FINITE-COHERENT PEANO SPACES

DISSERTATION

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By

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CHAPTER I

I.1 We present in this chapter a synopsis of the topological background. No proofs are given if others are available, but references are provided. The standard topological reference used is Whyburn's Analytic Topology, [9] in the Bibliography.

I.2 Notation

The following topological notations are used:

(a) By topological space or just space we shall mean a separable metric space.

(b) \( \mathcal{K}_X(A) \) will denote the closure of \( A \) with respect to the space \( X \).

(c) \( \mathcal{B}_X(A) \) will denote the boundary of \( A \) with respect to the space \( X \).

(d) \( X - A \) will denote the complement of \( A \) with respect to \( X \).

Where no confusion is likely to result we shall omit the subscript on \( \mathcal{K} \) and \( \mathcal{B} \).

(e) \( r(X) \) will denote the degree of multicoherence of \( X \) (see I.6).

(f) \( \delta(X) \) will denote the diameter of the set \( X \).
I.3 **Local connectedness**

The following characterization theorem for locally-connected spaces is important.

I.3.1 **Theorem** X is a locally-connected space if and only if the components of open subsets of X are open in X.

**Proof** [9, p. 20]

A point set M is said to have **property S** provided that for every $\epsilon > 0$, M is the sum of a finite number of connected sets of diameter less than $\epsilon$.

I.3.2 **Theorem** In order that a continuum M be locally connected it is necessary and sufficient that M should have property S.

**Proof** [9, p. 23]

I.4 **Peano spaces**

A **continuum** will be a compact connected space. A **Peano space** is a space which is the continuous image of the unit interval. The following results are useful.

I.4.1 **Theorem** (Hahn-Mazurkiewicz) In order that a space be a Peano space it is necessary and sufficient that it be a locally-connected continuum.

**Proof** [9, p. 33]
I.4.2 **Theorem** A Peano space is arcwise connected.

*Proof* [9, p. 36]

I.4.3 **Theorem** A connected open subset of a Peano space is arcwise connected.

*Proof* [9, p. 38]

I.4.4 **Lemma** If $X = A \cup \overline{B}$ where $A$ and $B$ are continua and $E$ is any connected subset of $X$ with $E \cap A \neq \emptyset$, $E \cap B \neq \emptyset$, then $E \cap A \cap B \neq \emptyset$.

*Proof* [2, p. 20]

I.4.5 **Lemma** Let $X$ be a Peano space and let $G$ be an open set in $X$. If $E$ is a connected subset of $X$ such that $E \cap G \neq \emptyset$, $E \cap (X - G) \neq \emptyset$, then $E \cap \overline{B}(G) \neq \emptyset$.

*Proof* [2, p. 21]

I.4.6 **Lemma** Let $X$ be a Peano space and let $G$ be a connected open subset of $X$ such that $\overline{B}(G)$ contains more than one point. Then there is a simple arc $\gamma$ with end points $p$ and $q$ in $\overline{B}(G)$ and $\gamma \subset G \cup \{p\} \cup \{q\}$.

*Proof* [2, p. 24]

I.5 **Dimension theory**

By $\dim X$ we shall mean the inductive dimension
defined in [1, p. 24]. We shall need the following results.

I.5.1 **Theorem** If $Y$ is a subspace of $X$ then $\dim Y \leq \dim X$.

**Proof** [1, p. 26]

I.5.2 **Theorem** Let $X$ be a locally compact space. Then $\dim X = 0$ if and only if $X$ is totally disconnected (the components of $X$ are points).

**Proof** [1, p. 22]

I.5.3 **Theorem** If $X$ is a locally compact metric space then $\dim X \leq n$ if and only if any two distinct points in $X$ can be separated by a closed set $W$ such that $\dim W \leq n - 1$.

**Proof** [1, p. 36]

I.6 **Degree of multicoherence**

Let $X$ be a continuum and let $X_1 \cup X_2 = X$ where $X_1$ and $X_2$ are continua. Let $r(X_1, X_2)$ be the number of components in $X_1 \cap X_2$ minus one. Let

$r(X) = \operatorname{lub} \{ r(X_1, X_2) \mid X_1 \cup X_2 = X \text{ and } X_1 \text{ and } X_2 \text{ are continua} \}$. Then $r(X)$ is the **degree of multicoherence** of $X$. If $r(X) = 0$, $X$ is said to be **unicoherent**.

I.6.1 **Lemma** Let $R$ be a subcontinuum of a Peano space $P$ with $r(P) = n$. If $C$ is a component of $P - R$ then $\mathcal{B}(C)$ can have at most $n + 1$ components.
Proof $P - C$ and $\mathcal{K}(C)$ are continua and

$$(P - C) \cup \mathcal{K}(C) = P. \quad (P - C) \cap \mathcal{K}(C) = \mathcal{B}(C)$$

and since $r(P) = n$, $\mathcal{B}(C)$ can have at most $n + 1$ components.

I.7 Cut points, end points, and separations

A cut point of a continuum $X$ is a point $x \in X$ such that $X - \{x\}$ is not connected.

An end point $x \in X$ is a point which has arbitrarily small neighborhoods whose boundaries are single points.

A set $Z$ separates sets $A$ and $B$ in $X$ if $A$ and $B$ are in separate components of $X - Z$.

I.7.1 Theorem If $A$ and $B$ are disjoint closed and connected subsets of a Peano space $X$, there exist three disjoint sets $R$, $F$ and $G$ where $R$ and $G$ are open connected sets containing $A$ and $B$, respectively; $X = R \cup G \cup F$ and $\mathcal{B}_X(R) = \mathcal{B}_X(G) = F$.

Proof [9, p. 49]

I.8 A-sets and B-sets

An A-set is a non-degenerate closed subset of a space $X$ such that every component of $X - A$ has a single boundary point.

A B-set is a non-degenerate closed subset of a
space $X$ such that every component of $X - A$ has a finite point set as a boundary. We note that $X$ is an A-set and a B-set, and an A-set is a B-set.

**1.8.1 Theorem** Let $X$ be a continuum.

(a) If $A \subseteq X$ is an A-set and $Z \subseteq X$ is connected, then $A \cap Z$ is connected.

(b) $r(A) \leq r(X)$ if $A \subseteq X$ is an A-set.

(c) If $X$ is a Peano space and $A$ is an A-set, then $A$ is a Peano space.

(d) If $X$ is a Peano space and $A$ is an A-set in $X$, and $C$ is a component of $X - A$, then $X(C)$ is an A-set and a Peano space, and $A \cup C$ is an A-set.

**Proof** [9, pp. 67-73]

**I.9 Cyclic elements and fine cyclic elements**

A cyclic element is an A-set which has no cut points in itself. A fine cyclic element is a B-set which is connected after the removal of any finite point set. A proper cyclic or fine cyclic element is non-degenerate.

**I.9.1 Theorem** If $X$ is a Peano space with true cyclic elements $E_1, E_2, \ldots$, we have

$$r(X) = \sum_{i=1}^{\infty} r(E_i)$$

**Proof** [9, p. 85]
I.9.2 **Theorem** If $p$ and $q$ are any two points of a proper cyclic element $C$ of a Peano space $X$, then $p$ and $q$ lie on a simple closed curve in $C$.

**Proof** [9, p. 78]

I.10 **Special functions**

A **monotone mapping** $f : X \to Y$ is a continuous function where $f^{-1}(y)$ is a continuum for $y \in Y$.

A **light mapping** $f : X \to Y$ is a continuous mapping where $f^{-1}(y)$ is totally disconnected for $y \in Y$.

A continuous function $f : X \to Y$ is a **two-point identification** if $f^{-1}(y)$ is non-degenerate for only one point $y_0 \in Y$ and $f^{-1}(y_0)$ consists of exactly two points.

I.10.1 **Theorem** If $X$ is a compact space and $f(X) = Y$ is a continuous function there exists a unique factorization (unique in that any two middle spaces of the same mapping are homeomorphic),

$$f(x) = f_2f_1(x) \quad x \in X$$

such that

$$f_1(x) = \mathcal{M}$$

is monotone and

$$f_2(\mathcal{M}) = Y$$

is light.

**Proof** [9, pp. 141-142]
\( \mathcal{M} \) is called the middle space of the factorization and the factorization will be referred to as the monotone-light factorization of \( f \). We shall have occasion to use the fact that the points of \( \mathcal{M} \) are really the components of the inverses of points of \( Y \) under \( f \).

1.10.2 Theorem If \( X \) is a continuum and \( f(X) = Y \) is a monotone mapping, then \( r(Y) \leq r(X) \).

Proof \[9, p. 154\]

1.10.3 Theorem If \( X \) is a compact space and \( f(X) = Y \) is monotone, then \( f^{-1}(A) \) is a continuum if and only if \( A \) is a continuum in \( Y \).

Proof \[8, p. 46\]

1.10.4 Theorem The product of two monotone mappings is monotone on compact spaces.

Proof Follows immediately from I.10.3.

1.10.5 Theorem If \( \mathcal{L}(X) = Y \) is a light mapping and \( A \subset Y \) is totally disconnected, then \( \mathcal{L}^{-1}(A) \) is totally disconnected.

Proof If \( B \) is a component of \( \mathcal{L}^{-1}(A) \), \( \mathcal{L}(B) \) is a connected subset of \( A \). \( A \) is totally disconnected so \( \mathcal{L}(B) = a \in A \) and \( B = \mathcal{L}^{-1}(a) \) which is totally disconnected.
I.10.6 Theorem Let \( f(X) = Y \) be a continuous mapping. Then \( f \) is light if and only if for every \( \epsilon > 0 \), \( \delta > 0 \) exists such that if \( B \) is a subcontinuum of \( Y \) of diameter less than \( \delta \), any component of \( f^{-1}(B) \) is of diameter less than \( \epsilon \).

Proof [9, p. 131]

I.10.7 Theorem The product of two light mappings is light.

Proof This is a direct consequence of I.10.5.

I.10.8 Theorem If \( f(X) = Y \) is a two-point identification where \( X \) and \( Y \) are compact spaces and \( f^{-1}(r) = \{ p, q \} \), then \( f(X - \{ p, q \}) = Y - \{ r \} \) is a homeomorphism.

Proof \( f \) is continuous and 1-1 between \( X - \{ p, q \} \) and \( Y - \{ r \} \). Let \( \{ y_n \} \) be a sequence of points in \( Y - \{ r \} \) such that \( y_n \to y, y \in Y - \{ r \} \). Now \( \{ f^{-1}(y_n) \} \subset (X - \{ p, q \}) \) and is an infinite point set in \( X \). Therefore there is a convergent subsequence \( \{ x_k \} \) since \( X \) is compact. Suppose \( x_k \to x \). Now \( f(x_k) \) converges to \( f(x) \) but \( f(x) = y \) so \( x \in (X - \{ p, q \}) \).

If \( x \) is any limit point of \( f^{-1}(y_n) \), then as above \( f(x) = y \) so \( f^{-1}(y_n) \to f^{-1}(y) \). This establishes that \( f^{-1}(Y - \{ r \}) = X - \{ p, q \} \) is continuous.
I.10.9 Lemma If \( m(X) = Y \) is a monotone mapping of a Peano space \( X \) onto a Peano space \( Y \), and \( A \) and \( B \) are continua such that \( A \cup B = X \), then
\[
m(A \cap B) = m(A) \cap m(B).
\]
**Proof** \( m(A \cap B) \subseteq m(A) \cap m(B) \) always.
Suppose \( y \in m(A) \cap m(B) \). Then \( m^{-1}(y) \cap A \neq \emptyset \) and \( m^{-1}(y) \cap B \neq \emptyset \). \( m^{-1}(y) \) is connected, so by I.4.7 \( m^{-1}(y) \cap A \cap B \neq \emptyset \). Let \( x \in m^{-1}(y) \cap A \cap B \). Then \( m(x) = y \in m(A \cap B) \).
Thus \( m(A) \cap m(B) \subseteq m(A \cap B) \) and \( m(A) \cap m(B) = m(A \cap B) \).

I.10.10 Lemma Let \( X \) and \( Y \) be topological spaces and let \( f(X) = Y \) be a two-point identification. Let \( p \) and \( q \) be the points in \( X \) identified by \( f \) and let \( f(p) = f(q) = r \in Y \). Let \( A \) and \( B \) be subsets of \( X \).
Then
(a) \( f^{-1}(f(A)) = A \) if and only if \( p, q \in A \) or \( p, q \in X - A \).
(b) \( p, q \in A \cap B \) implies that \( f(A \cap B) = f(A) \cap f(B) \) and \( f^{-1}(f(A) \cap f(B)) = A \cap B \).
**Proof**
(a) If \( p \in A \), \( q \in X - A \), then \( q \in f^{-1}(f(A)) \) so \( f^{-1}(f(A)) \neq A \). Now \( f^{-1}(f(x)) = x \) if \( x \neq p \) and \( x \neq q \) and \( f^{-1}(f(p)) = \{p, q\} = f^{-1}(f(q)) \). So if \( p, q \in A \), \( f^{-1}(f(A)) = A \) or if \( p, q \in X - A \), \( f^{-1}(f(A)) = A \).
(b) \( f(A \cap B) \subseteq f(A) \cap f(B) \) is a standard topological result (see I.10.9 above).

If \( y \in f(A) \cap f(B) \) then there exist \( x_1 \in A \) and \( x_2 \in B \) such that \( f(x_1) = f(x_2) = y \). However \( x_1 = x_2 \) unless \( y = r \) and if \( y = r \), \( x_1 = p \) and \( x_2 = q \).

Since \( p, q \in A \cap B, y \in f(A \cap B) \).

\( f^{-1}(f(A) \cap f(B)) = A \cap B \) follows from part (a).
CHAPTER II

We develop here some topological results. Some of these results are known, and references are given for proofs. Those with proofs have not been found in standard references and may even be new.

II.1 Some separation properties in locally connected spaces

II.1.1 Lemma Let $X$ be locally connected and let $Z$ be a closed subset of $X$. If $A$ is a component of $X - Z$, then $\mathcal{B}_X(A) \subseteq Z$.

Proof If $x \in \mathcal{B}_X(A)$ and $x \notin Z$, then $x \in C$ where $C$ is a component of $X - Z$. By I.3.1 $C$ is open and thus a neighborhood of $x$. But $C \cap A = \emptyset$ or $C \cap A = A$. Thus $x$ cannot be a boundary point of $A$. This is a contradiction so $x \in \mathcal{B}_X(A) \subseteq Z$.

II.1.2 Lemma Let $W$ be a closed subset of a locally connected set $X$ which separates $a$ and $b$ in $X$. Let $A$ be the component of $X - W$ containing $a$. Then $A$ is a component of $X - \mathcal{B}(A)$. Hence $\mathcal{B}(A)$ separates $a$ and $b$ in $X$. 

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Proof. A is an open, connected subset of $X - W$ (I.3.1) and $\mathcal{B}(A) \subseteq W$ (II.1.1). We must then conclude that $A$ is an open (in $X$), connected subset of $X - \mathcal{B}(A)$ containing $a$. If $x$ is a limit point of $A$ in $X - \mathcal{B}(A)$, then $x$ is a limit point of $A$ in $X$. This means that $x \in A$ or $x \in \mathcal{B}(A)$, but $x \in X - \mathcal{B}(A)$ forces us to conclude that $x \notin A$. Thus $A$ is an open and closed connected subset of $X - \mathcal{B}(A)$ and must be a component.

$b \notin A$ and $b \notin W$. Therefore $b \notin \mathcal{B}(A)$ and must be in another component of $X - \mathcal{B}(A)$.

II.1.3 Lemma. Let $W$ be a closed subset of a Peano space $X$ which separates the points $a$ and $b$ in $X$. Then there exist continua $X_1$ and $X_2$ such that $X = X_1 \cup X_2$, $a \in X_1$, $b \in X_2$, $Z = X_1 \cap X_2$ separates $a$ and $b$ in $X$ and $Z \subseteq W$.

Proof. Let $A$ be the component of $X - W$ which contains $a$. By II.1.1, $\mathcal{B}(A) \subseteq W$ and by II.1.2 $\mathcal{B}(A)$ separates $a$ and $b$ in $X$. Let $B$ be the component of $X - \mathcal{B}(A)$ containing $b$. Repeating the above argument we have that $\mathcal{B}(B)$ separates $a$ and $b$ in $X$ and $\mathcal{B}(B) \subseteq \mathcal{B}(A)$.

By II.1.2 $B$ is a component of $X - \mathcal{B}(B)$. Let $C$ be the component of $X - \mathcal{B}(B)$ containing $a$. By II.1.1 $\mathcal{B}(C) \subseteq \mathcal{B}(B) \subseteq \mathcal{B}(A)$. Suppose
x ∈ ℬ(B) ⊂ W and let N be a neighborhood of x. Then N ∩ A ≠ ∅. However, A is connected, a ∈ A and A ⊂ X - W ⊂ X - ℬ(B), so A ⊂ C. This means that N ∩ C ≠ ∅. Then x ∈ ℬ(C) and ℬ(C) = ℬ(B).

If D is a component of X - ℬ(B), we have, by II.1.1, ℬ(D) ⊂ ℬ(B). Since X is connected, ℬ(D) ≠ ∅. This means that C ∪ ℧(B) is connected.

Now X - C = ℧(B) ∪ ( ∪ { D | D a component of X - C, D ∩ C = ∅ } ), so X - C is a connected subset of X and is also closed since C is open by I.3.1. Hence X - C is a continuum. Let ℧(C) = X₁, X - C = X₂. Then X₁ and X₂ are continua. Z = X₁ ∩ X₂ = ℬ(B) = ℬ(C) is a closed subset of X separating a and b in X and Z ⊂ W.

II.2 **Topologizing the union of two topological spaces**

Further on we shall be splitting a topological space in a certain way. We shall take two subspaces whose union is the whole space, add some points to one of them and recombine them to give a new space. The following results define the topology for the new space.

II.2.1 **Definition** If d₁ and d₂ are metrics for the space X, then d₁ and d₂ agree on a subset W ⊂ X if and only if d₁(x,y) = d₂(x,y) for x, y ∈ W.
Let $d(x,y) = \begin{cases} d_1(x,y) & \text{if } x,y \in X \\ d_2(x,y) & \text{if } x,y \in Y \\ \inf \{d_1(x,w) + d_2(y,w)\} & \text{if } x \in X - W, y \in Y - W \end{cases}$

Since $W$ is compact, for $x \in X - W, y \in Y - W$ there is a $w_{xy} \in W$ such that

$$d(x,y) = d_1(x,w_{xy}) + d_2(y,w_{xy}).$$

From the definition of $d$ we have that $d(x,y) = d(y,x)$. Also $d(x,x) = 0$. If $d(x,y) = 0$ and $x \in X - W$ and $y \in Y - W$ then

$$d_1(x,w_{xy}) = 0 \text{ and } d_2(y,w_{xy}) = 0.$$ 

Since $W$ is compact and closed, this means $x \in W$ and $y \in W$. Thus $d(x,y) = 0$ implies that $x,y \in X$ or $x,y \in Y$. In either case, $d(x,y) = 0$ must imply that $x = y$.

Let $x,y,z \in Z$. If $x,y,z \in X$ we have

$$d(x,y) = d_1(x,y) \leq d_1(x,z) + d_1(y,z) = d(x,z) + d(y,z).$$

If $x,y,z \in Y$ we need only write $d_2$ for $d_1$ in the preceding sentence. The other possible locations of $x,y$ and $z$ can be reduced to $x \in X, y,z \in Y - W$ by renaming. In this case let $d(x,z) = d_1(x,w_{xz}) + d_2(z,w_{xz})$. Then
\[ d(x,y) = \operatorname{glb} \left\{ d_1(x,w) + d_2(y,w) \right\} \leq d_1(x,w_{xz}) + d_2(y,w_{xz}) . \]

From this we have
\[ d(x,y) \leq d_1(x,w_{xz}) + d_2(y,w_{xz}) \leq d_1(x,w_{xz}) + d_2(y,z) + d_2(z,w_{xz}) = d(x,z) + d(y,z) . \] This proves the triangle inequality.

Thus \( d \) is a metric for \( Z \).

II.2.3 **Definition** The set \( 0 \) is said to be \( d \)-open if and only if \( 0 \) is open with respect to the metric topology \( d \). \( d \)-closed, \( d \)-compact are similarly defined.

II.2.4 **Lemma** Let \( W, X, Y, Z, d_1, d_2 \) and \( d \) be as in II.2.2. If \( 0 \) is \( d \)-open, then \( 0 \cap X \) is \( d_1 \)-open and \( 0 \cap Y \) is \( d_2 \)-open.

**Proof** If \( 0 \cap X = \emptyset \) or \( 0 \cap Y = \emptyset \) there is nothing to prove.

Let \( x \in 0 \cap X \subset 0 \). Since \( 0 \) is open, there is an \( \epsilon > 0 \) such that \( N(x, \epsilon) = \{ y \in Z \mid d(x,y) < \epsilon \} \subset 0 \).

Since \( d(x,y) = d_1(x,y) \) for \( y \in X \) we have that \( N_1(x, \epsilon) = \{ y \in X \mid d_1(x,y) < \epsilon \} \subset N(x, \epsilon) \subset 0 \) and \( N_1(x, \epsilon) \subset X \) so \( N_1(x, \epsilon) \subset 0 \cap X \) and \( 0 \cap X \) is open with respect to the \( d_1 \)-topology. A similar argument shows that \( 0 \cap Y \neq \emptyset \) is open with respect to the \( d_2 \)-topology.
II.2.5 \textbf{Lemma} Let $W$, $X$, $Y$, $Z$, $d_1$, $d_2$ and $d$ be as in II.2.2. If $K \subseteq X (K \subseteq Y)$ is $d_1$-compact ($d_2$-compact), then $K$ is $d$-compact.

\textbf{Proof} Let $\mathcal{D}$ be a $d$-open family of sets which covers $K$. Then $\mathcal{D}_1 = \{ O \cap X \mid O \in \mathcal{D} \}$ is a $d_1$-open family of sets (II.2.4) which covers $K$. Since $K$ is $d_1$-compact, there is a finite subfamily $O_1 \cap X$, \ldots, $O_n \cap X$ which covers $K$. Then $O_1, \ldots, O_n$ is a finite $d$-open subfamily of $\mathcal{D}$ which covers $K$. Thus $K$ is $d$-compact. Similarly, if $K \subseteq Y$ is $d_2$-compact, then $K$ is $d$-compact.

II.2.6 \textbf{Corollary} Let $W$, $X$, $Y$, $Z$, $d_1$, $d_2$ and $d$ be as in II.2.2. Then $W$ is $d$-compact.

II.2.7 \textbf{Lemma} Let $W$, $X$, $Y$, $Z$, $d_1$, $d_2$ and $d$ be as in II.2.2. If $X$ and $Y$ are compact, then $Z$ is compact.

\textbf{Proof} By II.2.5 $Z = X \cup Y$ is the union of two $d$-compact subsets.

II.2.8 \textbf{Lemma} Let $W$, $X$, $Y$, $Z$, $d_1$, $d_2$ and $d$ be as in II.2.2. If $C \subseteq X (C \subseteq Y)$ is $d_1$-connected ($d_2$-connected), then $C$ is $d$-connected.

\textbf{Proof} Suppose $C$ is not $d$-connected. Then there is a separation $C = A \mid B$. There are $d$-open sets $A_1$ and $B_1$ so that $A = C \cap A_1$ and $B = C \cap B_1$. Now $A_1 \cap X$ and $B_1 \cap X$ are $d_1$-open by II.2.4. Also
A = C ∩ (A_1 ∩ X) and B = C ∩ (B_1 ∩ X), so 
C = A ∪ B would be a separation of C in the \( d_1 \)-topology. This contradicts the fact that C is \( d_1 \)-connected.

II.2.9 Lemma Let W, X, Y, Z, \( d_1 \), \( d_2 \), and \( d \) be as in II.2.2. Then Z is connected if X and Y are connected as topological spaces.

Proof By II.2.8 X and Y are \( d \)-connected and 
\( X \cap Y = W \neq \emptyset \) and Z = X ∪ Y so Z is the union of two non-disjoint connected subsets, hence connected.

II.2.10 Theorem Let W, X, Y, Z, \( d_1 \), \( d_2 \), and \( d \) be as in II.2.2. If X and Y are Peano spaces, then 
Z is a Peano space.

Proof By II.2.2, II.2.7 and II.2.9 we have that 
Z is a compact, connected, metric space. We must now show that Z is locally connected.

Let \( \varepsilon > 0 \) be given. Since X and Y are Peano spaces, there are finite sequences of connected sets 
\[ \{A_i\}_{i=1}^n, \{B_j\}_{j=1}^m \] such that 
\[ X = \bigcup_{i=1}^n A_i, \quad Y = \bigcup_{j=1}^m B_j, \]
\[ \delta_1(A_i) < \varepsilon, \quad i = 1, \ldots, n \] (\( \delta_1 \) denotes the diameter with respect to the \( d_1 \) metric, \( i = 1, 2 \)) and 
\[ \delta_2(B_j) < \varepsilon, \quad j = 1, \ldots, m. \] We define a new finite sequence \( \{C_k\}_{k=1}^{m+n} \) by 
\[ C_i = A_i, \quad i = 1, \ldots, n, \]
\[ C_{n+j} = B_j, \quad j = 1, \ldots, m. \] Then Z = \( \bigcup_{k=1}^{m+n} C_k \), each \( C_k \)
is connected (II.2.8) and
\[
\delta(C_k) = \begin{cases} 
\delta_1(A_k) & \text{if } 1 \leq k \leq n \\
\delta_2(B_{k-n}) & \text{if } n + 1 \leq k \leq m + n
\end{cases}
\]
so \( \delta(C_k) < \epsilon \). Thus \( Z \) is locally connected (\( Z \) has property S by I.3.2).

II.3 Light mappings and dimension

The main result of this section is that a light mapping on a compact space cannot lower the dimension. We first need a topological lemma.

II.3.1 Lemma Let \( A \) be an open subset of a topological space \( X \). If \( a \in A \) and \( B \) is the component of \( X - \mathcal{B}(A) \) which contains \( a \), then \( B \subseteq A \).

Proof \( A \cap B \neq \emptyset \) since \( a \in A \) and \( a \in B \).
Also \( A \cap B \subseteq B \). Since \( A \) is open in \( X \), \( A \cap B \) is open in \( B \).

If \( x \in B \) is a limit point of \( A \cap B \) considered as a subset of \( B \), then \( x \) is also a limit point of \( A \) considered as a subset of \( X \). Thus \( x \in A \) or \( x \in \mathcal{B}(A) \).
However, \( x \in B \subseteq X - \mathcal{B}(A) \) implies \( x \in A \) and \( x \in A \cap B \). This means that \( A \cap B \) is closed with respect to \( B \). Thus \( A \cap B \) is an open and closed subset of \( B \). Therefore \( B = [A \cap B] \subseteq A \), since \( B \) is connected.
II.3.2 Theorem  Let \( \mathcal{I}(X) = Y \) be a light mapping of the compact metric space \( X \) onto the (necessarily compact) metric space \( Y \). Then \( \dim X \leq \dim Y \).

Proof If \( \dim Y = \infty \) then \( \dim X \leq \infty \). So we assume that \( \dim Y \leq n < \infty \). The proof will be by induction on \( \dim Y \).

If \( \dim Y = -1 \), then \( Y = \emptyset \) and \( X = \emptyset \), so \( \dim X = -1 \). Thus the theorem holds for \( n = -1 \).

If \( \dim Y = 0 \), then \( Y \) is totally disconnected by I.5.2. By I.10.7, \( X = \mathcal{I}^{-1}(Y) \) is totally disconnected; so, by I.5.2, we have \( \dim X = 0 \). Thus the theorem holds for \( n = 0 \).

Assume the theorem is valid for all spaces satisfying the hypothesis such that \( \dim Y \leq k \).

Let \( X \) and \( Y \) be spaces satisfying the hypothesis such that \( \dim Y \leq k + 1 \).

Let \( a \) and \( b \) be distinct points in \( X \) and let \( \varepsilon = \frac{3}{2} d(a,b) \). Let \( \delta > 0 \) be as in I.10.6. Choose a neighborhood \( N \) of \( \mathcal{I}(a) \) such that the diameter of \( N \) is less than \( \frac{\delta}{2} \), i.e. \( \delta(N) < \frac{\delta}{2} \), and \( \dim B(N) \leq k \). Furthermore, if \( \mathcal{I}(a) \neq \mathcal{I}(b) \) we choose \( N \) so that \( \delta(N) < \frac{\delta}{2} \min(\mathcal{I}(a), \mathcal{I}(b)) \).

Now \( B(N) \) is a closed, hence compact, subset of \( Y \). Since \( \mathcal{I} \) is continuous, \( \mathcal{I}^{-1}(B(N)) = W \) is a closed, hence compact, subset of \( X \).

Now \( \mathcal{I}(W) = B(N) \), \( \mathcal{I} \) is light and \( \dim B(N) \leq k \).
By the induction hypothesis we have that \( \dim W \leq k \).

We have that \( a \in X - W \) and \( b \in X - W \),
\[ W = \mathcal{L}^{-1}(\mathcal{B}(N)), \]
for \( \mathcal{L}(a) = \mathcal{L}(b) \) implies that
\[ \mathcal{L}(a) = \mathcal{L}(b) \in N \] and \( N \cap \mathcal{B}(N) = \emptyset \). If
\[ \mathcal{L}(a) \neq \mathcal{L}(b), \]
then \( \mathcal{L}(a) \in N \) and if \( \mathcal{L}(b) \in \mathcal{B}(N) \),
we have that \( d(\mathcal{L}(a), \mathcal{L}(b)) \leq \mathcal{S}(N) < \frac{1}{2} d(\mathcal{L}(a), \mathcal{L}(b)) \).

If \( A \) is the component of \( X - W \) which contains \( a \),
then \( \mathcal{L}(A) \) is a connected subset of \( Y - \mathcal{B}(N) \) and
\[ \mathcal{L}(a) \in \mathcal{L}(A) \]. Then \( \mathcal{L}(A) \subset B \) where \( B \) is the
component of \( Y - \mathcal{B}(N) \) which contains \( \mathcal{L}(a) \). However,
by II.3.1, \( B \subset N \). Thus \( \mathcal{L}(A) \subset N \).

Then \( A \) is a connected subset of \( \mathcal{L}^{-1}(\mathcal{K}(N)) \).
Now \( \mathcal{S}(\mathcal{K}(N)) \leq \mathcal{S}/2 < \mathcal{S} \), so by I.10.6
\[ \mathcal{S}(A) < \mathcal{S} = \frac{1}{2} d(a, b) \]. Thus \( b \notin A \), so \( b \) must
be in some other component of \( X - W \). Thus \( W \) separates
\( a \) and \( b \) in \( X \).

We then have \( a \) and \( b \) separated in \( X \) by a closed
subset \( W \) and \( \dim W \leq k \). Thus by I.5.3, we have
\( \dim X \leq k + 1 \). This establishes the theorem.

II.4 More on Peano spaces

II.4.1 Theorem Let \( X \) be a Peano space, let \( X = P \cup Q \)
where \( P \) and \( Q \) are continua, and let \( x \) be a point in \( X \).
Then there are continua \( P' \) and \( Q' \) such that \( P \subset P' \),
\( Q \subset Q' \), \( x \in P' \cap Q' \) and \( P' \cap Q' \) has the same
number of components as \( P \cap Q \).
Proof Assume $x \in P - Q = X - Q$. Let $xy$ be an arc joining $x$ to a point $y \in P \cap Q$, and let $\alpha$ be the closure of the component of $(P - Q) \cap xy$ containing $x$. $\alpha$ is an arc and $\alpha \cap P \cap Q$ is an end point of $\alpha$.

Then $P' = P$ and $Q' = Q \cup \alpha$ are continua. $P' \cap Q' = (P \cap Q) \cup \alpha$, but $\alpha \cap P \cap Q$ is a point. We can then conclude that $P' \cap Q'$ has the same number of components as $P \cap Q$. 
CHAPTER III

In this chapter we study some classifications of one-dimensional continua. Most of the results are known, so we give references to the proofs. These results will be used later to extend some theorems on the classifications of middle-spaces.

III.1 Definitions

III.1.1 Definition A dendrite is a locally connected continuum which contains no simple closed curve.

III.1.2 Definition A continuum $X$ is said to be a regular curve provided every point $p \in X$ is contained in arbitrarily small neighborhoods with finite point sets as boundaries.

III.1.3 Definition A continuum $X$ is said to be a rational curve provided every point $p \in X$ is contained in arbitrarily small neighborhoods with countable point sets as boundaries.

III.1.4 Definition A continuum $X$ is said to be hereditarily locally connected provided every sub-continuum of $X$ is locally connected.
III.2 Characterization theorems

III.2.1 Theorem Let $M$ be a continuum. The following are equivalent:

(a) $M$ is a dendrite.
(b) Any two distinct points in $M$ can be separated by a third point.
(c) Every point of $M$ is a cut point or an end point.
(d) Every subcontinuum of $M$ contains uncountably many cut points of $M$.
(e) The intersection of any two connected subsets of $M$ is connected.
(f) $M$ is unicoherent and one-dimensional.

Proof For the proof of the equivalence of (a), (b), (c), (d), and (e), see [9, p. 88].

(b) and (d) imply (f) (see III.2.2) and (f) implies (e) will follow from a later (III.3.4), but independent, lemma.

III.2.2 Theorem In order that a continuum $M$ be

(1) a dendrite, (2) a regular curve, (3) hereditarily locally connected, (4) a rational curve, (5) of dimension less than or equal to $n$, ($n = 1, 2, \ldots$), it is necessary and sufficient that any two distinct points of any subcontinuum $N$ of $M$ should be separated in $N$ by a closed set which is (1) a single point,
(2) finite, (3) countable and contained in a finite number of arbitrarily small subcontinua of \( N \), (4) countable, (5) of dimension less than or equal to \( n - 1 \).

**Proof** See \([9, \text{pp. 88-99}]\).  

If \( D \) is the class of all dendrites, \( R \) the class of all regular curves, \( H \) the class of all hereditarily locally connected continua, \( R_a \) the class of all rational curves, \( C_n \) the class of all continua of dimension less than or equal to \( n \), then we have the following as an immediate result of III.2.3.

\[ D \subset R \subset H \subset R_a \subset C_1 \subset C_2 \ldots \subset C_n \subset C_{n+1} \ldots \]

There are examples that show that the inclusions are not equalities.

**III.2.3 Theorem** In order that a Peano space be a dendrite, it is necessary and sufficient that each cyclic element reduce to a point.

**Proof** \([9, \text{p. 89}]\)

**III.1 Some classification theorems**

**III.3.1 Theorem** If \( X \) is a one-dimensional Peano space and \( r(X) \) is countable, then \( X \) is a rational curve.

**Proof** Let \( a \) and \( b \) be distinct points in \( X \). By III.2.2 there is a closed set \( W \) with \( \dim W \leq 0 \) which separates \( a \) and \( b \) in \( X \). Since \( X \) is connected, \( \dim W \neq -1 \),
so dim $W = 0$. By I.5.2 $W$ is a totally disconnected set. By II.1.3 there are continua $X_1$ and $X_2$ such that $X = X_1 \cup X_2$, $a \in X_1$, $b \in X_2$. $X_1 \cap X_2 = Z \subset W$, and $Z$ separates $a$ and $b$ in $X$. But $Z$ can have at most a countable number of components, and the components of $Z$ are connected subsets of $W$. $W$ is totally disconnected, so the components of $Z$ are points.

This means that $a$ and $b$ are separated by a countable closed subset, so by III.2.2 $X$ is a rational curve.

III.3.2 Theorem If $X$ is a one-dimensional Peano space and $r(X)$ is finite, then $X$ is a regular curve.

Proof Using the same proof as III.3.1, we need note only that $X_1 \cap X_2 = Z$ can contain only a finite set of points. We then apply (b) of III.2.2.

III.3.3 Theorem If $X$ is a unicoherent one-dimensional Peano space, then $X$ is a dendrite.

Proof As in III.3.1, $X_1 \cap X_2 = Z$ can contain only a single point. We then apply (a) of III.2.2.

We note, however, that there are regular curves with infinite multicoherence.

III.3.4 Example Let $X$ be the perimeter of the triangle with vertices $(0,0), (1,1), (1,-1)$ together with the intersections of the lines $x = 1/2^n$, $n = 1, 2, \ldots$. 
with the interior of the triangle. Then $X$ is compact, connected and locally connected. Any two points can be separated by at most three points, so $X$ is a regular curve. However, if $X_1 = X \cap \{ (x,y) \mid y > 0 \}$ and $X_2 = X \cap \{ (x,y) \mid y \leq 0 \}$, then $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \{ (0,0) \} \cup \bigcup \{ (1/2^n,0) \mid n = 0, 1, 2, \ldots \}$ which is countably infinite.
CHAPTER IV

IV.1 We study here special properties of regular curves with a finite degree of multicoherence. We obtain two new characterization theorems for such spaces. These results are later used, along with the middle-space theory of the next chapter, to prove a characterization theorem for Peano spaces with finite multicoherence. We shall state the results here, and the remainder of the chapter will be devoted to the proofs.

IV.1.1 Definition An ordinary curve is a continuum which is the union of a finite collection of simple arcs which intersect pairwise in at most two points. We note that an ordinary curve is a regular curve.

IV.1.2 Definition A suitable system \((f, X, Y)\) is a continuous mapping \(f: X \rightarrow Y\) where \(X\) and \(Y\) are spaces, and \(f\) is either the identity mapping, or there exist spaces \(X = X_0, X_1, \ldots, X_k = Y\) and two-point identifications \(f_1, f_2, \ldots, f_k\) such that \(f_1: X_{i-1} \rightarrow X_i\) and \(f = f_k \circ f_{k-1} \circ \ldots \circ f_1\). The number
k of two-point identifications will be called the degree of f.

IV.1.3 **Theorem** Let X be a one-dimensional Peano space. Then X is a regular curve with \( r(X) = n \) if and only if there exists an ordinary curve \( A \subseteq X \) such that \( A \) is an \( A \)-set and \( r(A) = r(X) \).

IV.1.4 **Theorem** Let X be a Peano space. Then X is a regular curve with \( r(X) = n \) if and only if there exists a suitable system \((f, Z, X)\) such that \( Z \) is a dendrite and the degree of \( f \) is \( n \).

IV.2 **Subcontinua of regular curves with finite multicoherence**

IV.2.1 **Lemma** Let X be a regular curve with \( r(X) = n \) and let \( A \) be a subcontinuum of X. Then \( A \) is a B-set in \( X \).

**Proof** Let \( C \) be a component of \( X - A \) and suppose that \( \{a_1, \ldots, a_{n+2}\} \) are boundary points of \( C \).

Let \( \eta = \min \{ d(a_i, a_j) \mid i \neq j, i, j = 1, 2, \ldots, n+2 \} \).

Then \( \eta > 0 \). Let \( N_i \) be a connected, open neighborhood of \( a_i, i = 1, 2, \ldots, n+2 \), with \( \delta(N_i) < \eta/2 \) and let \( c_i \) be a point of \( C \) in \( N_i \).

\( C \) is an open, connected subset of a Peano space and is thus arcwise connected (I.4.3). Let \( c_{i+1}c_{i+1} \)
denote an arc in $C$ with $c_i$ and $c_{i+1}$ as end points and let $P = \bigcup \{c_i, c_{i+1} \mid i = 1, 2, \ldots, n + 1\}$. Then $P \subset C$ and is a subcontinuum of $C$ and of $X$. We also have $P \cap A = \emptyset$, so $\epsilon = d(P, A) > 0$.

For $x \in A$ let $M_x$ be a neighborhood of $x$ with a finite boundary and with $d(M_x) < \epsilon/2$. $A$ is compact, so there is a finite subset $M_1, \ldots, M_k$ of $\{M_x \mid x \in A\}$ which covers $A$. Let $B = \bigcup \{M_j \mid j = 1, \ldots, k\}$. Then $B$ is a closed subset of $X$, $A \subset B$ and $B(B) \subset \bigcup \{B(M_j) \mid j = 1, \ldots, k\}$, which is a finite point set. We also have $B \cap P = \emptyset$.

$P$ is a connected subset of $X - B$ and is contained in a component $D$ of $X - B$. Then $B(D) \subset B(B)$, so by I.6.2 $B(D)$ can contain at most $n + 1$ points.

Each $N_i$ chosen above is an open, connected subset of $X$. By I.4.3 there is an arc $\alpha_i$ in $N_i$ with $a_i$ and $c_i$ as end points. Then $\alpha_i \cap A \neq \emptyset$, $\alpha_i \cap D \neq \emptyset$ so $\alpha_i \cap B(D) \neq \emptyset$ (I.4.5).

However, $\alpha_i \cap \alpha_j = \emptyset$ for $i \neq j$, so $B(D)$ must contain $n + 2$ points. This is a contradiction, so we must conclude that $C$ has at most $n + 1$ boundary points.
IV.2.2 Example Consider example III.3.4. \( X_1 \) is a subcontinuum, and \( X - X_1 \) is connected.

\[
(X - X_1 = X \cap \{ (x,y) \mid y < 0 \})
\]

but does not have a finite boundary.

IV.2.3 Example Let \( I = [0,1] \) be the unit interval. Let \( C_i \) be the circle with center at \( (1/2^i, 1/4^i) \) and with radius \( 1/4^i \). Let \( X = I \cup ( \cup \{ C_i \mid i = 1, 2, \ldots \} ) \). Then \( X \) is a Peano space and every subcontinuum is a B-set. But \( r(X) = \infty \).

These examples show that IV.2.1 has no direct converse. However, we have the following lemma.

IV.2.4 Lemma If \( X \) is a Peano space and every subcontinuum of \( X \) is a B-set, then \( X \) is a regular curve.

Proof Let \( a \) and \( b \) be points in \( X \). By I.7.1 there are three disjoint sets \( R, \mathcal{F} \) and \( G \) such that \( R \) and \( G \) are open connected sets, \( a \in R \), \( b \in G \), \( X = R \cup \mathcal{F} \cup G \) and \( \mathcal{B}(R) = \mathcal{B}(G) = \mathcal{F} \). This means that \( \mathcal{F} \) separates \( a \) and \( b \) in \( X \). However, \( R \cup \mathcal{F} \) is a continuum and hence a B-set. But \( G = X - (R \cup \mathcal{F}) \) is a component of \( X - (R \cup \mathcal{F}) \) and hence has a finite point set as a boundary. Since \( \mathcal{F} = \mathcal{B}(G) \), \( \mathcal{F} \) is a finite set of points. By III.2.2, \( X \) is a regular curve.

We then have the following:
IV.2.5 **Theorem** Let $X$ be a Peano space with $r(X) = n$. Then $X$ is a regular curve if and only if every subcontinuum is a B-set.

**Proof** The proof follows immediately from IV.2.1 and IV.2.4.

IV.2.6 **Lemma** Let $X$ be a regular curve with $r(X) = n$, let $A$ be a subcontinuum of $X$, and let $\mathcal{P}(A)$ be the set of all components of $Z - A$ which have more than one boundary point or whose closures are not unicoherent. Then $\mathcal{P}(A)$ can contain at most $n$ elements.

**Proof** Let $D_0, D_1, \ldots, D_n$ be $n + 1$ elements of $\mathcal{P}(A)$, and let $E_i = \mathcal{K}(D_i)$, $i = 0, \ldots, n$. Then each $E_i$ is a regular curve and a Peano space. If $E_i$, $i = 0, \ldots, n$, has two or more boundary points, let $e_i$ be one of them and let $P_i$ be the remaining ones. Since $E_i - e_i$ is a connected, open (in $E_i$) subset of a Peano space, it is arcwise connected.

$P_i$ is a finite point set since $A$ is a B-set. Let $Q_i$ be the union of a finite number of arcs in $E - e_i$ joining the points of $P_i$. Then $e_i$ and $Q_i$ are disjoint closed connected sets in $E_i$, so by I.7.1 there are disjoint open connected sets $F_i$ and $G_i$ and a set $H_i$ such that $E_i = F_i \cup G_i \cup H_i$,

$$\mathcal{B}_{E_i}(F_i) = \mathcal{B}_{E_i}(G_i) = H_i \text{ and } e_i \subseteq F_i, Q_i \subseteq G_i.$$
Let $E_i = F_i \cup H_i, N_i = G_i \cup H_i$. Then

$$E_i = E_i \cup N_i, M_i \cap N_i = H_i \text{ and } H_i \cap B(E_i) = \emptyset.$$  

If $E_i$ has a single boundary point, then $E_i$ is not unicoherent. In this case let $E_i = M_i \cup N_i$ where $M_i$ and $N_i$ are continua and $M_i \cap N_i$ has more than one component. By II.4.1 we may assume the boundary point of $E_i$ is in $M_i \cap N_i$.

Let $K = X - \bigcup D_i$. $K$ is closed since $\bigcup D_i$ is open and it is connected since $A \subseteq K$ and if $x \in K - A$, $x$ is in a component of $X - A$ whose boundary is in $A$.

Let $M = K \cup (\bigcup M_i), N = K \cup (\bigcup N_i)$. Then $M$ and $N$ are continua and $X = M \cup N$. Furthermore, $M \cap N = K \cup (\bigcup (M_i \cap N_i))$ and so must contain at least $n + 2$ components since each $M_i \cap N_i$ contains at least one component in $D_i$. This contradicts the fact that $r(X) = n$. We must then conclude that $\mathcal{Y}(A)$ can contain at most $n$ elements.

IV.2.7 Lemma Let $X$ be a regular curve with $r(X) = n$. If $A$ is a subcontinuum of $X$ which is a B-set and not an A-set, then $r(A) < n$.

Proof

Let $\mathcal{Y}(A) = \{D_0, D_1, \ldots, D_k\}$. Since $A$ is a B-set and not an A-set, $\mathcal{Y}(A) \neq \emptyset$. By IV.2.6 $k \leq n$. 

Let \( A = B \cup C \) where \( B \) and \( C \) are continua. By II.4.1 we may assume that \( B(D_i) \subseteq B \cap C \), \( 0 \leq i \leq k \). Using the procedure of the previous lemma, there are continua \( F \) and \( G \) such that
\[
\mathcal{K}(D_0) = F \cup G, \quad F \cap G \subseteq D_0,
\]
\[
B(D_0) \cap F \neq \emptyset \quad \text{and} \quad B(D_0) \cap G \neq \emptyset.
\]

Now \( K = A \cup \bigcup \{ X_i \mid i = 0, \ldots, k \} \) is an \( A \)-set of \( X \). This implies \( r(K) \leq n \) (I.8.2).

Let \( X = B \cup F \cup \bigcup \{ X_i \mid i = 1, \ldots, k \} \) and let \( M = C \cup G \). Then \( X \) and \( M \) are continua and \( K = X \cup M \). However, \( X \cap M = (B \cap C) \cup (F \cap G) \) which has at least one component in \( D_0 \) and \( D_0 \cap (B \cup C) = \emptyset \). Therefore, \( B \cap C \) must have one less component than \( X \cap M \) which can have at most \( n + 1 \) components. This means that \( r(A) \leq n - 1 < n \).

IV.2.8 Lemma Let \( X \) be a Peano space with \( r(X) = n \) and let \( A \) be a non-degenerate \( A \)-set of \( X \). Then \( r(X) = r(A) \) if and only if the closure of every component of \( X - A \) is unicoherent.

Proof By I.8.1 \( r(A) \leq r(X) \).

Assume \( r(X) = r(A) \) and let \( C \) be a component of \( X - A \). Then \( \mathcal{K}(C) \) is a continuum and an \( A \)-set (I.8.1). Suppose \( \mathcal{K}(C) \) is not unicoherent. Let \( \{c\} = B(C) \) and let \( \mathcal{K}(C) = P \cup Q \) where \( P \cap Q \) has more than one component. By II.4.1 we can have \( c \in P \cap Q \).
Now let $E \cup F = A$ where $E \cap F$ has $n + 1$ components, and $E$ and $F$ are continua. Since $c \in A$ also, we can have $c \in E \cap F$ (II.4.1). Then $E' = E \cup P$ and $F' = F \cup Q$ are continua and $E' \cup F' = A \cup C$. Now $A \cup C$ is also an $A$-set (I.8.1). However, $E' \cap F' = (E \cap F) \cup (P \cap Q)$ which has at least $n + 2$ components. This contradicts $r(A \cup C) \leq n$ so we must conclude that $\mathcal{X}(C)$ is unicoherent.

Assume that $\mathcal{X}(C)$ is unicoherent for every component $C$ of $X - A$. If $r(X) = 0$, there is nothing to prove; so we assume $r(X) \geq 1$. By II.2.3 $X$ must contain at least one proper cyclic element. Let $E_1, E_2, \ldots, E_k$ be the non-unicoherent proper cyclic elements of $X$. I.9.1 implies there are only a finite number of these. If $E_i \subset \mathcal{X}(C)$ then $C$ is not unicoherent, since $E_i$ is an $A$-set of $X$ and hence of $\mathcal{X}(C)$. Therefore, $\bigcup_{i=1}^{k} E_i \subset A$. These are proper cyclic elements of $A$, so $n \geq r(A) \geq \sum r(E_i) = r(X) = n$.

IV.219 Lemma Let $X$ be a regular curve with $r(X) = n$. If $C$ is an ordinary curve in $X$ with $r(C) < n$ then there is an ordinary curve $B$ in $X$ such that $C \subset B$, $r(C) < r(B) \leq n$. 
Proof Let \( \mathcal{D}(C) \) be as in IV.2.6. If \( \mathcal{D}(C) = \emptyset \), then \( C \) is an A-set satisfying the conditions of IV.2.8. This means \( r(C) = n \). We must then conclude that \( \mathcal{D}(C) \neq \emptyset \).

If \( D \in \mathcal{D}(C) \) has at least two boundary points let \( \alpha \) be an arc in \( \mathcal{K}(D) \) joining two boundary points. Then \( B = C \cup \alpha \) is an ordinary curve and a subcontinuum of \( X \). This means that \( r(B) \leq n \) by IV.2.7. Also, \( C \) is a B-set of \( B \) which is not an A-set. Then \( r(C) < r(B) \) (IV.2.7).

If \( D \in \mathcal{D}(C) \) has only one boundary point, then \( \mathcal{K}(D) \) is not unicoherent. This means that \( \mathcal{K}(D) \) must contain a simple closed curve \( \gamma \). If \( \gamma \cap \mathcal{B}(D) = \emptyset \), let \( \alpha \) be an arc joining \( \mathcal{B}(D) \) and \( \gamma \). Otherwise \( \alpha = \mathcal{B}(D) \). Then \( \alpha \cup \gamma \) can be decomposed into three arcs, \( \beta \), \( \delta \) and \( \epsilon \). Setting \( B = C \cup \beta \cup \delta \cup \epsilon \) we have an ordinary curve \( B \), \( C \subset B \), and by IV.2.7 \( r(B) \leq n \). Since \( C \) is an A-set of \( B \) and \( \gamma \) is a cyclic element of \( B \) with \( r(\gamma) = 1 \), we have that \( r(C) < r(B) \).

IV.2.10 Lemma Let \( X \) be a regular curve with \( r(X) = n \). Then there is an ordinary curve \( A \subset X \) such that \( A \) is an A-set and \( r(A) = r(X) \).

Proof Let \( C_0 \) be an arc in \( X \). If \( r(X) = 0 \), \( X \) is a dendrite and \( C_0 \) is an A-set in \( X \) and \( r(C_0) = 0 \). We set \( A = C \) and we have our result.
Otherwise assume \( r(X) > 0 \). Then by IV.2.9 there is an ordinary curve \( C_1 \) in \( X \) such that \( C_0 \subset C_1 \), \( r(C_0) < r(C_1) \leq r(X) \). If \( r(C_1) = r(X) \), \( C_1 \) must be an \( A \)-set or else we can find \( C_1 \subset C_2 \) by the method of IV.2.9 with \( r(C_1) < r(C_2) \leq r(X) \). If \( r(C_1) < r(X) \) we find \( C_2 \) by IV.2.9 such that \( C_1 \subset C_2 \), \( r(C_2) > r(C_1) \). Continuing inductively in this manner we must find, after a finite number of steps, \( C_k, k \leq n \), such that \( C_k \) is an ordinary curve and \( r(C_k) = r(X) \). Now \( C_k \) is an ordinary curve and \( r(C_k) = r(X) \). Now \( C_k \) must be an \( A \)-set since it is a \( B \)-set and if there were a component of \( X - C_k \) with more than one point we could apply the construction of IV.2.9 to get \( C_{k+1} \) with \( r(C_{k+1}) > r(C_k) = r(X) \).

IV.3 Proof of IV.1.3

IV.3.1 Lemma Let \( A \) and \( B \) be subcontinua of a Peano space \( P \) such that \( A \cap B \neq \emptyset \) and \( A \cap (P - B) \neq \emptyset \) and let \( H \neq \emptyset \) be a component of \( A \cap B \). Then \( H \cap (B) \neq \emptyset \).

Proof Suppose \( H \subset B - B(B) \) which is an open subset of \( P \). Since \( H \) is an open and closed subset of \( A \cap B \), there is an open set \( \emptyset \) and a closed set \( C \) in \( P \) such that \( H = A \cap B \cap \emptyset = A \cap B \cap C \). \( A, B, \) and \( C \) closed in \( P \) implies that \( H \) is closed in
P and in A. Let \( O' = O \cap (B - B(B)) \subseteq O \cap B \). Then \( O' \) is open in \( P \) and \( \text{H} \subseteq O' \). Therefore \( \text{H} \subseteq O' \cap A \). However, \( O' \cap A \subseteq O \cap A \cap B = \text{H} \). Thus \( \text{H} = A \cap O' \) is an open subset in \( A \). This means that \( \text{H} = A \) since \( \text{H} \neq \emptyset \) and \( A \) is connected. But \( A \cap (P - B) \neq \emptyset \) and \( \text{H} = A \subseteq B - B(B) \) is a contradiction. Therefore, \( \text{H} \not\subseteq B \) and \( \text{H} \not\subseteq B - B(B) \) means that \( \text{H} \cap B(B) \neq \emptyset \).

**IV.3.2 Lemma** Let \( B \) be a subcontinuum of a Peano space \( P \) such that \( B(B) = \{ b_1, \ldots, b_k \} \). If \( A \) is any other subcontinuum of \( P \), then \( A \cap B \) can have at most \( k \) components.

**Proof** If \( \text{H} \) is a component of \( A \cap B \), then by the preceding lemma \( H = A \subset B, H = A \subset X - B, \) or \( H \cap B(B) \neq \emptyset \). Since there are only \( k \) boundary points of \( B \), there are at most \( k \) components of \( A \cap B \).

**IV.3.3 Lemma** Let \( X \) be a regular curve with \( r(X) = n \) and assume \( X = A \cup B \) where \( A \) is an arc, \( B \) is a continuum, and \( A \cap B = \{ a, b \} \). Then \( r(B) = n - 1 \).

**Proof** \( B \) is a \( B \)-set which is not an \( A \)-set. By IV.2.7 \( r(B) \leq n - 1 \).

We shall next show that \( r(B) \geq n - 1 \). Let \( X = P \cup Q \) where \( P \) and \( Q \) are continua and \( P \cap Q \) has \( n + 1 \) components. By II.4.1 we assume that
a, b \in P \cap Q. By IV.3.2 we note that P \cap A, Q \cap A, P \cap B, Q \cap B have at most two components and each component must contain a or b. We also note that if P \cap A (Q \cap A) is connected, P \cap A = A (Q \cap A = A since A is an arc and a, b \in P \cap A (Q \cap A).

Suppose P \cap A = H \cup K has two components. Then if P \cap B = M \cup N has two components, we have \{a\} = H \cap M, \{b\} = K \cap N and P = (H \cup M) \cup (K \cup N). Now H \cup M and K \cup N are continua. However, (H \cup M) \cap (K \cup N) = \emptyset which contradicts the connectedness of P. Thus P \cap B is connected.

Applying a similar argument we get that if P \cap B (Q \cap A, Q \cap B) has two components, then P \cap A (Q \cap B, Q \cap A) is connected. Since the components of Q \cap A and P \cap A are continua in an arc A, they must be arcs or points.

The remainder of the proof is contained in the following three cases.

Case 1 P \cap A = H \cup K, Q \cap A = M \cup N with a \in H \cap M and b \in K \cap N. Then P' = P \cap B and Q' = A \cap B are connected, hence continua, and P' \cup Q' = B. H, K, M, and N are arcs or points. H \cup M and K \cup N are continua and (H \cup M) \cup (K \cup N) = A. But A is an arc, so R = (H \cup M) \cap (K \cup N)
is connected, hence an arc or point. \((H \cup M) \cap (K \cup N)\) does not contain a or b, or else \(P \cap A\) or \(Q \cap A\) is connected. \(R\) is then a component of \(P \cap Q\) for it is contained in a component, a and b are in a component (or components) and if the component of \(P \cap Q\) containing \(R\) contained a point of \(B\) it would have to contain a or b by IV.3.2. Then \(P' \cap Q'\) contains exactly \(n\) components.

**Case 2** \(P \cap A = A, Q \cap A = M \cup N\). This also covers the case \(P \cap A = H \cup K, Q \cap A = A\). Now \(Q' = Q \cap B\) is connected. If \(P \cap B\) were connected, we would have \(Q' \cup (P \cap B) = B\) and \(P \cap B \cap Q'\) would have \(n + 1\) components. Thus \(P \cap B = H \cup K, a \in H, b \in K\). Let \(\alpha\) be an arc joining a and b in \(B\). Then \(B' = \alpha \cup (\bigcup \{C : C \in D_B(\alpha)\})\) is an \(A\)-set of \(B\) with \(r(B) = r(B')\). We can assume \(B = B'\) with no loss of generality. Then \(P \cap \alpha\) has at most \(n(N + 1)\) components. Since \(\alpha \subset P\) implies \(P \cap B\) connected, \(P \cap \alpha\) must have at least two components.

Let \(P \cap \alpha = C_1 \cup C_2 \ldots \cup C_k\), \(a \in C_1, b \in C_k\) be a decomposition of \(P \cap \alpha\) into its components. We must be able to find \(C_i, C_{i+1}\) such that \(C_i \subset H, C_{i+1} \subset K\). Let \(\beta\) be an arc in \(\alpha \cap Q\) with end points in \(C_i\) and \(C_{i+1}\). Then let \(P' = (P \cap B) \cup \beta\). Then
Consider $P' \cap Q'$. It must have less than $n$ components. Let $E_0, \ldots, E_n$ be the components of $P \cap Q$. Then we may suppose $E \subset E_0, N \subset E_1$. Then $E_2 \cup \ldots \cup E_n \subset P \cap Q \cap B$.

Now $\beta \subset Q - P$ except for the end points which are in $B(P) \subset P \cap Q$. Then $\beta$ can join at most two of the components $E_2, \ldots, E_n$ and $P' \cap Q'$ must have at least $n$ components.

**Case 3** $P \cap A = A, Q \cap A = A$. Then $A$ is contained in a component $E_0$ of $P \cap Q$. If $K \cap B$ is connected, $P \cap B$ and $Q \cap B$ are connected and $P \cap Q \cap B$ has $n + 1$ components. Thus $K \cap B$ is not connected. By IV.3.2 $K \cap B$ has two components and $(P \cap B) \cap (Q \cap B)$ has $n + 2$ components. Therefore one is not connected, say $P \cap B$. We use the argument above to find an arc $\beta$ such that $(P \cap B) \cup \beta = P'$ is connected. Then $P' \cap (Q \cap B)$ would have $n + 1$ components. So $Q \cap B$ must not be connected. Applying the preceding argument we find an arc $\gamma$ such that $Q' = (Q \cap B) \cup \gamma$ is connected. $P' \cap Q'$ has two less components than $P \cap Q \cap B$, which had $n + 2$ components. Then $P' \cap Q'$ has $n$ components.

This establishes the lemma.
IV.3.4 Lemma Let $X$ be an ordinary curve with 
$$X = \bigcup \alpha_i, \ i = 1, \ldots, n,$$
where each $\alpha_i$ is an arc and $\alpha_i \cap \alpha_j, i \neq j,$ is empty, a point or two points. Then $r(X) \leq n - 1.$

**Proof** By induction on $n.$

If $n = 1$, $X = \alpha_1$ is an arc and $r(X) = 1 - 1 = 0.$

Assume the theorem is valid for $n = k$. We must show it is valid for $n = k + 1$. Assume $X = \bigcup \alpha_i$, $i = 1, \ldots, k + 1$. If $X$ contains no simple closed curve it is a dendrite and $r(X) = 0$. Let $\gamma \subset X$ be a simple closed curve. If $p \in \gamma$, then there is an arc $\alpha_1$ (we assume $i = 1$) so that $p \in \alpha_1$. Furthermore, $\alpha_1 \subset \gamma$. $\gamma - \alpha_1 \neq \emptyset$ or else $\alpha_1$ is not an arc. Then $Y = \bigcup \alpha_i, i = 2, \ldots, k + 1$ is a subcontinuum of $X$ and by the induction hypothesis, $r(Y) \leq k - 1$. Then $Y \cup \alpha_1 = X$. If we have $X = P \cup Q$ where $P$ and $Q$ are continua and $P \cap Q$ has more than $k + 1$ components, we can repeat the proof of IV.3.3 to get $r(Y) > k - 1$. This is possible without the hypothesis of IV.3.3 that $r(X) = n$ because this part of the hypothesis was used only to show that the arc in $Y$ joining the end points of $\alpha_1$ had a finite number of boundary points while we have that fact here because $X$ is an ordinary curve. Thus $r(X) \leq k$ and we have the lemma by induction.
IV.3.5 **Proof of IV.1.3**

The necessity follows from IV.2.10.

If $A$ is an ordinary curve by IV.3.4, we have $r(A) = n < \infty$ and $r(X) = n$.

IV.4 **Two-point identifications of regular curves**

In this section we prepare for the proof of IV.1.4 by investigating the effect of a two-point identification on a regular curve.

IV.4.1 **Lemma** Let $f(X) = Y$ be a two-point identification where $X$ is a regular curve and $Y$ is a space. Then $Y$ is a regular curve.

**Proof** Let $a$ and $b$ be any distinct points in $Y$. We assume that $f^{-1}(b)$ is a single point in $X$ since $f^{-1}(y)$ is non-degenerate for only one point $r$ in $Y$.

Let $N$ be a neighborhood of $f^{-1}(b)$ with a finite boundary such that $N \cap f^{-1}(a) = \emptyset$ and $N \cap f^{-1}(r) = \emptyset$.

Now $N$ is an open subset of $X$ and $N \subseteq X - \{p,q\}$, where $p$, $q$ are the points identified under $f$. Then $N$ is an open subset of $X - \{p,q\}$. By I.10.8 $f(N)$ is an open subset of $Y - \{r\}$ and hence open in $Y$. Then $Z = B_Y(f(N)) = f(B_X(N))$ is a finite set of points. We also have that $a \notin f(N)$. 

Consider \(Y - Z\). This is not connected, for if it were we would have, by I.4.3, an arc \(\alpha\) in \(Y - Z\) joining \(a\) and \(b\) since \(a, b \in Y - Z\).

But by I.4.5 \(\alpha \cap B_Y(f(N)) \neq \emptyset\). This means that \(Z\) separates \(a\) and \(b\) in \(Y\) and \(Z\) is a finite point set. Thus \(Y\) is a regular curve by III.2.2.

IV.4.2 Lemma Let \(f(X) = Y\) be a two-point identification where \(X\) and \(Y\) are Peano spaces. Then \(r(Y) \geq 1\).

Proof Let \(f^{-1}(r) = \{p, q\}\). By I.7.1 there are continua \(P\) and \(Q\) (\(P = R \cup F\), \(Q = G \cup F\)) such that \(p \in P - Q\), \(q \in Q - P\) and \(X = P \cup Q\). Then \(f(P)\) and \(f(Q)\) are continua, \(f(P) \cup f(Q) = Y\) and \(f(P) \cap f(Q) = f(P \cap Q) \cup \{r\}\) and \(\{r\}\) is a component of \(f(P) \cap f(Q)\). Thus \(r(Y) \geq 1\).

IV.4.3 Lemma Let \(X\) be a regular curve, \(1 \leq r(X) = n\). Then there is a regular curve \(Y\), \(r(Y) = n - 1\) and a two-point identification \(f(Y) = X\).

Proof \(r(X) \geq 1\) implies that \(X\) contains a simple closed curve \(\gamma\). \(\gamma\) is a subcontinuum of \(X\), so by IV.2.6 \(X - \gamma\) has at most \(n\) components which have more than one boundary point or which have one boundary point and are not dendrites. Thus \(\gamma\) contains only a finite number of points which are boundary points of such continua. Let \(\alpha\) be an arc in \(\gamma\).
which does not contain any of these points. Let 
A be the union of $\alpha$ and all components of $X - \gamma$ 
which have boundaries in $\alpha$. Then the closure of 
each such component is a dendrite and A is a dendrite. 
Let $B = \mathcal{K}(X - A)$. Since $\gamma - \alpha$ is connected and 
$B = \mathcal{K}(\gamma - \alpha) \cup (\cup \{C\} C$ is a component 
of $X - \gamma$, $B(C) \subset (\gamma - \alpha)$ we have that 
$B$ is a continuum. Also, $A \cap B = \{a, b\}$ where a 
and b are the end points of $\alpha$.

The hypotheses of IV.3.4 are satisfied and we 
have that $r(B) = n - 1$. 
Let $C$ be $(A - \{b\}) \cup \{p\}$ where $p$ is a point 
not in $X$ and let the topology for $C$ be the same as 
that of $A$, replacing $b$ everywhere by $p$ (in the metric). 
Let $Y = B \cup C$. Then $B \cap C = \{a\}$. By II.2.10 
$Y$ is a Peano space. $B$ is an A-set of $Y$ satisfying 
IV.2.6. This means that $r(Y) = r(B) = n - 1$. Also 
$Y$ is a regular curve since $B$ and $C$ are regular curves 
which intersect at a point. For if $x$ and $y$ are any 
two points in $Y$, $x, y \in B$ can be separated by a finite 
set of points in $B$ and the same set separates $x$ and 
y in $Y$. $x, y \in C$ can be separated by a single point 
in $C$ which also separates $x$ and $y$ in $Y$. $x \in B - C$, 
y $\in C - B$ can be separated by a. By III.2.2 $Y$ is 
a regular curve.
Define \( f(Y) = X \) as follows:
\[
    f(x) = \begin{cases} 
        x & \text{if } x \neq p \\
        b & \text{if } x = p 
    \end{cases}
\]

To show that \( f \) is continuous let \( D \) be any closed subset of \( X \). If \( b \notin D \) then \( f^{-1}(D) = D \) and is closed. If \( b \in D \), then \( D \cap A \) and \( D \cap B \) are closed sets and \( f^{-1}(D \cap B) = D \cap B \) which is closed. Let \( D' = (D \cap (A - \{b\})) \cup \{p\} \). Then \( D' \) is a closed subset of \( C \) and hence of \( Y \).

\( f^{-1}(D) = (D \cap B) \cup D' \) which is closed, \( f \) is a homeomorphism between \( Y - \{b,p\} \) and \( X - \{b\} \).

\( f^{-1}(x) = x \) if \( x \neq b \) and \( f^{-1}(b) = \{b,p\} \) so \( f \) is a two-point identification.

**IV.4.4 Theorem** Let \( X \) be a regular curve, \( 1 \leq r(X) = n \). Then there are \( n \) regular curves \( Y_1, \ldots, Y_n \) and \( n \) two-point identifications \( f_1, \ldots, f_n \) such that \( Y_n \) is a dendrite, \( f_i(Y_i) = Y_{i-1}, i = 2, \ldots, n, f_1(Y_1) = X \).

**Proof** The proof is by induction on \( n \).

Let \( n = 1 \). By IV.4.3 there is a regular curve \( Y_1 \) and a two-point identification \( f_1 \) such that \( f_1(Y_1) = X \) and \( r(Y_1) = n - 1 = 1 - 1 = 0 \). This implies that \( Y_1 \) is a dendrite.

Assume the theorem valid for \( n = k \).

Let \( r(X) = k + 1 \). By IV.4.3 there is a regular
curve $Y_1$ and a two-point identification $f_1$ such that $f_1(Y_1) = X$ and $r(Y_1) = k$. By the induction hypothesis (relabeling as we proceed) there are $k$ regular curves, $Y_2, \ldots, Y_k$ and $k$ two-point identifications $f_2, \ldots, f_k$ such that $Y_k$ is a dendrite, $f_i(Y_i) = Y_{i-1}$ for $i = 2, \ldots, k+1$, $f_1(Y_1) = X$.

IV.4.5 Lemma If $f(X) = Y$ is a two-point identification of two continua $X$ and $Y$, if $p$ and $q$ are the points of $X$ identified by $f$, and if $B$ is a subcontinuum of $Y$, then $f^{-1}(B)$ is a continuum or $f^{-1}(B)$ has two components, one of which contains $p$ and the other $q$.

Proof If $f(p) \notin B$ then $f^{-1}(B)$ is a continuum in $X - \{p,q\}$. Assume $f(p) \in B$ and that $f^{-1}(B)$ is not connected. If $C$ is a component of $f^{-1}(B)$ and $p \notin C$ and $q \notin C$ then $f(p) \notin f(C)$. However $f(C)$ is a closed subset of $B$. Now $C$ is open in $f^{-1}(B)$ so there is an open subset $O$ of $X$ such that $p,q \notin O$ and $O \cap f^{-1}(B) = C$. This means that $p \notin O$ and $q \notin O$ so $f(O)$ is an open subset of $Y - \{f(p)\}$ and also of $Y$ since $Y - \{f(p)\}$ is open. However $f(O) \cap f(C) = f(O \cap C) = f(C)$ so $f(C)$ is open in $B$. This contradicts the connectedness of $B$ unless $C = B$. Thus if $C$ is a component of $f^{-1}(B)$ we must have $p \in C$ or $q \in C$ and this limits $f^{-1}(B)$ to at most two components.
IV.4.6 Lemma Let $X$ be a regular curve with $r(X) = n$ and let $f(X) = Y$ be a two-point identification where $Y$ is a space. Then $Y$ is a regular curve and $r(Y) = n + 1$.

$Y$ is a regular curve by IV.4.1.

We show that $r(Y) = n + 1$ by induction on $n$. Suppose $n = 0$. Then $X$ is a dendrite. Let $\alpha$ be the arc in $X$ joining $a$ and $b$. Then $f(\alpha)$ is a simple closed curve and is a subcontinuum, hence a B-set, of $Y$. Since $r(f(\alpha)) = 1$, by I.8.1 $r(Y) > 1$.

However, any component of $Y - \{f(\alpha)\}$ is homeomorphic to a component of $X - \alpha$ and its closure is a dendrite. This means that $f(\alpha)$ is an A-set of $Y$ and the closure of every component of $Y - \{f(\alpha)\}$ is a dendrite. By IV.2.6 $r(Y) = r(f(\alpha)) = 1$.

Assume that for $0 \leq k \leq n$, $r(Y) = k + 1$.

Let $n = k + 1$. We distinguish two cases.

Case 1 There is a dendrite $A$ such that $a \in A$, $\mathcal{B}(A) = \{d\} \neq \emptyset$. Then $X = A \cup B$, $B = \mathcal{K}(X - A)$.

Then $f(A)$ is a dendrite, $f(B)$, $r(f(B)) = k + 1$, is a continuum and $Y = f(A) \cup f(B)$, $f(A) \cap f(B) = \{c, f(d)\}$. By IV.3.3 $r(Y) = r(B) + 1 = k + 2$ if $r(Y)$ is finite. We shall show below that $r(Y)$ cannot be infinite. We note that the same argument applies if $b \in A$. 
Case 2 a lies on a simple closed curve in X. Using the arguments in IV.4.3 we find a regular curve Z and a two-point identification g such that g(Z) = X and r(Z) = k and X = C \cup D where C is a dendrite, D is a continuum, C \cap D = \{e,d\} and we can choose e and d distinct from a and b. Now Z = C' \cup D' where C' \cap D' = \{d'\}. C' is a dendrite and g^{-1}(e) consists of two points h and j.

Let h(Z) = W be a two-point identification of Z defined by identifying a and b. Then by the induction hypothesis r(W) = k + 1. Then j(W) = Y made by identifying h and j is a two-point identification which satisfies Case 1. Then r(Y) = k + 1.

To complete the proof we must show that r(Y) cannot be infinite. Let Y = H \cup K, H, K, continua, where H \cap K has infinitely many components. We note that f^{-1}(H), f^{-1}(K) have at most two components (IV.4.5), say f^{-1}(H) = M \cup N, f^{-1}(K) = P \cup Q. Then one of M \cap P, M \cap Q, N \cap P, or N \cap Q must have infinitely many components, say M \cap P. Then M \cup P is a subcontinuum of X. By IV.2.7 r(M \cup P) \leq r(X). This is a contradiction if M \cap P has infinitely many components. Therefore r(Y) is finite.
IV.4.7 Theorem Let $X_0$ be a dendrite, let $X_1, \ldots, X_n$ be spaces and let $f_1, \ldots, f_n$ be two-point identifications such that $f_i(X_{i-1}) = X_i$. Then $X_1, \ldots, X_n$ are regular curves and $r(X_i) = i, 1 \leq i \leq n$.

Proof By IV.4.6 $f_1(X_0) = X_1$ so $X_1$ is a regular curve and $r(X_1) = 1$, by lemma IV.4.6 $f_2(X_1) = X_2$, so $X_2$ is a regular curve and $r(X_2) = 2$. Continuing in this manner we get $f_n(X_{n-1}) = X_n$, $X_n$ is a regular curve and $r(X_n) = n$.

IV.4.1 Proof of IV.1.4

The necessity follows from IV.4.4 and the sufficiency from IV.4.7.
V.1 A classical result in analytic topology is the following:

Let \( f: X \to I \) be any continuous mapping of the Peano space \( X \) into the unit interval \( I \). Then \( X \) is unicoherent if and only if the middle space in the monotone-light factorization of \( f \) is a dendrite.

In this chapter we generalize this result somewhat.

V.1.1 Theorem Let \( X \) be a Peano space, let \( Y \) be a space such that \( \dim Y = 1 \), and let \( f: X \to Y \) be a continuous mapping of \( X \) into \( Y \). Let \( M \) be the middle space in the monotone-light factorization of \( f \).

Then:

(a) \( \dim (M) \leq 1 \).

(b) If \( r(X) \) is countable, then \( M \) is a rational curve.

(c) If \( r(X) \) is finite, then \( M \) is a regular curve.

(d) If \( r(X) = 0 \), then \( M \) is a dendrite.

Proof We assume \( M \) is non-degenerate, or else there is nothing to prove.
By I.5.1, dim \( f(X) \leq dim Y = 1 \), and by II.3.2
\[ \dim \mathcal{M} = \dim Y = 1. \]
Since \( \mathcal{M} \) is compact, connected, and non-degenerate, by I.5.2 \( \dim \mathcal{M} \geq 1 \). Thus
\[ \dim \mathcal{M} = 1. \] This establishes (a).

By I.10.2, \( r(\mathcal{M}) \leq r(X) \). From III.3.1, III.3.2, and III.3.4 we have (b), (c), and (d).

V.1.2 Example

Let \( X_0 \) be the boundary of the unit square.

Let \( X_n = I \times \{1/2^n\} \), \( n = 1, 2, \ldots \) (\( I = [0,1] \))
Let \( Y_n = \{1/2^n, 2/2^n, \ldots, 2^{n-1}/2^n\} \times [0, 1/2^n] \)

Now let \( X = X_0 \cup \left( \bigcup_{n=1}^{\infty} X_n \right) \cup \left( \bigcup_{n=1}^{\infty} Y_n \right) \).

\( X \) is a Peano space, a rational curve, and has a countable degree of multicoherence.

Define a mapping \( \mathcal{L}:X \rightarrow I \) by projecting \( p \in X \) orthogonally onto the diagonal line joining \((0,0)\) and \((1,1)\) and then projecting that point orthogonally onto the interval \([0,1]\). \( \mathcal{L} \) is continuous since "adjacent" points go into "adjacent" points. \( \mathcal{L} \) is light since the inverse of a point is a line \( y = -x + a, \)
\( 0 \leq a \leq 1 \) intersected with \( X \), and this intersection must be a countable set of points.

We then have a continuous mapping of a Peano space which has a countable degree of multicoherence onto the unit interval where the middle space is a rational curve and not a regular curve.
V.1.3 Example

Let \( X_0 \) be the boundary of the unit square and let \( C \) be the Cantor set. Let \( A_1 = [0, 1/3] \cup [2/3, 1] \), \( A_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1], \ldots \) be the intervals defining the Cantor set, i.e.
\[
C = \bigcap_{n=1}^{\infty} A_n.
\]

Let \( X_1 = I \times C \)
and \( Y_n = \{1/2^n, \ldots, 2^{n-1}/2^n\} \times A_n \).

Now let \( X = X_0 \cup (\bigcup_{n=1}^{\infty} X_n) \cup (\bigcup_{n=1}^{\infty} Y_n) \).

Then \( X \) is a Peano space and the degree of multicoherence of \( X \) is non-countable (\( r(X) = \text{card } C \)).

Define a mapping of \( X \) onto the unit interval as in V.1.2. Then we have \( \mathcal{L}:X \rightarrow I \) and \( \mathcal{L} \) is light.

This is an example of a Peano space which has a non-countable degree of multicoherence and a continuous mapping of this space onto the unit interval. The middle space is \( X \) and \( X \) is not a rational curve, but is one-dimensional.

V.1.4 Example

Let \( X \) be the unit circle and let \( Y \) be the interval \([-1,1]\). Let \( \mathcal{L}:X \rightarrow Y \) be the projection \( \mathcal{L}(p) = \mathcal{L}((x,y)) = x \). Then \( \mathcal{L} \) is continuous and light. \( r(X) = 1, r(Y) = 0 \) and the middle space is \( X \).
These examples show that the results of V.1.1 are the strongest under the given hypotheses.

V.1.5 Lemma Let $X$ be a Peano space and let $I^X$ be the class of all continuous mappings of $X$ into the unit interval $I$. Let $M_f$ be the middle space in the monotone-logarithm factorization of $f \in I^X$, $f = \ell m$. Then $r(X) = \max \{ r(M_f) \mid f \in I^X \}$.

Proof By I.10.1 $r(M_f) \leq r(X)$.

Let $X = A \cup B$ where $A$ and $B$ are continua $X - A \neq \emptyset \neq X - B$. Let $a = \inf \{ d(x, A \cap B) \mid x \in A \}$ $b = \inf \{ d(x, A \cap B) \mid x \in B \}$ where $d$ is the metric in $X$. Since $X - A \neq \emptyset \neq X - B$, $a > 0$ and $b > 0$.

Define a mapping of $X$ into the unit interval $I$ by

$$f(x) = \begin{cases} \frac{1}{2} + \frac{d(x, A \cap B)}{2a} & \text{for } x \in A \\ \frac{1}{2} - \frac{d(x, A \cap B)}{2b} & \text{for } x \in B \end{cases}$$

$f(x)$ is continuous because the distance function is continuous. Since $X$ and $A \cap B$ are compact, there is a $p \in A$ and a $q \in B$ such that $d(p, A \cap B) = a$, $d(q, A \cap B) = b$. Then $f(p) = 1$, $f(q) = 0$.

We note that $f^{-1}(\frac{1}{2}) = A \cap B$ and $m(A)$ and $m(B)$ are continua in $M_f$. Furthermore, $m(A \cap B) = m(A) \cap m(B) = \mathcal{L}^{-1}(\frac{1}{2})$ which has the
same number of components as $A \cap B$ since the com-
ponents of $A \cap B$ are the points of $f^{-1}(\emptyset)$.

If $r(X) = n < \infty$ then there must exist con-
tinua $A$ and $B$ such that $A \cup B = X$ and $A \cap B$ has
$n + 1$ components. Then for the mapping defined above
$r(M_f) = n$. Thus
\[
r(X) = \max \{ r(M_f) \mid f \in \mathcal{I}^X \}
\]

If $X$ has a countable degree of multicoherence,
then for every integer $n$ there exists a pair of con-
tinua $A_n$, $B_n$ such that $X = A_n \cup B_n$ and $A_n \cap B_n$
has at least $n + 1$ components. If $f_n$ is the mapping
defined above and $M_{f_n}$ is the associated middle
space $r(M_{f_n}) \geq n$. This means
\[
\max \{ r(M_f) \mid f \in \mathcal{I}^X \}
\]
must be countably infinite.

If $X$ has a non-countable degree of multicoherence
there must be a pair of continua $A$ and $B$ such that
$A \cup B = X$ and $A \cap B$ has a non-countable number
of components. Then if $f$ is the function defined above
$r(M_f)$ must be non-countable.

V.1.6 Lemma Let $X$ be a Peano space, let $Y$ be a
space, and let $Y^X$ denote the class of all continuous
mappings of $X$ into $Y$. For $f \in Y^X$ let $M_f$ denote
the middle space in the monotone-light factorization
of $f$. Then if $Y$ contains a simple arc
\[
r(X) = \max \{ r(M_f) \mid f \in \mathcal{Y}^X \}.
\]
Proof By 1.10.1 \( r(\mathcal{M}_f) \leq r(X) \) for \( f \in Y^X \)
so \( \max \{ r(\mathcal{M}_f) \mid f \in Y^X \} \leq r(X) \).

If \( g \in I^X \) and \( h \) is a homeomorphism from \( I \) into
the arc \( \alpha \subset Y \) then \( hg \in Y^X \). Since \( h \) is light,
\( \mathcal{M}_{hg} = \mathcal{M}_g \). By V.1.5,
\[
\max \{ r(\mathcal{M}_g) \mid g \in I^X \} = r(X)
\]
and so \( \max \{ r(\mathcal{M}_f) \mid f \in Y^X \} = r(X) \).

V.1.7 Theorem Let \( X, Y \) and \( Y^X \) be as in V.1.6, suppose
that \( \dim Y = 1 \) and assume that \( Y \) contains a simple
arc \( \alpha \). Then \( X \) is unicoherent if and only if for
any \( f \in Y^X \), \( \mathcal{M}_f \) is a dendrite.

Proof If \( X \) is unicoherent \( \mathcal{M}_f \) is a dendrite by
(d) of V.1.1.

If \( X \) is not unicoherent, i.e. \( r(X) > 0 \), by
V.1.6 there is an \( f_0 \in Y^X \) such that \( r(\mathcal{M}_{f_0}) = r(X) > 0 \)
and \( \dim (\mathcal{M}_{f_0}) = 1 \). Thus \( \mathcal{M}_{f_0} \) is not a dendrite.

V.1.8 Corollary \( X \) is a unicoherent Peano space if
and only if the middle space in the monotone-light
factorization of any continuous mapping of \( X \) into the
unit interval \( I \) is a dendrite.

V.1.9 Corollary \( X \) is a unicoherent Peano space if
and only if for any continuous mapping of \( X \) into the
unit circle \( S \) in the complex plane the middle space
in the monotone-light factorization of the mapping
is a dendrite.
V.1.10 Example Let $C_j$ be the circle in the plane with center at $(3/2^{j+1},0)$ and radius $1/2^{j+1}$, $j = 1, 2, \ldots$ and let $X = \bigcup_{j=1}^{\infty} C_j \cup (0,0)$. Then $X$ is a Peano space and a regular curve. $X_1 = X \cap \{(x,y) | y > 0\}$ and $X_2 = X \cap \{(x,y) | y \leq 0\}$ are closed, connected subsets of $X$ and $X = X_1 \cup X_2$. $X_1 \cap X_2 \subseteq \{(0,0), (1,0), (1/2,0), \ldots\}$ Thus $r(X) = \infty$ (but countable).

For any $n \geq 1$, let $A_n = \bigcup_{j=1}^{n} C_j$ and let $m_n(p) = \begin{cases} p & \text{if } p \in A_n \\ (1/2^n,0) & \text{if } p \notin A_n \end{cases}$

Then $m_n$ is a monotone retract ($A_n$ is actually an $A$-set of $X$). Let $\ell_n(p) = \ell_n(x,y) = x$ for $p \in A_n$. Then $\ell_n$ is a light mapping of $A_n$ onto the interval $[1/2^n,1]$. We have then that $f_n = \ell_n m_n$ is a continuous mapping of $X$ into $I = [0,1]$. The middle space in the monotone light factorization is $A_n$ and $r(A_n) = n$.

A monotone image of a regular curve is a regular curve. Then for every $f \in \mathcal{I}^X$ the middle space $\mathcal{M}_f$ is a regular curve, but $X$ does not have a finite degree of multicoherence. This means that V.1.7 would not be valid if we replace unicoherence ($r(X) = 0$) by $r(X) = n$ and dendrite by regular curve. However, the following theorem is valid.
V.1.11 Theorem Let $X, Y, Y^X$ be as in V.1.6. If $Y$ contains a simple arc $\mathcal{X}$ then

$$r(X) = n \text{ (is countable, is noncountable)}$$

if and only if for every $f \in Y^X$, $\mathcal{M}_f$ is a regular curve (rational curve, one-dimensional continuum) and

$$\max \{r(\mathcal{M}_f) \mid f \in Y^X\} = n \text{ (is countable, is noncountable)}.$$

Proof The necessity follows from V.1.1 and V.1.6.

The sufficiency follows from V.1.5 and V.1.6.
CHAPTER VI

In this chapter we prove a slight generalization of a proposal of J. W. T. Youngs made at the 1958 Summer Institute on surface area and related topics [7]. According to a recent private communication with Youngs, he has never published a proof of this result.

The proposal of Youngs is contained in VI.1.2 and VI.1.3.

VI.1.1 Definition A Peano space is said to be of type A or an A-space if all its B-sets are A-sets.

VI.1.2 Theorem $X$ is a Peano space with $r(X) = n$ if and only if there exists a suitable system $(f, Z, X)$ where $Z$ is a Peano space of type A, the degree of $f$ is $k$ with $k \leq n$ and $r(Z) = n - k$.

VI.1.3 Theorem Let $Y$ be a Peano space with $r(Y) = n$ and let $(f, X, Y)$ be the suitable system of VI.1.2. Then $C$ is a fine-cyclic element of $Y$ if and only if there exists a proper cyclic element $D$ of $X$ such that $f(D) = C$. Furthermore $f(D) = C$ is a homeomorphism.
Youngs also proposed that B was a B-set of X if and only if there existed an A-set of Z such that f(A) = B. However, this is not so. Consider the mapping f(x) = e^{2\pi i x}, 0 \leq x \leq 1. This is a two-point identification of the unit interval I onto the unit circle J. Any non-degenerate subarc of J is a B-set but the inverse of any proper subarc containing (1,0) as an interior point is the union of two arcs in I. However, these two arcs are A-sets in I.

The remainder of this chapter will be devoted to a proof of these results.

VI.2 Proof of VI.1.2

VI.2.1 Lemma If f(X) = Y is a two-point identification of Peano spaces X and Y, then r(X) = n if and only if r(Y) = n + 1.

Proof Let p and q be the two points of X identified by f. By I.10.8 X = \{p, q\} and Y = \{f(p)\} are homeomorphic.

Let g:Y \rightarrow I be a mapping of Y into the unit interval I. Then gf:X \rightarrow I is a mapping of X into I. Let g = \mathcal{M} m be the monotone-light factorization of g with \mathcal{M} as the middle-space. Then h = mf:X \rightarrow \mathcal{M} maps X onto \mathcal{M}. Let h = IM be the monotone-light
factorization of $h$ with $\mathcal{M}_1$ as middle-space. The following diagram illustrates the situation.

We note that $L$ is a light mapping by I.10.5.

We also have that $\mathcal{I} \mathcal{M}(x) = \mathcal{I} \mathcal{M}(x) = g(x)$ so that $g = ( \mathcal{I} \mathcal{I} M)$ is a monotone-light factorization and $\mathcal{M}_1$ is the middle space.

We shall now show that $L$ is a two-point identification or a homeomorphism.

Assume $L$ is not a homeomorphism. Then $L$ is not 1-1. If $L(x) = L(y) = t$ with $x \neq y$, we have that $M^{-1}(x)$ and $M^{-1}(y)$ are disjoint continua in $X$. However, $\mathcal{M}(M^{-1}(x)) = \mathcal{M}(M^{-1}(y)) = t$, but $M^{-1}(t)$ is a continuum in $Y$. Thus $f(M^{-1}(x)) \cup f(M^{-1}(y)) = M^{-1}(t)$. This can happen only if $f(M^{-1}(x)) \cap f(M^{-1}(y)) \neq \emptyset$ and this can happen only if $p \in M^{-1}(x)$ and $q \in M^{-1}(y)$. Thus $L$ is 1-1 between $\mathcal{M}_1 - \{M(p), M(q)\}$ and $\mathcal{M} - \{\mathcal{M}(p)\}$ and $L(M(P)) = L(M(Q))$.

By V.1.1, if $r(X) < \infty$, $\mathcal{M}_1$ is a regular curve and $r(\mathcal{M}_1) \leq r(X)$. By IV.4.6 $r(\mathcal{M}) \leq r(X) + 1$ and by V.1.11 $r(Y) \leq r(X) + 1$. 
We shall show next that $r(Y) \geq r(X) + 1$.

Let $X = P \cup Q$ where $P$ and $Q$ are continua and $P \cap Q$ has $n + 1$ components. We may assume (II.4.1) that $p \in P \cap Q$. Let $P \cap Q = E_0 \ldots E_n$ be a decomposition of $P \cap Q$ into components and assume $p \in E_0$.

Let $H = E_1 \cup \ldots \cup E_n$. Define a mapping $\phi : X \to I$ by

$$\phi(x) = \frac{d(x, p)}{d(x, p) + d(x, H)}$$

where $d$ is the metric in $X$. We note that $\phi^{-1}(0) = p$

$$\phi^{-1}(1) = H$$

and $\phi(p) \neq \phi(q)$. Let $j : I \to K$ be a two-point identification of the unit interval where $\phi(p)$ and $\phi(q)$ are the two points identified. Then $K$ will be homeomorphic to the unit circle if $\phi(q) = 1$ or homeomorphic to the unit circle with an arc attached at one point if $\phi(q) \neq 1$. Then $j \circ \phi : X \to K$.

Define a mapping $g : Y \to K$ by $g(y) = j(\phi(f^{-1}(y)))$. Since $f^{-1}(y)$ is single-valued or $p$ and $q$, and since $j(\phi(p)) = j(\phi(q))$, $g$ is single-valued. If $C$ is a closed set in $K$, $j^{-1}(C)$ is a closed subset of $I$,

$$\phi^{-1}(j^{-1}(C))$$

is a closed, hence compact, subset of $X$, and $f(\phi^{-1}(j^{-1}(C)))$ is a compact, hence closed, subset of $Y$. Now

$$g^{-1}(C) = f(\phi^{-1}(j^{-1}(C)))$$

so we conclude that $g$ is continuous.
Let $g = \mathcal{L} M$ be the monotone-light factorization of $g$ with $M$ as the middle space. Let $h(X) = m(f(X)) = M$ and let $h = IM$ be the monotone-light factorization of $h$ with $M_1$ as the middle space. Let $\varphi = \mathcal{L} M_2$ be the monotone-light factorization of $\varphi$ with middle-space $M_2$. Then $j \mathcal{L} 2$ is a light mapping, so $(j \mathcal{L} 2) M_2$ is a monotone light factorization of $j \varphi = gf$, so $M_2$ and $M_1$ are homeomorphic. The following diagram shows the mappings and spaces involved.

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{f} & & \downarrow{h} \\
M_1 & \xrightarrow{\varphi} & M_1 \\
\downarrow{L} & & \downarrow{L} \\
M & \xrightarrow{q} & M \\
\downarrow{j} & & \downarrow{j} \\
K & & K
\end{array}
$$

Since $\mathcal{L} L$ is light and $gf = j \varphi$, we have that $gf = (\mathcal{L} L) M$ is the monotone-light factorization of $gf$ and $M_1$ is the middle space. As before, $L$ is either a homeomorphism or a two-point identification. However, $p$ is a component of $(gf)^{-1}(gf(p))$ so $L$ is a two-point identification and not a homeomorphism.

By I.10.9 $M(P \cap Q) = M(P) \cap M(Q) = \bigcup_{i=0}^{n} M(E_i)$. We note that each $M(E_i)$ is a distinct point for $i = 1, \ldots, n$ since each $E_i$ is contained in a component of $\varphi^{-1}(1)$. Also $M(E_0)$ is a continuum in
\( M_1 \). If \( M(E_0) \) does not contain any \( K(E_1) \). For if it did, then \( E_2 M(E_0) \) would contain all of \( I \). Thus \( K(P) \cap K(Q) \) has at least \( n + 1 \) components. Since \( \lambda \) is monotone we have \( r(\ M_1) \leq n \) and so \( r(\ M_1) = n \).

The fact that \( \lambda \) is a two-point identification implies that \( r(M) = n + 1 \) by IV.4.6. Since \( r(Y) \geq r(M), r(Y) \geq n + 1 \).

We have \( r(Y) = n + 1 \), and the necessity is established.

To establish the sufficiency we need only show that \( r(X) \) is finite. Then, if \( r(X) \neq n \), we contradict the necessity.

Let \( X = P \cup Q \) where \( P \) and \( Q \) are continua and let \( P' = f(P) \) and \( Q' = f(Q) \). The \( P' \) and \( Q' \) are continua and \( P' \cup Q' = Y \). By II.4.1 we may assume \( p, q \in P \cap Q \). Then by I.10.10 \( f(P \cap Q) = f(P) \cap f(Q) = P' \cap Q' \) and \( f^{-1}(P' \cap Q') = P \cap Q \).

If \( E \) is a component of \( P \cap Q \), then \( f(E) \subset E' \) where \( E' \) is a component of \( P' \cap Q' \). If \( r \notin E' \), then \( f^{-1}(E') \) is a connected subset of \( P \cap Q \) (we use the homeomorphism of I.10.8) and is contained in a component \( E'' \) of \( P \cap Q \). However \( E \subset E'' \) so \( E = E'' \).

If \( r \in E' \) and \( f^{-1}(E) \) is connected, we have the same result. In addition, \( p, q \in E \). We assume \( r \in E' \) and \( f^{-1}(E') = F \cup G \) with \( p \in F, q \in G, F \cap G = \emptyset \),
by IV.4.5. Then \( E \cap F \neq \emptyset \) or \( E \cap G \neq \emptyset \).
Assume \( E \cap F \neq \emptyset \). \( F \) would be in a component of \( P \cap Q \), so \( F \subseteq E \). Now \( E \subseteq f^{-1}(E') \) and is connected. It follows that \( E \subseteq F \) and \( E = F \). Similarly \( G \) would be a component of \( P \cap Q \).

If \( P \cap Q \) were to have more than \( n + 3 \) components \( P' \cap Q' \) would have more than \( n + 2 \) components, contradicting \( r(Y) = n + 1 \). This means \( r(X) \leq n + 2 \), so \( r(X) = n \) or the necessity is contradicted.

VI.2.2 Theorem If \( Y \) is a Peano space with \( r(Y) = n \), then there is a suitable system \((f, X, Y)\) where \( X \) is a Peano space of type \( A \), the degree of \( f \) is equal to \( k \leq n \) and \( r(X) = n - k \).

Proof If every \( B \)-set of \( Y \) is an \( A \)-set, then we let \( f(Y) = Y \) be the identity map, \( X = Y \), and we have the theorem.

We prove the theorem by induction on \( n \).

Let \( B \) be a \( B \)-set in \( Y \) and let \( C \) be a component of \( Y - B \). If \( n = 0 \) then \( \mathcal{B}(C) \) can contain only one point so \( B \) is an \( A \)-set. This reduces to the case covered in the preceding paragraph.

Assume the theorem valid for \( n = k \).

Assume \( r(Y) = k + 1 \). If all \( B \)-sets of \( Y \) are \( A \)-sets, we have the case of the first paragraph.
If $Y$ contains a B-set $B$ which is not an $A$-set, then $Y - B$ must contain a component $C$ which has at least two (but less than $n + 1$) boundary points. Let $p$ be one of these boundary points and let

$$D = (K(C) - \{p\}) \cup \{q\}$$

where $q$ is a point not in $Y$. By defining the metric properly on $D$, it follows that $D$ is a Peano space isometric with $K(C)$. $Y - C$ is also a Peano space, and $(Y - C) \cap D = B(C) - \{p\}$. (However, $p \notin Y - C$).

If we let $Z = (Y - C) \cup D$ and give $Z$ the topology of II.2.10, we have immediately that $Z$ is a Peano space. Furthermore, $f_2(Z) = Y$ defined by

$$f_2(Z) = \begin{cases} z, & z \neq q \\ p, & z = q \end{cases}$$

is a two-point identification. By VI.2.1 we have $r(Z) = n - 1$.

If every B-set of $Z$ is an $A$-set, we have proven the theorem. If $Z$ contains a B-set which is not an $A$-set, we apply the induction hypotheses to $Z$ and get a suitable system $(g, X, Z)$ satisfying the conditions of the theorem. Then if $g = g_m g_{m-1} \cdots g_0$, $f = f_z g_m \cdots g_0$ is a mapping of $X$ onto $Y$ yielding a suitable system $(f, X, Y)$. 
VI.2.3 Theorem Let \((f, X, Y)\) be a suitable system where \(X\) and \(Y\) are Peano spaces, \(r(X) = n\), and the degree of \(f\) is \(k\). Then \(r(Y) = n + k\).

Proof Let \(X = X_0, X_1, \ldots, X_k = Y, f_i(X_{i-1}) = X_i\) be the factorization of \(f\) into two-point identifications. Then by VI.2.1, \(r(X_i) = r(X_{i-1}) + 1\) and \(r(Y) = r(X) + k = n + k\).

VI.2.4 The proof of VI.1.2 is established by VI.2.2 and VI.2.3.

VI.3 Proof of VI.1.3

VI.3.1 Lemma If \(P\) and \(Q\) are disjoint closed subsets of a Peano space \(X\), then there are at most a finite number of components of \(X - (P \cup Q)\) with boundary points in both \(P\) and \(Q\).

Proof Assume that \(\{C_n\}\) is an infinite sequence of components of \(X - (P \cup Q)\) which have boundary points in both \(P\) and \(Q\). Let \(a = \frac{1}{2}d(P, Q)\). In each \(C_n\) we choose a point \(x_n\) such that \(d(x_n, P) > a\) and \(d(x_n, Q) > a\). Then \(\{x_n\}\) is an infinite sequence of points in \(X\) and since \(X\) is compact we can assume that \(x_n \to x\) as \(n \to \infty\). Now \(d(x, P) > a\) and \(d(x, Q) > a\), so \(x\) is in some component \(C_x\) of \(X - (P \cup Q)\).

Let \(N\) be a connected neighborhood of \(x\) such that \(\delta(N) < \frac{1}{2}a\). Then there is at least one \(x_n\) such
that \( x_n \in N \). However, \( C_n \cap C_x = \emptyset \) so \( N \) cannot be connected. This contradicts the local connectedness of \( X \). Hence, there can be at most a finite number of components of \( X - (P \cup Q) \) which have boundary points in both \( P \) and \( Q \).

VI. 3.2 Lemma If \( f(X) = Y \) is a two-point identification where \( X \) and \( Y \) are Peano spaces, then:

(a) \( f(B) \) is a \( B \)-set of \( Y \) if \( B \) is a \( B \)-set of \( X \).

(b) \( f^{-1}(B) \) is a \( B \)-set of \( X \) if \( f^{-1}(B) \) is connected and \( B \) is a \( B \)-set of \( Y \). If \( f^{-1}(B) \) is not connected, then the non-degenerate components of \( f^{-1}(B) \) are \( B \)-sets of \( X \).

Proof Let \( p, q \in X \) be such that \( f(p) = f(q) = r \in Y \). Then \( f(X - \{p, q\}) = Y - \{r\} \) is a homeomorphism.

(a) We note that \( f(B) \) is a subcontinuum of \( Y \), so assume \( f(B) \neq Y \) and \( f(B) \) is non-degenerate. If \( f(B) \) is not a \( B \)-set, then there is a component \( C \) of \( Y - f(B) \) such that \( B_Y(C) \) contains an infinite number of points. Let \( \{y_n\} \) be an infinite sequence of such points such that \( r \) is not in the sequence. Then \( \{ x_n | x_n = f^{-1}(y_n) \} \) is an infinite sequence in \( X \). Let \( N \) be a neighborhood of \( x_n \) such that \( p \notin N \), \( q \notin N \). Then \( f(N) \) is a neighborhood of \( y_n \) and contains points of both \( C \) and \( f(B) \). Then \( N \) must contain
points of both \( f^{-1}(C) \) and \( B \). However, \( f^{-1}(C) \) breaks up into at most two components and each of these is contained in a component of \( X - B \). Thus each \( x_n \) is a boundary point of a component of \( X - B \). But there are at most two components of \( X - B \) involved, so one of them would have to contain an infinite number of points. This is a contradiction that \( B \) is a \( B \)-set of \( X \). Thus \( f(B) \) must be a \( B \)-set of \( Y \).

(b) Suppose \( f^{-1}(B) \) is connected. Suppose there is a component \( C \) of \( X - f^{-1}(B) \) which has an infinite number of boundary points. Let \( \{x_n\} \) be an infinite sequence in \( B_x(C) \) which does not contain \( p \) or \( q \). Now \( f(C) \subset Y - B \) and is in a component \( F \) of \( Y - B \) since \( f(C) \) is connected. If \( N \) is a neighborhood of \( x_n \) which does not contain \( p \) or \( q \), then \( f(N) \) is a neighborhood of \( f(x_n) \). However, \( N \) contains points of \( C \) and of \( f^{-1}(B) \) \((x_n \in f^{-1}(B))\) which means \( f(N) \) contains points of \( B \) and \( f(C) \subset F \). Thus \( f(x_n) \) is a boundary point of \( F \). But \( F \) can have only a finite number of boundary points, and \( \{f(x_n)\} \) is an infinite sequence of points. This is a contradiction, so \( B_x(C) \) must be a finite set of points.

Suppose \( f^{-1}(B) = P \cup Q \) where \( p \in P \) and \( q \in Q \) (IV.4.5). Assume \( P \) is non-degenerate. Let \( C \) be a component of \( X - P \). If \( C \) is also a component of \( X - (P \cup Q) \), we can apply the methods of the preceding
paragraph to show that the boundary of \( C \) is a finite point set. We assume \( C \) is not a component of \( X - (P \cup Q) \). Then we must have \( Q \subset C \). (\( Q \) is connected, and \( Q \subset X - P \)).

Let \( \{ F_i \mid i = 1, \ldots, k \} \) be the collection of components of \( X - (P \cup Q) \) such that \( \mathcal{B}_X(F_i) \cap P \neq \emptyset \) and \( \mathcal{B}_X(F_i) \cap Q \neq \emptyset \). (By VI.3.1 this collection is finite). Let \( \{ C_\alpha \mid \alpha \in \mathcal{A} \} \) be the collection of components of \( X - (P \cup Q) \) such that \( \mathcal{B}_X(C) \cap Q \neq \emptyset \) and \( \mathcal{B}_X(C) \cap P = \emptyset \).

Then
\[
G = Q \cup \left( \bigcup \{ F_i \mid i = 1, \ldots, k \} \right) \cup \left( \bigcup \{ C_\alpha \mid \alpha \in \mathcal{A} \} \right)
\]
is a connected subset of \( X - P \) and hence is contained in \( C \). However, if \( x \in C - G \), then \( x \in D \) where \( D \) is a component of \( X - (P \cup Q) \). But then \( \mathcal{B}_X(D) \subset P \), \( D \) would be a component of \( X - P \), \( D \neq P \), and we must have \( C = G \).

If \( \mathcal{B}_X(C) \) has infinitely many points, then one \( F_i \) must have infinitely many points since \( \mathcal{B}_X(C) \subset \bigcup \mathcal{B}_X(F_i) \). But each \( F_i \) has only a finite number of boundary points, so \( \mathcal{B}_X(C) \) must be a finite point set. This means that \( P \) is a B-set in \( X \).

VI.3.3 Lemma Let \( f(X) = Y \) be a two-point identification where \( X \) and \( Y \) are Peano spaces. If \( C \) is a fine-cyclic element of \( Y \), and \( f^{-1}(C) \) is connected,
then $f^{-1}(C)$ is a fine-cyclic element of $X$. If $f^{-1}(C) = P \cup Q$ where $P$ and $Q$ are the components of $f^{-1}(C)$, then one component is degenerate and the other is a fine-cyclic element.

**Proof** Let $f(p) = f(q) = r$ be as in the preceding lemmas. Suppose $f^{-1}(C)$ is connected. By VI.3.2 $f^{-1}(C)$ is a $B$-set of $X$. We need only note that if $A$ is a finite subset of $f^{-1}(C)$ disconnecting $f^{-1}(C)$, then $f(A) \cup \{r\}$ is a finite set disconnecting $C$ if $r \not\in C$. If $r \not\in C$, then $f(A)$ is a finite set disconnecting $C$.

If $P$ and $Q$ are non-degenerate, then $r$ is a cut point of $f(P \cup Q) = C$, and one of them must degenerate to a point, say $P$. Then $f|_Q : Q \to C$ is a 1-1 continuous map, hence a homeomorphism.

**VI.3.4 Lemma** Let $P$ be a Peano space with the property that all proper $B$-sets are $A$-sets. If $C$ is a cyclic element of $P$, then $C$ is a fine-cyclic element of $P$.

**Proof** Suppose $C$ is cyclic but not fine-cyclic. Then there is a finite subset $Z$ of $C$ such that $C - Z$ is not connected. Let $B$ be a component of $X - Z$.

Now $C$ is an $A$-set of $X$, so $B \cap C = \emptyset$ or $B \cap C \neq \emptyset$ and is connected. If $B \cap C = \emptyset$ then $B$ is a component of $X - C$ and $\mathcal{B}_X(B)$ is a single point. Let $B' = B \cap C$ if $B \cap C$ is connected. Then
$B_X(B) = B_G(B') \subseteq Z$, and $B_G(B')$ must have at least two points or else $C$ is not cyclic. But then $B$ would be a $B$-set in $X$ which is not an $A$-set. This contradicts the assumed property of $P$. Therefore $C$ is not disconnected by any finite set of points and is thus a fine-cyclic element.

VI.3.5 Lemma Let $f(X) = Y$ be a two-point identification of Peano spaces $X$ and $Y$ and let $C$ be a fine-cyclic element of $X$. Then $f(C)$ is a fine-cyclic element of $Y$.

Proof By VI.3.2 $f(C)$ is a $B$-set in $Y$. Let $K$ be any finite subset of $f(C)$. Then $f^{-1}(K)$ is a finite subset of $X$, and so $C - f^{-1}(K)$ is connected. However, $f(C - f^{-1}(K)) = f(C) - K$, so $f(C) - K$ is connected.

VI.3.6 We now prove VI.1.3.

Proof Let $X = X_0, X_1, \ldots, X_k = Y$ and $f_i(X_{i-1}) = X_i \quad i = 1, \ldots, k$ be the factorization of $f$ into two-point identifications. (If $f$ is the identity map we have nothing to prove). If $D$ is a proper cyclic element of $X$ such that $f(D) = C$, then $f_1(D) = C_1$ is a fine-cyclic element of $X_1$ by VI.3.5, $f_2(C_1) = C_2$ is a fine-cyclic element of $X_2$. Continuing
inductively we have that $f_k(C_{k-1}) = f(D) = C$ is a fine-cyclic element of $Y$.

Suppose $C$ is a fine-cyclic element of $Y$. By VI.3.3 $f_k^{-1}(C)$ contains a component $C_{k-1}$ such that $C_{k-1}$ is a fine-cyclic element of $X_{k-1}$, and $f_k(C_{k-1}) = C$ is a homeomorphism. Continuing inductively we have at each stage $f_i^{-1}(C_i)$ which contains a component $C_{i-1}$ which is a fine-cyclic element of $X_{i-1}$, and $f_i(C_{i-1}) = C_i$ is a homeomorphism. At the final stage we have $f_1^{-1}(C_1) = D$ containing a component $D$ which is a fine-cyclic element of $X = X_0$, and $f_1(D) = C_1$ is a homeomorphism. Then $f(D) = C$ is a homeomorphism. We also note that by VI.3.2 $D$ is a $B$-set of $X$ but must also be an $A$-set by VI.2.2. Thus $D$ is a proper cyclic element of $X$. 
LIST OF REFERENCES


7. T. Rado et al., Abstracts of Lectures Presented at the 1958 Summer Institute on Surface Area and Related Topics, (Private file of Prof. Earl J. Mickle), (mimeographed).
