A STUDY OF THE THREE-DIMENSIONAL STRESS
FUNCTIONS OF ELASTICITY WITH PARTICULAR APPLICATION
TO THE RECTANGULAR BEAM

DISSERTATION

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Introduction

The development of the theory of elasticity to its present state is due largely to the interest of mathematicians and engineers in two distinct directions. Mathematicians have been interested primarily in the formulation and manipulation of the exact mathematical relationships representing a general class of problems of an ideal nature for their own sake. Engineers, on the other hand, are interested in determining to a reasonable degree of accuracy the actual physical stresses and displacements present in a particular situation. The most important solutions to problems of elasticity have been made by those who are able to combine the training of the mathematician with the sense of urgency and application of the engineer.

It has been realized for about a century that the solution to any elasticity problem is simply the satisfaction of three sets of necessary and sufficient conditions -- the equilibrium, compatibility, and boundary conditions. Unfortunately, however, the solution of the three sets of partial differential equations representing these conditions has been found to be extremely difficult, with the result that comparatively few exact solutions have been found thus far. Usually, either mathematical approximations to the theory or experimental testing have had to be used to obtain quantitative data of reasonable accuracy.

The utilization of the concept of the stress function has resulted in most of the existing solutions to problems in elasticity. This procedure involves defining the stresses in terms of mathematical functions
which serve to transform the aforementioned three sets of conditions into more useable forms. This approach was first used on certain classes of three-dimensional problems which, owing to the particular nature of their geometry and external loading, could be considered as two-dimensional. Such a two-dimensional stress function, which bears his name, was introduced by G. B. Airy in 1862. The two-dimensional stresses are defined in terms of the Airy function $\phi$ in the following manner:

\[
\begin{align*}
\sigma_{xx} &= \frac{\partial^2 \phi}{\partial y^2}, \\
\sigma_{yy} &= \frac{\partial^2 \phi}{\partial x^2}, \\
\tau_{xy} &= -\frac{\partial^2 \phi}{\partial x \partial y}
\end{align*}
\] (1)

where $\sigma_{xx}$ and $\sigma_{yy}$ are the normal stresses in the $x$ and $y$ directions, respectively, and $\tau_{xy}$ is the shear stress coplanar with $\sigma_{xx}$ and $\sigma_{yy}$. It is seen that equations (1) identically satisfy the two-dimensional equilibrium equations without body forces:

\[
\begin{align*}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} &= 0.
\end{align*}
\] (2)

Upon substitution of equations (1) into the compatibility equation

\[
\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_{xx} + \sigma_{yy}) = 0,
\] (3)

---

1The numbers appearing as superscripts throughout this manuscript refer to the references given at the end of the paper.
there results
\[ \frac{\partial^2 \Phi}{\partial x^2} + 2 \frac{\partial^2 \Phi}{\partial x \partial y} + \frac{\partial^2 \Phi}{\partial y^2} = 0. \] (4)

The simultaneous solution of equation (4) and the boundary conditions for the particular problem will result in the solution of the two-dimensional problem. This approach has yielded a large number of solutions for rectangular beams under which the assumptions of generalized plane stress (the thickness of the beam is very small as compared with the length and depth) or plain strain (the thickness is very great in comparison to the length and depth) apply. The transformation of equation (4) into polar coordinates has resulted also in the solution of two-dimensional problems best suited to that coordinate system. Recent use of the complex variable approach, notably by Muskhelishvili, has further extended the scope of problem solution into curvilinear coordinates.

Although a large number of problems have been solved by use of the foregoing two-dimensional analysis, it must be remembered that it is very seldom that an actual physical problem is two-dimensional, so that these solutions usually are of practical value only as approximations to reality. The added complexity of the equations of equilibrium and compatibility in three dimensions precludes the possibility of expressing these equations in a form as simple as the biharmonic equation (4).

The first general, three-dimensional stress function approach was presented by Maxwell in 1870. According to this approach, let the
shear stresses be defined in terms of three arbitrary functions \( A, B, \) and \( C \) in the following manner:

\[
\tau_{xy} = -\frac{\partial^2 A}{\partial y \partial z}, \quad \tau_{xz} = -\frac{\partial^2 B}{\partial z \partial x}, \quad \tau_{yz} = -\frac{\partial^2 C}{\partial x \partial y} \tag{5}
\]

The equations of equilibrium are

\[
\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = 0 \tag{6a}
\]

\[
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0 \tag{6b}
\]

\[
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0 \tag{6c}
\]

Substituting equations (5) into (6a) yields

\[
\frac{\partial \tau_{xx}}{\partial x} = \frac{\partial^3 C}{\partial y^2 \partial z} + \frac{\partial^3 B}{\partial z^2 \partial x}
\]

which can be integrated with respect to \( x \) to give

\[
\tau_{xx} = \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 B}{\partial z^2} + f(y, z) \tag{7}
\]

In view of the arbitrariness of \( A, B, C \), the function \( f(y, z) \) can be included in the functions \( B \) and \( C \). Repeating the procedure for the remaining two equilibrium equations (6b) and (6c) results in the following permuting set of equations for the normal stresses in terms of the
Maxwell stress functions:

\[
\begin{align*}
\tau_{xx} &= \frac{\partial^2 B}{\partial y^2} + \frac{\partial^2 C}{\partial x \partial y} \\
\tau_{yy} &= \frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} \\
\tau_{xy} &= \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 B}{\partial x^2}.
\end{align*}
\]

(8)

It is evident from equations (5) and (8) that if \( A = B = 0 \), we have the Airy stress function as defined in (1).

Another approach along similar lines was made by Morera\(^5\) in 1892. Using this method, we may define the normal stresses in terms of the arbitrary functions \( L, M, \) and \( N \) according to

\[
\begin{align*}
\sigma_{xx} &= \frac{\partial^2 L}{\partial y \partial y}, \quad \sigma_{yy} = \frac{\partial^2 M}{\partial y \partial x}, \quad \sigma_{xy} = \frac{\partial^2 N}{\partial x \partial y},
\end{align*}
\]

(9)

which, upon substitution into the equilibrium equations (6), yield the following shear stresses:

\[
\begin{align*}
\tau_{xy} &= \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial L}{\partial y} - \frac{\partial M}{\partial y} - \frac{\partial N}{\partial y} \right) \\
\tau_{yx} &= \frac{1}{2} \frac{\partial}{\partial y} \left( -\frac{\partial L}{\partial x} + \frac{\partial M}{\partial x} - \frac{\partial N}{\partial x} \right) \\
\tau_{yx} &= \frac{1}{2} \frac{\partial}{\partial y} \left( -\frac{\partial L}{\partial x} - \frac{\partial M}{\partial x} + \frac{\partial N}{\partial x} \right).
\end{align*}
\]

(10)

Since any given state of stress must be expressible in terms of both the Maxwell and Morera stress functions, upon comparison of equa-
tions (5), (8), (9), and (10), the following relationships are seen to prevail between the two types of stress functions:

\[
\frac{\partial^2 B}{\partial y^2} + \frac{\partial^2 C}{\partial x^2} = \frac{\partial^2 L}{\partial y \partial x} \\
\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} = \frac{\partial^2 M}{\partial x \partial y} \\
\frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 B}{\partial x^2} = \frac{\partial^2 N}{\partial x \partial y} \\
\frac{\partial^2 A}{\partial y \partial x} = \frac{1}{2} \frac{\partial}{\partial y} \left( \frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} - \frac{\partial N}{\partial y} \right) \\
\frac{\partial^2 B}{\partial y \partial x} = \frac{1}{2} \frac{\partial}{\partial x} \left( -\frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial y} \right) \\
\frac{\partial^2 C}{\partial x \partial y} = \frac{1}{2} \frac{\partial}{\partial y} \left( \frac{\partial L}{\partial x} - \frac{\partial M}{\partial y} + \frac{\partial N}{\partial y} \right).
\]

It has been shown by Weber\textsuperscript{6} that the Maxwell and Morera stress functions actually supplement each other, and that together they are the components of a second order symmetric tensor, representable as the array shown below with the elements in the principal diagonal being the Maxwell functions and the others being the Morera functions.

\[
\begin{bmatrix}
A & N & M \\
N & B & L \\
M & L & C
\end{bmatrix}
\]

In 1906, Love\textsuperscript{7} introduced a strain function for solids of revolution subjected to loadings having axial symmetry. Although this approach
was limited to a certain class of problems, Galerkin in 1930 extended this idea with the conception of three strain functions for the general three-dimensional problem of elasticity. The strains are actually defined directly in terms of the strain functions, with the stresses being related to the strains by Hooke's Law:

\[
\begin{align*}
\varepsilon_{xx} &= \frac{1}{E} (\sigma_{xx} - \nu \sigma_{yy} - \nu \sigma_{zz}) \\
\varepsilon_{yy} &= \frac{1}{E} (\sigma_{yy} - \nu \sigma_{zz} - \nu \sigma_{xx}) \\
\varepsilon_{zz} &= \frac{1}{E} (\sigma_{zz} - \nu \sigma_{xx} - \nu \sigma_{yy}),
\end{align*}
\]

where \( \varepsilon_{xx} \) is the strain in the \( x \)-direction, and similarly for \( y \) and \( z \); \( \nu \) is Poisson's ratio, and \( E \) is the modulus of elasticity. The three Galerkin strain functions \( \psi_1, \psi_2, \) and \( \psi_3 \) lend themselves to interpretation as the three components of a vector according to the following definition:

\[
\psi = \psi_1 \hat{i} + \psi_2 \hat{j} + \psi_3 \hat{k}.
\]

If the \( xyz \)-coordinate system is rotated such that the individual coordinates of a point in space are changed, yet the vector as function of the position of the point remains unchanged, the three functions will change so as to represent the new components of the vector, while the stress and strain defined by the functions \( \psi_1, \psi_2, \) and \( \psi_3 \) remain the same. Because of this property, the functions \( \psi_1, \psi_2, \) and \( \psi_3 \) are called the components of the "Galerkin vector." The Maxwell and Morera stress functions discussed previously do not lend themselves
to interpretation as components of a vector, however.

In terms of the Galerkin functions, the displacements $u$, $v$, and $w$ in the $x$, $y$, and $z$-directions, respectively, have been recorded by Westergaard as

$$\begin{align*}
\delta G u &= z (1-v) \nabla^2 \psi_1 - \frac{\partial}{\partial x} \text{div} \psi \\
\delta G v &= z (1-v) \nabla^2 \psi_2 - \frac{\partial}{\partial y} \text{div} \psi \\
\delta G w &= z (1-v) \nabla^2 \psi_3 - \frac{\partial}{\partial z} \text{div} \psi
\end{align*}$$

(13)

where $\nabla^2$ denotes the differential operator $(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2)$ and $G$ is the modulus of rigidity, related to the modulus of elasticity by

$$G = \frac{E}{2(1+v)}. \quad (14)$$

The stresses are further given in terms of the strain functions by

Westergaard as

$$\begin{align*}
\tau_{xx} &= z (1-v) \nabla^2 \frac{\partial \psi_1}{\partial x} + (v \nabla^2 - \frac{\partial^2}{\partial x^2}) \text{div} \psi \\
\tau_{yy} &= z (1-v) \nabla^2 \frac{\partial \psi_2}{\partial y} + (v \nabla^2 - \frac{\partial^2}{\partial y^2}) \text{div} \psi \\
\tau_{zz} &= z (1-v) \nabla^2 \frac{\partial \psi_3}{\partial z} + (v \nabla^2 - \frac{\partial^2}{\partial z^2}) \text{div} \psi
\end{align*}$$

(15)
and

\[ \sigma_{\gamma \gamma} = (1 - \nu) \nabla^2 \left( \frac{\partial \psi_1}{\partial y} + \frac{\partial \psi_2}{\partial x} \right) - \frac{\partial^2}{\partial x \partial y} \text{div} \, \psi \]

\[ \sigma_{\gamma \delta} = (1 - \nu) \nabla^2 \left( \frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_3}{\partial x} \right) - \frac{\partial^2}{\partial y \partial z} \text{div} \, \psi \]

\[ \sigma_{\gamma \chi} = (1 - \nu) \nabla^2 \left( \frac{\partial \psi_3}{\partial y} + \frac{\partial \psi_1}{\partial x} \right) - \frac{\partial^2}{\partial z \partial x} \text{div} \, \psi \quad . \quad (16) \]

The last, general three-dimensional stress function approach was presented by Papkovitch\(^{10}\) and Neuber\(^{11}\) in the years 1932-1934. This method also gives the stresses indirectly, for the displacements are defined according to

\[ u = \phi_1 - \frac{1}{4(1 - \nu)} \frac{\partial}{\partial x} \left( \phi_0 + \phi_1 + \phi_2 + \phi_3 \right) \]

\[ v = \phi_2 - \frac{1}{4(1 - \nu)} \frac{\partial}{\partial y} \left( \phi_0 + \phi_1 + \phi_2 + \phi_3 \right) \]

\[ w = \phi_3 - \frac{1}{4(1 - \nu)} \frac{\partial}{\partial z} \left( \phi_0 + \phi_1 + \phi_2 + \phi_3 \right) , \quad (17) \]

where the functions \( \phi_0, \phi_1, \phi_2, \phi_3 \) are harmonic, i.e.,

\[ \nabla^2 \phi_0 = \nabla^2 \phi_1 = \nabla^2 \phi_2 = \nabla^2 \phi_3 = 0 . \]

In this form, it appears that four arbitrary harmonic functions are required. Actually, however, any one of the four functions may be dropped without loss of generality.

It is evident that the expressions for the stress in terms of the Papkovitch-Neuber functions must be determined by first employing the
equations relating strains and displacements:

\[ \epsilon_{xx} = \frac{\partial u}{\partial x}, \quad \epsilon_{yy} = \frac{\partial v}{\partial y}, \quad \epsilon_{zz} = \frac{\partial w}{\partial z} \]  

\[ (18a) \]

\[ \epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \]

\[ \epsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \]

\[ \epsilon_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \]  

and then using the relationships between the stresses and strains (11).

In addition to the general methods of attack for three-dimensional problems discussed thus far, there exist a sizeable number of particular stress function approaches for particular classes of problems usually having some form of symmetry in their geometry and/or loading. These various methods have been summarized excellently by Marguerre who presents an extensive bibliography on the three-dimensional stress function in the same paper.

The author became particularly interested in one facet of the overall picture as suggested in a paper by Langhaar and Stippes, wherein a method of generating Maxwell functions from harmonic functions is presented in such a manner so as to satisfy identically the equilibrium and compatibility equations. The pertinent part of this paper is reviewed in Appendix A. The class of problems involving the rectangular beam has been one of the most important and intriguing to elasticians during the past century, yet it has generally resisted the solution of
all but a relatively few problems. In the following work, the author presents a study of two possible methods of solving elasticity problems involving rectangular beams, one involving the use of transcendental harmonic functions and the other employing the harmonic polynomials.
The Stress Equations in Terms of Harmonic Functions

In Appendix A the derivation is presented which results in a set of equations expressed in rectangular coordinates giving the six stresses in terms of three harmonic functions $H_1$, $H_2$, and $H_3$. In this form, the equations identically satisfy the three-dimensional conditions of equilibrium and compatibility. The equations referred to are those given by (60). These equations, which arise in a natural manner from considerations of the Maxwell stress functions, can be simplified by omitting the common coefficient $-1/2(1-\nu)$, owing to the fact that the equilibrium and compatibility equations are homogeneous if there are no body forces. Equations (60) can then be rewritten as

\[
\begin{align*}
\tau_{xx} &= \alpha H_{xx} + \gamma H_{xyy} + \zeta H_{yy} \\
& \quad + 2\nu (H_{xx} - H_{xyy} - H_{yy}) - 2H_{xx} \\
& \quad (19a) \\
\tau_{yy} &= \alpha H_{yy} + \gamma H_{xyy} + \zeta H_{xy} \\
& \quad + 2\nu (-H_{xx} + H_{xyy} - H_{xy}) - 2H_{yy} \\
& \quad (19b) \\
\tau_{yz} &= \alpha H_{yz} + \gamma H_{xy} + \zeta H_{yy} \\
& \quad + 2\nu (-H_{xx} - H_{xy} + H_{yy}) - 2H_{yy} \\
& \quad (19c) \\
\tau_{xy} &= \alpha H_{xy} + \gamma H_{xyy} + \zeta H_{xy} \\
& \quad - (1 - 2\nu)(H_{xy} + H_{xyy}) \\
& \quad (19d)
\end{align*}
\]
The sum of the three normal stresses is seen to be

$$\sigma = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

$$= -2(1 + \nu)(H_{xx} + H_{yy} + H_{zz})$$  (20)

Two special cases of equations (19) are worth mentioning. The first of these involves letting \(H_2 = H_3 = 0\). The stresses then become:

$$\sigma_{xx} = \lambda H_{xx} - 2(1 - \nu)H_{xx}$$

$$\sigma_{yy} = \lambda H_{yy} - 2\nu H_{xx}$$

$$\sigma_{zz} = \lambda H_{zz} - 2\nu H_{xx}$$  (21a)

$$\sigma_{xy} = \lambda H_{xy} - (1 - 2\nu)H_{xy}$$

$$\sigma_{yz} = \lambda H_{yz}$$

$$\sigma_{xz} = \lambda H_{xz} - (1 - 2\nu)H_{xz}$$  (21b)

This operation simplifies the equations by the use of only one harmonic function, but also causes a loss of generality and symmetry in the equations.
Another special case arises from letting $H_1 = H_2 = H_3 = H$. The resulting equations are not only simpler in form, but their symmetry is increased.

\[
\begin{align*}
\tau_{xx} &= (xH_x + yH_y + zH_z)_{xx} - 4(1-v)H_{xx} \\
\tau_{yy} &= (xH_x + yH_y + zH_z)_{yy} - 4(1-v)H_{yy} \\
\tau_{zz} &= (xH_x + yH_y + zH_z)_{zz} - 4(1-v)H_{zz} \\
\tau_{xy} &= (xH_x + yH_y + zH_z)_{xy} - 4(1-v)H_{xy} \\
\tau_{yz} &= (xH_x + yH_y + zH_z)_{yz} - 4(1-v)H_{yz} \\
\tau_{xz} &= (xH_x + yH_y + zH_z)_{xz} - 4(1-v)H_{xz}.
\end{align*}
\]
The Transcendental Function Approach

One approach to the solution of elasticity problems by using the equations of the preceding section is to employ the harmonic transcendental functions obtained from Laplace's equation by a separation of variables. These functions are derived in Appendix D. The most general form obtained is

\[ H_i = (\cos \alpha_i x + A_i \sin \alpha_i x)(\cos \beta_i y + B_i \sin \beta_i y) \]

\[ \times \left( C_i e^{-i\alpha_i x + \beta_i y} + D_i e^{i\alpha_i x + \beta_i y} \right), \tag{23} \]

where \( A, B, C, D, \alpha, \) and \( \beta \) are constants to be determined from the boundary conditions, and the index \( i \) ranges over 1, 2, and 3 for the stress equations. The exponential functions in the last parentheses of (23) can be replaced by hyperbolic functions if desired.

In order to simplify the writing of the stress equations, the following definitions will be made:

\[ R_i \equiv \cos \alpha_i x + A_i \sin \alpha_i x \tag{24} \]

\[ R_i^* \equiv -\sin \alpha_i x + A_i \cos \alpha_i x = \frac{1}{\alpha_i} \frac{dR_i}{dx} \tag{25} \]

\[ S_i \equiv \cos \beta_i y + B_i \sin \beta_i y \tag{26} \]

\[ S_i^* \equiv -\sin \beta_i y + B_i \cos \beta_i y = \frac{1}{\beta_i} \frac{dS_i}{dy} \tag{27} \]
Using the above definitions, the stress equations (19) are written below in terms of the harmonic transcendental function (23):

\[
\sigma_{xx} = -\alpha_{x} R^* S_i T_i x - \alpha_{x} B_{x} R_{x} S_{x} T_{x} y - \alpha_{x} \sqrt{\alpha_{x}^{2} + \beta_{x}^{2}} R_{x} S_{x} T_{x} y \\
+ 2\nu \left[ -\alpha_{x}^2 R_{x} S_{x} T_{x} + \beta_{x}^2 R_{x} S_{x} T_{x} - (\alpha_{x}^2 + \beta_{x}^2) R_{x} S_{x} T_{x} \right] \\
+ 2\alpha_{x} \beta_{x} R_{x} S_{x} T_{x} 
\]

\[
\sigma_{yy} = -\alpha_{y} B_{y} R^* S_i T_i y - \beta_{y}^2 R_{y} S_{y} T_{y} y - \beta_{y} \sqrt{\alpha_{y}^{2} + \beta_{y}^{2}} R_{y} S_{y} T_{y} y \\
+ 2\nu \left[ -\alpha_{y}^2 R_{y} S_{y} T_{y} + \beta_{y}^2 R_{y} S_{y} T_{y} - (\alpha_{y}^2 + \beta_{y}^2) R_{y} S_{y} T_{y} \right] \\
+ 2\beta_{y} \beta_{y} R_{y} S_{y} T_{y} 
\]

\[
\sigma_{xz} = \alpha_{x} (\alpha_{x}^2 + \beta_{x}^2) R^* S_i T_i x + \beta_{x} (\alpha_{x}^2 + \beta_{x}^2) R_{x} S_{x} T_{x} y + (\alpha_{x}^2 + \beta_{x}^2) R_{x} S_{x} T_{x} y \\
+ 2\nu \left[ -\alpha_{x}^2 R_{x} S_{x} T_{x} + \beta_{x}^2 R_{x} S_{x} T_{x} + (\alpha_{x}^2 + \beta_{x}^2) R_{x} S_{x} T_{x} \right] \\
- 2(\alpha_{x}^2 + \beta_{x}^2) R_{x} S_{x} T_{x} 
\]

\[
\sigma_{yz} = -\alpha_{y} \beta_{y} R, S^* T_i y - \alpha_{x} \beta_{x} R_{x} S_{x} T_{x} y \\
+ \alpha_{x} \beta_{x} \sqrt{\alpha_{x}^{2} + \beta_{x}^{2}} R_{x} S_{x} T_{x} y \\
- (1-2\nu) \left[ -\alpha_{y} \beta_{y} R_{y} S_{y} T_{y} + \alpha_{x} \beta_{x} R_{x} S_{x} T_{x} \right] 
\]
These stress equations contain six undetermined coefficients (A, B, C, D, α, β) for each of the three \( H^e \)s. This gives a total of 18 coefficients to be determined from the boundary conditions. However, at the same time, for a rectangular beam with the stresses on all six faces known, three boundary conditions can be written for each face, giving a total of 18 equations to be used. Thus it would appear that there exists a class of rectangular beam problems for which a solution could be obtained by the simultaneous solution of 18 equations containing 18 unknowns. This method of solution will, in general, require a lengthy trial and error process, however. Furthermore, the class of problems is limited to the type of boundary stresses which this form of transcendental function can fit.

The most obvious direct attack upon a general rectangular beam problem having \( n \) non-homogeneous stress boundary conditions would be to represent the problem by \( n \) separate problems, each having all but one of the boundary conditions non-homogeneous. Then the complete solution would be effected by a superposition of the partial solutions. Since the stress on any face is generally a function of two variables, one would
expect that, in the end, the final coefficients would be determined by a double Fourier series analysis.

A somewhat similar problem is the classical problem of the temperature distribution throughout a rectangular prism with fixed temperatures on the six faces. Using the coordinate system shown in Figure 1 below, let the temperature on the face \( z = 0 \) be a function of \( x \) and \( y \), while on the other faces it is everywhere zero. For a steady-state distribution, the temperature function must be harmonic; i.e., it must satisfy Laplace's equation. If the dimensions in the \( x \), \( y \), and \( z \) directions are \( a \), \( b \), and \( c \), respectively, the transcendental function

\[
sin \left( \frac{m\pi}{a}x \right) \cdot \sin \left( \frac{n\pi}{b}y \right) \cdot \sinh \left[ \sqrt{\left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2} \right] (c-z)
\]

Fig. 1 - Coordinate system for the rectangular prism.
is seen to be harmonic and satisfy the five homogeneous boundary conditions as well. The remaining condition is satisfied by the temperature distribution \( u \) given by the following double series:

\[
    u(x, y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[ d_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right. \\
    \left. \sinh \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 (c-z)} \right],
\]

where \( d_{mn} \) are the Fourier coefficients given by

\[
    d_{mn} = \frac{4}{\pi^2 c \cdot \sinh \sqrt{(m\pi/a)^2 + (n\pi/b)^2}} \times \int_0^a \int_0^b f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy,
\]

and \( f(x, y) \) is the temperature distribution given by the non-homogeneous boundary condition.

The author was generally unsuccessful in achieving a method of solution by this procedure for the stress boundary conditions. The function (31) is actually derived by applying the five non-homogeneous boundary conditions of the temperature problem, one at a time, thereby dropping out the cosines and hyperbolic cosine, and determining the \( \alpha \) and \( \beta \). For the stress boundary value problem, however, complications arise. In this case, rather than being able to solve for the coefficients one at a time, they must come from a simultaneous solution of 18 equations. Furthermore, as was pointed out earlier, this gives only solutions to a class of problems having boundary conditions which the
function can fit. If the solution of a general type of problem is to be achieved, certain of the boundary conditions must necessarily be expressed as infinite series. This complicates the work enormously.

Some solutions can be obtained by substituting transcendental harmonic functions into the stress equations and observing what form of boundary stresses result. As an example, let us consider the stresses which arise by using the function given in (31). Referring to the stress equations (19), the normal stress \( \sigma_{xx} \) is seen to be zero for \( x = 0 \). This is because the first term is prefixed by \( x \) and the rest of the terms contain even numbered derivatives with respect to \( x \) of \( \sin \left( \frac{m \pi x}{a} \right) \). Similarly, the normal stresses on the faces \( y = 0 \) and \( z = 0 \) are zero. Normal stresses do exist on the faces \( x = a \), \( y = b \), and \( z = c \), however, because the odd numbered derivatives with respect to a single variable do not drop out. Shear stresses also exist on all faces. It is seen that the stresses obtained from this simple example are of a complex distribution, involving trigonometric and hyperbolic functions as well as linear functions of \( x \), \( y \), and \( z \). It appears, therefore, that solutions obtained in this manner are of little practical interest.
The Polynomial Approach

The Stress Equations Expressed in Terms of Polynomials

The following pages in this section contain the equations of normal and shearing stress for the polynomials of degree two through five. These equations were obtained by substituting the harmonic polynomials given on pages 74 and 75 of Appendix C into the stress equations (19). In this form, the stress equations identically satisfy the three-dimensional equations of equilibrium and compatibility.

The A's and B's are the coefficients of the various terms of the particular polynomial. The superscript 1, 2, or 3 on the coefficient refers to the particular harmonic function \( H_1 \), \( H_2 \), or \( H_3 \) for which it entered into the stress equation. The first of the two subscripts identifies the degree of the polynomial and the second identifies the particular term for which it arises in the polynomial. The exact procedure used in determining the second subscript is illustrated in Appendix C.
\[ \sigma_{\alpha\beta} = \nu \left( A_{21}^1 - A_{23}^2 + A_{23}^3 + A_{23}^3 \right) - A_{21}^1 \]

\[ \sigma_{\gamma\gamma} = \nu \left( -A_{21}^1 + A_{23}^2 + A_{23}^3 + A_{23}^3 \right) - A_{23}^2 \]

\[ \sigma_{\zeta\gamma} = \nu \left( -A_{21}^1 - A_{23}^2 - A_{23}^3 - A_{23}^3 \right) + (A_{21}^3 + A_{23}^3) \]

\[ \sigma_{\delta\gamma} = A_{22}^1 + A_{22}^2 \]

\[ \sigma_{\theta\gamma} = B_{22}^2 + B_{22}^3 \]

\[ \sigma_{\chi\chi} = B_{21}^3 + B_{21}^1 \]
\[ \sigma_{\xi\xi} = \left[ -3A_{31} + \nu (6A_{31} - 2A_{33} + 6A_{33} + 2A_{33}) \right] \xi \\
+ \left[ A_{33}^2 - 2A_{32} + \nu (2A_{33} - 6A_{33} + 2A_{33} + 6A_{33}) \right] \eta \\
+ \left[ B_{33}^3 - 2B_{31} + \nu (2B_{33} - 2B_{33} + 2B_{33} + 2B_{33}) \right] \zeta \]

\[ \sigma_{\eta\eta} = \left[ A_{33} - 2A_{33} + \nu (-6A_{31} + 2A_{33} + 6A_{33} + 2A_{33}) \right] \xi \\
+ \left[ -3A_{34} + \nu (-2A_{32} + 6A_{34} + 2A_{32} + 6A_{34}) \right] \eta \\
+ \left[ B_{33}^3 - 2B_{33} + \nu (-2B_{33} + 2B_{33} + 2B_{33} + 2B_{33}) \right] \zeta \]

\[ \sigma_{\zeta\zeta} = \left[ -3A_{31} - A_{33} + 6A_{31} + 2A_{33} + \nu (-6A_{31} - 2A_{33} - 6A_{33} - 2A_{33}) \right] \xi \\
+ \left[ -A_{32}^2 - 3A_{34} + 2A_{32} + 6A_{34} + \nu (-2A_{32} - 6A_{34} - 2A_{32} - 6A_{34}) \right] \eta \\
+ \left[ B_{31}^3 + B_{33}^3 + \nu (-2B_{31} - 2B_{33}^2 - 2B_{31} - 2B_{33}) \right] \zeta \]}
\[ \sigma_{xy} = \begin{bmatrix} -2A_{32}^2 + \nu(4A_{52}^1 + 4A_{32}^2) \\ -2A_{33}^1 + \nu(4A_{63}^1 + 4A_{33}^2) \\ B_{32}^3 - B_{32}^1 - B_{32}^2 + \nu(2B_{32}^1 + 2B_{32}^2) \end{bmatrix}_y \]

\[ \sigma_{yz} = \begin{bmatrix} B_{32}^1 - B_{32}^1 - B_{32}^3 + \nu(2B_{32}^2 + 2B_{32}^3) \\ -2B_{33}^1 + \nu(4B_{53}^2 + 4B_{33}^3) \\ 2A_{32}^2 + 6A_{33}^1 + \nu(-4A_{32}^2 - 12A_{33}^2 - 4A_{33}^3 - 12A_{33}^2) \end{bmatrix}_y \]

\[ \sigma_{zx} = \begin{bmatrix} -2B_{31}^3 + \nu(4B_{31}^1 + 4B_{31}^3) \\ -B_{32}^1 + B_{32}^2 - B_{32}^3 + \nu(2B_{32}^1 + 2B_{32}^3) \\ 6A_{30}^1 + 2A_{33}^1 + \nu(-12A_{30}^1 - 4A_{33}^1 - 12A_{30}^2 - 4A_{33}^2) \end{bmatrix}_y \]
\[ T_{hh} = \begin{bmatrix}
+ v (12 A_{41} - 2 A_{43}^2 + 12 A_{43}^3 + 2 A_{43}^4) \\
- 2 A_{43}^4 + 2 A_{43}^2 + v (2 A_{43}^4 - 12 A_{43}^2 + 2 A_{43}^3 + 12 A_{43}^3) \\
12 A_{41} + 2 A_{43} - 12 A_{41}^3 - 2 A_{43}^3 \\
+ v (-12 A_{41} + 2 A_{43}^2 + 12 A_{43}^2 - 12 A_{43}^3 - 4 A_{43}^3 - 12 A_{43}^3) \\
+ [3 A_{41} - 3 A_{43}^2 + v (-6 A_{42} + 6 A_{43}^2 + 6 A_{43}^3 + 6 A_{43}^4) \\
[3 A_{41} - 3 A_{43}^2 + v (-6 A_{42} + 6 A_{43}^2 + 6 A_{43}^3 + 6 A_{43}^4) \\
B_{43}^2 + B_{43}^3 - 2 B_{43}^2 + v (2 B_{43}^2 - 6 B_{43}^2 + 2 B_{43}^3 + 6 B_{43}^3) \\
-3 B_{41} + 3 B_{43} + v (6 B_{41}^3 - 2 B_{43}^2 + 6 B_{43}^2 + 2 B_{43}^3) \\
-3 B_{41} + 3 B_{43} + v (6 B_{41} - 2 B_{43}^2 + 6 B_{43}^2 + 2 B_{43}^3)
\end{bmatrix} \]

\[ \sigma_{4y} = \begin{bmatrix}
2 A_{43}^4 - 2 A_{43}^2 + v (-12 A_{41} + 2 A_{43}^2 + 12 A_{43}^3 + 2 A_{43}^4) \\
2 A_{43}^2 + 12 A_{43}^3 - 2 A_{43}^3 - 12 A_{43}^3 \\
+ v (12 A_{41} + 2 A_{43}^2 - 12 A_{43}^2 - 12 A_{41}^3 - 4 A_{43}^3 - 12 A_{43}^3) \\
3 A_{41} + 3 A_{43}^2 + v (-6 A_{42} + 6 A_{43}^2 + 6 A_{43}^3 + 6 A_{43}^4) \\
[3 A_{41} - 3 A_{43}^2 + v (-6 A_{42} + 6 A_{43}^2 + 6 A_{43}^3 + 6 A_{43}^4) \\
[B_{43}^2 + B_{43}^3 - 2 B_{43}^2 + v (2 B_{43}^2 - 6 B_{43}^2 + 2 B_{43}^3 + 6 B_{43}^3) \\
B_{43}^2 + B_{43}^3 - 2 B_{43}^2 + v (6 B_{41} + 2 B_{43}^2 + 6 B_{43}^2 + 2 B_{43}^3) \\
B_{43}^2 + B_{43}^3 - 2 B_{43}^2 + v (6 B_{41} + 2 B_{43}^2 + 6 B_{43}^2 + 2 B_{43}^3)
\end{bmatrix} \]
\[ \sigma_{33} = \left[ -12A_{41}^I - 2A_{43}^I + 12A_{41}^0 + 2A_{43}^0 + \nu \left( -12A_{41}^I - 2A_{43}^I - 12A_{41}^0 - 2A_{43}^0 + 12A_{43}^0 \right) \right] \chi^2 + \left[ -2A_{43}^2 - 12A_{45}^2 + 2A_{43}^2 + 12A_{45}^2 + \nu \left( -2A_{43}^2 - 2A_{45}^2 - 2A_{43}^2 - 12A_{43}^2 \right) \right] \gamma^2 + \left[ \nu \left( 12A_{41}^0 + 2A_{43}^0 + 2A_{43}^0 + 12A_{45}^0 + 4A_{43}^0 + 12A_{43}^0 \right) \right] \zeta^2 + \left[ -3A_{42}^I - 3A_{44}^I - 3A_{42}^0 - 3A_{44}^0 + 6A_{42}^0 + 6A_{44}^0 \right. \\
+ v \left( -6A_{42}^I - 6A_{44}^I - 6A_{42}^0 - 6A_{44}^0 \right) \right] \rho y + \left[ -B_{42}^2 - 3B_{44}^2 + B_{42}^2 + 3B_{44}^2 + \nu \left( -2B_{42}^2 - 6B_{44}^2 - 2B_{42}^2 - 6B_{44}^2 \right) \right] \gamma y + \left[ -3B_{42}^0 - B_{44}^0 + 3B_{42}^0 + B_{44}^0 \right. \\
+ v \left( -6B_{42}^0 - 2B_{44}^0 - 6B_{42}^0 - 2B_{44}^0 \right) \right] \zeta y \]

\[ 2\sigma_{1j} = \left[ 3A_{42}^I - 3A_{42}^0 + v \left( 6A_{42}^I + 6A_{42}^0 \right) \right] \chi^2 + \left[ -3A_{41}^I + 3A_{44}^I + v \left( 6A_{41}^I + 6A_{44}^I \right) \right] \gamma^2 + \left[ 3A_{42}^I + 3A_{44}^I + 3A_{42}^0 + 3A_{44}^0 - 6A_{42}^0 - 6A_{44}^0 \right. \\
+ v \left( -6A_{42}^I - 6A_{44}^I - 6A_{42}^0 - 6A_{44}^0 \right) \right] \gamma^2 + \left[ \nu \left( 8A_{42}^0 + 8A_{44}^0 \right) \right] \rho y + \left[ -2B_{42}^2 + 2B_{44}^2 + v \left( 4B_{42}^2 + 4B_{44}^2 \right) \right] \gamma y + \left[ -2B_{42}^0 + 2B_{44}^0 + v \left( 4B_{42}^0 + 4B_{44}^0 \right) \right] \gamma y.
\[ 2 \Phi_{y3} = \left[ 2 B_{42}^1 - B_{42}^2 - B_{42}^3 \right. \left. + v \left( 2 B_{42}^2 + 2 B_{42}^3 \right) \right] \kappa_x^2 \\
+ \left[ 3 B_{44}^2 - 3 B_{44}^3 \right. \left. + v \left( 6 B_{44}^2 + 6 B_{44}^3 \right) \right] \kappa_y^2 \\
+ \left[ B_{42}^2 + 3 B_{44}^2 - B_{42}^3 - 3 B_{44}^3 + v \left( -2 B_{42}^2 - 6 B_{44}^2 - 2 B_{42}^3 - 6 B_{44}^3 \right) \right] \gamma_x^2 \\
+ \left[ 2 B_{43}^1 - 2 B_{43}^2 \right. \left. + v \left( 4 B_{43}^1 + 4 B_{43}^2 \right) \right] \gamma_y^2 \\
+ \left[ v \left( -8 A_{45}^2 - 48 A_{45}^3 - 8 A_{45}^2 - 48 A_{45}^3 \right) \right] \gamma_y \\
+ \left[ -6 A_{42}^1 - 6 A_{42}^2 + 6 A_{42}^3 + 6 A_{42}^2 + 6 A_{42}^3 + v \left( -12 A_{42}^1 - 12 A_{42}^2 - 12 A_{42}^3 - 12 A_{42}^2 \right) \right] \gamma_x \\
\]

\[ 2 \Phi_{y4} = \left[ 3 B_{41}^1 - 3 B_{41}^3 \right. \left. + v \left( 6 B_{41}^1 + 6 B_{41}^3 \right) \right] \kappa_x^2 \\
+ \left[ - B_{43}^1 + 2 B_{43}^2 - B_{43}^3 \right. \left. + v \left( 2 B_{43}^1 + 2 B_{43}^3 \right) \right] \kappa_y^2 \\
+ \left[ 3 B_{41}^1 + B_{43}^1 - 3 B_{41}^3 - B_{43}^3 + v \left( -6 B_{41}^1 - 2 B_{43}^1 - 6 B_{41}^3 - 6 B_{43}^3 \right) \right] \gamma_x^2 \\
+ \left[ 2 B_{42}^2 - 2 B_{42}^3 \right. \left. + v \left( 4 B_{42}^1 + 4 B_{42}^3 \right) \right] \gamma_y^2 \\
+ \left[ 6 A_{42}^1 + 6 A_{42}^2 - 6 A_{42}^3 - 6 A_{42}^3 + v \left( -12 A_{42}^1 - 12 A_{42}^2 - 12 A_{42}^3 - 12 A_{42}^3 \right) \right] \gamma_y \\
+ \left[ v \left( -48 A_{41}^2 - 48 A_{41}^3 - 48 A_{41}^2 - 48 A_{41}^3 \right) \right] \gamma_x \]
\[ 
\sigma_{\alpha\beta} = 
\begin{cases} 
\begin{align*}
20 A_{51} & \quad + v \left( 40 A_{51}^1 - 4 A_{53}^2 + 40 A_{51}^3 - 4 A_{53}^3 \right) \alpha^3 \\
-4 A_{54} + 6 A_{54}^2 & \quad + v \left( 4 A_{54}^1 - 40 A_{54}^2 + 4 A_{54}^3 + 40 A_{54}^3 \right) \alpha^4 \\
8 B_{51}^1 + \frac{4}{3} B_{53}^1 - 12 B_{51}^3 - 2 B_{53}^3 & \quad + v \left( -8 B_{51}^1 - \frac{4}{3} B_{53}^1 + \frac{4}{3} B_{53}^2 + 8 B_{53}^2 - 8 B_{51}^3 - \frac{8}{3} B_{53}^3 - 8 B_{53}^3 \right) \alpha^3 \\
12 A_{52} & \quad + v \left( 24 A_{52}^1 - 12 A_{54}^2 + 24 A_{52}^3 + 12 A_{54}^3 \right) \alpha^4 \\
12 B_{51}^3 & \quad + v \left( 24 B_{51}^1 - 4 B_{53}^2 + 24 B_{51}^3 + 4 B_{53}^3 \right) \alpha^4 \\
-6 A_{53}^1 + 12 A_{53}^2 & \quad + v \left( 12 A_{53}^1 - 24 A_{53}^2 + 12 A_{53}^3 + 24 A_{53}^3 \right) \alpha^4 \\
-4 B_{53}^1 + 4 B_{53}^2 + 2 B_{53}^3 & \quad + v \left( 4 B_{53}^1 - 24 B_{53}^2 + 4 B_{53}^3 + 24 B_{53}^3 \right) \alpha^4 \\
60 A_{51} + 6 A_{53} + 120 A_{51}^3 + 12 A_{53}^3 & \quad + v \left( -120 A_{51}^1 - 12 A_{53}^1 + 12 A_{53}^2 + 24 A_{53}^3 - 12 A_{53}^3 \right) \alpha^3 \\
24 A_{52} + 12 A_{54} + 12 A_{54}^3 + 24 A_{52}^3 - 12 A_{54}^3 & \quad + v \left( -24 A_{52}^1 - 12 A_{54}^1 + 12 A_{54}^2 + 24 A_{52}^3 - 12 A_{54}^3 \right) \alpha^4 \\
-6 B_{52}^1 + 6 B_{52}^2 + 2 B_{52}^3 & \quad + v \left( 12 B_{52}^1 - 12 B_{52}^2 + 2 B_{52}^3 + 12 B_{52}^3 \right) \alpha^4 
\end{align*}
\end{cases} 
\]
\[ a_{yy} = \left[ \begin{array}{c}
6A_{53} - 4A_{53}^2 \\
+ \nu \left( -40A_{51} + 14A_{53} + 40A_{51}^3 + 4A_{53}^3 \right) \right] \kappa^3 \\
+ \left[ 20A_{56}^2 \\
+ \nu \left( -4A_{54} + 40A_{56}^2 + 4A_{54}^3 + 40A_{56}^3 \right) \right] \alpha^3 \\
+ \left[ \frac{4}{3}B_{53}^2 + 8B_{55}^2 - 2B_{53}^3 - 12B_{55}^3 \\
+ \nu \left( 8B_{51} + \frac{4}{3}B_{53}^2 - \frac{4}{3}B_{55}^2 - 8B_{51}^3 - \frac{4}{3}B_{53}^3 - 8B_{55}^3 \right) \kappa^3 \\
+ \left[ 12A_{54} - 6A_{54}^2 \\
+ \nu \left( -24A_{52} + 12A_{54} + 24A_{52}^3 + 12A_{54}^3 \right) \right] \kappa^3 \\
+ \left[ 4B_{53}^2 - 4B_{55}^2 + 2B_{53}^3 \\
+ \nu \left( -24B_{51} + 4B_{53}^2 + 24B_{53}^3 + 4B_{55}^3 \right) \alpha^3 \\
+ \left[ 12A_{55} \\
+ \nu \left( -12A_{53} + 24A_{55} + 12A_{53}^3 + 24A_{55}^3 \right) \right] \alpha^3 \\
+ \left[ 12B_{55}^3 \\
+ \nu \left( -4B_{53} + 24B_{55}^2 + 4B_{53}^3 + 24B_{55}^3 \right) \alpha^3 \\
+ \left[ -6A_{53} - 12A_{55} + 12A_{53}^2 + 24A_{55}^2 - 24A_{53}^3 - 24A_{55}^3 \\
+ \nu \left( 120A_{51} + 12A_{53}^2 + 12A_{55}^2 - 24A_{55}^3 - 24A_{53}^3 - 24A_{55}^3 \right) \right] \kappa^3 \\
+ \left[ 6A_{54}^2 + 60A_{56}^2 - 12A_{54}^3 - 120A_{56}^3 \\
+ \nu \left( 24A_{52}^2 + 12A_{54}^2 - 12A_{56}^2 - 24A_{52}^3 - 24A_{54}^3 - 120A_{56}^3 \right) \alpha^3 \\
+ \left[ 6B_{54}^2 - 6B_{54}^2 + 6B_{54}^3 \\
+ \nu \left( -12B_{52}^3 + 12B_{54}^2 + 12B_{52}^3 + 12B_{54}^3 \right) \right] \kappa^3 \\
\end{array} \right] \]
\[ \sigma_{yy} = \left[ -60A_1' - 6A_5 + 40A_1^2 + 4A_5^2 - 40A_1^2 - 4A_5^2 \right] \alpha^3
+ \left[ -6A_4^2 - 60A_6^2 + 4A_5^3 + 4A_6^3 - 4A_5^3 - 4A_6^3 \right] \gamma^3
+ \left[ \frac{4}{3} B_3 + 4B_5 \right. \\
\left. + \nu \left( \frac{8}{3} B_3 + \frac{8}{3} B_5 \right) \right] \beta^3
+ \left[ -24A_5' - 12A_5 - 4A_5^2 - 6A_5 + 24A_5^3 + 12A_5^4 \\
+ \nu \left( -24A_5' - 12A_5 - 24A_5^2 - 12A_5^3 \right) \right] \alpha^2 \gamma
+ \left[ -24B_5' - 4B_5 + 12B_5^2 + 2B_5^3 + \nu \left( -24B_5' - 4B_5 - 24B_5^2 - 4B_5^3 \right) \right] \alpha \gamma^2
+ \left[ -6A_5' - 12A_5 - 12A_5^2 - 24A_5^3 + 12A_5^4 + 24A_5^5 \\
+ \nu \left( -12A_5' - 24A_5^2 - 12A_5^3 - 24A_5^4 \right) \right] \alpha \beta
+ \left[ -4B_5' - 24B_5^2 + 12B_5^3 + 12B_5^4 + \nu \left( -4B_5' - 24B_5^2 - 4B_5^3 - 24B_5^4 \right) \right] \beta \gamma
+ \left[ 60A_5' + 12A_5 + 12A_5^2 \\
+ \nu \left( 120A_5' + 12A_5 + 12A_5^2 + 24A_5^3 + 120A_5^3 + 24A_5^4 + 24A_5^5 \right) \right] \alpha^2 \gamma
+ \left[ 12A_5 - 12A_5^2 + 60A_5^2 \\
+ \nu \left( 24A_5 - 12A_5^2 + 12A_5^3 + 120A_5^3 + 24A_5^3 + 120A_5^4 \right) \right] \gamma^2
+ \left[ -6B_5 - 6B_5 - 6B_5^2 + 6B_5^3 + 6B_5^4 \\
+ \nu \left( -6B_5 - 6B_5^2 + 12B_5^3 - 12B_5^4 \right) \right] \beta \gamma \]
\begin{align*}
T_{ij} &= \left[ 8A_{52}' - 4A_{52}^2 \right] x^3 \\
&\quad + v \left( 8A_{52}' + 8A_{52}^2 \right) x^3 \\
&\quad + \left[ -4A_{53}' + 8A_{53}^2 \right] y^3 \\
&\quad + v \left( 8A_{53}' + 8A_{53}^2 \right) y^3 \\
&\quad + \left[ B_{52}' + B_{54}' + B_{52}^2 + B_{54}^2 - 3B_{52}^3 - 3B_{54}^3 \right] \eta^3 \\
&\quad + v \left( -2B_{52}' - 2B_{54}' - 2B_{52}^2 - 2B_{54}^2 \right) \eta^3 \\
&\quad + \left[ 6A_{53}' \right] x^2 y \\
&\quad + v \left( 12A_{53}' + 12A_{53}^2 \right) x^2 y \\
&\quad + \left[ 3B_{52}' - 3B_{52}^2 + 3B_{52}^3 \right] x^2 z \\
&\quad + v \left( 6B_{52}' + 6B_{52}^2 \right) x^2 z \\
&\quad + \left[ 6A_{54}' \right] x^2 \eta \\
&\quad + v \left( 12A_{54}' + 12A_{54}^2 \right) x^2 \eta \\
&\quad + \left[ -3B_{54}' + 3B_{54}^2 + 3B_{54}^3 \right] x^2 \eta^2 \\
&\quad + v \left( 6B_{54}' + 6B_{54}^2 \right) x^2 \eta^2 \\
&\quad + \left[ 12A_{52}^2 + 6A_{52}' - 24A_{52}^3 - 12A_{52}^3 \right] y^2 \\
&\quad + v \left( -24A_{52}' - 12A_{52}^2 - 24A_{52}^2 \right) y^2 \\
&\quad + \left[ 6A_{53}' + 12A_{53}^2 - 24A_{53}^3 \right] y \eta \\
&\quad + v \left( -12A_{53}' - 24A_{53}^2 + 12A_{53}^2 - 24A_{53}^2 \right) y \eta \\
&\quad + \left[ 4B_{53}' \right] y \eta^2 \\
&\quad + v \left( 8B_{53}' + 8B_{53}^2 \right) y \eta^2
\end{align*}
\[ \sum_{N_2} = \left[ 3 B_{52} - B_{52}^2 - B_{52}^3 \right] \chi^3 \\
+ \left[ 8 B_{53} - 4 B_{53}^2 + \nu (8 B_{53} + 8 B_{53}^3) \right] \gamma^3 \\
+ \left[ -4 A_{52}^2 - 4 A_{54}^2 - 20 A_{52}^2 + 8 A_{52}^4 + 8 A_{54}^3 + 40 A_{56}^3 \right] \beta^3 \\
+ \nu (8 A_{52}^2 + 8 A_{54}^2 + 40 A_{52}^6 + 8 A_{54}^3 + 8 A_{54}^3 + 40 A_{56}^3) \gamma^3 \\
+ \left[ 4 B_{53} - B_{55}^2 + \nu (4 B_{53} + 4 B_{53}) \right] \chi^3 \gamma \\
+ \left[ -24 A_{52} - 12 A_{54} + 12 A_{54}^2 + 6 A_{56}^2 + \nu (-24 A_{52} - 12 A_{54} - 24 A_{52}^3 - 12 A_{54}^3) \right] \chi^3 \gamma \\
+ \left[ 3 B_{54} + 3 B_{54} - 3 B_{54}^3 + \nu (6 B_{54} + 6 B_{54}^3) \right] \chi^3 \gamma^2 \\
+ \left[ -6 A_{54} - 60 A_{56}^2 + \nu (-12 A_{54} - 120 A_{56}^2 - 12 A_{54}^3 - 120 A_{56}^3) \right] \chi^3 \gamma^2 \\
+ \left[ -3 B_{52} - 3 B_{54} + 3 B_{52} - 3 B_{52}^3 + 3 B_{52} - 3 B_{54}^3 + \nu (-6 B_{52} - 6 B_{54} - 6 B_{52}^3 - 6 B_{54}^3) \right] \chi^3 \gamma^2 \\
+ \left[ -2 B_{53} - 12 B_{53}^3 + \nu (-4 B_{53} - 24 B_{53}^3 - 9 B_{53} - 24 B_{53}^3) \right] \chi^3 \gamma^2 \\
+ \left[ -12 A_{53} - 24 A_{53}^3 + \nu (-24 A_{53} - 48 A_{53}^3 - 24 A_{53}^3 - 48 A_{53}) \right] \chi^3 \gamma^2 \]
\[ \bar{\eta}_k = \left[ 8 B_{s1}^i - 4 B_{s1}^3 + \nu (8 B_{s1}^i + 8 B_{s1}^3) \right] \chi^3 \\
+ \left[ - B_{s4}^i + 3 B_{s4}^2 - B_{s4}^3 + \nu (2 B_{s4}^i + 2 B_{s4}^3) \right] \eta^3 \\
+ \left[ - 20 A_{s1}^i - 4 A_{s3}^i - 4 A_{s5}^i + 40 A_{s1}^3 + 8 A_{s3}^3 + 8 A_{s5}^3 \\
+ \nu (40 A_{s1}^i + 8 A_{s3}^i + 8 A_{s5}^i + 40 A_{s1}^3 + 8 A_{s3}^3 + 8 A_{s5}^3) \right] \zeta^3 \\
+ \left[ 3 B_{s2}^i + 3 B_{s2}^2 - 3 B_{s2}^3 + \nu (6 B_{s2}^i + 6 B_{s2}^3) \right] \chi^2 \eta \\
+ \left[ - 60 A_{s2}^i - 6 A_{s5}^i \\
+ \nu (-120 A_{s2}^i - 12 A_{s5}^i - 120 A_{s2}^3 - 12 A_{s5}^3) \right] \chi \eta^2 \\
+ \left[ 4 B_{s3}^2 - 2 B_{s3}^3 + \nu (4 B_{s3}^i + 4 B_{s3}^3) \right] \chi^2 \eta^2 \\
+ \left[ 6 A_{s3}^i + 12 A_{s3}^2 - 12 A_{s3}^3 - 24 A_{s3}^5 + \nu (12 A_{s3}^i + 24 A_{s3}^3 + 12 A_{s3}^3 + 24 A_{s3}^5) \right] \chi \eta^3 \\
+ \left[ - 20 A_{s3}^i - 2 B_{s3}^3 \\
+ \nu (-24 B_{s3}^i - 4 B_{s3}^i - 24 B_{s3}^3 + 4 B_{s3}^3) \right] \eta^3 \\
+ \left[ 3 B_{s4}^i + 3 B_{s4}^2 - 3 B_{s4}^3 - 3 B_{s4}^2 - 3 B_{s4}^3 \\
+ \nu (-6 B_{s4}^i - 6 B_{s4}^3 - 6 B_{s4}^3 - 6 B_{s4}^3) \right] \eta^2 \\
+ \left[ - 24 A_{s4}^i - 12 A_{s4}^2 \\
+ \nu (-48 A_{s4}^i - 24 A_{s4}^2 - 48 A_{s4}^2 - 24 A_{s4}^3) \right] \eta \eta^2 \\
+ \left[ 6 B_{s5}^3 + 12 B_{s5}^2 - 12 B_{s5}^3 - 24 B_{s5}^5 + \nu (12 B_{s5}^i + 24 B_{s5}^3 + 12 B_{s5}^3 + 24 B_{s5}^5) \right] \eta^3 \\
+ \left[ - 20 B_{s5}^i - 2 B_{s5}^3 \\
+ \nu (-24 B_{s5}^i - 4 B_{s5}^i - 24 B_{s5}^3 + 4 B_{s5}^3) \right] \eta^2 \\
+ \left[ 3 B_{s6}^i + 3 B_{s6}^2 - 3 B_{s6}^3 - 3 B_{s6}^2 - 3 B_{s6}^3 \\
+ \nu (-6 B_{s6}^i - 6 B_{s6}^3 - 6 B_{s6}^3 - 6 B_{s6}^3) \right] \eta \\
+ \left[ - 24 A_{s6}^i - 12 A_{s6}^2 \\
+ \nu (-48 A_{s6}^i - 24 A_{s6}^2 - 48 A_{s6}^2 - 24 A_{s6}^3) \right] \eta^2 \\
+ \left[ 6 B_{s7}^3 + 12 B_{s7}^2 - 12 B_{s7}^3 - 24 B_{s7}^5 + \nu (12 B_{s7}^i + 24 B_{s7}^3 + 12 B_{s7}^3 + 24 B_{s7}^5) \right] \eta^3 \\
+ \left[ - 20 B_{s7}^i - 2 B_{s7}^3 \\
+ \nu (-24 B_{s7}^i - 4 B_{s7}^i - 24 B_{s7}^3 + 4 B_{s7}^3) \right] \eta^2 \\
+ \left[ 3 B_{s8}^i + 3 B_{s8}^2 - 3 B_{s8}^3 - 3 B_{s8}^2 - 3 B_{s8}^3 \\
+ \nu (-6 B_{s8}^i - 6 B_{s8}^3 - 6 B_{s8}^3 - 6 B_{s8}^3) \right] \eta
The Solution of Problems by Means of Polynomials

The equations of stress obtained from the homogeneous harmonic polynomials of degrees two through five have been presented in the previous section. It is obvious that since each stress equation identically satisfies the equilibrium equations (6) and the compatibility equations (46), linear combinations of these equations must also represent solutions. Furthermore, each of the A's and B's taken separately must represent a solution. Therefore, a number of random solutions to three-dimensional problems may be obtained by simply letting all coefficients be zero except one particular A or B. Further solutions may then be obtained by linear combinations of these. In this manner, a catalogue of solutions might be constructed which could be referred to if the solution to a particular problem is desired. Such a catalogue is partially given in Appendix D, where the solutions obtained, letting each coefficient, in turn, be non-zero, with the others all being zero, are presented. Some interesting examples of these solutions and linear combinations thereof will be discussed below.

The solutions obtained from $P_2$ as given in Appendix D are those found by simply substituting $P_2$ into the stress equations (19). Since the degree of stress will always be two less than the polynomial, these solutions will be all constants. However, it is easily seen that any constant stress field identically satisfies the equilibrium and compatibility equations. It is also seen, however, that any constant stress may be obtained by a linear combination of the solutions given. The ability to add constant stresses to any problem without altering its
validity is a particularly important tool for achieving useful solutions.

The polynomial $P_3$ gives linear variations in the stresses as solutions. Only the equilibrium conditions apply any restriction upon these stresses, for the compatibility conditions are identically satisfied by a linear stress variation. If the case of only $A_{34}^2$ being non-zero is considered, the following system of stresses is found:

\[
\begin{align*}
\sigma_{xx} &= 2vA_{xx}^z y \\
\sigma_{yy} &= (1-2v)A_{yy}^z y \\
\sigma_{yy} &= (1+2v)A_{yy}^z y
\end{align*}
\]

\[
\tau_{xy} = 0 \\
\tau_{yx} = -(1-2v)A_{xx}^z y \\
\tau_{xy} = 0.
\]

In terms of one rectangular coordinate system usually found particularly useful for cantilever beam problems, these stresses may be represented by the three orthographic views shown below.

![Diagram of stresses](image)

Fig. 2 - Stresses resulting from $A_{34}^2 \neq 0$. 
The normal stresses in the x and z directions are seen to be of the self-equilibrating, bending type, while the remaining one in the y-direction produces tension at the top and compression at the bottom. The shear stress varies linearly along the top and bottom and is a constant along the sides.

Now let the arbitrary coefficients $A_{34}^2$ and $A_{34}^3$ both be considered as the only non-zero ones, and let $A_{34}^2 = A_{34}^3 = A_{34}^4$. If the two resulting sets of stress equations are simply added, the following stresses are found:

\[
\begin{align*}
\sigma_{xx} &= 0 \\
\sigma_{yy} &= (1-4v)A_{34}^y \\
\sigma_{zz} &= -(1-4v)A_{34}^z \\
\tau_{xz} &= 0.
\end{align*}
\]

The above equations could be further simplified by omitting the $(1-4v)$ from each one. This procedure has eliminated the normal stress in the

![Fig. 3 - Stresses resulting from $A_{34}^2 = A_{34}^3 \neq 0$.](image-url)
x-direction, and has reduced the problem to a two-dimensional one. The stresses may then be represented in the single view above, for there is no stress variation in the x-direction. The picture can be further simplified by the addition or subtraction of a constant normal stress in the y-direction, as discussed previously, without invalidating the equilibrium or compatibility conditions. This would give normal stresses at either the top or bottom, without having to have both. The above solution can be considered useful, for Figure 3 could be construed to be a two-dimensional beam, with the length in the z-direction, subjected to a uniform loading in the y-direction, and constrained at the ends so as to produce the linearly varying bending stress shown by $\sigma_{yy}$ and the constant shear $\tau_{xy}$. Unfortunately, however, a varying shear stress remains along the top and bottom, which is unlikely to be found in a physical problem.

Another interesting problem is obtained by letting $B_{32}^3 \neq 0$. The stress equations for this case, as given in Appendix D, are

\[
\begin{align*}
\sigma_{xx} &= 0 \\
\sigma_{yy} &= (1-2v)B_{32}' y \\
\tau_{xy} &= 0 \\
\tau_{yz} &= -B_{32}' z \\
\tau_{zx} &= 0 \\
\sigma_{xx} &= (1-2v)B_{32}' y.
\end{align*}
\]

Furthermore, if $B_{32}^2$ and $B_{32}^3$ are also non-zero, by suitable linear combinations of the three sets of equations all having zero normal stresses, it is possible to make any two of the shear stresses zero. One such possibility would be $\tau_{yz}$ equal to a constant times x, and all other
stresses zero. This would satisfy the equilibrium equations because the derivatives of this stress are taken with respect to \( y \) and \( z \). This shear stress could be represented by the following figure.

![Figure 4 - Stresses resulting from \( B_{32}^1 = B_{32}^2 = B_{32}^3 \neq 0 \)](image)

Turning to the polynomial \( P_{ij} \), it is seen that here the compatibility equations first have their effects felt. If we let \( A_{41}^1 = A_{41}^3 = A_{41} \neq 0 \), and subtract one of the two sets of stresses from the other, a stress distribution consisting of only normal stresses results:

\[
\begin{align*}
\sigma_{xx} &= -A_{41} \xi^2 \\
\sigma_{yy} &= \nu A_{41} (\xi^2 - \eta^2) \\
\sigma_{zz} &= A_{41} \xi^2
\end{align*}
\]

\[
\begin{align*}
\sigma_{xy} &= 0 \\
\sigma_{yz} &= 0 \\
\sigma_{zx} &= 0.
\end{align*}
\]
These stresses may be pictured as shown below for the particular case of a rectangular prism having $x$, $y$, and $z$ dimensions of $a$, $2a$, and $2a$, respectively.

*Fig. 5 - Stress variation on the faces of a rectangular prism resulting from $A_{11}^1 = A_{11}^3 \neq 0$. 

Furthermore, if $A_{11}^1 = A_{11}^3 = A_{11} \neq 0$ again, but the resulting two sets of equation are this time added, they give:

\[
\begin{align*}
\sigma_{xx} &= -A_{11}(x^2-y^2) \\
\sigma_{yy} &= 0 \\
\sigma_{yz} &= A_{11}(x^2-y^2) \\
\sigma_{xy} &= 0 \\
\sigma_{xz} &= 2A_{11}yz
\end{align*}
\]
This set of equations represents the problem with the boundary stresses shown in Figure 6 on the following page. The complete stress distribution would be that obtained by the superposition of the normal stress and shear stress diagrams shown.

It is interesting to note that the problems obtained above from considerations of the coefficients \( A_{34} \) and \( A_{44} \) are actually problems governed by plane strain. That is, the resulting stress equations are of the form \( \sigma_{ii} = \nu (\sigma_{ij} + \sigma_{kk}) \) and \( \sigma_{ij} = \sigma_{kk} = 0 \). By studying the solutions given in Appendix D, it is evident that for each polynomial several solutions arise which are plane strain problems. A few trivial cases of plane stress arise, also. In general, however, the solutions given in Appendix D are three-dimensional in nature.

Thus far, solutions to problems by using polynomials have been presented in a most indirect manner. The substitution of the coefficients of the harmonic polynomials, one by one, into the stress equations, succeeds in achieving little more than a random collection of miscellaneous problems. Using this method, one might say that he has succeeded in finding an answer and has only to see what problem the answer determines. The engineer cannot work in this way. He has a problem for which he needs an answer. Using the preceding approach, it would be necessary for him to work with a catalogue, such as Appendix D, with the greatest facility and ingenuity in order to achieve the solution. Indeed, a solution to the proposed problem may even be impossible, owing to an unobvious set of incompatible boundary conditions which he may be attempting to fit. There is a need for a direct method of obtaining the solution to a given problem if a solution exists.
Fig. 6a - Variation in normal stress for $A_{ij}^1 = A_{ij}^3 = A_{ij} 
eq 0$.

Fig. 6b - Variation in shear stress for $A_{ij}^1 = A_{ij}^3 = A_{ij} 
eq 0$. 
One such direct approach consists of writing all the boundary conditions for the problem in terms of the harmonic polynomials and solve for the coefficients by a process of equating coefficients. The question then arises regarding what degree of polynomial is necessary in order to solve the given problem. Unfortunately, the author has yet been unable to determine the relationship governing this condition. However, the scope of the problem can be greatly reduced by considerations of symmetry and antisymmetry. Since any given problem can be broken into two problems, one which is symmetrical and the other which is antisymmetrical, the complete solution could be achieved by the superposition of these two solutions, once they are obtained.

Using this approach, the author investigated a particular problem which has attracted considerable interest for many years. This problem consists of the rectangular cantilever beam subjected to a parabolically distributed shearing stress on the free end as shown in Figure 7 below.

![Fig. 7 - Rectangular cantilever beam with a parabolically distributed shearing stress on the free end.](image)

This problem has been solved for an approximating answer by means of trigonometric series. It would seem that only a solution in the form of an infinite series could be expected if the boundary conditions on the
free end must be expressed as an infinite series. On the other hand, an exact solution in the form of a finite series of polynomials does seem plausible if the boundary conditions are expressible in terms of finite series of polynomials, as they obviously are.

For the coordinate system shown in Figure 7, with z coming out of the paper, let the dimensions of the rectangular beam in the x, y, and z directions be a, 2b, and 2c, respectively. Then the parabolic shearing stress on the free end is expressible as

\[ \tau_{xy} = \frac{3}{8} \frac{P}{b^3 c} \left( b^2 - y^2 \right). \]  

The function expressing the shear stress in equation (32) is an even function, and since the remainder of the boundary stresses on the five free faces are zero, the stress distribution throughout the body should be in terms of even functions. This agrees with the approximate solution obtained from strength of materials, which states that the shear stress remains constant along the length and the bending stress \( \sigma_{xy} \) varies linearly with both x and y.

Many of the answers obtained from strength of materials analyses indicate that they are actually approximations of solutions obtained from elasticity. Often these answers are the first term or two of a somewhat longer, yet exact, elasticity solution. This is particularly vividly demonstrated for many of the two-dimensional problems involving plane stress or plane strain.²

The author first attempted to obtain a solution to the aforementioned problem of the rectangular cantilever beam subjected to a para-
bolic shear distribution on the free end by considering the polynomials of degrees two, three, and four. It was obvious that four would be the minimum degree required, since even the fourth degree polynomial gives stresses of only the second degree, which are required by the strength of materials solution. It was decided to apply the three known stress boundary conditions on each of the five free sides and thereby determine the stress conditions at the wall and elsewhere in the beam.

As an example of the procedure involved, if the boundary condition to be applied is that the shear stress \( \tau_{x'y'} \) must conform to equation (32) when \( x = 0 \), the following equation arises:

\[
\frac{3}{8} \frac{P}{bc} \left( b^2 - y^2 \right) = (1-2v)A_{2x} + (1-2v)A_{22} + (1-2v)A_{33} y
\]

\[
- 2vA_{33} y + (1-2v)B_{32} z + (1-2v)B_{32} z - B_{32} z
\]

\[
+ 3(1-2v)A_{44} y^2 - 3(1+2v)A_{44} y^2 - 3(1-2v)A_{44} z^2
\]

\[
- 3(1-2v)A_{44} z^2 + 6A_{44} z^2 - 3(1-2v)A_{44} z^2 - 3(1-2v)A_{44} z^2
\]

\[
+ 6A_{44} z^2 + 2(1-2v)B_{43} y - 4vB_{43} y - 2B_{43} y.
\]

(33)

Equating coefficients results in six equations as follows:

\[
(1-2v)A_{22} + (1-2v)A_{22} = \frac{3}{8} \frac{P}{bc}
\]

(34)

\[
(1-2v)A_{33}' - 2vA_{33} = 0
\]

(35)
This process must be repeated for the three stress boundary conditions on each of the five sides, resulting in a total of $3 \times 5 \times 6$, or 90 equations, of the type given above in equations (34) through (39). Referring to Appendix C, we see that $P_2$ has five arbitrary $A$'s and $B$'s as coefficients, while $P_3$ has seven, and $P_4$ has nine. Furthermore, each coefficient runs over three superscripts, giving a total of $(5+7+9) \times 3$, or 63 coefficients. Thus at a first glance, it appears that one has an array of 90 linear equations containing 63 unknowns. In actually working through the procedure, however, one discovers that of the 63 coefficients, 13 of them do not appear in the equations, and that four more must be dropped to avoid duplicating columns in the matrix of coefficients, leaving a net total of 46 $A$'s and $B$'s to be determined. Of the ninety equations, 19 are exact duplications, owing to the symmetry of the $y$ and $z$ axes, and 11 are dependent upon the others in an obvious manner. Other dependencies also exist which are not as obvious as these. However, in the end, there are still more equations than unknowns -- 46 unknowns and approximately 50 equations.

The resulting coefficient matrix of the equations can become
broken into useful sub-matrices of a much lower order, however, for not a great deal of interdependency exists among the A's and B's. In fact, no row of the matrix has more than 12 of its 46 terms non-zero. Since all of the original 90 equations are homogeneous except (34) and (37), only those equations containing any of the coefficients found in (34) and (37) must be considered. The result is that $A_{22}^3$ is one of the four coefficients which must be dropped to avoid duplicating columns; therefore one can immediately solve for $A_{22}^2$ from equation (34). It is found to be

$$A_{22}^2 = \frac{3}{8} \frac{P}{bc(1-2v)}. \quad (40)$$

Considering the other non-homogeneous equation (37), it is discovered that six linearly independent equations are applicable, containing the five coefficients $A_{42}^1$, $A_{42}^2$, $A_{42}^3$, $A_{44}^1$, and $A_{44}^2$. These equations can be represented by the matrix equation shown below:

$$
\begin{bmatrix}
0 & 0 & 0 & 3(1-2v) & -3(1+2v) \\
-(1-2v) & -(1-2v) & 2 & -(1-2v) & -(1-2v) \\
1 & -(1-2v) & 2v & 1 & -(1-2v) \\
(1+2v) & 1 & -2(1-v) & 1 & (1+2v) \\
2v & 0 & -2v & -1 & (1-2v) \\
-(1-2v) & 1 & 2v & -(1-2v) & 1
\end{bmatrix}
\begin{bmatrix}
A_{42}^1 \\
A_{42}^2 \\
A_{42}^3 \\
A_{44}^1 \\
A_{44}^2
\end{bmatrix} = \begin{bmatrix}
-\frac{3}{8} \frac{P}{bc} \\
0 \\
0 \\
0 \\
0
\end{bmatrix} \quad (41)$$
Thus no solution for the particular problem can be obtained by considering polynomials only through degree four. However, other interesting problems can be found by using this set of six equations and five unknowns. Obviously, equation (37) must be retained for a non-trivial solution. Of the remaining five, the choice of any four does result in a problem having the parabolic shear stress at the free end, but other boundary stresses as well. For example, if we choose to neglect the equation given by the fourth row of the coefficient matrix in (41), and solve the remaining 5 x 5 set, the following solutions are found for the coefficients:

\[
\begin{align*}
A'_{42} & = - \frac{P(1-2v)}{64v^3c} \\
A''_{42} & = - \frac{P(1+2v)}{64v^3c} \\
A'''_{42} & = 0 \\
A'_{44} & = \frac{P(1-2v)}{64v^3c} \\
A''_{44} & = \frac{P(1+2v)}{64v^3c}.
\end{align*}
\]

In this case, the ignored fourth equation of (41) is obtained from the condition that the normal stress in the z direction \( \sigma_{zz} \) does not depend upon the xy term. Therefore, when the solutions (42) are substituted back into the equations for the stress, with the other
If all coefficients are zero, the resulting stress system is

\[
\begin{align*}
\sigma_{xx} &= \frac{3}{4} \frac{P}{bc} x y \\
\tau_{xy} &= \frac{3}{8} \frac{P}{bc} (b^2 - y^2) \\
\tau_{yy} &= 0 \\
\sigma_{yy} &= 0 \\
\tau_{yx} &= 0 \\
\sigma_{yx} &= 0.
\end{align*}
\]  

\[(43)\]

This set of stresses can be pictured by the orthographic views shown below.

Fig. 8 - One solution for the rectangular cantilever beam with a parabolically distributed shear stress on the free end.
The stress distribution at the wall is exactly that given by the strength of materials approach; that is, the bending stress varies linearly with $y$ and the shearing stress varies parabolically with $y$. However, the unsatisfied equation gives, in this case, a bending moment due to the $T_{yy}$ stresses along the sides of the beam which increases linearly with $x$. In fact, this additional transverse bending moment is $\nu$ times the longitudinal bending moment at all points in the beam.

Returning to the equations given by (4.1), if the second equation is neglected instead of the fourth, $T_{yy}$ shearing stresses which vary with $z^2$ arise on the top and bottom and the ends. Neglecting the third equation gives rise to $T_{yy}$ shearing stresses which vary with $zx$ on the top and bottom, and only with $x$ on the sides. Similarly, neglecting the fifth causes $T_{yy}$ normal stresses, varying linearly only with $x$ to appear on the top and bottom. Finally, omitting the sixth results in $T_{yx}$ shearing stresses, which vary linearly only with $y$ appearing on the sides and ends. In all of the above cases, it is seen that the additional stresses which arise are necessary only in order that the compatibility conditions be satisfied. The first case considered, which results in equations (4.3), is a problem which satisfies the conditions of plane strain. The other four cases give neither plane strain nor plane stress.

Although the original array containing the 46 undetermined coefficients in approximately 50 independent equations was set up in the attempt to solve the aforementioned particular problem of the rectangular cantilever beam with the parabolic shearing stress on the free end, the left sides of these equations must remain unchanged as long as the
coordinate system shown in the foregoing illustrations is used. The setting up of another problem having different boundary conditions only changes the constants which appear on the right sides. Therefore, the set of five coefficients used for the preceding problem results in all the solutions possible under the severe restriction of limiting the degree of the polynomial to the fourth.

One interesting set of coefficients obtained in this manner is the set containing $B_{31}^1$, $B_{31}^3$, $B_{33}^2$, and $B_{33}^3$. This set of four unknown coefficients is found to occur in four independent equations. Letting a particular three of the equations be equal to zero results in a pure couple on the ends; i.e., $\sigma_{mx}$ varies linearly only with $z$. Letting another of the equations be non-zero, instead, gives a pure couple along the top and bottom of the beam; i.e., $\sigma_{yy}$ varies linearly only with $z$. Another case has $\sigma_{yy}$ varying linearly along the sides ($z = \pm c$) and constant along the top and bottom ($y = \pm b$). The last solution results in $\sigma_{yy}$ being proportional to $x$ along the sides, zero on the free end, and a constant along the fixed end.

Another set of coefficients having an equal number of equations is that containing $B_{32}^1$, $B_{32}^2$, and $B_{32}^3$. This was mentioned earlier in this section while discussing the superposition of solutions obtained by letting certain coefficients be non-zero, in turn. As was indicated there, this set only gives linearly varying shear stresses on the various sides.

The author made one further attempt at solving the problem of the rectangular cantilever beam with the parabolic shear distribution on the free end. Since the earlier attempt with this problem, using the poly-
nomials only through the fourth degree, resulted in six equations containing only five unknowns, it was hoped that further extensions of the polynomials might give an equal number of equations and unknowns. Since the stress distribution must be expressible in terms of even functions for this problem, it was obvious that nothing could be obtained from $P_5$. It was hoped that $P_6$ would be adequate. To have attempted to set up the entire matrix of coefficients for all the polynomials through $P_6$ would have required 225 equations containing 135 undetermined coefficients before reducing their number because of repetitions, dependence, etc. This herculean task was avoided, however, by first rewriting the few equations which had applied to the problem in the earlier analysis. This set of equations contained three new coefficients obtained from $P_6$. Then all the remaining equations containing these three new coefficients were written. Since these latter equations were all homogeneous, however, it was found that strictly by row operations they could be reduced to only two independent ones. This appeared to solve the problem, for there were now eight equations and eight unknowns. Further examination revealed, however, that the left sides of the two new homogeneous equations were duplicated for the new coefficients in the previous six equations, thus resulting in nothing obtained from $P_6$.

That $P_6$ does not solve the problem does not appear strange to the author. The fact that it does not contribute to the solution is perplexing, however. It remains to be seen whether consideration of $P_8$ and $P_{10}$ contributes anything to the problem. In the first place, the problem may require either a finite or an infinite number of terms. The degree of polynomial required for a solution to any problem remains yet unde-
Another thought has even broader implications, however. It could be that no compatible solution exists for the problem in a mathematical sense, because the problem may be incompatible itself in the physical sense. The proof that the problem actually can exist would in itself be most enlightening.
Conclusions

In the foregoing study, two approaches are presented for the solution of three-dimensional problems involving the rectangular beam. The approach using the transcendental functions gives no method for handling the general type of problem but, rather, is limited to a class of problems having stresses on the boundaries which are expressible in terms of these transcendental functions. This does not mean that this approach cannot be useful in the solution of problems of a more general nature, but that the author was unable to discover a workable technique.

The other approach discussed is that using the harmonic polynomials. Two methods are considered which involve this approach. The first method consists of the superposition of known solutions in such a manner so as to give the complete solution for the problem in question. This involves using a catalogue of solutions such as the partial one presented in Appendix D. Various examples are presented in which these solutions are combined to give interesting results. Some of the results thus obtained are seen to be problems involving plane strain.

The more direct method of problem solution is also presented, wherein the equations are written expressing the restrictions imposed at the boundaries, and the coefficients are obtained by the simultaneous solution of a set of linear equations. Using this method, certain of the polynomials can be ignored, depending upon whether the boundary stresses are even or odd functions; however, the author was unable to
discover the relationship which determines the degree of polynomial needed for the solution of any particular problem.

As an example of the preceding method, the problem of the rectangular cantilever beam subjected to a parabolically distributed shearing stress on the free end is discussed. It is seen that consideration of the polynomials through the fourth degree does not solve the problem, but does give the solutions to five other problems of a similar nature. The further extension to the sixth degree polynomial gives no further results for this particular problem. It remains to be seen whether the continuation to higher degrees such as the eighth or tenth will give the solution. What other problems of interest are solved by using these slightly higher degrees of polynomials are also yet undetermined.

The application discussed in this dissertation is limited to bodies having rectangular prismatic shapes; therefore, the entire analysis was made in Cartesian coordinates. The study of other types of three-dimensional bodies would, no doubt, require the use of other coordinate systems such as the spherical, cylindrical, ellipsoidal, etc. The stress equations would have to be rewritten in terms of these coordinates and harmonic functions of other forms would be found useful for these problems.

The problems discussed in this work are limited to those for which the boundary conditions are expressed in terms of the stresses. Other problems having the boundary conditions given by displacements can be investigated by this three-dimensional stress function approach. Mixed boundary value problems having both stresses and displacements given are also treatable. The displacement equations are given in Appendix D.
In conclusion, the author wishes to state that the material presented in this dissertation is merely a beginning in the direction of solving three-dimensional elasticity problems of a useful nature, and that many avenues for future exploration remain, some of which are mentioned above. The author intends to carry this work further on in these directions in an effort to contribute something of value to man's knowledge.
Appendix A

The Solution of the Equations of Equilibrium and Compatibility in Terms of the Maxwell Stress Functions
The stresses may be defined in terms of the Maxwell stress functions according to:

\[
\begin{align*}
\sigma_{xx} &= B_{yy} + C_{yy}, \\
\sigma_{yy} &= C_{xx} + A_{xx} \\
\sigma_{yz} &= A_{yz} + B_{yz}, \\
\sigma_{yz} &= -A_{yz}, \\
\sigma_{yz} &= -B_{yz}, \\
\sigma_{yz} &= -C_{yz}
\end{align*}
\]

where \( A, B, \) and \( C \) are any arbitrary functions of \( x, y, \) and \( z, \) and the subscripts on \( A, B, \) and \( C \) indicate the partial derivatives with respect to \( x, y, \) and \( z. \) Defined in this manner, it has been shown (page 44) that the stresses identically satisfy the three-dimensional equations of equilibrium.

The Beltrami-Michell form of the compatibility equations is

\[
(1 + \nu) \nabla^2 \sigma_{xx} + \Theta_{xx} = 0
\]
\[
(1 + \nu) \nabla^2 \sigma_{yy} + \Theta_{yy} = 0
\]
\[
(1 + \nu) \nabla^2 \sigma_{yz} + \Theta_{yz} = 0
\]
\[
(1 + \nu) \nabla^2 \sigma_{yy} + \Theta_{yy} = 0
\]
\[
(1 + \nu) \nabla^2 \sigma_{yz} + \Theta_{yz} = 0
\]
\[
(1 + \nu) \nabla^2 \sigma_{yz} + \Theta_{yz} = 0,
\]

in which \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \) and \( \Theta = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}. \)

If the first three of equations (46) are added, we have

\[
(1 + \nu) \nabla^2 (\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) + (\Theta_{xx} + \Theta_{yy} + \Theta_{zz}) = 0,
\]
or

\[
(2 + \nu) \nabla^2 \Theta = 0.
\]
Therefore, \[ \nabla^2 \Theta = 0, \quad (47) \]
or \[ \Theta_{xx} = -\Theta_{yy} - \Theta_{zz} \quad (48) \]

Substituting equations (44) into the first three compatibility equations (46), and using equation (47) gives

\[ \begin{align*}
[ (1+v) \nabla^2 B - \Theta ]_{yy} + [ (1+v) \nabla^2 C - \Theta ]_{yy} &= 0 \\
[ (1+v) \nabla^2 C - \Theta ]_{xx} + [ (1+v) \nabla^2 A - \Theta ]_{yy} &= 0 \\
[ (1+v) \nabla^2 A - \Theta ]_{yy} + [ (1+v) \nabla^2 B - \Theta ]_{xx} &= 0. 
\end{align*} \quad (49) \]

Similarly, substituting equations (45) into the last three compatibility equations gives

\[ \begin{align*}
[ (1+v) \nabla^2 C - \Theta ]_{yy} &= 0 \\
[ (1+v) \nabla^2 A - \Theta ]_{yy} &= 0 \\
[ (1+v) \nabla^2 B - \Theta ]_{yy} &= 0. 
\end{align*} \quad (50) \]

When equations (49) and (50) are integrated, additive functions will arise on the right sides of the equations. In view of the arbitrariness of the functions A, B, and C, it is possible to include these additive functions in A, B, and C in such a manner that the right sides of the resulting equations will be zero. Then equations (49) and (50) become

\[ \nabla^2 A = \nabla^2 B = \nabla^2 C = \frac{\Theta}{1+v}. \quad (51) \]

Equations (51) were obtained by Maxwell in a somewhat different manner.
Now, returning to the original definitions (44) for the stress functions, we have

\[ \Theta = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \]
\[ = (B_{zz} + C_{yy}) + (C_{xx} + A_{zz}) + (A_{xy} + B_{xz}) \]
\[ = \nabla^2 A + \nabla^2 B + \nabla^2 C - (A_{xy} + B_{xy} + C_{yy}). \]  

(52)

Substituting equation (51) into this expression gives

\[ \Theta = \frac{3\Theta}{1 + \nu} - (A_{xx} + B_{yy} + C_{yy}). \]  

(53)

Solving this equation for \( \Theta \) results in

\[ \Theta = \frac{1 + \nu}{2 - \nu} (A_{xx} + B_{yy} + C_{yy}). \]  

(54)

Or, referring again to equation (51) we have

\[ \nabla^2 A = \nabla^2 B = \nabla^2 C = \frac{1}{2 - \nu} (A_{xx} + B_{yy} + C_{yy}). \]  

(55)

Now, since \( \nabla^2 A = \nabla^2 C \), and \( \nabla^2 B = \nabla^2 C \), we have that \( \nabla^2 (A-C) = 0 \) and \( \nabla^2 (B-C) = 0 \). That is, the functions \( A-C \) and \( B-C \) are harmonic functions. Therefore, \( A-C = \mathcal{H}_1 \), and \( B-C = \mathcal{H}_2 \).

Then

\[ \mathcal{H}_{xxx} + \mathcal{H}_{xyy} = (A_{xx} - C_{xx}) + (B_{yy} - C_{yy}) \]
\[ = (A_{xx} + B_{yy} + C_{yy}) - \nabla^2 C. \]  

(56)

But from equation (55) we have

\[ A_{xx} + B_{yy} + C_{yy} = (2 - \nu) \nabla^2 C. \]

Therefore, \( \mathcal{H}_{xxx} + \mathcal{H}_{xyy} = (1 - \nu) \nabla^2 C \)  

(57)

The general solution of equation (57) is

\[ C = \frac{\nu}{2(1-\nu)} \frac{\partial \mathcal{H}_x}{\partial x} + \frac{\nu}{2(1-\nu)} \frac{\partial \mathcal{H}_y}{\partial y} + \frac{\nu}{1-\nu}, \]  

(58)
where $H_3$ is another arbitrary harmonic function. But since $H_1 = A - C$ and $H_2 = B - C$, we have $A = H_1 + C$ and $B = H_2 + C$, which gives solutions for $A$, $B$, and $C$ which are not symmetric in appearance.

If the procedure from equation (55) through the last paragraph is repeated for $\nabla^2 (A - B) = 0$ and $\nabla^2 (C - B) = 0$, another solution is obtained for $B$ and, consequently, also for $A$ and $C$. Another repetition gives three more solutions for $A$, $B$, and $C$. If these sets of solutions for $A$, $B$, and $C$ are added, in turn, the following symmetric equations result,

$$A = \frac{1}{2(1-v)} \left[ \alpha \frac{\partial H_1}{\partial x} + \gamma \frac{\partial H_2}{\partial y} + \beta \frac{\partial H_3}{\partial z} \right] - H_2 - H_3$$

$$B = \frac{1}{2(1-v)} \left[ \alpha \frac{\partial H_1}{\partial x} + \gamma \frac{\partial H_2}{\partial y} + \beta \frac{\partial H_3}{\partial z} \right] - H_3 - H_1$$

$$C = \frac{1}{2(1-v)} \left[ \alpha \frac{\partial H_1}{\partial x} + \gamma \frac{\partial H_2}{\partial y} + \beta \frac{\partial H_3}{\partial z} \right] - H_1 - H_2 \quad (59)$$

where $H_1$, $H_2$, and $H_3$ are also harmonic functions of $x$, $y$, and $z$.

Thus equations (59) determine the Maxwell stress functions in such a manner that the three-dimensional equations of equilibrium and compatibility are satisfied. Previously, $A$, $B$, and $C$ could be any arbitrary functions to satisfy only one set of equations -- those determining equilibrium. Upon requiring the restriction that the functions must be constructed of harmonic functions in the prescribed manner above, the conditions of compatibility are found to be satisfied as well. Therefore, only the boundary conditions of a particular problem need be further satisfied in order to have a complete solution of the problem. Equations (59) were obtained by Langhaar and Stippes.\(^13\)
Substituting these expressions for the stress functions into the
stress equations (44) and (45) gives the following equations for the
stresses, in terms of the arbitrary harmonic functions $H_1$, $H_2$, and $H_3$:

$$
\sigma_{xx} = \frac{1}{2(1-v)} \left[ \alpha H_{1xx} + \eta H_{2xx} + \beta H_{3xx} + 2v (H_{1xx} - H_{2yy} - H_{3yy}) - 2H_{1yy} \right] \tag{60a}
$$

$$
\sigma_{yy} = \frac{1}{2(1-v)} \left[ \alpha H_{1yy} + \eta H_{2yy} + \beta H_{3yy} + 2v (-H_{1xx} + H_{2yy} + H_{3yy}) - 2H_{1yy} \right] \tag{60b}
$$

$$
\sigma_{yy} = \frac{1}{2(1-v)} \left[ \alpha H_{1yy} + \eta H_{2yy} + \beta H_{3yy} + 2v (-H_{1xx} + H_{2yy} + H_{3yy}) - 2H_{1yy} \right] \tag{60c}
$$

$$
\sigma_{xy} = \frac{1}{2(1-v)} \left[ \alpha H_{1xy} + \eta H_{2xy} + \beta H_{3xy} - (1-2v)(H_{1xy} + H_{2xy}) \right] \tag{60d}
$$

$$
\sigma_{yz} = \frac{1}{2(1-v)} \left[ \alpha H_{1yz} + \eta H_{2yz} + \beta H_{3yz} - (1-2v)(H_{2yz} + H_{3yz}) \right] \tag{60e}
$$

$$
\sigma_{xz} = \frac{1}{2(1-v)} \left[ \alpha H_{1xz} + \eta H_{2xz} + \beta H_{3xz} - (1-2v)(H_{1xz} + H_{2xz}) \right] \tag{60f}
$$

Next, the displacements will be found in terms of these stress
functions. From equations (51) we have

$$
(1+v) \nabla^2 A = \Theta,
$$

or

$$
(1+v)(A_{xx} + A_{yy} + A_{zz}) - (\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = 0. \tag{61}
$$
But \( A_{xx} + A_{yy} + A_{zz} = A_{yy} + (T_{yy} - B_{xx}) + (T_{yy} - C_{xx}) \) (62)

from the original definitions (44) of the stress functions. Therefore, substituting equation (62) into (61) gives:

\[
(1 + \nu)(A_{xx} - B_{xx} - C_{xx} + T_{yy} + T_{zz}) - (T_{xx} + T_{yy} + T_{zz}) = 0,
\]

or \((1 + \nu)(A_{xx} - B_{xx} - C_{xx}) - (T_{xx} - \nu T_{yy} - \nu T_{zz}) = 0. \) (63)

But the strain in the \( x \)-direction is related to the normal stresses according to Hooke's Law as

\[
\varepsilon_{xx} = \frac{1}{E} (T_{xx} - \nu T_{yy} - \nu T_{zz}),
\]

where \( E \) is the modulus of elasticity for the material. Substituting this relationship into equation (63) gives

\[
(1 + \nu)(A_{xx} - B_{xx} - C_{xx}) - E \cdot \varepsilon_{xx} = 0. \]

Or in terms of the displacement \( u \) in the \( x \)-direction,

\[
\frac{\partial u}{\partial x} = \frac{(1+\nu)}{E} (A_{xx} - B_{xx} - C_{xx}).
\]

Integrating this equation gives an additive function of \( y \) and \( z \) which is a rigid body displacement. Therefore, the relative displacement is given by

\[
u = \frac{(1+\nu)}{E} (A_{xx} - B_{xx} - C_{xx}).
\]

Repeating the process with the other two of equations (51) gives the complete set of equations for the displacements \( u, v, \) and \( w \) in the \( x, y, \) and \( z \) directions, respectively, in terms of the Maxwell stress functions \( A, B, \) and \( C \):
\[ u = \frac{1+v}{E} (A_x - B_x - C_x) \]
\[ v = \frac{(1+v)}{E} (-A_y + B_y - C_y) \]
\[ w = \frac{(1+v)}{E} (-A_z - B_z + C_z) \quad (67) \]

If the relationships (59) for the stress functions are substituted into these equations, the following equations result:

\[ u = \frac{1+v}{E} \left[ 2H_{1x} - \frac{1}{2(1-v)} (\alpha H_{1x} + \gamma H_{2y} + \beta H_{2y}) \right] \]
\[ v = \frac{1+v}{E} \left[ 2H_{1y} - \frac{1}{2(1-v)} (\alpha H_{1x} + \gamma H_{2y} + \beta H_{2y}) \right] \]
\[ w = \frac{1+v}{E} \left[ 2H_{2y} - \frac{1}{2(1-v)} (\alpha H_{1x} + \gamma H_{2y} + \beta H_{2y}) \right] . \quad (68) \]

Referring to equations (17) of the text, it is seen that the harmonic functions \( H_1, H_2, \) and \( H_3 \) are related to the Papkovich stress functions \( \phi_1, \phi_2, \) and \( \phi_3 \) according to the following equations:

\[ \phi_1 = \frac{2(1+v)}{E} \frac{\partial H_1}{\partial x} \]
\[ \phi_2 = \frac{2(1+v)}{E} \frac{\partial H_2}{\partial y} \]
\[ \phi_3 = \frac{2(1+v)}{E} \frac{\partial H_3}{\partial z} \quad . \quad (69) \]
Appendix B

Transcendental Harmonic Functions
A function $H$ is said to be a harmonic function if it satisfies Laplace's equation, which can be written in rectangular coordinates as

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{\partial^2 H}{\partial z^2} = 0. \quad (70)$$

One solution of this equation may be readily obtained by the method of separation of variables. This method entails the assumption that $H$ is expressible as a product of three separate functions in the form

$$H = X(x) \cdot Y(y) \cdot Z(z), \quad (71)$$

where $X$ is a function only of $x$, $Y$ is a function only of $y$, and $Z$ is a function only of $z$. If equation (71) is substituted into (70), there results

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0, \quad (72)$$

where the double primes indicate differentiation twice with respect to the argument of the function. Dividing through by $XYZ$ gives

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0, \quad (72')$$

which can be further rewritten as

$$-\frac{X''}{X} = \frac{Y''}{Y} + \frac{Z''}{Z}. \quad (73)$$

The left side of this equation can only be a function of $x$, whereas the right side can only be a function of $y$ and $z$. The only type of function that is able to satisfy these restrictions simultaneously is a constant. Therefore, the left side of the equation may be set equal to a constant, say $\alpha^2$.

$$-\frac{X''}{X} = \alpha^2,$$

or

$$X'' + \alpha^2 X = 0. \quad (74)$$
Since $X$ is a function only of $x$, equation (74) is an ordinary differential equation whose solution appears in one form as
\[ X = K \cos \alpha x + L \sin \alpha x, \quad (75) \]
where $K$ and $L$ are constants of integration.

Returning to equation (72), it is apparent that it may be rewritten in another form as
\[ -\frac{Y''}{Y} = \frac{X''}{X} + \frac{Z''}{Z}. \quad (76) \]
Repeating the argument used before, it may be concluded that the left side of this equation must also equal a constant, say $\beta^2$. Then
\[ Y'' + \beta^2 Y = 0, \quad (77) \]
whose solution may be written as
\[ Y = M \cos \beta y + N \sin \beta y, \quad (78) \]
with $M$ and $N$ again being constants of integration.

Finally, by substituting $-\alpha^2$ for $\frac{X''}{X}$ and $-\beta^2$ for $\frac{Y''}{Y}$ in equation (72), there results
\[ \frac{Z''}{Z} = \alpha^2 + \beta^2, \]
or
\[ Z'' - (\alpha^2 + \beta^2) Z = 0, \quad (79) \]
whose solution may be given in terms of the exponential function $e$ as
\[ Z = P \cdot e^{-\sqrt{\alpha^2 + \beta^2} \theta} + Q \cdot e^{\sqrt{\alpha^2 + \beta^2} \theta}, \quad (80) \]
where $P$ and $Q$ are constants of integration. Or the solution of (79) may be given in terms of hyperbolic sines and cosines.
Substituting (75), (78), and (80) into equation (71) gives the following harmonic function $H$ as a solution:

$$H = (K \cos \alpha + L \sin \alpha)(M \cos \beta y + N \sin \beta y)$$

$$\times \left( P \cdot e^{-\sqrt{a^2+b^2} \gamma} + Q \cdot e^{\sqrt{a^2+b^2} \gamma} \right).$$

This equation may be simplified by dividing the first quantity in parentheses by $K$ while multiplying the last by $K$, and dividing the second parenthetical quantity by $M$ while multiplying the last by $M$. This gives:

$$H = ( \cos \alpha + A \sin \alpha)(\cos \beta y + B \sin \beta y)$$

$$\times \left( C \cdot e^{-\sqrt{a^2+b^2} \gamma} + D \cdot e^{\sqrt{a^2+b^2} \gamma} \right),$$  \hspace{1cm} (81)$$

where the new constants $A$, $B$, $C$, and $D$ are combinations of $K$, $L$, $M$, $N$, $P$, and $Q$. In this form it is apparent that there are six constants to be evaluated from the conditions of the particular problem -- $A$, $B$, $C$, $D$, $\alpha$, and $\beta$. 
Appendix C

Development of the Harmonic Polynomials
and Examples Thereof
Let $P_m$ denote a homogeneous harmonic polynomial of degree $m$, in which a typical term is of the form $C_{ijk} x^i y^j z^k$, with $i + j + k = m$. Any such polynomial may be arranged in a triangular array as indicated below.

$$
\begin{array}{cccccc}
\alpha^m & \alpha y & \alpha^2 y^2 & \cdots & \alpha^m y^m \\
\alpha^m z & \alpha^2 yz & \alpha y^2 z & \cdots & \alpha^m y^m z \\
\alpha^2 z^2 & \alpha y^2 z^2 & \alpha y^2 z^2 & \cdots & \alpha^m y^m z^2 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\alpha^m \ z & \alpha^m y \ z & \alpha^m y \ z & \cdots & \alpha^m y \ z \\
\ z^m 
\end{array}
$$

(82)

The coefficients $C_{ijk}$ can be written in a corresponding array as follows.

$$
\begin{array}{cccccc}
C_{m,0,0} & C_{m-1,0} & C_{m-2,0} & \cdots & C_{2,0} & C_{1,0} & C_{0,0} \\
C_{m,1,0} & C_{m-1,1} & C_{m-2,1} & \cdots & C_{2,1} & C_{1,1} & C_{0,1} \\
C_{m,2,0} & C_{m-1,2} & C_{m-2,2} & \cdots & C_{2,2} & C_{1,2} & C_{0,2} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
C_{1,0,m-1} & C_{0,1,m-1} \\
C_{0,0,m} 
\end{array}
$$

(83)

It will be assumed that $m \geq 2$, for all polynomials of degrees 0 and 1 are identically harmonic.

The general harmonic polynomial $P_m$ may then be expressed by the following summation:

$$
P_m = \sum_{i=0}^{m} \sum_{j=0}^{m-i} \sum_{k=0}^{m-i-j} C_{ijk} x^i y^j z^k.
$$

(84)
In order that $P_m$ be harmonic, Laplace's equation,

\[ \frac{\partial^2 P_m}{\partial x^2} + \frac{\partial^2 P_m}{\partial y^2} + \frac{\partial^2 P_m}{\partial z^2} = 0, \tag{85} \]

must be satisfied. Upon carrying out the indicated differentiation upon $P_m$, the following are found:

\[
\frac{\partial^2 P_m}{\partial x^2} = \sum_{i=0}^{m} \sum_{j=0}^{m-i} \sum_{k=0}^{m-i-j} i(i-1) C_{ijk} x^{i-2} y^{j} z^{k},
\]

\[
\frac{\partial^2 P_m}{\partial y^2} = \sum_{i=0}^{m} \sum_{j=0}^{m-i} \sum_{k=0}^{m-i-j} j(j-1) C_{ijk} x^{i} y^{j-2} z^{k},
\]

\[
\frac{\partial^2 P_m}{\partial z^2} = \sum_{i=0}^{m} \sum_{j=0}^{m-i} \sum_{k=0}^{m-i-j} k(k-1) C_{ijk} x^{i} y^{j} z^{k-2} . \tag{86}
\]

Now let $i - 2 = \alpha$ in the first of these, $j - 2 = \beta$ in the second, and $k - 2 = \gamma$ in the last. Then these expressions can be put into the form

\[
\frac{\partial^2 P_m}{\partial x^2} = \sum_{\alpha=0}^{m-2} \sum_{\beta=0}^{m-\alpha-2} \sum_{\gamma=0}^{m-\alpha-\beta-2} (\alpha+2)(\alpha+1) C_{\alpha\beta\gamma} x^{\alpha} y^{\beta} z^{\gamma},
\]

\[
\frac{\partial^2 P_m}{\partial y^2} = \sum_{\alpha=0}^{m-2} \sum_{\beta=0}^{m-\alpha-2} \sum_{\gamma=0}^{m-\alpha-\beta-2} (\beta+2)(\beta+1) C_{\alpha\beta\gamma} x^{\alpha} y^{\beta} z^{\gamma},
\]

\[
\frac{\partial^2 P_m}{\partial z^2} = \sum_{\alpha=0}^{m-2} \sum_{\beta=0}^{m-\alpha-2} \sum_{\gamma=0}^{m-\alpha-\beta-2} (\gamma+2)(\gamma+1) C_{\alpha\beta\gamma} x^{\alpha} y^{\beta} z^{\gamma} . \tag{87}
\]

Now in order that equation (85) is satisfied, the sum of the coefficients of each term of degree $m-2$ as given by equations (87) above must
be zero. That is, the coefficients are related in the following manner:

\[(d+2)(d+1)C_{d+2,\beta c} + (\beta+2)(\beta+1)C_{d,\beta+2,\gamma} + (\gamma+2)(\gamma+1)C_{d,\beta,\gamma+2} = 0. \quad (88)\]

This expression may be regarded as a recursion formula, for if any two of the C's are known, the other may be determined from them. In particular, \(C_{d,\beta,\gamma+2}\) may be obtained from the other C's. Upon examination of the triangular array of the coefficients (83), it is seen that the third subscript indicates the row in which the coefficient stands. The coefficient \(C_{d,\beta,\gamma+2}\) can lie in no row earlier than the third, while \(C_{d+2,\beta,\gamma}\) and \(C_{d,\beta+2,\gamma}\) lie two rows preceding it. Thus the C's in the third row are determined by those lying in the first row, the C's in the fourth row are determined by those in the second row, the ones in the fifth row are prescribed by the ones already obtained for the third row, and so on. It is thus seen that the first two rows of the array (83) are arbitrary, with the rest of the C's in the remaining rows being determined from them. It is also seen then that every polynomial of degree m is determined from \(2m+1\) arbitrary coefficients.

As an example, \(P_3\) will be calculated below. This polynomial may be arranged in the triangular array

\[
\begin{array}{cccccc}
\alpha^3 & \alpha^2y & \alpha y^2 & y^3 \\
\alpha^2z & \alpha yz & y^2z \\
\alpha y^2 & y^2z \\
\gamma^3 \\
\end{array}
\]
with its coefficients in a similar array.

\[
\begin{align*}
C_{300} & \quad C_{210} & \quad C_{120} & \quad C_{030} \\
C_{210} & \quad C_{111} & \quad C_{021} \\
C_{120} & \quad C_{012} \\
C_{030} & \\
\end{align*}
\]

Now let the first two rows of C's be arbitrary. In particular, let them be denoted as

\[
\begin{align*}
A_{31} & \quad A_{32} & \quad A_{33} & \quad A_{34} \\
B_{31} & \quad B_{32} & \quad B_{33} \\
\end{align*}
\]

where the first subscript denotes the degree of polynomial.

The recursion formula (88) will be applied to determine the remainder of the C's in the following rows. In this case \( \gamma + \beta + \delta = m - 2 = 1. \)

\( \alpha = 1, \beta = 0, \gamma = 0 \)

\[
(3)(2)C_{300} + (2)(1)C_{120} + (2)(1)C_{102} = 0.
\]

\[
\therefore C_{102} = -(3A_{31} + A_{33}).
\]

\( \alpha = 0, \beta = 1, \gamma = 0 \)

\[
(2)(1)C_{210} + (3)(2)C_{030} + (2)(1)C_{012} = 0.
\]

\[
\therefore C_{012} = -(A_{32} + 3A_{34}).
\]

\( \alpha = 0, \beta = 0, \gamma = 1 \)

\[
(2)(1)C_{201} + (2)(1)C_{021} + (3)(2)C_{003} = 0.
\]

\[
\therefore C_{003} = -\frac{1}{3}(B_{31} + B_{33}).
\]
Thus the harmonic polynomial of degree three is found to be

\[ P_3 = A_{31} x^3 + A_{32} x^2 y + A_{33} x y^2 + A_{34} y^3 \\
+ B_{31} x^2 y^2 + B_{32} x y^2 + B_{33} y^3 \\
- (3A_{31} + A_{33}) y^2 - (A_{32} + 3A_{34}) y^3 \\
- \frac{1}{3} (B_{31} + B_{33}) y^3 , \]

(89)

with all of the A's and B's being arbitrary constants.

On the following pages is a list of the harmonic polynomials of degrees one through six. Since the size of the polynomials may be represented by the area of a triangular area, the number of terms in the polynomials increase with the square of m. The exact number of terms in a polynomial of degree m is given by

\[ \text{no. of terms} = \frac{(m + 1)(m + 2)}{2} . \]

That these polynomials are harmonic may be readily verified by substituting them into equation (85).
\[ P_0 = A_{01} \]

\[ P_1 = A_{11} \alpha + A_{12} \eta y + B_{11} \eta \]

\[ P_2 = A_{21} \alpha^2 + A_{22} \alpha \eta y + A_{23} \eta^2 y + B_{21} \eta \xi + B_{22} \eta \eta y - (A_{21} + A_{23}) \eta^2 \]

\[ P_3 = A_{31} \alpha^3 + A_{32} \alpha \eta y + A_{33} \eta^2 y + A_{34} \eta^3 y + B_{31} \eta^2 \xi + B_{32} \eta \eta y + B_{33} \eta \eta^2 y - (3A_{31} + A_{33}) \eta^3 y - (A_{32} + 3A_{34}) \eta^2 y^2 - \frac{1}{3} (B_{31} + B_{33}) \eta^3 \]

\[ P_4 = A_{41} \alpha^4 + A_{42} \alpha^3 \eta y + A_{43} \alpha^2 \eta^2 y + A_{44} \alpha \eta^3 y + A_{45} \eta^4 y + B_{41} \eta^2 \xi + B_{42} \eta \eta y + B_{43} \eta \eta^2 y + B_{44} \eta \eta^3 y - (6A_{41} + A_{43}) \alpha \eta^2 y^2 - 3(A_{42} + A_{44}) \eta \eta y^2 - (A_{43} + 6A_{45}) \eta^2 y^3 - \left( B_{41} + \frac{1}{3} B_{43} \right) \eta^3 y^3 - \left( \frac{1}{3} B_{42} + B_{44} \right) \eta \eta^2 y^2 + (A_{41} + \frac{1}{3} A_{43} + A_{45}) \eta^2 y^4 \]

\[ P_5 = A_{51} \alpha^5 + A_{52} \alpha^4 \eta y + A_{53} \alpha^3 \eta^2 y + A_{54} \alpha^2 \eta^3 y + A_{55} \alpha \eta^4 y + A_{56} \eta^5 y + B_{51} \alpha \eta y + B_{52} \eta^2 y + B_{53} \eta^3 y + B_{54} \eta^4 y + B_{55} \eta^5 y + B_{56} \eta^6 y - (10A_{51} + A_{53}) \alpha \eta^2 y^2 - 3(2A_{52} + A_{54}) \alpha \eta^3 y^2 - 3(A_{53} + 2A_{55}) \eta^2 y^3 - (A_{54} + 10A_{56}) \eta^3 y^3 - \left( \frac{1}{3} \right)(6B_{51} + B_{53}) \alpha \eta^2 y^2 + (B_{52} + B_{54}) \eta \eta y^2 - \left( \frac{1}{3} \right)(B_{53} + 6B_{55}) \eta \eta^2 y^2 + (5A_{57} + A_{59} + A_{55}) \eta^2 y^3 + \left( 3B_{51} + B_{53} + 3B_{55} \right) \eta^3 y^3 + \left( A_{52} + A_{54} + 5A_{56} \right) \eta \eta y^2 + \left( \frac{1}{15} \right)(3B_{51} + B_{53} + 3B_{55}) \eta^2 y^2 \]
Appendix D

Solutions Obtained from the Harmonic Polynomials of Degree Two Through Five by Letting All Other Coefficients but One Be Zero
The following pages are, in effect, a partial catalogue of the solutions obtained from the polynomials when all coefficients except the particular one under consideration are set equal to zero. These begin with the solutions derived from the second degree polynomial, which is the first to give non-trivial results, and are extended through the fifth degree polynomial, which gives stresses in the form of cubics. That the equations given do represent exact solutions of elasticity problems has been verified by checking to insure that each set satisfies the equations of equilibrium and compatibility. All of the cases for which only two of the shearing stresses are zero are cases of plane strain.
\( A_{21}^1 \neq 0 \)
\[
\begin{align*}
\sigma_{xx} &= (1 - 2v) A_{21}^1 \\
\sigma_{yy} &= 2v A_{21}^1 \\
\sigma_{zz} &= 2v A_{21}^1 \\
\sigma_{xy} &= 0 \\
\sigma_{xz} &= 0 \\
\sigma_{yz} &= 0
\end{align*}
\]

\( A_{21}^2 \neq 0 \)
\[
\begin{align*}
\sigma_{xx} &= 0 \\
\sigma_{yy} &= 0 \\
\sigma_{zz} &= 0 \\
\sigma_{xy} &= 0 \\
\sigma_{xz} &= 0 \\
\sigma_{yz} &= 0
\end{align*}
\]

\( A_{21}^3 \neq 0 \)
\[
\begin{align*}
\sigma_{xx} &= -2v A_{21}^3 \\
\sigma_{yy} &= -2v A_{21}^3 \\
\sigma_{zz} &= -(1 - 2v) A_{21}^3 \\
\sigma_{xy} &= 0 \\
\sigma_{xz} &= 0 \\
\sigma_{yz} &= 0
\end{align*}
\]
\( A'_{22} \neq 0 \)

\[
\begin{align*}
\tau_{xx} &= 0 \\
\tau_{yy} &= 0 \\
\tau_{zz} &= 0 \\
\tau_{xy} &= (1-2v)A'_{22} \\
\tau_{xz} &= 0 \\
\tau_{yx} &= 0
\end{align*}
\]

\( A''_{22} \neq 0 \)

\[
\begin{align*}
\tau_{xx} &= 0 \\
\tau_{yy} &= 0 \\
\tau_{zz} &= 0 \\
\tau_{xy} &= (1-2v)A''_{22} \\
\tau_{xz} &= 0 \\
\tau_{yx} &= 0
\end{align*}
\]

\( A'''_{22} \neq 0 \)

\[
\begin{align*}
\tau_{xx} &= 0 \\
\tau_{yy} &= 0 \\
\tau_{zz} &= 0 \\
\tau_{xy} &= 0 \\
\tau_{xz} &= 0 \\
\tau_{yx} &= 0
\end{align*}
\]
\[ A'_{23} \neq 0 \]

\[
\begin{align*}
\gamma_{xx} &= 0 \\
\gamma_{yy} &= 0 \\
\gamma_{zz} &= 0 \\
\gamma_{xy} &= 0 \\
\gamma_{yz} &= 0 \\
\gamma_{zx} &= 0 
\end{align*}
\]

\[ A^2_{23} \neq 0 \]

\[
\begin{align*}
\sigma_{xx} &= 2\nu A^2_{23} \\
\sigma_{yy} &= (1-2\nu) A^2_{23} \\
\sigma_{zz} &= 2\nu A^2_{23} \\
\sigma_{xy} &= 0 \\
\sigma_{yz} &= 0 \\
\sigma_{zx} &= 0 
\end{align*}
\]

\[ A^3_{23} \neq 0 \]

\[
\begin{align*}
\sigma_{xx} &= -2\nu A^3_{23} \\
\sigma_{yy} &= -2\nu A^3_{23} \\
\sigma_{zz} &= -(1-2\nu) A^3_{23} \\
\sigma_{xy} &= 0 \\
\sigma_{yz} &= 0 \\
\sigma_{zx} &= 0 
\end{align*}
\]
$B_{21} 
eq 0$

\[ \sigma_{xx} = 0 \]
\[ \sigma_{yy} = 0 \]
\[ \sigma_{zz} = 0 \]
\[ \tau_{xy} = 0 \]
\[ \tau_{xz} = 0 \]
\[ \tau_{yz} = 0 \]
\[ \sigma_{yx} = (1-2v)B_{21} \]

$B_{22} 
eq 0$

\[ \sigma_{xx} = 0 \]
\[ \sigma_{yy} = 0 \]
\[ \sigma_{zz} = 0 \]
\[ \tau_{xy} = 0 \]
\[ \tau_{xz} = 0 \]
\[ \tau_{yz} = 0 \]
\[ \sigma_{yx} = 0 \]

$B_{23} 
eq 0$

\[ \sigma_{xx} = 0 \]
\[ \sigma_{yy} = 0 \]
\[ \sigma_{zz} = 0 \]
\[ \tau_{xy} = 0 \]
\[ \tau_{xz} = 0 \]
\[ \tau_{yz} = 0 \]
\[ \sigma_{yx} = (1-2v)B_{23} \]
\( B_{22} \neq 0 \)

\[
\begin{aligned}
\sigma_{xx} &= 0 \\
\sigma_{yy} &= 0 \\
\sigma_{zz} &= 0 \\
\tau_{xy} &= 0 \\
\tau_{xz} &= 0 \\
\tau_{yx} &= 0 \\
\tau_{yz} &= 0 \\
\tau_{zx} &= 0 
\end{aligned}
\]

\( B_{22}^z \neq 0 \)

\[
\begin{aligned}
\sigma_{xx} &= 0 \\
\sigma_{yy} &= 0 \\
\sigma_{zz} &= 0 \\
\tau_{xy} &= 0 \\
\tau_{xz} &= 0 \\
\tau_{yx} &= 0 \\
\tau_{yz} &= (1-2v)B_{22}^z \\
\tau_{zx} &= 0 
\end{aligned}
\]

\( B_{22}^3 \neq 0 \)

\[
\begin{aligned}
\sigma_{xx} &= 0 \\
\sigma_{yy} &= 0 \\
\sigma_{zz} &= 0 \\
\tau_{xy} &= 0 \\
\tau_{xz} &= 0 \\
\tau_{yx} &= 0 \\
\tau_{yz} &= (1-2v)B_{22}^3 \\
\tau_{zx} &= 0 
\end{aligned}
\]
\[ A_{31} \neq 0 \]

\[ \sigma_{xx} = (1 - 2v) A_{31} x \]
\[ \sigma_{yy} = 2v A_{31} x \]
\[ \sigma_{zz} = (1 + 2v) A_{31} x \]
\[ \tau_{xy} = 0 \]
\[ \tau_{yz} = 0 \]
\[ \tau_{xz} = - (1 - 2v) A_{31} y \]

\[ A_{32} \neq 0 \]

\[ \sigma_{xx} = 0 \]
\[ \sigma_{yy} = 0 \]
\[ \sigma_{zz} = 0 \]
\[ \tau_{xy} = 0 \]
\[ \tau_{yz} = 0 \]
\[ \tau_{xz} = 0 \]

\[ A_{33} \neq 0 \]

\[ \sigma_{xx} = -v A_{33} x \]
\[ \sigma_{yy} = -v A_{33} x \]
\[ \sigma_{zz} = -(1 - v) A_{33} x \]
\[ \tau_{xy} = 0 \]
\[ \tau_{yz} = 0 \]
\[ \tau_{xz} = v A_{33} y \]
\[ A_{32}' \neq 0 \]
\[ \sigma_{xx} = A_{32}' y (1 - v) \]
\[ \sigma_{yy} = v A_{32}' y \]
\[ \sigma_{zz} = v A_{32}' y \]
\[ \tau_{xy} = -v A_{32}' x \]
\[ \tau_{yz} = 0 \]
\[ \tau_{zx} = 0 \]

\[ A_{32}^2 \neq 0 \]
\[ \sigma_{xx} = -A_{32}^2 y \]
\[ \sigma_{yy} = 0 \]
\[ \sigma_{zz} = A_{32}^2 y \]
\[ \tau_{xy} = (1 - 2v) A_{32}^2 x \]
\[ \tau_{yz} = -(1 - 2v) A_{32}^2 y \]
\[ \tau_{zx} = 0 \]

\[ A_{32}^3 \neq 0 \]
\[ \sigma_{xx} = -v A_{32}^3 y \]
\[ \sigma_{yy} = -v A_{32}^3 y \]
\[ \sigma_{zz} = -(1 - v) A_{32}^3 y \]
\[ \tau_{xy} = 0 \]
\[ \tau_{yz} = v A_{32}^3 y \]
\[ \tau_{zx} = 0 \]
\[ A'_{33} \neq 0 \]

\[ \sigma_{xx} = 0 \]
\[ \sigma_{yy} = -A'_{33} \alpha \]
\[ \sigma_{zz} = A'_{33} \alpha \]
\[ \sigma_{xy} = (1-2\nu)A'_{33} \gamma \]
\[ \sigma_{yz} = 0 \]
\[ \sigma_{zx} = -(1-2\nu)A'_{33} \gamma \]

\[ A^2_{33} \neq 0 \]

\[ \sigma_{xx} = \nu A^2_{33} \alpha \]
\[ \sigma_{yy} = (1-\nu)A^2_{33} \alpha \]
\[ \sigma_{zz} = \nu A^2_{33} \alpha \]
\[ \sigma_{xy} = -\nu A^2_{33} \gamma \]
\[ \sigma_{yz} = 0 \]
\[ \sigma_{zx} = 0 \]

\[ A^3_{33} \neq 0 \]

\[ \sigma_{xx} = -\nu A^3_{33} \alpha \]
\[ \sigma_{yy} = -\nu A^3_{33} \alpha \]
\[ \sigma_{zz} = -(1-\nu)A^3_{33} \alpha \]
\[ \sigma_{xy} = 0 \]
\[ \sigma_{yz} = 0 \]
\[ \sigma_{zx} = \nu A^3_{33} \gamma \]
\[ A_{34}^{1} \neq 0 \]

\[
\begin{align*}
\tau_{xx} & = 0 \\
\tau_{yy} & = 0 \\
\tau_{zz} & = 0 \\
\tau_{xy} & = 0 \\
\tau_{xz} & = 0 \\
\tau_{yz} & = 0
\end{align*}
\]

\[ A_{34}^{2} \neq 0 \]

\[
\begin{align*}
\tau_{xx} & = 2\nu A_{34}^{2} y \\
\tau_{yy} & = (1-2\nu) A_{34}^{2} y \\
\tau_{zz} & = (1+2\nu) A_{34}^{2} y \\
\tau_{xy} & = 0 \\
\tau_{xz} & = - (1-2\nu) A_{34}^{2} y \\
\tau_{yz} & = 0
\end{align*}
\]

\[ A_{34}^{3} \neq 0 \]

\[
\begin{align*}
\tau_{xx} & = -\nu A_{34}^{3} y \\
\tau_{yy} & = -\nu A_{34}^{3} y \\
\tau_{zz} & = - (1-\nu) A_{34}^{3} y \\
\tau_{xy} & = 0 \\
\tau_{xz} & = \nu A_{34}^{3} y \\
\tau_{yz} & = 0
\end{align*}
\]
\[
\begin{align*}
B_{31}' &\neq 0 \\
\tau_{yy} &= (1-v)B_{31}' y \\
\tau_{yy} &= vB_{31}' y \\
\tau_{yy} &= vB_{31}' y \\
\tau_{yy} &= 0 \\
\tau_{yy} &= 0 \\
\tau_{yy} &= -vB_{31}' x
\end{align*}
\]

\[
\begin{align*}
B_{31}^2 &\neq 0 \\
\tau_{xx} &= 0 \\
\tau_{yy} &= 0 \\
\tau_{yy} &= 0 \\
\tau_{yy} &= 0 \\
\tau_{yy} &= 0 \\
\tau_{yy} &= 0
\end{align*}
\]

\[
\begin{align*}
B_{31}^3 &\neq 0 \\
\tau_{xx} &= -(1+2v)B_{31}^3 x \\
\tau_{yy} &= -2vB_{31}^3 y \\
\tau_{yy} &= -(1-2v)B_{31}^3 y \\
\tau_{yy} &= 0 \\
\tau_{yy} &= 0 \\
\tau_{yy} &= (1-2v)B_{31}^3 x
\end{align*}
\]
\[ B'_{32} \neq 0 \]

\[
\begin{align*}
\sigma_{xx} &= 0 \\
\sigma_{yy} &= 0 \\
\sigma_{zz} &= 0 \\
\sigma_{xy} &= (1-2\nu) B'_{32} y \\
\sigma_{yz} &= -B'_{32} x \\
\sigma_{zx} &= (1-2\nu) B'_{32} y \\
\end{align*}
\]

\[ B^2_{32} \neq 0 \]

\[
\begin{align*}
\sigma_{xx} &= 0 \\
\sigma_{yy} &= 0 \\
\sigma_{zz} &= 0 \\
\sigma_{xy} &= (1-2\nu) B^2_{32} y \\
\sigma_{yz} &= (1-2\nu) B^2_{32} x \\
\sigma_{zx} &= -B^2_{32} y \\
\end{align*}
\]

\[ B^3_{32} \neq 0 \]

\[
\begin{align*}
\sigma_{xx} &= 0 \\
\sigma_{yy} &= 0 \\
\sigma_{zz} &= 0 \\
\sigma_{xy} &= -B^3_{32} y \\
\sigma_{yz} &= (1-2\nu) B^3_{32} x \\
\sigma_{zx} &= (1-2\nu) B^3_{32} y \\
\end{align*}
\]
\[ B_{33}' \neq 0 \]

\[ \begin{align*}
\sigma_{xx} &= 0 \\
\sigma_{yy} &= 0 \\
\sigma_{zz} &= 0 \\
\sigma_{xy} &= 0 \\
\sigma_{yz} &= 0 \\
\sigma_{zx} &= 0
\end{align*} \]

\[ B_{33}^2 \neq 0 \]

\[ \begin{align*}
\sigma_{xx} &= \nu B_{33}^2 \gamma \\
\sigma_{yy} &= B_{33}^2 \gamma \\
\sigma_{zz} &= \nu B_{33}^2 \gamma \\
\sigma_{xy} &= 0 \\
\sigma_{yz} &= -\nu B_{33}^2 \gamma \\
\sigma_{zx} &= 0
\end{align*} \]

\[ B_{33}^3 \neq 0 \]

\[ \begin{align*}
\sigma_{xx} &= -2\nu B_{33}^3 \gamma \\
\sigma_{yy} &= -(1-2\nu) B_{33}^3 \gamma \\
\sigma_{zz} &= -(1-2\nu) B_{33}^3 \gamma \\
\sigma_{xy} &= 0 \\
\sigma_{yz} &= (1-2\nu) B_{33}^3 \gamma \\
\sigma_{zx} &= 0
\end{align*} \]
\[ A_{x_1} \neq 0 \]

\[ \sigma_{xx} = [-v x^2 - (1-v) y^2] A'_{x_1} \]
\[ \sigma_{yy} = v (x^2 - y^2) A'_{x_1} \]
\[ \sigma_{zz} = [(1+v) x^2 - v y^2] A'_{x_1} \]
\[ \tau_{xy} = 0 \]
\[ \tau_{yz} = 0 \]
\[ \tau_{zx} = 2v y x A'_{x_1} \]

\[ A_{x_1} \neq 0 \]

\[ \sigma_{xx} = 0 \]
\[ \sigma_{yy} = 0 \]
\[ \sigma_{zz} = 0 \]
\[ \tau_{xy} = 0 \]
\[ \tau_{yz} = 0 \]
\[ \tau_{zx} = 0 \]

\[ A_{x_1} \neq 0 \]

\[ \sigma_{xx} = [-v x^2 + (1+v) y^2] A_{x_1} \]
\[ \sigma_{yy} = [-x^2 + y^2] v A_{x_1} \]
\[ \sigma_{zz} = [-(1-v) x^2 - v y^2] A_{x_1} \]
\[ \tau_{xy} = 0 \]
\[ \tau_{yz} = 0 \]
\[ \tau_{zx} = 2v y x A_{x_1} \]
\[ A'_{42} \neq 0 \]

\[ \begin{align*}
\tau_{xx} &= (1-2v) a_y A'_{42} \\
\tau_{yy} &= 2v a_y A'_{42} \\
\tau_{zz} &= (1+2v) a_y A'_{42} \\
\tau_{xy} &= -\frac{1}{2} \left[ (1+2v)x^2 + (1-2v)y^2 \right] A'_{42} \\
\tau_{yz} &= \gamma x A'_{42} \\
\tau_{zx} &= -(1-2v) y z A'_{42}
\end{align*} \]

\[ A''_{42} \neq 0 \]

\[ \begin{align*}
\tau_{xx} &= -a_y A''_{42} \\
\tau_{yy} &= 0 \\
\tau_{zz} &= a_y A''_{42} \\
\tau_{xy} &= \frac{1}{2} \left[ (1-2v) x^2 - (1-2v)y^2 \right] A''_{42} \\
\tau_{yz} &= -(1-2v) y z A''_{42} \\
\tau_{zx} &= y z A''_{42}
\end{align*} \]

\[ A'''_{42} \neq 0 \]

\[ \begin{align*}
\tau_{xx} &= -2v a_y A'''_{42} \\
\tau_{yy} &= -2v a_y A'''_{42} \\
\tau_{zz} &= -2 a_y A'''_{42} (1-v) \\
\tau_{xy} &= \gamma^2 A'''_{42} \\
\tau_{yz} &= 2v y z A'''_{42} \\
\tau_{zx} &= 2v y z A'''_{42}
\end{align*} \]
\[ A_{43} \neq 0 \]

\[
\begin{align*}
\tau_{xx} &= (1-v)(y^2-z^2)A_{43}^1 \\
\tau_{yy} &= \left[ -x^2 + v(y^2-z^2) \right] A_{43}^2 \\
\tau_{zz} &= \left[ x^2 + v(y^2-z^2) \right] A_{43}^2 \\
\tau_{xy} &= -2v \, y \, A_{43}^1 \\
\tau_{y} &= 0 \\
\tau_{xz} &= 2v \, x \, A_{43}^2
\end{align*}
\]

\[ A_{43} \neq 0 \]

\[
\begin{align*}
\tau_{xx} &= \left[ v(x^2-y^2-z^2) \right] A_{43}^2 \\
\tau_{yy} &= (1-v)\left[ x^2-z^2 \right] A_{43}^2 \\
\tau_{zz} &= \left[ v(x^2+y^2-z^2) \right] A_{43}^2 \\
\tau_{xy} &= -2v \, y \, A_{43}^2 \\
\tau_{yz} &= 2v \, x \, A_{43}^2 \\
\tau_{xz} &= 0
\end{align*}
\]

\[ A_{43} \neq 0 \]

\[
\begin{align*}
\tau_{xx} &= \left[ -v x^2 - v y^2 + (1+2v) z^2 \right] A_{43}^3 \\
\tau_{yy} &= \left[ -v x^2 - v y^2 + (1+2v) z^2 \right] A_{43}^3 \\
\tau_{zz} &= \left[ -(1-v)x^2 - (1-v)y^2 - 2v z^2 \right] A_{43}^3 \\
\tau_{xy} &= 0 \\
\tau_{yz} &= 2v \, y \, A_{43}^3 \\
\tau_{xz} &= 2v \, x \, A_{43}^3
\end{align*}
\]
\[ A_{++}^t \neq 0 \]

\[
\begin{align*}
\sigma_{xx} &= 0 \\
\sigma_{yy} &= -ny' A_{++}^t \\
\sigma_{zz} &= ny' A_{++}^t \\
\sigma_{xy} &= \frac{1}{2} [(1+2v)ny' - (1-2v)n^2] A_{++}^t \\
\sigma_{xz} &= ny A_{++}^t \\
\sigma_{zx} &= -(1-2v)ny A_{++}^t
\end{align*}
\]

\[ A_{++}^2 \neq 0 \]

\[
\begin{align*}
\sigma_{xx} &= 2v ny A_{++}^2 \\
\sigma_{yy} &= (1-2v)ny A_{++}^2 \\
\sigma_{zz} &= (1+2v)ny A_{++}^2 \\
\sigma_{xy} &= -\frac{1}{2} [(1+2v)ny + (1-2v)n^2] A_{++}^2 \\
\sigma_{xz} &= -(1-2v)ny A_{++}^2 \\
\sigma_{zx} &= ny A_{++}^2
\end{align*}
\]

\[ A_{++}^3 \neq 0 \]

\[
\begin{align*}
\sigma_{xx} &= -2v ny A_{++}^3 \\
\sigma_{yy} &= -2v ny A_{++}^3 \\
\sigma_{zz} &= -2(1-v)ny A_{++}^3 \\
\sigma_{xy} &= n^2 A_{++}^3 \\
\sigma_{xz} &= G A_{++}^3 \\
\sigma_{zx} &= 2v ny A_{++}^3 \\
\sigma_{yz} &= 2v ny A_{++}^3
\end{align*}
\]
\[ A_{15} \neq 0 \]

\[ \tau_{xx} = 0 \]

\[ \tau_{yy} = 0 \]

\[ \tau_{zz} = 0 \]

\[ \tau_{xy} = 0 \]

\[ \tau_{xz} = 0 \]

\[ \tau_{yz} = 0 \]

\[ A_{15}^2 \neq 0 \]

\[ \tau_{xx} = \nu \left[ y^2 - z^2 \right] A_{15}^2 \]

\[ \tau_{yy} = - \left[ \nu y^2 + (1-\nu)z^2 \right] A_{15}^2 \]

\[ \tau_{zz} = \left[ (1+\nu)y^2 - \nu z^2 \right] A_{15}^2 \]

\[ \tau_{xy} = 0 \]

\[ \tau_{xz} = 2\nu yz A_{15}^2 \]

\[ \tau_{yz} = 0 \]

\[ A_{15}^3 \neq 0 \]

\[ \tau_{xx} = -\nu \left[ y^2 - z^2 \right] A_{15}^3 \]

\[ \tau_{yy} = - \left[ \nu y^2 + (1-\nu)z^2 \right] A_{15}^3 \]

\[ \tau_{zz} = \left[ (1+\nu)y^2 + \nu z^2 \right] A_{15}^3 \]

\[ \tau_{xy} = 0 \]

\[ \tau_{xz} = 2\nu yz A_{15}^3 \]

\[ \tau_{yz} = 0 \]
\[ B_{41}^1 \neq 0 \]

\[
\begin{align*}
\bar{\Gamma}_{xx} &= -(1-2v) g_x B_{41}^1 \\
\bar{\Gamma}_{yy} &= 2v g_x B_{41}^1 \\
\bar{\Gamma}_{xz} &= v g_x B_{41}^1 \\
\bar{\Gamma}_{yz} &= 0 \\
\bar{\Gamma}_{zx} &= 0 \\
\bar{\Gamma}_{zy} &= -\frac{1}{2} \left[ (1+2v) t^2 + (1-2v) s^2 \right] B_{41}^1
\end{align*}
\]

\[ B_{41}^2 \neq 0 \]

\[
\begin{align*}
\bar{\Gamma}_{xx} &= 0 \\
\bar{\Gamma}_{yy} &= 0 \\
\bar{\Gamma}_{xz} &= 0 \\
\bar{\Gamma}_{yz} &= 0 \\
\bar{\Gamma}_{zx} &= 0 \\
\bar{\Gamma}_{zy} &= 0
\end{align*}
\]

\[ B_{41}^3 \neq 0 \]

\[
\begin{align*}
\bar{\Gamma}_{xx} &= -(1+2v) g_x B_{41}^3 \\
\bar{\Gamma}_{yy} &= -2v g_x B_{41}^3 \\
\bar{\Gamma}_{xz} &= -(1-2v) g_x B_{41}^3 \\
\bar{\Gamma}_{yz} &= 0 \\
\bar{\Gamma}_{zx} &= 0 \\
\bar{\Gamma}_{zy} &= \frac{1}{2} \left[ (1-2v) t^2 + (1+2v) s^2 \right] B_{41}^3
\end{align*}
\]
\[ B_{42}^1 \neq 0 \]
\[ \sigma_{xx} = 2 v y z (1-v) B_{42}^1 \]
\[ \sigma_{yy} = 2 v y z B_{42}^1 \]
\[ \sigma_{zz} = 2 v y z B_{42}^1 \]
\[ \sigma_{xy} = -2 v y x B_{42}^1 \]
\[ \sigma_{yz} = -v^2 B_{42}^1 \]
\[ \sigma_{zx} = -2 v x y B_{42}^1 \]

\[ B_{42}^2 \neq 0 \]
\[ \sigma_{xx} = -y z B_{42}^2 \]
\[ \sigma_{yy} = 0 \]
\[ \sigma_{zz} = y z B_{42}^2 \]
\[ \sigma_{xy} = (1-2v) y x B_{42}^2 \]
\[ \sigma_{yz} = \frac{1}{2} (1-2v) [x^2 - y^2] B_{42}^2 \]
\[ \sigma_{zx} = -d y B_{42}^2 \]

\[ B_{42}^3 \neq 0 \]
\[ \sigma_{xx} = -(1+2v) B_{42}^3 y z \]
\[ \sigma_{yy} = -2 v y z B_{42}^3 \]
\[ \sigma_{zz} = -(1-2v) y z B_{42}^3 \]
\[ \sigma_{xy} = -B_{42}^3 y x \]
\[ \sigma_{yz} = \frac{1}{2} [(1-2v) x^2 + (1+2v) y^2] B_{42}^3 \]
\[ \sigma_{zx} = (1-2v) y x B_{42}^3 \]
\[ B'_{43} \neq 0 \]

\[
\begin{align*}
\tau_{xx} &= 0 \\
\tau_{yy} &= -y' B'_{43} \\
\tau_{zz} &= x' B'_{43} \\
\tau_{xy} &= (1-2\nu) y' B'_{43} \\
\tau_{xz} &= -x y' B'_{43} \\
\tau_{yz} &= \frac{1}{2} (1-2\nu) [y'^2 - z'^2] B'_{43}
\end{align*}
\]

\[ B^2_{43} \neq 0 \]

\[
\begin{align*}
\tau_{xx} &= 2 \nu y' B^2_{43} \\
\tau_{yy} &= 2 (1-\nu) y' B^2_{43} \\
\tau_{zz} &= 2 \nu x' B^2_{43} \\
\tau_{xy} &= -2 \nu y' B^2_{43} \\
\tau_{xz} &= -2 \nu y' B^2_{43} \\
\tau_{yz} &= -y'^2 B^2_{43}
\end{align*}
\]

\[ B^3_{43} \neq 0 \]

\[
\begin{align*}
\tau_{xx} &= -2 \nu y' B^3_{43} \\
\tau_{yy} &= -(1+2\nu) y' B^3_{43} \\
\tau_{zz} &= -(1-2\nu) x' B^3_{43} \\
\tau_{xy} &= -y' B^3_{43} \\
\tau_{xz} &= (1-2\nu) x' y' B^3_{43} \\
\tau_{yz} &= \frac{1}{2} \left[ (1-2\nu) y'^2 + (1+2\nu) y'^2 \right] B^3_{43}
\end{align*}
\]
\[
\begin{align*}
B'_{44} & \neq 0 \\
\Gamma_{xx} &= 0 \\
\Gamma_{yy} &= 0 \\
\Gamma_{zz} &= 0 \\
\Gamma_{xy} &= 0 \\
\Gamma_{xz} &= 0 \\
\Gamma_{yx} &= 0 \\
\Gamma_{zx} &= 0 \\
\Gamma_{yz} &= 0 \\
\Gamma_{zy} &= 0 \\
\Gamma_{zx} &= 0 \\
\Gamma_{zy} &= 0 \
\end{align*}
\]

\[
\begin{align*}
B''_{44} & \neq 0 \\
\Gamma_{xx} &= 2 \nu \eta z B''_{44} \\
\Gamma_{yy} &= (1-2\nu) \eta z B''_{44} \\
\Gamma_{zz} &= (1+2\nu) \eta z B''_{44} \\
\Gamma_{xy} &= 0 \\
\Gamma_{xz} &= 0 \\
\Gamma_{yx} &= 0 \\
\Gamma_{zx} &= 0 \\
\Gamma_{yz} &= -\frac{1}{2} \left[ (1+2\nu) \eta^2 + (1-2\nu) z^2 \right] B''_{44} \\
\Gamma_{zy} &= 0 \\
\end{align*}
\]

\[
\begin{align*}
B''_{44} & \neq 0 \\
\Gamma_{xx} &= -2 \nu \eta z B''_{44} \\
\Gamma_{yy} &= -(1+2\nu) \eta z B''_{44} \\
\Gamma_{zz} &= -(1-2\nu) \eta z B''_{44} \\
\Gamma_{xy} &= 0 \\
\Gamma_{xz} &= 0 \\
\Gamma_{yx} &= 0 \\
\Gamma_{zx} &= 0 \\
\Gamma_{yz} &= \frac{1}{2} \left[ (1-2\nu) \eta^2 + (1+2\nu) z^2 \right] B''_{44} \\
\Gamma_{zy} &= 0 \\
\end{align*}
\]
\[ A_s' \neq 0 \]
\[
\sigma_{xx} = (1+2\nu)A_s^3 + 3(1-2\nu)Az^2]
\]
\[
\sigma_{yy} = [2\nu A_s^3 - 6(1+\nu)Az^2]
\]
\[
\sigma_{zz} = [2(1-\nu)A_s^3 + 6\nu A_z^2]
\]
\[
\sigma_{xy} = 0
\]
\[
\sigma_{xz} = 0
\]
\[
\sigma_{yz} = 0
\]
\[
\sigma_{zx} = 0
\]

\[ A_s^2 \neq 0 \]
\[
\sigma_{xx} = 0
\]
\[
\sigma_{yy} = 0
\]
\[
\sigma_{zz} = 0
\]
\[
\sigma_{xy} = 0
\]
\[
\sigma_{xz} = 0
\]
\[
\sigma_{yz} = 0
\]
\[
\gamma_{zx} = 0
\]

\[ A_s^3 \neq 0 \]
\[
\sigma_{xx} = [2\nu A_s^3 - 6(1+\nu)Az^2] A_s^3
\]
\[
\sigma_{yy} = [2\nu A_s^3 - 6\nu A_z^2] A_s^3
\]
\[
\sigma_{zz} = [2(1-\nu)A_s^3 + 6\nu A_z^2] A_s^3
\]
\[
\sigma_{xy} = 0
\]
\[
\sigma_{xz} = 0
\]
\[
\sigma_{yz} = 0
\]
\[
\sigma_{zx} = [2(1+\nu)A_z^3 - 6\nu A_z^2] A_s^3
\]
\( \frac{A_{\text{si}}}{A_{\text{si}} \neq 0} \)

\[ \sigma_{xx} = \left[ (1 + 2\nu)\varepsilon^3 + 3(1 - 2\nu)\varepsilon^2 \right] A_{\text{si}} \]
\[ \sigma_{yy} = \left[ -2\nu \varepsilon^3 + 6\nu \varepsilon^2 \right] A_{\text{si}} \]
\[ \sigma_{zz} = \left[ -(3 + 2\nu)\varepsilon^3 + 3(1 + 2\nu)\varepsilon^2 \right] A_{\text{si}}' \]
\[ \sigma_{xy} = 0 \]
\[ \sigma_{xz} = 0 \]
\[ \sigma_{yz} = 0 \]
\[ \tau_{yx} = \left[ -(1 - 2\nu)\varepsilon^3 - 3(1 + 2\nu)\varepsilon^2 \right] A_{\text{si}}' \]

\( \frac{A_{\text{si}}}{A_{\text{si}} \neq 0} \)

\[ \sigma_{xx} = 0 \]
\[ \sigma_{yy} = 0 \]
\[ \sigma_{zz} = 0 \]
\[ \sigma_{xy} = 0 \]
\[ \sigma_{xz} = 0 \]
\[ \sigma_{yz} = 0 \]
\[ \tau_{yx} = 0 \]

\( \frac{A_{\text{si}}}{A_{\text{si}} \neq 0} \)

\[ \sigma_{xx} = \left[ 2\nu \varepsilon^3 - 6(1 + \nu)\varepsilon^2 \right] A_{\text{si}}' \]
\[ \sigma_{yy} = \left[ 2\nu \varepsilon^3 - 6\nu \varepsilon^2 \right] A_{\text{si}}' \]
\[ \sigma_{zz} = \left[ 2(1 - \nu)\varepsilon^3 + 6\nu \varepsilon^2 \right] A_{\text{si}}' \]
\[ \sigma_{xy} = 0 \]
\[ \sigma_{xz} = 0 \]
\[ \sigma_{yz} = 0 \]
\[ \tau_{yx} = \left[ 2(1 + \nu)\varepsilon^3 - 6\nu \varepsilon^2 \right] A_{\text{si}}' \]
\( A_{s2} \neq 0 \)
\[
\begin{align*}
\Gamma_{xx} &= [3v x^2 y + 3(1-v) y^2] A_{s2}' \\
\Gamma_{yy} &= [-3 v x^2 y + 3 v y^2] A_{s2}' \\
\Gamma_{yz} &= [-3 (1+v) x^2 y + 3 v y^2] A_{s2}' \\
\Gamma_{zx} &= [(1+v) x^2 - 3 v y^2] A_{s2}' \\
\Gamma_{xy} &= -3 x^2 y A_{s2}' \\
\Gamma_{yz} &= -6 v x y z A_{s2}
\end{align*}
\]

\( A_{s2}^2 \neq 0 \)
\[
\begin{align*}
\Gamma_{xx} &= [3 x^2 y - 3 y^2] A_{s2}^2 \\
\Gamma_{yy} &= 0 \\
\Gamma_{yz} &= [-3 x^2 y + 3 y^2] A_{s2}^2 \\
\Gamma_{zx} &= [-(1-v) x^3 + 3 (1-2v) x y^2] A_{s2}^2 \\
\Gamma_{xy} &= [-(1-2v) z^3 + 3 (1-2v) x^2 y] A_{s2}^2 \\
\Gamma_{zx} &= -6 x y z A_{s2}^2
\end{align*}
\]

\( A_{s2}^3 \neq 0 \)
\[
\begin{align*}
\Gamma_{xx} &= [3 v x^2 y - 3 (1+v) y^2] A_{s2}^3 \\
\Gamma_{yy} &= [3 v x^2 y - 3 v y^2] A_{s2}^3 \\
\Gamma_{yz} &= [3 (1-v) x^2 y + 3 v y^2] A_{s2}^3 \\
\Gamma_{zx} &= -3 y^2 z A_{s2}^3 \\
\Gamma_{xy} &= [-3 x^2 y z A_{s2}^3 \\
\Gamma_{yz} &= [(1+v) z^3 - 3 v x^2 y] A_{s2}^3 \\
\Gamma_{zx} &= [-6 v x y z] A_{s2}^3
\end{align*}
\]
\[ \sigma_{xx} = \left[ -3(1-2\nu)xy + 3(1-2\nu)yz \right] A_{33} \]
\[ \sigma_{yy} = \left[ 3x^2 - 6\nu xy - 3(1-2\nu)yz \right] A_{33} \]
\[ \sigma_{zz} = \left[ -3x^2 - 3(1+2\nu)xy + 6(1+\nu)yz \right] A_{33} \]
\[ \tau_{xy} = \left[ 3(1+2\nu)x^2y + 3(1-2\nu)yz \right] A_{33} \]
\[ \tau_{yz} = \left[ -6\nu xy \right] A_{33} \]
\[ \tau_{zx} = \left[ -2(1-2\nu)x^2y - 3(1+2\nu)x^2z + 3(1-2\nu)yz \right] A_{33} \]

\[ \sigma_{xx} = \left[ -v x^3 + 3v y^2 + 3v z^2 \right] A_{33} \]
\[ \sigma_{yy} = \left[ -(1-v)x^3 + 3(1-v)yz \right] A_{33} \]
\[ \sigma_{zz} = \left[ -v x^3 - 3v y^2 + 3v z^2 \right] A_{33} \]
\[ \tau_{xy} = \left[ 3v x^2 y - 3v yz \right] A_{33} \]
\[ \tau_{yz} = -6\nu xy A_{33} \]
\[ \tau_{zx} = -3 v y^2 A_{33} \]

\[ \sigma_{xx} = \left[ v x^3 + 3v y^2 - 3(1+2\nu)yz \right] A_{33} \]
\[ \sigma_{yy} = \left[ v x^3 + 3v y^2 - 3(1+2\nu)yz \right] A_{33} \]
\[ \sigma_{zz} = \left[ (1-v)x^3 + 3(1-v)yz + 6\nu yz \right] A_{33} \]
\[ \tau_{xy} = -3v yz A_{33} \]
\[ \tau_{yz} = -6\nu xy A_{33} \]
\[ \tau_{zx} = \left[ 2(1+\nu)z^2 - 3v x^2 y - 3v yz \right] A_{33} \]
\[ A_{s4} \neq 0 \]
\[
\begin{align*}
\tau_{xx} &= \left[ -(1-v)\gamma^3 + 3(1-v)\gamma y^2 \right] A_{s4} \\
\tau_{yy} &= \left[ -v\gamma^3 + 3\alpha^2 y + 3v\gamma y^2 \right] A_{s4} \\
\tau_{zz} &= \left[ -v\gamma^3 - 3\alpha^2 y + 3v\gamma y^2 \right] A_{s4} \\
\tau_{xy} &= \left[ 3v\gamma \gamma^2 - 3v\alpha \gamma^2 \right] A_{s4} \\
\tau_{xz} &= -3 \alpha^2 z A_{s4} \\
\tau_{yz} &= -6 v v y z A_{s4}
\end{align*}
\]

\[ A_{s2} \neq 0 \]
\[
\begin{align*}
\tau_{xx} &= \left[ 3\gamma^3 - 6v v \gamma^2 - 3(1-2v)\gamma y^2 \right] A_{s2} \\
\tau_{yy} &= \left[ -3(1-2v)\alpha^2 y + 3(1-2v)\gamma y^2 \right] A_{s2} \\
\tau_{zz} &= \left[ -3\gamma^3 + 3(1+2v)\alpha^2 y + 6(1+2v)\gamma y^2 \right] A_{s2} \\
\tau_{xy} &= \left[ 3(1+2v)\gamma \gamma^2 + 3(1-2v)\alpha \gamma^2 \right] A_{s2} \\
\tau_{xz} &= \left[ -2(1-2v)\gamma^3 + 3(1-2v)\alpha^2 z - 3(1+2v)\gamma y^2 \right] A_{s2} \\
\tau_{yz} &= -6 v v y z A_{s2}
\end{align*}
\]

\[ A_{s3} \neq 0 \]
\[
\begin{align*}
\tau_{xx} &= \left[ v\gamma^3 + 3v \alpha^2 \gamma y - 3(1+2v)\gamma y^2 \right] A_{s3} \\
\tau_{yy} &= \left[ v\gamma^3 + 3v \alpha^2 \gamma y - 3(1+2v)\gamma y^2 \right] A_{s3} \\
\tau_{zz} &= \left[ (1-v)\gamma^3 + 3(1-v)\alpha^2 y + 6v v \gamma y^2 \right] A_{s3} \\
\tau_{xy} &= -3 \alpha^2 z A_{s3} \\
\tau_{xz} &= \left[ 2(1+v)\gamma^3 - 3v \alpha^2 z - 3v \gamma^3 y^2 \right] A_{s3} \\
\tau_{yz} &= -6 v v y z A_{s3}
\end{align*}
\]
\[ A_{ss}' \neq 0 \]

\[ \tau_{xx} = 0 \]
\[ \tau_{yy} = [3 \nu y^2 - 3 \nu z^2] A_{ss}' \]
\[ \tau_{zz} = [-3 \nu y^2 + 3 \nu z^2] A_{ss}' \]
\[ \tau_{xy} = [- (1-2\nu)y^3 + 3 (1-2\nu)yz^2] A_{ss}' \]
\[ \tau_{yz} = [-6 \nu yz] A_{ss}' \]
\[ \tau_{xz} = [- (1-2\nu)z^3 + 3 (1-2\nu)yz] A_{ss}' \]

\[ A_{ss}^2 \neq 0 \]
\[ \tau_{xx} = [-3 \nu y^2 + 3 \nu z^2] A_{ss}^2 \]
\[ \tau_{yy} = [3 \nu y^2 + 3 (1-\nu)z^2] A_{ss}^2 \]
\[ \tau_{zz} = [-3 (1+\nu)y^2 + 3 \nu z^2] A_{ss}^2 \]
\[ \tau_{xy} = [(1+\nu)y^3 - 3 \nu yz^2] A_{ss}^2 \]
\[ \tau_{yz} = -6 \nu yz A_{ss}^2 \]
\[ \tau_{xz} = -3 y^2 z A_{ss}^2 \]

\[ A_{ss}^3 \neq 0 \]
\[ \tau_{xx} = [3 \nu y^2 - 3 \nu z^2] A_{ss}^3 \]
\[ \tau_{yy} = [3 \nu y^2 - 3 (1+\nu)z^2] A_{ss}^3 \]
\[ \tau_{zz} = [3 (1-\nu)y^2 + 3 \nu z^2] A_{ss}^3 \]
\[ \tau_{xy} = [-3 yz^2] A_{ss}^3 \]
\[ \tau_{yz} = [-6 \nu yz] A_{ss}^3 \]
\[ \tau_{xz} = [(1+\nu)z^3 - 3 \nu yz^2] A_{ss}^3 \]
\[ A_{56} \neq 0 \]

\[ \sigma_{xx} = 0 \]
\[ \sigma_{yy} = 0 \]
\[ \sigma_{zz} = 0 \]
\[ \sigma_{xy} = 0 \]
\[ \sigma_{xz} = 0 \]
\[ \sigma_{yz} = 0 \]

\[ \overline{A}_{56} \neq 0 \]

\[ \sigma_{xx} = \left[ -2y + 6yz \right] A_{56}^2 \]
\[ \sigma_{yy} = \left[ (1 + 2y) + 3(1 - 2y)yz \right] A_{56}^2 \]
\[ \sigma_{zz} = \left[ -(3 + 2y) + 3(1 + 2y)yz \right] A_{56}^2 \]
\[ \sigma_{xy} = 0 \]
\[ \sigma_{xz} = \left[ -(1 - 2y)yz - 3(1 + 2y)yz \right] A_{56}^2 \]
\[ \sigma_{yz} = 0 \]

\[ \overline{A}_{56} \neq 0 \]

\[ \sigma_{xx} = \left[ y^3 - 3yz \right] A_{56}^3 \]
\[ \sigma_{yy} = \left[ y^3 - 3(1+y)yz \right] A_{56}^3 \]
\[ \sigma_{zz} = \left[ (1-y)yz + 3yz \right] A_{56}^3 \]
\[ \sigma_{xy} = 0 \]
\[ \sigma_{xz} = \left[ (1+y)yz - 3y^2z \right] A_{56}^3 \]
\[ \sigma_{yz} = 0 \]
\[ B_{31} \neq 0 \]

\[
\Gamma_{xx} = \left[ (1-v)g^3 + 3v \alpha^2 g \right] B_{31} \\
\Gamma_{yy} = \left[ v g^3 - 3v \alpha^2 g \right] B_{31} \\
\Gamma_{zz} = \left[ v g^3 - 3(1+v) \alpha^2 g \right] B_{31} \\
\Gamma_{xy} = 0 \\
\Gamma_{xz} = 0 \\
\Gamma_{yz} = \left[ (1+v) \alpha^3 - 3v \alpha^2 \right] B_{31} 
\]

\[ B_{31} \neq 0 \]

\[
\Gamma_{xx} = 0 \\
\Gamma_{yy} = 0 \\
\Gamma_{zz} = 0 \\
\Gamma_{xy} = 0 \\
\Gamma_{xz} = 0 \\
\Gamma_{yz} = 0 
\]

\[ B_{31} \neq 0 \]

\[
\Gamma_{xx} = \left[ -(3+2v)g^3 + 3(1+2v) \alpha^2 g \right] B_{31} \\
\Gamma_{yy} = \left[ -2v g^3 + 6v \alpha^2 g \right] B_{31} \\
\Gamma_{zz} = \left[ (1+2v) g^3 + 3(1-2v) \alpha^2 g \right] B_{31} \\
\Gamma_{xy} = 0 \\
\Gamma_{xz} = 0 \\
\Gamma_{yz} = \left[ -(1-2v) \alpha^3 - 3(1+2v) \alpha^2 g \right] B_{31} 
\]
\( B_{s2} \neq 0 \)

\[
\begin{align*}
\sigma_{xx} &= -6(1-2v)xyz B_{s2}^1 \\
\sigma_{yy} &= -12v xyz B_{s2}^1 \\
\sigma_{zz} &= -6(1+2v)xyz B_{s2}^1 \\
\sigma_{xy} &= \left[ 3(1+2v)x^2 y + (1-2v)y^3 \right] B_{s2}^1 \\
\sigma_{yz} &= \left[ 3x^3 - 3xz^2 \right] B_{s2}^1 \\
\sigma_{xz} &= \left[ 3(1+2v)x^2 y + 3(1-2v)y^2 z \right] B_{s2}^1
\end{align*}
\]

\( B_{s2}^2 \neq 0 \)

\[
\begin{align*}
\sigma_{xx} &= 6xyz B_{s2}^2 \\
\sigma_{yy} &= 0 \\
\sigma_{zz} &= -6xyz B_{s2}^2 \\
\sigma_{xy} &= \left[ -3(1-2v)x^2 y + (1-2v)y^3 \right] B_{s2}^2 \\
\sigma_{yz} &= \left[ -(1-2v)x^3 + 3(1-2v)y^2 z \right] B_{s2}^2 \\
\sigma_{xz} &= \left[ 3x^2 y - 3yz^2 \right] B_{s2}^2
\end{align*}
\]

\( B_{s2}^3 \neq 0 \)

\[
\begin{align*}
\sigma_{xx} &= 6(1+2v)xyz B_{s2}^3 \\
\sigma_{yy} &= 12v xyz B_{s2}^3 \\
\sigma_{zz} &= 6(1-2v)xyz B_{s2}^3 \\
\sigma_{xy} &= \left[ 3x^2 y - 3y^3 \right] B_{s2}^3 \\
\sigma_{yz} &= \left[ -(1-2v)x^3 - 3(1+2v)y^2 z \right] B_{s2}^3 \\
\sigma_{xz} &= \left[ -3(1-2v)x^2 y - 3(1+2v)y^2 z \right] B_{s2}^3
\end{align*}
\]
\[ B_{53}^0 \neq 0 \]

\[ \sigma_{xy} = \left[ \frac{1}{3} (1 - \nu) \xi^3 - (1 - \nu) \eta^3 \right] B_{53}^0 \]
\[ \sigma_{yy} = \left[ \frac{1}{3} \nu \eta^3 + \alpha^2 \eta - \nu \eta^3 \right] B_{53}^0 \]
\[ \sigma_{xz} = \left[ \frac{1}{3} \nu \eta^3 - \alpha^2 \eta - \nu \eta^3 \right] B_{53}^0 \]
\[ \tau_{xy} = 2 \nu \eta \xi \]
\[ \tau_{yz} = \nu \eta^2 B_{53}^0 \]
\[ \tau_{xz} = \nu \eta^2 B_{53}^0 \]

\[ B_{52}^2 \neq 0 \]

\[ \sigma_{xx} = \left[ \frac{1}{3} \nu \eta^3 - \nu \eta^2 \xi + \eta^2 \right] B_{53}^2 \]
\[ \sigma_{yy} = \left[ \frac{1}{3} (1 - \nu) \xi^3 - (1 - \nu) \xi^3 \right] B_{53}^2 \]
\[ \sigma_{xz} = \left[ \frac{1}{3} \nu \eta^3 - \nu \eta^2 \xi - \eta^2 \right] B_{53}^2 \]
\[ \tau_{xy} = 2 \nu \eta \xi \]
\[ \tau_{yz} = \nu \eta^2 B_{53}^2 \]
\[ \tau_{xz} = \nu \eta^2 B_{53}^2 \]

\[ B_{53}^3 \neq 0 \]

\[ \sigma_{xx} = \left[ - \left( 1 + \frac{2}{3} \nu \right) \xi^3 + 2 \nu \eta^2 \xi + \left( 1 + 2 \nu \right) \eta^2 \right] B_{53}^3 \]
\[ \sigma_{yy} = \left[ - \left( 1 + \frac{2}{3} \nu \right) \xi^3 + \left( 1 + 2 \nu \right) \eta^2 \xi + 2 \nu \eta^2 \right] B_{53}^3 \]
\[ \sigma_{xz} = \left[ \frac{2}{3} \left( 1 + 2 \nu \right) \xi^3 + \left( 1 - 2 \nu \right) \xi^2 \eta + (1 - 2 \nu) \eta^2 \right] B_{53}^3 \]
\[ \tau_{xy} = 2 \nu \eta \xi \]
\[ \tau_{yz} = \left[ -(1 - 2 \nu) \alpha^2 \eta - (1 + 2 \nu) \eta^2 \xi \right] B_{53}^3 \]
\[ \tau_{xz} = \left[ -(1 - 2 \nu) \alpha^2 \eta - (1 + 2 \nu) \eta^2 \xi \right] B_{53}^3 \]
\[ B_{5^4} \neq 0 \]

\[ \tau_{xx} = 0 \]
\[ \tau_{yy} = \left[ 6 \, dyz \right] B_{5^4} \]
\[ \tau_{zz} = -6 \, dyz \, B_{5^4} \]
\[ \tau_{xy} = \left[ (1-2u)z^3 - 3(1-2u)y^2 z \right] B_{5^4} \]
\[ \tau_{yz} = \left[ 3 \, dy^2 - 3 \, dz^2 \right] B_{5^4} \]
\[ \tau_{zx} = \left[ -(1-2u)z^3 + 3(1-2u)yz^2 \right] B_{5^4} \]

\[ B_{5^2} \neq 0 \]

\[ \tau_{xx} = -12 \, v \, dyz \, B_{5^2} \]
\[ \tau_{yy} = -6 \, (1-2v) \, dyz \, B_{5^2} \]
\[ \tau_{zz} = -6 \, (1+2v) \, dyz \, B_{5^2} \]
\[ \tau_{xy} = \left[ (1-2v)z^3 + 3(1+2v)y^2 z \right] B_{5^2} \]
\[ \tau_{yz} = \left[ 3 \, (1+2v) \, dy^2 + 3(1-2v) \, dz^2 \right] B_{5^2} \]
\[ \tau_{zx} = \left[ 3 \, y^3 - 3 \, yz^2 \right] B_{5^2} \]

\[ B_{5^3} \neq 0 \]

\[ \tau_{xx} = 12 \, v \, dyz \, B_{5^3} \]
\[ \tau_{yy} = 6 \, (1+2v) \, dyz \, B_{5^3} \]
\[ \tau_{zz} = 6 \, (1-2v) \, dyz \, B_{5^3} \]
\[ \tau_{xy} = \left[ 3 \, y^2 z - 3 \, z^2 \right] B_{5^3} \]
\[ \tau_{yz} = \left[ -3(1-2v) \, y^2 z - 3(1+2v) \, dz^2 \right] B_{5^3} \]
\[ \tau_{zx} = \left[ -(1-2v) \, y^3 - 3(1+2v) \, yz^2 \right] B_{5^3} \]
\[ B_{ss}^0 \]

\[ \bar{\tau}_{xx} = 0 \]
\[ \bar{\tau}_{yy} = 0 \]
\[ \bar{\tau}_{zz} = 0 \]
\[ \bar{\tau}_{yz} = 0 \]
\[ \bar{\tau}_{zx} = 0 \]

\[ B_{ss}^2 \neq 0 \]

\[ \bar{\tau}_{xx} = \left[ v y^3 - 3 v y^2 z \right] B_{ss}^2 \]
\[ \bar{\tau}_{yy} = \left[ (1-v) y^3 + 3 v y^2 z \right] B_{ss}^2 \]
\[ \bar{\tau}_{zz} = \left[ v y^3 - 3 (1+v) y^2 z \right] B_{ss}^2 \]
\[ \bar{\tau}_{yz} = 0 \]
\[ \bar{\tau}_{zx} = 0 \]

\[ B_{ss}^3 \neq 0 \]

\[ \bar{\tau}_{xx} = \left[ -2 v y^3 + 6 v y^2 z \right] B_{ss}^3 \]
\[ \bar{\tau}_{yy} = \left[ -(3+2v) y^3 + 3 (1+2v) y^2 z \right] B_{ss}^3 \]
\[ \bar{\tau}_{zz} = \left[ (1+2v) y^3 + 3 (1-2v) y^2 z \right] B_{ss}^3 \]
\[ \bar{\tau}_{yz} = 0 \]
\[ \bar{\tau}_{zx} = 0 \]
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