different types of anisotropy will be studied separately. The first kind is when the anisotropic constants are defined in the x and y directions, like plywood. The second kind is when the anisotropy constants are defined in radial and tangential directions, like a tree trunk.

(1) The anisotropic constants defined in the x and y directions; i.e.,
\[ e_y = a_{44} \tau_y + a_{45} \tau_x \]
\[ e_x = a_{45} \tau_y + a_{55} \tau_x \]

where \( e_y \) and \( e_x \) are tensorial strains; i.e., \( e_y \frac{\partial}{\partial y} + e_x \frac{\partial}{\partial x} \). Substitute Eq. (6-5) and Eq. (6-2) in Eq. (6-1*1*), yielding
\[ \beta \left( \frac{\partial \psi}{\partial y} + \xi \right) = -2 a_{44} \frac{\partial \psi}{\partial x} + 2 a_{45} \frac{\partial \psi}{\partial y} \]
\[ \beta \left( \frac{\partial \psi}{\partial x} - \xi \right) = -2 a_{45} \frac{\partial \psi}{\partial x} + 2 a_{55} \frac{\partial \psi}{\partial y} \]

Eliminating \( \beta \), Eq. (6-45) becomes
\[ a_{44} \frac{\partial^2 \psi}{\partial x^2} - 2 a_{45} \frac{\partial^2 \psi}{\partial x \partial y} + a_{55} \frac{\partial^2 \psi}{\partial y^2} = -\beta \]

The elastic constants \( a_{44}, a_{45} \) and \( a_{55} \) are not invariant, but depend on the directions of the coordinates. If we rotate
the x-y axes an angle $\theta$, we have new strain-stress relations

\[
\begin{align*}
\epsilon_{x3}' &= \alpha_{11}' \epsilon_{x3} + \alpha_{13}' \epsilon_{x3}' \\
\epsilon_{x3}' &= \alpha_{22}' \epsilon_{x3} + \alpha_{23}' \epsilon_{x3}' \\
\end{align*}
\]

...(6-47)

Due to the rotation of the coordinates, the strain or stress transformation can be represented by a matrix operation as

\[
\begin{bmatrix}
\epsilon_{x1}' & \epsilon_{x2}' & \epsilon_{x3}' \\
\epsilon_{y1}' & \epsilon_{y2}' & \epsilon_{y3}' \\
\epsilon_{z1}' & \epsilon_{z2}' & \epsilon_{z3}'
\end{bmatrix} =
\begin{bmatrix}
\tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 \\
\tilde{y}_1 & \tilde{y}_2 & \tilde{y}_3 \\
\tilde{z}_1 & \tilde{z}_2 & \tilde{z}_3
\end{bmatrix}
\begin{bmatrix}
\epsilon_{x1} & \epsilon_{x2} & \epsilon_{x3} \\
\epsilon_{y1} & \epsilon_{y2} & \epsilon_{y3} \\
\epsilon_{z1} & \epsilon_{z2} & \epsilon_{z3}
\end{bmatrix}
\begin{bmatrix}
\tilde{1} & \tilde{2} & \tilde{3}
\end{bmatrix}
\]

...(6-48)

where $l_1$, $m_1$, and $n_1$ are the directional cosines of the $x'$ axis with respect to $x$, $y$ and $z$ axes; and similarly for $l_2$, $m_2$, and $n_2$.\end{quote}
If we rotate $x$-$y$ through an angle $\theta$, then

\[ n_i = n_x = 1; \quad n_z = 1; \quad l_i = l_z = \cos \theta; \quad m_i = -l_z = \sin \theta \quad \ldots \ldots (6-49) \]

From Eq. (6-48) and Eq. (6-49) and noting that $\epsilon_{xx} = \epsilon_{yy} = \epsilon_{zz} = 0$, we obtain

\[ \begin{align*}
\epsilon'_{yz} &= \epsilon_{yz} \cos \theta - \epsilon_{xz} \sin \theta \\
\epsilon'_{yz} &= \epsilon_{yz} \sin \theta + \epsilon_{xz} \cos \theta
\end{align*} \quad \ldots \ldots (6-50) \]

Similarly, we obtain

\[ \begin{align*}
\tau'_{yz} &= \tau_{yz} \cos \theta + \tau_{xz} \sin \theta \\
\tau'_{xz} &= -\tau_{yz} \sin \theta + \tau_{xz} \cos \theta
\end{align*} \quad \ldots \ldots (6-51) \]

Substituting Eq. (6-44) and Eq. (6-51) in Eq. (6-50), we have

\[ \begin{align*}
e'_{yz} &= \left( a_{44} \cos^2 \theta - 2 a_{45} \sin \theta \cos \theta + a_{55} \sin^2 \theta \right) \tau'_{yz} + \\
&\left( (a_{44} - a_{55}) \cos \theta \sin \theta + a_{45} (\cos \theta - \sin \theta) \right) \tau'_{xz}
\end{align*} \quad \ldots \ldots (6-52) \]

In order that Eq. (6-47) and Eq. (6-52) be identically equal, requires

\[ \begin{align*}
a'_{44} &= a_{44} (\cos^2 \theta - 2 a_{45} \sin \theta \cos \theta + a_{55} \sin^2 \theta) \\
a'_{45} &= \left( a_{44} - a_{55} \right) \cos \theta \sin \theta + a_{45} (\cos \theta - \sin \theta) \\
a'_{55} &= a_{44} \sin^2 \theta + 2 a_{45} \sin \theta \cos \theta + a_{55} \cos^2 \theta
\end{align*} \quad \ldots \ldots (6-53) \]
By choosing $\theta$ properly, we can make $a_{44}' = 0$; i.e., choose

$$\theta = \frac{1}{2} \tan^{-1} \frac{a_{45}}{a_{55} - a_{44}} \quad \text{....(6-54)}$$

This we have reached the conclusion that an anisotropic torsional problem which is defined by Eq. (6-44) can always be reduced to an orthotropic torsional problem by rotation the $x$-$y$ axes through an angle $\theta$ as given by Eq. (6-54).

Eq. (6-46) becomes

$$a_{44}' \frac{\partial \tilde{\sigma}_x}{\partial x^2} + a_{55}' \frac{\partial \tilde{\sigma}_y}{\partial y^2} = -\beta \quad \text{....(6-55)}$$

Let $2G = \frac{1}{\sqrt{a_{44}'a_{55}'}}$, $\chi' = \frac{\sqrt{a_{44}'}}{a_{55}'} \xi$, $\eta' = \frac{\sqrt{a_{55}'}}{a_{44}'} \eta \quad \text{....(6-56)}$

where $G$ is the "average" of the shear modulus. Substitute Eq. (6-56) in Eq. (6-55), giving

$$\frac{\partial^2 \tilde{\sigma}_x}{\partial \xi^2} + \frac{\partial^2 \tilde{\sigma}_y}{\partial \eta^2} = -2G \beta \quad \text{....(6-57)}$$

Now, through a coordinate rotation and scaling, an anisotropic torsion defined by Eq. (6-44) was finally transformed into an equivalent isotropic torsional problem which we know all about.

The boundary conditions remain unchanged as defined by Eq. (6-10).

Final stresses are

$$\tau_{x\xi}' = \frac{\partial \tilde{\sigma}_x}{\partial y'} = \frac{\sqrt{2G}}{a_{55}'} \left( \frac{\partial \tilde{\sigma}_x}{\partial y'} + \frac{\partial \tilde{\sigma}_y}{\partial \eta'} \right) = 0 + \sqrt{\frac{a_{44}'}{a_{55}'}} \frac{\partial \tilde{\sigma}_x}{\partial \eta'} = \sqrt{2G a_{44}'} \frac{\partial \tilde{\sigma}_x}{\partial \eta'} \quad \text{....(6-58)}$$

$$\tau_{\eta\eta}' = \sqrt{2G a_{55}'} \frac{\partial \tilde{\sigma}_y}{\partial \xi'}$$
By substituting Eq. (6-58), Eq. (6-54) in Eq. (6-51) the shear stresses \( \tau_{33} \) and \( \tau_{33} \) can be obtained.

(ii) The anisotropic constants are defined in the radial and the tangential directions. (34) Let \( u, v \) and \( w \) represent the displacements in \( r, \theta \) and \( z \) directions, and let them be given as

\[
\begin{align*}
  u &= 0 \\
  v &= \beta r z \\
  w &= \beta \varphi (r, \theta)
\end{align*}
\]

Substitute Eq. (6-59) into the strain displacement relations, giving

\[
\begin{align*}
  \epsilon_{33} &= \frac{\partial u}{\partial z} + \frac{\partial v}{\partial z} = \beta \frac{\partial w}{\partial z} \\
  \epsilon_{\theta3} &= \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial z} = \beta \left( \frac{1}{r} \frac{\partial \varphi}{\partial \theta} + \frac{\partial \varphi}{\partial z} \right)
\end{align*}
\]

The anisotropic strain-stress relations are

\[
\begin{align*}
  \sigma_{33} &= a_{44} \epsilon_{33} + a_{45} \epsilon_{\theta3} \\
  \sigma_{\theta3} &= a_{45} \epsilon_{33} + a_{55} \epsilon_{\theta3}
\end{align*}
\]

It can be shown, by substituting the stress components

\[
\tau_{rr} = \tau_{\theta\theta} = \tau_{z\theta} = \tau_{rz} = 0, \quad \tau_{r\theta} (r, \theta) \quad \text{and} \quad \tau_{\theta\theta} (r, \theta)
\]

into the equilibrium equations in cylindrical coordinates, that Prantl's stress function is valid; i.e.,

\[
\begin{align*}
  \tau_{r\theta} &= \frac{1}{r} \frac{\partial \varphi}{\partial \theta} \\
  \tau_{\theta\theta} &= -\frac{\partial \varphi}{\partial r}
\end{align*}
\]
Substitute Eq. (6-60) and Eq. (6-62) in Eq. (6-61), giving

\[
\begin{align*}
\beta \frac{\partial^2 \phi}{\partial \lambda^2} &= -a_{45} \frac{\partial \phi}{\partial \lambda} + a_{55} \frac{1}{\lambda} \frac{\partial \phi}{\partial \theta} \\
\beta \left( \frac{1}{\lambda} \frac{\partial \phi}{\partial \theta} + \frac{\lambda}{\beta} \right) &= a_{44} \frac{\partial \phi}{\partial \lambda} + a_{45} \frac{1}{\lambda} \frac{\partial \phi}{\partial \theta} \\
\end{align*}
\]

\((6-63)\)

Eliminating \(\phi\) from Eq. (6-63), we obtain the differential equation governing this problem as

\[
\begin{align*}
a_{44} \frac{1}{\lambda} \frac{\partial}{\partial \lambda} \left( \lambda \frac{\partial \phi}{\partial \lambda} \right) - 2a_{45} \frac{1}{\lambda} \frac{\partial^2 \phi}{\partial \lambda \partial \theta} + a_{55} \frac{1}{\lambda^2} \frac{\partial^2 \phi}{\partial \theta^2} &= -2\beta \\
\end{align*}
\]

\((6-64)\)

We change the independent variables \(r\) and \(\theta\) by letting

\[
\begin{align*}
\xi &= \lambda^\alpha \\
\omega &= \theta + \delta \cdot \ln \lambda \\
\end{align*}
\]

\((6-65)\)

where

\[
\begin{align*}
\alpha &= \frac{1}{a_{44}} \frac{1}{G} \\
\delta &= \frac{a_{45}}{a_{44}} \\
\gamma &= \frac{1}{\sqrt{a_{44} a_{45} - a_{45}^2}} \\
\end{align*}
\]

\((6-66)\)

By this transformation, we can see that a radial line in the \(r-\theta\) plane will map into a spiral line in the \(\xi-\omega\) plane. From Eq. (6-65), it follows

\[
\begin{align*}
\lambda \frac{\partial \phi}{\partial \lambda} &= \alpha \delta \frac{\partial \phi}{\partial \xi} + \delta \frac{\partial \phi}{\partial \omega} \\
\frac{1}{\lambda} \frac{\partial}{\partial \lambda} \left( \lambda \frac{\partial \phi}{\partial \lambda} \right) &= \frac{\alpha \delta}{\lambda} \frac{\partial}{\partial \xi} \left( \lambda \frac{\partial \phi}{\partial \lambda} \right) + \delta \frac{2}{\lambda} \frac{\partial}{\partial \omega} \left( \lambda \frac{\partial \phi}{\partial \lambda} \right) \\
\end{align*}
\]
\[
\begin{align*}
\frac{x^2 \phi_4^2}{\beta^2} + \frac{x^4 \phi_4^4}{\beta^4} + \frac{2 \delta \phi \phi_4^2}{\beta^2} + \frac{\phi^2 \phi_4^2}{\beta^2} + \frac{\phi^2 \phi_4^2}{\beta^2} \\
= \frac{i}{\beta} \phi \phi_4^2 + \frac{1}{\beta} \phi \phi_4^2 + \frac{2}{\beta} \phi \phi_4^2 \\
+ \frac{2}{\beta} \phi \phi_4^2 + \frac{1}{\beta} \phi \phi_4^2 \\
+ \frac{1}{\beta} \phi \phi_4^2
\end{align*}
\]

Substitute these into Eq. (6-64), giving

\[
\frac{\phi^2}{\beta} \left( \frac{\phi^2}{\beta^2} + \frac{1}{\beta} \phi \phi_4^2 + \frac{1}{\beta} \phi \phi_4^2 \right) = -2 \beta
\]

or

\[
\phi = -2 \beta \phi_4^2 \phi - 2 \phi_4 \phi \phi_4^2 \phi
\]

The solution of Eq. (6-67) is

\[
\phi = \phi_4 (f, \omega) + \phi_4 \phi_4 (f, \omega)
\]

where the complementary solution \( \phi_4 \) is a harmonic function which can be either an infinite series or singular integral solution as discussed in section (A). The particular solution...
\( \Phi_p(\varphi, \omega) \) can be represented as

\[
\Phi_p(\varphi, \omega) = -\frac{\beta}{2\alpha_{44}} \varphi^2 \alpha_{44} \delta
\]

\[\ldots (6-69)\]

The boundary condition is still defined by Eq. (6-10). The final stresses are

\[
\lambda_{n_2} = \frac{1}{\alpha} \frac{\partial \Phi}{\partial \theta} = \frac{1}{\alpha} \frac{\partial \Phi(\varphi, \omega)}{\partial \omega}
\]

\[
\lambda_{\theta_2} = -\frac{\partial \Phi}{\partial \varphi} = -\frac{1}{\alpha} \left[ \alpha \varphi \frac{\partial \Phi(\varphi, \omega)}{\partial \varphi} + \delta \frac{\partial \Phi(\varphi, \omega)}{\partial \omega} \right]
\]

\[\ldots (6-70)\]

Note that \( r, \varphi \) and \( \omega \) are used in the same equation, but they can be transformed into each other by using Eq. (6-65).
CHAPTER VII

APPLICATION TO PLATE BENDING

A) Theory and Equations, Transverse and Terminal Loading

A thin plate of thickness h is considered, whose median plane lies in the x-y plane, with z denoting the distance from the plane. We assume that the transverse load acting on the plate is normal to its surface and the deflection is small in comparison with the thickness h. Then, the basic assumptions are that line elements which are perpendicular to the middle surface before loading remain so after loading; i.e., the strain-displacement relations are

\[ \varepsilon_{xx} = \frac{1}{h} \frac{\partial^2 W}{\partial x^2} \]

\[ \varepsilon_{yy} = \frac{1}{h} \frac{\partial^2 W}{\partial y^2} \]

\[ \gamma_{xy} = -2 \frac{1}{h} \frac{\partial^2 W}{\partial x \partial y} \]

The linear stress-strain relations are

\[ \sigma_{xx} = \frac{E}{1-\nu^2} \left( \varepsilon_{xx} + \nu \varepsilon_{yy} \right) \]

\[ \sigma_{yy} = \frac{E}{1-\nu^2} \left( \varepsilon_{yy} + \nu \varepsilon_{xx} \right) \]

\[ \tau_{xy} = \frac{E}{2(1+\nu)} \gamma_{xy} \]

Using Eq. (7-1) and Eq. (7-2), the moments per unit length can be obtained as.
where \( D = \frac{E h^3}{12(1-\nu^2)} \). Note that the sign convention for \( M_{xy} \) is different from that of reference (17). The equilibrium equations of forces in the \( z \) direction and of moments about the \( x \) and \( y \) axes acting on an element of volume \((dx\cdot dy\cdot h)\) of the plate as shown in Fig. 7-1 are, respectively, as follows

\[
\begin{align*}
\sum F_3 &= 0, \quad \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0 \\
\sum M_x &= 0, \quad \frac{\partial M_{xy}}{\partial y} - \frac{\partial M_y}{\partial y} + Q_x = 0 \\
\sum M_y &= 0, \quad \frac{\partial M_{xy}}{\partial y} - \frac{\partial M_x}{\partial x} + Q_y = 0
\end{align*}
\]

where \( q(x,y) \) is the transverse load per unit of area and \( Q_x \) and \( Q_y \) are the shear forces per unit of length acting on a surface whose normal is indicated by the subscript, they are represented by

\[
\begin{align*}
Q_x &= -D \frac{\partial^2 w}{\partial x^2} \\
Q_y &= -D \frac{\partial^2 w}{\partial y^2}
\end{align*}
\]

In Eq. (7-4), we neglected the influence due to the in-plane forces which are potentially very important in thermal plate problems. If we do consider the in-plane forces, then the solution for the equation will become awfully complicated. In Chapter VIII of
reference (1), the solution of the thermal plate problem with in-plane forces was discussed together with some numerical examples.

Substitute Eq. (7-5) and Eq. (7-6) in Eq. (7-4), giving

\[ \frac{\partial^2 M_x}{\partial x^2} + z \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -q \] \quad \ldots (7-8)

Substituting Eq. (7-3), we obtain the governing equation for plate bending with small deflection as (no in-plane forces)

\[ D \nabla^4 w = q(x, y) \] \quad \ldots (7-9)

This equation was obtained by Lagrange in 1811, when he was
examining the memoir presented to the French Academy of Science by Sophie Germain.

The boundary conditions of a plate are as follows:

(i) Clamped edge, $w=0$, $\frac{\partial^2 w}{\partial n^2}=0$ ....(7-10)
(ii) Simply supported edge, $w=0$, $M_n=0$ ....(7-11)
(iii) Free edge, $M_n=0$, $V_n=0$ ....(7-12)

where $V_n$ is known as the Kirchhoff edge reaction and it is represented by

$$V_n = Q_n + \frac{\partial M_{nt}}{\partial x} ....(7-13)$$

At any unsupported corner, the boundary condition must be supplemented by the requirement that

$$R = -M_n t_1 + M_n t_2 = -D(1-\nu)\left(\frac{\partial^2 w}{\partial n^2 t_2} - \frac{\partial^2 w}{\partial n^2 t_1}\right) = 0 ....(7-14)$$

where the subscripts 1 and 2 indicate the values of $M_{nt}$ on face 1 and face 2 of the corner.

In Eq. (7-3) and Eq. (7-7), moments and shear forces are represented in $x$ and $y$ directions, but the boundary conditions usually require that these components are represented in normal or tangential directions. The transformation of the moments or shear force components in $x$ and $y$ directions to the components in $n$ and $t$ directions are shown below

$$\begin{align*}
\frac{\partial w}{\partial n} &= \cos \nu \frac{\partial w}{\partial x} + \sin \nu \frac{\partial w}{\partial y} \\
\frac{\partial w}{\partial x} &= -\sin \nu \frac{\partial w}{\partial x} + \cos \nu \frac{\partial w}{\partial y}
\end{align*}$$

....(7-15)
where $\psi$ is the angle which the outer normal to the boundary makes with the $z$-axis.

Now, we discuss the thin plate subjected to thermal loading, which problem has become very important recently particularly in the field of high speed space vehicles and the atomic energy of nuclear reactors. The temperature distribution need not be steady-state, but we ignore the time variable. In other words, we assume the thermal shock is small. This is true unless the body is subjected to a steep transient temperature gradient so that large thermal stresses are produced instantaneously. If a plate is subjected to transverse and thermal loading simultaneously, it is advisable to solve the plate problem due to the transverse loading and that due to the thermal loading independently, because the

\[
\begin{align*}
M_n &= \frac{M_x + M_y}{2} + \frac{M_x - M_y}{2} \cos 2\psi + M_{xy} \sin 2\psi \\
M_t &= \frac{M_x + M_y}{2} - \frac{M_x - M_y}{2} \cos 2\psi - M_{xy} \sin 2\psi \\
M_{nt} &= -\frac{M_x - M_y}{\lambda} \sin 2\psi + M_{xy} \cos 2\psi \\
Q_n &= \cos \psi \cdot Q_x + \sin \psi \cdot Q_y \\
Q_t &= -\sin \psi \cdot Q_x + \cos \psi \cdot Q_y
\end{align*}
\]

\[\begin{equation}
\begin{aligned}
M_n &= \frac{M_x + M_y}{2} + \frac{M_x - M_y}{2} \cos 2\psi + M_{xy} \sin 2\psi \\
M_t &= \frac{M_x + M_y}{2} - \frac{M_x - M_y}{2} \cos 2\psi - M_{xy} \sin 2\psi \\
M_{nt} &= -\frac{M_x - M_y}{\lambda} \sin 2\psi + M_{xy} \cos 2\psi \\
Q_n &= \cos \psi \cdot Q_x + \sin \psi \cdot Q_y \\
Q_t &= -\sin \psi \cdot Q_x + \cos \psi \cdot Q_y
\end{aligned}
\end{equation}\]
solutions can be obtained in terms of the plate dimensions, transverse loading, thermal loading and other physical constants and that it is easy to check the solutions. Since our discussion will be limited to the linear theory, we assume that the temperature distribution is

$$T = t_0(x, y) + \frac{1}{3} \cdot \xi_1(x, y)$$  \hspace{1cm} (7-18)

Then the stress-strain relations becomes

$$
\begin{align*}
\sigma_{xx} &= \frac{E}{1-\nu^2} \left[ (\epsilon_{xx} + \nu \epsilon_{yy}) - (1+\nu) \alpha T \right] \\
\sigma_{yy} &= \frac{E}{1-\nu^2} \left[ (\epsilon_{yy} + \nu \epsilon_{xx}) - (1+\nu) \alpha T \right] \\
\tau_{xy} &= \frac{E}{2(1+\nu)} \tau_{xy}
\end{align*}
$$

\hspace{1cm} (7-19)

and the corresponding moments per unit length are

$$
\begin{align*}
M_x &= -D \left( \frac{\partial^3 w}{\partial x^3} + \nu \frac{\partial^3 w}{\partial y^3} \right) - \frac{M_T}{1-\nu} \\
M_y &= -D \left( \frac{\partial^3 w}{\partial y^3} + \nu \frac{\partial^3 w}{\partial x^3} \right) - \frac{M_T}{1-\nu} \\
M_{xy} &= -D \left( 1-\nu \right) \frac{\partial^2 w}{\partial x^2 \partial y} \\
\end{align*}
$$

\hspace{1cm} (7-20)

where $M_T = \alpha \int_{-\frac{h}{2}}^{\frac{h}{2}} \tilde{T} \, d\tau = \frac{\alpha E h^3}{12} k_1(x, y) = \alpha D (1-\nu^2) t_1$  \hspace{1cm} (7-21)

Substitute Eq. (7-20) in Eq. (7-4) to Eq. (7-6) giving

$$D \nabla^4 w = -\frac{1}{1-\nu} \nabla^2 M_T$$

\hspace{1cm} (7-22)

If we consider that(37)

$$q(x, y) = -\frac{1}{1-\nu} \nabla^2 M_T$$

\hspace{1cm} (7-23)

then, Eq.(7-22) and Eq. (7-9) are identically the same. The boundary conditions for the thermally loaded plate are also represented by Eq. (7-10), Eq. (7-11), Eq. (7-12) and Eq. (7-14) as for the isothermal plate problem. As for the physical
components, the deflection and slope are represented in the same way for thermal and isothermal problems, but the moments and shear forces are represented by Eq. (7-3) and Eq. (7-7) for the isothermal case and by Eq. (7-20) and

$$Q_x = -D \frac{\partial}{\partial x} \nabla^2 \bar{w} - \frac{1}{1-\nu} \frac{\partial}{\partial x} M_T$$

$$Q_y = -D \frac{\partial}{\partial y} \nabla^2 \bar{w} - \frac{1}{1-\nu} \frac{\partial}{\partial y} M_T$$

for the thermal case.

B) Singular Integral Solution

The solution of Eq. (7-9) usually consists of a particular solution $\bar{w}_p$ and a complementary solution $\bar{w}_c$; i.e.,

$$\bar{w} = \bar{w}_p + \bar{w}_c$$

...(7-25)

For the case of thermal loading, the solution of Eq. (7-22) can also be represented by Eq. (7-25). However, many times, it is advantageous to find the particular solution from

$$D \nabla^2 \bar{w}_p = -\frac{1}{1-\nu} M_T$$

...(7-26)

A particular solution of Eq. (7-26) is of course also a particular solution of Eq. (7-22). But, in general, it is easier to find a particular solution from a second order differential equation than from a fourth order differential
equation. If a particular solution is obtained from Eq. (7-26), then, for a simply supported straight edge, $\frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial x}$ will vanish on the boundary and the zero bending moment in the normal direction can be represented as

$$M_n = -D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \frac{M_T}{1-\nu}$$

$$= -D \nabla^2 w - \frac{M_T}{1-\nu} = -D \nabla^2 w_c - D \nabla^2 w_p - \frac{M_T}{1-\nu} = 0$$

Substitute Eq. (7-26) into this equation, giving

$$M_n = -D \nabla^2 w_c + \frac{M_T}{1-\nu} - \frac{M_T}{1-\nu} = -D \nabla^2 w_c = 0 \quad \ldots (7-27)$$

Eq. (7-27) indicates that the unknown coefficients in the complementary solution of Eq. (7-25) will be so adjusted that it satisfies a zero boundary value of Eq. (7-27). A similar thing happens for the Kirchhoff free edge shear force; i.e.,

$$V_n = -D \frac{\partial}{\partial \eta} \nabla^2 w - \frac{1}{1-\nu} \frac{\partial}{\partial \eta} M_T$$

$$= -D \frac{\partial}{\partial \eta} \nabla^2 w_c - D \frac{\partial}{\partial \eta} \nabla^2 w_p - \frac{1}{1-\nu} \frac{\partial}{\partial \eta} M_T$$

$$= -D \frac{\partial}{\partial \eta} \nabla^2 w_c + \frac{1}{1-\nu} \frac{\partial}{\partial \eta} M_T - \frac{1}{1-\nu} \frac{\partial}{\partial \eta} M_T$$

$$= -D \frac{\partial}{\partial \eta} \nabla^2 w_c = 0 \quad \ldots (7-28)$$
Along the whole boundary, if most of the boundary values which the complementary solution has to satisfy is zero or very small, the complete solution is obviously easy to reach with good accuracy. Several numerical examples of plate bending problems subjected to thermal loading were illustrated in Reference (1) with the particular solution obtained from Eq. (7-26).

The complementary solution can be represented by a complete series as

$$\omega_c = A_0 + B_c r^2 + C_0 \ln r + D_0 r^2 \ln r + \left( A_1 r + B_1 r^3 + C_1 r^{-1} + D_1 r \ln r \right) \cos \theta$$

$$+ \left( A'_1 r + B'_1 r^3 + C'_1 r^{-1} + D'_1 r \ln r \right) \sin \theta$$

$$+ \sum_{n=2}^{\infty} \left( A_n r^n + B_n r^{n+1} + C_n r^{-n} + D_n r^{-n+2} \right) \cos n \theta$$

$$+ \sum_{n=2}^{\infty} \left( A'_n r^n + B'_n r^{n+1} + C'_n r^{-n} + D'_n r^{-n+2} \right) \sin n \theta \quad \cdots (7-29)$$

In Reference (1), many numerical examples of thermal and isothermal plate problems with irregular boundary shape were solved by employing a series like Eq. (7-29) or other type series using the point matching method or least square approximation.

Now, we shall discuss how to represent the complementary solution of Eq. (7-25) by singular integral equations which we introduced in Chapter IV. Consider a point load $P$ acting at the origin of the coordinates, the radial shearing force at a small
distance \( r \) from the load \( P \) is

\[ Q_n = -\frac{P}{2\pi r} \]

Observing the expression Eq. (7-7) for \( Q \) we can readily verify that the respective deflection is given by

\[ \mathcal{W}_l = \frac{P}{8\pi D} \frac{\kappa^2}{r^2} \ln \frac{A}{C} \]

in which \( C \) is an arbitrary length. Eq. (7-30) is one of the singular solution of the biharmonic equation and it can be considered as a complete solution of a circular plate with radius \( C \), subjected to a concentrated load \( P \) at the origin and having boundary conditions \( w=0 \) and

\[ \frac{\partial^2 w}{\partial r^2} = \frac{P}{\partial r} \left( 2\ln \frac{C_1^2}{C} + 1 \right) \]

at \( r=C \). As we did for the membrane problems, we can assume that there is a density function called \( \rho(B) \) distributed along the boundary, and consider \( r^2 \ln \frac{r}{C} \) be a kernel of the integral equation, then the deflection at any point \( A \) in the region or on the boundary due to the density function \( \rho(B) \) (line load along the boundary) can be represented as

\[ \mathcal{W}_l = \frac{1}{8\pi D} \int_{L} \rho(B) r^2(B,A) \ln \left[ r(B,A)/C \right] dL \]  

\[ \cdots \cdots (7-31) \]

Since \( \rho(B) \) is an unknown function, we can merge it with \( \frac{1}{8\pi D} \) and call the combined term \( \rho(B) \). Accordingly, Eq. (7-31) becomes

\[ \mathcal{W}_l = \int_{L} \rho(B) r^2(B,A) \ln \left[ r(B,A)/C \right] dL \]

\[ \cdots \cdots (7-32) \]
From the last section, we know that there are two independent boundary conditions for the plate programs (at a corner there may be more than two), hence we need two unknown functions along the boundary in order to satisfy two independent boundary conditions. Now, to find the second independent singular solution, (see Reference (17), p. 325) we apply a single force \(-N/\Delta x\) at the origin and a single force \(+N/\Delta x\) at the point \((-\Delta x, 0)\), assuming that \(M\) is a known couple. From the previous results (Eq. (7-30)), the deflection due to the combined action of both forces is:

\[
\mathcal{W}_z = \lim_{\Delta x \to 0} \left\{ \frac{M}{8 \pi D} \left[ (x + \Delta x)^2 + y^2 \right] \ln \left( \frac{(x + \Delta x)^2 + y^2}{c} \right)^{1/2} \right. \\
\left. - \frac{M}{8 \pi D} \frac{x^2 + y^2}{\Delta x} \ln \left( \frac{x^2 + y^2}{c} \right)^{1/2} \right\} \\
= \frac{M x}{8 \pi D} \left( \ln \frac{x^2 + y^2}{c^2} + 1 \right)
\]

where \(c\) is an arbitrary constant which need not be the same as \(c\) in Eq. (7-32). If we combine the second term of unity within the parentheses with the constant \(c\) and use polar coordinates, this expression becomes

\[
\mathcal{W}_z = \frac{M}{4 \pi D} \rho \frac{\rho}{c} \cos \theta
\]  
\text{....(7-33)}
Now let \( \frac{M}{f_{a,b}} \) be replaced by \( M \) and \( \theta \) be replaced by \( \varphi \), where \( \varphi \) is the angle from an arbitrary \( n \) direction to the point \( OA \) (see Fig. 7-2), Eq. (7-33) becomes

\[
\mathcal{W}_n = M \ln \frac{h}{C} \cos \varphi
\]

Taking of \( r \ln \frac{h}{C} \cos \varphi \) as a kernel and the direction \( n \) as the normal direction on the boundary, the deflection at an arbitrary point \( A \) due to a density function \( m(B) \) (boundary normal moments) along the boundary will be

\[
\mathcal{W}_n = \int m(B) r(B,A) \ln \frac{h(B,A)}{C} \cos \varphi(n,A) dA
\]

Now, we have two independent integral equations represented by Eq. (7-32) and Eq. (7-35) with two independent unknown density
function $p(B)$ and $m(B)$ along the boundary, these two density functions can be determined by letting Eq. (7-25) satisfy two sets of boundary conditions. Numerically, we can further divide the boundary into $N$ segments and using step function approximation; i.e., there will be $2N$ unknowns $p_1, p_2, \ldots, p_n, m_1, m_2, \ldots, m_n$, and these unknowns can be determined by satisfying the boundary conditions (Eq. (7-10) to Eq. (7-11)) at $N$ or more discrete points (each point can generate two or more boundary equations).

The details of the application are straightforward and are the same as in Chapter IV and Chapter VI. Unfortunately, the matrix generated by this method may turn out to be ill-conditioned as we discussed in Chapter II. Now, there are two questions one may ask: First, why the matrices generated in Chapter IV are always well-conditioned, but the matrices generated from Eq. (7-32) and Eq. (7-35) may become ill-conditioned? Second, what modification we can do to Eq. (7-32) and Eq. (7-35) so that they will always generate a well-conditioned matrix.

To answer the first question, consider the singular integral function of the membrane problem again, which is represented by Eq. (4-5) as $w=kP \ln \frac{r}{c}$ and graphically shown in Fig. (7-3). From this figure, we see that the singular function of Eq. (4-5) will have a larger deflection for the points near the singular point than for those far away from the singular point if $\frac{h}{c}$ is less than 1. If $r$ becomes larger than
C, then the deflection will increase as r increases. This is the reason we pick C=2.0 in the sample problems in Chapter IV, since the largest distance between boundary points is 2.23. For the membrane problem, the boundary condition is defined by \( \hat{w} \) (or for the torsion problem), therefore, the boundary equation for a given boundary point will have large coefficient elements contributed by singular loads near that point and small coefficient elements contributed by those away from that point. Fig. 4-6 is a typical example of the kind boundary equations encountered, which apparently indicate a well-conditioned matrix.

\[ f = \frac{f}{f} \]

\[ f = -\ln \frac{r}{c} \]

\[ f = -(\frac{r}{c})^{\ln} \frac{r}{c} \]

Fig. 7-3
For the problem of steady-state conductive heat transfer or for the motion of a non-viscous fluid (potential flow), the governing equation is the same as that for the membrane problem. But their boundary conditions may be defined partly by \( w \) (here \( w \) denotes the temperature for the heat transfer problem or the potential function of fluid) and partly by the derivative of \( w \) in the normal or tangential direction along the boundary. The normal and the tangential derivatives of \( w \) on the boundary can be obtained from the derivatives of \( w \) in polar coordinates as

\[
\begin{align*}
\frac{\partial w}{\partial n} &= \cos \phi \frac{2w}{\partial \nu^n} + \frac{1}{n} \sin \phi \frac{\partial w}{\partial \theta}, \\
\frac{\partial w}{\partial \nu^n} &= -\sin \phi \frac{3w}{\partial \nu^n} + \frac{1}{n} \cos \phi \frac{\partial w}{\partial \theta}
\end{align*}
\]

where \( \phi \) is the angle between the line joining the singular point and the boundary point and the normal direction of the boundary point. For the singular solution of the membrane problem (Eq. (4-5)), the derivatives of \( w \) in polar coordinates are

\[
\begin{align*}
\frac{\partial w}{\partial \nu^n} &= k \frac{w}{n}, \\
\frac{1}{n} \frac{\partial w}{\partial \theta} &= 0
\end{align*}
\]

where \( \nu^n/\nu_h \) is also shown in Fig. 7-3. We see that \( \frac{\partial w}{\partial \nu^n} \) will decrease as \( r \) increases, no matter whether \( \frac{h}{c} \) is less or greater than 1. But due to \( \frac{\partial w}{\partial \theta} = 0 \) and \( \phi \) of Eq. (7-36), the boundary equation of \( \frac{\partial w}{\partial n} \) or \( \frac{\partial w}{\partial \theta} \) for a given boundary point may not always have large coefficient elements contributed by singular
loads near that point, but always have small coefficient elements contributed by those away from that point. The corresponding coefficient matrix, therefore, will not be as well-conditioned as the coefficient matrices of the membrane or torsion problems. This is one of the major reasons, from the numerical analysis viewpoint that it is more difficult to solve a mixed boundary value problem than to solve a problem with the boundary conditions not mixed. For the problems of plate bending, the boundary conditions are usually mixed; i.e. the boundary conditions may be defined by any combination of \( w \), and its first, second and third derivatives.

Due to a single load, the singular solution for a plate is represented by Eq. (7-30) and shown in Fig. 7-3. One can see that the deflection of the plate at the point of the singular load is zero. As a result, if the boundary condition for a plate problem is defined by \( w \), then the matrix generated by the method of singular integral equations with Eq. (7-30) as the kernel will have zero or very small elements along the main diagonal. Similarly, if Eq. (7-34) is used as a kernel, the diagonal elements of the generated matrix will be zero or very small. This explains why the matrix generated in Chapter IV is always well-conditioned, and those generated by using Eq. (7-30) and Eq. (7-34) may become ill-conditioned. Actually,
Eq. (7-30) and Eq. (7-34) are unbounded at r=0 for stress components (derivatives of w), but not unbounded for w itself.

Now, we shall study how to modify Eq. (7-30) and Eq. (7-34) so that they will always generate a well-conditioned matrix. For a singular solution, if we add some other singular or non-singular function to it, the combined function will still be a singular solution provided that all the new terms satisfy the homogeneous differential equation. Now, we add several terms to Eq. (7-30), and let the singular solution become

\[ w_i = \left[ -0.5 - z(z) \cos \varphi - 1.5 \left( \frac{r}{2} \right)^2 + \left( \frac{r}{2} \right)^2 \ln \frac{r}{2} \right] P \quad \ldots (7-38) \]

where c is an arbitrary constant. If we scale all the linear dimensions by c, then Eq. (7-38) becomes

\[ w_i = \left[ -0.5 - z \cos \varphi - 1.5 r^2 + r^2 \ln r \right] P \quad \ldots (7-39) \]

The physical meaning of Eq. (7-39) is that it is a solution of a circular plate subjected to a point load at the origin with certain boundary values at r=1, in which we are not interested. The different orders derivatives of \( w_i \) with respect to \( r \) and \( \varphi = 180^\circ \) are shown in Fig. 7-4, from which we see that, if \( r<1 \), the singular solution of Eq. (7-39) always has a larger influence upon the near points in the components of \( w_i, \frac{\partial^2 w_i}{\partial r^2} \frac{\partial w_i}{\partial r}, \frac{\partial^3 w_i}{\partial r^3} \) and \( \frac{\partial^3 w_i}{\partial r^3} \) than to those points away from the singularity. The scale factor C should be chosen approximately equal to the
largest dimension of the boundary. Using Eq. (7-39) as a kernel, the generated matrix will be well-conditioned no matter what the boundary conditions may be.

Fig. 7-4
As for the second singular solution, physically it represents a solution due to a concentrated moment. We add some terms of the biharmonic functions to Eq. (7-3l') and let the combined singular solution be

\[ w(x) = \left(1 + \frac{A}{r} \cos \phi + (\frac{A}{r}) \ln \frac{A}{r} \cos \phi \right) M \quad \cdots (7-40) \]

Scale the linear dimension \( r \) by the constant \( C \). Eq. (7-40) then becomes

\[ w(x) = \left(1 + \frac{C}{r} \cos \phi + \ln r \cos \phi \right) M \quad \cdots (7-41) \]

For \( \phi = 180^\circ \), \( w, \partial w/\partial x, \partial^2 w/\partial x^2 \), and \( \partial^3 w/\partial x^3 \) are shown in Fig. 7-5, which imply that the generated matrix will be well-conditioned.

Now, with step function approximation, we divide the boundary into \( N \) segments, each segment having a density function value \( p_\lambda \) and \( m_\lambda \). Due to the density function \( p_\lambda \) and \( m_\lambda \) on segment \( S_\lambda \), with Eq. (7-39) and Eq. (7-41) as kernels respectively, the deflection and its derivatives (\( w, \partial w/\partial x, \partial^2 w/\partial x^2 \), \( \partial^3 w/\partial x^3 \cdots \)) at an arbitrary point in the region or on the boundary can be obtained. These expressions are lengthy and they are shown in the Appendix B. From the boundary conditions (Eq. 7-10) to Eq. (7-14), the unknowns \( p_1 \ldots p_N \), \( m_1 \ldots m_N \) can be obtained. Then from the equations in the Appendix B, all the physical components of the plate problem can be evaluated. One can see that, this method treats the multiply connected boundary the same way as it does the simply connected boundary. At the time
this writing, the computer program for this method will not yet be completed. Hence, no numerical example will be included here.

C) Plates On Elastic Foundations

A laterally loaded plate may rest on an elastic foundation, as in the case of a concrete highway pavement or an airport runway. If the intensity of the reaction of the ground is proportional to the deflection $w$ of the plate, this intensity is then given by the expression $kw$, where the constant $k$ is called the modulus of foundation. The $k$ values for various subgrades are given in Reference (17), p. 259.

Due to the reaction of the subgrade, the governing equation Eq. (7-9) for plates becomes

$$D \nabla^4 w = q - \bar{k} w$$  

...(7-12)

For a plate with polar symmetry and subjected to a concentrated load $P$ at the origin, the applied transverse load is zero everywhere except at the origin. Therefore, Eq. (7-12) becomes

$$D \left( \frac{d^4 w}{dr^4} + \frac{1}{h} \frac{d w}{dh} \right) = 0$$

or

$$\left( \frac{d^4}{dr^4} + \frac{1}{h} \frac{d}{dh} + \xi^4 \xi \right) \left( \frac{d^4 w}{dr^4} + \frac{1}{h} \frac{d w}{dh} - \xi^2 \xi \right) = 0$$  

...(7-13)

where $\xi^4 = \frac{k}{D}$  

...(7-14)
The solution of Eq. (7-43) are readily written down in modified Bessel function as
\[ W = c_1 I_0(\xi R) + c_2 I_1(\xi R) + c_3 K_0(\xi R) + c_4 K_1(\xi R) \]
\[ = c_1 \text{ber}_0(\xi R) + c_2 \text{bei}_0(\xi R) + c_3 \text{ker}_0(\xi R) + c_4 \text{kei}_0(\xi R) \]  
where the constants c_1, c_2, c_3 and c_4 will be determined by the boundary conditions. If the plate is infinitely large, then c_1=c_2=c_3=0 since ber_0 (\xi R) and bei_0 (\xi R) become unbounded when r increases, and ker_0 (\xi R) becomes infinitely large at the origin.

The solution of Eq. (7-45) becomes
\[ W = c_1 \text{ker}_0(\xi R) \]  
\[ ... (7-46) \]

For small values of argument we have
\[ \text{ber}_0(\xi R) = -\frac{(\xi R)^2}{4} \ln(\xi R) - \frac{\pi}{4} + (1+\ln 2-\gamma) \frac{(\xi R)^2}{4} + ... (7-47) \]
where \( \gamma = 0.5772157 \). For large values of the argument the following asymptotic expressions hold:
\[ \text{ker}_0(\xi R) \sim -\frac{\xi \eta^2}{\sqrt{2} \xi \eta / \pi} \sin(\frac{\xi \eta}{\sqrt{2}} + \frac{\gamma}{\eta}) \]  
\[ ... (7-48) \]

In order to determine the constant c_4, we calculate, by means of Eq. (7-7), the shearing force
\[ Q_n = -D \frac{d}{dR} \left( \frac{dW}{dR} + \frac{1}{R} \frac{dW}{dR} \right) = c_4 D \xi^3 \left( \frac{1}{\xi R} - \frac{\pi}{8} + ... \right) \]
As \( r \) decreases, the value of \( Q_R \) tends to \( \frac{C_4 D \xi^2}{L} \). On the other hand, upon distributing the load \( P \) uniformly over the circumference with radius \( r \), we have \( Q_R = \frac{P}{2 \pi R} \).

Equating both expressions obtained for \( Q_R \), we have

\[
C_4 = - \frac{P}{2 \pi D \xi^2}
\]

So, Eq. (7-46) becomes

\[
\omega_1' = - \frac{P}{2 \pi D \xi^2} k \ell_0' (\xi R)
\]

Now, consider a concentrated couple \( M \) acting at the origin of an infinite plate. As we derived Eq. (7-33), we can obtain the deflection function due to \( M \) easily by differentiating Eq. (7-49); i.e.,

\[
\omega_2 = M \frac{2}{\pi} \left[ - \frac{M}{2 \pi D \xi^2} k \ell_0 (\xi R) \right]
\]

\[
= - \frac{M}{2 \pi D \xi^2} \frac{3}{\pi} k \ell_0 (\xi R) \frac{d}{dn}
\]

\[
= - \frac{M}{2 \pi D \xi^2} \cos \varphi \ k \ell_0 (\xi R)
\]

where \( \varphi \) is the angle from a reference line \( n \) to the radial line as shown in Fig. 7-6.
Now, consider a finite plate with arbitrary shape resting on the elastic foundation and subjected to a finite number of concentrated forces, say K. Obviously, Eq. (7-49) can be used as a particular solution as

\[ M = \lim_{\Delta n \to 0} (P \cdot \Delta n) \]

where \( P \) is the concentrated force and \( r_i \) is the distance measured from the load \( P_i \) to an arbitrary point \( A \). To represent the complementary solution by a singular integral equation, we let \( \text{kei}_\nu (\xi r) \) of Eq. (7-49) and \( \cos \nu \cdot \text{kei} (\xi r) \) of Eq. (7-50) be two singular kernels, and \( p(B) \) and \( m(B) \) be two density functions along the boundary. The complementary solution can be written as
\[ W_c = \int p(B) \beta \omega_0 [E \eta(B, A)] dt + \int m(B) \cos \varphi(n, A) \beta \omega_0 [E \eta(B, A)] dt \quad \ldots (7-52) \]

where \( p(B) \) and \( m(B) \) can be determined by using the step function approximation. Substitute Eq. (7-51) and Eq. (7-52) into Eq. (7-25) and then satisfying the boundary conditions (Eq. (7-10) to Eq. (7-11)). Note that, for the numerical results, one need not actually perform the integration of Eq. (7-52). A good approximation can always be obtained by the summation of the numerical values at many boundary points as was done in section C, Chapter VI for the torsional rigidity.
A) Theory and Equations. Thermal and Isothermal Loading

If a thin plate is loaded by the boundary forces and the stress components \( \tau_{xx}, \tau_{yy}, \) and \( \tau_{xy} \) are zero, the state of stress is called generalized plane stress. On the other hand, consider a long prismatical bar subjected to the boundary forces which are perpendicular to the longitudinal axis. If the boundary forces do not vary along the length and the strain components \( \epsilon_{xx}, \epsilon_{yy}, \) and \( \gamma_{xy} \) are zero, the problem is called plane strain. The theory which applies to the generalized plane stress problems applies also to the plane strain problem.

Consider first the case of the generalized plane stress. The basic formulas of the equilibrium equations and the strain-displacement equations are

\[
\begin{align*}
\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} &= 0 \\
\epsilon_{xx} &= \frac{\partial u}{\partial x} \\
\epsilon_{yy} &= \frac{\partial v}{\partial y} \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}
\end{align*}
\]

\[\ldots (8-1)\]

\[\ldots (8-2)\]
From Eq. (8-2), we obtain the compatibility equation

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \quad \cdots (8-3)$$

By using Hooke's law, the stress-strain relations are represented as

\[
\begin{align*}
\varepsilon_{xx} &= \frac{1}{E} \left( \sigma_{xx} - \nu \sigma_{yy} \right) \\
\varepsilon_{yy} &= \frac{1}{E} \left( \sigma_{yy} - \nu \sigma_{xx} \right) \\
\gamma_{xy} &= \frac{1}{G} \tau_{xy} = \frac{2(1+\nu)}{E} \tau_{xy} \\
\end{align*}
\]

Substituting Eq. (8-4) in Eq. (8-3) and using the relation of Eq. (8-1), the compatibility equation becomes

$$\nabla^2 (\sigma_{xx} + \sigma_{yy}) = 0 \quad \cdots (8-5)$$

The stresses remain represented by Airy stress function as

\[
\begin{align*}
\sigma_{xx} &= \frac{\partial^2 \Phi}{\partial y^2} \\
\sigma_{yy} &= \frac{\partial^2 \Phi}{\partial x^2} \\
\tau_{xy} &= -\frac{\partial^2 \Phi}{\partial x \partial y} \\
\end{align*}
\]

Note that Airy stress function satisfies the equilibrium equations of Eq. (8-1). Then, Eq. (8-5) can be written as

$$\nabla^4 \Phi = 0 \quad \cdots (8-7)$$
Now, the generalized plane stress problem is to find a solution of Eq. (8-7) that satisfies certain boundary conditions defined by stress components and/or displacement components.

Let \( z = x + iy, \ z = x - iy, \ \text{where} \ i = \sqrt{-1}. \)

Then \( \nabla^4 \Phi = 16 \frac{\partial^4 \Phi}{\partial x^4} \) \( \ldots (8-8) \)

By successive integration of Eq. (8-8), we obtain

\[ \Phi(z, \bar{z}) = \frac{f_1(z) + \bar{f}_1(\bar{z})}{3} + \frac{f_2(z) + \bar{f}_2(\bar{z})}{3} + f_3(z) + f_4(\bar{z}) \] \( \ldots (8-9) \)

where \( f_1, f_2, f_3 \) and \( f_4 \) are arbitrary complex functions.

For the class of real functions, we rewrite Eq. (8-9) as

\[ \Phi(z, \bar{z}) = \Phi(x, y) = \frac{1}{2} \int \frac{\phi(z) + \overline{\phi(\bar{z})}}{3} + \frac{\chi(z) + \overline{\chi(\bar{z})}}{3} \] \( \ldots (8-10) \)

where \( \overline{\phi(\bar{z})} \) and \( \overline{\chi(\bar{z})} \) are the complex conjugate of \( \phi(z) \) and \( \chi(z) \), respectively.

Let

\[
\begin{align*}
\phi(z) &= \phi_1(x, y) + i \phi_2(x, y) \\
\overline{\phi(z)} &= \phi_1(x, y) - i \phi_2(x, y) \\
\chi(z) &= \chi_1(x, y) + i \chi_2(x, y) \\
\overline{\chi(z)} &= \chi_1(x, y) - i \chi_2(x, y)
\end{align*}
\] \( \ldots (8-11) \)

Substituting Eq. (8-11) and Eq. (8-12) in Eq. (8-10) gives

\[ \Phi = \chi_1 \phi_1(x, y) + \chi_2 \phi_2(x, y) + \chi_1(x, y) + \chi_2(x, y) \] \( \ldots (8-13) \)

where \( \chi_1 \) is a harmonic function, \( \phi_1 \) and \( \phi_2 \) are harmonic and conjugate; e.g., they satisfy the Cauchy-Riemann relations.
The idea of using two independent harmonic functions to represent a biharmonic function was first suggested by E. Goursat in 1898. According to Novozhilov(31), A.E.H. Love applied this idea to the plane elasticity.

From Eq. (8-13), we have

\[
\frac{\partial^2 \varphi}{\partial x^2} = 2 \frac{\partial^2 \varphi}{\partial y^2} + \kappa \frac{\partial^2 \varphi}{\partial x^2} + \gamma \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2},
\]

\[
\frac{\partial^2 \varphi}{\partial y^2} = 2 \frac{\partial^2 \varphi}{\partial y^2} + \kappa \frac{\partial^2 \varphi}{\partial x^2} + \gamma \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \psi}{\partial y^2},
\]

\[
\frac{\partial^2 \psi}{\partial x \partial y} = \kappa \frac{\partial^2 \psi}{\partial y^2} + \gamma \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y}.
\]

Substitute Eq. (8-15) in Eq. (8-6) and then in Eq. (8-4), we obtain

\[
\varepsilon_{xx} = \frac{1}{E} (\sigma_{xx} - \nu \sigma_{yy}) = \frac{1}{E} \left( \frac{\partial^2 \psi}{\partial y^2} - \nu \frac{\partial^2 \psi}{\partial y^2} \right)
\]

\[
= \frac{1}{E} \left[ (2 \frac{\partial^2 \varphi}{\partial y^2} + \kappa \frac{\partial^2 \varphi}{\partial x^2} + \gamma \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2} \right]
\]

\[
- \nu \left( 2 \frac{\partial^2 \varphi}{\partial y^2} + \kappa \frac{\partial^2 \varphi}{\partial x^2} + \gamma \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2} \right) \]

\[
= \frac{2}{E} \left( \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \psi}{\partial y^2} \right) - \frac{1 + \nu}{E} \left( 2 \frac{\partial^2 \varphi}{\partial y^2} + \kappa \frac{\partial^2 \varphi}{\partial x^2} + \gamma \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2} \right).
\]

Similarly, we have

\[
\varepsilon_{yy} = \frac{2}{E} \left( \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \psi}{\partial y^2} \right) - \frac{1 + \nu}{E} \left( 2 \frac{\partial^2 \varphi}{\partial y^2} + \kappa \frac{\partial^2 \varphi}{\partial x^2} + \gamma \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2} \right)
\]

\[
\gamma_{xy} = - \frac{2(1 + \nu)}{E} \left( \kappa \frac{\partial^2 \varphi}{\partial x \partial y} + \gamma \frac{\partial^2 \varphi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial x \partial y} \right).
\]
Let \( \omega = \omega_j = \frac{1}{2} \left( \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} \right) \)

\( \ldots (8-17) \)

where \( \omega \) is the rigid body rotation. From Eq. (8-2), Eq. (8-16) and Eq. (8-17), and with using Eq. (8-11), we have

\[
\frac{\partial \omega}{\partial x} = \frac{1}{2} \left( \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} \right) = \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}
\]

\[
= - \frac{1+\nu}{E} \left( \frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial y} \right) + \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{2}{E} \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)
\]

\[
+ \frac{1+\nu}{E} \left( \frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial y} \right) - \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^3 \psi}{\partial x^2 \partial y} \ldots (8-18)
\]

Similarly, we have

\[
\frac{\partial \omega}{\partial y} = \frac{1}{2} \left( \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 u}{\partial x \partial y} \right) = - \frac{1}{2} \frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial x^2}
\]

\[
\ldots (8-19)
\]

From Eq. (8-18) and Eq. (8-19), we obtain

\[
\omega = - \frac{4}{E} \frac{\partial \psi}{\partial y} + \omega_c = \frac{4}{E} \frac{\partial \psi}{\partial x} + \omega_c \ldots (8-20)
\]

where \( \omega_c \) is the constant of integration. From Eq. (8-2) and Eq. (8-16), we have

\[
\frac{\partial u}{\partial x} = \frac{E}{4} \frac{\partial^2 \psi}{\partial x^2} - \frac{1+\nu}{E} \left( \frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial y} \right) \ldots (8-21)
\]

From Eq. (8-17), Eq. (8-2) with using the relation of Eq. (8-16) and Eq. (8-20), we have

\[
\frac{\partial u}{\partial y} = \frac{1}{2} \frac{\partial^2 \psi}{\partial y^2} - \omega = \frac{4}{E} \frac{\partial^2 \psi}{\partial y^2} - \frac{1+\nu}{E} \left( \frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial y} \right) \ldots (8-22)
\]
From Eq. (8-21) and Eq. (8-22), we can find the displacement $u$ as

$$u = \frac{1}{E} \varphi_1 - \frac{1 + \nu}{E} \frac{\partial \varphi_2}{\partial x} - \omega_0 y + \xi_0$$  \hspace{1cm} \text{(8-23)}

Similarly, we obtain

$$v = \frac{1}{E} \varphi_2 - \frac{1 + \nu}{E} \frac{\partial \varphi_1}{\partial y} + \omega_0 x + \zeta_0$$  \hspace{1cm} \text{(8-24)}

where $\xi_0$ and $\zeta_0$ are constant of integration. In conclusion, the isothermal generalised plane stress problem is to find the harmonic functions $\varphi_1$, $\varphi_2$ and $\xi_1$ of Eq. (8-13), and these functions satisfy the boundary conditions defined by Eq. (8-6), Eq. (8-23) and Eq. (8-24).

Now, consider the plane thermal problem. As we discussed in Chapter VII, the temperature distribution need not be steady-state, but it does not vary along the $z$ axis. Let $T(x,y)$ be the temperature and the thermal stress-strain relations are

$$\varepsilon_{xx} = \frac{1}{E} \left( \frac{\partial T}{\partial x} - \nu \frac{\partial T}{\partial y} \right) + \alpha T$$

$$\varepsilon_{yy} = \frac{1}{E} \left( \frac{\partial T}{\partial y} - \nu \frac{\partial T}{\partial x} \right) + \alpha T$$

$$\gamma_{xy} = \frac{1}{G} \varepsilon_{xy} = \frac{2(1+\nu)}{E} \varepsilon_{xy}$$  \hspace{1cm} \text{(8-25)}

Substituting Eq. (8-25) into the compatibility equation Eq. (8-3) and using the relation Eq. (8-1) and Eq. (8-6),
the governing equation of thermal stress problem is found to be

$$\nabla^2 \Phi = -\alpha E \nabla^2 T \quad \ldots (8-26)$$

As we said before, $T(x,y)$ need not be steady state, therefore, $\nabla^2 T$ need not be zero. The solution of Eq. (8-26) consists of a complementary solution $\Phi_c$ and a particular solution $\Phi_p$.

$$\Phi = \Phi_p + \Phi_c \quad \ldots (8-27)$$

Note that $\Phi_c$ is the complete solution for the plane isothermal stress problem which is represented by Eq. (8-13). We further assume that $\Phi_p$ is the particular solution of

$$\nabla^2 \Phi = -\alpha E T \quad \ldots (8-28)$$

Any particular solution of Eq. (8-28) is certainly a particular solution of Eq. (8-26), but not vice versa.

To obtain $\Phi_p$ from Eq. (8-28) instead of Eq. (8-26) itself enables us to find the displacement functions $u$ and $v$ in a simple differential form of $\Phi_p$. Substituting Eq. (8-6) in Eq. (8-25) and using the relations of Eq. (8-27) and Eq. (8-28), we obtain
Differentiating Eq. (8-17) with respect to \( x \) and \( y \), and substituting Eq. (8-25) and Eq. (8-29) in it, we obtain

\[
\frac{\partial \omega}{\partial x} = \frac{1}{2} \left( \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \psi}{\partial x \partial y} \right) = \frac{1}{2} \frac{\partial \xi_{xx}}{\partial x} - \frac{\partial \xi_{xy}}{\partial y}
\]

\[
= -\frac{1}{2} \left[ \frac{\partial^3 \psi}{\partial x^3} + \frac{\partial^2 \psi}{\partial x^2 \partial y} + \frac{\partial^2 \psi}{\partial x \partial y^2} + \frac{\partial^2 \psi}{\partial y^3} \right] + \frac{1}{2} \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \psi}{\partial x \partial y}
\]

\[
= -\frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 \phi}{\partial x^2} + \frac{1}{2} \frac{\partial^2 \psi}{\partial x \partial y}
\]

\[
= -\frac{2}{\partial x} \frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 \phi}{\partial x^2}
\]

\[
\frac{\partial \omega}{\partial y} = \frac{1}{2} \left( \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \phi}{\partial y^2} \right) = -\frac{1}{2} \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{4 \partial^2 \phi}{\partial x \partial y}
\]

From Eq. (8-30), we find \( \omega \) to be the same function as in the isothermal case.
\[ 
\omega = -\frac{4}{E} \frac{\partial \phi_1}{\partial \gamma} + \omega_o = \frac{4}{E} \frac{\partial \phi_2}{\partial \gamma} + \omega_o \]  \quad \cdots (8-35)

Substituting Eq. (8-29) in Eq. (8-2), we obtain \( \frac{\partial u}{\partial \gamma} \), and from Eq. (8-2) and Eq. (8-17) by using the relations of Eq. (8-29) and Eq. (8-31), \( \frac{\partial u}{\partial \gamma} \) can be obtained. They are

\[
\frac{\partial u}{\partial \gamma} = \epsilon_{,\gamma} = \frac{4}{E} \frac{\partial \phi_1}{\partial \gamma} - \frac{1}{E} \left( \zeta \frac{\partial \phi_1}{\partial \gamma} + \frac{2}{E} \frac{\partial \phi_2}{\partial \gamma} \right) - \frac{1}{E} \frac{\partial \phi_1}{\partial \gamma^2} \]  \quad \cdots (8-32)

\[
\frac{\partial u}{\partial \gamma} = \frac{1}{2} \gamma_{,\gamma} - \omega = \frac{4}{E} \frac{\partial \phi_1}{\partial \gamma} - \frac{1}{E} \left( \zeta \frac{\partial \phi_1}{\partial \gamma} + \frac{2}{E} \frac{\partial \phi_2}{\partial \gamma} \right) - \frac{1}{E} \frac{\partial \phi_1}{\partial \gamma^2} - \omega - \frac{1}{E} \frac{\partial \phi_1}{\partial \gamma^2} \]  \quad \cdots (8-33)

Integrating Eq. (8-32) and Eq. (8-33), it yields

\[
u = \frac{4}{E} \phi_1 - \frac{1}{E} \frac{\partial \phi_1}{\partial \gamma} - \frac{1}{E} \frac{\partial \phi_1}{\partial \gamma^2} + \omega_o + \omega - \frac{1}{E} \frac{\partial \phi_1}{\partial \gamma} \]  \quad \cdots (8-34)

Similarly, we find

\[
u = \frac{4}{E} \phi_1 - \frac{1}{E} \frac{\partial \phi_1}{\partial \gamma} + \omega_o \gamma + \gamma - \frac{1}{E} \frac{\partial \phi_1}{\partial \gamma} \]  \quad \cdots (8-35)

\[
u = \frac{4}{E} \phi_1 - \frac{1}{E} \frac{\partial \phi_1}{\partial \gamma} + \omega_o \gamma + \gamma - \frac{1}{E} \frac{\partial \phi_1}{\partial \gamma} \]  \quad \cdots (8-35)

where \( \phi_c \) is \( \phi \) of Eq. (8-13), and \( \phi_p \) is the particular solution of Eq. (8-28). Comparing Eq. (8-23) and Eq. (8-24) with Eq. (8-34) and Eq. (8-35), we see that they differ only in that the latter equations each contains one additional term which represents the contribution of the thermal particular solution.
As for the case of plane strain, all of the equations which we have derived are valid if we let $E$ be replaced by $E_1$ and $\nu$ be replaced by $\nu_1$, where

\[
E_1 = \frac{E}{1-\nu^2} \quad \text{and} \quad \nu_1 = \frac{\nu}{1-\nu}
\] ....(8-36)

B) Series Solution and Singular Integral Solution

The complete series solution of Eq. (8-7) was obtained by Michell\(^{(15)}\) in 1899. Considering that the isothermal case is a special case of the thermal plane problem with $\Phi_p = 0$, we let $\Phi_c$ be the function represented by Eq. (8-13)

\[
\Phi_c = A_0 \ln r + B_0 r^2 + C_0 r^3 \ln r + D_0 r^2 \theta + A'_c \theta
\]

\[
+ \frac{A_1}{2} r \theta \sin \theta + (B_1 r^{-1} + C_1 r^3 + D_1 r \ln r) \cos \theta
\]

\[
- \frac{A'_1}{2} r \theta \cos \theta + (B'_1 r^{-1} + C'_1 r^3 + D'_1 r \ln r) \sin \theta
\]

\[
+ \sum_{n=2}^{\infty} \left( A_n r^n + B_n r^{-n} + C_n r^{n+2} + D_n r^{-n+2} \right) \cos n\theta
\]

\[
+ \sum_{n=2}^{\infty} \left( A'_n r^n + B'_n r^{-n} + C'_n r^{n+2} + D'_n r^{-n+2} \right) \sin n\theta \] ....(8-37)

Its corresponding harmonic and harmonic conjugate functions of $\chi_1$, $\phi_1$, and $\phi_2$ are found to be...
\[ \Psi_1 = A_0 \ln r + A_0' \theta + \frac{A_1}{4}(r \theta \sin \theta - r \ln r \cos \theta) - \frac{A_0'}{4}(r \theta \cos \theta + r \ln r \sin \theta) \]

\[ + \frac{D_1}{r}(-r \theta \sin \theta + r \ln r \cos \theta) + B_1 r^2 \cos \theta + B_1' r^{-2} \sin \theta + \frac{D_1'}{r}(r \theta \cos \theta + r \ln r \sin \theta) \]

\[ + \sum_{n=2}^{\infty} \left( A_n r^n + B_n r^{-n} \right) \cos n \theta + \sum_{n=2}^{\infty} \left( A_n' r^n + B_n' r^{-n} \right) \sin n \theta \quad \ldots \quad (8-38) \]

\[ q_1 = B_0 r \cos \theta + C_0(r \ln r \cos \theta - r \theta \sin \theta) + D_0(r \ln r \sin \theta + r \theta \cos \theta) \]

\[ + \frac{A_1}{4} \ln r - \frac{A_1'}{4} \theta + C_1 r^2 \cos 2 \theta + C_1' r^2 \sin 2 \theta + \frac{D_1}{2} \ln r - \frac{D_1'}{2} \theta \]

\[ + \sum_{n=2}^{\infty} \left[ C_n r^{n+1} \cos (n+1) \theta + D_n r^{-n+1} \cos (n-1) \theta \right] \]

\[ + \sum_{n=2}^{\infty} \left[ C_n' r^{n+1} \sin (n+1) \theta + D_n' r^{-n+1} \cos (n-1) \theta \right] \quad \ldots \quad (8-39) \]

\[ c_2 = B_0 r \sin \theta + C_0(r \ln r \sin \theta + r \theta \cos \theta) + D_0(r \theta \sin \theta - r \ln r \cos \theta) \]

\[ + \frac{A_1}{4} \theta + \frac{A_1'}{4} \ln r + C_1 r^2 \sin 2 \theta - C_1' r^2 \cos 2 \theta + \frac{D_1}{2} \theta + \frac{D_1'}{2} \ln r \]

\[ + \sum_{n=2}^{\infty} \left[ C_n r^{n+1} \sin (n+1) \theta - D_n r^{-n+1} \sin (n-1) \theta \right] \]

\[ + \sum_{n=2}^{\infty} \left[ C_n' r^{n+1} \cos (n+1) \theta + D_n' r^{-n+1} \cos (n-1) \theta \right] \quad \ldots \quad (8-40) \]
If boundary conditions are defined by stresses, then we have to find the components of the normal and the shear stresses in the normal direction in terms of the series Eq. (8-37). The components of the normal and shear stresses can be obtained from the transformation of the components of the stresses in the radial and the tangential directions.

\[
\sigma_{nn} = \frac{\partial^2 \bar{r}}{\partial x^2} = \sigma_n \cos^2 \phi + \sigma_\theta \sin^2 \phi + \sigma_{n\theta} \sin 2\phi
\]

\[
= \left(\frac{1}{\eta^2} \frac{\partial^2 \bar{r}}{\partial \theta^2} + \frac{1}{\eta^2} \frac{\partial \bar{r}}{\partial \theta} \right) \cos^2 \phi + \frac{\partial^2 \bar{r}}{\eta \partial \theta} \sin^2 \phi
\]

\[
- \frac{\partial}{\partial \eta} \left(\frac{1}{\eta} \frac{\partial \bar{r}}{\partial \theta} \right) \sin 2\phi
\]

\[
\tau_{nt} = -\frac{1}{2} (\sigma_n - \sigma_\theta) \sin 2\phi + \sigma_{n\theta} \cos 2\phi
\]

\[
= \frac{1}{2} \left( \frac{\partial^2 \bar{r}}{\partial \eta^2} - \frac{1}{\eta^2} \frac{\partial \bar{r}}{\partial \eta} - \frac{1}{\eta^2} \frac{\partial^2 \bar{r}}{\partial \theta^2} \right) \sin 2\phi
\]

\[
- \frac{\partial}{\partial \eta} \left(\frac{1}{\eta} \frac{\partial \bar{r}}{\partial \theta} \right) \cos 2\phi
\]

where \(\phi\) is the angle between the radial line and the normal line. Substituting Eq. (8-38), Eq. (8-39) and Eq. (8-40) in Eq. (8-37), and then in Eq. (8-27) and Eq. (8-41), we obtain...
\[ T_{nn} = \frac{\alpha_0}{2} \frac{\pi^2}{4} + A_0 \pi^{-2} \cos 2\phi + B_0 2 + C_0 (1 + \ln n) - \cos 2\phi + D_0 (2\theta - \sin 2\phi) \]

\[ + A'_0 \pi^{-2} \sin 2\phi + A_1 \pi^{-2} \cos^2 \phi \cos \theta - B_1 2 \pi^{-2} \cos (\theta - 2\phi) \]

\[ + C_1 2 \pi \left[ 2 \cos \theta - \cos (\theta + 2\phi) \right] + D_1 \pi^{-1} \left[ \cos \theta + \frac{1}{2} \cos (\theta - 2\phi) - \frac{1}{2} \cos (\theta + 2\phi) \right] \]

\[ + A'_1 \pi^{-1} \cos^3 \phi \sin \theta - B'_1 2 \pi^{-3} \sin (\theta - 2\phi) + C'_1 2 \pi \left[ 2 \sin \theta - \sin (\theta + 2\phi) \right] \]

\[ + D'_1 \pi^{-1} \left[ \sin \theta + \frac{1}{2} \sin (\theta - 2\phi) - \frac{1}{2} \sin (\theta + 2\phi) \right] \]

\[ + \sum_{n=2}^{\infty} \left\{ -A_n n \left( n-1 \right) \pi^{n-2} \cos (n\theta + 2\phi) - B_n n \left( n+1 \right) \pi^{-n+1} \cos (n\theta - 2\phi) \right\} \]

\[ + C_n \left( n+1 \right) \pi^n \left[ 2 \cos n\theta - n \cos (n\theta + 2\phi) \right] \]

\[ - D_n \left( n-1 \right) \pi^{-n} \left[ 2 \cos n\theta + n \cos (n\theta - 2\phi) \right] \}

\[ + \sum_{n=2}^{\infty} \left\{ -A'_n n \left( n-1 \right) \pi^{n-2} \sin (n\theta + 2\phi) - B'_n n \left( n+1 \right) \pi^{-n+1} \sin (n\theta - 2\phi) \right\} \]

\[ + C'_n \left( n+1 \right) \pi^n \left[ 2 \sin n\theta - n \sin (n\theta + 2\phi) \right] \]

\[ - D'_n \left( n-1 \right) \pi^{-n} \left[ 2 \sin n\theta + n \sin (n\theta - 2\phi) \right] \}\] \( 8-12 \)
\[ \tau_{nt} = -\frac{\partial^2 \phi}{\partial n^2} - A_0 n^{-2} \sin 2\phi + C_0 \sin 2\phi - D_0 \cos 2\phi + A_0' n^{-2} \cos 2\phi \]

\[ - A_1 \frac{1}{2} n^{-1} \cos \theta \sin 2\phi - B_1 2 n^{-3} \sin (\theta - 2\phi) + C_1 2 n \sin (\theta + 2\phi) \]

\[ + D_1 \frac{1}{2} n^{-1} [\sin(\theta + 2\phi) + \sin(\theta - 2\phi)] \]

\[ - A_1' \frac{1}{2} n^{-1} \sin \theta \sin 2\phi + B_1' 2 n^{-3} \cos (\theta - 2\phi) - C_1' 2 n \cos (\theta + 2\phi) \]

\[ - D_1' \frac{1}{2} n^{-1} [\cos(\theta + 2\phi) + \cos(\theta - 2\phi)] \]

\[ + \sum_{n=2}^{\infty} \left\{ A_n n^{-2} n(n-1) \sin(n\theta + 2\phi) - B_n n^{-2} n(n+1) \sin(n\theta - 2\phi) \right. \]

\[ + C_n n(n+1) n^{-1} n(n-1) \sin(n\theta - 2\phi)] \}

\[ + \sum_{n=2}^{\infty} \left\{ -A_n n^{-2} n(n-1) \cos(n\theta + 2\phi) + B_n n^{-2} n(n+1) \cos(n\theta - 2\phi) \right. \]

\[ - C_n n(n+1) n^{-1} n(n+1) \cos(n\theta - 2\phi)] \}

\[ \ldots (8-43) \]

As for the components of displacement, they can be obtained by substituting Eq. (8-38), Eq. (8-39) and Eq. (8-40) in Eq. (8-37) and Eq. (8-27), and then in Eq. (8-34) and Eq. (8-35)
\[ u = -\frac{1+\nu}{E} \frac{2\pi^3}{3\kappa} - \omega_0 y + u_o \]

\[ + \frac{1}{E} \left\{ -A_0 (1+\nu) \Lambda^{-1} \cos \theta + B_0 (1-\nu) \Lambda \cos \theta \right\} \]

\[ + C_0 \frac{h}{E} \left\{ 2 (1-\nu) \ln \cos \theta - \left[ 4 \theta \sin \theta + (1+\nu) \cos \theta \right] \right\} \]

\[ + D_0 \frac{h}{E} \left\{ 2 (1-\nu) \theta \cos \theta + (4 \ln \Lambda + \nu) \sin \theta \right\} \]

\[ + A_0' (1+\nu) \Lambda^{-1} \sin \theta + A_1 \left\{ \ln \Lambda + \frac{1}{2} \left[ 1+\nu \right] \sin^2 \theta \right\} + B_1 (1+\nu) \Lambda \cos \theta \]

\[ + C_1 \frac{h}{E} \left\{ (3-\nu) \cos 2\theta - 2 (1+\nu) \right\} + D_1 \left\{ (1-\nu) \ln \Lambda - (1+\nu) \cos \theta \right\} \]

\[ - A_1' \left\{ \frac{1}{2} (1-\nu) \theta + \frac{1}{2} (1+\nu) \sin 2\theta \right\} + B_1' (1+\nu) \Lambda \sin 2\theta + \]

\[ + C_1' (3-\nu) \Lambda^2 \sin 2\theta - D_1' \left\{ 2 \theta + \frac{1}{2} (1+\nu) \sin 2\theta \right\} \]

\[ + \sum_{n=2}^{\infty} \left\{ -A_n (1+\nu) \Lambda \cos (n-1) \theta + B_n (1+\nu) \Lambda \cos (n+1) \theta \right\} \]

\[ - D_n \left\{ (3-\nu) \cos (n-1) \theta + (1+\nu) \cos (n+1) \theta \right\} \Lambda^{-n+1} \}

\[ + D_n \left\{ (3-\nu) \cos (n-1) \theta + (1+\nu) \cos (n+1) \theta \right\} \Lambda^{-n+1} \}

\[ + \sum_{n=2}^{\infty} \left\{ -A_n' (1+\nu) \Lambda \sin (n-1) \theta + B_n' (1+\nu) \Lambda \sin (n+1) \theta \right\} \]

\[ - C_n' \left\{ (3-\nu) \sin (n+1) \theta - (1+\nu) \sin (n-1) \theta \right\} \Lambda^{-n+1} \}

\[ + D_n' \left\{ (3-\nu) \sin (n-1) \theta + (1+\nu) \sin (n+1) \theta \right\} \Lambda^{-n+1} \}

\[ \ldots \ldots \ldots (8-44) \]
\[ \nu = -\frac{1+\nu}{E} \frac{\partial^2 \varphi}{\partial \theta^2} + \omega_0 \nu + \nu_0 \]

\[ + \frac{1}{E} \left[ -A_0 (1+\nu) \lambda^{-1} \sin \theta + B_0 \frac{2}{(1-\nu)} \lambda \sin \theta + C_0 \left[ (1-\nu)(2\ln \lambda - 1) \sin \theta + 4\theta \cos \theta \right] \right] \]

\[ + D_0 \left[ (1-\nu) z \sin \theta - (1+\nu + 4 \ln \lambda) \cos \theta \right] \]

\[ - A'_0 (1+\nu) \lambda^{-1} \cos \theta + A_1 \left[ (1-\nu) \frac{\theta}{2} - \frac{1}{4} (1+\nu) \sin 2\theta \right] + B_1 (1+\nu) \lambda^{-2} \sin 2\theta \]

\[ + C_1 (3-\nu) \lambda^2 \sin 2\theta + D_1 \left[ 2 \theta - \frac{1}{2} (1+\nu) \sin 2\theta \right] \]

\[ + A'_1 \left[ -\ln \lambda + \frac{1}{4} (1+\nu) \cos 2\theta \right] - B'_1 (1+\nu) \lambda^{-2} \cos 2\theta \]

\[ - C'_1 \lambda^2 \left[ (3-\nu) \cos 2\theta + 2(1+\nu) \right] + D'_1 \left[ 2 \ln \lambda - (1+\nu) \left( \frac{1}{2} - \frac{1}{2} \cos 2\theta + \ln \lambda \right) \right] \]

\[ + \sum_{n=2}^{\infty} \left\{ A_n (1+\nu) \lambda^{n-1} \sin (n-1) \theta + B_n (1+\nu) \lambda^{-(n+1)} \sin (n+1) \theta \right. \]

\[ \quad + C_n \lambda^{n+1} \left[ (3-\nu) \sin (n+1) \theta + (1+\nu) (n+1) \sin (n-1) \theta \right] \]

\[ \quad + D_n \lambda^{-n+1} \left[ - (3-\nu) \sin (n-1) \theta + (1+\nu) (n-1) \sin (n+1) \theta \right] \}

\[ + \sum_{n=2}^{\infty} \left\{ -A'_n (1+\nu) \lambda^{n-1} \cos (n-1) \theta - B'_n (1+\nu) \lambda^{-(n+1)} \cos (n+1) \theta \right. \]

\[ - C'_n \lambda^{n+1} \left[ (3-\nu) \cos (n+1) \theta + (1+\nu) (n+1) \cos (n-1) \theta \right] \]

\[ \quad + D'_n \lambda^{-n+1} \left[ (3-\nu) \cos (n-1) \theta - (1+\nu) (n-1) \cos (n+1) \theta \right] \} \]

\[ \cdots \cdots \text{(8.45)} \]
The constants \( u_0 \), \( v_0 \) and \( \omega_0 \) represent the rigid body motion. If a party of the boundary is clamped and no rigid body motion will be allowed, then we can let \( u_0 = v_0 = \omega_0 = 0 \) at the boundary.

When the unknowns of the series have been determined, we can substitute them back into Eq. (8-42) to Eq. (8-45) to obtain the stresses and displacements of any point in the region. However, for stresses, it is more convenient to represent them in \( x \) and \( y \) components instead of in the normal components. Therefore, we obtain the stress functions in the \( x \) and \( y \) directions in terms of the series by substituting Eq. (8-37) to Eq. (8-40) in Eq. (8-27), and then in Eq. (8-6).

\[
G_{xx} = \frac{\partial^2 u}{\partial y^2} + A_0 \lambda^{-2} \cos \theta + B_0 \lambda^{2} + C_0 \left[2(1+|n|)\cos 2\theta\right] + D_0 (2\theta + \sin 2\theta)
\]

\[
- A_0 \lambda^{-2} \sin 2\theta + A_1 \lambda^{-1} \cos 3\theta - B_1 2\lambda^{-3} \cos 3\theta + C_1 2\lambda \cos \theta
\]

\[
+ D_1 \frac{1}{2} \lambda^{-1} (\cos \theta + \cos 3\theta) + A'_1 \lambda^{-1} \sin \theta \cos 3\theta - B'_1 2\lambda^{-3} \sin 3\theta
\]

\[
+ C'_1 2\lambda \sin \theta + D'_1 \frac{1}{2} \lambda^{-1} (3 \sin \theta + \sin 3\theta)
\]

\[
+ \sum_{n=2}^{\infty} \left\{-A_n \lambda^{-n-2} n(n-1) \cos(n-2)\theta - B_n \lambda^{-n+2} n(n+1) \cos(n+2)\theta \right.
\]

\[
+ C_n (n+1) \lambda^{-n} \left[2 \cos n \theta - n \cos(n-2)\theta\right]
\]

\[
+ D_n (n-1) \lambda^{-n} \left[-2 \cos n \theta - n \cos(n+2)\theta\right]\}
\]

\[
+ \sum_{n=2}^{\infty} \left\{-A'_n \lambda^{-n-2} n(n-1) \sin(n-2)\theta - B'_n \lambda^{-n+2} n(n+1) \sin(n+2)\theta \right.
\]

\[
+ C'_n (n+1) \lambda^{-n} \left[2 \cos n \theta - n \cos(n-2)\theta\right]
\]

\[
+ D'_n (n-1) \lambda^{-n} \left[-2 \cos n \theta - n \cos(n+2)\theta\right]\}
\]
\[ \sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2} - A_0 \lambda^{-2} \cos 2 \theta + B_0 \lambda + C_0 \left[ 2(1+\ln \lambda) + \cos 2 \theta \right] \]

\[ + A_0 \left( 2 \theta - \sin 2 \theta \right) + A_0 \lambda^{-2} \sin 2 \theta + A_1 \lambda^{-1} \cos \theta \sin \theta \]

\[ + B_1 \lambda^{-3} \cos 3 \theta + C_1 \lambda \cos \theta + D_1 \lambda^{-3} \left( 3 \cos \theta - \cos 3 \theta \right) \]

\[ + A_1 \lambda^{-1} \sin 3 \theta + B_1 \lambda^{-3} \sin 3 \theta + C_1 \lambda \sin \theta + D_1 \lambda^{-1} \left( \sin \theta - \sin 3 \theta \right) \]

\[ + \sum_{n=1}^{\infty} \left\{ A_n \lambda^{-n-1} n (n-1) \cos(n-2) \theta + B_n \lambda^{-n+1} n (n+1) \cos(n+2) \theta \right. \]

\[ + C_n \lambda^n (n+1) \left[ 2 \cos n \theta + n \cos (n+2) \theta \right] \]

\[ + D_n \lambda^{-n} (n-1) \left[ -2 \cos n \theta + n \cos (n-2) \theta \right] \]

\[ + \sum_{n=2}^{\infty} \left\{ A_{n-1} \lambda^{-n-2} n (n-1) \sin(n-2) \theta + B_{n-1} \lambda^{-n+2} n (n+1) \sin(n+2) \theta \right. \]

\[ + C_{n-1} \lambda^n (n+1) \left[ 2 \sin n \theta + n \sin(n-2) \theta \right] \]

\[ + D_{n-1} \lambda^{-n} (n-1) \left[ -2 \sin n \theta + n \sin(n+2) \theta \right] \]

\[ \ldots (B-47) \]
\[
\left\{ \begin{align*}
\theta &+ \sum_{n=0}^{\infty} \left( \cos(n+2)^2 \right) \\
&- \sum_{n=0}^{\infty} \left( \cos(n)^2 \right) \\
&+ \sum_{n=0}^{\infty} \left( \sin(n+2)^2 \right) \\
&- \sum_{n=0}^{\infty} \left( \sin(n)^2 \right)
\end{align*} \right.
\]

\[
- \frac{\pi}{2} \cos \theta - \frac{\pi}{2} \sin \theta
\]

\[
+ \pi \frac{1}{2} \sin \theta - \frac{1}{2} \cos \theta
\]

\[
- \frac{1}{2} \frac{\pi}{2} \sin \theta - \frac{1}{2} \cos \theta
\]

\[
+ \pi \frac{1}{2} \cos \theta + \frac{1}{2} \sin \theta - \frac{1}{2} \sin \theta - \frac{1}{2} \cos \theta
\]

\[
+ \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2}
\]
Now, for a given problem of plane elasticity, all we need to do is to select the proper terms of Eq. (8-37) and determine them by satisfying the boundary conditions of Eq. (8-42) to Eq. (8-43) at selected boundary points. However, as to which terms of Eq. (8-37) should be selected in a particular case, therefore, we shall give a detailed discussion of this question.

In Eq. (8-37), the low order terms (the first three lines of Eq. (8-37) have some special physical representation which are shown in Table (8-1), (16) and the high order terms $(n > 2)$ usually represent the Fourier components on the boundary. For a given problem, we should start to select the terms from the lowest order (the first line of Eq. (8-37)) and keep only those terms which do not violate the physical conditions of the problem until the total number of the terms reaches a certain arbitrary number. We can set this arbitrary number large if we want a rather accurate solution and small if we merely want a rough solution. For $n > 2$, it is easy to see which terms should be selected. For instance, if the problem has a geometrical symmetry with respect to the $x$-axis, we should select only the cosine terms, if the problem has symmetry with respect to both the $x$-axis and the $y$-axis, we should retain only the cosine terms with $n=2, 4, 6, ...$. If the origin of the series is at an
Table 8-1. Physical Meaning of Various Terms in Eq. (8-37)

\[ \bar{F}_c = A_0 n h + B_0 h^3 \]

\[ \bar{F}_c = A_0 \ln h + B_0 h + C_0 h^3 \ln h \]

\[ \bar{F}_c = (B_1 h^3 + C_1 h^5 + D_1 h \ln h) \sin \theta \]

\[ \bar{F}_c = (B_1 h^3 + C_1 h^5 + D_1 h \ln h) \cos \theta \]

\[ \bar{F}_c = \frac{A_1}{h} h^3 \sin \theta \]

\[ \bar{F}_c = \frac{A_1'}{h} h^3 \cos \theta \]

\[ \bar{F}_c = A_0' \theta \]

\[ \bar{F}_c = D_0 (h' \theta - A_1' \theta_1) \]
arbitrary point inside the region, then the negative order terms like $B_n$ and $D_n$ should not be retained since they introduce unwanted singularities at the origin. On the other hand, the positive order terms like $A_n$ and $C_n$ should be thrown away if the problem has an infinitely distance boundary with stress components bounded at infinity.

Among the terms in the first three lines of Eq. (8-37), $A_0 \ln r$, $B_0 r^{-1} \cos \theta$ and $B'_0 r^{-1} \sin \theta$ can be considered as the ordinary Fourier components with the negative order and $n=0$ for $A_0$, and $n=1$ for $B_0$ and $B'_0$. Similarly, $B_0 r^2$, $C_1 r^3 \cos \theta$ and $C'_1 r^3 \sin \theta$ can be considered as the ordinary Fourier components with the positive order and $n=0$ for $B_0$ and $n=1$ for $C_1$ and $C'_1$. The physical representation of $A_0 \theta$ is shown in Table (8-1) and it can be selected even though the problem does not have a polar symmetry. This term gives infinite shear stress $\tau_{\theta \phi}$ at $r=0$, so it should not be retained if the origin of the series is inside the region. The rest of the terms in the first three lines of Eq. (8-37) yield multivalued stresses, or displacements, or both. These terms and the components with the multivaluedness are shown in Table (8-2). The function $z$ itself has no physical meaning, so it can be multivalued. But the components of stresses or displacements can not be multivalued in the region. From Table (8-2), one can see that the $D_0$ term gives multivaluedness for $\epsilon_{m0}$, $u$ and $v$, $C_0$ term gives multivaluedness for $u$ and $v$ and $\omega$, $A_1$ and $D_1$ term give multivaluedness for $v$ only while $A'_1$ and $D'_1$ give multi-
Table 8-2 Terms of series Eq. (8-37) which give multiple valued stresses or displacements \( u, v, \omega \) in \( \frac{1}{E} \)

<table>
<thead>
<tr>
<th>Term</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_0 )</td>
<td>( \frac{\pi}{2} \ln r )</td>
</tr>
<tr>
<td>( \frac{\tau_{xx}}{} )</td>
<td>( * )</td>
</tr>
<tr>
<td>( \tau_{yy} )</td>
<td>( R { 2(1-\nu) \ln \cos \theta - 4\theta \sin \theta + (1+\nu) \cos \theta } )</td>
</tr>
<tr>
<td>( \tau_{zz} )</td>
<td>( (1-\nu)(2\ln r - 1) \sin \theta + 4\theta \cos \theta r )</td>
</tr>
<tr>
<td>( \omega )</td>
<td>( 4\theta )</td>
</tr>
<tr>
<td>( D_0 )</td>
<td>( \frac{\pi}{2} \theta \sin \theta )</td>
</tr>
<tr>
<td>( \frac{\tau_{xx}}{} )</td>
<td>( * )</td>
</tr>
<tr>
<td>( \tau_{yy} )</td>
<td>( R { 2(1-\nu) \theta \cos \theta + (4\ln r + 1+\nu) \sin \theta } )</td>
</tr>
<tr>
<td>( \tau_{zz} )</td>
<td>( (1-\nu)(2\ln r - 1) \sin \theta - (1+\nu + 4\ln r) \cos \theta r )</td>
</tr>
<tr>
<td>( \omega )</td>
<td>( / )</td>
</tr>
<tr>
<td>( A_1 )</td>
<td>( \frac{\pi}{2} \theta \cos \theta )</td>
</tr>
<tr>
<td>( \frac{\tau_{xx}}{} )</td>
<td>( * )</td>
</tr>
<tr>
<td>( \tau_{yy} )</td>
<td>( / )</td>
</tr>
<tr>
<td>( \tau_{zz} )</td>
<td>( / )</td>
</tr>
<tr>
<td>( \omega )</td>
<td>( / )</td>
</tr>
<tr>
<td>( D_1 )</td>
<td>( 2\theta - \frac{1}{2}(1+\nu) \sin \theta )</td>
</tr>
<tr>
<td>( \frac{\tau_{xx}}{} )</td>
<td>( / )</td>
</tr>
<tr>
<td>( \tau_{yy} )</td>
<td>( / )</td>
</tr>
<tr>
<td>( \tau_{zz} )</td>
<td>( / )</td>
</tr>
<tr>
<td>( \omega )</td>
<td>( / )</td>
</tr>
<tr>
<td>( A'_1 )</td>
<td>( -\frac{\pi}{2} \theta \cos \theta )</td>
</tr>
<tr>
<td>( \frac{\tau_{xx}}{} )</td>
<td>( * )</td>
</tr>
<tr>
<td>( \tau_{yy} )</td>
<td>( / )</td>
</tr>
<tr>
<td>( \tau_{zz} )</td>
<td>( / )</td>
</tr>
<tr>
<td>( \omega )</td>
<td>( / )</td>
</tr>
<tr>
<td>( D'_1 )</td>
<td>( -R \ln r \sin \theta )</td>
</tr>
<tr>
<td>( \frac{\tau_{xx}}{} )</td>
<td>( * )</td>
</tr>
<tr>
<td>( \tau_{yy} )</td>
<td>( / )</td>
</tr>
<tr>
<td>( \tau_{zz} )</td>
<td>( / )</td>
</tr>
<tr>
<td>( \omega )</td>
<td>( / )</td>
</tr>
</tbody>
</table>

* The dash indicates a single valued function
valuedness for \( u \) only. In order that all the components of stress or displacement be singlevalued in a multiply connected region when the origin is inside a hole, \( C_o \) and \( D_o \) terms should not be retained, but \( A_1, D_1, A'_1 \) and \( D'_1 \) can be retained with the following supplementary conditions

\[
\begin{align*}
A_1 &= -\frac{4}{1-\nu} D_1 \\
A'_1 &= -\frac{4}{1-\nu} D'_1 \\
\end{align*}
\]

\[\cdots (8.49)\]

With the conditions of Eq. (8.49), the combined term of \( A_1 \) and \( D_1 \) yields the components of stresses and displacements with the property of symmetry with respect to the \( x \)-axis while the components of the combined term of \( A'_1 \) and \( D'_1 \) have the property of symmetry with respect to the \( y \)-axis. As to the physical meaning, the former represents a concentrated force applied at the origin in the \( x \)-direction in an infinite plate, while the latter represents a concentrated force in the \( y \)-direction. These two combined terms should not be retained if the origin of the series is inside the region unless a force actually acts at the origin.

On the other hand, if the origin of the series is outside the region and is not enclosed by the region, all of the terms in Table 8-2 can be retained. If the problem has some geometrical symmetry, certainly only the terms which possess the same symmetry property should be selected. As to obtaining the exact solution for plane elasticity problem having a multiply-Connected region, the analytical properties of the multivaluedness of the function
and its stress on displacement components have to be studied. This was discussed by N.I. Muskhelishvilli. (36)

If the region is simply connected or doubly connected, we can use one series to represent the solution as we did in Chapter I. If the plate contains more than one hole, then there should be at least one series with the origin in each hole. This multiple pole scheme was generalized by A.E. Green (39,40) (1940) when he analytically solved the problem of a plate with an arbitrary number of circular holes. For the numerical solution, L.E. Hulbert (6) (1963) applied this scheme together with the point matching method or least square method so to make the analysis simple and straightforward. Hulbert investigated the problem of plates with non-circular holes. Both of them studied only the plane stress problem of the first kind; i.e., the boundary conditions are defined by stresses only. The work presented in Section A and here is an extension of this method to solve the plane problem with non-circular holes and mixed boundary conditions, together with a discussion on the selection of the terms of the series. (Hulbert has recently extended his work to the plane problem with mixed boundary conditions also although this work has not been published). It may be noted that, when more than one series are used for a plate with a finite outer boundary, Green suggested that only one of the series should have the positive order terms. However, for the non-circular outer boundary, if we satisfy the boundary conditions approximately by using
a finite number of terms, we may use several series with the positive order terms. But no matter which way we do, experience shows that the problem of ill-conditioned matrices always occurs if the holes or the outer boundary are far from being circular.

In order to solve problems having irregularly shaped boundaries, the singular integral solution will be introduced in the following.

As discussed above, if the condition of Eq. (8-49) is satisfied, the combined term of \( \frac{A_1}{x} \sin\theta \) and \( D_1 h |\ln h \cos\theta| \), and the combined term of \( -\frac{A'_1}{x} \cos\theta \) and \( D'_1 h |\ln h \sin\theta| \) represent the concentrated forces in an infinite plate in the x and y directions respectively.

If we use these two combined terms as kernels, physically they represent certain forces applied along certain contours (boundaries) in an infinite plate so that the resulting stresses or displacements on these contours are identically equal to the desired boundary conditions of a plane problem. Now, we let \( f_1 \) and \( f_2 \) represent two singular kernels for plane elasticity as

\[
\begin{align*}
\{f_1 &= -\frac{2}{1-\nu} h \theta \sin\theta + h |\ln h \cos\theta| \\
\{f_2 &= \frac{2}{1-\nu} h \theta \cos\theta + h |\ln h \sin\theta|
\end{align*}
\]

...(8-50)

Due to \( f_1 \), the normal and shear stresses in the \( n \) direction and the displacements \( u_1 \) and \( v_1 \) can be obtained from Eq. (8-12) to Eq. (8-15) with using the relation of Eq. (8-49)
Similarly, the normal and shear stresses and the displacements due to \( f_2 \) can be found as

\[
\sigma_1 = \frac{1}{h} \left[ -\frac{4}{1-\nu} \cos\phi \cos \theta + \cos \theta + \frac{1}{4} \cos(\theta - 2\phi) - \frac{1}{2} \cos(\theta + 2\phi) \right]
\]

\[
\tau_1 = \frac{1}{h} \left[ \frac{2}{1-\nu} \cos \theta \sin \phi + \frac{1}{2} \sin(\theta + 2\phi) + \frac{1}{2} \sin(\theta - 2\phi) \right]
\]

\[
\nu_1 = (1 - \nu) \ln h - \ell(1 + \nu) \left( \cos^2 \theta + \frac{2}{1-\nu} \sin^2 \theta \right)
\]

Then, we can determine two density functions \( p_1(t) \) and \( p_2(t) \) along the boundary by satisfying any combination of the boundary conditions of Eq. (8-53) to Eq. (8-56).
where \( (\sigma_{mn})_p, (\tau_{nt})_p \), \( u \) and \( v \) are the components of stresses and displacements of the particular solution due to thermal loading, and \( \sigma_{nn}, \tau_{nt} \), \( u \) and \( v \) are values on the boundary.

After \( p_1(t) \) and \( p_2(t) \) have been obtained, we can compute the stresses or displacements at any point in the region or on the boundary from Eq. (8-53) to Eq. (8-56). If we like to have the stresses in \( x \) and \( y \) coordinates, we can use Eq. (8-57) as

\[
\begin{align*}
\sigma_{xx} &= \int_0^\infty \left( (\sigma_{x1} \, \rho_1 + \sigma_{x2} \, \rho_2) \, dt + (\sigma_{xx})_p \right) \\
\sigma_{yy} &= \int_0^\infty \left( (\sigma_{y1} \, \rho_1 + \sigma_{y2} \, \rho_2) \, dt + (\sigma_{yy})_p \right) \\
\tau_{xy} &= \int_0^\infty \left( (\tau_{x1} \, \rho_1 + \tau_{x2} \, \rho_2) \, dt + (\tau_{xy})_p \right)
\end{align*}
\]

where \( \sigma_{x1}, \sigma_{x2}, ... \) can be obtained from Eq. (8-50) and Eq. (8-52) to Eq. (8-55). They are

\[
\begin{align*}
\sigma_{x1} &= \frac{1}{\mu} \left[ -\frac{4}{1-\nu} \cos^3\theta + \frac{1}{2} \cos\theta + \frac{1}{2} \cos 3\theta \right] \\
\sigma_{x2} &= \frac{1}{\mu} \left[ -\frac{4}{1-\nu} \sin\theta \cos^2\theta + \frac{3}{2} \sin\theta \cos^2\theta + \frac{1}{2} \sin 3\theta \right] \\
\sigma_{y1} &= \frac{1}{\mu} \left[ -\frac{4}{1-\nu} \cos \sin^2\theta + \frac{3}{2} \cos \sin^2\theta - \frac{1}{2} \cos 3\theta \right] \\
\sigma_{y2} &= \frac{1}{\mu} \left[ -\frac{4}{1-\nu} \sin \sin^2\theta + \frac{1}{2} \sin \sin^2\theta - \frac{1}{2} \sin 3\theta \right] \\
\tau_{xy1} &= \frac{1}{\mu} \left[ -\frac{2}{1-\nu} \cos \sin 2\theta + \frac{1}{2} \sin 2\theta - \frac{1}{2} \sin 3\theta \right] \\
\tau_{xy2} &= \frac{1}{\mu} \left[ -\frac{2}{1-\nu} \sin \sin 2\theta - \frac{1}{2} \cos - \frac{1}{2} \cos 3\theta \right]
\end{align*}
\]
It may be noted that, from Eq. (8-51) and Eq. (8-52), the singular force will not have a big influence for v and u at the near points. This will make the condition of the coefficient matrix suffer if the boundary conditions are defined by displacements.

C Numerical Example

Consider a short cantilever beam subjected to the uniform load as shown in Fig. 8-1. If the length of the beam is great, then the elementary strength of materials theory can be used to find the stresses at the points away from the ends of the beam. However, if the beam is short, the strength of materials is no longer valid. But it is interesting to see how much difference exists between the results obtained from the elementary theory and those obtained from the elasticity theory. For this sample problem, Young's modulus $E$ is $30 \times 10^6$ psi, and Poisson's ratio $\nu$ is 0.3.

We use the series of Eq. (8-37) retaining $B_0$, $C_1$, $C'_1$, $A_2$, $A_3...A_{16}$, $C_2$, $C_3...C_{16}$ for a total of 33 terms, and determine those unknowns by approximately satisfying 82 boundary equations which are generated from 40 equally spaced points (each of the free corner points has three boundary conditions). The results of stresses and displacements along the middle of the beam and along the clamped edge are shown in Fig. 8-1 and Fig. 8-2. The results obtained from the strength of materials
theory are also shown in the same figures for comparison.
Except for the shear stresses along the middle section, their results are quite different.
displacement in \(10^{-7} \text{m} \cdot \text{p} \), Stress in \(\text{p} \)

\[ -10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8 \]

\[ V \]

\[ \Gamma_y \]

\[ \Gamma_{xy} \text{ (SOM)} \]

\[ V_{(SOM)} \text{ (Bending only)} \]

\[ \sigma_x \]

\[ y \]

\[ y \]

\[ \text{SECTION A-A} \]

\[ \text{Fig. 8-1} \]

Note: SOM = Strength of materials
displacement in $10^{-7}$ m, stress in $\sigma$

SECTION B-B  Fig. 8-2
APPENDIX A

FREDHOLM INTEGRAL EQUATION AND GREEN'S FUNCTION

An integral equation is one which contains the unknown function behind the integral sign. Its importance for physical problems lies in the fact that most differential equations together with their boundary conditions may be reformulated to give a single integral equation. The theory of integral equations also furnishes a uniform method for the study of the eigenvalue problems of mathematical physics.

A linear integral equation has the following forms:

\[ f(x) = \int_{a}^{b} k(x,z) \phi(z) \, dz \]  
\[ \phi(z) = f(z) + \lambda \int_{a}^{b} k(x,z) \phi(z) \, dz \]
\[ g(x) \phi(z) = f(x) + \lambda \int_{a}^{b} k(x,z) \phi(z) \, dz \]

where \( g(x) \), \( f(x) \) and \( k(x,z) \) are known functions, the latter is called the kernel. Eq. (A-1) and Eq. (A-2) are known as the first kind and the second kind of Fredholm integral equations. Actually, they are the special cases of the third kind of integral equation Eq. (A-3) for \( g(x) = 0 \) and \( g(x) = 1 \) respectively. If we replaced the lower and upper limits \( a \) and \( b \) of the integral by 0 and \( x \), the corresponding integral equations are called
Volterra's integral equations of the respective kind and can be similarly treated. These names are used to honor the principal founders of the theory of integral equations Vito Volterra (1862-1940) and Ivar Fredholm (1866-1927). Among the others, the contributions by David Hilbert (1862-1943) and Erhard Schmidt (1876-1960) are the most important.

Though as stated by Tricomi (1957): "Some mathematicians still have a kind of fear whenever they encounter a Fredholm integral equation of the first kind. Today, this fear is no longer justified...." However, most of the authors of works on integral equations still limit their discussion to integral equations of the second kind. In the recent years, many methods have appeared concerning to the solution of integral equations. Here, we shall introduce briefly the most important one: successive approximations by the Neumann's series.\(^{(20)}\) (26)

For the Fredholm integral equation of the second kind Eq. (A-2), we set

\[
\phi(x) = \gamma_0(x) + \lambda \gamma_1(x) + \lambda^2 \gamma_2(x) + \lambda^3 \gamma_3(x) + \cdots \quad \text{....(A-4)}
\]

Eq. (A-4) is called Neumann series. Substituting Eq. (A-4) into Eq. (A-2) and equating coefficients of equal powers of \(\lambda\), we obtain

\[
\begin{align*}
\gamma_0(x) &= \int_a^b K(x, z) \gamma_0(z) \, dz \\
\gamma_1(x) &= \int_a^b K(x, z) \gamma_1(z) \, dz \\
\gamma_2(x) &= \int_a^b K(x, z) \gamma_2(z) \, dz \\
&\vdots \\
\gamma_n(x) &= \int_a^b K(x, z) \gamma_n(z) \, dz
\end{align*}
\] \quad \text{....(A-5)}
where \( x \) and \( z \) are restricted to lie between \( a \) and \( b \). We see that the kernel and \( f(x) \) must have maximum values, for we assumed them to be continuous. Let these maxima be given by \(|K(x, z)| \leq M_j\), \(|f(x)| \leq N\). Then it follows that \(|\psi_0(x)| \leq N; |\psi_1(x)| \leq NM(b-a); \ldots |\psi_n(x)| \leq N[M(b-a)]^n\).

If \(|\lambda| < \frac{1}{M(b-a)}\) the series Eq. (A-4) converges uniformly and is the unique continuous solution of Eq. (A-2) within the range \(a \leq x \leq b\).

In order to obtain the solution in more convenient form, we define the iterated kernels:

\[
\begin{align*}
K_1(x, z) &= K(x, z) \\
K_2(x, z) &= \int_a^b k(x, y) K(y, z) \, dy \\
& \quad \vdots \\
K_n(x, z) &= \int_a^b k(x, y) K_{n-1}(y, z) \, dy \\
& \quad \vdots \\
& = \int_a^b \cdots \int_a^b k(x, y_1) k(y_1, y_2) \cdots k(y_{n-1}, y_n) \, dy_1 \cdots dy_n.
\end{align*}
\]

Introducing these functions into Eq. (A-5) we may write

\[
\begin{align*}
\psi_1(x) &= \int_a^b K_1(x, z) f(z) \, dz \\
\psi_2(x) &= \int_a^b K_2(x, z) f(z) \, dz \\
& \quad \vdots \\
\psi_n(x) &= \int_a^b K_n(x, z) f(z) \, dz
\end{align*}
\]

As before, we see that \(|K_n(x, z)| \leq M^n(b-a)\); hence if Eq. (A-6) is fulfilled we can construct a uniformly convergent series.
called the resolvent.

\[ K(x, \delta; \lambda) = \sum_{n=0}^{\infty} \lambda^n k_{n+1}(x, \delta) \quad \ldots \quad (A-9) \]

From Eq. (A-4), Eq. (A-7) and Eq. (A-9), it follows that the solution of the integral equation Eq. (A-2) is

\[ \phi(x) = f(x) + \lambda \int_{a}^{b} K(x, \delta; \lambda) f(\delta) \, d\delta \quad \ldots \quad (A-10) \]

Eq. (A-10) represents the classic method for the solution of integral equations of the second kind.

At the beginning of this appendix, we said that most differential equations can be reformulated as integral equations so that we may state physical problems in either form at will. Here, we shall show a simple example to reformulate a second order differential equation

\[ y'' = f(x, y) \quad \ldots \quad (A-11) \]

into an integral equation. Integrate Eq. (A-11), giving

\[ y'(x) = \int_{0}^{x} f \{ \delta, y(\delta) \} \, d\delta + c_1 \]

\[ y(x) = \int_{0}^{x} \left[ \int_{0}^{\delta} f \{ \zeta, y(\zeta) \} \, d\zeta + c_1 \delta \right] \, d\delta + c_1 x + c_2 \]

An alternative form of the last expression is

\[ y(x) = \int_{0}^{x} (x - \delta) f \{ \delta, y(\delta) \} \, d\delta + g(x) \quad \ldots \quad (A-12) \]

where \[ g(x) = c_1 x + c_2 \]

One can see that a second order non-linear differential equation Eq. (A-11) has been reformulated as a non-linear Volterra integral equation of the second kind Eq. (A-12). The boundary conditions which are needed to determine the two integration constants \( c_1 \) and \( c_2 \), may be of either of two types:
(a) \( y \) and \( y' \) are fixed at one point, say at \( x = 0 \) (initial value problems);

(b) \( y \) is fixed at two points (boundary value problems).

The first case is simple, for if \( y(0) = a, y'(0) = b \),

Eq. (A-12) becomes

\[
 y(x) = \int_0^x (x - \xi) f \left[ \int_0^\xi \frac{d\xi}{(\xi - \xi')} \right] d\xi + b x + a 
\]

...(A-13)

The second case leads to greater difficulties. Suppose \( y(0) = a, y(1) = b \); then \( c_i = a \) as before.

For \( x = 1 \), we have

\[
 b = y(1) = \int_0^1 (1 - \xi) f d\xi + c_i + a 
\]

\[
 c_i = (b - a) - \int_0^1 (1 - \xi) f d\xi 
\]

where \( f = \{ z, y(z) \} \). Substituting the values of \( c_i \) and \( c_z \) into Eq. (A-12) we obtain

\[
 y(x) = h(x) + \int_0^x (x - \xi) f d\xi + \int_0^1 (\xi - 1) f d\xi 
\]

\[
 = h(x) + \int_0^x (x - \xi) f d\xi + \int_0^x (\xi - 1) f d\xi 
\]

where \( h(x) = a + (b - a)x \). We thus see that in this case, if we are willing to divide the range of \( x \) into two parts with a different kernel for each part,

\[
 K(x, \xi) \left\{ \begin{array}{ll}
  = \xi (\xi - 1) & \xi > \xi \\
  = x (\xi - 1) & \xi \leq \xi 
\end{array} \right. 
\]

Eq. (A-12) becomes a Fredholm integral equation of the second kind

\[
 y(x) = h(x) + \int_0^1 K(x, \xi) f \left[ \int_0^{\xi} g(\eta) \right] d\xi 
\]

...(A-15)
From this example, we see that there is a lot of work involved to find the kernel if we want to reform a differential equation together with the boundary condition into an integral equation.

There is a special kernel which can be treated systematically which is known as a Green's function. To introduce a Green's function, we consider a tightly stretched string which is governed by the differential equation

\[ y'' = - \frac{f(y)}{T} \]  \hspace{1cm} \text{(A-16)}

with a boundary condition \( y=0 \) at \( x=0 \) and \( x=L \). Now, if we consider a load \( f(z) \) acting at point \( z \) of the string, then the deflection, at point \( x \) is proportional to

\[ G(x,z) = \begin{cases} \frac{1}{2} - \frac{1}{2} \left( \frac{x}{L} \right)^2 & \text{for } 0 \leq x \leq \frac{L}{2} \\ \frac{1}{2} - \frac{1}{2} \left( \frac{x}{L} \right)^2 & \text{for } \frac{L}{2} \leq x \leq L \end{cases} \]

The function \( G(x,z) \) is called the Green's function for this problem. The deflection at any given point of the string is

\[ y(x) = \frac{1}{T} \int_0^L G(x,z) f(z) \, dz \]  \hspace{1cm} \text{(A-17)}

One may say that Eq. (A-17) is an ordinary integral, not an integral equation. However, if we consider the vibration of the string, then the transverse load becomes

\[ f(x) = - \mu \frac{d^2 y}{dx^2} \]  \hspace{1cm} \text{(A-18)}

Where \( \mu \) is the mass per unit length of the string and \( t \) is time. Consider only the harmonic vibration; i.e.,

\[ y = y_0(x) e^{i\omega t} \]  \hspace{1cm} \text{(A-19)}
and substitute Eq. (A-19) and Eq. (A-18) in Eq. (A-17), giving

\[ y_0 = \lambda \int_0^1 G(x, \zeta) y_{1\zeta} d\zeta \quad \ldots \quad (A-20) \]

where \( \lambda = \frac{m}{\pi} \). Eq. (A-20) is the second kind of the Fredholm integral equation with Green's function as a kernel. In general, a Green's function is defined by the following requirements:

1. It satisfies the homogeneous part of the given \( n \)th order differential equation

   \[ L(y) = \sum_{i=0}^{n-1} p_i(x) y^{(i)}(x) \equiv 0 \quad \text{i.e.,} \]

   \[ L(G) = 0 \]

   everywhere within the domain except at point \( z \).

2. It satisfies all the boundary conditions imposed on the problem.

3. It is continuous and possesses continuous derivatives with respect to \( x \) in the domain of all orders up to the \( (n-2) \)th, while the \( (n-1) \)st derivative is allowed to admit of a discontinuity at \( x=z \) such that

   \[ \lim_{\epsilon \to 0} \left[ G^{(n-1)}(x+\epsilon, \zeta) - G^{(n-1)}(x-\epsilon, \zeta) \right] = \frac{1}{p_n(\zeta)} \]

   In physical problems, Green's function has been used very successfully in many one dimensional problems. However, it is very difficult to construct a Green's function in two dimensional or three dimensional problems with arbitrary boundary shape. For a membrane or plate bending problems, a Green's function is just the complete solution of a membrane or a plate subjected to a concentrated load for a given boundary shape.
In the preceding paragraphs, we have shown that how an integral equation is equivalent to a differential equation. As to whether the linear boundary value problems will lend themselves to numerical solution more readily in their present metamorphosis as integral equations rather than in their differential formulation; the answer will, as stated by Kopal (1955)\textsuperscript{25}, be anything but unequivocal. Since Fredholm's solution (1900) of the Dirichlet problem first attracted the mathematical world to the theory of integral equations, the literature on analytical properties of integral equations has been enormous - in contrast to the literature on the numerical treatment of such equations (which, in most cases, remains the only way of learning anything specific about their solution), which is as yet almost non-existent.

The method of solving the boundary value problems as presented in Chapter IV actually uses a singular Fredholm's integral equation of the first kind. However, the kernel we used was

\[ K(P, Q) = \ln \left( \frac{h(P, Q)}{c} \right) \]

As compared to the kernel of the Dirichlet problem presented by Fredholm

\[ K(P, Q) = \frac{d}{dn} \ln h(P, Q) \]

Furthermore, the integral equation we obtained is the first kind, the one which Fredholm obtained is the second kind. As to the numerical treatment, most authors of works on integral
equations suggested replacing the kernel by a step function
(so called PG - kernel), instead, we replace the unknown density
function by a step function or polygonal function.
APPENDIX B

EQUATIONS OF SINGULAR INTEGRAL SOLUTION FOR PLATE BENDING

By using the coordinates as shown in Fig. 4-3, the deflections, slopes, moments and shear forces due to a constant uniform line load \( f_x \) and moment \( m_{x} \) distributed on length \( S_{x} \) (Eq. (8-39) and Eq. (8-11)) are obtained as following

\[
\frac{1}{F_{l}} (\mathcal{N}_{l})_{x} = \int_{-\frac{S_{l}}{2}}^{\frac{S_{l}}{2}} \frac{h^{2}(B,A) \ln h(B,A)}{A} \, dx - \int_{-\frac{S_{l}}{2}}^{\frac{S_{l}}{2}} f_{x} \cos \phi(n,A) \, dx + \int_{-\frac{S_{l}}{2}}^{\frac{S_{l}}{2}} \frac{m_{x}}{A} \, dx
\]

\[
= \left\{ \begin{array}{l}
\frac{1}{3} S_{c}^{3} - \frac{1}{3} S_{c} R_{c}^{2} (1 + C N_{c}^{2}) + \left[ \frac{R_{c}}{3} \left( S_{c}^{2} + 2 S_{c} R_{c} - 2 R_{c}^{2} - 2 S_{c}^{2} - 2 R_{c}^{2} \right) \ln \left( \frac{R_{c}^{2} - R_{c} S_{c} + S_{c}^{2}}{R_{c}^{2} + R_{c} S_{c} + S_{c}^{2}} \right) \right] + \frac{1}{3} R_{c}^{3} C N_{c}^{3} \left[ \tan \frac{S_{c}}{2} - \frac{R_{c} S_{c}}{C N_{c}} \right] + \tan \frac{S_{c}}{2} \frac{R_{c} S_{c}}{C N_{c}} \right. \\
- \frac{1}{3} S_{c} (1 + 3 R_{c}^{2} C N_{c}^{2}) - \frac{1}{3} S_{c}^2 R_{c} S_{c} - R_{c} C N_{c} S_{c} \end{array} \right. \quad \text{(B-1)}
\]

*The functions inside the {} corresponds to \( r^2 \ln r \) or \( r \ln r \cos \phi \), and those outside the {} corresponds to the other biharmonic functions.
\[
\frac{1}{R_c} \left( \frac{\partial \psi}{\partial y} \right)_k = \int_{-S_k}^{S_k} (R_k \cos \theta_k + \lambda \sin \gamma_k) (2 \ln \ln (B/A) + 1) \, dt
\]

\[
- \int_{-S_k}^{S_k} \left[ \cos \gamma_k + 3 (R_k \cos \theta_k + \lambda \sin \gamma_k) \right] \, dt
\]

\[
= \left\{ -S_k R_k \cos \theta_k - S_k R_k SN_k \sin \gamma_k
\right.
\]

\[
+ 2 \cos \gamma_k R_k^2 \frac{CN_k}{R_k} \left( \tan^{-1} \frac{S_k - R_k SN_k}{R_k CN_k} + \tan^{-1} \frac{S_k + R_k SN_k}{R_k CN_k} \right)
\]

\[
+ \left[ R_k (\cos \theta_k + SN_k \sin \gamma_k) \left( \frac{S_k}{2} - R_k SN_k \right) + \frac{\sin \gamma_k}{2} (R_k^2 - R_k S_k SN_k + \frac{S_k^2}{2}) \right].
\]

\[
\ln \left( R_k^2 - R_k S_k SN_k + \frac{S_k^2}{4} \right)
\]

\[
+ \left[ R_k (\cos \theta_k + SN_k \sin \gamma_k) \left( \frac{S_k}{2} + R_k SN_k \right) - \frac{\sin \gamma_k}{2} (R_k^2 + R_k S_k SN_k + \frac{S_k^2}{4}) \right].
\]

\[
\ln \left( R_k^2 + R_k S_k SN_k + \frac{S_k^2}{4} \right)
\]

\[
- \cos \gamma_k S_k - 3 S_k R_k \cos \theta_k
\]

\[\text{.............. (B-2)\]}

\[
\frac{1}{R_c} \left( \frac{\partial \psi}{\partial y} \right)_k = \int_{-S_k}^{S_k} \left\{ (R_k \sin \theta_k - \lambda \cos \gamma_k) (2 \ln \ln (B/A) + 1) \right\} \, dt
\]

\[
- \int_{-S_k}^{S_k} \left[ \sin \gamma_k + 3 (R_k \sin \theta_k - \lambda \cos \gamma_k) \right] \, dt
\]

\[
= \left\{ -S_k R_k \sin \theta_k + R_k S_k SN_k \cos \gamma_k
\right.
\]

\[
+ 2 \sin \gamma_k R_k^2 \frac{CN_k}{R_k} \left( \tan^{-1} \frac{S_k - R_k SN_k}{R_k CN_k} + \tan^{-1} \frac{S_k + R_k SN_k}{R_k CN_k} \right)
\]

\[
+ \left[ R_k (\sin \theta_k - SN_k \cos \gamma_k) \left( \frac{S_k}{2} - R_k SN_k \right) - \frac{\cos \gamma_k}{2} (R_k^2 - R_k S_k SN_k + \frac{S_k^2}{4}) \right].
\]

\[
\ln \left( R_k^2 - R_k S_k SN_k + \frac{S_k^2}{4} \right)
\]

\[\text{.............. (B-3)\]}

\[
\frac{1}{R_c} \left( \frac{\partial \psi}{\partial y} \right)_k = \int_{-S_k}^{S_k} \left\{ (R_k \sin \theta_k - \lambda \cos \gamma_k) (2 \ln \ln (B/A) + 1) \right\} \, dt
\]

\[
- \int_{-S_k}^{S_k} \left[ \sin \gamma_k + 3 (R_k \sin \theta_k - \lambda \cos \gamma_k) \right] \, dt
\]

\[
= \left\{ -S_k R_k \sin \theta_k + R_k S_k SN_k \cos \gamma_k
\right.
\]

\[
+ 2 \sin \gamma_k R_k^2 \frac{CN_k}{R_k} \left( \tan^{-1} \frac{S_k - R_k SN_k}{R_k CN_k} + \tan^{-1} \frac{S_k + R_k SN_k}{R_k CN_k} \right)
\]

\[
+ \left[ R_k (\sin \theta_k - SN_k \cos \gamma_k) \left( \frac{S_k}{2} - R_k SN_k \right) - \frac{\cos \gamma_k}{2} (R_k^2 - R_k S_k SN_k + \frac{S_k^2}{4}) \right].
\]

\[
\ln \left( R_k^2 - R_k S_k SN_k + \frac{S_k^2}{4} \right)
\]

\[\text{.............. (B-3)\]
\[ R_i (\sin \theta_i - SN_i \cos \frac{\gamma_i}{2}) \left( \frac{S_i}{2} + R_i SN_i \right) + \cos \frac{\gamma_i}{2} \left( \frac{R_i}{2} + R_i S_i SN_i + \frac{S_i^2}{4} \right) \ln \left( \frac{R_i^2 + R_i S_i SN_i + \frac{S_i^2}{4}}{R_i^2} \right) \]

\[ - S_i \sin \gamma_i - 3 S_i R_i \sin \theta_i \]

\[ \frac{1}{P_i} \left( \frac{\partial \omega_i}{\partial x_i} \right) = \int_{-\frac{S_i}{2}}^{\frac{S_i}{2}} \left[ 2 \ln (B, A) + 2 \cos \gamma_i \right] dt - 3 \int_{-\frac{S_i}{2}}^{\frac{S_i}{2}} d\dot{x_i} \]

\[ \frac{1}{P_i} \left( \frac{\partial \omega_i}{\partial \theta_i} \right) = \int_{-\frac{S_i}{2}}^{\frac{S_i}{2}} \left[ 2 \ln (B, A) + 2 \sin \theta_i \right] dt - 3 \int_{-\frac{S_i}{2}}^{\frac{S_i}{2}} d\dot{\theta_i} \]
\[
\frac{1}{\rho} \left( \frac{2\omega}{\rho} \nabla \phi \right) = \frac{1}{\rho} \left( \frac{\partial \phi}{\partial t} \right) = 4 \int_{-\frac{S_i}{2}}^{\frac{S_i}{2}} \frac{\cos \theta(B,A)}{\kappa(B,A)} \, dt
\]

\[
= \left\{ \begin{array}{l}
4 \cos \phi_i \left[ \tan^{-1} \frac{S_i}{2} - R_i S N_i \frac{\pi}{R_i C N_i} + \tan^{-1} \frac{S_i}{2} + R_i S N_i \frac{\pi}{R_i C N_i} \right] \\
+ 2 \sin \phi_i \left[ \ln \left( R_i^2 - R_i S_i S N_i + \frac{S_i^2}{4} \right) - \ln \left( R_i^2 + R_i S_i S N_i + \frac{S_i^2}{4} \right) \right]
\end{array} \right\} \quad \ldots \ldots \ldots \ldots (B-7)
\]

\[
\frac{1}{\rho} \left( \frac{2\omega}{\rho} \nabla \phi \right) = \frac{1}{\rho} \left( \frac{\partial \phi}{\partial t} \right) = 4 \int_{-\frac{S_i}{2}}^{\frac{S_i}{2}} \frac{\sin \theta(B,A)}{\kappa(B,A)} \, dt
\]

\[
= \left\{ \begin{array}{l}
4 \sin \phi_i \left[ \tan^{-1} \frac{S_i}{2} - R_i S N_i \frac{\pi}{R_i C N_i} + \tan^{-1} \frac{S_i}{2} + R_i S N_i \frac{\pi}{R_i C N_i} \right] \\
- 2 \cos \phi_i \left[ \ln \left( R_i^2 - R_i S_i S N_i + \frac{S_i^2}{4} \right) - \ln \left( R_i^2 + R_i S_i S N_i + \frac{S_i^2}{4} \right) \right]
\end{array} \right\} \quad \ldots \ldots \ldots \ldots (B-8)
\]

\[
\frac{1}{\rho} \left( \frac{2\omega}{\rho} \nabla \phi \right) = \frac{1}{\rho} \left( \frac{\partial \phi}{\partial t} \right) = \int_{-\frac{S_t}{2}}^{\frac{S_t}{2}} \left[ 2 \sin \theta(B,A) - 4 \cos^2 \theta(B,A) \sin \theta(B,A) \right] \, dt
\]

\[
= \ldots \ldots \ldots \ldots (B-9)
\]

\[
\frac{1}{\rho} \left( \frac{2\omega}{\rho} \nabla \phi \right) = \frac{1}{\rho} \left( \frac{\partial \phi}{\partial t} \right) = \int_{-\frac{S_t}{2}}^{\frac{S_t}{2}} \left[ 2 \cos \theta(B,A) - 4 \cos \theta(B,A) \sin^2 \theta(B,A) \right] \, dt
\]

\[
= \ldots \ldots \ldots \ldots (B-10)
\]
\[
\begin{align*}
\frac{1}{m_c} \left( \frac{\partial \mathbf{w}_{\mathbf{L}}}{\partial \mathbf{L}} \right)_\perp &= -\int_{-\frac{S_i}{2}}^{\frac{S_i}{2}} \left[ \ln(R(B,A) \cdot \cos^2(n, A) - \int_{-\frac{S_i}{2}}^{\frac{S_i}{2}} \left[ R(B,A) \cos^2(n, A) + 1 \right] \, \mathrm{d}A \right. \\
&= -R_A \cdot \mbox{CN}_A \left\{ \left( \frac{S_i}{2} - R_A \cdot \mbox{SN}_A \right) \ln \left( R_A^2 - R_A \cdot \mbox{SN}_A \cdot S_A + \frac{S_i}{4} \right) \right. \\
&\quad + \left( \frac{S_i}{2} + R_A \cdot \mbox{SN}_A \right) \ln \left( R_A^2 + R_A \cdot \mbox{SN}_A \cdot S_A + \frac{S_i}{4} \right) \\
&\quad + R_A \cdot \mbox{CN}_A \left( \frac{\tan^{-1} \frac{S_i - R_A \cdot \mbox{SN}_A}{R_A \cdot \mbox{CN}_A}}{R_A \cdot \mbox{CN}_A} + \frac{\tan^{-1} \frac{S_i + R_A \cdot \mbox{SN}_A}{R_A \cdot \mbox{CN}_A}}{R_A \cdot \mbox{CN}_A} \right) \\
&\left. \left( \frac{R_A^2 - R_A \cdot \mbox{SN}_A \cdot S_A - S_A}{R_A \cdot \mbox{CN}_A} + S_A \right) \right\} - R_A \cdot \mbox{CN}_A \cdot S_A + S_A \quad \ldots \ldots \ldots (B-11)
\end{align*}
\]

\[
\begin{align*}
\frac{1}{m_c} \left( \frac{\partial \mathbf{w}_{\mathbf{Y}}}{\partial \mathbf{Y}} \right)_\perp &= \int_{-\frac{S_i}{2}}^{\frac{S_i}{2}} \left[ \cos \varphi_Y \left\{ \ln(R(B,A) \cdot \cos \theta(B,A)) \cdot \cos \theta(B,A) \right\} \, \mathrm{d}A \right. \\
&= \left\{ \cos \varphi_Y \cdot R_A \left[ \mbox{CN}_A + \cos \left( \varphi_Y - \theta(Y) \right) \right] \left( \frac{\tan^{-1} \frac{S_i - R_A \cdot \mbox{SN}_A}{R_A \cdot \mbox{CN}_A}}{R_A \cdot \mbox{CN}_A} + \frac{\tan^{-1} \frac{S_i + R_A \cdot \mbox{SN}_A}{R_A \cdot \mbox{CN}_A}}{R_A \cdot \mbox{CN}_A} \right) \\
&\quad + \frac{1}{2} \left[ \cos \varphi_Y \left( \frac{S_i}{2} - R_A \cdot \mbox{SN}_A \right) + \sin \varphi_Y \cdot R_A \cdot \cos \left( \varphi_Y - \theta(Y) \right) \right] \ln \left( R_A^2 - R_A \cdot \mbox{SN}_A \cdot S_A + \frac{S_i}{4} \right) \\
&\quad + \frac{1}{2} \left[ \cos \varphi_Y \left( \frac{S_i}{2} + R_A \cdot \mbox{SN}_A \right) - \sin \varphi_Y \cdot R_A \cdot \cos \left( \varphi_Y - \theta(Y) \right) \right] \ln \left( R_A^2 + R_A \cdot \mbox{SN}_A \cdot S_A + \frac{S_i}{4} \right) \\
&\left. \left( \frac{R_A^2 - R_A \cdot \mbox{SN}_A \cdot S_A - S_A}{R_A \cdot \mbox{CN}_A} + S_A \right) \right\} - \cos \varphi_Y \cdot S_A \quad \ldots \ldots \ldots (B-12)
\end{align*}
\]
\[
\frac{1}{m_e} \left( \frac{3/2}{2 \mu} \right)_1 = \int_{-S \overline{c}}^{S \overline{c}} \left\{ \ln(R_B R_A) \sin \gamma_c + \cos[\gamma_c - \theta(B,A)] \sin \theta(B,A) \right\} dt - \int_{-S \overline{c}}^{S \overline{c}} \sin \gamma_c dt \\
= \left\{ \sin \gamma_c R_c \left[ -CN_c + \cos(\gamma_c - \theta_c) \right] \left( \tan^{-1} \frac{S \overline{c} - R \overline{c} SN_c^2}{R \overline{c} CN_c} + \tan^{-1} \frac{S \overline{c} + R \overline{c} SN_c^2}{R \overline{c} CN_c} \right) \right. \\
- \frac{1}{2} \left[ \sin \gamma_c \left( \frac{S \overline{c}}{2} - R \overline{c} SN_c \right) + \cos \gamma_c \cos(\gamma_c - \theta_c) \right] \ln \left( R \overline{c}^2 - R \overline{c} S \overline{c} SN_c + \frac{S \overline{c}}{2} \right) \\
- \frac{1}{2} \left[ \sin \gamma_c \left( \frac{S \overline{c}}{2} + R \overline{c} SN_c \right) - \cos \gamma_c \cos(\gamma_c - \theta_c) R \overline{c} \right] \ln \left( R \overline{c}^2 + R \overline{c} S \overline{c} SN_c + \frac{S \overline{c}}{2} \right) \\
+ S \overline{c} \sin \gamma_c \left\} - \sin \gamma_c S \overline{c} \hspace{1cm} \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (B-13) \\

\frac{1}{m_e} \left( \frac{3/2}{2 \mu} \right)_2 = \int_{-S \overline{c}}^{S \overline{c}} \left\{ \frac{2 \cos \gamma_c \cos \theta(B,A)}{\ln(B, A)} - \frac{\cos 2\theta(B,A) \cos[\gamma_c - \theta(B,A)]}{\ln(B, A)} \right\} dt \\
= \left\{ \frac{2 \cos \gamma_c}{\tan^{-1} \frac{S \overline{c} - R \overline{c} SN_c^2}{R \overline{c} CN_c} + \tan^{-1} \frac{S \overline{c} + R \overline{c} SN_c^2}{R \overline{c} CN_c}} \right. \\
- R \overline{c} \cos 2 \gamma_c \cos(\gamma_c - \theta_c) \left( \frac{S \overline{c} - R \overline{c} SN_c^2}{R \overline{c}^2 - R \overline{c} S \overline{c} SN_c + \frac{S \overline{c}}{2}} + \frac{S \overline{c} + R \overline{c} SN_c^2}{R \overline{c}^2 + R \overline{c} S \overline{c} SN_c + \frac{S \overline{c}}{2}} \right) \\
- \sin 2 \gamma_c [R \overline{c}^2 \ln \cos(\gamma_c - \theta_c) - \frac{1}{2} \ln \left( R \overline{c}^2 - R \overline{c} S \overline{c} SN_c + \frac{S \overline{c}}{2} \right) \\
- \ln \left( R \overline{c}^2 + R \overline{c} S \overline{c} SN_c + \frac{S \overline{c}}{2} \right) \left\} \hspace{1cm} \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (B-14) \\
\]
\[
\frac{1}{m_i} \left( \frac{\partial^2 \mathbf{W}_\perp}{\partial y^2} \right)_i = \int \frac{2 \sin \eta \sin \theta (B, A)}{\eta (B, A)} + \frac{\cos 2 \theta (B, A) \cos \left[ \eta \frac{1}{2} - \theta (B, A) \right]}{\eta (B, A)} \, d \eta \\
= \left\{ 2 \sin ^2 \eta \left[ \tan ^{-1} \frac{\frac{5}{2} - R_i S i}{R_i C N} + \tan ^{-1} \frac{\frac{5}{4} + R_i S N_i}{R_i C N} \right] \\
+ R_i \cos 2 \eta \cos \left( \eta \frac{1}{2} - \theta (B, A) \right) \left( \frac{\frac{5}{2} - R_i S i}{R_i^2 R_i S i S N_i + \frac{5}{4}} + \frac{\frac{5}{4} + R_i S N_i}{R_i^2 R_i S i S N_i + \frac{5}{4}} \right) \\
+ \sin ^2 \eta \left[ R_i^2 C N_i \cos \left( \eta \frac{1}{2} - \theta (B, A) \right) - \frac{1}{2} \right] \left[ \ln \left( R_i - R_i S i S N_i + \frac{5}{4} \right) - \ln \left( R_i^2 + R_i S i S N_i + \frac{5}{4} \right) \right] \right\} \ldots \ldots (B-15)
\]

\[
\frac{1}{m_i} \left( \frac{\partial^2 \mathbf{W}_\perp}{\partial y^2} \right)_i = \int \frac{\sin \left[ \eta \frac{1}{2} - \theta (B, A) \right] \cos 2 \theta (B, A)}{\eta (B, A)} \, d \eta \\
= \left\{ \frac{1}{R_i^2 R_i S i S N_i + \frac{5}{4}} \left[ a \cdot b \frac{\frac{5}{2} - R_i S N_i}{2 R_i^2 C N} - \frac{a c - b}{2} + \frac{ad - c}{2} \left( R_i S N_i - \frac{5}{2} \right) - \frac{d}{2} R_i^2 C N_i \right] \\
+ \frac{1}{R_i^2 R_i S i S N_i + \frac{5}{4}} \left[ a \cdot b \frac{\frac{5}{2} + R_i S N_i}{2 R_i^2 C N} + \frac{a c - b}{2} - \frac{ad - c}{2} \left( R_i S N_i + \frac{5}{2} \right) + \frac{d}{2} R_i^2 C N_i \right] \\
- \frac{1}{2} \left[ \ln \left( R_i^2 \right) - R_i S i S N_i \right] - \ln \left( R_i S i S N_i + \frac{5}{4} \right) \right\} \ldots \ldots (B-16)
\]

where
\[
\begin{align*}
\alpha &= -R_i \left[ \sin \left( \eta \frac{1}{2} - \theta (B, A) \right) + S N_i \right] \\
\beta &= -R_i^2 \left[ \cos \theta + 2 \sin \left( \eta \frac{1}{2} + \theta (B, A) \right) S N_i \right] - \cos 2 \eta \cdot S N_i \\
\gamma &= -R_i \left[ \sin \left( \eta \frac{1}{2} + \theta (B, A) \right) - \cos 2 \eta \cdot S N_i \right] \\
\delta &= \cos 2 \eta \frac{1}{2}
\end{align*}
\]
\[ \frac{1}{m_c} \left( \frac{2}{3} x \nabla^2 \gamma \right)_c = \frac{1}{m_c} (G_x)_c = \int -\frac{\frac{5}{2} 2 \cos \gamma_x - 4 \cos \theta(B,A) \cos \gamma_x - \theta(B,A)}{\lambda^2(B,A)} dx \]

\[ = \left\{ \frac{2 \cos \gamma_x}{R_c C N_i} - \frac{2 \cos \gamma_x \cos (\gamma_x - \theta_c)}{R_c C N_i} \left[ \tan^{-1} \frac{S_i - R_c S N_i}{R_c C N_i} + \tan^{-1} \frac{S_i + R_c S N_i}{R_c C N_i} \right] - 2 \frac{2 \cos \gamma_x \cos (\gamma_x - \theta_c)}{C N_i} \cdot S_i \cdot R_c \right. \]

\[ + \frac{R_c^2 + \frac{5}{4} \frac{S_i - R_c S N_i}{R_c C N_i} - 2 R_c S N_i^2}{R_c^2 + \frac{5}{4} \frac{S_i - R_c S N_i}{R_c C N_i} + \frac{S_i^2}{R_c C N_i} + \frac{S_i^2}{16}} \cos \left( \gamma_x - \theta_c \right) \} \ldots \ldots (B-17) \]

\[ \frac{1}{m_c} \left( \frac{\partial^{2}}{\partial y^{2}} \nabla^2 \gamma \right)_c = \frac{1}{m_c} (G_y)_c = \int -\frac{\frac{5}{2} 2 \sin \gamma_x - 2 \sin \theta(B,A) \cos \gamma_x - \theta(B,A)}{\lambda^2(B,A)} dx \]

\[ = \left\{ \frac{2 \sin \gamma_x}{R_c C N_i} - 2 \frac{2 \sin \gamma_x \cos (\gamma_x - \theta_c)}{R_c C N_i} \left[ \tan^{-1} \frac{S_i - R_c S N_i}{R_c C N_i} + \tan^{-1} \frac{S_i + R_c S N_i}{R_c C N_i} \right] \right. \]

\[ - \frac{2 \sin \gamma_x \cos (\gamma_x - \theta_c) \cdot S_i \cdot R_c}{C N_i^2} + \frac{R_c^2 + \frac{5}{4} \frac{S_i - R_c S N_i}{R_c C N_i} - 2 R_c S N_i^2}{R_c^2 + \frac{5}{4} \frac{S_i - R_c S N_i}{R_c C N_i} + \frac{S_i^2}{R_c C N_i} + \frac{S_i^2}{16}} \cos \left( \gamma_x - \theta_c \right) \} \ldots \ldots (B-18) \]
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