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PROPOSAL FOR INTEGRATING THE CONCEPTS OF PLANE AND SOLID GEOMETRY

BASED ON STUDENT THINKING ABOUT THE CONCEPT OF DIMENSION

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Helen Florence Kriegsman, B.S., M.S.

* * * * *

The Ohio State University
1964

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CHAPTER I

INTRODUCTION

Origin of the Problem

For nearly one hundred years the combining of plane and solid geometry into one subject has been advocated by teachers and textbook writers in the United States as well as in several European countries. Although a number of these people have actually tested their ideas, the uniting of these two phases of geometry has not been universally accepted. The plan of offering one year of plane geometry followed, a year or two later, by one semester of solid geometry is still widely accepted in the American secondary school.

Permitting students to spend a full year in the study of two-dimensional geometry without mentioning three-dimensional space, when they are living in a world of the latter form, seems somewhat unrealistic. Since many of these students will not continue their mathematical study, they never have the guided opportunity to extend the concepts and ideas of two dimensions to their own surroundings. For three-quarters of a century educators have questioned this division of geometry and the limitations to a study of "Flatland" which are imposed upon those students who study only one year of high school geometry.\(^1\) The need to acquaint

the students with the world in which they live is probably the strongest argument for teaching geometry as a unified course.

More recently the great advances in scientific and technological knowledge have made it desirable to encourage high school students with ability in mathematics to progress in this field as far as feasible before entering college. Not only has new subject matter arisen within the field of mathematics, itself, but other areas, such as the life sciences, the social sciences, and business administration, have also found more application for certain phases of mathematics. The development of computing machines in addition to creating new types of positions has also caused a lessening of the need for manipulators and placed greater emphases on basic understanding. All of these changes have made a revision of the mathematics curriculum mandatory. To spend a full semester on solid geometry, when the important aspects of the material could be incorporated in the earlier year of geometry study, appears to be an unwise use of the student's time. Furthermore, some curriculums, such as engineering, which formerly required a course in solid geometry for entrance in their programs are eliminating this requirement.

From a less basic viewpoint, but perhaps more appealing to school officials in the light of increased enrollments and teacher shortage,
is the fact that the combining of these two courses would provide the means for increasing the mathematics offerings of the school without an increase in the number of teachers. This factor may continue to have added impetus as more and more colleges expect the student to enter with the work in three-dimensional geometry already completed.

These factors, then, indicate the need for the secondary school to make some provision for combining the important aspects of two- and three-dimensional space into one course.

**Methods of Combining Plane and Solid Geometry**

The methods which have been tested or suggested in attempts to combine plane and solid geometry fall into three categories:

1) Teach solid geometry as a separate unit in the tenth grade much as it is taught in the twelfth grade now, except that it would be more concentrated.

2) Fuse the solid geometry with the plane wherever such fusion seems desirable and possible.

3) Expect the student to become acquainted with solid geometry intuitively.\(^1\)

The first of these methods is frequently referred to as a tandem course, and as indicated here, the material is presented in much the same order as a plane geometry course offered in one year and followed by a one-semester course in solid geometry a year or two later. It is

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usually necessary to postulate a few more propositions and to omit some that may be included in the longer sequence, but other than condensing the course to one year the tandem method presents little change that is fundamental.

In the case of the fused course, a pre-planned program is followed in which analogous ideas and concepts from solid geometry are presented along with the plane geometry. In this manner the two phases are integrated throughout the year, and emphasis is given to the similarities rather than treating the material as two separate courses.

The third method, the intuitive approach, is a very informal one. From time to time, a knowledge of certain terms from solid geometry is acquired by the student, and he feels that a particular relationship exists, but he is not expected to justify his feeling; neither does he necessarily relate these feelings to similar plane geometry experiences.

**Statement of the Hypotheses**

Since the major difference between plane and solid geometry lies in emphasis placed on two-dimensional space in the former and three-dimensional in the latter, the concept of dimension should be the logical unifying feature. This study, then, attempts to test the following hypotheses:

1) If students receive an early awareness of the concept of dimension, they will tend to extend the ideas studied in two-dimensional space to spaces of other dimensions, particularly three-space.

2) Through these extensions to other dimensions the student will integrate the concepts of the various spaces, and this integration
will lead to a program of spatial insights initiated primarily by the student.

3) The null hypothesis, that the emphasis placed on the concept of dimension will not affect the student's achievement in the material normally included in the traditional two-space study, is also tested.

For this study it should be explained that in extending the concept of dimension to other spaces, the extension is not limited to an upward or increased number of dimensions, but it also considers the inclusion of one-space or zero-space as an extension of the concept. Thus an idea may originate in any one of these space concepts, and then through his awareness of the greater possibilities existing when dimensions are added and the limitations imposed by fewer dimensions, the student will be curious to investigate the idea in these other spaces. Rather than limiting his thinking to a world of two dimensions he will have a more complete picture of a particular relationship ranging from a world of no dimension to one of many dimensions; however, in most instances no attempt was made, in this experiment, to go beyond four-space unless a generalization was made for n-space.

Procedures for Testing the Hypotheses

The best place to test hypotheses such as those advanced here is in the classroom. The results of any method of instruction may be most effectively observed where the experimenter actually works with and notes the reactions of the students. No scientific laboratory can be constructed to produce the same situation—the classroom is the laboratory. The day-to-day contact with the students helps the exper-
imenter to be more cognizant of the attitudes and thinking of the class along the lines of the procedures being studied. Therefore, to test the ability of students to extend the concepts of two-dimensional space to that of other dimensions, the author taught a geometry class of the laboratory school at Kansas State College of Pittsburg as the experimental group. The procedure used was different from the three methods of combining plane and solid geometry previously mentioned although there were some features similar to those of the fusion method and the intuitive approach. Through the use of study guides and class discussion the students became conscious of the limitations imposed by a world of few dimensions and the freedom of added dimensions. During the course of the year the students were encouraged to extend the ideas of two-dimensional space to other spaces (1) by direct presentation on the part of the teacher; (2) by suggestions made by the students but pursued by the teacher; and (3) by ideas originating entirely from the students of the class. Records of these extensions were kept by the students in their class notebooks along with their lists of undefined terms, definitions, and assumptions, and the theorems proved. The teacher also noted the extensions in an anecdotal record book. A maximum of student participation was encouraged in an effort to determine how high school students think about dimension.

In addition to these records, a test on the extension of geometric concepts was constructed by the author and administered to this class and to four other classes selected at random from four neighboring communities. In two of these communities a teacher-directed course in plane and solid geometry was used, and in two of them no attempt was made to combine these
courses. The results of this test were compared as well as the results of a standardized geometry achievement test also given to these five groups.

Summary

In this introduction the major factors encouraging a combined course in plane and solid geometry have been presented. The hypotheses that by emphasizing the concept of dimension high school students may tend to extend the ideas and concepts of two-dimensional space to other dimensions and thus integrate plane and solid geometry without loss in the geometric achievement expected at this level were advanced, and the procedures for testing these hypotheses were presented.
CHAPTER II

HISTORICAL BACKGROUND

The idea of combining plane and solid geometry is not a new one nor is it limited to the secondary schools of the United States. Even before the public high school became prominent or educators had concerned themselves with standardizing its offerings, the thought of fusing these two subjects had taken root in some of the European countries.

Stamper\(^1\) reported that in the early part of the nineteenth century, Lacroix, a Frenchman, had called attention to the analogies between plane and solid geometry, and a fellow countryman, Gergone, in about 1825, questioned the separation of geometry into these two parts. In 1866, Valet, another Frenchman, published a pamphlet in which he arranged the subject matter in parallel columns, the books of plane geometry appearing on the left and the corresponding books from solid geometry listed on the right. This arrangement provided variety for the teacher using it as a textbook, since the entire group of plane geometry books could be studied and then followed by the solid, in a tandem fashion, or after studying a book from plane geometry the corresponding solid geometry book could be studied, or the plane geometry books could be used and the solid ignored without any interruption in continuity. It was not

until some twenty years after Méray published his book, *Nouveaux éléments de géométrie*, in 1873 that the teachers of France really showed interest, however, in presenting solid geometry along with plane. This book, unlike Valet's, presented the propositions from the two subjects at the same time and in this way fused the two together. The movement appeared to be well received by the normal schools, and teachers using the text were favorably impressed. One of them, M. Billiet, concluded that:

1) The simultaneous teaching of plane and solid geometry saves time.
2) The new method restores the agreement between the various subjects in the mathematical program and those of theoretical and practical teaching with which it is connected.
3) It appeals to intelligence more than to memory.
4) It accustoms the pupils to think for themselves.²

In 1904 the French Association for the Advancement of Science voted in favor of adopting Méray's method, and the Minister of Education approved its use in the lycees.

During this same period of time several Italian textbook writers were also producing books which employed the fusion idea. Although not the first to advocate the method, *Elementi di geometria* by Lazzeri and Bassani, published in 1891, made a more favorable impression than earlier texts. This book was followed by texts by Veronese and Reggio, both of which used the fusion method. In Italy, as in France, the method received the approval, in 1898, of the Association Mathesis, an organization interested in improving the teaching of mathematics.³

Other prominent mathematicians of this period who advocated fusion were D'Alembert and Bertrand of France,\textsuperscript{4} who ignored the classification of geometry into plane and solid, and Klein of Germany,\textsuperscript{5} who encouraged the fusion of plane and solid geometry. Furthermore, a report prepared for the British Mathematical Association on The Teaching of Geometry stated, "The view held is, in brief, that there ought to be no formal separation between plane and solid geometry and that, although, for practical reasons, it is often necessary to deal with tri-dimensional properties of figures apart from the corresponding plane properties, the two kinds should always be studied in close relation with one another."\textsuperscript{6}

One of the first indications of combining plane and solid geometry in the schools of the United States appeared in the Report of the Committee of Ten of the National Educational Association in 1894.\textsuperscript{7} This group recommended that a course in concrete geometry should be introduced into the grammar school and that it should be the object of this course to "familiarize the pupil with the facts of plane and solid geometry." They stated further that "a course of study in demonstrative geometry properly begins with a careful and exhaustive enumeration of those

\begin{flushleft}
\textsuperscript{5}Ibid., p. 18.
\textsuperscript{7}National Educational Association, "Report of the Committee of Ten on Secondary School Studies with the Reports of the Conferences Arranged by the Committee" (New York: American Book Company, 1894), p. 106.
\end{flushleft}
properties of space which do not admit of being deduced from still simpler properties; that space is continuous and of three dimensions; that figures may be moved about in it without change of size or shape; that straight lines and planes may exist in space, determined by two and three points respectively . . ."^8

By the close of the nineteenth century the great diversity of secondary school programs was a matter of grave concern to the colleges and universities of this country, and in order to establish some standard for preparation to enter college, the National Education Association established the Committee on College Entrance Requirements. Regarding its views on the mathematics curriculum the Report of the Committee stated: "There is serious doubt with many whether solid geometry should be forced into every preparatory curriculum. The chairman sympathizes with this view, and hopes the colleges will give earnest heed to this question before they insist making anything a constant in preparatory mathematics beyond algebra and plane geometry."^9

A subcommittee report of this group which was given as a report of the Committee of the Chicago Section of the American Mathematical Society indicated that there was some feeling for requiring solid geometry for college entrance by combining it with plane geometry:

"The attempt has been successfully made to teach geometry by interweaving solid and plane geometry from the outset. While

^8 Ibid., p. 112.

the committee is not prepared to commend this, there are advantages to be gained by beginning solid geometry before plane geometry is completed. In the opinion of the committee, the restriction of the study in geometry in many secondary schools to plane geometry is unfortunate, and it is desirable that the school course and the college-entrance requirement in geometry should cover both plane and solid geometry.\(^1\)

A study of sixty-one geometry textbooks published during the nineteenth century showed that several of the books contained some solid geometry material.\(^11\) By the close of the century, however, there was a decline in the proportion of the authors including three-dimensional subject matter in these textbooks.

In 1917 the report of the Committee on Geometry of the Central Association of Science and Mathematics Teachers expressed its feelings:

> The separation of solid geometry from plane geometry is unfortunate. In the preceding courses early training in the promotion of space conception has been afforded. Similarly a real advance in solid geometry should be made during the study of plane geometry. For, many theorems of solid geometry are closely related to the corresponding theorems in plane geometry. If they are proved in this course the student will have appropriate exercise in both two- and three-dimensional thinking.\(^12\)

One of the most influential committees to study the improvement of the teaching of mathematics in the United States has been the National Committee on Mathematical Requirements, appointed under the auspices of The Mathematical Association of America, Inc., in 1916. In its recom-

\(^{10}\text{Ibid.}, \text{p. 773.}\)

\(^{11}\text{John Donald Wilson, "An Analysis of the Plane Geometry Content of Geometry Textbooks Published in the United States before 1900" (unpublished Ed.D dissertation, University of Pittsburgh, Pittsburgh, Pa., 1959), p. 59.}\)

mendations for the senior high school program, this committee proposed four plans to assist teachers in arranging their courses. Two of these proposals, Plans B and D, suggested that plane and solid geometry be taught as a one year course.\textsuperscript{13} This suggestion followed recognition that some schools would find such a plan desirable. In discussing the difference between the work in mathematics abroad and in the United States, they noted that, "The distinction between plane and solid geometry [in Europe] is much less marked than in this country."\textsuperscript{14}

During the period from 1927 to 1935 a great deal of attention was given to the teaching of geometry and consideration of a combination of plane and solid geometry received its share of discussion. Since it had been customary in the United States to teach plane geometry for a year, followed by solid geometry for a half year, many teachers felt it would be impossible to cover the material in a one year course. Two outstanding teachers of mathematics expressed their views of this criticism. David Eugene Smith stated,

The proper question for us to ask to-day is whether the legitimate claims of geometry as a method rather than as a body of facts can be met in less time than was formerly allowed, and the answer that will probably appeal to any unbiased mind is in the affirmative. . . . While it is, mathematically speaking, desirable to be familiar with a considerable body of material, the method of geometry, the ability to reason geom-

\textsuperscript{13}National Committee on Mathematical Requirements, \textit{The Reorganization of Mathematics in Secondary Education}, A Report of the National Committee on Mathematical Requirements under the Auspices of the Mathematical Association of America, Inc. (Oberlin, Ohio: The Mathematical Association of America, 1923), p. 40.

\textsuperscript{14}Ibid., p. 173.
etricaly, the power to transfer this reasoning to non-matematical regions—this can be quite satisfactorily acquired in a single year.\(^{15}\)

A short time later William David Reeve expressed much the same view when he noted the decline in the teaching of solid geometry and expressed the view that if solid geometry were to remain in the curriculum it would probably have to be combined with plane geometry and be taught in the tenth year. He, too, emphasized that as far as covering the material was concerned, it could not be completed in a lifetime, and the purpose of teaching geometry was not to study a given list of propositions but to develop the ability to demonstrate.\(^{16}\)

In 1929 the Committee on College Entrance Requirements in Geometry was appointed jointly by the Mathematical Association of America and the National Council of Teachers of Mathematics "to study the feasibility of a proposal that college entrance requirements in geometry should be modified so as to bring about the more extensive introduction of courses including the essentials of plane and solid geometry in a single year's work, in place of the traditional year of plane geometry."\(^{17}\) This committee decided that where teachers wished to plan such a program they


should be given support and encouragement and that participation by
the College Entrance Examination Board would be effective and essential
toward accomplishment of the purpose. They cited instances where the
program had been adopted, but also expressed the opinion a second course
should be provided for mathematics and science students who might wish
to study material not covered in the one-year course.18

In order to supplement and assist the work of this committee the
Fifth Yearbook of the National Council of Teachers of Mathematics was
devoted to the teaching of geometry and included comments on the com-
bining of plane and solid geometry and presented descriptions of two
experiments which were being carried on. One of these programs was
reported by Allen19 at the University High School at Oakland, California.
She suggested four methods for making solid geometry a part of elementary
geometry.

1) Introduce the abstract phase of geometry through the students' con-
crete experiences in their real world of three dimensions.

2) Extend the facts and theorems of plane geometry to the anal-
ogous part of solid.

3) Select varied positions for the planes in which figures are
drawn rather than limiting them to the blackboard or writing
pad.

4) Spend the final month of the year on mensuration.

18 Ibid., p. 302.

19 Gertrude E. Allen, "An Experiment in Redistribution of Material
for High School Geometry," The Teaching of Geometry, Fifth Yearbook of
the National Council of Teachers of Mathematics (New York: Bureau of
Publications, Teachers College, Columbia University, 1930), pp. 73-75.
No formal proofs in solid geometry were given in the tenth grade, but the twenty per cent of the students who continued their study of mathematics through the senior year did have the experience then. As a result of this experiment, the teachers who participated were unanimous in recommending a more fundamental basis for organizing elementary and advanced geometry than a division into plane and solid geometry; an increased number of postulates and a decreased number of theorems to be proved in the elementary tenth year course; close correlation of plane and solid geometry, trigonometry, algebra, and arithmetic; significant originals and applied problems; organization of subject matter into a coherent whole made up of self-consistent interdependent units.

Wilt of the University High School, West Virginia University, Morgantown, West Virginia, found that in teaching plane and solid geometry simultaneously it was necessary to eliminate about one-third of the content of the two courses and that the more fundamental parts of the two should be introduced simultaneously. For example, in defining the concept, "angle," the student should also become acquainted with plane angles and dihedral and polyhedral angles.

As a result of the report of Jackson's committee, a second committee on geometry was appointed with C. M. Austin as chairman, for the purpose of presenting a clear statement on whether the third semester of geometry should be eliminated. This committee obtained statements from several teachers who had had experience with a combined course or

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20 Ibid., p. 81.
21 Ibid., pp. 84-85.
who had given serious consideration to teaching one. Excerpts from these statements indicated that most of them had little, if any, formal proofs in solid geometry, and the committee found this phase of the course expressed as largely imagination.\textsuperscript{23} The group also reported on questionnaires which had been sent out by two different individuals interested in this problem to teachers of mathematics. Professor Seymour, Mathematics Supervisor of New York State, found from 131 replies that 26 favored a combined course, 92 definitely opposed the combined course, while the remaining 13 had various qualifications in their answers.\textsuperscript{24} C. A. Stone of the University High School of Chicago, reported that from 140 replies, 108 favored solid geometry in high school; 88 opposed a one-year course; 27 favored such; 27 for fusion; 18 a tandem or parallel organization, and from this he concluded that teachers were strongly opposed to a combined course.\textsuperscript{25} From these results it appears that feeling for a combined course was not always favorable.

Austin's committee presented reasons for a combined course, a summary of which is given here:

1) Give students a conscious knowledge of three-dimensional space.

2) More interesting—more illustrations.

3) Geometry is a single subject and should not be separated into two parts.


\textsuperscript{24}Ibid., p. 374.

\textsuperscript{25}C. A. Stone, "Combined Course in Plane and Solid Geometry?" \textit{The Mathematics Teacher}, \textbf{XXIV} (March, 1931), p. 163.
4) Many parts of geometry were worked out for mature minds.
The tenth grade should include the simple parts of both plane
and solid and leave the more difficult for a later course.
5) Definitions and concepts are limited, for example angle and
loci.
6) Formulas of solid geometry should be part of the general
knowledge.
7) Part of the third semester may be devoted to calculus.

On the other side of the picture a summary of reasons against a
combined course is presented:

1) Tenth-year students are not mature enough and have difficulty
visualizing drawings.
2) Experimental basis is not strong enough.
3) Sufficient intuitive geometry is not given in the elementary
schools as it is in Europe.
4) Present course would be wrecked and sufficient solid geometry
material could not be introduced.
5) Too many students of lower ability in the geometry courses.
6) Present solid geometry course is the best place to mature
students mathematically.
7) Combined course might interfere with cultivation of postu-
lational thinking.
8) Too many assumptions are necessary.
9) Difficult for students to transfer schools.
10) Suitable texts or syllabi are not available.\(^{26}\)

\(^{26}\) C. M. Austin, \textit{op. cit.}, pp. 374-376.
Since most combination courses of plane and solid geometry are planned as tandem or as fused courses, the committee suggested tentative courses following each of these. The tandem course would require fifty-six theorems of plane geometry and twenty-eight of solid. In the fused course the solid geometry concepts would be introduced at appropriate places by imagination questions posed by the teacher. Rigid proofs are required only in plane geometry which includes thirty-eight theorems; however, eighteen corollaries of plane geometry and fifteen theorems of solid are also included. Most of the formulas are assumed or developed intuitively.\textsuperscript{27}

At least two members of this second committee favored a third semester of geometry. A summary of the committee conclusions follows:

1) The committee is not ready to recommend the adoption of a one-year course for all tenth-grade students although they recognize some value in knowledge of three-dimensional geometry.
2) Most people feel that solid geometry should be intuitive.
3) Some intuitive geometry is needed in the junior high school.
4) Experimental basis is not strong enough to warrant a one-year course.
5) Third semester of geometry needed.
6) Best procedure under present conditions is to introduce in plane geometry course the analogous solid geometry.

\textsuperscript{27}Ibid., p. 376.
7) Include study of angles, parallels, circles, and loci.
8) Teacher needs careful preparation.\textsuperscript{28}

Interest in a fused course in geometry seemed strong enough to warrant the construction of an outline for such a course by Reeve and his students. The course assumed intuitive geometry in junior high school to acquaint the student with vocabulary, geometric ideas, and construction. They also recognized the value of a unit of demonstrative geometry in the ninth year. The outline gave a list of proposed postulates and then presented the theorems of plane geometry with the analogous theorems of solid in each instance.\textsuperscript{29}

At about this same period Evans\textsuperscript{30} presented a proposed syllabus for a tandem course which included 104 propositions for plane and solid geometry, comprising the same total content as the 181 statements given by the College Entrance Examination Board's definition of these subjects.

In addition to determining the interest of teachers in a combined course by means of the questionnaire discussed in Austin's report, Stone also experimented with such a course. He selected two tenth-grade classes of similar ability and a twelfth-grade solid geometry class from the University High School, University of Chicago. For the unit on "Parallel and Perpendicular Lines" he taught a fused course to the more superior of the two tenth-grade classes. The results of a teacher-

\textsuperscript{28} Ibid., p. 378.

\textsuperscript{29} W. D. Reeve, "Tenth Year Mathematics Outline," \textit{The Mathematics Teacher, XXIII} (October, 1930), pp. 343-357.

\textsuperscript{30} George W. Evans, "Proposed Syllabus in Plane and Solid Geometry." \textit{The Mathematics Teacher, XXIII} (February, 1930), pp. 87-94.
made test showed that the fused group lost ground in plane geometry, whereas they had normally been superior to the other group, and made a very poor showing in solid. He felt the one year course was an "impossible program," but would propose further experimentation.\textsuperscript{31}

Another activity of national impact which occurred during this period was the appointment in 1932 of the Committee on Geometry by the National Council of Teachers of Mathematics. This group was to "initiate and carry forward a study of the whole question of geometry in our schools."\textsuperscript{32} A sub-committee of three examined the literature and prepared synopses recording the important suggestions pertaining to the teaching of geometry. From these, two major ideas apply to the combining of plane and solid geometry:

1) Pertinent reference ought to be made to solid geometry during the course in plane geometry.

2) The usual year and a half devoted to plane and solid geometry ought to be reordered on the basis of difficulty rather than dimension.\textsuperscript{33}

A questionnaire based on the ideas gleaned from these synopses was sent to each member of the committee and to members of regional associations of teachers of mathematics with the result that 101 replies were returned. Nearly all of these people favored occasional reference to

\textsuperscript{31}Stone, \textit{op. cit.}, pp. 164-165.


solid geometry, mainly of a mensurational sort, in the plane geometry course. About one-third of the committee wished to merge plane and solid more closely and the 101 other teachers were about equally divided on this point.\textsuperscript{34}

For the next twenty years very little was accomplished toward the unification of plane and solid geometry. In 1940 the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics published its final report in which two mathematics curriculum plans were proposed. The first of these proposals endorsed the lists of theorems issued by examining bodies such as the College Entrance Examination Board and the Regents of the University of the State of New York or by groups such as the National Committee of 1923. The program did include, however, as one of its objectives of the tenth grade course the "development of elementary spatial insight, and the habit of noticing geometric relations in three dimensions, and of comparing such relations with those in a plane. . . . Such a development can be secured by discussion and generalization of certain propositions or problems of plane geometry."\textsuperscript{35} The second plan was less inclined to include solid geometry in the tenth grade. Recognizing that more able pupils could profit from extending the theorems of plane geometry to related situations in three dimensions, the commit-

\textsuperscript{34}Ibid., p. 335.

tee felt, however, that this was not sufficient treatment for engineering schools, and so they recommended that the time be spent on algebraic or non-mathematical material.\footnote{\textit{Tbid.}, p. 111.}

Experimentation with fused courses was not completely dormant, since the Los Angeles City school system taught such a course\footnote{H. R. Douglass, "Current Trends in the Secondary-School Mathematics Curriculum," \textit{National Association of Secondary School Principals Bulletin}, XXVII (February, 1943), p. 22.} and an experimental course was carried on by Price at University High School, Iowa City, Iowa.\footnote{H. Vernon Price, "An Experiment in Fusing Plane and Solid Geometry," \textit{School Science and Mathematics}, XLIX (March, 1949), p. 200.} While this latter experiment had been going on for four years, the original course was based on the plane and solid geometry theorems considered fundamental by the National Committee on Mathematical Requirements. At the end of the first year several students who planned to enter technical schools requested a full year and one-half devoted to the study of geometry. To meet this demand, a third semester was established whereby the Grade 11 was devoted to a study of the more important concepts and relations and the first half of the twelfth year provided a review and extension of the previous year's work. The final result was a three-semester course which fused the entire body of material ordinarily included in plane and solid geometry. Although the students developed their own relationships, the content was covered in much the same order as in standard texts, with the work going from two to three dimensions and vice versa. Evaluation of such a course is difficult because there is no standardized test for those who
have studied geometry in this manner. "In general, it is probably true
that the plane geometry scores are spuriously high and the solid geo-
metry scores spuriously low, particularly for the one-year groups." 39
Only seventy-nine students were involved in this four-year study but
there are certain indications that a complete fusion of plane and solid
gometry may produce better results than is now obtained through sep-
parate courses and using proper content, organization, and methodology,
one year of instruction may provide sufficient training for all but the
most exacting curricula. 40 In conclusion Price recommended a one-year
course in plane and solid geometry in the eleventh grade preceded by
junior high school courses whose goal is functional competency.

The last ten years have seen a great increase in the number of
projects devoted to the improvement of teaching of mathematics and at
least two outstanding committee reports have dealt with this subject.
Included in the nine-point program of the Commission on Mathematics of
the College Entrance Examination Board of 1959 is the "incorporation
with plane geometry of some coordinate geometry and essentials of solid
gometry and space perception." 41 In discussing the shortcomings of
teaching geometry based on Euclid's Elements, the Appendices of the
Report of the Commission on Mathematics points out that Euclid was in-
terested in formulating a course which would serve as an introduction

39 Ibid., pp. 202-203.
40 Ibid., p. 203.
41 Commission on Mathematics, Program for College Preparatory
Mathematics, Report of the Commission on Mathematics (New York:
College Entrance Examination Board, 1959), p. iii.
to general philosophical studies and that he had no thought of preparing a high school course such as is given in the secondary schools today. His failure to fuse instruction in plane and solid geometry is one correction which should be made in present-day geometry courses.\footnote{42} These Appendices present an outline for a unit in solid and spherical geometry to be used with the mathematics of grade 10. The unit is intended to tie together the work in three dimensions and informal proofs are suggested.\footnote{43}

The Report of the Secondary-School Curriculum Committee of the National Council of Teachers of Mathematics, published in 1959, indicates that superior classes should find it possible to include a considerable amount of solid geometry along with plane geometry and that the references to space geometry will enhance the pupils' understanding of relations that were considered in plane. The Committee feels that the semester of solid geometry should be eliminated from the curriculum and necessary parts of this course should be woven into other subjects in the curriculum. Some of these concepts will be used as early as the junior high school and others will be incorporated as late as the twelfth grade when certain topics are considered in trigonometry.\footnote{44}


\footnote{43}{\textit{Ibid.}, pp. 140-158.}

Although the current studies of the mathematics curriculum do not make a major issue of combining plane and solid geometry, none of them contains a course called solid geometry. This does not mean that the study of space geometry is ignored but simply that each of these groups has been more interested in integrating the material wherever feasible. Moise, a member of the School Mathematics Study Group, in reporting on the geometry program, indicated that although this group had a different viewpoint in many respects from that of the Commission on Mathematics, he felt one of the best proposals of the Commission was the encouragement of an integration of plane and solid geometry. He believed the early introduction of solid geometry provided an opportunity to use three-dimensional problems as exercise material and that the students received valuable intuitive experience with figures in space without having to give complete deductive proofs.

Within the last few years several experimental programs have been offered in which plane and solid geometry have been integrated. Included among the programs is the one at Tulsa, Oklahoma, Seattle's

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program among superior students, and Harris' study at Winnetka, Illinois.

One deterrent to the teaching of a combined course in plane and solid geometry has been the lack of a suitable outline or list of basic propositions. To help alleviate this situation a number of individuals have prepared lists of postulates and theorems to guide the teacher who wishes to attempt such a course. Small prepared a list of 109 postulates, definitions, and theorems found in solid geometry texts and submitted this list to forty educators to obtain their opinions on whether the item should be included in a one-year course combining two- and three-dimensional geometry. From the thirty-one usable replies, he found fifty-eight items selected by seventy-five per cent or more of the participants for inclusion in such a course. A list of topics was formulated by Doxon, and others, such as O'Donnell, have prepared outlines for fusing the plane and solid concepts. In addition to these

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49 Dorothy Harris, "Combining the Study of Two and Three Dimensional Space" (unpublished Master of Education project, The Ohio State University, Columbus, 1959).


aids, most recently published first course geometry textbooks have included some material from three-dimensional space; however, the sections are usually included at the end of the unit and are frequently noted as optional.

Summary

For more than one hundred years there have been advocates of the combining of plane and solid geometry into a one-year course. Early in the nineteenth century European teachers of mathematics recognized advantages in using either a tandem or fused course in the study of two- and three-dimensional space.

While these schools in Europe accepted the idea of teaching these facts at the same time more readily than in the American schools, committees appointed to study the improvement of teaching geometry have recommended the unification of these two phases of geometry. These recommendations began as early as the Report of the Committee on College Entrance Requirements in 1899 and have continued to the present time as seen in such reports as the Commission on Mathematics of the College Entrance Examination Board and the Secondary School Curriculum Committee of the National Council of Teachers of Mathematics.

Teachers and textbook writers, however, have been slow to incorporate the recommendations of these groups, and as a result many schools still present plane and solid geometry as separate subjects. Several experiments have been carried on during the past thirty years and now many writers have included some three-dimensional ideas in their textbooks, but these are usually optional. Thus, although interest has been
shown for some time, the actual teaching of two- and three-dimensional concepts as one subject is still in the experimental stage in the American secondary school.
CHAPTER III

CONDITIONS OF THE EXPERIMENT

Nature of the Research

Since the teaching of plane and solid geometry as two separate courses has been questioned by every major committee studying the geometry offering in the mathematics curriculum, and at least some members of each of these groups have recognized the feasibility of integrating these subjects in a one-year course, experimentation should be continued to determine how these revisions may be incorporated into the high school program. The need for such experimentation in any phase of the school's program has been emphasized by Corey when he stated, "Research is of great importance to a profession. It is only through research that knowledge is increased and a basis for improved practice provided. Without the continuing impact of research findings, procedures become stereotyped, and the profession rapidly takes on the characteristics of a trade."¹

As the Report of the Secondary-School Curriculum Committee² has indicated, various groups are attempting to improve the content and


organization of the geometry course but there must be concern for what can be done now even before these programs are widely available or extensively accepted. The role of the individual teacher in improving the high school geometry program has been stressed by Wren when he indicated, "We can engage in experimentation on such problems as those of grade placement of geometric materials, motivation procedures, curriculum organization, evaluation patterns, and teaching techniques. Many of these problems are too big to be tackled with any degree of finality by individuals, but such individual efforts can have great potential as pilot studies for the much needed more elaborate investigations."³

Because of the many pitfalls encountered when an attempt is made to carry on traditional research with students in the classroom, "The process by which practitioners attempt to study their problems scientifically in order to guide, correct, and evaluate their decisions and actions . . ."⁴ and referred to by a number of people as action research has received favorable attention in recent years. This form of research encourages the attitude that

Most of the study of what should be kept in the schools and what should go and what should be added must be done in hundreds of thousands of classrooms and thousands of American communities. The studies must be undertaken by those who may have to change the way they do things as a result of the studies. Our schools cannot keep up with the life they are supposed to sustain and improve unless teachers, pupils, supervisors, administrators,


⁴Corey, op. cit., p. 6.
and school patrons continuously examine what they are doing. Singly and in groups, they must use their imaginations creatively and constructively to identify the practices that must be changed to meet the needs and demands of modern life, courageously try out those practices that give better promise, and methodically and systematically gather evidence to test their worth.\(^5\)

There appears to be rather widespread agreement among leaders in the field of teaching high school mathematics to integrate plane and solid geometry, but the procedure for accomplishing this unification has not been established. The purpose of the research reported in this paper was to determine how effectively students would tend to integrate these two aspects of geometry if they studied the concepts of two, three, or other space through an emphasis on dimension; therefore, the method of action research seemed most suitable for this experiment. The group of students selected to form the experimental class for this study consisted of the members of the 1962-1963 geometry class of the College High Laboratory School, Kansas State College of Pittsburg, Pittsburg, Kansas. The class numbered sixteen.

**Philosophy and Purpose of the School**

The laboratory school is a six-year, accredited, secondary school, comprising grades seven through twelve. The primary purpose of the school is to provide a laboratory for college students preparing to enter the secondary teaching field. The school provides a setting for these people to observe high school students, to participate in some of the high school classes and activities, and to perform some of their student teaching experience. While working toward this primary

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\(^5\)Ibid., p. viii.
function, the staff of the school is still cognizant of their responsibility to the secondary school students who are attending the school. This staff is well aware of the needs and problems of high school students as all have had some experience teaching in public schools.

**The School Population**

In order to present as nearly as possible the atmosphere of a public school, the students are required to apply for admission, and the selection is then made to secure a typical cross-section of the school population of the community. Most of the students come to the school from the elementary laboratory school which is part of the city elementary school system—the building being owned and maintained by the city, but staffed by the College. It serves as one of the elementary schools of the city system and is assigned its district on the same basis as the other schools in the city; however, due to its unique function of training elementary teachers, no student is required to attend this school. Practically no one, however, sends his child to an elementary school outside the district. On the other hand the secondary school is entirely separate from the city school system and is permitted to make its own regulations regarding admission.

Once the student seeks admission to the laboratory school his application is studied by the principal and a committee of the staff. Since all of the students from the elementary laboratory school who apply are admitted, the majority of the students in the secondary school live within the vicinity of the College. The group living in this area would fall within the upper-lower to upper-middle class. At the present
time very few are refused admission unless they live outside the area and appear to be transferring merely because they are not interested in adapting themselves to the public high school community. The two systems have generally agreed that students will not be allowed to transfer back and forth for petty reasons, so that once the student makes his selection at the beginning of the seventh grade, the school population is relatively stable except for those who move in or out of the city. At the time of the experiment the school population totaled 284, consisting of 41 seventh graders, 52 eighth graders, 50 freshmen, 47 sophomores, 52 juniors, and 42 seniors.

The Curriculum

The students of the laboratory school are required to meet the minimum standards set by the state of Kansas and a few additional requirements are specified by the school. The work of the seventh and eighth grades is fairly standardized except for a few variations in areas such as art and music. As far as the mathematics for these two grades is concerned, each student is required to spend one period per day in such a course.

Beginning with the ninth grade or freshman year, some electives are permitted. The graduation requirement for the high school, grades nine through twelve, is comprised of a total of seventeen units. A unit, in this school, is defined to be the credit received for successfully pursuing a course for a period of fifty-five minutes per day, five days per week for thirty-six weeks. There are one or two exceptions, such as the health and physical education courses and some music
courses which allow only a fractional unit of credit for this period. The seventeen units are distributed as shown in Table 1.

**TABLE 1**

GRADUATION REQUIREMENTS FOR COLLEGE HIGH LABORATORY SCHOOL

<table>
<thead>
<tr>
<th>Subject Area</th>
<th>Number of Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Social Studies</td>
<td>4</td>
</tr>
<tr>
<td>English</td>
<td>3</td>
</tr>
<tr>
<td>Mathematics</td>
<td>1</td>
</tr>
<tr>
<td>Laboratory Science</td>
<td>1</td>
</tr>
<tr>
<td>Fine Arts</td>
<td>1</td>
</tr>
<tr>
<td>Practical Arts</td>
<td>1</td>
</tr>
<tr>
<td>Health and Physical Education</td>
<td>1</td>
</tr>
<tr>
<td>Electives</td>
<td>5 or more</td>
</tr>
</tbody>
</table>

The mathematics offerings for these last four years of high school are given in Table 2.

**TABLE 2**

MATHEMATICS COURSES FOR GRADES NINE THROUGH TWELVE OFFERED BY COLLEGE HIGH LABORATORY SCHOOL

<table>
<thead>
<tr>
<th>Grade</th>
<th>Courses Offered</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>General Mathematics</td>
</tr>
<tr>
<td></td>
<td>Elementary Algebra</td>
</tr>
<tr>
<td>10</td>
<td>Plane Geometry</td>
</tr>
<tr>
<td>11</td>
<td>Intermediate Algebra</td>
</tr>
<tr>
<td>12</td>
<td>Senior Mathematics</td>
</tr>
</tbody>
</table>

Although the courses are usually taken at the time specified in Table 2, a student with the necessary prerequisites may take the courses at any
time during his last four years of high school. Elementary algebra is not a required prerequisite for plane geometry.

Students normally meet the requirement of one unit of mathematics by taking either general mathematics or elementary algebra in the freshman year. Although the students are given freedom of choice between these two courses, an algebraic aptitude test is administered toward the end of the eighth grade; the student is informed of the results of this test and encouraged to consider it in making his selection. For the remaining work in mathematics the courses are elective; however, those students who have performed well in previous courses or indicate ability by their standardized test scores are encouraged to continue in mathematics. Since the school is located on the college campus, qualified senior students are permitted to enroll in one or two college courses for credit. In this manner some of the seniors have elected to take trigonometry or college algebra for college credit. The number following this procedure in any one semester is quite small, less than five. When the demand is such that a class of five or more students wishes a senior mathematics course, such a class is provided in the laboratory school. For the past three years, a course in trigonometry has been offered, but solid geometry has not been offered. Furthermore, since this latter course is not required in any curriculum of the college and is being dropped from its offerings, very few students have contact with the geometry of three dimensions except as it arises in later mathematics courses in college.

Because of the accessibility of the College and because some students acquire a few hours of college credit during their senior
year, a rather high percentage of the students continue their formal education beyond the high school. Those who continue in either a four-year institution or the vocational school which is operated by the College comprise about 70 per cent of the graduating class.

**Selection of Control Groups**

Although the objective of this experiment was to report the achievements of one class based upon a particular emphasis in the method of instruction, in order to determine whether these reactions were different from those which might be expected if some other method had been used, the experimental group was compared with two other groups. The control groups consisted of selected students from four classes, each from a high school located in one of four communities somewhat comparable in size and in the same geographic area as that of the experimental group. Institutions of higher learning are located in three of these communities—one of the schools being in the same city as the experimental group and two others having junior colleges as a part of the public school system. Two of these schools were selected because their tenth-grade geometry classes were taught as plane geometry with practically no reference to solid geometry which was offered as a separate course in the senior year. The other two schools were selected because their tenth-grade geometry courses incorporated planned work in three-dimensional space along with that of two-dimensional. All of these classes were taught by teachers who had at least five years experience in teaching geometry. Each class was selected at random from several sections offered by the cooperating
teachers. All the students studying plane geometry only were placed in one group and those enrolled in the combined course were placed in a second group. Sixteen students, the same number as in the experimental class, were selected at random from each of the two groups, so that the final comparison was made with three groups of sixteen each. Although the groups were similar in many ways, no attempt was made to match the students of the individual groups on specific items, such as sex, aptitude, or ability, since it is probably not possible to find two groups of human beings who are really equivalent.

The curriculum for the students of the public schools was similar to that of the laboratory school, and the offerings and arrangement of the mathematics curriculum were practically the same. The students' principal contact with geometric material prior to the time of the study had occurred in the eighth grade where they had studied geometry from an intuitive approach. Although different textbooks had been used, they included similar topics such as recognizing common geometric figures, being familiar with units of measure, determining the measures of the area and volume of geometric figures by using specified formulas, performing the basic constructions.


Comparison of the Three Groups

To give some indication of the similarities of the groups of this study, several comparisons were made. In these comparisons the laboratory school students will be referred to as Group A, those in the combined plane and solid geometry group will be designated as Group B, and those studying plane geometry, only, will appear as Group C.

In each of these schools, as in the experimental school, geometry was offered as an elective course, primarily to tenth grade students, although students from other grades were found in each group.

<table>
<thead>
<tr>
<th>Year in School</th>
<th>Group A</th>
<th>Group B</th>
<th>Group C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grade 9</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Grade 10</td>
<td>14</td>
<td>10</td>
<td>13</td>
</tr>
<tr>
<td>Grade 11</td>
<td>0</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>Grade 12</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

In each of the three groups there were more boys than girls enrolled in the geometry course as shown in Table 4.
TABLE 4
DISTRIBUTION OF BOYS AND GIRLS IN GROUPS A, B, AND C

<table>
<thead>
<tr>
<th>Sex</th>
<th>Group A</th>
<th>Group B</th>
<th>Group C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boys</td>
<td>9</td>
<td>10</td>
<td>11</td>
</tr>
<tr>
<td>Girls</td>
<td>7</td>
<td>6</td>
<td>5</td>
</tr>
</tbody>
</table>

The average age of the students in the laboratory school class was slightly less than that of the students in the other two groups, but the difference was less than one-half year, and the major portion of each group was in the fifteen-sixteen year age group as shown in Table 5.

TABLE 5
AGE OF STUDENTS IN GROUPS A, B, AND C

<table>
<thead>
<tr>
<th>Age</th>
<th>Group A</th>
<th>Group B</th>
<th>Group C</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>14</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>15</td>
<td>9</td>
<td>7</td>
<td>7</td>
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<tr>
<td>16</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>17</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>18</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

The mental ability of each of these students was determined by administering the Henmon-Nelson Test of Mental Ability, and the geomet-
ric aptitude of each student at the beginning of the fall semester was indicated by the Orleans Geometry Prognosis Test. Table 6 shows the individual geometric aptitude scores, arranged in descending order, for each of the groups and the corresponding I.Q. as determined by the mental ability test. The aptitude test has a possible score of 96.

### Table 6

**ORLEANS GEOMETRY PROGNOSIS TEST AND HENMON–NELSON TEST OF MENTAL ABILITY SCORES OF STUDENTS IN GROUPS A, B, AND C**

<table>
<thead>
<tr>
<th>Orleans Geometry Prognosis Test</th>
<th>Henmon-Nelson Test of Mental Ability</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Group A</td>
</tr>
<tr>
<td></td>
<td>161</td>
</tr>
<tr>
<td>76</td>
<td>76</td>
</tr>
<tr>
<td>67</td>
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</tr>
</tbody>
</table>

Copies of these tests are given in Appendix C.
Table 7 gives the mean, median, and standard deviation for each group on the Henmon-Nelson Test of Mental Ability. These statistics show that while the means on the mental ability test differed by several points, the median scores for Groups A and C were more nearly the same. The standard deviations indicate that the students in Group B were more similar in ability than the students in either of the other groups.

**TABLE 7**

DATA ON THE SCORES FOR THE HENMON—NELSON TEST OF MENTAL ABILITY OF STUDENTS IN GROUPS A, B, AND C

<table>
<thead>
<tr>
<th>Item</th>
<th>Group A</th>
<th>Group B</th>
<th>Group C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>118</td>
<td>106</td>
<td>112</td>
</tr>
<tr>
<td>Median</td>
<td>114</td>
<td>105.5</td>
<td>112</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>13.2</td>
<td>7.1</td>
<td>10.5</td>
</tr>
</tbody>
</table>

All of the students in these groups had had a course in elementary algebra, but none of the students had been enrolled in a formal course in geometry prior to the year of the study. The mean, median, and standard deviation of each group for the Orleans Geometry Prognosis Test are given in Table 8. The results of the prognosis test indicate the students in Groups A and C showed about the same aptitude for geometry, but the students in Group B appeared to show less aptitude for this subject than either of the other groups. The variability among the students in Group A and B was greater than for those in Group C.
TABLE 8

DATA ON THE SCORES FOR THE ORLEANS GEOMETRY PROGNOSIS TEST
OF STUDENTS IN GROUPS A, B, AND C

<table>
<thead>
<tr>
<th>Item</th>
<th>Group A</th>
<th>Group B</th>
<th>Group C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>52</td>
<td>39</td>
<td>53</td>
</tr>
<tr>
<td>Median</td>
<td>47.5</td>
<td>39</td>
<td>49.5</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>13.7</td>
<td>13.1</td>
<td>10.9</td>
</tr>
</tbody>
</table>

All of these students were enrolled in classes which met five
days each week, but the length of period varied from forty-five minutes
to fifty-five minutes.

Summary

Since this research deals with a method of presenting concepts
of various spaces to high school students, the method known as action
research was employed in the study. Three groups of students were con-
sidered—those in the experimental group where special emphasis was
given to the concept of dimension, one group who studied two-dimensional
space, and another group who followed a pre-arranged course in two- and
three-dimensional space. These groups were compared according to grade
placement, sex, age, mental ability, and geometry aptitude.
CHAPTER IV

THE EXPERIMENT

Procedures of Instruction for the Experimental Group

Since the purpose of the study is to determine how well students are able to extend the ideas of two-space geometry to that of other spaces, the method of instruction must be one which is conducive to such an extension. Two recent committee reports on the improvement of the mathematics curriculum have emphasized the desirability of the discovery approach to the development of mathematical concepts. The Commission on Mathematics indicates that, "Most if not all the Commission members would prefer to see a developmental approach, which would encourage the student to discover as much of the mathematical subject matter for himself as his ability and the time available (for this is a time-consuming method) will permit."¹ The Secondary-School Curriculum Committee states, "There is much more emphasis upon giving students certain types of learning experiences that serve not only to enhance their understanding of mathematics but also to enliven their interests and increase their appreciation. Experiences that lead to 'discovery' of mathematical properties and relations are highly favored for this purpose."²


Several groups which are working on new mathematics curriculums have advocated and used the discovery method. In describing the work of the University of Illinois Committee on School Mathematics, McCoy stated, "As experimentation continued, it became evident to the curriculum makers that there were two primary considerations to keep in mind in the development of the courses. These considerations were (a) that the language of textbook and teacher be made as unambiguous as possible; and (b) that discovery of generalizations by the student be encouraged and fostered." The School Mathematics Study Group felt, "Teaching which emphasizes understanding, insight, and imagination, without neglecting basic skills, is the best for all students of whatever ability and makes the best preparation for any vocation which uses mathematics."  

Previous experiments in the field of mathematics, such as those of Fawcett, Schaaf, and Cummins have found the discovery method to be an effective means of instruction. All of these factors contributed

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6Oscar Schaf, "Student Discovery of Algebraic Principles as a Means of Developing Ability to Generalize" (unpublished Ph.D. dissertation, The Ohio State University, Columbus, 1952).

to the selection of this method as the principal type of instruction to be utilized in the instruction of the class reported in this experiment. Throughout the course the students were encouraged to develop their own mathematical structure of space by selecting certain undefined terms, developing definitions, and assuming certain statements which appear obvious. To provide the same basis for further study, common agreement was reached on the list of undefined terms, the definitions, and the assumptions, although there were instances where definitions and assumptions of minor importance were included by only a portion of the members of the class. Finally from the agreed upon terms and assumptions, implied propositions were proved.

No book was adopted as the official textbook for the class, but each student compiled his own by recording in a notebook the undefined terms needed, the definitions agreed upon, the assumptions which seemed feasible, the generalizations proved, and other information pertinent to the concept of space. To insure that the students were acquiring at least a minimum sequence of two-space geometry, the instructor introduced the principal concepts through study guides or class discussion, although frequently comments or questions from individual members of the class actually brought important ideas to the attention of the students. The students often recognized generalizations which were not required as a necessary part of the development of the course, but each was encouraged to incorporate his own ideas in the notebook, and as a result the books varied somewhat in content as each reflected the initiative of the individual.
To present an outline of the course developed by the experimental group and offer it as a proposal for future classes would be contrary to one of the purposes of this study—that of determining how successfully students might be expected to initiate their own ideas with respect to the integration of two- and three-dimensional space. One of the objections advanced by critics of a combination course, however, has been that too much material, ordinarily included in the separate courses, would be omitted. It may be profitable, therefore, for educators, interested in a one-year course in geometry, to know the actual two- and three-dimensional material studied in this particular course. Lists of undefined terms, definitions, assumptions, theorems for two-space, proposed ideas for three-space, and basic constructions are given in Appendix A. The number of theorems proved varied, but all students included, through proof or discussion, the twenty essential theorems of plane geometry which Christofferson considered as comprising a minimum list, and the thirty-one most important theorems listed in the 1933 syllabus published by the College Entrance Board.

The extension of the concept of space to dimensions other than two-space was accomplished primarily in three ways. At the beginning of the course, the idea of other dimensions was introduced by the teacher and from time-to-time during the year additional ideas were suggested, but in several instances the extension was made by the students without

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8Halbert Carl Christofferson, Geometry Professionalized for Teachers (Menasha, Wis.: George Banta Publishing Co., 1933), pp. 77-88.

leadership or other impetus from the teacher, and then there were also those occasions when the suggestion came as a combination of comments from the students and teacher. Frequently some question would arise or an idea would be presented by a student which would lead to difficulty in arriving at an answer or crystallizing the concept. When such a situation arose, it was sometimes expedient for the teacher to use the student's suggestion as the basis for a study guide. The discussion on the measurement of a polyhedral angle developed in this manner as will be shown later in this chapter as part of the illustrations of class activities. To give some indication of the origin of the content of the course, Appendix A also categorizes the items as (1) teacher-directed ideas, (2) student-teacher proposals, or (3) student developed ideas.

On a number of the study guides, the questions were worded in such a manner that the student's answer readily indicated whether he was limiting his replies to two-space or making the extension to other dimensions. If the concept being stressed in the guide were one which the teacher felt should be given consideration in more than two-space, later questions would frequently ask the student for his opinion concerning this concept in other spaces.

Many of the extensions to other dimensions were made by a process of analogy, a method commonly used in solving mathematical problems and one which lends itself well to a study of concepts involving different dimensions. By this process an individual looks for similarity in simpler situations and through a consideration of this more easily seen situation, he is able to come to a solution of the more complex problem.
Polya has defined the process, "Analogy is a sort of similarity. Similar objects agree with each other in some respect, analogous objects agree in certain relations of their respective parts."\(^{10}\)

It is rather simple to illustrate problems in two-space by an illustration on a plane, a sheet of paper, and problems in one-space may also be illustrated on paper, but those of three-space or more require other models and more imagination. To assist the students in their study of three-space, a set of three-dimensional models was always available in the classroom, along with other supplies which could be used to form models of ideas which might arise in the course of the year. For the discussion of planes, particularly intersecting planes, a set of dividers such as those found in packing cases for glasses proved most useful. Dowel pins and string also were helpful in many instances.

Since it was not the intent of the course to provide a rigorous treatment of three-space geometry, very few proofs were written in their entirety for three-space geometry. On occasion, through class discussion, the group outlined the procedure to follow in establishing a proof for a particular relationship. In other instances the students presented, in outline or paragraph form, their thoughts on the extension of certain concepts to other spaces, and in other cases they merely agreed that some generalizations appeared to follow. The students maintained a list of these generalizations in their notebooks.

To indicate to the reader the way in which some of the ideas of other space arose, the remainder of the chapter will be devoted to illustrations describing the actual situation. At the end of each section a list of two-space ideas will be given with the corresponding extensions to other spaces. Also included at the close of each section are the study guides which apparently had direct bearing on the extensions given in the section. Since not all of the study guides appeared to stimulate ideas in other spaces, the complete set of guides for the course is given in Appendix B.

Introduction to the Concept of Dimension

The first indication of students' thoughts on the concept of dimension appeared early in the course before any instruction on this topic had actually been presented to the class. In order to determine the ideas relating to the concept of dimension which the members of the class held before beginning a formal study of space, a test was constructed and administered during the early part of the semester.11

Two of the students remained after the regular class period and discussed the test. One of the boys had had his curiosity aroused and wondered how such things as fastening sheets of paper together might be accomplished in other dimensions. His question was, "How could you staple two sheets of paper together in a space of four dimensions?" He felt that as more dimensions were added a more complicated object might be needed, and the comment seemed to indicate that a world of more dimensions might create a change in even commonplace objects. The second

11A copy of the test is found in Appendix C.
student commented that although he found several of the questions confusing he did find that they stimulated his thinking and he wondered what it would be like to live in a world of some other dimension.

By the use of some non-mathematical situations and a few mathematical ideas from both two and three space, the class established the need for undefined terms, definitions, and assumptions in discovering and proving generalizations and thus found one of the primary purposes for studying geometry. The students were oriented to the idea of thinking in different dimensions by exercises presented in three study guides. At the conclusion of each guide, the material was discussed by the members of the class.

The purpose of the first guide (see p. 55) was to create an awareness in the student that as the range of activity increased more and more information was needed to pinpoint a situation. The exercises were limited to one-, two-, and three-dimensional experiences and no attempt was made to have the student go beyond these spaces.

Although there were various answers to the questions, in most instances the students recognized when one, two, or three pieces of information were needed. There was controversy on the exercise concerning the location of the basketball backboard. One student said, "You only need to know one dimension for locating the backboard." Several students challenged his answer and he replied by stating, "There is a regulation height, so all you need is the distance from one wall." Another student replied, "You may not be placing it according to regulations and anyhow you need to know the regulation, also the backboards
are not always against the wall—they aren't in our gym." By this time the class agreed that in some manner three dimensions were needed.

The situation concerning the car which had mechanical difficulty created some discussion. A couple of students said that, "All a person would know would be the highway number as no one checks the mileage for every town." Other members of the class responded with, "At least you would know which direction from town. The number of the highway and the direction would be like two dimensions."

The second of these guides (see pp. 56-57) was designed to emphasize the limitations which would be imposed if freedom of movement were restricted to certain spaces and that removal of these limitations would broaden a person's range of activity. In the first series of exercises all of the students recognized that the first two sets of triangles could be moved in a two-dimensional space so that they matched but that the third pair of triangles could not match with the two-space limitation. Only one student failed to recognize that by allowing the figure to move in a third dimension the two figures could be matched.

In answering question three, the students noted that one glove could fit inside the other only if one glove could be turned inside out. One student did qualify his statement by saying there would be no need to alter the glove if the front and back of the glove were identical. When this qualification was mentioned in class, several students disagreed with the idea since they called attention to the fact that the thumb was normally placed on the palm side. No student suggested a method for fitting the gloves unless some alteration or qualification were placed on the gloves.
All of the students agreed that the rubber band could be turned inside out either by cutting it or by allowing it to be lifted from the table and thus removing the two-space limitations. In the case of the hollow rubber ball, however, most of the students indicated only one method— that of cutting or piercing a hole in the ball. Two students did recognize the similarity between this situation and the rubber band and suggested that the ball would not need to be cut if it were possible to have a fourth dimension. It was observed in the class discussion that the majority of the class thought of time as the only possible fourth dimension. The teacher introduced the class to the ideas presented in the book, Flatland by Abbott,\(^{12}\) and some of them recalled stories seen on television where people could not be confined by the walls of a room but could seemingly pass through them without benefit of doors or windows. As a result the students began to conceive an extension of three dimensions into a new fourth dimension, and it was suggested by the teacher that the gloves in the earlier exercise might be likened to the third set of triangles so that movement into another dimension would allow them to be matched without alteration.

After this discussion, the teacher recommended that the students read the book, Flatland, or the condensation of it in The World of Mathematics.\(^{13}\) Within a week all but two or three students had read at least a portion of one of these books.

\(^{12}\)Edwin A. Abbott, Flatland (Boston: Little, Brown, 1926).

The last study guide of this group was one in which the student was to begin with the most simple element, a point, and attempt to build the elements of other dimensions from the ones previously determined, that is, from a point, the next element, a line, might be traced, from this one, the plane, and so on (see pp. 58-59). The students were instructed to attempt to answer the questions, then read Schnell and Crawford, Plane Geometry: A Clear Thinking Approach,114 and go back to the exercises to answer any which had been omitted.

All of the students answered the first three questions correctly and after reading the reference, they had no difficulty with the last question in the first group. The second group of questions presented little difficulty, but only four students were able to go beyond the first question in the third set even after reading the material suggested. With guidance from the teacher and by discussing and illustrating the figures, the students seemed to gain some insight into the development of more complex figures from the more simple elements.

Summary of Extensions of Concept of Dimension

<table>
<thead>
<tr>
<th>Zero-space</th>
<th>One-space</th>
<th>Two-space</th>
<th>Three-space</th>
<th>Four-space</th>
</tr>
</thead>
<tbody>
<tr>
<td>point</td>
<td>line</td>
<td>plane</td>
<td>solid</td>
<td>tessaract</td>
</tr>
</tbody>
</table>

1. Assume you have a friend who has just moved to a new city and you wish to visit him, but do not know the location of his home. It is necessary for you to inquire where he lives under the following conditions:

a. He lives in a very small town which is composed of one-family dwellings and all located on one street of the town. What information would you need?

b. He lives in a small city which is composed of one-family dwellings. What information would you need?

c. He lives in an apartment building in a large city. What information would you need?

2. a. You have been asked to hang a picture on the north wall of a particular room. What information will you need to hang the picture in the proper place?

b. A man has been asked to determine the location of a basketball backboard in a gymnasium. What information will he need to locate this backboard in the proper place?

3. a. You are driving on the highway and your car develops some type of trouble for which you need a mechanic. You place a call to a garage in the nearest town. What information will you give the garage attendant, so that he will be able to locate your car?

b. An airplane pilot needs to inform an airport of the location of his plane while he is in flight. What information will he need to radio to the airport?
Dimensions—2

1. Assume in each of the following cases that two triangles are formed by joining strips of cardboard and these are placed on a table in a position similar to the illustrations below. (Those parts which are lettered the same are equal in size.) Assume also that the figures may be moved by sliding from one position to another, but they cannot be lifted from the table.

a. Is it possible, in Fig. 1, to move one triangle so that it matches the other? 

![Fig. 1]

b. Is it possible to match the triangles in Fig. 2?

![Fig. 2]

c. Is it possible to match the triangles in Fig. 3?

![Fig. 3]

2. If the restriction of keeping the figures on the table is removed, would this change the answer to (a)? to (b)? to (c)? If it changes any answer, explain the change.
3. Assume you have a pair of gloves. Is it possible to place one glove so it will fit exactly inside the other (ignore thickness of material)? ___________ If your answer is "no," can you imagine any way in which it would be possible? ___________ Explain your answer.

4. Assume a rubber band is placed on a table so that one edge always remains on the table. Is it possible to turn this rubber band inside out? ___________ Assume you can cut the band in one place and fit it back together. Is it possible under these conditions to turn the band inside out? ___________ Assume the rubber band can be lifted from the table. Is it possible to turn the band inside out? ___________ If your answers to the first and last questions are different, what accounts for this difference?

5. Assume you have a rubber ball which is hollow. Is it possible to turn it inside out? ___________ If you can imagine certain conditions under which the ball may be turned inside out, explain them.
Dimensions -- 3

1. If we began with a zero-dimensional element, which we ordinarily call a point, and moved it in one direction only, its path would trace a __________. It would have the dimension, __________.

2. If we began with this new figure and moved it in one direction only, its path would trace a __________. It would have the dimension(s), __________.

3. If we began with the figure generated in exercise 2 and moved it in one direction only, its path would trace a __________. It would have the dimension(s), __________.

4. If we began with the figure generated in exercise 3 and moved it in one direction only, its path would trace a __________. It would have the dimension(s), __________.

Let us now consider the elements of which each of these figures which you have generated is composed.

1. The figure of zero dimension is composed of a __________.

2. The figure of one dimension is a __________ bounded by the beginning and ending or __________.

3. The figure of two dimensions is a __________ bounded by __________.

4. The figure of three dimensions is a __________ bounded by __________.

If you will notice in each case, you obtain a "new thing" between the previous "things."

1. Let the element of zero dimension be denoted by "p" and that of one dimension be denoted by "l." The elements of the figure of one dimension are __________ between __________.

2. Let the figure of two dimensions be denoted by "A." The elements of this figure are __________ between __________ but each of the latter is composed of __________; therefore, the elements of A include __________.

3. In similar manner analyze the figure of three dimensions, denoting it by "V."
Dimensions--3 (contd.)

4. In a manner similar to the above, analyze the figure of four dimensions. When a new figure is obtained, indicate some letter to denote this figure.

5. Can you extend this type of analysis to five dimensions?
Points, Lines, and Planes

Once the purposes of studying geometry had been established and the students had been introduced to the elements used in studying this subject, study guides presenting some of the elementary ideas concerning points and lines were considered (see pp. 67-68). Among these ideas the students agreed that two straight lines may intersect in one and only one point, but there are some types of lines, such as curved lines, which may intersect in several points, and others, such as parallel lines, which never intersect. At this time it seemed appropriate to establish definitions for these various types of lines and to take the opportunity to introduce a three-dimensional concept—skew lines. Since in the class discussion the students had more or less given their definition of parallel lines, a brief exercise was used to determine whether they felt they had carefully defined this term (see p. 69).

When the students submitted the completed guide, all of the students replied that they could give an illustration of two lines which would not intersect even if the lines were extended beyond the limits of the sheet of paper. On the second part of the guide, most of the class felt the definition of parallel lines as being "two lines which will not intersect no matter how far extended" was satisfactory, and they could not give "an illustration of two lines which do not intersect but neither do they appear as parallel." Three students, however, expressed other ideas which are described here.

1) The student drew two lines, \( \overrightarrow{AB} \) and \( \overrightarrow{CD} \), as shown at the top of the next page, and indicated that \( \overrightarrow{CD} \) was five feet behind \( \overrightarrow{AB} \).
2) One girl wrote, "If you had two lines, one about two feet below the other, but one is going east and west and the other north and south, they would not intersect, but neither would they be parallel lines."

3) A boy wrote, "If one line was five inches behind another line, and they were running up and down, they wouldn't intersect." He presented this illustration:

Since the majority of the class seemed satisfied with the definition presented in the study guide, the teacher pursued the problem by demonstrating the second illustration given above by holding two meter-sticks in the manner described. The students agreed that this representation fit their definition--one pointing out, "This is the way we de-
fined parallel lines in the eighth grade"—but was not what they really thought of as parallel lines. One girl wanted to know what to call the lines if they were not parallel lines. This comment allowed the teacher to introduce the term "skew lines." The class decided that the addition of the phrase "in the same plane" inserted in the definition would clarify the definition of parallel lines.

Upon further questioning it appeared the first two students who gave illustrations had a concept of skew lines, but the third student did not recognize that the situation he described would still be in one plane. It was pointed out to him that the plane need not be in horizontal position, and the figure might appear thus:

```
  +-----+
  |     |
  +-----+
  |     |
  +-----+
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For a definition for skew lines the proposal was "Skew lines are lines which do not intersect," but immediately the fallacy of this definition was shown by a student who commented that, "Then parallel lines are skew lines." Finally the revised definitions were:

Parallel lines are lines in the same plane which do not intersect no matter how far they are extended, and

Skew lines are lines, not in the same plane, which do not intersect no matter how far they are extended.
It was sometime later before the students realized that the last phrase in their definition of skew lines was redundant, since the lines could not intersect if they were in different planes. Likewise the phrase, "no matter how far they are extended," is not needed since the concept of a line implies no limit to the length.

The discussions in class had involved several ideas of points and lines in different planes, so that a study guide on "Planes" was distributed for study, to be followed by discussion the next day (see pp. 70-71). Many of the ideas presented in this guide followed closely ones which had been developed previously for points and lines in one plane, so the generalizations followed the same pattern as these earlier statements (see p. 67).

Before these generalizations were finally formulated, the replies to the individual questions were discussed by the group. On the earlier guide the students had stated that "any number of lines (infinite number of lines) may contain a given point;" while on the guide discussing planes, all but one student recognized that an infinite number of planes may contain either one or two points, and if a plane contains two points, the line determined by these points lies entirely within the plane. The students also found that unless the points are in a straight line, only one plane may contain three points. Consequently, they reached the generalizations that:

Any number of planes (infinite number of planes) may contain a given point.

One and only one plane may contain three points not in a straight line.
If two distinct points of a line are contained in a plane, then the line lies entirely in the plane.

When the group considered the case of two intersecting lines, most of them replied that two intersecting lines would lie in one plane, but two members of the class thought the lines could lie in many planes. An illustration with two sticks quickly convinced them of their error. Only one-third of the class felt that it would make no difference if one of the lines did not lie on the sheet of paper as they explained that the lines would still be intersecting lines and be contained in only one plane. Those who replied "yes" to the second portion of exercise 4 explained that if one line did not lie in the plane of the sheet of paper the two lines might not intersect, but if the lines continued to intersect, then one plane would contain both lines. Three students continued their explanations by stating that the lines might be skew lines and, therefore, not lie in the same plane. All of these comments led to the agreement that,

If two lines intersect, then they may be contained in one and only one plane.

In reference to the number of planes which may contain a given line and a given point, one student gave no reply. One-fifth of the remainder of the class stated that only one plane could contain these two elements regardless of whether the point were on the line or not. The other four-fifths indicated if the point were on the line, an infinite number of planes would contain these elements, but if the point were not on the line, only one plane would contain them. In the discussion one student noted the results of exercises 3 and 5 with respect
to each other by pointing out that two of the three points in exercise 3 would determine a line and then the two situations would be the same. Other students then mentioned the similarity between exercises 4 and 5, if one of the intersecting lines were considered along with a point, not the intersection, on the intersecting line.

The students found that only one plane would contain two lines which did not intersect, provided the lines were parallel; however, as one student expressed it, "If the lines were skew lines, there would be no plane." These last two exercises resulted in these conclusions:

A line and a point, not on the line, may be contained in one and only one plane, and

Two parallel lines may be contained in one and only one plane.

Earlier the class had agreed that "two straight lines may intersect in one and only one point." When the assumption, the intersection of two planes is a line, was reached, one boy said spontaneously, "and the intersection of two solids is a plane." A second boy answered, "Sure, that's easy to see," and stood two books on end indicating the plane they had in common. (Of course, these solids were only portions of solids, but this illustration seemed logical to the class.) The first student followed these comments with, "then this could just keep going and two four-dimensional objects would form a three-dimensional solid when they intersect."
Summary of Extensions of Points, Lines, and Planes

<table>
<thead>
<tr>
<th>Two-space</th>
<th>Three-space</th>
<th>Four-space</th>
<th>N-space</th>
</tr>
</thead>
<tbody>
<tr>
<td>Definition of parallel lines</td>
<td>Definition of skew lines</td>
<td>The intersection of two three-space solids is a</td>
<td>The intersection of two n-space solids is a</td>
</tr>
<tr>
<td>An infinite number of lines may contain a given point.</td>
<td>An infinite number of planes may contain a given point.</td>
<td>straight line.</td>
<td>(n - 1)-space figure.</td>
</tr>
<tr>
<td>Two distinct points determine one and only one straight line.</td>
<td>An infinite number of planes may contain two distinct points (a straight line).</td>
<td></td>
<td></td>
</tr>
<tr>
<td>If two distinct points of a line are contained in a plane, then the line lies entirely in the plane.</td>
<td>If two lines intersect, then they are contained in one and only one plane.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>One and only one plane may contain three non-collinear points.</td>
<td>A line and a point not on the line may be contained in one and only one plane.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Two parallel lines may be contained in one and only one plane.</td>
<td>Two parallel lines may be contained in one and only one plane.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Two-space</th>
<th>Three-space</th>
<th>Four-space</th>
<th>N-space</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two straight lines may intersect in one and only one point.</td>
<td>The intersection of two planes is a straight line.</td>
<td>The intersection of two three-space solids is a plane.</td>
<td>The intersection of two n-space solids is a (n - 1)-space figure.</td>
</tr>
</tbody>
</table>
Points—1

1. Mark a point, A, in the space below. Draw a straight line which has A as one of its points. Can you draw other lines through point A? ________ How many such lines can you draw? ________

2. Mark a point, A, and a second point, B, in the space below. Can you draw a straight line which contains both A and B? ________ If your answer is "yes," illustrate. How many lines can you draw such as this? ________ If one of these points were not on the paper would it be possible to connect these points by means of a string or similar device? ________

3. Mark three points, A, B, and C, in the space below. Can you draw a straight line which contains all three of these points? ________ If your answer is "yes," illustrate. If your answer is "no," how many straight lines can be located using these three points? ________ Illustrate.

4. Mark a point, A, and a second point, B, in the space below. Draw a line which contains both A and B. Draw three other such lines. (These need not be straight lines.) How many other lines could you draw? ________ Name your lines \( l_1, l_2, l_3 \), \ldots \. Which of your lines is the shortest? ________

5. Draw two straight lines which cross or intersect. Call this intersection, \( P \). What geometric figure does this intersection represent? ________ Using the same two straight lines, how many other intersections can you obtain? ________

6. In geometry we say a given set of conditions determine a geometric figure if one and only one such figure results from these conditions. Make a general statement for each of the ideas considered in the above exercises.
1. Two ways of naming a straight line have been suggested in class. These were using a lower case letter for the line and naming two points on the line. According to this agreement, may the line be named $AB$? $AC$? $BC$? 

How far may line be extended without changing its name? 

2. Sometimes we are concerned with only the part of the line between two specific points, such as A and B in the above figure. We call this a line segment, $AB$. If we speak of the line segment, $BC$, may this segment be extended? Write a definition for line segment.

3. Not all lines in geometry are straight lines. Two other types of lines are called broken line and curved line.

The figure below is an example of a broken line.

Define broken line.

The figures below are examples of curved lines.

Define curved line.

4. May two curved lines intersect in more than one point? In how many?
Lines--2

1. In a previous exercise you were asked to draw two lines which intersect. Can you draw a line which will not intersect $AB$? If your answer is "yes," illustrate.

$\text{A} \quad \text{B}$

When the line $\leftrightarrow AB$ is considered beyond the limits of this paper, will the lines intersect? 

2. Parallel lines are sometimes defined as two lines which will not intersect no matter how far extended. Does this fit your idea of parallel lines? Can you give an illustration of two lines which do not intersect but neither do they appear as parallel lines?
Planes—1

1. Mark a point, A, in the space below. How many planes may contain this point? ________

2. Mark two points, A and B, in the space below. How many planes may contain both these points? ________ Draw the straight line connecting these two points. How many of the planes contain this line? ________ How many planes contain only a portion of this line? ________

3. Mark three points, A, B, and C, in the space below. How many planes may contain all three points? ________ Does it make any difference if the points are in a straight line or not? ________ If so, how? If one of the points were not on this paper, how would it affect your answer? ____________________________________________________________

4. Draw two intersecting lines in the space below. How many planes may contain both these lines? ________ Does it make any difference if one of these lines does not lie on this paper? ________ If so, how? ____________________________________________________________

5. Draw a line and mark a point in the space below. How many planes may contain both the line and the point? ________ Does it make any difference if the point is on the line? ________ If so, how? If the point were not on this paper, how would it affect your answer? ____________________________________________________________

6. Draw two lines which do not meet regardless of their length. How many planes may contain both these lines? ________ If one of these lines were not on this paper, how would it affect your answer?

____________________________________
Planes--l (contd.)

7. Draw two planes which intersect. What is the geometric figure formed by this intersection? ________ Using the same two planes, how many other intersections can you obtain? ________

8. Make a general statement for each of the ideas considered in the above exercises.
Along with study guides on angles on a plane, a guide was presented introducing angles formed by intersecting planes (see pp. 79-80). Consideration was given to two intersecting planes and a definition for dihedral angle was formed. The first one suggested was that "a dihedral angle is the angle formed by the line of intersection of two planes," and the second was that "a dihedral angle is the angle formed by two intersecting planes." Some of the members of the class pointed out that the first definition implied that the angle was a line, and the second definition was adopted. The class also defined the edge and the faces of a dihedral angle.

From additional questions posed on the guide, the class observed that while lines may intersect in a point only, planes may intersect in a line or a point. The students recognized that two planes would intersect in a line, but three or more planes might intersect in either a line or a point, and to care for those which intersect in a point, a new type of angle—a polyhedral angle—needed to be defined. The suggested definition, "A polyhedral angle is an angle formed by the intersection of three or more planes at a given point," appeared satisfactory until it was illustrated that if these planes did not enclose space, no angle was formed. The definition was modified to clarify the statement.

Once the students had established the definition they were able to see numerous illustrations of polyhedral angles. Some of those suggested were a tepee, two walls and the ceiling of a room, and a pyramid. It was left to the students to determine how to name a polyhedral angle.
Several of them were uncertain, but two students listed the generally adopted form; however, one of these admitted he had seen the method used in a book. The more common suggestion was to name the intersection of a face with the plane of the base, then the vertex, and finally another intersection. For example,

![Diagram of a polyhedron with vertices A, B, C, D, and E, and vertex V, with notation AB-V-BC.]

The students making this suggestion felt this notation was more in line with the notation for dihedral angles—where the name of a point on one of the intersecting planes is given, then the intersection of the two planes, and finally the name of a point on the second plane—than the more conventional method. As the two methods were compared it was noted that the usual notation, V-ABCDE, did indicate the number of faces of the polyhedral angle while the method suggested by these students, AB-V-BC, did not make this point clear; therefore, it was agreed that the conventional method would be used.

The optional exercise which asked the student to determine the number of dihedral angles formed by special polyhedral angles was answered correctly by most of the students.
At the close of class one girl asked how a polyhedral angle was measured, so this was incorporated in a study guide for the following day. Rather than limiting the measurement to that of polyhedral angles, measurement of dihedral angles was also included.

The students were already familiar with measuring angles on a plane and with the names for angles depending on the size of the angle, so this was briefly reviewed (see p. 85). Most of the students showed good insight into the measurement of dihedral angles. They were asked to fold a piece of cardboard, forming a 60° angle, and were to explain how to determine that the angle contained the specified number of degrees. A sample of their answers includes the following:

1) Determine the size of the angle by its edges.
2) The angle would be determined by the angle of the lines forming the plane.
3) Measure it from the side.
4) The angle would be determined by the side view, and it would be measured like an angle on a plane.
5) Measure the angle between the two faces of the angle.
6) Draw just the two planes at a side view and it would look just like a plane angle and measure it.

The assumption the students had made without realizing it was that the edges of the cardboard would be perpendicular to the fold. This point was clarified so that the class finally agreed that a dihedral angle would be measured by the plane angle. They also recognized that neither the length of the edge nor the size of the faces would affect the size of the dihedral angle.
In response to the question, "What parts of a polyhedral angle affect its size?" the answers fell in the following categories:

1) Number of faces - 2 responses
2) Size of the face angles - 1 response
3) Number of faces and the size of the face angles - 7 responses
4) No idea expressed - 6

Some suggested patterns were given and the students were to make models using these suggestions. One of the patterns suggested face angles having measures of $20^\circ$, $30^\circ$, $50^\circ$. Several students commented that this did not give them a polyhedral angle. One of the boys stated that in order to have a polyhedral angle there must be at least three face angles and each face angle must measure less than the sum of the measures of the other angles. One of the girls thought this was not the case, but as she described her idea it resulted that the figure would be inverted and this brought out the point that the sum of the measures of the face angles would be less than $360^\circ$ if a polyhedral angle were formed.

When the students were studying adjacent angles (see pp. 82-84), the question, "Are there similar dihedral angles? Polyhedral angles? Illustrate." was raised. Only about one-fourth of the class wrote an opinion of these questions, but in class one boy suggested that a good illustration of adjacent dihedral angles could be shown using an open book. The class quickly recognized the similarity of using "common edge" for "common vertex" and "common face" for "common side." They did not see a similarity in polyhedral angles.
At the time definitions were proposed for complementary and supplementary angles, several students noted that similar situations could exist for dihedral angles, so they included definitions for complementary dihedral and supplementary dihedral angles in their notebooks (see pp. 86-87).

In consideration of the guide on vertical angles (see pp. 88-90), the students arrived at two definitions when the angles were on a plane:

1) Vertical angles are two non-adjacent angles formed by the intersection of two lines.
2) Vertical angles are angles with a common vertex but not a common side.

Obviously the second definition was too broad to accept so the first one was selected.

At this point in the discussion, one of the boys asked, "Why can't we have vertical dihedral angles when two planes intersect?" This idea was readily acceptable to the class, and they considered under what conditions two polyhedral angles may be classified as vertical. Someone referred back to the earlier discussion of the situation where the polyhedral angle was inverted. In the instance of vertical polyhedral angles it was decided the edges would be extended through the vertex; each face angle would have a corresponding angle in the same plane; and these new face angles would form the vertical polyhedral angle, its face angles being in the same order.
Among the first propositions proved, these were included:

1) Complements of equal angles are equal.
2) Supplements of equal angles are equal.
3) Vertical angles are equal.

Most of the students explained the similar situations for dihedral and polyhedral angles.

Summary of Extensions of Angles

<table>
<thead>
<tr>
<th>Two-space</th>
<th>Three-space</th>
</tr>
</thead>
<tbody>
<tr>
<td>angle (plane)</td>
<td>dihedral angle</td>
</tr>
<tr>
<td>vertex of an angle</td>
<td>edge of a dihedral angle</td>
</tr>
<tr>
<td>sides of an angle</td>
<td>faces of a dihedral angle</td>
</tr>
<tr>
<td>measure of an angle</td>
<td>face angle</td>
</tr>
<tr>
<td>adjacent angles</td>
<td>measure of dihedral angle</td>
</tr>
<tr>
<td>complementary angles</td>
<td>measure of polyhedral angle</td>
</tr>
<tr>
<td>supplementary angles</td>
<td>adjacent dihedral angles</td>
</tr>
<tr>
<td>vertical angles</td>
<td>complementary dihedral angles</td>
</tr>
<tr>
<td></td>
<td>supplementary dihedral angles</td>
</tr>
<tr>
<td></td>
<td>vertical dihedral angles</td>
</tr>
<tr>
<td></td>
<td>vertical polyhedral angles</td>
</tr>
</tbody>
</table>

The sum of the measures of the face angles of a polyhedral angle is less than $360^\circ$, and the measure of each face angle is less than the sum of the measures of the other face angles.

Complements of equal angles are equal.
Two-space (contd.)

Supplements of equal angles are equal.

Vertical angles are equal.

Three-space (contd.)

Supplements of equal dihedral angles are equal.

Vertical dihedral angles are equal.

Vertical polyhedral angles are equal.
Angles--3

1. We found that two rays which have one endpoint in common form an angle. When two half-planes have a line in common, they also form an angle. This angle is called a **dihedral angle**. Define *dihedral angle*.

Both of the figures below show illustrations of dihedral angles. Will your definition apply to these figures? ______ If not, revise your definition.

2. The common endpoint of the two rays forming an angle is the vertex of the angle. In a dihedral angle what part of the angle corresponds to this vertex? ______ This is called the **edge** of the dihedral angle. Define *edge of a dihedral angle*.

3. The two rays are the sides of the angle. In a dihedral angle what figures correspond to the sides of the angle? ______ These are called **faces** of the dihedral angle. Define *face of a dihedral angle*.

4. The angle (Fig. 1) we have agreed to call $\angle AOB$. The dihedral angle (Fig. 2) is called $\angle M\text{-}AB\text{-}N$. Is M common to both planes? ______ Is AB? ______ Is N? ______

Fig. 1

Fig. 2
5. Name the dihedral angles below.

6. We found that when two geometric figures intersect they form the figure with one less dimension. We also found an infinite number of planes may contain the same point. May three planes intersect in a line? ________ If "yes," give an illustration. May three planes intersect in a point? ________ If "yes," give an illustration.

What is the minimum number of planes that may meet in a point? ________ The maximum? ________

7. When a number of planes intersect in a point, the angle formed is a **polyhedral angle**. Define polyhedral angle.

8. What parts of a polyhedral angle need to be defined?

9. The figure below represents a polyhedral angle. How would you name this angle? ________
Optional:

Polyhedral angles are named according to the number of planes that meet to form the angle as:

- trihedral angle: 3 planes
- tetrahedral angle: 4 planes
- pentahedral angle: 5 planes
- hexahedral angle: 6 planes

10. How many dihedral angles would be formed when a trihedral angle is formed? __________

How many dihedral angles would be formed when a tetrahedral angle is formed? __________

How many dihedral angles would be formed when a pentahedral angle is formed? __________

How many dihedral angles would be formed when a hexahedral angle is formed? __________

State the relationship between the number of planes which form the polyhedral angle and the number of dihedral angles.
Angles—4

Study each of the diagrams below and answer the accompanying exercises.

1. Name the vertex of angle 1. ________
   Name the vertex of angle 2. ________
   Name the sides of angle 1. ________
   Name the sides of angle 2. ________

   Use vertical lines to shade the interior of angle 1. Use horizontal lines to shade the interior of angle 2.

   Do these two angles have any elements in common? ________
   If "yes," what are they? __________________________

2. Name the vertex of angle 3. ________
   Name the vertex of angle 4. ________
   Name the sides of angle 3. ________
   Name the sides of angle 4. ________

   Use vertical lines to shade the interior of angle 3. Use horizontal lines to shade the interior of angle 4.

   Do these two angles have any elements in common? ________
   If "yes," what are they? __________________________
3. Name the vertex of angle 5. ______
   Name the vertex of angle 6. ______
   Name the sides of angle 5. ______
   Name the sides of angle 6. ______

   Use vertical lines to shade the interior of angle 5. Use horizontal lines to shade the interior of angle 6.

   Do these two angles have any elements in common? ______
   If "yes," what are they? ________________________________

4. Name the vertex of angle 7. ______
   Name the vertex of angle 8. ______
   Name the sides of angle 7. ______
   Name the sides of angle 8. ______

   Use vertical lines to shade the interior of angle 7. Use horizontal lines to shade the interior of angle 8.

   Do these two angles have any elements in common? ______
   If "yes," what are they? ________________________________
5. Name the vertex of angle 9. _______
   Name the vertex of angle 10. _______
   Name the sides of angle 9. _______
   Name the sides of angle 10. _______

   Use vertical lines to shade the interior of angle 9. Use horizontal lines to shade the interior of angle 10.

   Do these two angles have any elements in common? _______
   If "yes," what are they? ______________________________________

6. Pairs of angles such as angle 9 and angle 10 are called adjacent angles, but the pairs in the other drawings do not satisfy all the necessary conditions. Write a definition of adjacent angles.

Angles—5

1. The unit of measure used to express the size of an angle is called a degree. A complete rotation of a ray about its endpoint is said to contain 360 degrees. What part of a complete rotation is one degree? ________ One-fourth of a complete rotation contains ________ degrees. What part of a complete rotation is 75 degrees? ________ 135 degrees? ________ What effect does the length of the sides of an angle have on its size? ________

2. If a series of adjacent angles were placed to just fill the space about the endpoint of a ray, what would be the sum of the measures of these angles? ________

3. Do you think dihedral angles could be measured in a manner similar to that of angles on a plane? ________ Suppose you folded a piece of cardboard to form a dihedral angle of 60 degrees. Explain how to determine the measure of the angle is 60 degrees. (Does the length of the edge affect the size? Does the size of the faces affect the size of the angle?)

4. If a series of dihedral angles having a common edge just filled the space about the common edge, what would be the sum of these angles? ________

5. What parts of a polyhedral angle affect its size? ________ Does the number of faces affect the size? ________ Does the size of the face angles affect the size? ________

6. On a piece of cardboard or stiff paper, draw figures similar to the patterns below. Make the straight lines about 3 inches long and the angles the sizes indicated.

Cut out your figures and fold on the lines. Attach the two outside edges with scotch tape to form polyhedral angles. Can the face angles be any size? Explain your answer.
Angles—6

Measure each of the angles to the nearest degree and record the measurement near the vertex of the angle.

![Fig. 1](image1)

![Fig. 2](image2)

![Fig. 3](image3)

![Fig. 4](image4)

![Fig. 5](image5)

![Fig. 6](image6)

![Fig. 7](image7)

Angles such as those in figures 1, 2, 4, 6, and 7 are called complementary angles. Do you see any common relationship existing among these pairs of angles? If so, what is it?

Write what you consider to be an acceptable definition of complementary angles.
Angles—7

Measure each of the angles to the nearest degree and record the measurement near the vertex of the angle.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Angles such as those in figures 1, 2, 3, and 6 are called supplementary angles. Do you see any common relationship existing among these pairs of angles? ______ If so, what is it? ________________________________

Write what you consider to be an acceptable definition of supplementary angles.
Angles—8

1. Name the vertex of angle 1. ______

2. Name the vertex of angle 2. ______

3. Name the sides of angle 1. ______

4. Name the sides of angle 2. ______

5. Do these angles have a common vertex? If so, what? ______

6. Do any sides of angles 1 and 2 lie in a straight line? If so, which ones? ______

7. Name the vertex of angle 3. ______
   Of angle 4. ______ 
   Of angle 5. ______ 
   Of angle 6. ______

8. Name the sides of angle 3. ______
   Of angle 4. ______
   Of angle 5. ______
   Of angle 6. ______

9. Do angles 3 and 5 have a common vertex? If so, what? ______

10. Do angles 4 and 6 have a common vertex? If so, what? ______

11. Do any of the sides of angles 3 and 5 lie in a straight line? If so, which ones? ______

12. Do any of the sides of angles 4 and 6 lie in a straight line? If so, which ones? ______
Angles--8 (contd.)

13. Name the vertex of angle 7. ______
   Of angle 8. ______
   Of angle 9. ______
   Of angle 10. ______

14. Name the sides of angle 7. ______
   Of angle 8. ______
   Of angle 9. ______
   Of angle 10. ______

15. Do these four angles have a common vertex? ________
    If so, what? ________

16. Do angles 7 and 9 have a common side? ________
    If so, what? ________

17. Do angles 8 and 10 have a common side? ________
    If so, what? ________

18. Do any of the sides of angles 7 and 9 lie in a straight line? ________
    If so, which ones? ________

19. Do any of the sides of angles 8 and 10 lie in a straight line? ________
    If so, which ones? ________

Angles 7 and 9 are called vertical angles, as well as angles 8 and 10. None of the other figures have vertical angles. Define vertical angles.

20. Measure the angles in the above three figures.

   Angle 1 ________ Angle 6 ________
   Angle 2 ________ Angle 7 ________
   Angle 3 ________ Angle 8 ________
   Angle 4 ________ Angle 9 ________
   Angle 5 ________ Angle 10 ________
21. Draw two straight lines $AB$ and $CD$, intersecting at $O$. Measure each of the angles formed. Are any of these angles equal? If so, which ones?

Perpendicular Lines

Since a definition for perpendicular lines had been agreed upon, the class was given a few exercises to consider which would eventually lead to some assumptions concerning perpendicular lines (see p. 95). The questions did not specify whether the lines were to be considered in two- or three-dimensional space. As the class worked, numerous students were observed going through motions to determine the results if three dimensions were considered. More than three-fourths of the class recognized the limitations of two dimensions without suggestions from the other members of the class or the teacher. As soon as these people made their comments in the class discussion the remainder of the group also realized that three dimensional space gives a wider breadth of ideas.

For the exercise, "Draw a line AB and select a point P on the line. How many lines can be perpendicular to the line AB at point P?" one of the boys pointed out the answer depended on whether a plane or all space was to be considered. It was agreed that two statements were needed since on a plane only one line could be perpendicular to a given line at a given point on the line, while, in three-space, an infinite number could be perpendicular. In determining where these lines would be located, one boy replied, "They would be on a circle." Several students disagreed because the lines would not be limited in length, and the class finally decided the lines would be on a plane perpendicular to the line at the point. This conclusion brought out the necessity of defining when a line and plane were perpendicular. It finally appeared that in order for a line and plane to be perpendicular the
line would have to be perpendicular to all the lines in the plane which passed through the point of intersection of the line and the plane.

From this first exercise, two assumptions developed.

1) On a plane one and only one line may be perpendicular to a given line at a given point.

2) All lines perpendicular to a given line at a given point lie on a plane perpendicular to the line at the point.

In the second exercise, "Draw a line AB and select a point P not on the line. How many lines through the point would be perpendicular to the line AB?" it appeared that whether P were on the plane or not the number of lines perpendicular to line AB through P would still be one. The assumption agreed upon, therefore, was:

One and only one line may be perpendicular to a given line from a given point not on the line.

The third exercise, "Consider a plane, M, and select a point, P, on the plane. How many lines can be drawn perpendicular to the plane at P?" led to the statement:

One and only one line may be perpendicular to a given plane at a given point on the plane.

The students expressed the following additional observations on perpendicularity:

1) Two girls mentioned that if two planes formed a right dihedral angle the planes would be perpendicular. This comment raised the question, "When are two planes perpendicular?"

The girls explained their statement by referring to the earlier statement that the lines which meet to form a right angle on a plane are perpendicular-
ular lines, so they extended this idea to the two planes which form a right dihedral angle and looked upon these planes as being perpendicular. Their explanation satisfied the other members of the class.

2) Two boys asked if a line could be perpendicular to a solid.

One of them had illustrated the idea with a drawing.

![Diagram](https://via.placeholder.com/150)

The class felt this drawing really showed a line perpendicular to a plane of a solid. The second boy decided there was no way to have the point outside the solid unless one were in fourth dimensional space. It was agreeable to the class that in four-space it would probably be possible to have a line perpendicular to a three-space solid.

3) Another boy suggested that a line perpendicular to one of two parallel lines was perpendicular to the other. This statement was later proved by some members of the class.

4) A fourth boy attempted to visualize a line perpendicular to two skew lines, but found this rather difficult to explain. Finally a classmate suggested a vertical line at the corner of the classroom would be perpendicular to the intersection of the ceiling and one end of the classroom and also perpendicular to the intersection of the floor and a wall adjacent to the first wall mentioned.
5) Two other boys suggested that one and only one line may be perpendicular to a given plane from a point outside the plane.

Summary of Extensions of Perpendicular Lines

<table>
<thead>
<tr>
<th>Two-space</th>
<th>Three-space</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>perpendicular lines</strong></td>
<td><strong>line perpendicular to a plane</strong></td>
</tr>
<tr>
<td><strong>distance from a point to a line</strong></td>
<td><strong>perpendicular planes</strong></td>
</tr>
<tr>
<td><strong>On a plane one and only one line may be perpendicular to a given line at a given point on the line.</strong></td>
<td><strong>distance from a point to a plane</strong></td>
</tr>
<tr>
<td><strong>One and only one line may be perpendicular to a given line from a point not on the line.</strong></td>
<td><strong>All lines perpendicular to a given line at a given point lie on a plane perpendicular to the line at the point.</strong></td>
</tr>
<tr>
<td><strong>If two lines meet to form right angles, the lines are perpendicular.</strong></td>
<td><strong>One and only one line may be perpendicular to a given plane at a point on the plane.</strong></td>
</tr>
<tr>
<td><strong>If one of two parallel lines is perpendicular to a third coplanar line, the other is also perpendicular to the third line.</strong></td>
<td><strong>If two planes meet to form right dihedral angles, the planes are perpendicular.</strong></td>
</tr>
<tr>
<td><strong>A line may be perpendicular to each of two skew lines.</strong></td>
<td></td>
</tr>
</tbody>
</table>
Perpendicular Lines—2

Recall the definition for perpendicular lines and consider the assumptions which may follow from the exercises below.

1. Draw a line AB and select a point P on the line. How many lines can be perpendicular to the line AB at point P?

2. Draw a line AB and select a point P not on the line (about one inch above). How many lines through this point would be perpendicular to the line AB?

3. Consider a plane, M, and select a point, P, on the plane. How many lines can be drawn perpendicular to the plane at P?

4. Can you think of other conditions which might arise concerning perpendicular lines? If so, discuss them.
Reaching Conclusions

From time to time the students were assigned series of exercises to prepare. These exercises did not necessarily serve as a portion of the sequence of the geometry course, but were designed to give the student the opportunity to apply some of the definitions, assumptions, or generalizations which he had formulated to specific applications. In some instances a student might make some observation in his notebook.

Before introducing the student to a formal type of proof, it seemed feasible to acquaint him with situations in which he would be expected to justify a particular conclusion by stating a definition or assumption which was applicable. In this specific set of exercises a statement was given and a conclusion followed from it which was to be justified (see pp. 97-99). Although most of the exercises dealt with situations pertaining to two-space geometry, some dealing with three-space were included in the regular assignment as well as in the optional section. Although no new assumptions were discovered, the class recognized the assumptions needed in the three-space situations as readily as those for the two-space exercises.
Definitions and Assumptions

In each of the following exercises quote, in full, the definition or assumption which explains why the conclusion follows from the specific statement. If you do not have a definition or assumption for a specific problem, first write "None" and then attempt to frame a new definition or assumption for the list prepared in class.

1. \( x = 12 \)
   \[ .\ 5x = 60 \]
   \[ .\ 5x = 60 \]

2. \( 4a + 7 = 19 \)
   \[ .\ 4a = 12 \]
   \[ .\ 4a = 12 \]

3. \( y - 8 = 14 \)
   \[ .\ y = 22 \]
   \[ .\ y = 22 \]

4. \( a = b \), and \( b = 6 \)
   \[ .\ a = 6 \]
   \[ .\ a = 6 \]

5. \( \triangle AC = FE \)
   \[ \angle A = \angle F \]
   \[ \angle C = \angle E \]
   \[ .\ \triangle ACB \cong \triangle FED \]

6. \( \angle 1 + \angle 2 = 90^\circ \)
   \[ \angle 3 + \angle 2 = 90^\circ \]
   \[ .\ \angle 1 = \angle 3 \]

7. \( \angle 1 + \angle 2 = 180^\circ \)
   \[ \angle 3 + \angle 2 = 180^\circ \]
   \[ .\ \angle 1 = \angle 3 \]

8. \( \triangle RTS \)
   \( RT + TS > RS \)
Definitions and Assumptions (contd.)

9. \( \overrightarrow{MN} \perp \overrightarrow{AB} \)
   \[ \therefore \angle 1 = \angle 2 \]

10. \( \overrightarrow{PQ} \) and \( \overrightarrow{RS} \) are intersecting lines
    \[ \therefore \angle 1 = \angle 2 \]

11. \( \overrightarrow{PD} \perp \overrightarrow{AB} \)
    \[ \therefore \overrightarrow{PC} \) is not perpendicular to \( \overrightarrow{AB} \)

12. \( a = c \) and \( b = a \)
    \[ \therefore \ b = c \]

13. \( \overrightarrow{PB} \perp \) plane \( M \)
    \[ \therefore \overrightarrow{PA} \) is not perpendicular to \( \) plane \( M \)

14. Intersecting planes \( M \) and \( N \)
    \[ \therefore \overrightarrow{AB} \) is a straight line \]
Definitions and Assumptions (contd.)

15. \( AB = CB \)

\[ \therefore \triangle ABC \text{ is isosceles} \]

Optional:

In most of the following problems there is not enough specific information to reach the conclusion. Supply the minimum amount of information which will make it possible to reach the given conclusion and then tell why the conclusion follows.

1. \( AB = LM \)

(two possibilities)

\[ \therefore \triangle ABC \cong \triangle LMN \]

2. \( AB = BC \)

\[ \therefore \triangle ABC \text{ is equilateral} \]

3. \( \angle A = \angle X \text{ and } \angle B = \angle Y \)

\[ \therefore \triangle ABC \cong \triangle XYZ \]

4. Plane angle \( \triangle ABC \) of dihedral angle \( M-DE-N \) and plane angle \( \triangle RST \) of dihedral angle \( P-XY-Q \)

\[ \therefore \text{dihedral angles } M-DE-N \text{ and } P-XY-Q \text{ are equal} \]

5. Isosceles triangles \( \triangle ABC \) and \( \triangle RST \)

\( AB = BC \text{ and } RS = ST \) and

\[ \angle B = \angle S \]

\[ \therefore \triangle ABC \cong \triangle RST \]
Parallel Lines

Now that the class had established a number of needed definitions and assumptions and had experienced some opportunity to prove simple generalizations, a study guide on relationships which exist between different types of angles with respect to parallel lines was distributed (see pp. 110-111). Before establishing these relationships there were a few terms to be defined such as transversal, exterior angles, interior angles, alternate-exterior and alternate-interior angles, and corresponding angles.

In the discussion concerning these definitions, the comment was made, "If a transversal is a line which intersects two other lines, then you could have two perpendicular lines in a plane and then a line in another plane perpendicular to the given plane at the point of intersection would be a transversal—indeed the lines would not need to be perpendicular. For that matter, couldn't the lines all be in the same plane?" In other words the question really became whether one of three lines intersecting in the same point may be considered a transversal with respect to the other two lines. Along with these comments, a girl asked, "Could the transversal intersect two lines which were not in the same plane?" One of the boys immediately replied that in this case the lines would need to be skew lines because intersecting or parallel lines determine a plane; therefore, except in the case of skew lines, the two lines would lie in the same plane. Examples were given which showed that a line may intersect two non-coplanar lines. One of the illustrations suggested was that the line formed by the intersection of two walls of the classroom would intersect the line formed by the inter-
section of one of the walls and the floor and would also intersect the
line formed by the intersection of the other wall and the ceiling.

One other question which a few students asked was whether a
transversal could cut more than two lines. It was pointed out that
this was the situation in Fig. 5 of the exercise, and since there
appeared to be no reason to limit the transversal to intersecting two
lines only, the definition agreed upon was:

A transversal is a line which intersects two or more lines in
the same plane.

When this definition was used it seemed the question concerning the
three lines intersecting in a point would be a special case and since,
on the whole, the relationships were concerned with parallel lines, it
would not alter these statements.

The other definitions and relationships did not present any
notable controversy, so each student was expected to prove as many of
the relationships stated in exercises 8, 12, 14, and 15 as he could.

One other question did arise, however, when several of the stu-
dents suggested that it might be possible to have a transversal plane
which would intersect two or more planes, and then it appeared that all
the statements made concerning lines and plane angles could be dupli-
cated using planes and dihedral angles. One student mentioned that the
class had not established conditions under which two planes were par-
allel, and it would, therefore, be necessary to define parallel planes
before the suggested extensions could be made.

Since the problem of parallel planes and lines parallel to planes
had arisen on one or two other occasions when it was not feasible to
pursue the subject, it appeared that now was the time to give some
attention to questions on parallel lines and planes (see p. 112).

Due to several days vacation the answers to the questions were
collected and when class reconvened a list of the answers submitted was
distributed to the class (see pp. 113-114).

The first two questions of the study guide dealt primarily with
determining the number of lines which may be drawn parallel to a given
line through a point not on the given line, although the question simply
asked for the number which would not intersect the given line. Most of
the students recognized two situations—if the lines were to be on a
plane, then there would be only one line which would not intersect the
given line, but if the situation were not limited to a plane, then there
would be an infinite number of lines which would not intersect the given
line. The students realized that in the latter case one line would be
parallel and the others would be skew lines. As a matter of fact all
the students mentioned the second case, but one student failed to note
the limitations of considering lines on a plane only. After some mod-
ification of the suggested statements, agreement was reached on the
assumption:

On a plane one and only one line may be parallel to a given line
and pass through a point not on the line. However, an infinite
number of non-intersecting lines may be drawn through the given
point if the lines are not in the same plane as the given line.

The students then considered a definition for parallel planes and
also the conditions under which a line would be parallel to a plane.
For the former they agreed:

If two planes do not intersect no matter how far they are extended, then they are parallel.

But one student suggested that perhaps the planes could be skew planes and that the phrase, "in the same solid," should be added, making the definition,

If two planes, in the same solid, do not intersect no matter how far they are extended, then they are parallel.

Another student interrupted to ask, "How could you have skew planes?--but maybe you could in the fourth dimension." Although the students could not visualize this situation, they did feel it was important enough to consider in the formulation of the statement. For the second situation they decided on the statement:

If a line and a plane do not intersect and the line is not on the plane, they will be parallel.

The portion, "and the line is not on the plane," may seem redundant but several members of the class felt it necessary to emphasize this point, since some may consider that a line which lies in a plane does not intersect it. It was finally agreed that both statements would be more clearly expressed as,

If two planes (or a line and a plane), in the same solid, have no points in common, then they are parallel.

Further observations of the class were that if two planes were parallel then every line in the one plane would be parallel to the other plane. One student specified that "they would not necessarily run in the same direction." The class challenged him to explain what he meant
by the direction of a plane since a plane did not have limits in either direction. It then became evident that the student was actually considering the relationship between lines in the two planes rather than the entire plane in the one case. This clarification actually led to the next question where the students were to comment on the relationship between the lines of two parallel planes. The summary of comments here indicated that all of the students concluded that the lines of the two planes would not have points in common, that is, they would be skew or parallel lines.

In answer to the question, "If a line is parallel to a plane, what is the relationship between the given line and the lines in the given plane?" a variety of ideas were expressed, including:

1) The given line would be parallel to all the lines in the given plane.
2) The given line and the lines in the given plane would be either parallel or skew lines.
3) The given line would be parallel to only one line in the plane.

It was a fairly simple matter for those who had given condition (2) to give illustrations which would show that (1) and (3) did not hold for all cases.

The students were also to comment on "If a line is parallel to a plane, what relationship exists between the plane and any plane containing the given line?" Here again several ideas were submitted, including:

1) The planes would be parallel.
2) There would be one and only one plane containing the given line and parallel to the given plane.

3) The planes would either be parallel or intersect.

Those who had expressed (2) or (3) readily agreed with each other and gave illustrations to show those who gave (1) that they had considered only one case. It was also noted by several members of the class that:

Through any point outside a given plane, there is one and only one plane parallel to the given plane, and

If a line, not in a plane, is parallel to a line in the plane, it is parallel to the plane.

The next day a few additional ideas were proposed by the teacher.

Using a portion of a cardboard packing divider as a model, she held it before the class in such a manner that two planes were parallel and the third was perpendicular to them. The students were to note any relationship which appeared to exist between the lines of intersection. The students recognized that in this position the lines would be parallel, but several doubted this relationship would hold if the third plane were not perpendicular to the parallel planes. By collapsing the model, it was easily shown this limitation of a perpendicular plane was not necessary. Furthermore, one boy explained that if the two lines of intersection on the perpendicular plane were to intersect, the planes would intersect and, therefore, the planes would no longer be parallel.

When the model was collapsed, the parallel planes accidentally slipped and intersected. Immediately three of the students noticed the lines were still parallel, so they were interested in seeing whether the
intersections of three planes, taken in pairs, would always be parallel.

A representation of this model would appear somewhat as follows:

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It was reasoned that there would be three lines of intersection, but if the planes were to intersect in pairs, only, then one of the three lines of intersection would not be contained in one of the planes making up the pair of intersecting planes, so it would be parallel to that plane.

At this point in the discussion, one student reminded the class of the original question which had started the class on the study of parallel planes, and noted that here was a model of two parallel planes intersected by a third, or transversal, plane. If one considered the conditions which had been established for parallel planes and also recalled the manner in which dihedral angles are measured, then statements comparable to those proved for a plane (see pp. 110-111) could also be made for three-dimensional space.
Several days later a student mentioned that on at least two occasions he had proved that two lines perpendicular to the same line are parallel by showing two corresponding or alternate-interior angles equal, and he thought it would be profitable for the class to consider formulating a theorem along these lines. Some of his classmates adopted his suggestion, but one of them recognized that two lines might be perpendicular to the same line, but not be parallel, if the lines were skew lines. On the other hand if the lines were in the same plane, they would be parallel. The statements, therefore, were phrased:

If two lines, in a plane, are perpendicular to the same line, then they are parallel.

If two planes were perpendicular to the same line, they would be parallel.

Summary of Extensions of Parallel Lines

<table>
<thead>
<tr>
<th>Two-space</th>
<th>Three-space</th>
</tr>
</thead>
<tbody>
<tr>
<td>parallel lines</td>
<td>parallel planes</td>
</tr>
<tr>
<td>distance between parallel lines</td>
<td>skew planes</td>
</tr>
<tr>
<td>transversal (line)</td>
<td>line parallel to a plane</td>
</tr>
<tr>
<td>alternate angles</td>
<td>distance between parallel planes</td>
</tr>
<tr>
<td>corresponding angles</td>
<td>transversal (plane)</td>
</tr>
<tr>
<td>exterior angle</td>
<td>alternate dihedral angles</td>
</tr>
<tr>
<td>interior angle</td>
<td>corresponding dihedral angles</td>
</tr>
<tr>
<td></td>
<td>exterior dihedral angle</td>
</tr>
<tr>
<td></td>
<td>interior dihedral angle</td>
</tr>
</tbody>
</table>
Two-space (contd.)

On a plane one and only one line may be parallel to a given line and pass through a point not on the line.

Three-space (contd.)

An infinite number of non-intersecting lines may be drawn through a given point if the lines are not in the same plane as the given line.

Through any point outside a given plane, there is one and only one plane parallel to the given plane.

If two parallel lines are cut by a transversal, the corresponding angles are equal.

If two parallel lines are cut by a transversal, then it is parallel to every line in the plane having the same direction and skew to all other lines in the plane.

If two parallel lines are cut by a transversal making the corresponding angles equal, then the lines are parallel.

If three planes intersect in pairs, then the line of intersection of two planes is parallel to the third plane.

If two parallel planes are cut by a transversal, the corresponding dihedral angles are equal.

If two planes in the same space are cut by a transversal making the corresponding dihedral angles equal, then the planes are parallel.

If two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are supplementary.

If two parallel planes are cut by a transversal, the alternate-interior dihedral angles are equal.

If two parallel lines are cut by a transversal, the alternate-interior angles are equal.
Two-space (contd.)

If two lines in the same plane are cut by a transversal and the interior angles on the same side of the transversal are supplementary, the lines are parallel.

If two lines in the same plane are cut by a transversal and the alternate-interior angles are equal, the lines are parallel.

If two coplanar lines are perpendicular to the same line, they are parallel.

If one of two parallel lines is perpendicular to a third coplanar line, the other is also perpendicular to the third line.

Three-space (contd.)

If two planes in the same space are cut by a transversal and the interior dihedral angles on the same side of the transversal are supplementary, the planes are parallel.

If two planes in the same space are cut by a transversal and the alternate-interior dihedral angles are equal, the planes are parallel.

If two lines are perpendicular to the same plane, they are parallel.

If two planes are perpendicular to the same line, they are parallel.

If one of two parallel lines is perpendicular to a plane, the other is also perpendicular to the plane.
Parallel Lines—1

In most of the above drawings lines AB and CD are cut by line EF. In Fig. 1-5, EF is called a transversal, but in Fig. 6 it is not. Can you distinguish the difference and write a definition for transversal?

In these drawings you will recognize vertical angles, such as a and c or f and h, and supplementary angles, such as a and b or f and g.

1. In Figs. 1-5, angles a, b, g, and h are exterior angles. Define exterior angles.

2. In Figs. 1-5, angles c, d, e, and f are interior angles. Define interior angles.

3. In Figs. 1-5, angles, such as a and e or c and g, are called corresponding angles. Define corresponding angles.
Parallel Lines—1 (contd.)

4. In Figs. 1-4 angles, such as d and f or c and e are alternate-interior angles. Define alternate-interior angles.

5. Other than the vertical angles in Fig. 1 and 2, do any of the angles appear to be equal? If so, which ones?

6. In Figs. 3 and 4, do any of the angles in these figures, other than the vertical angles, appear to be equal? If so, which ones?

7. If your answer to exercise 6 is different from your answer to exercise 5, what appears to account for the difference?

8. If you found angles, other than the vertical angles, in either exercises 5 or 6 which were equal, summarize the situation in one or two general statements.

9. Other than the adjacent angles in Figs. 1 and 2, do any of the angles appear to be supplementary? If so, which ones?

10. In Figs. 3 and 4, do any of the angles, other than the adjacent ones, appear to be supplementary? If so, which ones?

11. If your answer to exercise 9 is different from your answer to exercise 10, what appears to account for the difference?

12. If you found angles, other than the adjacent angles, in either exercises 9 or 10 which were supplementary, summarize the situation in one or two general statements.

13. If the transversal were perpendicular to line AB in Fig. 3, what would be its relationship to line CD?


15. Give a converse for each of your general statements in exercises 8, 12, 14.

16. Prove deductively the generalizations formed in exercises 8, 12, 14, and 15.
Parallel Lines and Planes--2

Consider each of the following questions and give your answer as a statement on a separate sheet of paper.

1. If you are given a line and a point not on the line, how many lines may be drawn through the point, not intersecting the line?

2. How many of the lines in exercise 1 will be parallel to the given line?

3. If some of the lines are not parallel to the given line, what is their relationship to it?

4. Under what conditions do you think two planes would be parallel?

5. Under what conditions do you think a line and a plane would be parallel?

6. If two planes were parallel, what relationship would exist between one of the planes and lines in the second plane?

7. If two planes were parallel, what relationship would exist between the lines of the two planes?

8. If a line is parallel to a plane, what is the relationship between the given line and lines in the given plane?

9. If a line is parallel to a plane, what relationship exists between the plane and any plane containing the given line?

10. List other ideas you may have on parallel planes or lines and planes.
Answers to Parallel Lines and Planes

1. If you are given a line and a point not on the line, how many lines may be drawn through the point, not intersecting the line?

1.1 An infinite number of lines may be drawn through a given point not on a given line.
1.2 If a line and a point are on the same plane, only one line may be drawn through this point which will not intersect with the first line.
1.3 An infinite number of lines may be drawn through a point not on another line without intersecting the line if the lines are on different planes.
1.4 On a plane one and only one line may pass through a point and not intersect a line when the point is not on the line.

2. How many of the lines in exercise 1 will be parallel to the given line?

2.1 One and only one line may be parallel to a given line and pass through a point not on that line.
2.2 One and only one line passing through a given point not on a given line on the same plane may be parallel to the given line.
2.3 All lines on the same plane as another line when drawn through a point not on that line do not intersect the line are parallel.

3. If some of the lines are not parallel to the given line, what is their relationship to it?

3.1 They are skew.
3.2 If the line is not parallel to the given line then the line will intersect with the given line.
3.3 They will intersect it if they are on the same plane.
3.4 When two lines are not parallel, then they will either intersect or be skew lines.

4. Under what conditions do you think two planes would be parallel?

4.1 When they don't intersect.
4.2 If two planes are parallel, then they will not intersect.
4.3 If two planes do not intersect no matter how far they are extended, then they are parallel.
4.4 Two non-intersecting planes are parallel.

5. Under what conditions do you think a line and a plane would be parallel?

5.1 When a line and a plane do not intersect, they are parallel.
5.2 If a line and a plane are parallel, they will not intersect.
5.3 If they don't intersect and the line is not on the plane they would be parallel.
Answers to Parallel Lines and Planes (contd.)

5. If a line and a plane are parallel if they would never intersect regardless of how far they are extended.

6. If two planes were parallel, what relationship would exist between one of the planes and lines in the second plane?

6.1 If two planes were parallel then one of the planes and the lines contained in the other plane would never intersect no matter how far they were extended.

6.2 If two planes are parallel, then the lines of one plane and the other plane would be parallel.

6.3 If two planes were parallel, the one plane and the lines in another plane would be parallel, but would not necessarily run in the same direction.

7. If two planes were parallel, what relationship would exist between the lines of the two planes?

7.1 If two planes are parallel, then the lines of one plane would be parallel to the lines of the other plane.

7.2 They would not intersect with the lines on the second plane.

7.3 They would be skew or parallel.

7.4 If two planes were parallel, then the lines of the two planes would never intersect no matter how far they were extended.

7.5 If two planes are parallel, a number of lines may be parallel in the two planes.

8. If a line is parallel to a plane, what is the relationship between the given line and lines in the given plane?

8.1 If a line and a plane are parallel, then they would not intersect.

8.2 They are skew or parallel. They would not intersect.

8.3 All the lines are parallel and will not intersect.

8.4 If a line is parallel to a given plane, the line is parallel to all the lines of the given plane.

8.5 If a line is parallel to a plane it is parallel to one of the lines in the plane.

9. If a line is parallel to a plane, what relationship exists between the plane and any plane containing the given line?

9.1 If a line is parallel to a plane, then the plane and any plane that may contain the line will either be parallel or will intersect.

9.2 If a line is parallel to a plane, then one and only one plane containing the line may be parallel to the other plane.

9.3 If a line is parallel to a plane, then the plane and any plane containing the given line will not intersect.

9.4 If a line is parallel to a plane, a plane containing that line may be parallel to the given plane.
Perpendicular Bisector of a Line Segment

On a few occasions the class had encountered the need for proving that one line segment bisected another line segment and was also perpendicular to it, so it seemed wise to form a generalization which could be proved and thereby eliminate needless repetition. A line segment, AB, was given and two points, P and Q, were to be located so that each was the same distance from A as it was from B (see p. 118). A few students recognized only that the two points determined a line which would bisect the segment AB, but the remainder of the group realized it was the perpendicular bisector of this segment.

When the students were to locate a third point, all but one of them selected a point on the segment PQ; however, one boy stated that the third point did not need to be in the plane determined by AB intersecting PQ. As a consequence of these ideas, all but the one student answered exercise 4 by specifying the points would lie on a line. In exercise 5, however, when it was called to their attention to consider the set of all points, more than three-fourths of the students realized the set of all points would lie on a plane which would be perpendicular to and bisect the line segment.

Considering only the points in a plane, the class formulated the following generalization:

If two points are equidistant from the extremities of a line segment, then the points determine the perpendicular bisector of the line segment.
This statement was proved as well as its converse:

A point on the perpendicular bisector of a line segment is equidistant from the extremities of the line segment.

Several members of the class also wrote statements which would extend these ideas to three-dimensional space:

All points which are equidistant from the extremities of a line segment lie on the plane which is the perpendicular bisector of the line segment.

If a point lies on the plane which is the perpendicular bisector of a line segment, then it is equidistant from the extremities of the segment.

After these proofs were recorded in the students' notebooks, a few exercises were assigned applying these statements, and two optional exercises were given involving three-dimensional space (see p. 119).

While these two proofs were informal, those students who had prepared statements for three-dimensional space had no difficulty in presenting a plan for proving these two problems.

Summary of Extensions of Perpendicular Bisector of a Line Segment

<table>
<thead>
<tr>
<th>Two-space</th>
<th>Three-space</th>
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</thead>
<tbody>
<tr>
<td>If two points are equidistant from the extremities of a line segment, then the points determine the perpendicular bisector of the line segment.</td>
<td>All points which are equidistant from the extremities of a line segment lie on the plane which is the perpendicular bisector of the line segment.</td>
</tr>
</tbody>
</table>
Two-space

A point on the perpendicular bisector of a line segment is equidistant from the extremities of the line segment.

Three-space

If a point lies on the plane which is the perpendicular bisector of a line segment, then it is equidistant from the extremities of the segment.
Perpendicular Lines—3

1. Using the line segment, AB, locate a point, P, which appears to be the same distance from A as from B. Locate a second point, Q, which is the same distance from A as from B.

A __________________________ B

2. The two points, P and Q, determine a line. What relationship, if any, does this line appear to have to the segment AB?

3. Locate a third point, R, which is the same distance from A as from B. Is this point on the line PQ? __________

4. These three points, P, Q, and R, have been described as being the same distance from A as from B. Give another description of the location of the set of all points which are equidistant from the extremities of the line segment AB.

5. Would your description in exercise 4 place all the points on line PQ? __________ If your answer is "yes," have you considered the set of all points? __________

6. From your description form a proposition in the "If . . . , then . . . ." form which will apply when the points all lie in the same plane.

7. State the converse of your proposition in exercise 6.
Exercises

Prove each of the following:

1. Hypothesis: Isosceles triangle $ABC$ with $AC = BC$. $CD$ is the bisector of angle $C$.
   Conclusion: $CD \perp AB$
   $DA = DB$

2. Hypothesis: Isosceles triangle $ABC$ with $AC = BC$. $CD$ is the bisector of $AB$.
   Conclusion: $CD \perp AB$
   $\angle ACD = \angle BCD$

   Conclusion: $AE = BD$

Optional:

4. Hypothesis: Line segment $AB \perp$ plane, $P$ and the plane intersects $AB$ at its midpoint, $M$. $R$ is any point on $P$.
   Conclusion: $AR = BR$

5. Hypothesis: $AB \perp$ plane $P$ at point, $M$. $MR = MT$.
   Conclusion: $AR = AT$
Parallelograms

While some of the group completed proofs for the basic theorems in the unit on parallelograms, others worked on special types of parallelograms (see pp. 124-125). In reply to exercises 5 and 6 which referred to three-space solids which may be bounded by special types of parallelograms, a number of the students described figures which would be classed as rectangular prisms and a few considered oblique prisms. The group talked about the characteristics which these figures had in common, such as the lateral edges and their being equal and parallel, and the bases congruent. One boy had described a figure which could best be illustrated by a deck of cards which had been stacked and then given a twist, and he inquired if this sort of figure would be classed as an oblique prism. The terms, "polyhedron" and "prism," were mentioned and the group discussed definitions of these terms. The student who had asked about the "twisted figure" immediately recognized that his figure would not be classed as a prism since its faces would be curved, rather than plane, surfaces.

Another boy suggested the possibility of having a solid whose lateral faces would be bounded by trapezoids, and this comment stimulated some discussion of other polyhedrons such as pyramids and frustum of a pyramid.

Among the generalizations for plane figures which were mentioned as also applying for solid figures was that, "The diagonals of a prism bisect each other." Immediately someone pounced on the term, "diagonal of a prism," and asked for the definition. The students who had made the statement indicated that they were not thinking of a diagonal as
it had been defined for a polygon—a line joining two non-consecutive vertices. They used the classroom as a model and indicated that they thought of the diagonal as a line from one corner of the room to the farthest corner of the room. The explanation was acceptable and it was decided to consider the diagonal of a prism as the line joining two vertices not in the same face.

The students who had made the original generalization were challenged further when it was brought to their attention that they would need to determine whether or not the diagonals would necessarily intersect. A study of some illustrations revealed that in general the diagonals of a polyhedron would not intersect, but if the polyhedron were a prism with parallelograms as bases they would intersect, and the group discovered informally that a plane could be passed through each pair of parallel edges, so that the diagonals of this particular type of prism (called a parallelepiped) would be the diagonals of the parallelogram formed by the plane intercepted by the edges of the prism; therefore, these diagonals would bisect each other.

When the proposal for the theorem, if three or more parallel lines cut off equal segments on one transversal, they cut off equal segments on all transversals, was presented (see p. 126), some students extended their statements to include the phrase, "parallel planes," along with "parallel lines." To justify this extension, they recommended passing a plane through the known transversal and the transversal on which the segments were to be proved equal. They reasoned that this plane would intersect the parallel planes in a series of parallel
lines and the remainder of the proof would follow that used to prove the statement in two-space.

One student who had not considered the parallel planes said, "That's O.K. if the transversals will intersect or are parallel, but what if they are skew lines—you can't have a plane." This comment brought on a long pause until one of the original group showed that if, in the figure below, \(a\) and \(b\) are skew lines, then a transversal, \(c\), may be drawn parallel to \(a\) through \(P\), the point where \(b\) intersects the plane. Then the suggested procedure could be used by passing a plane through parallel lines \(a\) and \(c\), showing these segments equal, then passing a plane through intersecting lines, \(b\) and \(c\), and showing the required segments equal.
**Summary of Extensions of Parallelograms**

<table>
<thead>
<tr>
<th><strong>Two-space</strong></th>
<th><strong>Three-space</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>polygons</td>
<td>polyhedrons</td>
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<tr>
<td>parallelogram</td>
<td>prism, parallelepiped</td>
</tr>
<tr>
<td>rectangle</td>
<td>rectangular solid</td>
</tr>
<tr>
<td>square</td>
<td>cube</td>
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<tr>
<td>rhombus</td>
<td>pyramid</td>
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<tr>
<td>trapezoid</td>
<td>frustum of a pyramid</td>
</tr>
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<td>lateral faces</td>
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<tr>
<td>diagonal of a polygon</td>
<td>diagonal of a prism</td>
</tr>
<tr>
<td>The opposite sides of a parallelogram are equal.</td>
<td>The lateral edges of a prism are equal and parallel.</td>
</tr>
<tr>
<td></td>
<td>The bases of a prism are congruent polygons.</td>
</tr>
</tbody>
</table>

The diagonals of a parallelogram bisect each other.

The diagonals of a parallelepiped bisect each other.

If three or more parallel lines cut off equal segments on one transversal, they cut off equal segments on all transversals.

If three or more parallel planes cut off equal segments on one line, they cut off equal segments on all lines which intersect the planes.
Parallelograms—2

By defining a parallelogram as "a quadrilateral with the two pairs of opposite sides parallel," the following conclusions were reached:

a. In a parallelogram the opposite angles are equal.
b. In a parallelogram the opposite sides are equal.
c. In a parallelogram the consecutive angles are supplementary.
d. The diagonals of a parallelogram bisect each other.

1. Complete the proofs for the above statements for your notebook.

There are three types of parallelograms which are common enough to have special names. The figures below represent these special parallelograms.

![Fig. a](image1)

Figure a is a parallelogram with one right angle and is called a rectangle.

Figure b is a parallelogram with two adjacent sides equal and is called a rhombus.

Figure c is a rectangle with two adjacent sides equal and is called a square.

2. Write the definitions for these three terms in your notebook.

3. All the conclusions proved for the parallelogram will apply to the special types, but there are some additional conclusions which apply only to one or more of these special parallelograms. (For example: All the angles of a rectangle are right angles.) List as many of these conclusions as you can.
4. Prove at least four of your statements in exercise 3.

5. What types of solids would be obtained if parallelograms, rectangles, squares, and/or rhombuses were used to form the edges of three-space figures? Give illustrations.

6. Which of the conclusions concerning plane figures do you think would still follow for solid figures?
Parallel Lines--3

a, b, and c are parallel lines.

1. Measure the segments of transversal d which are intercepted by these lines and record your measurement in the table below. Are these segments equal? 

2. Draw at least two other transversals of a, b, and c, each in a different position.

3. Measure the segments of each of these transversals cut off by the parallel lines. Record your measurements in the table below.

<table>
<thead>
<tr>
<th>Transversal</th>
<th>a and b</th>
<th>b and c</th>
</tr>
</thead>
<tbody>
<tr>
<td>d</td>
<td></td>
<td></td>
</tr>
<tr>
<td>e</td>
<td></td>
<td></td>
</tr>
<tr>
<td>f</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4. How do the segments of each of these compare? 

5. Assuming you have one transversal divided into equal parts by three or more parallel lines, make a statement about any other transversal cutting these lines.

6. Prove this statement deductively.
Circles and Spheres

When the class began the study of the unit on circles, a study guide was used to introduce several terms associated with the unit, although most of the students had some idea of these terms from earlier mathematics courses (see pp. 133-134). The first three questions of the guide referred to locating the points which would be a given distance, \( d \), from a given point, \( P \). An indication that a number of students had become dimension-conscious was apparent when several read the first question and inquired if they were to consider all points or only those points on a plane. One student countered with the statement that the answer to exercise 1 would be an infinite number of points regardless of whether or not the answer was limited to a plane. Although most of the students simply replied to the first exercise by stating that the number of points would be infinite, one did qualify his answer by stating that all the points would need to lie on a sphere. From this comment and from several of the answers given to exercise 3, it was evident the students were familiar with the term, sphere. It was also evident, however, that some did not have the correct interpretation of either "circle" or "sphere" since they looked upon these figures as including the points within the figure as well as the figure itself. This lack of understanding was apparent when remarks such as "a chord is a line joining two points on the outer edge of a circle" and "a radius is the distance from the center of a circle to the edge" appeared in later definitions. Another indication of this incorrect interpretation arose when the teacher suggested the students check the qualification specified at the beginning of the study guide to see if it would have meaning.
in a one-space world. A few students stated that in one-space the points meeting the condition would form the diameter; others the radius; but finally one girl suggested that it would be only the endpoints of the diameter. She justified her answer by showing that any other point on the diameter would be closer to the center of the circle, or to the given point, P, than the endpoint of the diameter. Following this discussion, another student concluded that in a zero-space there would be only the given point so that the situation suggested by this guide would not exist.

At this time one of the students suggested starting with two points in a one-dimensional space and then moving to two-space by "stacking" points above and below the original two points, with the distance across becoming continuously smaller, and thus forming a circle as illustrated below.

One could then move to three-space by "stacking" circles above and below the circle obtained in two-space, with the radii of these circles becoming continuously smaller, and the final result would be a sphere as shown.
By such a process, one could proceed to four-space by "stacking" spheres in a manner similar to that described for points and circles above, and in this way a four-dimensional object which would be comparable to a circle in two-space or a sphere in three-space would develop.

The class found that their definitions for radius, diameter, and chord applied to a sphere as well as to a circle. When an arc had been defined as a part of the line forming a circle, and it was suggested that this applied to a sphere, one student argued that to define an arc of a sphere as part of a line on the sphere would lead to some confusion as a line on a sphere could be shown in various shapes as illustrated by line $l$.

Another student then suggested that if a plane intersected a sphere the intersection would be a circle and then a portion of this circle could be considered as an arc on the sphere. This suggestion also led to discussion of the terms "small circle" and "great circle" which were subsequently defined.

When the theorem that chords equidistant from the center of the circle are equal was proved, five members of the class also showed that if two planes equidistant from the center of a sphere intersect the sphere, then the circles formed by the intersections would be equal (see pp. 136-137). They reasoned that in each instance a right triangle would be formed, having the perpendicular distance from the center of
the sphere to the center of the circle and the radius of the circle as legs and the radius of the sphere as hypotenuse. The right triangles would be congruent and thus the radii of the circles equal, and consequently the circles would be equal.

Also while working on this guide some comments arose about whether the distance from the center had any affect on the results, particularly the extreme distances. If the chords passed through the center of the circle would the relationship hold? If the distance between the points joined by each of the chords became smaller so that the points would coincide, would the chords still be of equal length?

In the first instance the chords would be diameters of the circle, and the class had concluded in a previous assumption that the radii of the same or equal circles were equal (see p. 135). This assumption had been extended to include diameters of a circle as well as both radii and diameters of the same or equal spheres. Because of this assumption, the conclusion that these diameters, whose distances from the center would be the same, zero in this case, would be equal chords would still apply.

The second case, however, presented a different situation. If the distance between the points decreased until the points coincided, then there would be nothing to limit the length of the line as it would have only one point in common with the circle. Actually there would no longer be a chord since by definition it is the line joining two points. From these comments, a new term, "tangent," was encountered. After this term was defined for a circle, it was desirable to see whether a similar situation would apply to a sphere. Since in three-space the direction
of the lines would not be limited, there would be an infinite number of lines tangent to a sphere at a specific point. In fact these lines would all lie on a plane which would have only the specific point in common with the sphere; therefore, it was decided that the plane could be called a "tangent plane." This result posed another problem—would all of the lines in the plane passing through the point of tangency be tangent to a circle of the sphere that passed through the same point? From the definition agreed upon, all of the lines would be tangent to the circle. The teacher suggested that in order to avoid the confusion of having many tangents to a circle at one point, it was common practice to agree to refer to the coplanar tangent as the tangent of the circle. This suggestion sounded sensible and it was agreed to follow this same practice.

Still referring to the guide which prompted this discussion, several members of the class thought that since the distance of a chord from the center of the circle would be measured along the perpendicular from the center to the chord, perhaps the tangent would be perpendicular to the radius drawn to the point of tangency. These people attempted to prove this relationship and some of them completed the proof satisfactorily. Four of the students also explained how this relationship could be extended to show that the radius of a sphere drawn to the common point would be perpendicular to the tangent plane.
### Summary of Extensions of Circles

<table>
<thead>
<tr>
<th>Two-space</th>
<th>Three-space</th>
</tr>
</thead>
<tbody>
<tr>
<td>circle</td>
<td>sphere</td>
</tr>
<tr>
<td>radius of a circle</td>
<td>radius of a sphere</td>
</tr>
<tr>
<td>diameter of a circle</td>
<td>diameter of a sphere</td>
</tr>
<tr>
<td>center of a circle</td>
<td>center of a sphere</td>
</tr>
<tr>
<td>semicircle</td>
<td>hemisphere</td>
</tr>
<tr>
<td>chord of a circle</td>
<td>chord of a sphere</td>
</tr>
<tr>
<td>arc of a circle</td>
<td>small circle</td>
</tr>
<tr>
<td>tangent</td>
<td>great circle</td>
</tr>
<tr>
<td>tangent line</td>
<td>tangent plane</td>
</tr>
</tbody>
</table>

- The same, or equal, circles have equal radii and equal diameters.
- Two circles can intersect in at most two points.
- Chords equidistant from the center of a circle are equal.
- A tangent of a circle is perpendicular to the radius drawn to the point of tangency.
- The same, or equal, spheres have equal radii and equal diameters.
- Two spheres may intersect in a circle.
- If two planes, equidistant from the center of a sphere intersect the sphere, then the circles formed by the intersections are equal.
- If a plane is tangent to a sphere, then the radius drawn to the common point is perpendicular to the plane.

### Other ideas

- Conditions for a circle in one-space are satisfied by two points.

- Define circle by "stacking" points; define sphere by "stacking" circles; define four-space figure by "stacking" spheres.
Circles—1

Given the point P shown below. Consider the location of all points a given distance d from this point in answering the following questions:

- P

1. How many points would satisfy this given condition? ________

2. Would all of these points lie on this sheet of paper? ________

3. If your answer to exercise 2 is "no," how would you describe the surface on which these points lie?

4. If you consider only those points which would be located on this sheet of paper, would they lie on a straight line? ________ A broken line? ________ A curved line? ________

5. When we consider all the points which satisfy the above condition the geometric solid is called a sphere. When we consider only those points on a plane, such as the sheet of paper represents, the geometric figure is called a circle. The point P is the center of the sphere or circle and a straight line from point P to one of the points satisfying the given condition is called the radius. Write a definition for each of these four terms.
Circles--1 (contd.)

6. The drawings below illustrate some of the common terms associated with circles. Study the drawings carefully and write a definition for each of the terms.

- Diameter
- Chord
- Minor arc
- Major arc
A circle is usually named by the letter naming the point which is the center. In circle 0, above, 0A is called a .

What is the length of OA? 

Draw two other straight line segments from 0 to other points of the circle and measure their lengths. How do these lengths compare to OA? 

Measure the radius of each of the circles below, recording the measurement on the line below the circle. The circle in Fig. 1 is said to equal the circle in Fig. 4, while that in Fig. 2 equals the one in Fig. 3.

Write a general statement concerning the conclusion formed by considering the radii of the same circle or of equal circles.
Circles--4

Measure the radius of each circle; the length of a chord, AB; and the perpendicular distance, OP. Record your measurements in the table on the following page.
Circles—4 (contd.)

<table>
<thead>
<tr>
<th>Circle</th>
<th>Radius OA</th>
<th>Chord AB</th>
<th>Perpendicular OP</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
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<td>3</td>
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<td>8</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1. Do any of the circles have equal radii? _____ If so, list them (all possibilities). ________________________________

2. Do any of the circles have equal chords? _____ If so, list them. ________________________________

3. Are any of the chords equidistant from the center of the circle? _____ If so, list them. ________________________________

4. In those cases where the radii are equal, are the chords necessarily equal? _____

5. When the chords are the same distance from the center of the circle, are the chords always equal? _____

6. If the radii are equal and the chords are equal, are the chords always the same distance from the center of the circle? _____

7. In equal circles, if you see a relationship between the chords and the distance from the center of the circle, state this relationship as a theorem.

8. Prove your statement deductively.
Measurement of Angles on a Sphere

Relationships had been discovered for measuring angles formed by lines intersecting inside, on, or outside a circle, so that several students became curious about the measurement of angles formed by lines on a sphere.

To assist in visualizing the appearance of such angles, students drew lines on a chalk globe and experimented with a set of embroidery hoops. A couple of suggestions were made which somewhat approached the usual methods used to measure spherical angles, but neither method was sufficiently explained to be used satisfactorily. The proposal was made to flatten the sphere by some means, but this met with several objections:

1) If the sphere were a hard surface, it could not be flattened without breaking it.
2) Even if the sphere were pliable, flattening it would distort the shape.
3) If the sphere were flattened, it might not be possible to return it to its original form.

The second procedure was based upon the use of arcs of circles in measuring angles related to circles. This proposal suggested measuring the arc intercepted by the sides of the angle. The difficulty encountered with this method was in deciding how to select the arc because drawings on the globe showed various arcs which could be drawn. Since the period ended without resolving the problem, the teacher prepared a study guide to direct the students' thinking toward the methods used to
measure spherical angles, and the guide also included the idea of spherical polygons (see pp. 141-142).

As the students worked on the guide one boy pointed out to the class that vertical spherical angles could occur just as vertical angles existed on a plane. He also noted that they could be shown to be equal when the spherical angles were measured by their plane angles because the corresponding plane angles would be formed by two intersecting lines on a plane.

Each student was able to answer the questions on spherical polygons and to determine the limitations on the sum of the measures of the sides of a spherical triangle and also the sum of the measures of the angles of the triangle. Earlier in the year the teacher had introduced the students to some of the differences between Euclidean and non-Euclidean geometries, but it was not until they completed this exercise that some of the students could fully accept the idea that under certain conditions the sum of the measures of the angles of a triangle may be more than 180 degrees.

Summary of Extensions of Measurement of Angles on a Sphere

<table>
<thead>
<tr>
<th>Two-space</th>
<th>Three-space</th>
</tr>
</thead>
<tbody>
<tr>
<td>inscribed angle</td>
<td>spherical angle</td>
</tr>
</tbody>
</table>

An inscribed angle is measured by one-half its intercepted arc.

The angle formed by a tangent and a chord is measured by one-half its arc.
Two-space (contd.)

The angle formed by lines intersecting outside a circle is measured by one-half the difference of the intercepted arcs.

The angle formed by chords intersecting inside a circle is measured by one-half the sum of its arc and the arc of its vertical angle.

Three-space (contd.)

If two great circles intersect, the vertical spherical angles are equal.

The sum of the measures of the sides of a spherical triangle is less than 360 degrees.

The sum of the measures of the angles of a spherical triangle is less than 540 degrees.
Spheres—1

In the figure below the angle ABC is formed by the arcs of two great circles that intersect at a point. This angle is called a **spherical angle**.

![Spherical Angle Diagram](image)

1. These two great circles will intersect in how many points? ____
2. Name another spherical angle formed by the same two great circles. ________
3. When two lines intersect on a plane, how many angles are formed? ______
4. When two great circles intersect on a sphere, how many spherical angles are formed? ______

While the suggestion of measuring an angle on a sphere by flattening the sphere did not prove practical, consider somewhat this same effect by recalling that a plane which has only one point in common with a sphere will be tangent to the sphere. In the figure above, let B, the vertex of the angle, be the common point. Now draw the tangents to the two great circles at point B (BC' and BA'). These tangents will form a plane angle. Can angle A'BC' be measured? ______

Let us consider a second method of measuring angle ABC. The two planes which intersect the sphere to form circles ABD and CBD will intersect each other and form a line, BD. If a plane is passed perpendicular to this intersection, it will intersect the sphere to form a circle. A portion of this circle (arc AC) will lie between the two great circles which form the spherical angle. Can the arc AC be measured? ______

5. In the figure above, therefore, the spherical angle ABC may be measured by angle ____ or arc _____. The spherical angle ADC may be measured by angle ____ or _____.
Further let us consider the closed figure which is formed by three or more arcs of great circles intersecting on a sphere. This figure is called a spherical polygon and is illustrated below. In the figure, polygon $ABC$ is a spherical triangle. When the vertices of the spherical triangle are connected to the center of the sphere a trihedral angle is formed. The sides of the spherical triangle correspond to the face angles and the angles of the spherical triangle correspond to the dihedral angles of the polyhedral angle.

6. When we discussed the face angles of a trihedral angle earlier, it was decided that the sum of the face angles would need to be $\ldots$ The sum of the sides of a spherical triangle, therefore, would be __________.

7. How large may the dihedral angle $C-OB-A$ be? __________

8. How large may the dihedral angle $B-OA-C$ be? __________

9. How large may the dihedral angle $A-OC-B$ be? __________

10. Since the spherical angles correspond to the dihedral angles, the sum of the measures of the angles of a spherical triangle cannot be greater than __________ degrees.
Locus

The principal locus statements were introduced by a group of exercises, each representing one of the basic relationships (see pp. 1147-1149). All of the answers to the illustration suggesting the locus of points a given distance from a given point specified that on a plane the points would be located on a circle and in three-space on a sphere. Nearly half the class made the distinction of the two situations in their answers to the first exercise; the remainder extended their description when they considered the point in space. Three of the students mentioned that the idea of locating these points was the same as that used in defining circles and spheres. Some members of the class noted that if there were a limitation to one-space, the locus would be two points, collinear with the given point, and the two points would be the given distance on either side of the given point.

Describing the location of the points in three-space which would be one-half inch from a given line proved to be difficult due to the lack of good descriptive terms. Among the suggestions were included these:

1) an infinite number of lines parallel to the given line and one-half inch from it;
2) a cylinder around the line having a radius of one-half inch;
3) an infinite number of one-half inch circles around the given line.

Those who suggested the cylinder objected to the first comment because it seemed as if it implied that there could be lines or points between the parallel lines which would not satisfy the condition, an
idea with which they would not agree, and a similar criticism was made for the description involving a series of circles. They insisted the points would comprise a surface. The others argued that a cylinder had two bases and this description would be placing a limit on the length of the line. The discussion was finally resolved by describing the location of the points as a circular cylindrical surface.

Previously the class had proved that all the points on the plane which is the perpendicular bisector of a line segment are equidistant from the endpoints of the line segment. The majority of the group recalled this proposition and could readily see that such a plane would contain all the points which are the same distance from two given points.

The description of the locus of points the same distance from two parallel lines was well stated by all the students for both two- and three-dimensional space, but they did not respond as well when the lines were intersecting. These replies fell into three categories:

1) Seven students replied that in two-space the points would lie on two lines which were the bisectors of the vertical angles formed by the intersecting lines; while in three-space the points would lie on the two planes which bisect the dihedral angles corresponding to the plane angles. One boy noted that the lines, forming the locus, would be perpendicular and likewise the planes would be perpendicular.

2) Seven others indicated only one line and one plane bisecting the angles as above. This answer provided a good illustration, for explaining the need to meet two conditions in a locus problem—namely that every point on the locus satisfy
the condition and that every point which meets the condition be included in the locus.

3) Two pupils recognized the two lines, only, as satisfying the condition in three-space as well as in two-space.

Those in the first group were able to convince the remaining two groups that they had overlooked certain points in their descriptions, and general statements were agreed upon for all five exercises. As a group the class proved the proposition concerning the location of points equidistant from two intersecting lines, limiting the formal proof to the two-space situation.

To provide further practice in determining the locus of points for specified conditions, the teacher assigned other exercises, some describing situations which could arise in two- or three-space (see pp. 150-152). One exercise which many students found difficult to describe in three-space was "the locus of a point which is always three inches from a circle whose diameter is ten inches." Showing that, on a plane, this locus would be comprised of two circles, concentric with the original, and having diameters of sixteen inches and four inches, was rather simple, although some neglected the inside circle at first. After numerous complicated descriptions, one student finally thought of the figure in three-space as being a shape similar to the surface of a doughnut whose cross-sectional diameter would be six inches.
Summary of Extensions of Locus

<table>
<thead>
<tr>
<th>One-space</th>
<th>Two-space</th>
<th>Three-space</th>
</tr>
</thead>
<tbody>
<tr>
<td>The locus of points in one-space equidistant from a given point on the line is two collinear points, each the given distance from the given point.</td>
<td>The locus of points on a plane equidistant from a given point in the plane is a circle with the given point as center and the given distance as radius.</td>
<td>The locus of points in three-space equidistant from a given point is a sphere with the given point as center and the given distance as radius.</td>
</tr>
</tbody>
</table>

The remainder of the situations had no meaning in one-space.

<table>
<thead>
<tr>
<th>Two-space</th>
<th>Three-space</th>
</tr>
</thead>
<tbody>
<tr>
<td>The locus of points on a plane equidistant from a given line in the plane is a pair of lines parallel to the given line and the given distance from it.</td>
<td>The locus of points in three-space equidistant from a given line is a right cylindrical surface having the given line as axis and the given distance as radius.</td>
</tr>
<tr>
<td>The locus of points on a plane equidistant from two parallel lines in the plane is a line parallel to the given lines and midway between them.</td>
<td>The locus of points in three-space equidistant from two parallel planes is a plane parallel to the given planes and midway between them.</td>
</tr>
<tr>
<td>The locus of points on a plane equidistant from two given points in the plane is the line which is the perpendicular bisector of the line segment joining the given points.</td>
<td>The locus of points in three-space equidistant from two given points is the plane which is the perpendicular bisector of the line segment joining the given points.</td>
</tr>
<tr>
<td>The locus of points on a plane equidistant from two intersecting straight lines in the plane is the pair of lines which bisects the angles formed by the given lines.</td>
<td>The locus of points in three-space equidistant from two intersecting planes is the pair of planes which bisects the dihedral angles formed by the intersecting planes.</td>
</tr>
</tbody>
</table>
Locus—1

1. Locate six points each one inch from point P.

   • P

   Are there other such points? ______ If so, where do they appear to be located?

   If this point were suspended in space, would this change your answer to the previous question? ______ If so, what would your answer be?

2. Locate six points each one-half inch from line 1.

   \[ \text{\[ image \]} \]

   Are there other such points? ______ If so, where do they appear to be located?

   If this line were suspended in space, would this change your answer to the previous question? ______ If so, what would your answer be?

3. Locate six points each the same distance from point A as from point B.

   \[ \text{\[ image \]} \]

   Are there other such points? ______ If so, where do they appear to be located?
If these points were suspended in space, would this change your answer to the previous question? _____ If so, what would your answer be?

4. Locate six points each the same distance from line \( m \) as from line \( n \).

\[ \begin{array}{c}
\text{\textarrow{m}} \\
\text{\textarrow{n}} \\
\end{array} \]

Are there other such points? _____ If so, where do they appear to be located?

If these lines were suspended in space, would this change your answer to the previous question? _____ If so, what would your answer be?

5. Locate six points each the same distance from line \( m \) as from line \( n \).

\[ \begin{array}{c}
\text{\textarrow{m}} \\
\text{\textarrow{n}} \\
\end{array} \]

Are there other such points? _____ If so, where do they appear to be located?

If these lines were suspended in space, would this change your answer to the previous question? _____ If so, what would your answer be?
The path through which a point moves to satisfy certain conditions such as those in the above exercises is known as the locus of points.

6. Write a statement which will describe the locus of points for each general situation illustrated by the specific examples in exercises 1-5.
Exercises on Locus

Describe the following loci. Sketch when desirable.

1. What is the locus of points equidistant from two fixed points 6 inches apart?

2. How many circles can touch a fixed line segment AB at a point P on the segment? Describe the locus of their centers.

3. Describe the locus of the center of a circle that has a given radius and touches a given segment.

4. Describe the locus of the center of a circle that touches a given circle at a given point.

5. Find the locus of the middle point of a ladder 12 ft. long as the ladder is pulled away from a house, one end remaining on the house and the other end on the ground, assuming the ground to be level.

6. Find the locus of the middle points of all chords that can be drawn through a fixed point P within a circle.

7. Find the locus of the middle points of all chords that can be drawn parallel to a given chord of a circle.

8. Circles are drawn tangent to a given line at the same point A. From another point P on the same line, tangents are drawn to these circles. What is the locus of the points of contact of these tangents?

9. Circles are drawn so that they are always wholly within a given triangle and tangent to two and only two of its sides. What is the locus of the centers of these circles?

10. Find the locus of the center of a circle which is tangent to a given line at a given point.

11. The points A and B are 4 inches apart. Find all points 3 inches from A and 2 inches from B.

12. Find all points which are a given distance from a fixed point P and a fixed line l.

13. Find a point equidistant from two given intersecting lines, a and b, and equidistant from two points, A and B.

14. Find the points equidistant from two given intersecting lines, a and b, and a given distance, d, from a given line, c.
15. Three lines intersect at A, B, and C. Find all points equidistant from A and C which are also equidistant from line BA and line BC.

16. A circle lies between two parallel lines, a and b. Find all points on the circle which are equidistant from a and b.

17. Find the locus of the vertex of all right triangles having a given hypotenuse and a given altitude to the hypotenuse.

18. Find all points equidistant from two points, A and B, and also a distance, d, from a third point, C.
Exercises on Locus

Describe the loci; draw an illustration when desirable.

1. ABCD is a rectangular sheet of paper. Describe the set of all points on the paper which are 2 inches from AB.

2. Describe the set of all points on the paper of exercise 1 which are 1 inch or less than 1 inch from a side of the paper.

3. A rectangular sheet of paper has edges 12 inches and 8 inches long. Describe the set of all points on the paper which are (a) equidistant from the 12 inch sides, (b) equidistant from the 8 inch sides. Is there a point equidistant from all the sides?

4. A given circle has center O and radius 6 inches. What is the locus of the midpoints of radii of the circle?

5. A given sphere has center O and radius 6 inches. What is the locus of the midpoints of radii of the sphere?

6. What is the locus of a point which is always 3 inches from a circle of diameter 10 inches?

7. What is the locus of points which are one-half inch from a circle of radius 2 inches?

8. What is the locus of points which are equidistant from two concentric circles with radii 6 in. and 10 in.?

9. What is the locus of points in a plane which are more than 3 inches and less than 4 inches from a fixed point P in the plane?

10. What is the locus of points which are n inches from a circle of radius 2n inches? Is it the same as the locus of points which are 2n inches from a circle of radius n inches?

11. If exercise 9 is not limited to a plane, how will this affect your answer?

12. Did you limit your answer to exercise 10 to a plane? If so, consider the situation in three-space.
Proportions

During the study of proportions, a study guide leading to the generalization that a line parallel to one side of a triangle divides the other two sides proportionally, was used (see pp. 154-155). After the class had proved this generalization, the teacher suggested that a line be drawn through the vertex opposite the parallel lines and parallel to these lines. Then she asked whether any new generalization could be formed. After some thought, the class recognized a situation similar to one they had had earlier when three parallel lines cut off equal segments on transversals, and they realized they could show that three or more parallel lines would divide two transversals proportionally. Now they also recalled that the previous statement had been extended to parallel planes and decided that in a similar manner this one could be extended.

"Then, would a plane parallel to the base of a pyramid divide its edges proportionally?" asked a member of the class. Almost at once he answered his own question by realizing that the plane would intersect the face in a line parallel to the edge of the base, and this would yield a plane figure similar to that of the figure which introduced this discussion.

Summary of Extensions of Proportions

<table>
<thead>
<tr>
<th>Two-space</th>
<th>Three-space</th>
</tr>
</thead>
<tbody>
<tr>
<td>Three or more parallel lines cut off proportional segments on two or more transversals.</td>
<td>Three or more parallel planes cut off proportional segments on two or more lines intersecting the planes.</td>
</tr>
<tr>
<td>A line parallel to one side of a triangle divides the other sides proportionally.</td>
<td>A plane parallel to the base of a pyramid divides the edges of the pyramid proportionally.</td>
</tr>
</tbody>
</table>
1. Make the measurements for each figure as requested in the table below; also indicate the ratios requested.

<table>
<thead>
<tr>
<th>Fig.</th>
<th>BD</th>
<th>DA</th>
<th>BD : DA</th>
<th>BE</th>
<th>EC</th>
<th>BE : EC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
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<td></td>
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</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
2. Are there any figures where $BD : DA = BE : EC$? If so, which figures? How do these figures differ from those figures where the ratios are not equal?

3. Make a general statement from your results in exercise 2.

4. In Fig. 1 use $AD$ as a unit of measure. Mark off $BD$ into segments equal to $AD$. Draw lines through the points of division parallel to $AC$ and terminating in $EC$. Why is this possible?

5. What relationship holds among the segments into which $BC$ is divided? Why?

6. $BD$ is how many times as long as $DA$? Therefore, $BD : DA =$

7. $BE$ is how many times as long as $EC$? Therefore, $BE : EC =$


9. If a small unit which divided $BD$ and $DA$ evenly were used as the unit of measure in place of $AD$, would the proportion $BD : DA = BE : EC$ still be true?

10. Prove the statement in exercise 3 deductively. Also write the proportion in exercise 9 by inversion, alternation, and addition.
Pythagorean Theorem

When the Pythagorean theorem was proved, a series of exercises involving applications of this theorem was studied. Included among the exercises was an illustration of a rectangular parallelepiped with dimensions as indicated on the drawing below, and the students were to determine the length of a diagonal.

Since three members of the class were not sure they had solved this exercise correctly, one of the other students offered to explain his solution. He had found the diagonal of the rectangular base by using a right triangle having legs of 4 and 12 units, each, and the diagonal of the base as the hypotenuse. Then using this distance as a leg and the height of the solid as the second leg, he determined the length of the hypotenuse of this right triangle, which would be the required diagonal.

Two other students indicated they had obtained the same final result, but they started with the right triangle having legs of 3 and 4 units, each, because they realized the diagonal of this face would be 5 units. At this point one of the boys stated that it really would not...
matter which face diagonal was found first and that if the three dimensions were denoted by $a$, $b$, and $c$, and the diagonal of the rectangular parallelepiped by $d$, the formula, $a^2 + b^2 + c^2 = d^2$, could be used.

One of the girls then raised the question of whether this formula would be satisfactory for any parallelepiped or was it limited to rectangular solids. By looking at a model of a parallelepiped, it was quickly seen that just as repeated application of the Pythagorean theorem, plus more information than just the dimensions of the sides of the parallelogram, would be needed to determine its diagonal, ordinarily the dimensions of a parallelepiped would be insufficient information for determining its diagonal.

Later in the year some attention was given to coordinate geometry, and the class did some work on determining distances on a number line and a coordinate plane (see p. 159). In finding the distances on a number line or in instances where either the $x$ or $y$ coordinate were the same, as illustrated in exercises 1, 3, and 4 of the guide, the students had no difficulty in subtracting the proper coordinates and determining the required distance. When they attempted to find the distance between points, $(-4, -3)$ and $(6, 21)$, most of the class said they did not have enough information, but each was encouraged to study his drawing carefully, and, without exception, each one soon realized he could draw a right triangle and determine the length of the segment.

After recognizing this application of the Pythagorean theorem, the class suggested that the procedure could be extended as previously to find distances in three-space. Then someone showed that the extension could also be made to finding the distance in one-space because
the one dimension could be squared and the square root of the result would yield the original dimension. Of course, the student recognized this was not a practical procedure, but it appeared that, regardless of the space one was using, the distance between two points could be found by finding the sum of the squares of the dimensions, taken at right angles to each other, and then taking the square root of this sum. It was also mentioned that the method would also hold for zero-space since all distances would be zero.

Summary of Extensions of Pythagorean Theorem

<table>
<thead>
<tr>
<th>Two-space</th>
<th>Three-space, one-space, n-space</th>
</tr>
</thead>
<tbody>
<tr>
<td>In a right triangle the square on the hypotenuse is equal to the sum of the squares on the legs.</td>
<td>The distance between two points in any space is equal to the square root of the sum of the squares of the dimensions, taken at right angles to each other.</td>
</tr>
</tbody>
</table>
Coordinates - 1

1. Using the line above to represent a number line, locate a point, +6. Call this point A. Locate a point, +10; call it B. How long is the segment AB? __________. Locate a point, -5; call it C. How long is the segment CA? __________. Locate a point, -7; call it D. How long is the segment DA? __________

2. When two points are located on a number line, explain how to find the length of the segment connecting the two points.

3. On a sheet of graph paper draw a set of axes and locate the following points: A = (2, 8); B = (11, 8); C = (-2, 3); D = (7, 3); E = (-11, -6); F = (-2, -6). Find the length of AB = __________; CD = __________; EF = __________.

4. Do the same for the points: M = (6, 3); N = (6, 11); O = (3, -7); P = (3, 10); Q = (-6, -12); R = (-6, -4). Find the length of MN = __________; OP = __________; QR = __________.

5. Explain how to find the length of the segments as requested in exercises 3 and 4.

6. On a sheet of graph paper draw a set of axes and locate (-4, -3) and (6, 21). Find the length of the segment connecting these two points.

7. Explain how to find the length of the segment in exercise 6.

8. Would the method of exercise 7 work for exercises 3 and 4? ______

9. Would this method work for finding the distance between any two points on a plane? ______

10. Sketch the quadrilateral determined by (-3, -2), (5, -2), (5, 9), and (-3, 9). What kind of quadrilateral is it? ______
Area of Polygons

The students were able to verbalize most of the formulas for determining the area of two-dimensional and volume of three-dimensional figures from their contact with these formulas in eighth grade mathematics, but they had not had the opportunity to start with one basic formula and develop the others from it. By starting with the assumption that the number of square units in the area of a rectangle is equal to the product of the number of linear units in the base and altitude, the class developed the formulas for the area of a triangle and a parallelogram; most of the class included the trapezoid in their work; and as a group the formulas relating to the circumference and area of a circle were shown. The class, then, decided to see what they could do with the volume formulas. Since the area of a rectangle had been the starting point for plane figures, they assumed the formula for the volume of a rectangular prism as the beginning of this series. Most of the students obtained the formula for the volume of a right cylinder easily, but only a small portion of the class continued, with help from the teacher, to reach the formulas for the volume of a pyramid, cone, and sphere. By directing the students' thinking toward determining the area of the walls of a room, the amount of material needed for constructing a cylinder, and similar situations, the teacher introduced the term, lateral area. Following this introduction some of the pupils added the development of the formulas for the total and lateral areas of rectangular prisms, cylinders, pyramids, cones, and spheres.

Throughout the discussion of these formulas, emphasis had been placed on the type of unit used in measuring length, area, and volume.
Two study guides were used to assist in recognizing the comparison of similar polygons, particularly the area, and how these comparisons could be related to the linear measures (see pp. 164-167).

While working these exercises the students found the two lengths could be compared by the ratio of their lengths and the two areas of similar polygons by the ratio of the squares of their corresponding linear measures. It appeared that the volumes of two similar polyhedrons might have the same ratio as the cubes of their corresponding lengths and this relationship was checked for rectangular prisms, right cylinders, and spheres. Some of the students, then, generalized that two similar figures would have a ratio equal to their corresponding linear measures raised to the same power as the number of the dimensions of the figure.

Summary of Extensions of Area of Polygons

<table>
<thead>
<tr>
<th>Two-space</th>
<th>Three-space</th>
</tr>
</thead>
<tbody>
<tr>
<td>area</td>
<td>volume</td>
</tr>
<tr>
<td>altitude of polygon</td>
<td>lateral area</td>
</tr>
<tr>
<td>The area of a rectangle is</td>
<td>The volume of a rectangular</td>
</tr>
<tr>
<td>equal to the product of the</td>
<td>prism is equal to the product</td>
</tr>
<tr>
<td>length and the width.</td>
<td>of the length, width, and</td>
</tr>
<tr>
<td></td>
<td>height.</td>
</tr>
<tr>
<td>The area of a parallelogram</td>
<td>The volume of a prism is equal to</td>
</tr>
<tr>
<td>is equal to the product of its</td>
<td>the product of the altitude</td>
</tr>
<tr>
<td>base and altitude.</td>
<td>and the area of the base.</td>
</tr>
<tr>
<td>The area of a triangle is</td>
<td>The volume of a pyramid is equal</td>
</tr>
<tr>
<td>equal to one-half the product</td>
<td>to one-third the area of the base and its altitude.</td>
</tr>
</tbody>
</table>
The area of a trapezoid is equal to one-half the product of the altitude and the sum of the bases.

The area of a circle is equal to \( \pi \) times the square of the radius.

The circumference of a circle is equal to \( \pi \) times the diameter.

The volume of a sphere is equal to \( \frac{4}{3} \pi \) times the cube of the radius.

The volume of a circular cylinder is equal to the product of the altitude and the area of the base.

The volume of a circular cone is equal to one-third the product of the altitude and the area of the base.

The lateral area of a rectangular prism is equal to the product of the perimeter of the base and the altitude.

The lateral area of a circular cylinder is equal to the product of the circumference of the base and the altitude.

The lateral area of a pyramid is equal to one-half the product of the perimeter of the base and the altitude of a face.

The lateral area of a circular cone is equal to one-half the product of the circumference of the base and the slant height.

The surface area of a sphere is equal to \( 4 \pi \) times the square of the radius.
Two-space (contd.)

The areas of two similar polygons are proportional to the squares of the corresponding sides.

Three-space (contd.)

The volumes of two similar three-dimensional solids are proportional to the cubes of the corresponding edges.

Other ideas

Two similar figures are in the ratio which is equal to their corresponding linear measures raised to the same power as the number of the dimensions of the figures.
1. If the base of one parallelogram is 18 in. and the altitude is 15 in., how will its area compare to that of a parallelogram having a base of 1.5 ft. and an altitude of 15 in.?

2. If the base of one parallelogram is 18 in. and the altitude is 30 in., how will its area compare to that of a parallelogram having a base of 1 ft. and altitude of 2.5 ft.?

3. If \( b \) represents the base of a parallelogram and \( \text{a represents its altitude and } \) \( m \) represents the base of a second parallelogram and \( \text{n its altitude, how do the areas compare?} \)

4. In exercise 3 if \( b = m \) and \( \text{a = n, how do the areas compare?} \)

5. What statement can be made concerning the areas of two parallelograms when the bases and the altitudes are respectively equal?

6. If the base of one parallelogram is 24 in. and the altitude is 7 in., how will its area compare to that of a parallelogram having a base of 17 in. and an altitude of 5 in.?

7. If the base of one parallelogram is 3 ft. and the altitude is 30 in., how will its area compare to that of a parallelogram having a base of 23 in. and an altitude of 11 in.?

8. What statement may be made concerning the ratio between the areas of two parallelograms?
Area of Polygons—1 (contd.)

9. Two parallelograms each have an altitude of 60 feet, but one has a base of 100 feet, and the other has a base of 125 feet. How do the areas of the two parallelograms compare?

10. One parallelogram has an altitude of 6 feet and a base of 9 feet. A second parallelogram has an altitude of 2 yards and a base of 5 feet. How do the areas of the two parallelograms compare?

11. One parallelogram has an altitude of \(a\) units and a base of \(b\) units. A second parallelogram has an altitude of \(a\) units and a base of \(b'\) units. How do the areas of the two parallelograms compare?

12. What statement may be made concerning the ratio between the areas of two parallelograms having equal altitudes?

13. Make a similar statement concerning the ratio between the areas of two parallelograms having equal bases.

14. Prove deductively the statements in exercises 5, 8, 12, and 13.
Area of Polygons—2

Use the following triangles for exercises 1 - 7.

1. What relationship exists between triangle ABC and triangle DEF?

2. What relationship exists between triangle LMN and triangle RST?

3. Compare the area of triangle ABC to the area of triangle DEF.
   \[ \frac{A_1}{A_2} = \] _______

4. Compare the area of triangle LMN to the area of triangle RST.
   \[ \frac{B_1}{B_2} = \] _______

5. Is the comparison of \( A_1 : A_2 = BC : EF \)?
   \[ \frac{A_1}{A_2} = 2 \cdot BC : 2 \cdot EF \]
   \[ \frac{A_1}{A_2} = 3 \cdot BC : 3 \cdot EF \]
   \[ \frac{A_1}{A_2} = BC^2 : EF^2 \]
   \[ \frac{A_1}{A_2} = BC^3 : EF^3 \]

6. If another pair of corresponding sides, such as AC and DF were used, would your answers to the parts of exercise 5 be the same? _______
   If the answer is "no," what are the differences?

7. According to the results of exercises 5 and 6, make a statement concerning the ratio of the areas of two triangles in terms of corresponding sides.
Area of Polygons--2 (contd.)

Use the following triangles for exercises 8 - 14. Assume triangle ABC is similar to triangle A'E'C' with corresponding altitudes h and h'. The triangles are not right triangles.

8. Complete the proportionality: \( \frac{a}{b} = \frac{b}{c} \)

9. Complete the proportionality: \( \frac{h}{c} \)

10. Compare the area of triangle ABC to area of triangle A'E'C'.
    \( C_1 : C_2 = \) __________

11. From exercise 8, does \( b : b' = c : c' \)? _______

12. From exercise 9, does \( h : h' = c : c' \)? _______

13. From the results of exercises 10, 11, and 12, \( C_1 : C_2 = \) __________

14. Would your statement in exercise 7 be true for any pair of similar triangles or is it limited to right triangles? __________

15. Prove deductively your statement for exercise 7.
Summary

This chapter outlined the procedures used in teaching the course proposed to help students integrate plane and solid geometry through an emphasis on the concept of dimension.

The actual experiences described here illustrate the manner in which this integration developed for one particular group of students.
CHAPTER V

EVALUATION

Types of Evaluation

In a study of the nature carried on in this research, several types of evaluation may be employed. There is, on the one hand, the outcomes experienced in the classroom as described in Chapter IV and reflected in the individual notebooks of the students. Tests which give some indication of the students' growth in the area emphasized provide an opportunity to compare the results of these individuals with other groups who have followed a different procedure. The reaction of the participants and comments of people closely associated with the group also has merit in determining the degree of success attained. All of these procedures have been used to some extent here.

Extension of Ideas Experienced in the Classroom

The principal value of this study probably lies in determining the actual extensions the students made of concepts from one dimension to other dimensions. Here one finds the results of the student's thinking, what appeared to prompt the extension, how the student thought about the situation, and the extent to which he was able to conceive an idea in various dimensional environments.

The facility with which the students made these extensions varied from those which were teacher-directed, through those suggested by either
the students or the teacher and carried on by contributions of each, to
the culmination of those initiated and developed entirely by the stu-
dents. Although any attempt to classify the experiences into one of
these three categories is somewhat arbitrary as the contribution of
each person was in varying degree, it may be of value to see which
illustrations needed the impetus of teacher direction and which came
spontaneously from the members of the class.

It may be noted that those illustrations at the one extreme
resulted from direct questioning or detailed presentation of some idea
which the teacher felt should be part of the student's experience. At
the other end are found those ideas which some student or group recog-
nized as existing in several different dimensional spaces and were able
to explain or justify to the satisfaction of other classmates.

Teacher-Directed Ideas

**Undefined Terms**

| solid          | tesseract |

**Definitions**

| axis             | half planes  |
| dihedral angle   | lateral area |
| distance between parallel planes | polyhedral angle |
| distance from a point to a plane | skew lines |
| edge of dihedral angle | face of dihedral angle |
Assumptions

An infinite number of planes may contain one point.

An infinite number of planes may contain a given line.

One and only one plane may contain three non-collinear points.

If two lines intersect, then they may be contained in one and only one plane.

A line and a point, not on the line, determine one and only one plane.

Two parallel lines determine one and only one plane.

The intersection of two planes is a straight line.

Three or more planes may intersect in a point.

One and only one line may be perpendicular to a given plane at a given point on the plane.

If a line is parallel to a plane, then it is parallel to every line in the plane having the same direction and skew to all other lines in the plane.

Theorems

The sum of the measures of the sides of a spherical triangle is less than 360 degrees.

The sum of the measures of the angles of a spherical triangle is less than 540 degrees.

Student-Teacher Proposals

Undefined terms

volume

Definitions

adjacent dihedral angles chord of sphere

center of sphere cone
Definitions (contd.)

- coplanar lines and points
- cube
- cylinder
- diameter of sphere
- extension of conditions for a circle to one-space
- face angle
- frustum of a pyramid
- great circle
- line of intersection
- line parallel to plane
- line perpendicular to plane
- measurement of dihedral, polyhedral and spherical angles
- parallel planes
- parallelepiped
- polyhedron
- prism
- pyramid
- radius of sphere
- rectangular solid
- small circle
- sphere
- spherical angle
- tangent to sphere

Assumptions

If a straight line lies in a plane, then all the points on the line are in the plane.

A line intersects a plane in only one point.

Each face angle of a polyhedral angle must measure less than the sum of the measures of the other angles.

A line is perpendicular to a plane if it is perpendicular to the intersecting lines on the plane at the point of intersection.

On a plane one and only one line may be parallel to a given line through a point not on the line. An infinite number of non-intersecting lines, however, may be drawn through the given point if the lines are not in the same plane as the given line.

Through any point outside a given plane, there is one and only one plane parallel to the given plane.
**Assumptions (contd.)**

If a line, not in a plane, is parallel to a line in the plane, it is parallel to the plane.

If three planes intersect in pairs, then the line of intersection of two planes is parallel to the third plane.

**Theorems**

All points which are equidistant from the extremities of a line segment lie on the plane which is the perpendicular bisector of the segment.

If a point lies on the plane which is the perpendicular bisector of a line segment, then it is equidistant from the extremities of the segment.

The square on the diagonal of a rectangular parallelepiped is equal to the sum of the squares on the three dimensions of the parallelepiped.

The volume of a pyramid is equal to one-third the product of the area of the base and its altitude.

The volume of a sphere is equal to \(\frac{4}{3}\pi r^3\) times the cube of the radius.

The lateral area of a rectangular prism is equal to the product of the perimeter of the base and the altitude.

The lateral area of a regular pyramid is equal to one-half the product of the perimeter of the base and the altitude of a face.

The lateral area of a circular cone is equal to one-half the product of the circumference of the base and the slant height.

The surface area of a sphere is equal to \(\frac{4}{3}\pi r^2\) times the square of the radius.

**Student Developed Ideas**

**Definitions**

- alternate dihedral angles
- complementary dihedral angles
- diagonal of a prism
- exterior dihedral angles
- hemisphere
- interior dihedral angles
Definitions (contd.)

- perpendicular planes: vertical dihedral, polyhedral, and spherical angles
- skew planes: "stecking" points to form circles, circles to form spheres, and spheres to form four-space objects
- supplementary dihedral angles

Assumptions

The intersection of three-dimensional solids is a plane.

The intersection of two four-dimensional objects is a three-dimensional solid.

All lines perpendicular to a given line at a given point lie on a plane perpendicular to the line at the point.

In four-space it may be possible to have a line perpendicular to a three-dimensional solid.

If two planes form a right dihedral angle the planes are perpendicular.

A line may be perpendicular to each of two skew lines.

One and only one line may be perpendicular to a given plane from a point outside the plane.

The intersection of a sphere and a plane is a circle.

The locus of points in three-space equidistant from a given point is a sphere with the given point as center and the given distance as radius.

The locus of points in three-space equidistant from a given line is a right cylindrical surface having the given line as axis and the given distance as radius.

The locus of points in three-space equidistant from a given plane is a pair of planes parallel to the given plane and the given distance from it.

The locus of points in three-space equidistant from two parallel planes is a plane parallel to the given planes and midway between them.
Assumptions (contd.)

The locus of points in three-space equidistant from two given points is the plane which is the perpendicular bisector of the line segment joining the given points.

The locus of points in three-space equidistant from two intersecting planes is the pair of planes which bisect the dihedral angles formed by the intersecting planes.

The volume of a rectangular prism is equal to the product of the length, width, and height.

Theorems

The sum of the measures of the face angles of a polyhedral angle is less than 360 degrees.

Vertical dihedral angles are equal.

Vertical polyhedral angles are equal.

Complements of equal dihedral angles are equal.

Supplements of equal dihedral angles are equal.

If two parallel planes are cut by a third plane, the alternate-interior and corresponding dihedral angles are equal.

If two parallel planes are cut by a third plane, the interior dihedral angles on the same side of the transversal plane are supplementary.

If two lines are perpendicular to the same plane, they are parallel.

If two planes are perpendicular to the same line, they are parallel.

The lateral edges of a prism are equal and parallel.

The bases of a prism are congruent polygons.

The diagonals of a parallelepiped bisect each other.

If three or more parallel planes cut off equal segments on one line, they cut off equal segments on all lines which intersect the planes.
Theorems (contd.)

If two planes, equidistant from the center of a sphere intersect the sphere, then the circles formed by the intersections will be equal.

If a plane is tangent to a sphere, then the radius drawn to the common point is perpendicular to the plane.

If two great circles intersect, the vertical spherical angles are equal.

A plane parallel to the base of a pyramid divides the edges of the pyramid proportionally.

The distance between two points in any space is equal to the square root of the sum of the squares of the dimensions, taken at right angles to each other.

The volume of a circular cylinder is equal to the product of the altitude and the area of the base.

The volume of a circular cone is equal to one-third the product of the altitude and the area of the base.

The volumes of two similar three-dimensional solids are proportional to the cubes of the corresponding edges.

Two similar figures are in the ratio which is equal to their corresponding linear measures raised to the same power as the number of the dimensions of the figures.

Among the classifications—undefined terms, definitions, assumptions, and theorems—the student developed ideas appeared to be greatest in the category of theorems. This greater extension may have been due to a need for greater knowledge of the terms and basic assumptions before the student could express his ideas.

Test on the Concept of Dimension

In many instances the extensions just recorded were a culmination of the thinking and discussion of the entire class or at least a portion of the group, rather than one individual's effort alone. Un-
doubtedly the comment of one member of the class would serve as a stimulus to another member to see how much farther he could extend an idea. While this group effort has many values, there is also interest in the effect the emphasis on dimension may have had in changing the thinking of each member of the class as compared to changes which would occur among students who followed a different approach.

To assist in the individual evaluation of those participating as a member of the experimental group, as a student in the classes which studied plane geometry only, or as a pupil in the planned, combined courses in plane and solid geometry, the author constructed a test on the extension of the dimension concept. No commercial test was available stressing the extension of this concept.

As a basis for determining the material to use in formulating questions which should be included in a test which was to give some indication of student thinking in moving from one dimension to another, the material suggested by the Report of the Commission on Mathematics for a unit in solid and spherical geometry served as background. This outline recommends principles concerned with

1) determination of a plane and the relationships that exist between lines and planes, and between parallel planes and their lines of intersection with a third plane,

2) a line perpendicular to a plane and two planes perpendicular to each of them,

\[\text{1A copy of this test appears in Appendix C.}\]
3) common solids, and
4) extension of loci to three dimensions.²

Early in the spring of 1962, the first formulation of the test was administered to a plane geometry class in a school not connected with the study. This test was too long to be completed in one class period, and some of the questions were difficult to interpret. Although there was no intent to place a time limit on the test, it was desirable to have it administered in one class period so it would reflect the individual thinking of the students. After the test was shortened and the questions clarified, it was administered to another non-participating geometry class. It now appeared to be appropriate in length and clear in interpretation. These two classes followed a traditional program of geometry, and at least one of the participating schools was planning to use one of the recently developed mathematics programs, so for this reason the test was then administered to a geometry class in each of the schools which had agreed to participate in the study the following year. At the suggestion of the teachers in these schools, a few minor changes were made in the test before it was considered to be in final form.

The test was administered to the students in each of the three categories when school opened in the fall and again in the spring when a year's study of geometry was almost complete. In order to give some indication of the reliability of this test, the results obtained from the experimental group and the four classes from which the other partic-

Participants were selected were used to determine the reliability coefficient. As stated by Thorndike and Hagen, "A test with relatively low reliability will permit us to make useful studies of and draw accurate conclusions about groups, but relatively high reliability is required if we are to have precise information about individuals."\(^3\)

The Kuder-Richardson (Formula 21),

\[
   r_{11} = \frac{n}{n - 1} \left[ 1 - \frac{M_t \left( 1 - \frac{M_t}{n} \right)}{s_t^2} \right]
\]

was used. In the formula \(r_{11}\) represents the estimate of reliability; \(n\), the number of items in the test; \(M_t\), the mean score of the group; and \(s_t\), the standard deviation of the test.\(^4\) When the substitutions, \(n = 55\), \(M_t = 8.8\), and \(s_t^2 = 32.81\), were made the test was found to have a reliability coefficient of 0.78.

The test was constructed so that a series of questions began with a situation in zero or one-space, then proceeded to two-space and three-space, and finally, in a few cases, to four-space. The student has a physical model to which he might relate some of these items and the acceptableness of his reply be determined, but other ideas exist only in his imagination or as he interprets the analogous problem. Because of this difference, the results of the test were considered in two parts. The first analysis was made with respect to those items


\(^4\)Ibid., p. 181.
pertaining to three-space or less for which correct replies might be determined according to the usual course in geometry, and then the results of the questions relating to the opinion items were compared separately.

It will be recalled from Chapter III that the experimental group is referred to as Group A, the students in the combined course as Group B, and the students enrolled in plane geometry only as Group C. There were fifty-five items which applied to zero through three-space. Table 9 shows the number of questions answered correctly for each of the three groups on the initial test and also the final test.

<table>
<thead>
<tr>
<th>Table 9</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>INITIAL AND FINAL SCORES ON THE TEST ON THE CONCEPT OF DIMENSION FOR EACH STUDENT IN GROUPS A, B, AND C</strong></td>
</tr>
</tbody>
</table>

| Group A | | Group B | | Group C |
|---------|---------|---------|---------|
| Initial | Final   | Initial | Final   | Initial | Final   |
| 40      | 55      | 6       | 27      | 6       | 26      |
| 16      | 55      | 11      | 26      | 12      | 22      |
| 7       | 14      | 11      | 23      | 9       | 22      |
| 6       | 14      | 11      | 22      | 13      | 19      |
| 7       | 14      | 7       | 21      | 9       | 19      |
| 10      | 14      | 7       | 20      | 10      | 18      |
| 15      | 38      | 12      | 17      | 9       | 18      |
| 13      | 38      | 14      | 15      | 6       | 17      |
| 7       | 38      | 3       | 14      | 5       | 17      |
| 18      | 37      | 16      | 13      | 1       | 15      |
| 6       | 37      | 11      | 13      | 16      | 13      |
| 15      | 31      | 5       | 13      | 13      | 13      |
| 7       | 30      | 8       | 9       | 7       | 13      |
| 17      | 29      | 8       | 7       | 0       | 12      |
| 16      | 29      | 8       | 4       | 0       | 10      |
| 5       | 26      | 13      | 3       | 8       | 5       |
Since the three groups were not matched, Fisher's methods of analysis of covariance as described by Lindquist were used to compare the results of these tests. The statistical controls used in the analysis of covariance make allowances for initial differences in the groups and thus the inconveniences of matching procedures in educational research are eliminated. Table 10 shows the total score and mean for each group for the initial and the final tests separately.

**TABLE 10**

TOTAL SCORE AND MEAN FOR GROUPS A, B, AND C
FOR THE INITIAL AND THE FINAL TESTS

<table>
<thead>
<tr>
<th>Group</th>
<th>Initial Scores</th>
<th>Final Scores</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Total</td>
<td>Mean</td>
</tr>
<tr>
<td>Group A</td>
<td>205</td>
<td>12.8</td>
</tr>
<tr>
<td>Group B</td>
<td>151</td>
<td>9.1*</td>
</tr>
<tr>
<td>Group C</td>
<td>124</td>
<td>7.3</td>
</tr>
</tbody>
</table>

Before the final means were corrected for initial differences the methods of analysis of variance were used to find the variance for methods and the variance within groups. These results are given in Table 11.

Using the information from these two tables, the reduced variance for methods and the adjusted error variance (within groups) were deter-

---


\(^6\) Ibid., pp. 86-95.
mined and the ratio of these two gave the variance ratio. The ratio is 44.0. This variance ratio indicates that the difference between methods is significant.\textsuperscript{7}

\textbf{TABLE 11}

\textbf{SUM OF SQUARES FOR METHODS AND WITHIN GROUPS}

\begin{table}[h]
\centering
\begin{tabular}{lcc}
\hline
 & Initial Scores & Final Scores \\
\hline
Methods & 212.6 & 5365.2 \\
Within Groups & 1625.4 & 2301.4 \\
Total & 1838.0 & 7666.6 \\
\hline
\end{tabular}
\end{table}

Next the adjusted final mean for each method was computed. The average regression within groups was found to be 0.3212 and the general mean for initial test scores was 10.0.\textsuperscript{8} Table 12 lists the initial mean, the final mean, and the adjusted final mean for each group.

\textbf{TABLE 12}

\textbf{INITIAL MEAN, FINAL MEAN, AND ADJUSTED FINAL MEAN FOR GROUPS A, B, AND C}

\begin{table}[h]
\centering
\begin{tabular}{lccc}
\hline
 & Initial Mean & Final Mean & Adjusted Final Mean \\
\hline
Group A & 12.8 & 38.3 & 37.50 \\
Group B & 9.4 & 15.4 & 15.59 \\
Group C & 7.8 & 16.2 & 16.91 \\
\hline
\end{tabular}
\end{table}

\textsuperscript{7}\textit{Ibid.}, pp. 192-193.

\textsuperscript{8}\textit{Ibid.}, p. 186 and p. 194.
Since the difference between methods was significant, the standard error of a methods mean and the standard error of the difference between adjusted methods means were determined. The standard error of the difference was computed by the formula:

\[
\sigma_{\text{diff.}}^2 = \left[ \frac{2}{n} + \frac{(\bar{X}_A - \bar{X}_B)^2}{\sum x^2} \right] s^2
\]

where \( s^2 \) is the adjusted error sum of squares, \( n \) is the number of pupils per class, \( \bar{X}_A \) is the initial A-mean, \( \bar{X}_B \) is the initial B-mean, and \( \sum x^2 \) is the initial sum of squares within groups.\(^9\) For the three methods:

\[
\sigma_{A - B} = 2.53 \\
\sigma_{A - C} = 2.61 \\
\sigma_{B - C} = 2.18
\]

Since a \( t \) of 2.70\(^{10}\) would be required at the 1 per cent level of significance, a difference between adjusted methods means for A and B would have to be greater than 6.83; for A and C, greater than 7.05; and for B and C, 6.70. The difference in adjusted methods means for A and B is 21.81; for A and C, 20.49; and for B and C, 1.32. Thus there is a significant difference between the experimental group and each of the other groups on the Test on the Concept of Dimension, but the difference

---

\(^9\)Ibid., p. 195.

\(^{10}\)Merle W. Tate and Richard C. Clelland, Nonparametric and Shortcut Statistics in the Social, Biological, and Medical Sciences (Danville, Illinois: Interstate Printers and Publishers, Inc., 1957), Table C, p. 128.
between the group studying the pre-planned plane and solid geometry course and those studying plane geometry only is not significant.

Table 13 gives a summary of the opinions expressed on those items which extended the dimension concept beyond three-space. In this table the number and item are listed with an indication of the number of students in each category giving a reply. Just below the item is listed (a) the most frequent reply on the initial test, with the number giving the reply and (b) the most frequent reply on the final test. In the initial test, three items, Nos. 15, 30, and 33, had more than one most frequent reply so each is listed.

From the summaries in Table 13, these findings follow:

1) The number of students venturing a reply increased on the final test for each item for Group A; for all but one, where it remained the same, for Group C; but for Group B, the number decreased on seven of the eighteen items.

2) The most frequent reply given on the final test changed in thirteen instances from that given on the initial test.

3) The most frequent reply given on the final test is the one to expect as an extension of the analogous situation in two- and three-space except for the last item where the expected reply would be "infinite."

4) At least five-eighths of Group A expressed an opinion on each item of the final test. The number was as low as one in sixteen for Group B and three in sixteen for Group C.

5) There was one most frequent answer for each item on the final test; however, on the initial test, three items had more than one most frequent reply.
### TABLE 13

**SUMMARY OF TYPE AND NUMBER OF REPLIES TO ITEMS EXTENDING THE CONCEPT OF DIMENSION BEYOND THREE-SPACE FOR GROUPS A, B, AND C**

<table>
<thead>
<tr>
<th>Test Item No., Question and Most Frequent Reply</th>
<th>Group A</th>
<th>Group B</th>
<th>Group C</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>No. of replies</td>
<td>No. of replies</td>
<td>No. of replies</td>
</tr>
<tr>
<td></td>
<td>Initial</td>
<td>Final</td>
<td>Initial</td>
</tr>
<tr>
<td><strong>10. Consider a point P in space. How many distinct solids may contain this point?</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a. one</td>
<td>7</td>
<td>14</td>
<td>9</td>
</tr>
<tr>
<td>b. infinite</td>
<td>4</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Discuss this situation for a four-dimensional space.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a. one</td>
<td>7</td>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td>b. infinite</td>
<td>2</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td><strong>11. If you are given two distinct points P and Q, how many distinct solids of three-space may contain these two points?</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a. one</td>
<td>7</td>
<td>14</td>
<td>6</td>
</tr>
<tr>
<td>b. infinite</td>
<td>2</td>
<td>2</td>
<td>11</td>
</tr>
<tr>
<td>Discuss this situation for a four-dimensional space.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a. infinite</td>
<td>7</td>
<td>10</td>
<td>4</td>
</tr>
<tr>
<td>b. infinite</td>
<td>4</td>
<td>0</td>
<td>6</td>
</tr>
</tbody>
</table>

*The most frequent reply on the initial test, with the number of students in each group giving the reply.

b*The most frequent reply on the final test, with the number of students in each group giving the reply.
<table>
<thead>
<tr>
<th>Test Item No., Question, and Most Frequent Reply</th>
<th>Group A</th>
<th>Group B</th>
<th>Group C</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>No. of replies</td>
<td>No. of replies</td>
<td>No. of replies</td>
</tr>
<tr>
<td>Initial</td>
<td>Final</td>
<td>Initial</td>
<td>Final</td>
</tr>
<tr>
<td>12. If you are given three points P, Q, and R, not on the same line, how many solids of three-space may contain these three points?</td>
<td>7</td>
<td>15</td>
<td>5</td>
</tr>
<tr>
<td>a. one</td>
<td>4</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>b. infinite</td>
<td>12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Discuss this situation in a four-dimensional space.</td>
<td>6</td>
<td>10</td>
<td>3</td>
</tr>
<tr>
<td>a. infinite</td>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>b. infinite</td>
<td>6</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>13. If you are given four points P, Q, R, and S, not in the same plane, how many solids of three-space may contain all these points?</td>
<td>5</td>
<td>16</td>
<td>5</td>
</tr>
<tr>
<td>a. one</td>
<td>0</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>b. infinite</td>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Discuss this situation for a solid in a four-dimensional space.</td>
<td>5</td>
<td>11</td>
<td>4</td>
</tr>
<tr>
<td>a. one</td>
<td>1</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>b. infinite</td>
<td>6</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>
TABLE 13—Continued

<table>
<thead>
<tr>
<th>Test Item No., Question, and Most Frequent Reply</th>
<th>Group A</th>
<th>Group B</th>
<th>Group C</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>No. of replies</td>
<td>No. of replies</td>
<td>No. of replies</td>
</tr>
<tr>
<td></td>
<td>Initial</td>
<td>Final</td>
<td>Initial</td>
</tr>
<tr>
<td>15. If you recognize any relationship between the number of points needed to determine a geometric figure and the number of dimensions the figure has, express this relationship.</td>
<td>6</td>
<td>14</td>
<td>4</td>
</tr>
<tr>
<td>a. no relationship number points equal number dimensions</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>b. number points equal number dimensions plus one</td>
<td></td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>16. From the general statement it would appear the number of points needed to determine a four-dimensional figure would be _______.</td>
<td>6</td>
<td>15</td>
<td>8</td>
</tr>
<tr>
<td>a. four</td>
<td>0</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>b. five</td>
<td>11</td>
<td></td>
<td>2</td>
</tr>
</tbody>
</table>

*The three replies were given an equal number of times on the initial test.*
### TABLE 13—Continued

<table>
<thead>
<tr>
<th>Test Item No., Question, and Most Frequent Reply</th>
<th>Group A</th>
<th>Group B</th>
<th>Group C</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>No. of replies</td>
<td>No. of replies</td>
<td>No. of replies</td>
</tr>
<tr>
<td></td>
<td>Initial</td>
<td>Final</td>
<td>Initial</td>
</tr>
<tr>
<td>16 (continued). If there is any limitation on the location of these points, state this limitation.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a. no limitation</td>
<td>3</td>
<td>11</td>
<td>3</td>
</tr>
<tr>
<td>b. not on same solid</td>
<td></td>
<td></td>
<td>6</td>
</tr>
<tr>
<td>19. Consider a three-dimensional solid s. Assume a point P, not contained in s, and discuss the question as to how many lines may be drawn perpendicular to s through P.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a. one</td>
<td>5</td>
<td>14</td>
<td>6</td>
</tr>
<tr>
<td>b. one</td>
<td>2</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>22. Consider a solid in a three-dimensional space. Assume a point P, not contained in the solid, and discuss the question as to how many planes may be drawn perpendicular to the solid through P.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a. one</td>
<td>3</td>
<td>12</td>
<td>5</td>
</tr>
<tr>
<td>b. infinite</td>
<td>1</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>Test Item No., Question, and Most Frequent Reply</td>
<td>Group A</td>
<td>Group B</td>
<td>Group C</td>
</tr>
<tr>
<td>-----------------------------------------------</td>
<td>---------</td>
<td>---------</td>
<td>---------</td>
</tr>
<tr>
<td></td>
<td>No. of replies</td>
<td>No. of replies</td>
<td>No. of replies</td>
</tr>
<tr>
<td></td>
<td>Initial Final</td>
<td>Initial Final</td>
<td>Initial Final</td>
</tr>
<tr>
<td>26. Describe the intersection of two distinct three-dimensional solids.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a. three-dimensional solid</td>
<td>7 14</td>
<td>2 3</td>
<td>4 5</td>
</tr>
<tr>
<td>b. plane</td>
<td>3 6</td>
<td>1 2</td>
<td>2 1</td>
</tr>
<tr>
<td>27. Describe the intersection of two distinct four-dimensional objects.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a. no two answers the same</td>
<td>4 12</td>
<td>1 1</td>
<td>3 3</td>
</tr>
<tr>
<td>b. solid</td>
<td>6 0</td>
<td>0 0</td>
<td></td>
</tr>
<tr>
<td>30. Describe the intersection of a three-dimensional solid and a plane not contained in the solid.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a. none</td>
<td>3 13</td>
<td>5 1</td>
<td>3 4</td>
</tr>
<tr>
<td>solid</td>
<td>0 2</td>
<td>0 0</td>
<td>0 0</td>
</tr>
<tr>
<td>b. line</td>
<td>5 0</td>
<td>0 0</td>
<td>0 0</td>
</tr>
</tbody>
</table>

d. The two replies were given an equal number of times on the initial test.
<table>
<thead>
<tr>
<th>Test Item No., Question, and Most Frequent Reply</th>
<th>Group A</th>
<th>Group B</th>
<th>Group C</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>No. of replies</td>
<td>No. of replies</td>
<td>No. of replies</td>
</tr>
<tr>
<td></td>
<td>Initial</td>
<td>Final</td>
<td>Initial</td>
</tr>
<tr>
<td>33. Consider a three-dimensional solid s and a point P, not contained in s. Discuss the question as to how many lines may contain P and be parallel to s.</td>
<td>2</td>
<td>12</td>
<td>0</td>
</tr>
<tr>
<td>a. infinite</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>b. infinite</td>
<td>7</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Discuss the question as to how many planes may contain P and be parallel to s.</td>
<td>2</td>
<td>12</td>
<td>1</td>
</tr>
<tr>
<td>a. two</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>three infinite</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>b. one</td>
<td>8</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

*The three replies were given an equal number of times on the initial test.*
The range, mean, and median number of total replies suggested on these eighteen items for both the initial test and the final test are shown for all three groups in Table 14.

### Table 14

**Range, Mean, and Median for Groups A, B, and C for the Items Extended Beyond Three-Space on the Test on Concept of Dimension for Initial and Final Scores**

<table>
<thead>
<tr>
<th></th>
<th>Initial Scores</th>
<th></th>
<th>Final Scores</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Range</td>
<td>Mean</td>
<td>Median</td>
</tr>
<tr>
<td>Group A</td>
<td>0 - 17</td>
<td>5.8</td>
<td>3.0</td>
</tr>
<tr>
<td>Group B</td>
<td>1 - 16</td>
<td>4.8</td>
<td>2.0</td>
</tr>
<tr>
<td>Group C</td>
<td>0 - 18</td>
<td>5.3</td>
<td>4.5</td>
</tr>
</tbody>
</table>

**Shaycoft Plane Geometry Test**

Toward the close of the school year, the Shaycoft Plane Geometry Test, Form Am was administered to all of the participating classes. The test was used to measure the achievement in two-space geometry since material of this dimension, only, was common to all the participants.

The highest score which may be attained on this test is 60, and the range of scores for the forty-eight students, three groups of sixteen each, compared was 26 - 59. In order to compare the three groups, the Kruskal-Wallis sum of ranks or H test was used. This test is the

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11 A copy of this test appears in Appendix C.
non-parametric analogue of the variance-ratio test of differences among means of three or more samples in single classification.\footnote{12} The forty-eight scores were ranked with the highest score receiving a rating of "1" and the lowest score, a rank of "48." Tied scores were ranked by the average. The scores with the corresponding rank are given by groups in Table 15. The null hypothesis, that the populations are identical, is tested by comparing the observed sum of ranks with their expected values by

$$H = \left[ \frac{12}{N(N + 1)} \right] \left[ \sum \frac{R_i^2}{n_i} \right] - 3(N + 1)$$

where $R_i$ is the sum of ranks, $n_i$ the number in each sample, and $N$ is the total number.\footnote{13}

By substituting in the formula, $H = 4.08$. At 2 degrees of freedom 4.08 is significant at the 10 per cent level.\footnote{14}

From Table 15 it may be observed that the experimental sum of ranks is smaller than either of the other groups, indicating that as a whole this group ranked higher.

This higher rank in achievement may be due to any one or a combination of several factors, such as mental ability, geometric aptitude, attitude, method of instruction, or other variables, but the achievement of the experimental group does compare favorably with that of all the participants.

\footnote{12}Ibid., pp. 109-111.\footnote{13}Ibid., p. 110.\footnote{14}Ibid., Table B, p. 127.
### TABLE 15

**SCORE AND RANK FOR EACH STUDENT IN GROUPS A, B, AND C ON THE SHAYCOFT PLANE GEOMETRY TEST**

<table>
<thead>
<tr>
<th>Group A</th>
<th>Group B</th>
<th>Group C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Score</td>
<td>Rank</td>
<td>Score</td>
</tr>
<tr>
<td>59</td>
<td>1</td>
<td>54</td>
</tr>
<tr>
<td>54</td>
<td>2 ½</td>
<td>47</td>
</tr>
<tr>
<td>48</td>
<td>6</td>
<td>46</td>
</tr>
<tr>
<td>47</td>
<td>7 ½</td>
<td>46</td>
</tr>
<tr>
<td>46</td>
<td>10 ½</td>
<td>45</td>
</tr>
<tr>
<td>45</td>
<td>11 ½</td>
<td>41</td>
</tr>
<tr>
<td>44</td>
<td>18</td>
<td>40</td>
</tr>
<tr>
<td>43</td>
<td>18 ½</td>
<td>38</td>
</tr>
<tr>
<td>41</td>
<td>28</td>
<td>38</td>
</tr>
<tr>
<td>40</td>
<td>28</td>
<td>36</td>
</tr>
<tr>
<td>40</td>
<td>28</td>
<td>35</td>
</tr>
<tr>
<td>40</td>
<td>28 ½</td>
<td>35</td>
</tr>
<tr>
<td>39</td>
<td>31 ½</td>
<td>29</td>
</tr>
<tr>
<td>36</td>
<td>39 ½</td>
<td>26</td>
</tr>
</tbody>
</table>

### Comments of Students and Observers

No instrument was prepared nor was any formal procedure used to obtain comments from students or observers concerning their reaction to the stressing of ideas in various dimensions, but a record of casual remarks or chance reports from those outside the regular class was maintained. Although some of the reactions pertain to the course in...
general, these still reflect the overall feeling of the students for whom the concept of dimension played a major role. A few of these informal incidents are described here to give the reader the benefit of these comments.

The only senior who was a member of the experimental group was overheard in conversation with some friends. His adviser and parents had urged him to enroll in geometry as he planned to attend college, although he had a dislike for mathematics. Now that the course was almost complete he found he had enjoyed it very much and felt his friends had missed a good opportunity by not enrolling also.

Throughout the year two students, particularly, carried on some friendly competition to see which one would be able to think of an extension of an idea first and then whether others could be added. Many times they left the classroom still engrossed in their discussion. Near the close of the term one of them observed that some of his ideas had been very unsound, but he liked the attempt to defend them.

One girl remarked she preferred this course to all her previous mathematics courses because she liked the opportunity to express her ideas rather than just work a routine set of problems.

On the less enjoyable side of the picture there were the two boys who thought the notebook was tedious and they would rather just try to retain the material in their minds. There was also the girl who seemed to have little imagination and needed a physical model for all situations and somewhat resented the suggestions which she did not easily comprehend.
Although there were no regular observers in the class, occasionally during the year, a college student preparing to become a secondary mathematics teacher visited the class. One of these observers was a mature woman who had taught several college freshman mathematics courses and was now becoming certified for public school teaching. She observed the day the students were discussing the definition of a prism. Her reaction was one of amazement at the imagination of high school students which she felt had not been apparent in her college classes. She remarked, "I had rather dreaded teaching high school students after working with older students, but if they can think like this and make class this exciting, I will find it even more interesting than teaching college students."

One young man who was to present a written report of his observations for a psychology class informed the teacher of the geometry class that he received a "B" on his observation report because he became so absorbed in the students' work he forgot to note some of the routine matters he had been assigned to observe.

A young woman who observed some of the students working on extending ideas of parallelism expressed the hope she could use some of the ideas in her student teaching. Later in the year the writer was discussing this girl's work with her critic teacher when he said, "Miss P--- had an interesting lesson on perpendicular lines yesterday. She even had the students discussing perpendicular planes. I don't know where she got the idea for doing this as we had not mentioned it in her pre-planning."
The parent of one student who was receiving higher marks in her other classes wrote on her nine weeks' evaluation report, "Probably T--- will never be a math expert but I feel the geometry is developmental and that she has received quite a lot from it."

The teacher of one of the plane geometry classes participating in the study reported that, when the dimension test was given at the end of the year, one student who had shown little interest in the class all year came to her after school and discussed the test for more than a half-hour. Of course there were also some students who thought the test contained some ridiculous questions as they did not think the fourth dimension existed.

The experimenter may have been prone to note the good comments and ignore somewhat the less favorable; however, no major criticisms of an adverse nature were received during the year. It is worthy to note that the incidents mentioned above were all voluntary contributions, and not made especially to impress someone.

Summary

The results of the various types of evaluation used in measuring the outcomes of stressing the concept of dimension in the study of geometry are described in this chapter. The evaluation procedures include:

1) a record of the extension of ideas which were experienced in the classroom,
2) a test on the concept of dimension,
3) a test on geometric achievement, and
4) comments from students, observers, and others.
CHAPTER VI

SUMMARY

Review of the Experiment

For several decades mathematics educators have realized that far more time is spent by students in the study of two-dimensional geometry than in three-dimensional. This emphasis seems unrealistic when one recognizes that the world in which the student lives is of the latter nature. To add to the problem, rapid developments in mathematics have created a need for students to make greater progress in their mathematics preparation while in high school than has ever been expected previously. If students are to become familiar with more material in less time, some change in the geometry program appears necessary.

Since the similarities and differences surrounding the aspects of plane and solid geometry seem to revolve around the limitations of two-space as compared to the greater freedom of three-space, the concept of dimension should provide a fruitful basis for the integration of fundamental spatial relations. Furthermore, if students are to appreciate fully the influences of this idea, they should not be bound to two and three space situations but should be encouraged to think as readily in worlds of other dimensions. This study proposed, therefore, to test the following hypotheses:

1) If students receive an early awareness of the concept of
dimension, they will tend to extend the ideas studied in two-dimensional space to spaces of other dimensions, particularly three-space.

2) Through such extensions to other dimensions the student will integrate the concepts of the various spaces, and this integration will lead to a program of spatial insights initiated primarily by the student.

3) The emphasis placed on the concept of dimension will not effect student achievement in the material normally included in the traditional two-space study.

While investigating these hypotheses, the procedures used by the pupils were of greater significance than the content. The illustrations given in Chapter IV reflected these procedures and the associated results:

1) Early in the course emphasis was placed on the extension of ideas from one dimension to another by means of study guides and class discussion.

2) The extension of a concept was considered in a world of fewer dimensions as well as in one of more dimensions.

3) No textbook was officially adopted.

4) Students were encouraged to formulate their own definitions and assumptions and to indicate undefined terms, but for purposes of use later in the course, general agreements were reached by the class on these items.

5) Each pupil had his own list of proved theorems and these varied among the students; however, all of them proved those
essential for developing the sequence of major theorems in the course.

6) Some pertinent ideas in more than one dimension were presented directly by the teacher.

7) Suggestions and questions from the students which would extend concepts to several dimensions were encouraged. These were then supplemented by comments from other pupils and the teacher.

8) Several extensions of definitions, assumptions, or theorems were formulated and explained entirely by the students.

9) Each student developed his own course by preparing a notebook showing his undefined terms, definitions, assumptions, and theorems in both two- and three-space along with the extension of his ideas to various other dimensional spaces.

**Generalizations from Experiment**

Although in testing the stated hypotheses this study reports the activities of only one class, it appears that the students in this group did tend to extend the ideas of two-space to other dimensional spaces and thus integrate these various concepts. These first two hypotheses are very closely related as shown by evidence of these tendencies of extension and integration in:

1) the descriptions of actual class activities reported in Chapter IV,

2) the material included in the notebooks of the students as indicated in the geometric content listed in Appendix A,
3) the results of the Test on the Concept of Dimension which showed a significant difference between methods for those students in the experimental group and those who studied a pre-planned, combination course in plane and solid geometry or those who studied a course in plane geometry only; however, the difference between methods for the last two groups was not significant. Furthermore, the test also indicated a greater tendency on the part of the experimental group to venture extensions to spaces for which the student had no physical model compared to those in the other two groups.

Regarding the third hypothesis, the emphasis on extension of ideas to various dimensions did not appear to have an adverse effect on the student achievement in two-space geometry. By applying the Kruskal-Wallis sum of ranks test to the results of the Shaycoft Plane Geometry Test, it was found that as a total the experimental group ranked higher than the students in either the combined or plane geometry courses.

Comments from observers and students showed that apparently the class found the extension of the concept of dimension interesting and imaginative.

**Recommendations for Further Study**

1) Since this research involved only one class as the experimental group, further study of other comparable groups would indicate whether or not the results tended to substantiate those observed here.
2) Instruments for measuring the tendency of students to extend concepts studied to analogous situations in various dimensional spaces have not been developed. The preparation of devices by someone not directly involved in the teaching of the course would be beneficial. Tests from this source would tend to eliminate any advantages due to the selection of materiel or phrasing of questions.

3) One of the desired outcomes is that the students who integrate the ideas of two- and three-space geometry by means of emphasizing the extension of the concept of dimension be able to achieve at least as well in later mathematics courses involving three-space as those studying a pre-planned combined course in plane and solid geometry or as those who study the two phases separately. A follow-up study of those participating in the three methods of studying geometry and continuing the study of mathematics should be made at the time these people complete calculus to determine how well each of the three groups achieves in materials related to three-space concepts.

4) The results of the Test on the Concept of Dimension showed that the students of the pre-planned combined course had less tendency to venture extending ideas beyond the realm of their study than those students in the other two groups. Does this formal instruction tend to stifle the imagination of the learner? Further information is needed concerning
the effect of such pre-planned, relatively rigid, programs on the creativeness of the student.
APPENDIX A

To give some indication of the material studied in the course, lists of the undefined terms, definitions, assumptions, and theorems are given. Preceding each item is an indication of the manner in which the item became a part of the course. The three classifications include (1) teacher-directed ideas, (2) student-teacher proposals, (3) student developed ideas. In some instances it was difficult to determine the proper category since a few students may have developed the idea independently while most of the class needed to exchange thoughts with other classmates and the teacher.

As in any study of geometry it was necessary to begin with certain undefined terms, and, as the year progressed, others were added as needed. These terms are listed in Table 16.
TABLE 16

UNDEFINED TERMS ACCEPTED FOR THE COURSE

<table>
<thead>
<tr>
<th>Category²</th>
<th>Item</th>
<th>Category</th>
<th>Item</th>
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</tr>
<tr>
<td>x</td>
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<td>x</td>
<td>between</td>
<td>x</td>
<td>line</td>
</tr>
<tr>
<td>x</td>
<td>dimension</td>
<td>x</td>
<td>straight line</td>
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<tr>
<td>x</td>
<td>direction</td>
<td>x</td>
<td>plane</td>
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<tr>
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<td>distance</td>
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<td>tesseract</td>
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<tr>
<td>x</td>
<td>inside</td>
<td>x</td>
<td>volume</td>
</tr>
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</table>

²(1) teacher-directed ideas, (2) student-teacher proposals, (3) student-developed ideas.

Definitions for many of the terms and concepts used were developed through the use of study guides, such as that for "adjacent angles," as illustrated in Chapter IV. Others were the outgrowth of extending other definitions or concepts as shown by the introduction of the term "tangent." Some terms were developed as the need arose in class discussion. These defined terms are listed in Table 17.
TABLE 17
DEFINITIONS

<table>
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<th>Item</th>
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<tbody>
<tr>
<td>(1)</td>
<td>x  acute angle</td>
</tr>
<tr>
<td>(2)</td>
<td>x  acute triangle</td>
</tr>
<tr>
<td>(3)</td>
<td>x  adjacent angle</td>
</tr>
<tr>
<td>(1)</td>
<td>x  adjacent dihedral angle</td>
</tr>
<tr>
<td>(1)</td>
<td>x  alternate angles</td>
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<td>(1)</td>
<td>x  alternate dihedral angles</td>
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<tr>
<td>(1)</td>
<td>x  altitude of polygon or polyhedron</td>
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<td>(1)</td>
<td>x  angle-plane arc-major and minor</td>
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<td>(1)</td>
<td>x  axis</td>
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<td>x  base of polygon or polyhedron</td>
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<td>(1)</td>
<td>x  bases of isosceles trapezoid</td>
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<td>x  bisect</td>
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<td>x  bisector</td>
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<td>x  broken line</td>
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<td>x  center of a circle or sphere</td>
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<td>x  central angle</td>
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<td>x  chord of a circle or sphere</td>
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<td>x  circumference</td>
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<td>x  circumscribe</td>
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<td>x  complement</td>
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<td>x  complementary angles</td>
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<td>x  cones</td>
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<td>(1)</td>
<td>x  congruent figures</td>
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<sup>a</sup>(1) teacher-directed ideas, (2) student-teacher proposals, (3) student-developed ideas.
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<td>inscribe</td>
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<td>projection</td>
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<tr>
<td>x</td>
<td>proportion—by inversion, by alternation</td>
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<td>pyramid</td>
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<td>ray</td>
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<tr>
<td>x</td>
<td>skew planes</td>
</tr>
<tr>
<td>x</td>
<td>slant height</td>
</tr>
<tr>
<td>x</td>
<td>small circle</td>
</tr>
<tr>
<td>x</td>
<td>sphere</td>
</tr>
<tr>
<td>x</td>
<td>spherical angle</td>
</tr>
<tr>
<td>x</td>
<td>square</td>
</tr>
<tr>
<td>x</td>
<td>straight angle</td>
</tr>
</tbody>
</table>
TABLE 17—Continued

<table>
<thead>
<tr>
<th>Category</th>
<th>Item</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) (2) (3)</td>
<td></td>
</tr>
<tr>
<td>x</td>
<td>supplement</td>
</tr>
<tr>
<td>x</td>
<td>supplementary angles</td>
</tr>
<tr>
<td>x</td>
<td>supplementary dihedral angles</td>
</tr>
<tr>
<td>x</td>
<td>tangent of a circle</td>
</tr>
<tr>
<td>x</td>
<td>tangent of a sphere (line and plane)</td>
</tr>
<tr>
<td>x</td>
<td>terminal side</td>
</tr>
<tr>
<td>x</td>
<td>transversal</td>
</tr>
<tr>
<td>x</td>
<td>trapezoid</td>
</tr>
<tr>
<td>x</td>
<td>triangle</td>
</tr>
<tr>
<td>x</td>
<td>vertex of an angle</td>
</tr>
<tr>
<td>x</td>
<td>vertical angles</td>
</tr>
<tr>
<td>x</td>
<td>vertical dihedral angles</td>
</tr>
<tr>
<td>x</td>
<td>vertical polyhedral angles</td>
</tr>
<tr>
<td>x</td>
<td>vertical spherical angles</td>
</tr>
</tbody>
</table>

In a manner similar to the procedures used for determining definitions, agreement was reached on statements to be accepted without proof. No attempt was made to keep the number of assumed statements to a minimum. Due to the nature of the course many of the generalizations agreed upon for three-space were analogous to statements in two-space and were not proved deductively. Consequently, several of the propositions which are normally given a more rigorous treatment in a solid geometry course are classified as assumptions in this study. The assumed statements are listed in Table 18.
<table>
<thead>
<tr>
<th>Category(^a)</th>
<th>Item</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>x</td>
<td></td>
</tr>
<tr>
<td>x</td>
<td></td>
</tr>
<tr>
<td>x</td>
<td></td>
</tr>
<tr>
<td>x</td>
<td></td>
</tr>
<tr>
<td>x</td>
<td></td>
</tr>
<tr>
<td>x</td>
<td></td>
</tr>
<tr>
<td>x</td>
<td></td>
</tr>
<tr>
<td>x</td>
<td></td>
</tr>
<tr>
<td>x</td>
<td></td>
</tr>
<tr>
<td>x</td>
<td></td>
</tr>
<tr>
<td>x</td>
<td></td>
</tr>
<tr>
<td>x</td>
<td></td>
</tr>
<tr>
<td>x</td>
<td></td>
</tr>
</tbody>
</table>

\(^a\) (1) teacher-directed ideas, (2) student-teacher proposals, (3) student-developed ideas.
<table>
<thead>
<tr>
<th>Category</th>
<th>Item</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>x</td>
<td>If a straight line lies in a plane, then all the points on the line are in the plane.</td>
</tr>
<tr>
<td>x</td>
<td>The intersection of two planes is a straight line.</td>
</tr>
<tr>
<td>x</td>
<td>Three or more planes may intersect in a point.</td>
</tr>
<tr>
<td>x</td>
<td>A line intersects a plane in only one point.</td>
</tr>
<tr>
<td>x</td>
<td>Each face angle of a polyhedral angle must measure less than the sum of the measures of the other angles.</td>
</tr>
<tr>
<td>x</td>
<td>On a plane one and only one line may be perpendicular to a given line at a given point on the line.</td>
</tr>
<tr>
<td>x</td>
<td>In space one and only one plane may be perpendicular to a given line at a given point on the line.</td>
</tr>
<tr>
<td>x</td>
<td>All lines perpendicular to a given line at a given point lie on a plane perpendicular to the line at the point.</td>
</tr>
<tr>
<td>x</td>
<td>A line is perpendicular to a plane if it is perpendicular to two or more intersecting lines on the plane at the point of intersection.</td>
</tr>
<tr>
<td>x</td>
<td>One and only one line may be perpendicular to a given line from a point not on the line.</td>
</tr>
<tr>
<td>x</td>
<td>One and only one line may be perpendicular to a given plane at a point on the plane.</td>
</tr>
<tr>
<td>x</td>
<td>A line may be perpendicular to each of two skew lines.</td>
</tr>
<tr>
<td>x</td>
<td>If two lines meet to form right angles the lines are perpendicular.</td>
</tr>
<tr>
<td>Category</td>
<td>(1)</td>
</tr>
<tr>
<td>----------</td>
<td>-----</td>
</tr>
<tr>
<td>x</td>
<td>If two planes meet to form right dihedral angles the planes are perpendicular.</td>
</tr>
<tr>
<td>x</td>
<td>One and only one line may be perpendicular to a given plane from a point outside the plane.</td>
</tr>
<tr>
<td>x</td>
<td>On a plane one and only one line may be parallel to a given line through a point not on the line. An infinite number of non-intersecting lines, however, may be drawn through the given point if the lines are not in the same plane as the given line.</td>
</tr>
<tr>
<td>x</td>
<td>If a line is parallel to a plane, then it is parallel to every line in the plane having the same direction and skew to all other lines in the plane.</td>
</tr>
<tr>
<td>x</td>
<td>Through any point outside a given plane there is one and only one plane parallel to the given plane.</td>
</tr>
<tr>
<td>x</td>
<td>If a line, not in a plane, is parallel to a line in the plane, it is parallel to the plane.</td>
</tr>
<tr>
<td>x</td>
<td>If three planes intersect in pairs, then the line of intersection of two planes is parallel to the third plane.</td>
</tr>
<tr>
<td>x</td>
<td>If the three sides of one triangle are equal, respectively, to the three sides of another triangle, then the triangles are congruent.</td>
</tr>
<tr>
<td>x</td>
<td>If two sides and the angle between them of one triangle are equal, respectively, to two sides and the angle between them of another triangle, then the triangles are congruent.</td>
</tr>
<tr>
<td>x</td>
<td>If a side and the two adjacent angles of one triangle are equal, respectively, to a side and the two adjacent angles of another triangle, then the triangles are congruent.</td>
</tr>
<tr>
<td>Category</td>
<td>Item</td>
</tr>
<tr>
<td>----------</td>
<td>------</td>
</tr>
<tr>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>x</td>
<td>Any finite quantity equals itself.</td>
</tr>
<tr>
<td>x</td>
<td>Quantities equal to the same quantity are equal.</td>
</tr>
<tr>
<td>x</td>
<td>A quantity may be substituted for its equal.</td>
</tr>
<tr>
<td>x</td>
<td>If equal quantities are added to equal quantities, the sums are equal.</td>
</tr>
<tr>
<td>x</td>
<td>If equal quantities are subtracted from equal quantities, the differences are equal.</td>
</tr>
<tr>
<td>x</td>
<td>If equal quantities are multiplied by equal quantities, the products are equal.</td>
</tr>
<tr>
<td>x</td>
<td>If equal quantities are divided by equal quantities, except zero, the quotients are equal.</td>
</tr>
<tr>
<td>x</td>
<td>The corresponding elements of congruent figures are equal.</td>
</tr>
<tr>
<td>x</td>
<td>All right angles are equal.</td>
</tr>
<tr>
<td>x</td>
<td>An angle has one and only one bisector.</td>
</tr>
<tr>
<td>x</td>
<td>The bisector of an angle of a triangle intersects the opposite side.</td>
</tr>
<tr>
<td>x</td>
<td>The sum of the measures of the angles about a point on one side of a straight line is 180 degrees.</td>
</tr>
<tr>
<td>x</td>
<td>The whole of a finite quantity is equal to the sum of its parts and greater than any of the parts.</td>
</tr>
<tr>
<td>x</td>
<td>A line segment has one and only one midpoint.</td>
</tr>
<tr>
<td>x</td>
<td>Two parallel lines are everywhere equidistant.</td>
</tr>
<tr>
<td>x</td>
<td>If two parallel lines are cut by a transversal, the corresponding angles are equal.</td>
</tr>
</tbody>
</table>
### TABLE 18--Continued

<table>
<thead>
<tr>
<th>Category</th>
<th>Item</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
</tr>
<tr>
<td>x</td>
<td>If two lines in the same plane are cut by a transversal making the corresponding angles equal, then the lines are parallel.</td>
</tr>
<tr>
<td>x</td>
<td>The same or equal circles, or spheres, have equal radii and equal diameters.</td>
</tr>
<tr>
<td>x</td>
<td>A straight line and a circle may intersect in at most two points.</td>
</tr>
<tr>
<td>x</td>
<td>Two circles may intersect in at most two points.</td>
</tr>
<tr>
<td>x</td>
<td>The intersection of a sphere and a plane is a circle.</td>
</tr>
<tr>
<td>x</td>
<td>A central angle has the same measure as its arc.</td>
</tr>
<tr>
<td>x</td>
<td>In the same or equal circles, equal central angles have equal arcs.</td>
</tr>
<tr>
<td>x</td>
<td>In the same or equal circles, equal arcs have equal central angles.</td>
</tr>
<tr>
<td>x</td>
<td>The locus of points on a plane equidistant from a given point in the plane is a circle with the given point as center and the given distance as radius.</td>
</tr>
<tr>
<td>x</td>
<td>The locus of points in three-space equidistant from a given point is a sphere with the given point as center and the given distance as radius.</td>
</tr>
<tr>
<td>x</td>
<td>The locus of points on a plane equidistant from a given line is a pair of lines parallel to the given line and the given distance from it.</td>
</tr>
<tr>
<td>x</td>
<td>The locus of points in three-space equidistant from a given line is a right cylindrical surface having the given line as axis and the given distance as radius.</td>
</tr>
<tr>
<td>Category</td>
<td>Item</td>
</tr>
<tr>
<td>----------</td>
<td>------</td>
</tr>
<tr>
<td>x</td>
<td>The locus of points on a plane equidistant from two parallel lines in the plane is a line parallel to the given lines and midway between them.</td>
</tr>
<tr>
<td>x</td>
<td>The locus of points in three-space equidistant from a given plane is a pair of planes parallel to the given plane and the given distance from it.</td>
</tr>
<tr>
<td>x</td>
<td>The locus of points in three-space equidistant from two parallel planes is a plane parallel to the given planes and midway between them.</td>
</tr>
<tr>
<td>x</td>
<td>The locus of points on a plane equidistant from two given points in the plane is the line which is the perpendicular bisector of the line segment joining the given points.</td>
</tr>
<tr>
<td>x</td>
<td>The locus of points in three-space equidistant from two given points is the plane which is the perpendicular bisector of the line segment joining the given points.</td>
</tr>
<tr>
<td>x</td>
<td>The locus of points in three-space equidistant from two intersecting planes is the pair of planes which bisect the dihedral angles formed by the intersecting planes.</td>
</tr>
<tr>
<td>x</td>
<td>If the three angles of a triangle are equal, respectively, to the three angles of another triangle, the triangles are similar.</td>
</tr>
<tr>
<td>x</td>
<td>If an angle of one triangle is equal to an angle of another triangle and the including sides are proportional, the triangles are similar.</td>
</tr>
<tr>
<td>x</td>
<td>If the sides of two triangles are respectively proportional, the triangles are similar.</td>
</tr>
<tr>
<td>x</td>
<td>The area of a rectangle is equal to the product of the length and the width.</td>
</tr>
</tbody>
</table>
Although each student recorded practically all of the definitions and assumptions in his notebook as they were agreed upon by the class, the number of propositions for which proofs were written varied. Only two students included all of the theorems listed in Table 19 while at the other extreme one student completed only half of them.

The students were not expected to write detailed, formal proofs for their extensions to three-space, but for some of the statements, some members of the class did outline or present a plan for proving the respective statements. The list given in Table 20 indicates those propositions concerning three-space for which at least some students justified their thinking. In several instances the proposition was also proposed by other students who did not explain their line of reasoning. Some of these ideas may not normally be expressed in a solid geometry textbook, but they are given here to show the thinking of this particular group of students.

The basic constructions listed in Table 21, along with their proofs, were included in each student's notebook.
### TABLE 19

THEOREMS FOR TWO-SPACE ESTABLISHED BY THE STUDENTS

<table>
<thead>
<tr>
<th>Category&lt;sup&gt;a&lt;/sup&gt;</th>
<th>Item</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) (2) (3)</td>
<td></td>
</tr>
<tr>
<td>x</td>
<td>Complements of equal angles are equal.</td>
</tr>
<tr>
<td>x</td>
<td>Supplements of equal angles are equal.</td>
</tr>
<tr>
<td>x</td>
<td>Vertical angles are equal.</td>
</tr>
<tr>
<td>x</td>
<td>If two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are supplementary.</td>
</tr>
<tr>
<td>x</td>
<td>If two parallel lines are cut by a transversal, the alternate-interior angles are equal.</td>
</tr>
<tr>
<td>x</td>
<td>If two lines in the same plane are cut by a transversal and the interior angles on the same side of the transversal are supplementary, the lines are parallel.</td>
</tr>
<tr>
<td>x</td>
<td>If two lines in the same plane are cut by a transversal and the alternate-interior angles are equal, the lines are parallel.</td>
</tr>
<tr>
<td>x</td>
<td>If two coplanar lines are perpendicular to the same line, they are parallel.</td>
</tr>
<tr>
<td>x</td>
<td>If one of two parallel lines is perpendicular to a third coplanar line, the other is also perpendicular to the third line.</td>
</tr>
<tr>
<td>x</td>
<td>The sum of the measures of the interior angles of any triangle is 180 degrees.</td>
</tr>
</tbody>
</table>

<sup>a</sup>(1) teacher-directed ideas, (2) student-teacher proposals, (3) student-developed ideas.
<table>
<thead>
<tr>
<th>Category</th>
<th>Item</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>x</td>
<td>If two angles of one triangle are equal to two angles of another triangle, the third angles are equal.</td>
</tr>
<tr>
<td>x</td>
<td>The acute angles of a right triangle are complementary.</td>
</tr>
<tr>
<td>x</td>
<td>If two angles and one side of a triangle are equal, respectively, to two angles and one side of another triangle, the triangles are congruent.</td>
</tr>
<tr>
<td>x</td>
<td>The measure of the exterior angle of a triangle is equal to the sum of the measures of the two non-adjacent interior angles.</td>
</tr>
<tr>
<td>x</td>
<td>When two sides of a triangle are equal, the angles opposite the sides are equal.</td>
</tr>
<tr>
<td>x</td>
<td>When two angles of a triangle are equal, the sides opposite the angles are equal.</td>
</tr>
<tr>
<td>x</td>
<td>Equilateral triangles are equiangular.</td>
</tr>
<tr>
<td>x</td>
<td>When two sides of a triangle are unequal, the angles opposite them are unequal and the larger angle is opposite the longer side.</td>
</tr>
<tr>
<td>x</td>
<td>Two points equidistant from the endpoints of a line segment determine the perpendicular bisector of the segment.</td>
</tr>
<tr>
<td>x</td>
<td>Any point on the perpendicular bisector of a line segment is equidistant from the endpoints of the segment.</td>
</tr>
<tr>
<td>x</td>
<td>The sum of the measures of the interior angles of any polygon is ((n - 2) 180) degrees.</td>
</tr>
<tr>
<td>x</td>
<td>The opposite sides of a parallelogram are equal.</td>
</tr>
<tr>
<td>x</td>
<td>The opposite angles of a parallelogram are equal.</td>
</tr>
<tr>
<td>Category</td>
<td>Item</td>
</tr>
<tr>
<td>----------</td>
<td>------</td>
</tr>
<tr>
<td>x</td>
<td>The consecutive angles of a parallelogram are supplementary.</td>
</tr>
<tr>
<td>x</td>
<td>If the opposite sides of a quadrilateral are equal, the quadrilateral is a parallelogram.</td>
</tr>
<tr>
<td>x</td>
<td>If the consecutive angles of a quadrilateral are supplementary, the quadrilateral is a parallelogram.</td>
</tr>
<tr>
<td>x</td>
<td>If the opposite angles of a quadrilateral are equal, the quadrilateral is a parallelogram.</td>
</tr>
<tr>
<td>x</td>
<td>If one pair of sides of a quadrilateral is equal and parallel, the quadrilateral is a parallelogram.</td>
</tr>
<tr>
<td>x</td>
<td>The diagonals of a parallelogram bisect each other.</td>
</tr>
<tr>
<td>x</td>
<td>If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.</td>
</tr>
<tr>
<td>x</td>
<td>Lines parallel to the same line are parallel.</td>
</tr>
<tr>
<td>x</td>
<td>If three or more parallel lines cut off equal segments on one transversal, they cut off equal segments on all transversals.</td>
</tr>
<tr>
<td>x</td>
<td>All angles of a rectangle are right angles.</td>
</tr>
<tr>
<td>x</td>
<td>All sides of a rhombus are equal.</td>
</tr>
<tr>
<td>x</td>
<td>The diagonals of a rectangle are equal.</td>
</tr>
<tr>
<td>x</td>
<td>A diameter perpendicular to a chord bisects the chord and its arcs.</td>
</tr>
<tr>
<td>x</td>
<td>A diameter that bisects a chord, not a diameter, is perpendicular to the chord.</td>
</tr>
<tr>
<td>x</td>
<td>In the same or equal circles, equal arcs have equal chords.</td>
</tr>
</tbody>
</table>
### TABLE 19—Continued

<table>
<thead>
<tr>
<th>Category</th>
<th>Item</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>x</td>
<td>In the same or equal circles, equal chords have equal arcs.</td>
</tr>
<tr>
<td>x</td>
<td>In the same or equal circles, equal chords are equidistant from the center of the circle.</td>
</tr>
<tr>
<td>x</td>
<td>In the same or equal circles, chords equidistant from the center of the circle are equal.</td>
</tr>
<tr>
<td>x</td>
<td>A tangent of a circle is perpendicular to the radius drawn to the point of tangency.</td>
</tr>
<tr>
<td>x</td>
<td>An inscribed angle is measured by one-half its intercepted arc.</td>
</tr>
<tr>
<td>x</td>
<td>Parallel lines cut off equal arcs on a circle.</td>
</tr>
<tr>
<td>x</td>
<td>The angle formed by a tangent and a chord is measured by one-half its arc.</td>
</tr>
<tr>
<td>x</td>
<td>The angle formed by lines intersecting outside a circle is measured by one-half the difference of the intercepted arcs.</td>
</tr>
<tr>
<td>x</td>
<td>The angle formed by chords intersecting inside a circle is measured by one-half the sum of its arc and the arc of its vertical angle.</td>
</tr>
<tr>
<td>x</td>
<td>If two tangents intersect, the angle formed is bisected by the line joining the point of intersection to the center of the circle, and the tangents from the point to the circle are equal.</td>
</tr>
<tr>
<td>x</td>
<td>If the hypotenuse and a leg of a right triangle are equal to the hypotenuse and corresponding leg of another right triangle, the triangles are congruent.</td>
</tr>
<tr>
<td>x</td>
<td>In a 30-60-90 triangle, the shorter leg is equal to one-half the hypotenuse.</td>
</tr>
<tr>
<td>Category</td>
<td>Item</td>
</tr>
<tr>
<td>----------</td>
<td>------</td>
</tr>
<tr>
<td>x</td>
<td>Three or more parallel lines cut off proportional segments on two or more transversals.</td>
</tr>
<tr>
<td>x</td>
<td>In a proportion, the product of the means is equal to the product of the extremes.</td>
</tr>
<tr>
<td>x</td>
<td>If four terms are in proportion, they are in proportion by inversion, alternation, and addition.</td>
</tr>
<tr>
<td>x</td>
<td>A line parallel to one side of a triangle divides the other sides proportionally.</td>
</tr>
<tr>
<td>x</td>
<td>If two angles of a triangle are equal, respectively, to two angles of another triangle, the triangles are similar.</td>
</tr>
<tr>
<td>x</td>
<td>In a right triangle the altitude to the hypotenuse divides the triangle into similar triangles which are also similar to the original triangle.</td>
</tr>
<tr>
<td>x</td>
<td>In a right triangle the square of the altitude to the hypotenuse is equal to the product of the projections of the legs on the hypotenuse.</td>
</tr>
<tr>
<td>x</td>
<td>In a right triangle the square of a leg is equal to the product of the hypotenuse and the projection of the leg on the hypotenuse.</td>
</tr>
<tr>
<td>x</td>
<td>In a right triangle the square on the hypotenuse is equal to the sum of the squares on the legs.</td>
</tr>
<tr>
<td>x</td>
<td>The locus of points on a plane equidistant from two intersecting straight lines is the pair of lines which bisects the angles formed by the given lines.</td>
</tr>
<tr>
<td>x</td>
<td>The perpendicular bisectors of the sides of a triangle are concurrent in a point equidistant from the vertices.</td>
</tr>
<tr>
<td>Category</td>
<td>Item</td>
</tr>
<tr>
<td>----------</td>
<td>------</td>
</tr>
<tr>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td></td>
<td>The bisectors of the angles of a triangle are concurrent in a point equidistant from the sides.</td>
</tr>
<tr>
<td>x</td>
<td>The altitudes of a triangle are concurrent.</td>
</tr>
<tr>
<td>x</td>
<td>The medians of a triangle are concurrent in a point which is two-thirds of the distance from any vertex to the midpoint of the opposite side.</td>
</tr>
<tr>
<td>x</td>
<td>The area of a parallelogram is equal to the product of its base and altitude.</td>
</tr>
<tr>
<td>x</td>
<td>The area of a triangle is equal to one-half the product of a base and its altitude.</td>
</tr>
<tr>
<td>x</td>
<td>The area of a trapezoid is equal to one-half the product of the altitude and the sum of the bases.</td>
</tr>
<tr>
<td>x</td>
<td>The area of a circle is equal to pi times the square of the radius.</td>
</tr>
<tr>
<td>x</td>
<td>The circumference of a circle is equal to pi times the diameter.</td>
</tr>
<tr>
<td>x</td>
<td>The areas of two parallelograms are proportional to the products of their respective bases and altitudes.</td>
</tr>
<tr>
<td>x</td>
<td>The areas of two similar polygons are proportional to the squares of the corresponding sides.</td>
</tr>
<tr>
<td>Category&lt;sup&gt;a&lt;/sup&gt;</td>
<td>Item</td>
</tr>
<tr>
<td>---------------------</td>
<td>-------------------------------------------------------------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>(1)</td>
<td>x The sum of the measures of the face angles of a polyhedral angle is less than 360 degrees.</td>
</tr>
<tr>
<td>(2)</td>
<td>x Vertical dihedral angles are equal.</td>
</tr>
<tr>
<td>(3)</td>
<td>x Vertical polyhedral angles are equal.</td>
</tr>
<tr>
<td></td>
<td>x Complements of equal dihedral angles are equal.</td>
</tr>
<tr>
<td></td>
<td>x Supplements of equal dihedral angles are equal.</td>
</tr>
<tr>
<td></td>
<td>x If two parallel planes are cut by a third plane, the alternate-interior and corresponding dihedral angles are equal.</td>
</tr>
<tr>
<td></td>
<td>x If two parallel planes are cut by a third plane the interior dihedral angles on the same side of the transversal plane are supplementary.</td>
</tr>
<tr>
<td></td>
<td>x All points which are equidistant from the extremities of a line segment lie on the plane which is the perpendicular bisector of the segment.</td>
</tr>
<tr>
<td></td>
<td>x If a point lies on the plane which is the perpendicular bisector of a line segment, then it is equidistant from the extremities of the segment.</td>
</tr>
<tr>
<td></td>
<td>x If two lines are perpendicular to the same plane, they are parallel.</td>
</tr>
<tr>
<td></td>
<td>x If two planes are perpendicular to the same line, they are parallel.</td>
</tr>
<tr>
<td></td>
<td>x The diagonals of a parallelepiped bisect each other.</td>
</tr>
</tbody>
</table>

<sup>a</sup>(1) teacher-directed ideas, (2) student-teacher proposals, (3) student-developed ideas.
<table>
<thead>
<tr>
<th>Category</th>
<th>Item</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>The lateral edges of a prism are equal and parallel.</td>
</tr>
<tr>
<td>x</td>
<td>The bases of a prism are congruent polygons.</td>
</tr>
<tr>
<td>x</td>
<td>If three or more parallel planes cut off equal segments on one line, they cut off equal segments on all lines which intersect the planes.</td>
</tr>
<tr>
<td>x</td>
<td>If two planes, equidistant from the center of a sphere, intersect the sphere, then the circles formed by the intersections will be equal.</td>
</tr>
<tr>
<td>x</td>
<td>If a plane is tangent to a sphere, then the radius drawn to the common point is perpendicular to the plane.</td>
</tr>
<tr>
<td>x</td>
<td>If two great circles intersect, the vertical spherical angles are equal.</td>
</tr>
<tr>
<td>x</td>
<td>The sum of the measures of the sides of a spherical triangle is less than 360 degrees.</td>
</tr>
<tr>
<td>x</td>
<td>The sum of the measures of the angles of a spherical triangle is less than 540 degrees.</td>
</tr>
<tr>
<td>x</td>
<td>A plane parallel to the base of a pyramid divides the edges of the pyramid proportionally.</td>
</tr>
<tr>
<td>x</td>
<td>Three or more parallel planes cut off proportional segments on two or more transversals.</td>
</tr>
<tr>
<td>x</td>
<td>The distance between two points in any space is equal to the square root of the sum of the squares of the dimensions, taken at right angles to each other.</td>
</tr>
<tr>
<td>x</td>
<td>The volume of a circular cylinder is equal to the product of the altitude and the area of the base.</td>
</tr>
<tr>
<td>x</td>
<td>The volume of a pyramid is equal to one-third the area of the base and its altitude.</td>
</tr>
<tr>
<td>Category</td>
<td>Item</td>
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<tr>
<td>----------</td>
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</tr>
<tr>
<td>x</td>
<td>The volume of a circular cone is equal to one-third the product of the altitude and the area of the base.</td>
</tr>
<tr>
<td>x</td>
<td>The volume of a sphere is equal to ( \frac{4}{3}\pi ) times the cube of the radius.</td>
</tr>
<tr>
<td>x</td>
<td>The lateral area of a rectangular prism is equal to the product of the perimeter of the base and the altitude.</td>
</tr>
<tr>
<td>x</td>
<td>The lateral area of a circular cylinder is equal to the product of the circumference of the base and the altitude.</td>
</tr>
<tr>
<td>x</td>
<td>The lateral area of a pyramid is equal to one-half the product of the perimeter of the base and the altitude of a face.</td>
</tr>
<tr>
<td>x</td>
<td>The lateral area of a circular cone is equal to one-half the product of the circumference of the base and the slant height.</td>
</tr>
<tr>
<td>x</td>
<td>The surface area of a sphere is equal to ( 4\pi ) pi times the square of the radius.</td>
</tr>
<tr>
<td>x</td>
<td>The volumes of two similar three-dimensional solids are proportional to the cubes of the corresponding edges.</td>
</tr>
<tr>
<td>Category&lt;sup&gt;a&lt;/sup&gt;</td>
<td>Item</td>
</tr>
<tr>
<td>-----------------</td>
<td>------</td>
</tr>
<tr>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>x</td>
<td>Bisect a line segment.</td>
</tr>
<tr>
<td>x</td>
<td>Construct a perpendicular to a line at a point on the line.</td>
</tr>
<tr>
<td>x</td>
<td>Construct a perpendicular from a point to a line.</td>
</tr>
<tr>
<td>x</td>
<td>Construct an angle equal to a given angle.</td>
</tr>
<tr>
<td>x</td>
<td>Bisect an angle.</td>
</tr>
<tr>
<td>x</td>
<td>Construct a line parallel to a given line through a point not on the given line.</td>
</tr>
<tr>
<td>x</td>
<td>Construct a circle through three non-collinear points.</td>
</tr>
</tbody>
</table>

<sup>a</sup>(1) teacher-directed ideas, (2) student-teacher proposals, (3) student-developed ideas.
APPENDIX B

Many of the concepts considered in the geometry course for the experimental group were introduced or expanded by use of study guides, although several items arose and were developed through class discussion only. Those study guides which had direct bearing on the illustrations were included in Chapter IV. It cannot be assumed, however, that the other guides did not influence the thinking of the students in their integration of two- and three-space concepts; therefore, the complete set of guides used during the year is given in Appendix B.

Since the students shared the responsibility for developing the course and each student's notebook was a reflection of the material he was able to study, it should be emphasized that some of the guides were studied on an optional basis.
Discuss the exercises on another sheet of paper.

1. During the last presidential campaign it was pointed out that the taller of the two candidates in the elections of the twentieth century has been elected. Mr. Nixon's height is 5 ft. 11 in. and Mr. Kennedy's is 6 ft.; therefore, Mr. Kennedy would win the election. As you know Mr. Kennedy did win the election. Do you feel the taller candidate will win the 1964 election? State reasons for your answer.

2. In this same campaign it was also noted that presidents elected in years divisible by twenty have died in office. The year of President Kennedy's election is divisible by twenty. Do you think he is more likely to die in office than those elected in years not divisible by twenty? State reasons for your answer.

3. Several months ago the comic strip, "Peanuts," showed Charlie Brown saying, "Rain, rain, go away, come again some other day." On two occasions it stopped raining after he made this statement, and he assumed he possessed some special power. One of his friends said, "It's only happened twice. If you can do it once more, then we'll know for sure." Do you agree with this argument? How many cases do you feel would be needed before knowing Charlie Brown had a special ability? State reasons for your answer.

4. Halley, an English astronomer, studied the paths of twenty-four comets and found that twenty-one of them appeared from outer space and left the areas of their observers by different paths. Three of the comets appeared to follow the same orbit, so Halley questioned if the same comet might be revisiting the earth. He assumed that if this were the same comet, then it should return to the visibility of observers on the earth at approximately the same time intervals. The three comets had been nearest the sun on August 2l, 1531; October 16, 1607 (an interval of 76.2 years); September 24, 1682 (an interval of 74.9 years). From these data Halley concluded that the comet would complete one revolution in its orbit about every 75 or 76 years, and he predicted the comet's return in 1758 or 1759. After his death, the comet did return in March, 1759 and again in 1835 and 1910. Do you think there is justification for saying the comet will return in 1985 or 1986? (adapted from the Twenty-fourth Yearbook of the National Council of Teachers of Mathematics)

5. The algebraic expression, \( n^2 - n + 41 \), gives prime numbers if you substitute 0, 1, 2, 3, 4, and so on for \( n \). How many cases would be needed to support the argument that this expression always gives results which are prime numbers? Can you find an exception?
Definitions

Write your answers to each of the following exercises on a separate sheet of paper.

1. A few years ago a young Arab sold his farm to finance a trip to the United States where he expected to receive a free college education. He had applied for entrance to several colleges in the United States and one of the larger universities had sent him a form letter which included the statement, "Campus life at State is full, free and friendly."

If you were an official in this university would you expect this young man to pay tuition? Give reasons for your answer.

2. The North High School had a proctor system, sponsored by the student council, which was in effect during the noon hour to help maintain proper conduct among the students. Students were stationed in the halls and were to give "traffic tickets" to any student who violated the rules of conduct. After a student received three tickets for the same type offense he was called before the student council officers for penalty.

One of the rules stated that running in the halls was not permitted. A freshman boy received three tickets for violation of this rule. When called before the student council officers he maintained he had not been running in the halls. Further investigation showed that more specifically he had been running on the stairs. He still said he was not guilty, but did admit that when he went up or down stairs, he hopped from one step to the next.

If you had been a proctor, would you have given this boy a ticket? Give reasons for your answer.

3. A group of students enrolled in a three-week course at a nearby university. Those who lived in the dormitory were to pay $1.50 a day and furnish their own bed linen or this would be furnished for a "nominal fee." After these students had lived in the dorm a week they received bills for the term. Those who furnished their own sheets were charged $31.50 while those who used the university's sheets were charged $42.00. A number of the latter group objected stating they could buy new sheets for the amount charged. The university argued they were charging only 50¢ per night and this was not expensive. Do you think the students had a good argument? Do you agree with the university?

4. Find the meaning of the word "kopeck" in the dictionary. If the definition contains any words with which you are not familiar, find the definition of these words.
5. What does the sentence, "She is fair," mean to you?

6. The assistant state attorney general of a neighboring state has handed down the ruling that school authorities may prohibit pupils from wearing attire "that might expose too much of their person."

   Would this ruling permit the wearing of bermuda shorts to school?

   Would girls be permitted to wear "knee ticklers" (just above the knees skirts)?
Assumptions

In our day-to-day living we are constantly assuming statements which at the time we are unable to prove, in fact we may never be able to prove them. Any sound argument which is presented will be based on certain assumptions. Those of you who have studied debate will recall that the participants base their presentation on definitions of major words and on assumptions associated with the question for discussion.

Many important documents clearly indicate that they are based on assumed statements. Illustrations of these may be found in the Declaration of Independence and the United Nations Charter. Quotations from both these famous documents are given below.

EXERCISE: List at least six assumptions from each of these quotations.

Declaration of Independence

We hold these truths to be self-evident—that all men are created equal; that they are endowed by their Creator with certain unalienable rights; that among these are life, liberty, and the pursuit of happiness. That, to secure these rights, governments are instituted among men, deriving their just powers from the consent of the governed; that, whenever any form of government becomes destructive of these ends, it is the right of the people to alter or to abolish it, and to institute a new government, laying its foundation on such principles, and organizing its powers in such form, as to them shall seem most likely to effect their safety and happiness.

Preamble of the United Nations Charter

We, the peoples of the United Nations
Determined to save succeeding generations from the scourge of war, which twice in our lifetime has brought untold sorrow to mankind, and
To reaffirm faith in fundamental human rights, in the dignity and worth of the human person, in the equal rights of men and women and of nations large and small, and
To establish conditions under which justice and respect for the obligations arising from treaties and other sources of international law can be maintained, and
To promote social progress and better standards of life in larger freedom, and for these ends
To practice tolerance and live together in peace with one another as good neighbors, and
Assumptions (contd.)

To unite our strength to maintain international peace and security, and
To insure, by the acceptance of principles and the institution of methods, that armed force shall not be used, save in the common interest, and
To employ international machinery for the promotion of the economic and social advancement of all peoples, have resolved to combine our efforts to accomplish these aims.
Accordingly, our respective governments, through representatives assembled in the city of San Francisco, who have exhibited their full powers found to be in good and due form, have agreed to the present Charter of the United Nations and do hereby establish an international organization to be known as the United Nations.
1. Assume you have a friend who has just moved to a new city and you wish to visit him, but do not know the location of his home. It is necessary for you to inquire where he lives under the following conditions:

   a. He lives in a very small town which is composed of one-family dwellings and all located on one street of the town. What information would you need?

   b. He lives in a small city which is composed of one-family dwellings. What information would you need?

   c. He lives in an apartment building in a large city. What information would you need?

2. a. You have been asked to hang a picture on the north wall of a particular room. What information will you need to hang the picture in the proper place?

   b. A man has been asked to determine the location of a basketball backboard in a gymnasium. What information will he need to locate this backboard in the proper place?

3. a. You are driving on the highway and your car develops some type of trouble for which you need a mechanic. You place a call to a garage in the nearest town. What information will you give the garage attendant, so that he will be able to locate your car?

   b. An airplane pilot needs to inform an airport of the location of his plane while he is in flight. What information will he need to radio to the airport?
1. Assume in each of the following cases that two triangles are formed by joining strips of cardboard and these are placed on a table in a position similar to the illustrations below. (Those parts which are lettered the same are equal in size.) Assume also that the figures may be moved by sliding from one position to another, but they cannot be lifted from the table.

a. Is it possible, in Fig. 1, to move one triangle so that it matches the other? _________

[Diagram of two triangles in Fig. 1]

b. Is it possible to match the triangles in Fig. 2? _________

[Diagram of two triangles in Fig. 2]

c. Is it possible to match the triangles in Fig. 3? _________

[Diagram of two triangles in Fig. 3]

2. If the restriction of keeping the figures on the table is removed, would this change the answer to (a)? _________ to (b)? _________ to (c)? _________ If it changes any answer, explain the change.
3. Assume you have a pair of gloves. Is it possible to place one glove so it will fit exactly inside the other (ignore thickness of material)? If your answer is "no," can you imagine any way in which it would be possible? Explain your answer.

4. Assume a rubber band is placed on a table so that one edge always remains on the table. Is it possible to turn this rubber band inside out? Assume you can cut the band in one place and fit it back together. Is it possible under these conditions to turn the band inside out? Assume the rubber band can be lifted from the table. Is it possible to turn the band inside out? If your answers to the first and last questions are different, what accounts for this difference?

5. Assume you have a rubber ball which is hollow. Is it possible to turn it inside out? If you can imagine certain conditions under which the ball may be turned inside out, explain them.
Dimensions--3

1. If we began with a zero-dimensional element, which we ordinarily call a point, and moved it in one direction only, its path would trace a _______. It would have the dimension, _______.

2. If we began with this new figure and moved it in one direction only, its path would trace a _______. It would have the dimension(s), _______.

3. If we began with the figure generated in exercise 2 and moved it in one direction only, its path would trace a _______. It would have the dimension(s), _______.

4. If we began with the figure generated in exercise 3 and moved it in one direction only, its path would trace a _______. It would have the dimension(s), _______.

Let us now consider the elements of which each of these figures which you have generated is composed.

1. The figure of zero dimension is composed of a _______.

2. The figure of one dimension is a _______ bounded by the beginning and ending or _______.

3. The figure of two dimensions is a _______ bounded by _______.

4. The figure of three dimensions is a _______ bounded by _______.

If you will notice in each case, you obtain a "new thing" between the previous "things."

1. Let the element of zero dimension be denoted by "p" and that of one dimension be denoted by "l." The elements of the figure of one dimension are _______ between _______.

2. Let the figure of two dimensions be denoted by "A." The elements of this figure are _______ between _______ but each of the latter is composed of _______; therefore, the elements of A include _______.

3. In similar manner analyze the figure of three dimensions, denoting it by "V."
4. In a manner similar to the above, analyze the figure of four dimensions. When a new figure is obtained, indicate some letter to denote this figure.

5. Can you extend this type of analysis to five dimensions?
Points—1

1. Mark a point, $A$, in the space below. Draw a straight line which has $A$ as one of its points. Can you draw other lines through point $A$? How many such lines can you draw? 

2. Mark a point, $A$, and a second point, $B$, in the space below. Can you draw a straight line which contains both $A$ and $B$? If your answer is "yes," illustrate. How many lines can you draw such as this? If one of these points were not on the paper would it be possible to connect these points by means of a string or similar device? 

3. Mark three points, $A$, $B$, and $C$, in the space below. Can you draw a straight line which contains all three of these points? If your answer is "yes," illustrate. If your answer is "no," how many straight lines can be located using these three points? Illustrate.

4. Mark a point, $A$, and a second point, $B$, in the space below. Draw a line which contains both $A$ and $B$. Draw three other such lines. (These need not be straight lines.) How many other lines could you draw? Name your lines $l_1, l_2, l_3, \ldots$. Which of your lines is the shortest? 

5. Draw two straight lines which cross or intersect. Call this intersection, $P$. What geometric figure does this intersection represent? Using the same two straight lines, how many other intersections can you obtain? 

6. In geometry we say a given set of conditions determine a geometric figure if one and only one such figure results from these conditions. Make a general statement for each of the ideas considered in the above exercises.
Lines--1

1. Two ways of naming a straight line have been suggested in class. These were using a lower case letter for the line and naming two points on the line. According to this agreement, may the line 1 be named $\overline{AB}$? $\overline{AC}$? $\overline{BC}$? 

\[ \overline{1} \quad A \quad B \quad C \]

How far may line 1 be extended without changing its name? 

2. Sometimes we are concerned with only the part of the line between two specific points, such as A and B in the above figure. We call this a line segment, $\overline{AB}$. If we speak of the line segment, $\overline{BC}$, may this segment be extended? Write a definition for line segment.

3. Not all lines in geometry are straight lines. Two other types of lines are called broken line and curved line.

The figure below is an example of a broken line.

Define broken line.

The figures below are examples of curved lines.

Define curved line.

4. May two curved lines intersect in more than one point? In how many?
1. In a previous exercise you were asked to draw two lines which intersect. Can you draw a line which will not intersect $AB$? If your answer is "yes," illustrate.

\[ \overleftrightarrow{A} \quad \overleftrightarrow{B} \]

When the line $\overleftrightarrow{AB}$ is considered beyond the limits of this paper, will the lines intersect? 

2. Parallel lines are sometimes defined as two lines which will not intersect no matter how far extended. Does this fit your idea of parallel lines? Can you give an illustration of two lines which do not intersect but neither do they appear as parallel lines?
Planes--1

1. Mark a point, A, in the space below. How many planes may contain this point? 

2. Mark two points, A and B, in the space below. How many planes may contain both these points? 
   Draw the straight line connecting these two points. How many of the planes contain this line? 
   How many planes contain only a portion of this line?

3. Mark three points, A, B, and C, in the space below. How many planes may contain all three points? 
   Does it make any difference if the points are in a straight line or not? 
   If so, how? 
   If one of the points were not on this paper, how would it affect your answer?

4. Draw two intersecting lines in the space below. How many planes may contain both these lines? 
   Does it make any difference if one of these lines does not lie on this paper? 
   If so, how?

5. Draw a line and mark a point in the space below. How many planes may contain both the line and the point? 
   Does it make any difference if the point is on the line? 
   If so, how? 
   If the point were not on this paper, how would it affect your answer?

6. Draw two lines which do not meet regardless of their length. How many planes may contain both these lines? 
   If one of these lines were not on this paper, how would it affect your answer?
Planes—1 (contd.)

7. Draw two planes which intersect. What is the geometric figure formed by this intersection? Using the same two planes, how many other intersections can you obtain?

8. Make a general statement for each of the ideas considered in the above exercises.
Angles--1

1. In the space below draw a line and locate a point A on the line. How far may the line be extended to the right? ______________ How far to the left? ______________. This point A divides the line into two parts with A as an endpoint. A and the part of the line to the right or A and the part of the line to the left is called a ray. Define ray.

2. Two rays which have the same endpoint form a geometric figure called an **angle**. Define angle.

3. In the figure below, point 0 is called the **vertex** of the angle. Rays OA and OB are the sides of the angle. Define vertex and side of an angle.

4. The set of points between the rays is the **interior** of the angle; all other points except the set of points which form the angle are the **exterior** of the angle. Define these two terms.

5. Angles are named in three ways:
   a. The letter at the vertex, as \( \angle 0 \) in the figure at the right.
   b. Place the letter at the vertex in the middle and one point from each ray on either side as \( \angle AOB \) or \( \angle BQA \).
   c. A lower case letter or number written inside the angle as \( \angle m \).
Angles—1 (contd.)

6. Name each of the angles below in two other ways.
   a. \( \angle O \) _______ _______
   b. \( \angle l \) _______ _______
   c. \( \angle m \) _______ _______
   d. \( \angle O \) _______ _______ Do you notice any difficulty in naming this angle? Explain.
   e. \( \angle a \) _______ _______
      \( \angle b \) _______ _______

   \textbf{Fig. a} \hspace{1cm} \textbf{Fig. b} \hspace{1cm} \textbf{Fig. c}

   \textbf{Fig. d} \hspace{1cm} \textbf{Fig. e}
Angles--2

1. Another way in which the term "angle" may be defined is to consider a ray, such as AB shown below, keep point "A" in its present position and rotate the ray about the point. Write a new definition for angle.

2. In mathematics courses it is usually agreed to rotate the ray in a counter-clockwise direction. The first position of the ray is referred to as the initial side and its final position as the terminal side. The point about which the ray rotates is called the vertex. Place these definitions in your notebook.

3. In each of the figures below indicate the vertex, the initial side, and the terminal side.

4. In naming the angle state the initial side first, as AO in Fig. 1, followed by the terminal side, OB. This would normally read AO-OB, but since the "O" is a repetition, place the vertex in the middle and call the angle in Fig. 1 \( \angle AOB \). Name the other three angles. Fig. 2 ________ Fig. 3 ________ Fig. 4 ________

5. Previously you have studied that angles have specific names depending upon the size of the angle. Name and define as many of these as you can recall.
Angles--3

1. We found that two rays which have one endpoint in common form an angle. When two half-planes have a line in common, they also form an angle. This angle is called a **dihedral angle**. Define dihedral angle.

Both of the figures below show illustrations of dihedral angles. Will your definition apply to these figures? ______ If not, revise your definition.

![Diagram 1](image1)

![Diagram 2](image2)

2. The common endpoint of the two rays forming an angle is the vertex of the angle. In a dihedral angle what part of the angle corresponds to this vertex? ______ This is called the **edge** of the dihedral angle. Define edge of a dihedral angle.

3. The two rays are the sides of the angle. In a dihedral angle what figures correspond to the sides of the angle? ______ These are called **faces** of the dihedral angle. Define face of a dihedral angle.

4. The angle (Fig. 1) we have agreed to call $\angle AOB$. The dihedral angle (Fig. 2) is called $\angle M-AB-N$. Is $M$ common to both planes? ______ Is $AB$? ______ Is $N$? ______
5. Name the dihedral angles below.

6. We found that when two geometric figures intersect they form the figure with one less dimension. We also found an infinite number of planes may contain the same point. May three planes intersect in a line? If "yes," give an illustration. May three planes intersect in a point? If "yes," give an illustration.

What is the minimum number of planes that may meet in a point? The maximum?

7. When a number of planes intersect in a point, the angle formed is a polyhedral angle. Define polyhedral angle.

8. What parts of a polyhedral angle need to be defined?

9. The figure below represents a polyhedral angle. How would you name this angle?
Polyhedral angles are named according to the number of planes that meet to form the angle as:

- trihedral angle 3 planes
- tetrahedral angle 4 planes
- pentahedral angle 5 planes
- hexahedral angle 6 planes

10. How many dihedral angles would be formed when a trihedral angle is formed? _________

How many dihedral angles would be formed when a tetrahedral angle is formed? _________

How many dihedral angles would be formed when a pentahedral angle is formed? _________

How many dihedral angles would be formed when a hexahedral angle is formed? _________

State the relationship between the number of planes which form the polyhedral angle and the number of dihedral angles.
Angles—4

Study each of the diagrams below and answer the accompanying exercises.

1. Name the vertex of angle 1. _____
   Name the vertex of angle 2. _____
   Name the sides of angle 1. _____
   Name the sides of angle 2. _____

   Use vertical lines to shade the interior of angle 1. Use horizontal lines to shade the interior of angle 2.

   Do these two angles have any elements in common? _____
   If "yes," what are they? __________________________

2. Name the vertex of angle 3. _____
   Name the vertex of angle 4. _____
   Name the sides of angle 3. _____
   Name the sides of angle 4. _____

   Use vertical lines to shade the interior of angle 3. Use horizontal lines to shade the interior of angle 4.

   Do these two angles have any elements in common? _____
   If "yes," what are they? __________________________
3. Name the vertex of angle 5. ______
   Name the vertex of angle 6. ______
   Name the sides of angle 5. ______
   Name the sides of angle 6. ______

   Use vertical lines to shade the interior of angle 5. Use horizontal lines to shade the interior of angle 6.

   Do these two angles have any elements in common? ______
   If "yes," what are they? ________________________________________

4. Name the vertex of angle 7. ______
   Name the vertex of angle 8. ______
   Name the sides of angle 7. ______
   Name the sides of angle 8. ______

   Use vertical lines to shade the interior of angle 7. Use horizontal lines to shade the interior of angle 8.

   Do these two angles have any elements in common? ______
   If "yes," what are they? ________________________________________
5. Name the vertex of angle 9. _______
   Name the vertex of angle 10. _______
   Name the sides of angle 9. _______
   Name the sides of angle 10. _______

   Use vertical lines to shade the interior of angle 9. Use horizontal lines to shade the interior of angle 10.

   Do these two angles have any elements in common? _______
   If "yes," what are they? _______________________________________

6. Pairs of angles such as angle 9 and angle 10 are called adjacent angles, but the pairs in the other drawings do not satisfy all the necessary conditions. Write a definition of adjacent angles.

1. The unit of measure used to express the size of an angle is called a degree. A complete rotation of a ray about its endpoint is said to contain 360 degrees. What part of a complete rotation is one degree? __________ One-fourth of a complete rotation contains ________ degrees. What part of a complete rotation is 75 degrees? ________ What part of a complete rotation is 135 degrees? ________ What effect does the length of the sides of an angle have on its size? ________

2. If a series of adjacent angles were placed to just fill the space about the endpoint of a ray, what would be the sum of the measures of these angles? ________

3. Do you think dihedral angles could be measured in a manner similar to that of angles on a plane? ________ Suppose you folded a piece of cardboard to form a dihedral angle of 60 degrees. Explain how to determine the measure of the angle is 60 degrees. (Does the length of the edge affect the size? Does the size of the faces affect the size of the angle?)

4. If a series of dihedral angles having a common edge just filled the space about the common edge, what would be the sum of these angles? ________

5. What parts of a polyhedral angle affect its size? ________ Does the number of faces affect the size? ________ Does the size of the face angles affect the size? ________

6. On a piece of cardboard or stiff paper, draw figures similar to the patterns below. Make the straight lines about 3 inches long and the angles the sizes indicated.

![Diagram of angles]

Cut out your figures and fold on the lines. Attach the two outside edges with scotch tape to form polyhedral angles. Can the face angles be any size? Explain your answer.
Angles—6

Measure each of the angles to the nearest degree and record the measurement near the vertex of the angle.

Angles such as those in figures 1, 2, 4, 6, and 7 are called "complementary angles." Do you see any common relationship existing among these pairs of angles? If so, what is it? 

Write what you consider to be an acceptable definition of complementary angles.
Angles—7

Measure each of the angles to the nearest degree and record the measurement near the vertex of the angle.

![Fig. 1](image1)

![Fig. 2](image2)

![Fig. 3](image3)

Angles such as those in figures 1, 2, 3, and 6 are called supplementary angles. Do you see any common relationship existing among these pairs of angles? ____ If so, what is it? _______________________

Write what you consider to be an acceptable definition of supplementary angles.
Angles—8

1. Name the vertex of angle 1. _____
2. Name the vertex of angle 2. _____
3. Name the sides of angle 1. _____
4. Name the sides of angle 2. _____
5. Do these angles have a common vertex? If so, what? _____
6. Do any sides of angles 1 and 2 lie in a straight line? If so, which ones? _____

7. Name the vertex of angle 3. _____
   Of angle 4. _____ Of angle 5. _____ Of angle 6. _____
8. Name the sides of angle 3. _____
   Of angle 4. _____ Of angle 5. _____ Of angle 6. _____
9. Do angles 3 and 5 have a common vertex? If so, what? _____
10. Do angles 4 and 6 have a common vertex? If so, what? _____
11. Do any of the sides of angles 3 and 5 lie in a straight line? If so, which ones? _____
12. Do any of the sides of angles 4 and 6 lie in a straight line? If so, which ones? _____
Angles—8 (contd.)


15. Do these four angles have a common vertex? If so, what? 

16. Do angles 7 and 9 have a common side? If so, what? 

17. Do angles 8 and 10 have a common side? If so, what? 

18. Do any of the sides of angles 7 and 9 lie in a straight line? If so, which ones? 

19. Do any of the sides of angles 8 and 10 lie in a straight line? If so, which ones? 

Angles 7 and 9 are called **vertical angles**, as well as angles 8 and 10. None of the other figures have vertical angles. Define vertical angles.

20. Measure the angles in the above three figures.

<table>
<thead>
<tr>
<th>Angle 1</th>
<th>Angle 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>_______</td>
<td>_______</td>
</tr>
<tr>
<td>Angle 2</td>
<td>Angle 7</td>
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<tr>
<td>_______</td>
<td>_______</td>
</tr>
<tr>
<td>Angle 3</td>
<td>Angle 8</td>
</tr>
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<td>_______</td>
<td>_______</td>
</tr>
<tr>
<td>Angle 4</td>
<td>Angle 9</td>
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<td>_______</td>
<td>_______</td>
</tr>
<tr>
<td>Angle 5</td>
<td>Angle 10</td>
</tr>
<tr>
<td>_______</td>
<td>_______</td>
</tr>
</tbody>
</table>
Angles—8 (contd.)

21. Draw two straight lines AB and CD, intersecting at O. Measure each of the angles formed. Are any of these angles equal? ___________
If so, which ones? ___________

22. What relationship appears to exist between two vertical angles?

Prove your statement deductively.
Perpendicular Lines--1

In the figure at the right, assume that AB and CD are two straight lines intersecting at point O.

1. Name the angles formed. ____________

2. Name the equal angles. ____________

3. What relationship exists between \( \angle AOD \) and \( \angle AOC \)? ____________

4. With respect to their position, what kind of angles are \( \angle AOD \) and \( \angle AOC \)? ____________

5. If \( \angle AOD \) is an acute angle, what kind of angle is \( \angle AOC \)? ____________

6. If \( \angle AOD \) increases in size, what happens to \( \angle AOC \)? ____________

7. Is it possible for \( \angle AOD \) to be equal to \( \angle AOC \)? ____________

8. If your answer to exercise 7 is "yes," what kind of angles will these two angles become? ____________

9. When \( \angle AOD = \angle AOC \), the lines AB and CD are said to be perpendicular lines. Write a definition of perpendicular lines.
Perpendicular Lines--2

Recall the definition for perpendicular lines and consider the assumptions which may follow from the exercises below.

1. Draw a line AB and select a point P on the line. How many lines can be perpendicular to the line AB at point P?

2. Draw a line AB and select a point P not on the line (about one inch above). How many lines through this point would be perpendicular to the line AB?

3. Consider a plane, M, and select a point, P, on the plane. How many lines can be drawn perpendicular to the plane at P?

4. Can you think of other conditions which might arise concerning perpendicular lines? If so, discuss them.
In most of the above drawings lines AB and CD are cut by line EF. In Fig. 1-5, EF is called a transversal, but in Fig. 6 it is not. Can you distinguish the difference and write a definition for transversal?

In these drawings you will recognize vertical angles, such as a and c or f and h, and supplementary angles, such as a and b or f and g.

1. In Figs. 1-4, angles a, b, g, and h are exterior angles. Define exterior angles.

2. In Figs. 1-4, angles c, d, e, and f are interior angles. Define interior angles.

3. In Figs. 1-4, angles such as a and e or c and g are called corresponding angles. Define corresponding angles.
Parallel Lines—1 (contd.)

4. In Figs. 1-4 angles such as d and f or c and e are alternate-interior angles. Define alternate-interior angles.

5. Other than the vertical angles in Fig. 1 and 2, do any of the angles appear to be equal? If so, which ones?

6. In Figs. 3 and 4, do any of the angles in these figures, other than the vertical angles, appear to be equal? If so, which ones?

7. If your answer to exercise 6 is different from your answer to exercise 5, what appears to account for the difference?

8. If you found angles, other than the vertical angles, in either exercises 5 or 6 which were equal, summarize the situation in one or two general statements.

9. Other than the adjacent angles in Figs. 1 and 2, do any of the angles appear to be supplementary? If so, which ones?

10. In Figs. 3 and 4, do any of the angles, other than the adjacent ones, appear to be supplementary? If so, which ones?

11. If your answer to exercise 9 is different from your answer to exercise 10, what appears to account for the difference?

12. If you found angles, other than the adjacent angles, in either exercises 9 or 10 which were supplementary, summarize the situation in one or two general statements.

13. If the transversal were perpendicular to line AB in Fig. 3, what would be its relationship to line CD?


15. Give a converse for each of your general statements in exercises 8, 12, 14.

16. Prove deductively the generalizations formed in exercises 8, 12, 14, and 15.
Consider each of the following questions and give your answer as a statement on a separate sheet of paper.

1. If you are given a line and a point not on the line, how many lines may be drawn through the point, not intersecting the line?

2. How many of the lines in exercise 1 will be parallel to the given line?

3. If some of the lines are not parallel to the given line, what is their relationship to it?

4. Under what conditions do you think two planes would be parallel?

5. Under what conditions do you think a line and a plane would be parallel?

6. If two planes were parallel, what relationship would exist between one of the planes and lines in the second plane?

7. If two planes were parallel, what relationship would exist between the lines of the two planes?

8. If a line is parallel to a plane, what is the relationship between the given line and lines in the given plane?

9. If a line is parallel to a plane, what relationship exists between the plane and any plane containing the given line?

10. List other ideas you may have on parallel planes or lines and planes.
1. In triangle ABC measure each interior angle and then find the sum of the angles.

\[ \angle A = \ldots; \quad \angle B = \ldots; \quad \angle C = \ldots; \]

Sum = ___

2. Draw several different triangles in the space below.

3. Measure the interior angles of each triangle; find the sum of these angles for each triangle; record the data below.

<table>
<thead>
<tr>
<th>Triangle</th>
<th>(\angle A)</th>
<th>(\angle B)</th>
<th>(\angle C)</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
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</tbody>
</table>

4. From this information, what conclusion would you draw concerning the sum of the interior angles of a triangle?

5. Draw two triangles. In each case cut out the interior angles and place them together, having their vertices at a common point. What kind of angle is formed? _____

6. Do your findings indicate that a change should be made in your statement concerning the sum of the measures of the interior angles of a triangle? _____ If so, write a new statement here.
Triangles--2

1. Using an unmarked straightedge and compasses construct a triangle that has the same shape and size as the triangle given below. Use as few operations as possible. If there is more than one way in which to do this construction, show all the methods.

2. Construct a triangle whose sides will be equal to the line segments, a, b, c.

3. Construct an isosceles triangle whose base will be equal to n and whose equal sides will each equal m.
h. Using \( m \) and \( n \) in exercise 3, construct an isosceles triangle, the base of which will be equal to \( m \) and the equal sides each equal to \( n \).

5. Construct a triangle the sides of which will be equal to \( x \), \( y \), and \( z \).

\[
\begin{align*}
x \\
y \\
z
\end{align*}
\]

6. Construct a triangle so that the sides will be equal to line segments \( a \) and \( b \) and the angle formed by these two segments will be equal to \( \angle x \).

\[
\begin{align*}
a \\
b
\end{align*}
\]

7. Construct a triangle, two of whose angles will be equal to \( \angle a \) and \( \angle b \) and so that the side between the two angles will be equal to \( s \).

\[
\begin{align*}
s \\
a \\
b
\end{align*}
\]
Triangles--2 (contd.)

8. Construct a triangle whose angles will be equal to $\angle a$, $\angle b$, and $\angle c$.

9. If you now know additional ways to construct the triangle in exercise 1, do so.

10. When two geometric figures have the same size and shape, they are called congruent figures. Describe the conditions which will make two triangles congruent.
Triangles—3

Measure the sides and angles of the following triangles and record the information in the table below.

<table>
<thead>
<tr>
<th>No. of Δ</th>
<th>Type of Δ according to sides</th>
<th>AB</th>
<th>AC</th>
<th>BC</th>
<th>∠C</th>
<th>∠B</th>
<th>∠A</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
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<td>4</td>
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<td>7</td>
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</tr>
</tbody>
</table>
Triangles—3 (contd.)

1. Name, by number, the triangles which have at least two equal sides.

2. Name, by number, the triangles which have at least two equal angles.

3. Write general statements concerning the relationship between the length of the sides of a triangle and the size of its angles. Also write the converse of each of your statements.

4. Prove your statements deductively.
Triangles—4

Previously angles within a polygon have been considered, but those formed by extending a side of the polygon have not been discussed. An angle, such as angle \( a \) in the figure below, is called an *exterior angle* of a polygon.

![Diagram of a triangle with an exterior angle](image)

1. What is the vertex of angle \( a \)?

2. Name the sides of angle \( a \).

3. What is the vertex of the angle within the triangle (interior angle) and adjacent to angle \( a \)?

4. Name the sides of this adjacent angle.

5. Define an exterior angle of a polygon.

6. In the triangle above the exterior angle of the triangle is larger than the interior angle. Can an interior angle of a triangle and its exterior angle be equal? If so, illustrate.

7. Does the same hold for other polygons? If not, explain.

8. Can an interior angle of a triangle be larger than its exterior angle? If so, illustrate.
9. Does the same hold for other polygons? ________ If not, explain.

10. Do you find any relationship which holds between an interior and an exterior angle of a polygon? ________ If so, state it.

11. In a triangle is there any relationship between an exterior angle and the other two interior angles? ________ If so, state it.

12. Prove this statement deductively.

Optional:

If you have not proved the statement: "If one side of a triangle is greater than a second side, the angle opposite the first side is greater than the angle opposite the second side," try to write a deductive proof.

(Hint: On the longer side, lay off a distance equal to the shorter side. Join this point to the endpoint of the shorter side and form an isosceles triangle.)
Perpendicular Lines—3

1. Using the line segment, AB, locate a point, P, which appears to be the same distance from A as from B. Locate a second point, Q, which is the same distance from A as from B.

   A _______________ B

2. The two points, P and Q, determine a line. What relationship, if any, does this line appear to have to the segment AB?

3. Locate a third point, R, which is the same distance from A as from B. Is this point on the line PQ? ______

4. These three points, P, Q, and R, have been described as being the same distance from A as from B. Give another description of the location of the set of all points which are equidistant from the extremities of the line segment AB.

5. Would your description in exercise 4 place all the points on line PQ? ______ If your answer is "yes," have you considered the set of all points? ______

6. From your description form a proposition in the "If . . . , then . . . " form which will apply when the points all lie in the same plane.

7. State the converse of your proposition in exercise 6.
Parallelograms

\[ \begin{array}{c}
A & B \\
D & C
\end{array} \]

1. Measure with a ruler the sides of parallelogram \(ABCD\).

\[ AB = \quad ; \quad BC = \quad ; \quad CD = \quad ; \quad DA = \quad \]

2. Measure with a ruler the lengths of the sides of several of the parallelograms provided and record your measurements in the table below.

<table>
<thead>
<tr>
<th></th>
<th>AB</th>
<th>BC</th>
<th>CD</th>
<th>DA</th>
</tr>
</thead>
<tbody>
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</tbody>
</table>

3. If your measurements indicate any relationship between the opposite sides of a parallelogram, write a statement of your conclusion.

4. With a compass measure the side \(AB\) of several parallelograms and determine how it compares with \(CD\) less than, equal to, or greater than. Do the same with \(BC\) and \(DA\). Record your results in the table at the top of the next page.
Parallelograms—1 (contd.)

<table>
<thead>
<tr>
<th></th>
<th>AB ? CD</th>
<th>BC ? DA</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
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</tbody>
</table>

5. As far as this information is concerned, is your previous statement still correct?  

6. List other generalizations which appear to exist for parallelograms.
Parallelograms--2

By defining a parallelogram as "a quadrilateral with the two pairs of opposite sides parallel," the following conclusions were reached:

a. In a parallelogram the opposite angles are equal.
b. In a parallelogram the opposite sides are equal.
c. In a parallelogram the consecutive angles are supplementary.
d. The diagonals of a parallelogram bisect each other.

1. Complete the proofs for the above statements for your notebook.

There are three types of parallelograms which are common enough to have special names. The figures below represent these special parallelograms.

Figure a is a parallelogram with one right angle and is called a rectangle.

Figure b is a parallelogram with two adjacent sides equal and is called a rhombus.

Figure c is a rectangle with two adjacent sides equal and is called a square.

2. Write the definitions for these three terms in your notebook.

3. All the conclusions proved for the parallelogram will apply to the special types, but there are some additional conclusions which apply only to one or more of these special parallelograms. (For example: All the angles of a rectangle are right angles.) List as many of these conclusions as you can.
4. Prove at least four of your statements in exercise 3.

5. What types of solids would be obtained if parallelograms, rectangles, squares, and/or rhombuses were used to form the edges of three-space figures? Give illustrations.

6. Which of the conclusions concerning plane figures do you think would still follow for solid figures?
Trapezoids

1. We have previously defined a trapezoid as a quadrilateral with one and only one pair of parallel sides. The figures below are all trapezoids, but figures "b" and "e" possess a special property and are called isosceles trapezoids. Determine this property and define the term.

```
   a            b            c
```

```
   e            d
```

2. What do you notice about the angles of the isosceles trapezoid? Prove your statement deductively.

3. In the figure below E is the midpoint of AD and F is the midpoint of BC. The line segment EF is called the median of a trapezoid. The sides AB and CD are called the bases. Define these terms.

```
   D               C
   E               F
   A               B
```


5. How does the length of EF appear to compare with the lengths of AB and CD? Prove your statement.
Parallel Lines—3

a, b, and c are parallel lines.

1. Measure the segments of transversal d which are intercepted by these lines and record your measurement in the table below. Are these segments equal? _________

2. Draw at least two other transversals of a, b, and c, each in a different position.

3. Measure the segments of each of these transversals cut off by the parallel lines. Record your measurements in the table below.

<table>
<thead>
<tr>
<th>Length between Lines</th>
</tr>
</thead>
<tbody>
<tr>
<td>transversal</td>
</tr>
<tr>
<td>-------------</td>
</tr>
<tr>
<td>d</td>
</tr>
<tr>
<td>e</td>
</tr>
<tr>
<td>f</td>
</tr>
</tbody>
</table>

4. How do the segments of each of these compare? ________________

5. Assuming you have one transversal divided into equal parts by three or more parallel lines, make a statement about any other transversal cutting these lines.

6. Prove this statement deductively.
Circles—1

Given the point P shown below. Consider the location of all points a given distance d from this point in answering the following questions:

1. How many points would satisfy this given condition? ________

2. Would all of these points lie on this sheet of paper? ________

3. If your answer to exercise 2 is "no," how would you describe the surface on which these points lie?

4. If you consider only those points which would be located on this sheet of paper, would they lie on a straight line? ________ A broken line? ________ A curved line? ________

5. When we consider all the points which satisfy the above condition the geometric solid is called a sphere. When we consider only those points on a plane, such as the sheet of paper represents, the geometric figure is called a circle. The point P is the center of the sphere or circle and a straight line from point P to one of the points satisfying the given condition is called the radius. Write a definition for each of these four terms.
6. The drawings below illustrate some of the common terms associated with circles. Study the drawings carefully and write a definition for each of the terms.

- Diameter
- Chord
- Arc
- Minor arc
- Major arc
A circle is usually named by the letter naming the point which is the center. In circle O, above, OA is called a ________.

What is the length of OA? ________

Draw two other straight line segments from O to other points of the circle and measure their lengths. How do these lengths compare to OA? ________

Measure the radius of each of the circles below, recording the measurement on the line below the circle. The circle in Fig. 1 is said to equal the circle in Fig. 4, while that in Fig. 2 equals the one in Fig. 3.

Write a general statement concerning the conclusion formed by considering the radii of the same circle or of equal circles.
1. In Fig. 1, chord AB, not a diameter, is perpendicular to chord CD. How does CP compare in length to DP? BC compare to BD? Arc BC compare to arc BD? AC compare to AD? Arc AC compare to arc AD? 

2. In Fig. 2, diameter AB is perpendicular to diameter CD. How does OC compare in length to OD? BC compare to BD? Arc BC compare to arc BD? AC compare to AD? Arc AC compare to arc AD? 

3. In Fig. 3, diameter AB is perpendicular to chord CD. How does CP compare in length to DP? OC compare to OD? BC compare to BD? Arc BC compare to arc BD? AC compare to AD? Arc AC compare to arc AD? 

4. What general statements appear to be true when a diameter is perpendicular to a chord?

5. Draw another chord perpendicular to diameter AB in Fig. 3. Do your general statements still appear to be true? 

6. Do your general statements appear to be true even if the chord is also a diameter? 

7. Prove your statements deductively.
8. In Fig. 4, diameter AB intersects chord CD at R. How does CR compare in length to DR? _________ AC compare to AD? _________ Arc AC compare to arc AD? _________ BC compare to BD? _________ Arc BC compare to arc BD? _________

9. In Fig. 5, diameter AB bisects diameter CD at O. How does AC compare in length to AD? _________ Arc AC compare to arc AD? _________ BC compare to BD? _________ Arc BC compare to arc BD? _________

10. In Fig. 6, diameter AB bisects chord CD at R. How does AC compare in length to AD? _________ Arc AC compare to arc AD? _________ BC compare to BD? _________ Arc BC compare to arc BD? _________

11. Write any general statement which appears to be true when a diameter bisects a chord.

12. Draw another chord, not a diameter, in Fig. 6 which will be bisected by AE. Would you make any changes in your statements in the previous exercise? _________ If yes, what changes?

13. Do your general statements appear to be true if the chord is also a diameter? _________

14. Prove your statements deductively.
Circles--4

Measure the radius of each circle; the length of a chord, AB; and the perpendicular distance, OP. Record your measurements in the table on the following page.
1. Do any of the circles have equal radii? ______ If so, list them (all possibilities). __________________________

2. Do any of the circles have equal chords? ______ If so, list them. ________________________________

3. Are any of the chords equidistant from the center of the circle? ______ If so, list them. ___________________

4. In those cases where the radii are equal, are the chords necessarily equal? ______

5. When the chords are the same distance from the center of the circle, are the chords always equal? ______

6. If the radii are equal and the chords are equal, are the chords always the same distance from the center of the circle? ______

7. In equal circles, if you see a relationship between the chords and the distance from the center of the circle, state this relationship as a theorem.

8. Prove your statement deductively.
Circles—5

The assumption was made that a central angle will contain as many angle degrees as its arc contains arc degrees, or a central angle is measured by its arc. There are, however, angles which are formed by lines other than radii. The following exercise considers one of these.

Make the measurements requested using the given drawing.

1. Central angle AOB = __________
2. Intercepted arc AB = __________
3. Angle APB = __________
4. Angle AQB = __________
5. Angle ARB = __________
6. As the vertex of an angle moves from the center of a circle to the circle and away from the intercepted arc, it becomes _________ in size.
7. How does \( \angle \text{ARB} \) compare in size to central \( \angle \text{AOB} \)? _________
8. How does the measure of \( \angle \text{ARB} \) compare to the measure of the intercepted arc AB? _________
9. The sides of \( \angle \text{ARB} \) are _________ of the circle.
10. Angle ARB is called an **inscribed angle**. Define this term.
Circles--6

1. Measure each central angle $BOC$ and each inscribed angle $BAC$ and record the data below.

<table>
<thead>
<tr>
<th>Fig.</th>
<th>$\angle BOC$</th>
<th>$\angle BAC$</th>
<th>$\widehat{BC}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<tr>
<td>2</td>
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<tr>
<td>3</td>
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</tr>
</tbody>
</table>

2. Draw other circles, each containing an inscribed angle. Measure the inscribed angle and the central angle which intercepts the same arc. Record the information in the table below.

<table>
<thead>
<tr>
<th>Fig.</th>
<th>Inscribed $\angle$</th>
<th>Central $\angle$</th>
<th>Intercepted arc</th>
</tr>
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</tbody>
</table>


3. How does the number of degrees in the inscribed angle compare to the number of degrees in the central angle? In the arc?

4. Make a statement concerning the relationship between the measurement of an inscribed angle and its intercepted arc.

5. Prove the statement deductively. The illustrations at the beginning indicate three cases. Are there others? Prove all cases.
Circles—7

1. In the figure below \( m \angle APB = \) ________.

\[ \text{Diagram: A circle with points P, A, and B} \]

2. If chord BP is rotated in a counter-clockwise direction about P until it becomes a tangent as shown below, then \( m \angle APB = \) ________ and \( m \overline{AP} = \) ________. How does the measure of the angle formed by the chord and tangent compare to the measure of its arc? ________

\[ \text{Diagram: A circle with points P, A, and B with line BP tangent to the circle at P} \]

3. Prove your results from exercise 2.

4. It has now been determined that:

   a. An angle formed by two radii is measured by \( \) ________

   b. An angle formed by two chords intersecting at a point on the circle is measured by \( \) ________

   c. An angle formed by a chord and a tangent intersecting at the point of tangency is measured by \( \) ________

   d. Illustrate other ways angles may be formed by the intersection of chords, tangents, secants, or some combination of these. State the relationship between the angle and its arc. Prove these statements deductively.
Given two unequal circles O and R.

1. How many points do these circles have in common? _____________

2. Draw the line segment OR. This segment is called the line of centers. Write a definition for this term.

3. Name the point where the segment intersects circle O, point A, and the point where it intersects circle R, point B. How does the length of the line of centers compare to the sum of OA and RB? _____________

4. How many tangents to circle O are also tangent to circle R? 
   _____________ Show these on the above figure.

5. The common tangents which intersect the line of centers are called common internal tangents. How many of the tangents in exercise 4 are common internal tangents? _____________

6. The common tangents which do not intersect the line of centers are called common external tangents. How many of the tangents in exercise 4 are common external tangents? _____________

In the space below draw a circle equal to circle O and one equal to circle R, but with the line of centers equal to OA + RB. Call these circles M and N.

Fig. 2
Circles—8 (contd.)

7. How many points do the circles in Fig. 2 have in common?  

8. How many tangents to circle M are also tangent to circle N? 
   Show these on the figure.

9. How many of the tangents in exercise 8 are common internal tangents?  

10. How many of the tangents in exercise 8 are common external tangents?  

In the space below draw a circle equal to circle O and one equal to 
circle R, but with the line of centers less than OA + RB. Call these 
circles P and Q.

Fig. 3

11. How many points do the circles in Fig. 3 have in common?  

12. How many tangents to circle P are also tangent to circle Q? 
   Show these on the figure.

13. How many of the tangents in exercise 12 are common internal tangents?  

14. How many of the tangents in exercise 12 are common external tangents?  

In the space below draw a circle equal to circle O and one equal to 
circle R, but with the line of centers equal to OA - RB. Call these 
circles C and D.

Fig. 4
Circles—8 (contd.)

15. How many points do the circles in Fig. 4 have in common? 

16. How many tangents to circle C are also tangent to circle D? Show these on the figure.

17. How many of the tangents in exercise 16 are common internal tangents? 

18. How many of the tangents in exercise 16 are common external tangents?

In the space below draw a circle equal to circle O and one equal to circle R, but with the line of centers equal to zero. Call these circles E and F.

Fig. 5

19. How many points do the circles in Fig. 5 have in common? 

20. How many tangents to circle E are also tangent to circle F? Show these on the figure.

21. How many of the tangents in exercise 20 are common internal tangents? 

22. How many of the tangents in exercise 20 are common external tangents?

23. If you were to consider these same questions for two spheres rather than two circles, what changes would you make in your answers?
Spheres--1

In the figure below the angle ABC is formed by the arcs of two great circles that intersect at a point. This angle is called a **spherical angle**.

![Diagram of a spherical angle](image)

1. These two great circles will intersect in how many points? _____

2. Name another spherical angle formed by the same two great circles. __________

3. When two lines intersect on a plane, how many angles are formed? __________

4. When two great circles intersect on a sphere, how many spherical angles are formed? __________

While the suggestion of measuring an angle on a sphere by flattening the sphere did not prove practical, consider somewhat this same effect by recalling that a plane which has only one point in common with a sphere will be tangent to the sphere. In the figure above, let B, the vertex of the angle, be the common point. Now draw the tangents to the two great circles at point B (BC' and BA'). These tangents will form a plane angle. Can angle A'BC' be measured? __________

Let us consider a second method of measuring angle ABC. The two planes which intersect the sphere to form circles AED and CBD will intersect each other and form a line, BD. If a plane is passed perpendicular to this intersection, it will intersect the sphere to form a circle. A portion of this circle (arc AC) will lie between the two great circles which form the spherical angle. Can the arc AC be measured? ______

5. In the figure above, therefore, the spherical angle ABC may be measured by angle __________ or arc __________. The spherical angle ADC may be measured by angle __________ or __________.
Further let us consider the closed figure which is formed by three or more arcs of great circles intersecting on a sphere. This figure is called a spherical polygon and is illustrated below. In the figure, polygon ABC is a spherical triangle. When the vertices of the spherical triangle are connected to the center of the sphere a trihedral angle is formed. The sides of the spherical triangle correspond to the face angles and the angles of the spherical triangle correspond to the dihedral angles of the polyhedral angle.

6. When we discussed the face angles of a trihedral angle earlier, it was decided that the sum of the face angles would need to be _______. The sum of the sides of a spherical triangle, therefore, would be _______.

7. How large may the dihedral angle C-OB-A be? _______

8. How large may the dihedral angle B-OA-C be? _______

9. How large may the dihedral angle A-OC-B be? _______

10. Since the spherical angles correspond to the dihedral angles, the sum of the measures of the angles of a spherical triangle cannot be greater than _______ degrees.
Locus—1

1. Locate six points each one inch from point P.

Are there other such points? ______ If so, where do they appear to be located?

If this point were suspended in space, would this change your answer to the previous question? ______ If so, what would your answer be?

2. Locate six points each one-half inch from line l.

Are there other such points? ______ If so, where do they appear to be located?

If this line were suspended in space, would this change your answer to the previous question? ______ If so, what would your answer be?

3. Locate six points each the same distance from point A as from point B.

Are there other such points? ______ If so, where do they appear to be located?
Locus—1 (contd.)

If these points were suspended in space, would this change your answer to the previous question? ______  If so, what would your answer be?

4. Locate six points each the same distance from line m as from line n.

ₘ

ₙ

Are there other such points? ______  If so, where do they appear to be located?

If these lines were suspended in space, would this change your answer to the previous question? ______  If so, what would your answer be?

5. Locate six points each the same distance from line m as from line n.

ₘ

ₙ

Are there other such points? ______  If so, where do they appear to be located?

If these lines were suspended in space, would this change your answer to the previous question? ______  If so, what would your answer be?
The path through which a point moves to satisfy certain conditions such as those in the above exercises is known as the locus of points.

6. Write a statement which will describe the locus of points for each general situation illustrated by the specific examples in exercises 1-5.
Proportions—1

1. Given line segment \( m \) and line segment \( n \) as shown below. In comparing segment \( m \) to segment \( n \) we might speak of segment \( m \) as being 2 in. shorter than segment \( n \) or of segment \( n \) as being 2 in. longer than segment \( m \). Another way of comparing these segments would be to say segment \( n \) is three times as long as \( m \) or line segment \( m \) is \( \frac{1}{3} \) of \( n \). When we compare quantities by the latter method we speak of the ratio of one quantity to another and we express this ratio in lowest terms.

\[
\frac{m}{n} = \frac{\text{shorter segment}}{\text{longer segment}}
\]

What is the ratio of:

a. 8 in. to 12 in.? _______

b. 1 ft. to 2 yds.? _______

c. 50 cents to $5.00? _______

d. \( 7\frac{1}{2} \) to 15? _______

e. \( 15a \) to \( 45a \)? _______

2. If two ratios are equal, we express this fact as a statement called a proportion. This statement may be written as \( \frac{2}{3} = \frac{10}{15} \) or \( 2 : 3 = 10 : 15 \). These statements are read "2 is to 3 as 10 is to 15." In the following statements check (X) those which are proportions.

a. \( 3 : 4 = 9 : 12 \) _______

b. \( 9 : 15 = 12 : 18 \) _______

c. \( 5 : x = 20 : 4x \) _______

d. \( 5 : x = x : 5 \) _______

e. \( 6x : 15x^2 = 24 : 60x \) _______

3. In considering a proportion, such as \( 2 : 3 = 10 : 15 \), the first and fourth terms are called extremes, and the second and third terms are called means. In the following proportions name the extremes and means.

a. \( 3 : 4 = 6 : 8 \) extremes _______; means _______

b. \( 7 : 14 = 5 : 10 \) extremes _______; means _______

c. \( a : b = c : d \) extremes _______; means _______
There are a number of useful relationships which may be developed concerning proportions.

4. Consider the statements in exercise 2. Find the product of the extremes and the product of the means.

   a. product of the extremes ______; product of the means ______
   b. product of the extremes ______; product of the means ______
   c. product of the extremes ______; product of the means ______
   d. product of the extremes ______; product of the means ______
   e. product of the extremes ______; product of the means ______

Do you notice relationships between the two products for those statements which are proportions which do not hold for the statements which are not proportions? ________ If so, state the relationships.

5. Consider the three proportions: 2 : 3 = 8 : 12
   1 : 2 = 6 : 12
   a : b = c : d.

Now consider the statements: 3 : 2 = 12 : 8
   2 : 1 = 12 : 6
   b : a = d : c.

If the first three statements are proportions, are the last three proportions? ________. If your answer is "yes," what could be done to the first three to obtain the last three, respectively?

6. If the first group of statements in exercise 5 are proportions are the following statements proportions? 2 : 8 = 3 : 12
   1 : 6 = 2 : 12
   a : c = b : d  ans. ______

   If your answer is "yes," what could be done to the first three to obtain these three, respectively?
7. If the first group of statements in exercise 5 are proportions are the following statements proportions? 5 : 3 = 20 : 12
   3 : 2 = 16 : 12
   (a + b) : b = (c + d) : d
   ans. _______

   If your answer is "yes," what could be done to the first three to obtain the last three, respectively?

8. If the first group of statements in exercise 5 are proportions are the following statements proportions? 8 : 3 = 2 : 12
   6 : 2 = 1 : 12
   c : b = a : d
   ans. _______

   If your answer is "yes," what could be done to the first three to obtain these three, respectively?

9. In the proportion a : b = c : d, if a = c, what relationship would exist between b and d? _______

10. Consider the series of equal ratios 3/4 = 6/8 = 9/12 = 12/16. What is the ratio of the sum of the numerators to the sum of the denominators? _______ How does this ratio compare to the individual ratios? _______

11. Given that 10 x 3 = 5 x 6, write as many proportions using these numbers as you can.

12. Given that a x b = c x d, write as many proportions using these numbers as you can.
Proportions--2

1. Make the measurements for each figure as requested in the table below; also indicate the ratios requested.

<table>
<thead>
<tr>
<th>Fig.</th>
<th>BD</th>
<th>DA</th>
<th>BD : DA</th>
<th>BE</th>
<th>EC</th>
<th>BE : EC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>4</td>
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</tbody>
</table>
Proportions—2 (contd.)

2. Are there any figures where BD : DA = BE : EC? If so, which figures? How do these figures differ from those figures where the ratios are not equal?

3. Make a general statement from your results in exercise 2.

4. In Fig. 1 use AD as a unit of measure. Mark off BD into segments equal to AD. Draw lines through the points of division parallel to AC and terminating in BC. Why is this possible?

5. What relationship holds among the segments into which BC is divided? Why?

6. BD is how many times as long as DA? Therefore, BD : DA = .

7. BE is how many times as long as EC? Therefore, BE : EC = .


9. If a small unit which divided BD and DA evenly were used as the unit of measure in place of AD, would the proportion BD : DA = BE : EC still be true? 

10. Prove the statement in exercise 3 deductively. Also write the proportion in exercise 9 by inversion, alternation, and addition.
Similar Polygons--1

Measure the angles and sides of the polygons below and record the information in the table on the following page.
Triangles 1, 2, and 3 are similar polygons. Quadrilaterals 5 and 6 are similar polygons. Pentagons 9 and 10 are similar polygons.

1. What properties exist among the similar polygons that are not common to the other polygons of the same type?

2. Define similar polygons.

3. It is possible to show two triangles are similar without proving as many common properties as in the case of other polygons. List properties which you think would be sufficient to prove two triangles similar.
1. Using the line above to represent a number line, locate a point, +6. Call this point A. Locate a point, +10; call it B. How long is the segment AB? Locate a point, -5; call it C. How long is the segment CA? Locate a point, -7; call it D. How long is the segment DA?

2. When two points are located on a number line, explain how to find the length of the segment connecting the two points.

3. On a sheet of graph paper draw a set of axes and locate the following points: A = (2, 8); B = (11, 8); C = (-2, 3); D = (7, 3); E = (-11, -6); F = (-2, -6). Find the length of AB = ________; CD = ________; EF = ________.

4. Do the same for the points: M = (6, 3); N = (6, 11); O = (3, -7); P = (3, 10); Q = (-6, -12); R = (-6, -14). Find the length of MN = ________; OP = ________; QR = ________.

5. Explain how to find the length of the segments as requested in exercises 3 and 4.

6. On a sheet of graph paper draw a set of axes and locate (-11, -3) and (6, 21). Find the length of the segment connecting these two points.

7. Explain how to find the length of the segment in exercise 6.

8. Would the method of exercise 7 work for exercises 3 and 4? ______

9. Would this method work for finding the distance between any two points on a plane? ______

10. Sketch the quadrilateral determined by (-3, -2), (5, -2), (5, 9), and (-3, 9). What kind of quadrilateral is it? ________
Area of Polygons--1

1. If the base of one parallelogram is 18 in. and the altitude is 15 in., how will its area compare to that of a parallelogram having a base of 1.5 ft. and an altitude of 15 in.?

2. If the base of one parallelogram is 18 in. and the altitude is 30 in., how will its area compare to that of a parallelogram having a base of ¾ ft. and altitude of 2.5 ft.?

3. If \( b \) represents the base of a parallelogram and \( a \) represents its altitude and \( m \) represents the base of a second parallelogram and \( n \) its altitude, how do the areas compare?

4. In exercise 3 if \( b = m \) and \( a = n \), how do the areas compare?

5. What statement can be made concerning the areas of two parallelograms when the bases and the altitudes are respectively equal?

6. If the base of one parallelogram is 2¾ in. and the altitude is 7 in., how will its area compare to that of a parallelogram having a base of 17 in. and an altitude of 5 in.?

7. If the base of one parallelogram is 3 ft. and the altitude is 30 in., how will its area compare to that of a parallelogram having a base of 23 in. and an altitude of 11 in.?

8. What statement may be made concerning the ratio between the areas of two parallelograms.
Area of Polygons—l (contd.)

9. Two parallelograms each have an altitude of 60 feet, but one has a base of 100 feet, and the other has a base of 125 feet. How do the areas of the two parallelograms compare?

10. One parallelogram has an altitude of 6 feet and a base of 9 feet. A second parallelogram has an altitude of 2 yards and a base of 5 feet. How do the areas of the two parallelograms compare?

11. One parallelogram has an altitude of $a$ units and a base of $b$ units. A second parallelogram has an altitude of $a$ units and a base of $b'$ units. How do the areas of the two parallelograms compare?

12. What statement may be made concerning the ratio between the areas of two parallelograms having equal altitudes?

13. Make a similar statement concerning the ratio between the areas of two parallelograms having equal bases.

14. Prove deductively the statements in exercises 5, 8, 12, and 13.
Area of Polygons—2

Use the following triangles for exercises 1 - 7.

1. What relationship exists between triangle ABC and triangle DEF?

2. What relationship exists between triangle LMN and triangle RST?

3. Compare the area of triangle ABC to the area of triangle DEF.
   \[ \frac{A_1}{A_2} = \frac{BC}{EF} \]

4. Compare the area of triangle LMN to the area of triangle RST.
   \[ \frac{B_1}{B_2} = \frac{LM}{NR} \]

5. Is the comparison of \( A_1 : A_2 = BC : EF \)?
   
   \[ \frac{A_1}{A_2} = \frac{2 \cdot BC}{2 \cdot EF} \]
   \[ \frac{A_1}{A_2} = \frac{3 \cdot BC}{3 \cdot EF} \]
   \[ \frac{A_1}{A_2} = \frac{BC^2}{EF^2} \]
   \[ \frac{A_1}{A_2} = \frac{BC^3}{EF^3} \]

6. If another pair of corresponding sides, such as AC and DF were used, would your answers to the parts of exercise 5 be the same?
   If the answer is "no," what are the differences?

7. According to the results of exercises 5 and 6, make a statement concerning the ratio of the areas of two triangles in terms of corresponding sides.
Area of Polygons—2 (contd.)

Use the following triangles for exercises 8 - 11. Assume triangle ABC is similar to triangle A'B'C' with corresponding altitudes \( h \) and \( h' \). The triangles are not right triangles.

8. Complete the proportionality: \( \frac{a}{b} = \frac{b}{c} \)

9. Complete the proportionality: \( \frac{h}{c} = \frac{C_1}{C_2} \)

10. Compare the area of triangle ABC to area of triangle A'B'C'.
    \( C_1 : C_2 = \) _________

11. From exercise 8, does \( b : b' = c : c' \)? _________

12. From exercise 9, does \( h : h' = c : c' \)? _________

13. From the results of exercises 10, 11, and 12, \( C_1 : C_2 = \) _________

14. Would your statement in exercise 7 be true for any pair of similar triangles or is it limited to right triangles? _________

15. Prove deductively your statement for exercise 7.
APPENDIX C

Copies of the tests administered to the three groups of students who participated in this experiment are given in Appendix C. These tests include:

1) Test on the Concept of Dimension
2) Henmon-Nelson Test of Mental Ability
3) Orleans Geometry Prognosis Test
4) Shaycoft Plane Geometry Test
TEST ON THE CONCEPT OF DIMENSION

DIRECTIONS: This is a test to see what ideas and concepts you have acquired concerning one-, two-, and three-dimensional space.
When questions refer to "a space of one-dimension," consider a straight line. When questions refer to "a space of two-dimensions," consider a plane. When questions refer to "a space of three-dimensions," consider a solid.

Some of the questions may not have a "right" answer. You are encouraged to use your imagination and give what you consider to be a reasonable answer, although the situation may not exist in the world as we know it. Write clear, but brief, answers.

TEST:

In ex. 1-7, let us assume A, B, and C are individuals (points) differing only in the fact that A lives in a world of one dimension, B in a world of two dimensions, and C in a world of three dimensions.

1. In the respective world of each, where are all the individuals 5 in. from A?

   5 in. from B?

   5 in. from C?

2. Suppose A, B, and C are joined by their twins X, Y, and Z, respectively. Where are all the individuals the same distance from A as from X?

   The same distance from B as from Y?

   The same distance from C as from Z?

3. Suppose that in each of these worlds an infinitely large number of individuals, making a continuous line, are lined up to buy tickets to the next sports event. Where are all the individuals in A's world who are 5 in. from the line?

   In B's world?

   In C's world?

4. Suppose that each of the lines in ex. 3 were 12 in. long. Where are all the individuals in A's world who are 5 in. from this line segment?

   In B's world?

   In C's world?
5. In A's world, where are all the individuals 6 in. from a given plane?

__________________________________________________________________________

In B's world?

__________________________________________________________________________

In C's world?

__________________________________________________________________________

6. In A's world, where are all the individuals the same distance from two parallel lines?

__________________________________________________________________________

In B's world?

__________________________________________________________________________

In C's world?

__________________________________________________________________________

7. In A's world, where are all the individuals the same distance from two parallel planes?

__________________________________________________________________________

In B's world?

__________________________________________________________________________

In C's world?

__________________________________________________________________________

8. Where are all the points on this paper which are the same distance from the sides of the angle at the right?

__________________________________________________________________________

9. Where are all the points located which are the same distance from two adjacent sides of this classroom?

__________________________________________________________________________

10. Consider a point, P, in space. How many other distinct (different) points contain this point? ________ How many distinct lines may contain this point? ________ How many distinct planes may contain this point? ________ Extend this idea to solids of a three-dimensional space, and discuss the question as to how many such solids may contain this point. ____________________________________________________________________________

Discuss this situation for a four-dimensional space. ___________

11. If you are given two distinct points, P and Q, how many distinct points may contain both of these points? ________ How many distinct lines may contain both of these points? ________ How many distinct planes may contain both of these points? ________ Extend this idea to solids of a three-dimensional space, and discuss the question as to how many such solids may contain these two points. ____________________________________________________________________________

Discuss this situation for a solid in a four-dimensional space. __
12. If you are given three points, P, Q, and R, not on the same line, how many distinct points may contain all these points? ________
   How many distinct lines may contain these three points? ________
   How many distinct planes may contain these three points? ________
   Extend this idea to solids of a three-dimensional space, and discuss the question as to how many such solids may contain these three points. ______________________________

Discuss this situation for a solid in a four-dimensional space.

13. If you are given four points, P, Q, R, and S, not in the same plane, how many distinct points may contain all these points? ________
   How many distinct lines may contain these four points? ________
   How many distinct planes may contain these four points? ________
   Extend this idea to solids of a three-dimensional space, and discuss the question as to how many such solids may contain these four points. ______________________________

Discuss this situation for a solid in a four-dimensional space.

14. From questions 10-13, it would appear that the number of points needed to determine one and only one line is ________.
   The number of points needed to determine one and only one plane is ________.
   The number of points needed to determine one and only one solid is ________.

15. If you recognize any relationship between the number of points needed to determine a geometric figure and the number of dimensions the figure has, write a general statement expressing this relationship. ______________________________

16. From the general statement it would appear the number of points needed to determine a four-dimensional figure would be ________.
   If there is any limitation on the location of these points, state this limitation. ______________________________

17. Consider a line, a, and a point, P, not on a. How many lines may be drawn perpendicular to a through P? ________

18. Consider a plane, m, and a point, P, not on m, how many lines may be drawn perpendicular to m through P? ________

19. Consider a three-dimensional solid, s. Assume a point, P, not contained in s and discuss the question as to how many lines may be drawn perpendicular to s through P? ________
20. Consider a line, c, and a point, P, not on c. How many planes may be drawn perpendicular to c through P? ________

21. Consider a plane, r, and a point, P, not on r. How many planes may be drawn perpendicular to r through P? ________

22. Consider a solid in a three-dimensional space. Assume a point, P, not contained in the solid, and discuss the question as to how many planes may be drawn perpendicular to the solid through P?

Describe the following situations by indicating the geometric figure which would be formed:

23. The intersection of two distinct points in space. __________

24. The intersection of two distinct lines in space. __________

25. The intersection of two distinct planes in space. __________

26. The intersection of two distinct three-dimensional solids. ____

27. The intersection of two distinct four-dimensional objects. ____

28. The intersection of a line and a point not contained in the line. __________

29. The intersection of a plane and a line not contained in the plane. __________

30. The intersection of a three-dimensional solid and a plane not contained in the solid. __________

31. Consider a line, c, and a point, P, not on c. How many distinct lines may be drawn through P, parallel to and in the same plane as c? ________ If the limitation of being in the same plane is removed, how many lines through P may be drawn parallel to c? ________ How many planes may pass through P and be parallel to c? ________

32. Consider a plane, m, and a point, P, not on m. How many lines may contain P and be parallel to m? ________ How many planes may contain P and be parallel to m? ________

33. Extend this idea to a three-dimensional solid, s, and a point, P, not contained in s. Discuss the question as to how many lines may contain P and be parallel to s. __________
Discuss the question as to how many planes may contain P and be parallel to s. 

34. Mr. and Mrs. Lineman, who live in a one-dimensional space, have decided to enlarge their home. Discuss the possibilities for doing this. 

35. Frank, John, and Tom, who live in a two-dimensional space, are discussing the size of their country's latest rocketite. Frank says it contains 2500 feet; John claims it contains 2500 square feet; but Tom is positive it contains 2500 cubic feet. If you were a citizen of this country, with whom would you agree and explain why?

36. If you were asked to describe the size of a rocket in a three-dimensional space, what would be the minimum number of dimensions you would consider? Explain your answer.
The Henmon-Nelson Tests of Mental Ability
Reusable Edition

Revised by Tom A. Lamke, Ph.D., and M. J. Nelson, Ph.D.
Iowa State Teachers College, Cedar Falls, Iowa
PRACTICE EXERCISES

The three practice exercises below are given so that you may see how to do the test.

Practice 1. Boys like to:

(1) run  (2) hat  (3) lost  (4) red  (5) same

Which word tells what boys like to do? Yes, run is the right answer. What is the number of the word run? The number is 1. Answer space number 1 has been marked to show that word number 1, run, is the right answer. You are to mark your answers in the same way.

Practice 2. I saw a . . . tree. A word for the blank is:

(1) quite  (2) care  (3) big  (4) so  (5) and

Mark the answer space that you think is right. Your mark should be in the answer space numbered 3.

Practice 3. □ is to □ as △ is to:

(1) ○  (2) □  (3) ○  (4) □  (5) △

What is the number of the right answer? The answer, of course, is number 5, since a square is to a smaller square as a triangle is to a smaller triangle. Mark the answer space numbered 5.
1. If the letters I e t r a were arranged properly, they would spell:
   (1) later  (2) elated  (3) rattle  (4) elevate  (5) relate

2. □ is to □ as □ is to:  
   (1) □  (2) □  (3) □  (4) □  (5) □

3. Tall is to short as day is to:
   (1) long  (2) night  (3) week  (4) day  (5) morning

4. ⊙ is to ⊙ as ⊙ is to:  
   (1) ψ  (2) □  (3) □  (4) □  (5) □

5. My sister’s daughter is my father’s:
   (1) niece  (2) cousin  (3) granddaughter  (4) sister-in-law  (5) aunt

6. 2, 9, 16, 23, 30, . . . . . . . . . . . What two numbers should come next?
   (1) 35 and 42  (2) 39 and 46  (3) 37 and 44  (4) 36 and 40  (5) 31 and 32

7. Which word does not belong with the others?
   (1) house  (2) factory  (3) residence  (4) home  (5) dwelling place

8. If the letters ln aled were arranged properly, they would spell:
   (1) delayed  (2) lament  (3) inlaid  (4) denial  (5) elapse

9. 5, 9, 13, 17, 21, 25, . . . . . . . . . . . . What two numbers should come next?
   (1) 29 and 30  (2) 29 and 31  (3) 29 and 33  (4) 25 and 27  (5) 27 and 29

10. The outline is too vague to . . . . the shape. A word for the blank is:
    (1) summon  (2) resist  (3) indicate  (4) cause  (5) ordain

11. ⊙ is to ⊙ as ⊙ is to:  
    (1) ⊙  (2) □  (3) □  (4) □  (5) □

12. Poem is to poet as portrait is to:
    (1) sculptor  (2) architect  (3) painter  (4) musician  (5) historian

13. 1, 7, 13, 19, . . . . . . . . . . . What two numbers should be on the dotted lines?
    (1) 27 and 33  (2) 25 and 31  (3) 24 and 30  (4) 26 and 29  (5) 28 and 35

14. mistakes do students careful make not If these words were arranged to make a good sentence, what would be the third letter of the second word?
    (1) m  (2) d  (3) s  (4) c  (5) u

15. is to as is to:  
    (1) —  (2) /  (3) □  (4) □  (5) —

16. Which word does not belong with the others?
    (1) novice  (2) accomplice  (3) partner  (4) associate  (5) helper

17. X is to X as — is to:  
    (1) —  (2) —  (3) —  (4) —  (5) —

18. “Give every man thine ear, but few thy voice” means about the same as:
    (1) Few words, many deeds. (2) Full vessels give the least sound.
    (3) Much talk, little work. (4) The tongue is not steel, yet it cuts.
    (5) A man of sense talks little but listens much.

19. # is to □ as # is to:  
    (1) ⊙  (2) □  (3) □  (4) □  (5) □
20. Concentrate is the opposite of:
   (1) think (2) taste (3) owe (4) rebuild (5) disperse

21. \( \square \) is to \( \square \) as \( \bigtriangleup \) is to:
   (1) \( \square \) (2) \( \square \) (3) \( \bigtriangleup \) (4) \( \square \) (5) \( \bigtriangleup \)

22. 19, 19, 16, 16, 13, 13, ...
   What two numbers should come next?
   (1) 9 and 9 (2) 11 and 9 (3) 10 and 10 (4) 12 and 12 (5) 13 and 8

23. Which word does not belong with the others?
   (1) dainty (2) fastidious (3) delicate (4) exquisite (5) hearty

24. A commendable person is:
   (1) beginning (2) talkative (3) praiseworthy (4) formidable (5) important

25. \( \bigtriangleup \) is to \( \bigtriangleup \) as \( \bigtriangleup \) is to:
   (1) \( \bigtriangleup \) (2) \( \bigtriangleup \) (3) \( \bigtriangleup \) (4) \( \bigtriangleup \) (5) \( \bigtriangleup \)

26. 1, 6, 11, 16, ..., 31
   What two numbers should be on the dotted lines?
   (1) 21 and 26 (2) 17 and 25 (3) 26 and 29 (4) 22 and 27 (5) 20 and 25

27. Pride is to victory as humility is to:
   (1) defeat (2) modesty (3) achievement (4) conceit (5) violence

28. 4, 8, 16, 32, ...
   What number should come next?
   (1) 36 (2) 48 (3) 40 (4) 54 (5) 64

29. \( A \) is to \( \bigtriangleup \) as \( \times \) is to:
   (1) \( A \) (2) \( \bigtriangleup \) (3) \( \times \) (4) \( \times \) (5) \( \times \)

30. A drizzling rain fell without .... A word for the blank is:
   (1) beginning (2) opposite (3) intermission (4) length (5) moisture

31. \( \bigcirc \) is to \( \square \) as \( \bigtriangleup \) is to:
   (1) \( \bigcirc \) (2) \( \square \) (3) \( \bigtriangleup \) (4) \( \bigtriangleup \) (5) \( \bigtriangleup \)

32. substance made a bricks called and clay are from pottery
   If these words were arranged to make a good sentence, what would be the word after substance?
   (1) bricks (2) clay (3) called (4) are (5) pottery

33. Truth is to falsehood as pride is to
   (1) fear (2) crime (3) honor (4) humility (5) truth

34. 10, 7, 9, 6, 8, 5, 7, ...
   What number should come next?
   (1) 6 (2) 8 (3) 10 (4) 4 (5) 5

35. A desert always has:
   (1) a lack of vegetation (2) an oasis (3) nomads (4) camels (5) palm trees...

36. A rosette is a:
   (1) banner (2) decoration (3) seat (4) scepter (5) baton

37. 512, 256, 128, 64, 32, ...
   What two numbers should come next?
   (1) 8 and 4 (2) 31 and 30 (3) 33 and 34 (4) 16 and 8 (5) 24 and 16

38. Water seeks its own .... A word for the blank is:
   (1) money (2) weight (3) cold (4) level (5) length

39. \( \bigtriangleup \) is to \( \bigtriangleup \) as \( \bigtriangleup \) is to:
   (1) \( \bigtriangleup \) (2) \( \bigtriangleup \) (3) \( \bigtriangleup \) (4) \( \bigtriangleup \) (5) \( \bigtriangleup \)
40. Which word does not belong with the others?
   (1) publication (2) discourse (3) journal (4) periodical (5) magazine

41. My father's son's sister may be my daughter's . . . . .
   (1) uncle (2) aunt (3) cousin (4) grandmother (5) niece

42. How many pints are there in 1 gallon and \(1\frac{1}{2}\) quarts?
   (1) 7 (2) 9 (3) 11 (4) 15 (5) 19

43. Vague is the opposite of:
   (1) ambitious (2) poor (3) opaque (4) definite (5) insincere

44. 8, 4, 2, 1, \(\frac{1}{2}\), . . . . , . . . . What two numbers should come next?
   (1) \(\frac{1}{4}\) and \(\frac{1}{8}\) (2) \(\frac{1}{3}\) and \(\frac{1}{4}\) (3) 1 and \(\frac{1}{2}\) (4) \(\frac{1}{4}\) and \(\frac{1}{8}\) (5) \(\frac{3}{8}\) and \(\frac{3}{4}\)

45. "An ounce of prevention is worth a pound of cure" means about the same as:
   (1) Don't cry over spilt milk. (2) A miss is as good as a mile.
   (3) Discretion is the better part of valor. (4) Don't cross a bridge until you come to it.
   (5) A stitch in time saves nine.

46. An inaccessible place cannot be:
   (1) reached (2) seen (3) described (4) pierced (5) carried

47. The daughter of my uncle has a brother. My father is her brother's . . . .
   (1) grandfather (2) great-uncle (3) cousin (4) uncle (5) nephew

48. 729, 243, 81, 27, . . . . . What two numbers should come next?
   (1) 9 and 3 (2) 26 and 25 (3) 20 and 13 (4) 19 and 11 (5) 9 and 7

49. with busy filled the bees air of was hum the If these words were arranged to make a good sentence, what would be the last letter of the third word?
   (1) m (2) y (3) d (4) s (5) e

50. A tremulous leaf is:
   (1) green (2) brown (3) parched (4) wilted (5) quivering

51. The ranks were . . . . . by desertions. A word for the blank is:
   (1) depleted (2) accustomed (3) arranged (4) joyful (5) assumed

52. Two pints equal one liter. Three liters equal one rabek. What is the cost of 5 rabeks of milk at 6¢ a pint?
   (1) 90¢ (2) 30¢ (3) $1.50 (4) 60¢ (5) $3.60

53. \(\triangle\) is to \(\triangledown\) as \(\nabla\) is to: (1) \(\triangle\) (2) \(\nabla\) (3) \(\nabla\) (4) \(\triangledown\) (5) \(\nabla\)

54. When you multiply together the length, the width, and the height of a room, you find its:
   (1) perimeter (2) diagonal (3) area (4) volume (5) circumference

55. Listless means:
   (1) systematic (2) accurate (3) loathsome (4) enthusiastic (5) indifferent

56. Add is to subtract as humble is to:
   (1) rich (2) happy (3) haughty (4) mild (5) ill

57. fruit children good cereals are for and If these words were arranged to make a good sentence, what would be the second word?
   (1) fruit (2) children (3) good (4) cereals (5) and

58. 74, 63, 52, . . . . . , 19. What two numbers should be on the dotted lines?
   (1) 41 and 29 (2) 41 and 30 (3) 42 and 31 (4) 43 and 32 (5) 39 and 28
59. "Many cooks spoil the broth" means about the same as:
(1) A good fire makes a good cook. (2) Every cook praises his own broth.
(3) Two captains sink the ship. (4) Civilized man cannot live without cooks.
(5) All lay loads on the willing horse.

60. Club is to member as hand is to:
(1) arm (2) feel (3) finger (4) body (5) work

61. His scientific admired taste he knowledge for his artistic and for was. If these words were arranged to make a good sentence, what would be the word after artistic?
(1) taste (2) admired (3) scientific (4) knowledge (5) was

62. Napoleon said a French soldier was equal to 3 Austrians or to 5 Russians. A dozen Austrian soldiers were equal to how many Russians?
(1) 20 (2) 6 (3) 15 (4) 60 (5) 36

63. 7, 5, 8, 6, 9, 7, 10, .... What number should come next?
(1) 7 (2) 10 (3) 8 (4) 11 (5) 5

64. The uncle of my father's grandson is my:
(1) nephew (2) cousin (3) grandfather (4) brother (5) son

65. "Still waters run deep" means about the same as:
(1) Water is the best of all things. (2) After rain comes sunshine.
(3) Rashness is not valor. (4) The more understanding, the fewer words.
(5) He that seeks finds.

66. Electricity is to wire as gas is to:
(1) flame (2) spark (3) hot (4) pipe (5) stove

67. 3, 9, 12, 36, 39, 117, .... What two numbers should come next?
(1) 120 and 360 (2) 120 and 234 (3) 234 and 236 (4) 351 and 354
(5) 121 and 363

68. A demure person is always:
(1) modest (2) buoyant (3) intelligent (4) ill (5) dependable

69. In a spelling test a girl had 18 words right, giving her an accuracy of 75%. How many words did she miss?
(1) 8 (2) 6 (3) 3 (4) 4 (5) 9

70. Sorrow is to misfortune as joy is to:
(1) grief (2) happiness (3) hatred (4) success (5) pride

71. An obvious fact is:
(1) assumed (2) hateful (3) clear (4) hidden (5) doubtful

72. From a class of 20, 15% of the pupils were absent one day. How many were present?
(1) 15 (2) 17 (3) 3 (4) 7 (5) 5

73. Onyx is a:
(1) limestone (2) quartz (3) glass (4) granite (5) metal

74. 625, 125, 25, 5, .... What number should come next?
(1) 1 (2) 2 (3) 3 (4) 4 (5) 0

75. A palette is used by:
(1) carpenters (2) lawyers (3) musicians (4) artists (5) physicians

Go on to the next page.
76. **Vivacious** means about the same as:
   (1) intelligent  (2) animated  (3) sarcastic  (4) courageous  (5) moody

77. A certain kind of wood is \( \frac{1}{2} \) as heavy as water. Iron is about 7 times as heavy as water. Iron is how many times as heavy as the wood?
   (1) 21  (2) 7  (3) 3\( \frac{1}{2} \)  (4) 14  (5) 7\( \frac{1}{2} \)

78. "It is indeed an ill wind that blows no one good" means about the same as:
   (1) Birds of a feather flock together.
   (2) Correspondence is half a presence.
   (3) Patience is the key to glory.
   (4) The tongue is the neck's enemy.
   (5) The calamities of one nation turn to the benefit of another.

79. **Discreet** means:
   (1) wasteful  (2) attentive  (3) continuous  (4) remorseful  (5) prudent

80. **Depressed** is the opposite of:
   (1) repressed  (2) elated  (3) apathy  (4) anxious  (5) eager

81. A terse style of writing is:
   (1) emotional  (2) mechanical  (3) unsatisfactory  (4) concise  (5) ironical

82. When 3 pupils were absent from a class, the attendance was 94%. How many pupils were there in the class?
   (1) 30  (2) 90  (3) 50  (4) 15  (5) 31

83. **Subsequent** means:
   (1) small  (2) attached  (3) following  (4) irregular  (5) important

84. **Anger** is to **violence** as **love** is to:
   (1) caressing  (2) hate  (3) tempter  (4) hope  (5) happiness

85. **A craven** is a:
   (1) bird  (2) fish  (3) vase  (4) coward  (5) desire

86. "No sweetness in a cabbage twice boiled or in a tale twice told" means about the same as:
   (1) A good tale ill told is a bad one.
   (2) A tale never loses in the telling.
   (3) A good tale is not worse for being twice told.
   (4) There is much good sleep in an old story.
   (5) A tame tongue is a rare bird.

87. **A recreant individual** is:
   (1) young  (2) reborn  (3) smug  (4) happy  (5) cowardly

88. If a franc were worth \( 2\frac{1}{2} \)¢, how many francs would one receive for $100.00?
   (1) 2500  (2) 400  (3) 40,000  (4) 250  (5) 4000

89. **A menial person** is:
   (1) servile  (2) cunning  (3) cross  (4) deceitful  (5) severe

90. **Ambiguous** is the opposite of:
   (1) definite  (2) small  (3) genuine  (4) enigmatic  (5) perpetual

The End. Look back over your work.
Do not open this booklet, or turn it over, until you are told to do so.
Fill in these blanks, giving your name, age, etc. Write plainly.

Name......................................
Last name First name
Age last birthday............. years. Date.............. 19...
Grade or class......................Teacher......................
School.................................City......................

Have you ever studied geometry before?............. If so, how long?

This is a test to see whether you can learn geometry easily. It contains a number of lessons in geometry, each followed by a test to see what you have learned in the lesson. You will be given a certain time to study each lesson, then a certain time for the test. Study each lesson carefully so that you can do the test on it. Give your complete attention to your work. Do not waste any time.

If you finish any lesson or test before the time for it is up, go back over it until you are sure you have learned the lesson correctly, or have done all the examples in the test correctly. Do not turn to the following lesson or test until you are told to.

Ask no questions after the test begins.
Do not turn the page yet.
LESSON 1

DIRECTIONS. Read the following statements carefully and be sure that you understand what they mean. Then you will know how to do the test on the next page.

(1) If things that are equal are added to things that are equal, the sums are equal.

For example: If two bags of flour have the same weight and 5 pounds of flour are added to each bag, their weights will still be equal.

If John is as old as Tom, their ages will still be equal at the end of the next 10 years.

If \( a = b \), then (adding 5 to both \( a \) and \( b \)) \( a + 5 = b + 5 \). (You know from algebra that letters are used to represent numbers.)

(2) If things that are equal are subtracted from things that are equal, the remainders are equal.

For example: If each of two loaves of bread weighs 16 ounces and a 2-ounce piece is cut from each loaf, the remaining parts will be equal in weight.

(3) Doubles of equals are equal.

For example: If the cost of a 5-pound bag of sugar equals 35 cents, the cost of a bag containing twice as much sugar, or 10 pounds, will equal twice as much, or 70 cents.

(4) Halves of equals are equal.

For example: The line \( AB \) (from the point \( A \) to the point \( B \)) is as long as the line \( CD \) (from the point \( C \) to the point \( D \)). Then \( AR \), which is one half of the line \( AB \), is equal to \( CS \), which is one half of the line \( CD \). (See note below.)

(5) Things equal to the same thing are equal to each other.

For example: If Mary is as old as Ann and Sarah is as old as Ann, then Mary and Sarah are equal in age.

If the length of one stick is 3 feet and the length of a second stick is also 3 feet, then the two sticks are equal in length.

NOTE. In geometry it is customary to name the ends of a line by means of letters. Thus, in Figure 1 line \( KL \) means the length of the line from the point \( K \) to the point \( L \).

In Figure 2 line \( MN \) means the part of the line from point \( M \) to point \( N \).

Line \( MP \) means the whole line from point \( M \) to point \( P \).
Part A

DIRECTIONS. Read each of the statements below and decide whether it is true or false; then on your answer sheet, opposite the question number, make a heavy black mark under the letter T if a statement is true and under the letter F if it is false.

1. A straight angle always contains exactly 180 degrees.
2. The diagonals of a parallelogram bisect each other.
3. An obtuse triangle is one which has more than one obtuse angle.
4. In an equilateral triangle, the altitudes equal the sides.
5. All equilateral triangles are similar.
6. Corresponding medians of two similar triangles are in the same ratio as corresponding sides.
7. If a quadrilateral is equiangular, it is also equilateral.
8. A circle can be inscribed in any rhombus.
9. If two adjacent sides of a quadrilateral are equal, the other two sides are equal.
10. If two minor arcs are the same length, the one in the larger circle will have a smaller central angle.

DIRECTIONS. In questions 11 to 14, read each statement at the right; then decide which of the expressions at the left belongs in the blank to complete the statement best. Mark the corresponding space (a, b, c, d, or e) on your answer sheet. (The same answer may be chosen for more than one statement.)

11. If two sides of a triangle are ____, the triangle is isosceles.
   a. similar
   b. equal
   c. parallel
   d. perpendicular
   e. proportional
12. A line which passes through the vertex angle of an isosceles triangle and is ____ to the base bisects the base.
13. A trapezoid always has two bases which are ____.
14. If two rectangles are similar, the dimensions of one are ____ to the corresponding dimensions of the other.

DIRECTIONS. In questions 15 to 20, continue as in the preceding questions.

15. If two figures can coincide completely, they are ____ each other.
   a. congruent to
   b. similar to
   c. equal to
   d. equidistant from
   e. proportional to
16. All points on a circle are ____ the center.
17. All circles are ____ one another.
18. Corresponding angles of two similar quadrilaterals are ____ each other.
19. The areas of similar polygons are ____ the squares of corresponding sides.
20. Two polygons are ____ each other if corresponding sides are equal and corresponding angles are equal.

DIRECTIONS. In questions 21 and 22, read each statement at the right; then decide which of the four statements at the left applies to it, and mark the corresponding space (a, b, c, or d) on your answer sheet. (The same answer may be chosen for more than one statement.)

21. If two angles of a triangle are unequal, the sides opposite are unequal.
   a. Both the statement and its converse are true.
   b. The statement is true but its converse is false.
   c. The statement is false but its converse is true.
   d. Neither the statement nor its converse is true.
22. Similar figures are always congruent.

Directions. In questions 23 to 25, look at each figure at the right; then decide which of the terms at the left applies to it, and mark the corresponding space (a, b, c, d, or e) on your answer sheet. (The same answer may be chosen for more than one figure.)

23.

a. quadrilateral
b. hexagon
c. obtuse triangle
d. regular polygon
e. pentagon

24.

25.

Directions. In questions 26 and 27, continue as in the preceding questions.

26.

a. equilateral triangle
b. isosceles triangle
c. right triangle
d. obtuse triangle
e. none of the above

27.

Directions. In question 28, continue as in the preceding questions.

28.

a. rhombus
b. rectangle
c. equilateral polygon
d. regular hexagon
e. none of the above

Directions. In questions 29 and 30, continue as in the preceding questions.

29.

a. sector of a circle
b. segment of a circle
c. concentric circles
d. tangent circles
e. none of the above

30.

Directions. For each of the following questions, there are five possible answers. You are to decide which answer is the best one; then mark the corresponding space on your answer sheet.

31. A theorem is a
   a. complex postulate.
   b. purely geometric axiom.
   c. statement which is proved.
   d. geometric process.
   e. definition of a geometric figure.

32. "If two sides of a triangle are equal, the angles opposite are equal."
The hypothesis of the above statement is
   a. "if two sides of a triangle are equal."
   b. "the angles opposite are equal."
   c. "opposite angles."
   d. "equal angles."
   e. "sides of a triangle."

33. Pythagoras is credited with the theorem dealing with the relationship among the
   a. squares of the sides of any triangle.
   b. squares of the sides of any right triangle.
   c. square roots of the sides of any right triangle.
   d. squares of the sides of an isosceles right triangle.
   e. square roots of the sides of any triangle.

34. In general, using only compass and straightedge, it is not possible to divide an angle into
   a. 2 equal parts.
   b. 3 equal parts.
   c. 4 equal parts.
   d. 8 equal parts.
   e. 16 equal parts.

35. The feature that all squares, rhombuses, rectangles, parallelograms, and trapezoids have in common is that they have
   a. at least 2 angles equal.
   b. at least 2 sides equal.
   c. equal diagonals.
   d. perpendicular diagonals.
   e. at least 2 sides parallel.

36. The common chord of two intersecting circles is at right angles to the line joining the centers of the circles
   a. except when the center of one circle lies on the other circle.
   b. only if one circle passes through the center of the other.
   c. only if the two circles have equal radii.
   d. in all cases.
   e. in no case.
LESSON 2

DIRECTIONS. Read the following statements carefully and be sure you understand what they mean. Then you will know how to do the test on the next page.

(1) If a line is held fast at one end and turned around that end, then the amount the line is turned from its old to its new position is called an angle. Thus, if line $AB$ in Figure 1 (with the end $A$ held fast) is turned to the new position $AC$, then the amount that the line has been turned from $AB$ to $AC$ is called angle $BAC$ or angle $CAB$. The same angle would be formed if the line were at $AC$ and turned to the new position $AB$. (A curved arrow is used to show the amount the line is turned.)

In Figure 2 the angle is $RST$ or $TSR$, meaning that the line $RS$ was rotated to the new position $TS$, or that the line $TS$ was rotated to the new position $RS$.

In Figure 3 the angle is $MHP$ or $PHM$.

Note that in naming the angle, the letter at the point where the lines meet is placed between the letters that refer to the other ends of the lines.

(2) Look at Figure 4. You see that the two lines $AB$ and $DC$ cross at $E$. Four angles are formed: angle $AED$ or $DEA$ (with 1 in it); angle $DEB$ or $BED$ (with 2 in it); angle $BEC$ or $CEB$ (with 3 in it); and angle $CEA$ or $AEC$ (with 4 in it).

You may think of angle $CEB$ as being formed by the rotation of line $EB$ to the position $EC$ (or of line $EC$ to the position $EB$). In the same way you may think of angle $CEA$ as being formed by the rotation of the line from $EA$ to $EC$ (or from $EC$ to $E1$).

You may also think of the angles $BEC$ and $AED$ being formed at the same time by the rotation of the line $AB$ around to the new position $DC$ about the point $E$, which is kept fixed. In the same way you may also think of angles $AEC$ and $DEB$ being formed at the same time by the further rotation of the line $AB$ around to the position $DC$ about the point $E$, which is kept fixed.

(3) An angle is formed, therefore, when two lines meet or cross each other. Thus, in Figure 5 you see several lines that meet or cross, and a number of angles are formed at the points marked $E$, $F$, $G$, $K$, $P$, and $T$.

For example:
angle $EFT$ or $TFE$ (with 1 in it)
angle $EPT$ or $TPE$ (with 2 in it)
angle $TGK$ or $KGT$ (with 3 in it)
angle $KEG$ or $GEK$ (with 4 in it)
angle $FEG$ or $GEF$ (with 5 in it), etc.

Note that angle $FEK$ is made up of angles 5 and 4. In the same way angles 1 and $PFK$ together make up angle $EFK$. 

[ 4 ]
TEST 2

DIRECTIONS. The exercises in this test are based on the figure at the right, in which there are many angles. In the parentheses after each question, write the answer to that question. You may refer back to Lesson 2 if you need to.

1. Name the angle which contains the number 1.

2. Name the angle which contains the number 2.

3. Name the angle which contains the number 3.

4. What number does angle \(DEA\) contain?

5. What number does angle \(EAC\) contain?

6. Name an angle which does not contain any number.

7. Name the angle that is made up of angles 5 and 8.

8. Angle \(DAB\) is made up of what two angles?

Answers

1. ( )
2. ( )
3. ( )
4. ( )
5. ( )
6. ( )
7. ( )
8. ( and )

Number right. .......... (Score, Test 2)

LESSON 3

DIRECTIONS. Read the following statements very carefully so that you will understand what they mean. Then you will know how to do the test on the next page.

(1) Just as lines are measured by means of a unit of length (e.g., an inch), so angles are measured by means of degrees. An angle is said to have a certain number of degrees. You will have occasion to read about angles of 30 degrees, 60 degrees, 45 degrees, 90 degrees, 180 degrees, and others. (These may also be written \(30^\circ\), \(60^\circ\), \(45^\circ\), \(90^\circ\), \(180^\circ\)).

(2) If the line \(AB\) is held fast at the end \(A\) in the figure and is turned through a complete circle, the angle formed has 360°.

(3) An angle which is one fourth of a complete turn (or rotation) has 90° and is called a right angle (like angle \(BAC\)).

(4) An angle which is one half of a complete turn has 180° and is called a straight angle (like angle \(BAD\)).

(5) An angle which is less than a fourth of a complete turn has less than 90° and is called an acute angle (like angle \(BAE\)).

(6) An angle which is more than a fourth but less than half of a complete turn has between 90° and 180° and is called an obtuse angle (like angle \(BAF\)).
# TEST 3

**DIRECTIONS.** You may look back to Lesson 3 if you need to. In the parentheses after each of the following angles, write what kind of an angle it is. Use the letter **r** for right angle, **ac** for acute angle, **s** for straight angle, **ob** for obtuse angle.

<table>
<thead>
<tr>
<th></th>
<th>Angle of 45°</th>
<th>Answers</th>
<th></th>
<th>Angle of 115°</th>
<th>Answers</th>
<th></th>
<th>Angle of 90°</th>
<th>Answers</th>
<th></th>
<th>Angle of 180°</th>
<th>Answers</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Angle of 45°</td>
<td>(        ) 1</td>
<td>7.</td>
<td></td>
<td>( ) 7</td>
<td></td>
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</tr>
<tr>
<td>2.</td>
<td>Angle of 72°</td>
<td>(        ) 2</td>
<td>8.</td>
<td></td>
<td>( ) 8</td>
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</tr>
<tr>
<td>3.</td>
<td>Angle of 180°</td>
<td>(        ) 3</td>
<td>9.</td>
<td></td>
<td>( ) 9</td>
<td></td>
<td></td>
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<td></td>
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</tr>
<tr>
<td>4.</td>
<td>Angle of 115°</td>
<td>(        ) 4</td>
<td>10.</td>
<td></td>
<td>( ) 10</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>5.</td>
<td>Angle of 90°</td>
<td>(        ) 5</td>
<td>11.</td>
<td></td>
<td>( ) 11</td>
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<tr>
<td>6.</td>
<td></td>
<td>(        ) 6</td>
<td>12.</td>
<td></td>
<td>( ) 12</td>
<td></td>
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</tbody>
</table>

*Number right. .......... (Score, Test 3)*
LESSON 4

DIRECTIONS. Study this lesson very carefully. Then you will be able to do the test on the next page.

(1) If the sum of two angles is a right angle, or 90°, they are called complementary angles.

For example:
- Angles of 70° and 20° are complementary (because 70° + 20° = 90°).
- Angles of 35° and 55° are complementary (because 35° + 55° = 90°).
- Angles of 63° and 27° are complementary (because 63° + 27° = 90°).

How would you find the complement of 24°?
Answer. Subtract 24° from 90°. The complement of 24° is 66°.

Figure 1 is an illustration of two complementary angles, numbered 1 and 2, forming together the right angle CRM (or MRC).

(2) If the sum of two angles is a straight angle, or 180°, they are called supplementary angles.

For example:
- Angles of 140° and 40° are supplementary (because 140° + 40° = 180°).
- Angles of 25° and 155° are supplementary (because 25° + 155° = 180°).
- Angles of 62° and 118° are supplementary (because 62° + 118° = 180°).

How would you find the supplement of 130°?
Answer. Subtract 130° from 180°. The supplement of 130° is 50°.

Figure 2 is an illustration of two supplementary angles, numbered 1 and 2, together forming the straight angle GED (or DEG).

Figure 3 also shows two supplementary angles, numbered 3 and 4, together forming the straight angle KDS (or SDK).

Figure 4 also shows two supplementary angles, numbered 5 and 6, together forming the straight angle EBV (or VBE).
**TEST 4**

**DIRECTIONS.** In the parentheses after each question, write the answer to that question. You may refer back to Lesson 4 if you need to.

**Answers**

1. What is the complement of 60°? ( ) 1
2. What is the complement of 23°? ( ) 2
3. What is the supplement of 120°? ( ) 3
4. What is the supplement of 147°? ( ) 4
5. What is the supplement of 39°? ( ) 5

6. Name two angles in Figure 1 that are supplementary. ( ) 6

7. Name an angle in Figure 1 that is the supplement of angle 4. ( ) 7

8. Name an angle in Figure 2 that is the supplement of angle 8. ( ) 8

9. In Figure 2, angle 8 contains 130°. How many degrees are there in angle 5? ( ) 9

10. How many degrees are there in angle 7? ( ) 10

11. Angle $\angle ADC$ in Figure 3 is a right angle. If angle 9 equals 30°, how many degrees are there in angle 10? ( ) 11

*Number right............. (Score, Test 4)*
LESSON 5

DIRECTIONS. Read the following statements very carefully so that you will understand what they mean. Then you will be able to do the test on the next page.

(1) Figure 1 is called a triangle. It is named triangle $ABC$. It has three sides, called $AB$, $BC$, and $CA$, which may or may not be equal in length. It also has three angles, numbered 1, 2, and 3. They may or may not be equal; that is, they may or may not have the same number of degrees. Each angle is said to be opposite a side, and each side is said to be opposite an angle. For example, side $AB$ is opposite angle 3, and angle 2 is opposite side $AC$.

Each angle is said to be included by two sides, and each side is said to be included by two angles. For example, angle 1 is included by the sides $AB$ and $AC$, and side $BC$ is included by angles 2 and 3.

(2) Figure 2 is called a square. It is read $RSTV$. It has four equal sides, read $RS$, $ST$, $TV$, and $VR$. It also has four angles which are right angles.

Draw a line from $S$ to $V$. This line is called a diagonal.
Draw a line from $R$ to $T$. This line is also called a diagonal.

(3) In Figure 3 the two lines cross, forming four angles marked 1, 2, 3, 4. The two angles that are opposite each other are called a pair of vertical angles. Thus, angles 1 and 3 are one pair of vertical angles, and angles 2 and 4 are another pair of vertical angles.

Two vertical angles may be shown to be equal; that is, angle 1 has the same number of degrees as angle 3, and angle 2 has the same number of degrees as angle 4.
TEST 5

DIRECTIONS. In the parentheses after each question, write the answer to that question. You may refer to Lesson 5 if you need to.

**Answers**

1. In Figure 1, what line is opposite angle 2? ( ) 1
2. In Figure 1, what angle is opposite line ST? ( ) 2
3. In Figure 1, what angle is included between lines RT and TS? ( ) 3
4. In Figure 1, line ST is included between what angles? ( and ) 4

In Figure 2, it is true that if two sides are equal, the angles opposite those sides are equal.

5. If side MN equals side MO, which are the equal angles? ( and ) 5
6. If side MN equals side NO, which are the equal angles? ( and ) 6

7. The expression *two sides and the included angle* is used with reference to Figure 2. If MN is one of the sides referred to and angle 5 is the angle, which is the other of the two sides? ( ) 7
8. The expression *two angles and the included side* is used with reference to Figure 2. If angles 5 and 6 are the angles referred to, which is the included side? ( ) 8

In Figure 3, KLMN is a square. It is true that the diagonals of a square are equal.

9. Which lines in the figure are equal for that reason? ( and ) 9
10. There are two pairs of equal angles in Figure 3 (other than the angles at the corners of the square). Name a pair of angles that are equal. ( and ) 10

Number right............ (Score, Test 5)


**LESSON 6**

**DIRECTIONS.** Study this lesson very carefully.

(1) To *bisect* means to divide into two equal parts. Thus, in the line \(AB\) (Fig. 1) point \(O\) is the midpoint; that is, the distance from \(A\) to \(O\) is the same as from \(O\) to \(B\) or the line \(AO\) equals the line \(OB\). Therefore any line that would cross the line \(AB\) at point \(O\) would *bisect* the line \(AB\).

![Fig. 1](image1)

(2) If the two lines that cross each other in Figure 2 are each divided into two equal parts, they are said to *bisect* each other.

Line \(RS\) is bisected at point \(T\) and \(KL\) is bisected at point \(T\). Then \(KL\) bisects \(RS\), and \(RS\) bisects \(KL\); that is, \(KL\) and \(RS\) bisect each other. Therefore \(RT = TS\) and \(KT = TL\).

![Fig. 2](image2)

(3) In the angle \(ABC\) (Fig. 3) the line \(BD\) makes angles 1 and 2 equal; that is, angle 1 and angle 2 have the same number of degrees. Therefore \(BD\) is said to bisect angle \(ABC\).

![Fig. 3](image3)

**TEST 6**

**DIRECTIONS.** In the parentheses after each question, write the answer to that question. You may refer back to Lesson 6 if you need to.

**Answers**

1. In Figure 1, line \(IV\) bisects line \(PT\). What two lines are therefore equal? ( and ) 1

   In Figure 2, \(CD\) bisects angle \(ACB\) and the side opposite angle \(C\).

2. Which angles do you know are equal? ( and ) 2

3. Which lines do you know are equal? ( and ) 3

   In the square in Figure 3, the diagonal \(KH\) bisects angles \(GHJ\) and \(GKJ\).

4. Name one pair of angles that are equal for this reason. ( and ) 4

   Figure 4 contains two diagonals. It is true that diagonals *bisect each other.*

5. Name one pair of lines that are equal for this reason. ( and ) 5

   Figure 5 is a *circle.* Point \(A\) in the circle is called the *center.* The line \(CD\) is bisected at the center.

6. Name the lines that are therefore equal. ( and ) 6

LESSON 7

DIRECTIONS. Study this lesson very carefully. Then you will be able to do the test on the next page.

(1) It is customary to indicate that two lines in a figure are equal by marking them in the same way. The same is done with angles that are equal. Thus Figure 1 shows that angle $B$ in triangle $ABC$ is equal to angle $E$ in triangle $DEF$. (They are both marked with one cross line.)

![Figure 1](image1)

Figure 2 shows that side $AB$ is equal to side $DE$, since each is marked with one short line; and side $AC$ equals side $DF$, since each is marked with two short lines.

In Figure 3

- side $AB$ = side $DE$
- angle $BAC$ = angle $EDF$
- side $BC$ = side $EF$
- angle $ACB$ = angle $DFE$
- side $AC$ = side $DF$
- angle $CBA$ = angle $FED$

![Figure 3](image2)

(2) Figure 4 shows two triangles in which two sides and the included angle in one triangle are equal respectively to two sides and the included angle in the other triangle. That is, sides $AB$ and $BC$ in the first are respectively equal to sides $ED$ and $EF$ in the second (as shown by the way they are marked), and angle $ABC$ equals angle $DEF$ (as shown by the way they are marked).

In Figure 5 the diagonals bisect each other. The two parts of each diagonal are marked in the same way, showing that they are equal.

![Figure 5](image3)

NOTE that when two triangles are being compared, they may be separated from one another as in Figure 6, or they may touch in various ways as in Figures 7 and 8. Note that when they touch as in Figure 8, one of the sides ($RT$ in this figure) belongs to both triangles at the same time.

![Figure 6](image4)

![Figure 7](image5)

![Figure 8](image6)
TEST 7

DIRECTIONS. Each of the following statements is illustrated by one or more of the diagrams below. (Consider carefully the way in which the sides and angles are marked.) In the parentheses after each diagram write the letter of the statement which it illustrates. For example, statement a is, Two straight lines crossing each other and making opposite angles that are equal. The first diagram does not illustrate this statement, but the second diagram does. So write the letter a in the parentheses after the second diagram. Look at each of the other diagrams to see if it also illustrates this first statement and put the letter a in the parentheses after each one that does. Then do the same with statement b, and then with each of the other statements. Some diagrams illustrate more than one statement, so you may write more than one letter after a diagram.

a. Two straight lines crossing each other and making opposite angles that are equal.
b. Two straight lines crossing at right angles.
c. Two triangles in which two sides and the included angle of one triangle are equal to two sides and the included angle of the other triangle.
d. Two triangles in which two angles and the included side of one triangle are equal to two angles and the included side of the other triangle.
e. Two triangles in which the three sides of one triangle are equal to the three sides of the other triangle.
f. Two triangles in which two angles and the side opposite one of them of one triangle are equal to two angles and the side opposite one of them of the other triangle.

---

Number right............. (Score, Test 7)

[ 18 ]
LESSON 8

DIRECTIONS. Study this lesson very carefully. Then you will know how to do the problems on the following page.

1. First read and learn the following facts:
   a. If a side of one triangle is the same line as the side of another triangle, then these two sides are equal.
   b. If a triangle has two equal sides, it is called an isosceles triangle. Therefore, in any isosceles triangle two of the sides are equal.
   c. If a line bisects an angle, it divides the angle into two equal angles.
   d. If two sides and the included angle of one triangle are equal to two sides and the included angle of another triangle, then the two triangles are equal.

B. Now study the following problem and the way in which it is done. If you understand it, you will be able to do the problems on the next page.

In the accompanying figure, the following facts are true:
- Triangle $MNP$ is an isosceles triangle, with $MN$ equal to $MP$.
- Line $MR$ bisects angle $NMP$.
- Two triangles are formed; namely, triangle $MRN$ and triangle $MRP$.

Problem. To show that triangle $MRN$ is equal to triangle $MRP$.

Four statements about the above figure are given in the first column below. In the parentheses after each statement, you are to write the letter that is in front of the reason (at the top part of the page) that tells why the statement is true.

For example, the first statement below, “Line $MN$ equals line $MP$,” is true because an isosceles triangle has two equal sides. This is the second reason at the top of the page, with the letter $b$ in front of it. So a letter $b$ is written in the parentheses after the first statement.

The second statement, “Angle 1 equals angle 2,” is true because of the third reason given at the top of the page. So a letter $c$ is written in the parentheses after the second statement below.

The third statement, “Line $MR$ in triangle $MRN$ equals line $MR$ in triangle $MRP$,” is true because of the first reason given at the top of the page. So write a letter $a$ in the parentheses after the third statement.

The last statement is true because you have shown that two sides and the included angle in triangle $MRN$ are equal to two sides and the included angle in triangle $MRP$. This is the last reason given at the top of the page; so write a letter $d$ in the parentheses after the fourth statement.

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Line $MN$ equals line $MP$. Why?</td>
<td>($b$) 1</td>
</tr>
<tr>
<td>2. Angle 1 equals angle 2. Why?</td>
<td>($c$) 2</td>
</tr>
<tr>
<td>3. Line $MR$ in triangle $MRN$ equals line $MR$ in triangle $MRP$. Why?</td>
<td>($a$) 3</td>
</tr>
<tr>
<td>4. Therefore triangle $MRN$ equals triangle $MRP$. Why?</td>
<td>($d$) 4</td>
</tr>
</tbody>
</table>

Read this lesson again very carefully. You must understand it and know how to write your answers, if you are to be able to do the test on the next page.
TEST 8

DIRECTIONS. You may look back to Lesson 8 if you need to.

PROBLEM I. Read the following problem carefully. In the parentheses after each statement, write the letter of the fact (given in the box below) that tells the reason why the statement is true.

In the figure, $DB$ and $AC$ are perpendicular to each other. Point $B$ is the midpoint of $AC$.

Problem. To show that triangle $ABD$ is equal to triangle $CBD$.

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $DB$ in triangle $ABD$ equals $DB$ in triangle $CBD$. Why?</td>
<td>(1)</td>
</tr>
<tr>
<td>2. Angle 1 and angle 2 are right angles. Why?</td>
<td>(2)</td>
</tr>
<tr>
<td>3. Angle 1 equals angle 2. Why?</td>
<td>(3)</td>
</tr>
<tr>
<td>4. Line $AB$ equals line $BC$. Why?</td>
<td>(4)</td>
</tr>
<tr>
<td>5. Triangle $ABD$ equals triangle $CBD$. Why?</td>
<td>(5)</td>
</tr>
</tbody>
</table>

PROBLEM II. Read the following problem carefully. In the parentheses after each statement, write the letter of the fact (given in the box below) that tells the reason why the statement is true.

In the figure, $AD$ bisects $BE$ and $BE$ bisects $AD$.

Problem. To show that triangle $ACB$ equals triangle $ECD$.

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>6. Angle 3 and angle 4 are vertical angles. Why?</td>
<td>(6)</td>
</tr>
<tr>
<td>8. Line $AC$ equals line $CD$. Why?</td>
<td>(8)</td>
</tr>
<tr>
<td>9. Line $BC$ equals line $CE$. Why?</td>
<td>(9)</td>
</tr>
<tr>
<td>10. Triangle $ACB$ equals triangle $ECD$. Why?</td>
<td>(10)</td>
</tr>
</tbody>
</table>

a. If two lines are perpendicular to each other, they form right angles.
b. If a side of one triangle is the same line as the side of another triangle, then these two sides are equal.
c. When two straight lines intersect, vertical angles are formed.
d. The midpoint of a line divides the line into two equal parts.
e. All right angles are equal.
f. If two sides and the angle between them in one triangle are equal to two sides and the angle between them in another triangle, then the triangles are equal.
g. Vertical angles are equal.
h. When one line bisects another, it divides it into two equal parts.
i. If two angles and the side between them in one triangle are equal to two angles and the side between them in another triangle, then the triangles are equal.
SUMMARY TEST 9

DIRECTIONS. This is a test to see how well you have learned all the lessons in this booklet. You may look back to any lesson if you need to.

1. In Figure 1, lines AB and CD are perpendicular. They cross at E to form right angles. Line FG is drawn through E.

2. Name one right angle in this figure.

3. Name an angle that is equal to angle 3.

4. Name a pair of vertical angles.

5. If angle 1 equals angle 4, then angle FED equals angle 60°. Look back to page 2 and find the statement that tells why this is true. Then write the number of this reason in the parentheses for the answer.

6. What kind of an angle is angle 1 (see Lesson 1)?

7. In Figure 2 the four sides are equal. Line BD bisects angle ABC, and it also bisects angle ADC.

8. Name a pair of angles that are therefore equal.

9. In triangle BCD, what side is opposite angle BCD?

10-11. There are two reasons why triangle ABD equals triangle CBD. Look back to Test 7 (page 10) and find the two statements that tell these reasons. Then write the letters of these statements in the two parentheses for the answers.

Number right ........... (Score, Test 9)
DIRECTIONS:

Do not open this booklet until you are told to do so.

This is a test of your knowledge of geometry. The test includes several different types of questions; the exact directions for each kind are given within the test. You are to work each question, determine your answer, and then record the answer on the answer sheet. You may answer a question even when you are not perfectly sure that your answer is correct, but you should avoid wild guessing. Do not spend too much time on any one question. Study the sample questions below, and notice how the answers are to be marked on the separate answer sheet.

The directions and Sample A below tell you how to answer one type of question.

Read the statement below and decide whether it is true or false; then on your answer sheet, opposite the question number, make a heavy black mark under the letter T if the statement is true and under the letter F if it is false.

Sample A. All straight angles are equal.

The statement is, of course, true. Now look at your answer sheet. In the upper left-hand column there is a box marked SAMPLES. In the two answer spaces after Sample A, a heavy mark has been made, filling the space (the pair of dotted lines) under T to show that the statement is true.

The directions in the next paragraph and Sample B tell you how to answer another type of question.

Read the statement at the right; then decide which of the expressions at the left belongs in the blank to complete the statement best. Mark the corresponding space (a, b, c, d, or e) on your answer sheet.

Sample B. All straight angles are ___.

For Sample B the correct answer is “equal;” which is answer c. Now look at Sample B on your answer sheet. A heavy mark has been made in the space under c to show that c is the correct answer.

The following directions and Sample C show you how to answer a third type of question.

For the following question there are five possible answers. You are to decide which answer is the best one; then mark the corresponding space on your answer sheet.

Sample C. All straight angles are

a. congruent.
b. similar.
c. equal.
d. parallel.
e. perpendicular.

The correct answer is “equal,” which is answer d; so you would answer Sample C by making a heavy black mark that fills the space under the letter d. Do this now.

Read each question carefully and decide which one of the answers is best. Notice what letter your choice is. Then, on your separate answer sheet, make a heavy black mark in the space under that letter. In marking your answers, always be sure that the question number in the test booklet is the same as the question number on the answer sheet. Erase completely any answer you wish to change and be careful not to make any stray marks on your answer sheet. When you have finished a page, go on to the next page. If you finish the entire test before time is up, go back and check your answers. Work as rapidly and as accurately as you can. This test is divided into two parts. If you have not finished Part A when time is called, stop work on Part A and go on to Part B. When you finish Part A, go right on to Part B. The total working time is 40 minutes.
Directions. In questions 23 to 25, look at each figure at the right; then decide which of the terms at the left applies to it, and mark the corresponding space (a, b, c, d, or e) on your answer sheet. (The same answer may be chosen for more than one figure.)

23.

a. quadrilateral  
b. hexagon  
c. obtuse triangle  
d. regular polygon  
e. pentagon

Directions. In questions 26 and 27, continue as in the preceding questions.

26.

a. equilateral triangle  
b. isosceles triangle  
c. right triangle  
d. obtuse triangle  
e. none of the above

27.

a. sector of a circle  
b. segment of a circle  
c. concentric circles  
d. tangent circles  
e. none of the above

Directions. In question 28, continue as in the preceding questions.

28.

a. rhombus  
b. rectangle  
c. equilateral polygon  
d. regular hexagon  
e. none of the above

Directions. In questions 29 and 30, continue as in the preceding questions.

29.

a. sector of a circle  
b. segment of a circle  
c. concentric circles  
d. tangent circles  
e. none of the above

30.

a. theorem is a  
b. purely geometric axiom.  
c. statement which is proved.  
d. geometric process.  
e. definition of a geometric figure.

31. "If two sides of a triangle are equal, the angles opposite are equal."

The hypothesis of the above statement is

a. "if two sides of a triangle are equal."
   b. "the angles opposite are equal."
   c. "opposite angles."
   d. "equal angles."
   e. "sides of a triangle."

32. "Pythagoras is credited with the theorem dealing with the relationship among the
   a. squares of the sides of any triangle.
   b. squares of the sides of any right triangle.
   c. square roots of the sides of any right triangle.
   d. squares of the sides of an isosceles right triangle.
   e. square roots of the sides of any triangle.

33. In general, using only compass and straightedge, it is not possible to divide an angle into
   a. 2 equal parts.
   b. 3 equal parts.
   c. 4 equal parts.
   d. 8 equal parts.
   e. 16 equal parts.

34. The feature that all squares, rhombuses, rectangles, parallelograms, and trapezoids have in common is that
   a. at least 2 angles equal.
   b. at least 2 sides equal.
   c. equal diagonals.
   d. perpendicular diagonals.
   e. at least 2 sides parallel.

35. The common chord of two intersecting circles is at right angles to the line joining the centers of the circles
   a. except when the center of one circle lies on the other circle.
   b. only if one circle passes through the center of the other.
   c. only if the two circles have equal radii.
   d. in all cases.
   e. in no case.
37. Under what conditions is the third side of a triangle equal to the sum of the other two sides?
   a. in an acute triangle
   b. in an obtuse triangle
   c. in all right triangles
   d. in some but not all right triangles
   e. never

38. One exterior angle is formed at each vertex of a polygon by extending one of the two sides meeting there. If the sum of these exterior angles is 360°, how many sides has the polygon?
   a. 3
   b. 4
   c. 5
   d. impossible to determine
   e. no such polygon exists

39. A right triangle is divided into two similar triangles by
   a. the median to the hypotenuse.
   b. the altitude to the hypotenuse.
   c. the perpendicular bisector of the hypotenuse.
   d. the bisector of the right angle.
   e. none of the above.

40. The method by which the perpendicular bisector of a line segment is constructed is based on the theorem that
   a. two points each equidistant from the ends of a line determine the perpendicular bisector of the line.
   b. all points on the perpendicular bisector of a line are equidistant from the ends of the line.
   c. two triangles are congruent if two angles and the included side of one are equal respectively to two angles and the included side of the other.
   d. the bisector of a straight angle divides it into two right angles.
   e. two right triangles are congruent if the hypotenuse and an arm of one are equal respectively to the hypotenuse and an arm of the other.

41. Under what conditions are two triangles necessarily congruent?
   a. If three angles of one triangle are equal respectively to three angles of the other.
   b. If two sides and the angle opposite one of them in one triangle are equal respectively to two sides and the corresponding angle of the other.
   c. If two sides and the smallest angle of one triangle are equal respectively to two sides and the corresponding angle of the other.
   d. If two sides of one triangle are proportional to two sides of the other and the included angles are equal.
   e. None of the above.

42. If d is the diagonal of a square, \( \frac{d}{2} \) equals
   a. the radius of the inscribed circle.
   b. the radius of the circumscribed circle.
   c. the perimeter of the square.
   d. the area of the square.
   e. none of the above.

43. Lines AB and CD in the figure above intersect at an angle of 60°. The locus of the points equidistant from these two lines is
   a. a circle.
   b. two lines which intersect at a 30° angle.
   c. two lines which intersect at a 60° angle.
   d. two lines which intersect at a 90° angle.
   e. one line.

When you have finished Part A, go on to Part B without waiting.

Part B

Directions. Each of the questions below (44 to 48) consists of a set of measurements applying to a triangle (\( \triangle ABC \)). You are to decide whether it is possible to construct a triangle with these measurements. For each of the questions choose from the following phrases the one which applies, and mark the corresponding answer space. For instance, if a triangle with these measurements is impossible, mark answer space "a"; otherwise mark "b," "c," "d," or "e"—whichever is descriptive of triangles with the given measurements.

a. No such triangle exists.
   b. All such triangles are congruent.
   c. All such triangles are similar, but not necessarily congruent.
   d. Exactly two distinct triangles (neither congruent nor similar) can be constructed with these characteristics.
   e. More than two distinct triangles (neither congruent nor similar) can be constructed with these characteristics.

44. \( \angle A = 30^\circ, \angle B = 60^\circ, \angle C = 90^\circ \)
45. \( \angle A = 30^\circ, \angle B = 60^\circ, \angle C = 90^\circ \)
46. \( AB = AC = BC = 6 \text{ in.}, \angle C = 90^\circ \)
47. \( AC = BC = 10 \text{ in.}, \angle A = \angle B = 30^\circ \)
48. \( AB = 6 \text{ in.}, \angle A = 150^\circ \)
49. What is the perimeter of an equilateral triangle whose side is 10?
   a. $5\sqrt{3}$
   b. $15\sqrt{3}$
   c. 15
   d. 30
   e. none of the above

50. In the figure above, if $ABCD$ were a parallelogram, it would follow that
   a. $\angle x = \angle y$
   b. $\angle x = \angle z$
   c. $AD = CD$
   d. $BD \perp AC$
   e. $BD = AC$

51. What is the mean proportional between 40 and 90?
   a. 60
   b. 65
   c. 70
   d. 75
   e. none of the above

52. Given $\angle ACB$, what is the locus of the centers of circles inscribed in the given angle?
   a. an arc whose center is point $C$
   b. a circle through points $A$, $C$, and $B$
   c. an angle whose sides are parallel to the sides of $\angle ACB$
   d. line $AB$
   e. the bisector of $\angle ACB$

53. In parallelogram $ABCD$ above, $AB = BC = CD = DA = 6$. What is the area of the parallelogram?
   a. 18
   b. 24
   c. 36
   d. impossible to determine answer without additional information
   e. none of the above

54. If three angles of a quadrilateral equal $50^\circ$, $50^\circ$, and $100^\circ$ respectively, what does the fourth angle equal?
   a. $50^\circ$
   b. $75^\circ$
   c. $100^\circ$
   d. $150^\circ$
   e. none of the above

55. In the figure above, $AB = AC$ and $\angle x = \angle y$. To prove by the analytic method that $AD$ is a diameter of the circle, one could begin as follows:
   a. "$AB = AC$ if . . . ."
   b. "If $AB \neq AC$, . . . ."
   c. "$AD$ is a diameter if . . . ."
   d. "If $AD$ is a diameter, . . . ."
   e. "Assume $AD$ is not a diameter."

56. The above diagram is a scale drawing. If $AB = 1$ foot, the area of triangle $ABC$ is closest to
   a. $\frac{1}{2}$ sq. ft.
   b. 1 sq. ft.
   c. 2 sq. ft.
   d. 3 sq. ft.
   e. 4 sq. ft.

57. In the figure above, if $\angle E < \angle A$, what conclusion can be drawn?
   a. $AB$ is not parallel to $DE$.
   b. $AE$ is not perpendicular to $DB$.
   c. $\angle D < \angle B$
   d. $\triangle ABC$ is not similar to $\triangle DEC$.
   e. none of the above

Go on to the next page.
58. Some Boy Scouts hiked north 4 miles, then west 3 miles, then south 1 mile, and then west 1 mile. How far were they from where they started?
   a. 3 miles
   b. 5 miles
   c. 7 miles
   d. 9 miles
   e. none of the above

59. The shadow of a 33-foot tree is 70 feet long. To determine the angle of elevation of the sun, one would find the angle whose
   a. sine is \( \frac{33}{70} \)
   b. cosine is \( \frac{33}{70} \)
   c. tangent is \( \frac{33}{70} \)
   d. sine is \( \frac{70}{33} \)
   e. cosine is \( \frac{70}{33} \)

60. In the figure above, \( AB \) is parallel to \( CD \). What theorem proves \( \angle x = \angle y \)?
   a. If parallel lines are cut by a transversal, alternate interior angles are equal.
   b. If parallel lines are cut by a transversal, corresponding angles are equal.
   c. If two inscribed angles are equal, their arcs are equal.
   d. Inscribed angles intercepting the same arc in a circle are equal.
   e. Corresponding angles of congruent triangles are equal.

Go back and check your answers on both Part A and Part B.
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