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CRITICAL EVALUATION OF THE THEORY OF SHALLOW SHELLS AND APPLICATION OF DIGITAL COMPUTERS TO MIXED BOUNDARY VALUE PROBLEMS OF SHELLS

DISSertation

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

by

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0. INTRODUCTION

0.1 Love's First Approximation

The theory most commonly used for the analysis of thin shells is based upon what came to be called "Love's first approximation." This is closely parallel to Kirchoff's plate theory and the approximations made in the derivation of the equilibrium and compatibility equations correspond to the following physical assumptions:

a) The deformations are small and therefore the equilibrium equations may be referred to the unstrained shell.

b) Normal stresses acting on planes parallel to the middle surface and the strains arising from these are negligible.

c) Transverse shear deformations are negligible.

d) The thickness of the shell is small as compared with its other dimensions.

Of the above four assumptions (a) is used to linearize the problem; (b) and (c) make it possible to reduce the study of the three-dimensional shell to that of its middle-surface, a two-dimensional region imbedded in the three-dimensional space; and finally (d) is used in simplifying relations between strains and displacements or curvature changes and similarly between the forces and strains of the middle surface. (1)(2)(3). ¹

¹ Numbers in parentheses in the text refer to items in the Bibliography
There was disagreement on some terms arising in the derivation of the shell equations on the basis of the foregoing assumptions. Some authors neglected terms as being small which were retained by others and vice versa.

In the present work the formulae originally developed by Love will be accepted as a starting point (See Eqns. 1.63 through 1.66 in Art. 1.3).

### 0.2 Membrane Theory of Shells

The equations of equilibrium and compatibility for a shell of general shape which result from Love's first approximation are rather involved, and methods for solving them and meeting physically meaningful boundary conditions are not known.

A further approximation, very frequently used when practical solutions are sought, consists of neglecting bending and transverse shear entirely and thus, in effect, replacing the shell by a membrane. This so-called membrane theory of shells, especially in its form introduced by Pucher (4), has yielded many important and useful results. Indeed, one generally finds that in any book on the subject, after the first introductory chapters on general theory, the author will deal with cases actually solved which will almost invariably mean membrane solutions.

The membrane theory has several advantages as well as shortcomings. In many cases it leads to simplifications so great that formulae easily
applicable in everyday engineering practice result (5) or that rather unusual loadings may be considered (6). On the other hand, the simplifications introduced in its derivation reduce the order of the governing differential equation to two, and since the shell problem has four boundary conditions, the number of arbitrary constants in the membrane solution is inadequate. As a result of this, for a given shell with some prescribed load, the membrane solution will provide the boundary conditions for which the bending moments in the shell vanish rather than provide a solution for the actually prevailing boundary conditions. This has often been ignored and has led to the tacit acceptance of fictitious boundary conditions.

Another, possibly even more important, objection to the membrane theory is that it completely ignores compatibility and this may lead to quite obvious contradictions, such as indeterminate non-zero stresses due to zero load (3) or to discontinuous edge-effects producing "...a discontinuity in the stress resultants which is propagated along certain lines right across the shell." (7)

This may be an appropriate place to reflect on a curious remark by Flügge in Ref. (7), pp. 168-169 where the question of compatible displacements is brushed aside by the statement that (in the membrane theory of shells) "...u, v, w can always be chosen so that a given set of strains will be produced." While this is, in itself correct, it ignores the fact that the inextensional deformations form a homogeneous solution of the membrane
problem -- since they lead to no strains of the middle surface and hence no membrane forces -- and thus, unless the boundary conditions are carefully chosen to suit the membrane solution, the displacements arising from the membrane stresses will produce bending strains -- a contradiction inherent in the membrane theory.

The above theoretical considerations have been verified by many experimental investigations (19)(20), especially with regard to shells of negative Gauss curvature. All of this indicates the desirability of a shell theory which would allow bending to be considered.

0.3 Theory of Shallow Shells

Such a theory is the so-called "theory of shallow shells," proposed by Marguerre (8) and Mushtari (9), possibly independently of each other.

Marguerre uses the strain-energy of the shell to derive his equations and gives formulae for large deflections as well as small ones. The approximation introduced in either case is stated to be only that the squares or products of the slopes of the shell are neglected in comparison with unity. No attempt is made to set a limit on the numerical values of the slope at which this may be done or to investigate the effect of this on the results. This aspect of the problem will be explored further in Chapter 1 of the present work.
Mushtari, as described by Novozhilov (2), based his derivation on assumptions less directly applicable to physical properties of the shell, such as the requirement that the normal deflection of the shell be a "rapidly changing function" in the entire region of interest, i.e., a derivative of order n should be negligible when compared with a derivative of order (n+1).

Vlasov later introduced a Pucher-type stress function into Mushtari's equations and thus brought them into a form identical with Mcguerre's for the case of small deflections. Vlasov sets a limit on what can be called a shallow shell, but gives no explanation about how this figure (rise to span ratio not exceeding one-fifth) was arrived at.

The best known applications of the theory of shallow shells are due to Reissner (10)-(15) and indeed it is not unusual to read references to "Reissner's shell theory." It should be noted that most of Reissner's work on shallow shells is on spherical caps for which the operator in the governing fourth-order equation separates into two second-order operators and for which closed form solutions exist in terms of Bessel functions. Reissner sets a limit of one-eighth rise to span ratio on the applicability of the theory of shallow shells but offers no explanation about where this number came from.

The stress function applied by the preceding three authors has later been modified (18) and, although this has not been universally accepted
yet, it will be used in the present work in its modified form since this leads to greater accuracy and more general application.

Applications of the theory of shallow shells to non-spherical shapes are not too numerous. This scarcity is probably due to the algebraic and computational difficulties arising from the non-symmetric fourth-order differential operator and to the problems of matching boundary conditions with the functions that result. Nevertheless the work of Duddeck, Apeland and Bouma should be mentioned, all of whom use cartesian coordinates to solve some rather specialized boundary value problems of rectangular shallow shells (21) (22)(23).

0.4 Statement of the Problem

It is desired to obtain some means by which the effects of the approximations introduced during the derivation of the equations of the theory of shallow shells may be assessed. Since the problems that are to be solved have no known exact solution, the errors may not be obtained by comparing the approximate solutions with the exact ones. However, shallow shell theory may provide some means by which the magnitude of such errors may be estimated in any particular case.

It is also proposed to express the solution of the shallow shell equations in a form easily adaptable to high speed digital computers. The class of problems to which this is applicable is chosen to include those shells whose equation is
\[ z = \frac{1}{2} \left( z_{\xi \xi} \xi^2 + 2 z_{\xi \eta} \xi \eta + z_{\eta \eta} \eta^2 \right) \]  

where \( \xi \) and \( \eta \) are oblique coordinate axes in the horizontal plane and \( z \) is measured vertically. The shell shall have boundaries parallel to the \( \xi \) lines and the \( \eta \) lines respectively, i.e., its plan shall be a parallelogram.

It should be noted that there is no limitation in the above that the direction of the principal curvatures be also coordinate directions.

It is felt that the boundary conditions for which solutions are generally available in the shell literature are so restrictive that they become almost fictitious when applied to a real structure. To overcome this, boundary conditions will be formulated for a rather general case of elastic support, such as an edgebeam supported by columns or a series of isolated springs.

By properly choosing the constants involved in such a boundary condition, the free and completely restrained boundaries may be included as special cases.

Again, to utilize the potentials of the electronic computer, the point matching approach is chosen to satisfy the boundary conditions in a least square sense. This method has been used with great success recently for plane stress problems (24). Since in the present formulation certain external equilibrium equations are to be satisfied exactly, an extension of the point matching method is proposed to allow for restraints on the minimization of the least square errors.
All the above is carried to a point where direct application to a digital computer becomes merely a question of writing a program to execute the precisely formulated and detailed mathematical steps. Indeed, a program is currently being written accordingly, however, it does not form an integral part of the present work.

While the problems relating specifically to the computer program will be discussed separately in the final chapter, it is apparent that several decisions on how to approach certain questions have been influenced by the knowledge that solution will be effected by a digital computer. Some references to alternative possibilities, less suitable for treatment by computers, will be found in the text.
1. THE SHALLOW SHELL EQUATIONS

1.1 Fundamental Properties of a Shallow Surface

We shall start by applying some of the concepts of differential geometry to a particular surface.

Let \((x, y, z)\) be a right-handed rectangular cartesian coordinate system with the unit vectors \(i, j, k\) along the axes. Let there be a region \(R\) in the \(x\)-\(y\) plane such that the maximum absolute value of either coordinate in \(R\) is \(L\).

The surface \(S\) we will be dealing with has the equation

\[
Z = \frac{1}{2} \left( z_{xx} x^2 + 2 z_{xy} xy + z_{yy} y^2 \right) \tag{1.1}
\]

where \(z_{xx}, z_{xy}\) and \(z_{yy}\) are constants.

Let the maximum of the three numbers \(|z_{xx}|, |z_{xy}|, |z_{yy}|\) be denoted by \(c\). We shall limit \(L\) so that the dimensionless quantity \((Lc)\) will be small enough for its square to be negligible with respect to unity, i.e.,

\[
(Lc)^2 \ll 1 \tag{1.2}
\]

If the inequality (1.2) is satisfied, we shall call the surface \(S\) shallow within the region \(R\).

It is easy to see the motivation for the above definition if one finds the slopes of \(S\). For example, in the \(x\) direction

\[
\frac{\partial Z}{\partial x} = z_{xx} x + z_{xy} y \tag{1.3}
\]

The expression for the slope is made up of two terms, each having the same structure as \((Lc)\) but, by definition, within the region \(R\), neither
exceeding it. The above statement then restricts the region to that part of S where the square of its slope is of smaller order of magnitude than unity.

At the end of the present article we shall have occasion to point out why the limitation was placed on L as a function of the given c and not vice versa.

We shall now explore some properties of the surface S which will be important for the following articles. Throughout this derivation, quantities of the order of magnitude of \((Lc)^2\) will be neglected in comparison with unity.

First consider a vertical section through S, taken parallel to the x axis. Take a point \(P(x)\) anywhere along this section and let a new coordinate axis \(x_1\) be defined to coincide with the tangent to \(S^2\) in the vertical plane at \(P\) (Figure 1).

Let \(dx\) be an infinitesimal increment in the x direction and \(dx_1\) be the corresponding increment along \(x_1\). Then

\[
dx_1 = \sqrt{dx^2 + \left(\frac{\partial z}{\partial x}\right)^2} = dx\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2} \approx dx\tag{1.4}
\]

Similarly, for \(y_1\) defined in an analogous manner

\[
dy_1 \approx dy\tag{1.5}
\]

In the small, \(x_1\) and the arc along the vertical section of \(S\) are identical.
Figure 1
Vertical section through a shallow surface
The equation of $S$ in vectorial form is

$$\mathbf{r} = xi + yj + zk$$

(1.6)

and the derivatives of the position vector $\mathbf{r}$ along the tangents $x_1$ and $y_1$:

$$\frac{\partial \mathbf{r}}{\partial x_1} = \frac{\partial \mathbf{r}}{\partial x} \cdot \frac{dx}{dx_1} = i + (z_{xx}x + z_{xy}y) k$$

(1.7)

$$\frac{\partial \mathbf{r}}{\partial y_1} = \frac{\partial \mathbf{r}}{\partial y} \cdot \frac{dy}{dy_1} = j + (z_{xy}x + z_{yy}y) k$$

We shall define the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ as the unit vectors in the direction of $x_1$ and $y_1$ respectively and $\mathbf{e}_n$ to complete a right-handed triad with $\mathbf{e}_1$ and $\mathbf{e}_2$. The derivatives in (1.7) have the directions of $\mathbf{e}_1$ and $\mathbf{e}_2$ and their magnitude is

$$\left| \frac{\partial \mathbf{r}_1}{\partial x_1} \right| = \sqrt{1 + (z_{xx}x + z_{xy}y)^2} \approx \sqrt{1} = 1$$

(1.8)

$$\left| \frac{\partial \mathbf{r}_2}{\partial y_1} \right| = \sqrt{1 + (z_{xy}x + z_{yy}y)^2} \approx \sqrt{1} = 1$$
Thus we may write

$$\begin{align*}
\bar{e}_1 & = \frac{\partial \hat{r}}{\partial x_1} = \frac{\partial \hat{r}}{\partial x_1} = i + (z_{xx} x + z_{xy} y) \hat{k} \\
\bar{e}_2 & = \frac{\partial \hat{r}}{\partial y_1} = \frac{\partial \hat{r}}{\partial y_1} = j + (z_{xy} x + z_{yy} y) \hat{k}
\end{align*}$$

(1.9)

In the theory of surfaces the length $ds$ of an infinitesimal arc, when expressed in terms of the increments of the coordinates is called the "first fundamental form" of the surface (25). It is customary to use the notation

$$ds^2 = E \, dx^2 + 2F \, dx \, dy + G \, dy^2$$

(a)

where

$$E = \frac{\partial \hat{r}}{\partial x_1} \cdot \frac{\partial \hat{r}}{\partial x_1} \approx 1$$

$$F = \frac{\partial \hat{r}}{\partial x_1} \cdot \frac{\partial \hat{r}}{\partial y_1} = (z_{xx} x + z_{xy} y) (z_{yy} y + z_{xy} x) \approx 0$$

$$G = \frac{\partial \hat{r}}{\partial y_1} \cdot \frac{\partial \hat{r}}{\partial y_1} \approx 1$$

and therefore,

$$ds^2 = dx^2 + dy^2$$

(b)

It is now seen that for a shallow surface an arc on the surface may be replaced by its projection on the $x$-$y$ plane.
To find \( \bar{e}_n \), the normal unit vector, we use the cross product of the two tangent unit vectors.

\[
\bar{e}_n = \bar{e}_1 \times \bar{e}_2 = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
1 & 0 & (Z_{xx}x + Z_{xy}y) \\
0 & 1 & (Z_{xy}x + Z_{yy}y)
\end{vmatrix}
\]

\[
= -(Z_{xx}x + Z_{xy}y)\hat{i} - (Z_{xy}x + Z_{yy}y)\hat{j} + \hat{k}
\]

(1.10)

We shall also need the derivatives of these unit vectors with respect to \( x_1 \) and \( y_1 \):

\[
\frac{\partial \bar{e}_1}{\partial x_1} = Z_{xx} \hat{k} \quad \frac{\partial \bar{e}_2}{\partial x_1} = Z_{xy} \hat{k} \quad \frac{\partial \bar{e}_n}{\partial x_1} = -Z_{xx}\hat{i} - Z_{xy}\hat{j}
\]

\[
\frac{\partial \bar{e}_1}{\partial y_1} = Z_{xy} \hat{k} \quad \frac{\partial \bar{e}_2}{\partial y_1} = Z_{yy} \hat{k} \quad \frac{\partial \bar{e}_n}{\partial y_1} = -Z_{xy}\hat{i} - Z_{yy}\hat{j}
\]

(1.11)

Equations (1.11) will be more useful if they are given in terms of \( \bar{e}_1, \bar{e}_2, \) and \( \bar{e}_3 \) rather than \( \hat{i}, \hat{j}, \) and \( \hat{k} \). This is achieved by taking the scalar product of each of the derivatives in (1.11) with \( \bar{e}_1, \bar{e}_2, \) and \( \bar{e}_n \).

A peculiarity of our approximation is that, when the approximation (1.2) is consistently applied, the components of \( \partial \bar{e}_1/\partial x_1 \) are the same in the \( \bar{e}_1, \bar{e}_2, \) and \( \bar{e}_n \) system as in the \( \hat{i}, \hat{j}, \hat{k} \) system. Thus, we have

\[
\frac{\partial \bar{e}_1}{\partial x_1} = Z_{xx} \bar{e}_n \quad \frac{\partial \bar{e}_2}{\partial x_1} = Z_{xy} \bar{e}_n \quad \frac{\partial \bar{e}_n}{\partial x_1} = -Z_{xx}\bar{e}_1 - Z_{xy}\bar{e}_2
\]

\[
\frac{\partial \bar{e}_1}{\partial y_1} = Z_{xy} \bar{e}_n \quad \frac{\partial \bar{e}_2}{\partial y_1} = Z_{yy} \bar{e}_n \quad \frac{\partial \bar{e}_n}{\partial y_1} = -Z_{xy}\bar{e}_1 - Z_{yy}\bar{e}_2
\]

(1.12)
Since $\vec{e}_1$ and $\vec{e}_2$ are the first derivatives of the position vector $\vec{r}$

the above quantities are the second derivatives and hence they are

characteristic of the curvatures of $S$. \(^4\)

We have now derived all the formulae that we shall need, namely

(1.9), (1.10) and (1.12). However, there are a few remarks to be made.

In footnote 3 it was pointed out that the shallowness of $S$ resulted

in replacing the length of an arc on $S$ by its projection on the $x$-$y$ plane.

One could also think of this as simply replacing the local coordinate system

$x_1$-$y_1$ by the fixed system $x$-$y$ as far as all magnitudes measured along

the axes are concerned. Thus, for example, a force along a local axis

\[ L \, dx^2 + 2M \, dx \, dy + N \, dy^2 = 0 \]  

where

\[
L = \vec{e}_n \cdot \frac{\partial^2 \vec{r}}{\partial x_1^2} = Z_{xx}
\]

\[
M = \vec{e}_n \cdot \frac{\partial^2 \vec{r}}{\partial x_1 \partial y_1} = Z_{xy}
\]

\[
N = \vec{e}_n \cdot \frac{\partial^2 \vec{r}}{\partial y_1^2} = Z_{yy}
\]

Although the second fundamental form is usually introduced in

connection with the asymptotic curves on a surface (see, for example,

(25)) for our purposes its importance lies in that the normal curvature in

the direction defined by $dx$ and $dy$ is

\[
\frac{1}{R} = \frac{dx^2 + 2M \, dx \, dy + N \, dy^2}{dx^2 + 2F \, dx \, dy + G \, dy^2} = \frac{Z_{xx} \, dx^2 + 2Z_{xy} \, dx \, dy + Z_{yy} \, dy^2}{dx^2 + dy^2}
\] (d)
and its component along the corresponding fixed axis are approximately equal. The same cannot be said for components perpendicular to these axes, at least not on the strength of neglecting second degree quantities only.

This last point can readily be shown by considering a vector \( \vec{F} \) and an axis \( t \) such that \( \theta \), the angle between them, is a small quantity. Let \( F_t \) and \( F_n \) denote the components of \( \vec{F} \) along the axis \( t \) and perpendicular to it. Using the Taylor expansion for cosines and sines we obtain

\[
F_t = F \cos \delta = F \left( 1 - \frac{\delta^2}{2!} + \frac{\delta^4}{4!} + \ldots \right)
\]

\[
F_n = F \sin \delta = F \left( \delta - \frac{\delta^3}{3!} + \frac{\delta^5}{5!} - \ldots \right) \tag{1.13}
\]

It is now seen that if the smallness of \( \theta \) is interpreted as meaning \( \theta^2 \ll 1 \) then the bracketed part of both of the above equations reduces to unity and

\[
F_t = F \quad F_n = F \delta \tag{1.14}
\]

This means that the component along the \( t \) axis approximately equals the force itself, yet its normal component is not zero. Only when \( \theta \) itself -- as opposed to its square -- can be neglected in comparison with unity does the normal component vanish.

The other observation we wish to make requires some familiarity with the theory of surfaces. Since our remarks are not necessary for the
understanding of the following chapters we will not give any detailed explanation here.

Let there be any non-shallow surface $S'$ and let $x, y$ be rectangular axes through a point $P$ of $S'$ and lying in the tangent plane at $P$. Further, let $z$ be such that $x, y$ and $z$ form a right-handed triad. Then, if $z_{xx}, z_{yy}$ and $z_{xy}$ stand for the normal curvatures and the twist of $S'$ at $P$ in the directions $x$ and $y$ the surface

$$z = \frac{1}{2} \left( z_{xx} x^2 + 2z_{xy} xy + z_{yy} y^2 \right)$$

has second order contact with $S'$ at $P$. This means that if a region $R$ is defined in the $x$-$y$ plane such that its maximum measure $L$, as defined at the beginning of the present article, together with $c$ -- the maximum of $|z_{xx}|, |z_{xy}|$ and $|z_{yy}|$ -- satisfies inequality (1.2) we may replace the equation of $S'$ by Eqn. (1.15) within $R$.

Thus, for any surface, regardless of its curvatures, a "region of shallowness" may be defined. The question arises then, whether the theory of shallow shells could not be applied "piece wise" to a number of small enough regions of any shell, the solution of the whole being thus built up from the solutions of individually shallow panels. In some cases this is indeed possible, but it will also be seen later that there is another limitation that requires $L$ to be larger than some multiple of the shell thickness.

If these two requirements cannot be met simultaneously there exists no region of the shell to which the theory of shallow shells is applicable.
1.2 Strain-displacement Relations

The following derivation is, in some respects, more specialized than that which is usually found in the literature (1) (2), because it deals not with curvilinear coordinates on a general surface but only with a shallow surface as defined above. In another sense, however, it is more general because the coordinate system is not limited to lines of curvature but may have any orientation.

The assumptions made by Love (See Art. 0.1) and the shallowness of the shell will be the only criteria used in deciding what terms are negligible and what terms are not. We shall refrain from making any other assumptions such as, for example, the usual one about the relation between the magnitude of in-plane and normal displacements, since it is among our aims to investigate the effect of such assumptions.

Let the shell have a thickness \( h \), small in relation to \( L \), i.e.,

\[
\left( \frac{h}{L} \right)^2 \ll 1 \tag{1.16}
\]

The surface equidistant from the two faces of the shell will be referred to as the middle surface of the shell. Its equation is the same as that of \( S \) in Art. 1.1 (Eqn. 1.1) and it will be assumed to be shallow in the sense of Eqn. (1.2). Let the position vector \( \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \) define some point \( P \) on the unstrained middle surface, \( \vec{e}_1, \vec{e}_2, \vec{e}_n \) be the unit vectors as defined in the previous article and \( \xi \) be the coordinate along \( \vec{e}_n \), i.e., in the direction
of the thickness of the shell. Thus, the point \( P^* \), on the normal originating from \( P \) is defined by the position vector \( \bar{r}^* = \bar{r} + \oint \bar{e}_n \).

It will be assumed that the shell undergoes some displacement \( u, v, w \) in the directions of \( \bar{e}_1, \bar{e}_2 \) and \( \bar{e}_n \) due to the load and that the displacements are small and smooth, that is, the products of the displacements or their derivatives with each other may be neglected in relation to the displacements or their derivatives; further that their products with other small quantities, such as the slope of the middle surface or the thickness of the shell times one of the curvatures, are also negligible. Thus, without mentioning it again, we shall replace terms like

\[
\left[ w + Z_{xy} \left( \frac{\partial w}{\partial x} \right) \right] Z_{xx} \text{ by } Z_{xx} w
\]

and so on. These are the practical consequences of assumption (a) in Art. 0.1, as applied to a shallow shell.

Let \( \bar{e}_1' \) and \( \bar{e}_2' \) be the unit tangent vectors on the strained surface, corresponding to \( \bar{e}_1 \) and \( \bar{e}_2 \) on the unstrained surface and \( \bar{e}_n' \) be the normal vector of the strained surface. Then assumption (c) allows us to say that \( \bar{e}_n \), originally on \( \bar{e}_n' \), will move into \( \bar{e}_n' \) located on \( \bar{e}_n' \) and further, assumption (b) tells us that its distance from the middle surface is unchanged.

\[5\]

It is, of course, implied here that all derivatives needed in the following derivation exist.
We will need expressions for the vectors $\vec{e}_1', \vec{e}_2', \vec{e}_n'$ of the strained surface in terms of their original counterparts and the displacements. Let $\bar{R}$ denote the position vector of $P'$ on the strained surface then

$$\bar{R} = \bar{r} + u\vec{e}_1 + v\vec{e}_2 + w\vec{e}_n$$  \hspace{1cm} (1.17)

and the derivatives of $\bar{R}$ are

$$\frac{\partial \bar{R}}{\partial x} = \frac{\partial \bar{r}}{\partial x} + \frac{\partial u}{\partial x} \vec{e}_1 + \frac{\partial v}{\partial x} \vec{e}_2 + \frac{\partial w}{\partial x} \vec{e}_n$$

$$+ u \frac{\partial \vec{e}_1}{\partial x} + v \frac{\partial \vec{e}_2}{\partial x} + w \frac{\partial \vec{e}_n}{\partial x}$$  \hspace{1cm} (1.18)

Utilizing Eqns. (1.9) and (1.12) and collecting components along $\vec{e}_1', \vec{e}_2', \vec{e}_n'$ we may write

$$\frac{\partial \bar{R}}{\partial x} = (1 + \frac{\partial u}{\partial x} - Z_{xx} w) \vec{e}_1 + (\frac{\partial v}{\partial x} - Z_{xy} w) \vec{e}_2$$

$$+ (\frac{\partial w}{\partial x} + Z_{xx} u + Z_{xy} v) \vec{e}_n$$  \hspace{1cm} (1.19)

This vector is in the direction of $\vec{e}_1'$ -- just as $\frac{\partial \bar{r}}{\partial x}$ was in the direction of $\vec{e}_1$ -- but to turn it into a unit vector, we must divide it by its own magnitude:

$$\left| \frac{\partial \bar{R}}{\partial x} \right| = \sqrt{\frac{\partial \bar{R}}{\partial x} \cdot \frac{\partial \bar{R}}{\partial x}} = 1 + \frac{\partial u}{\partial x} - Z_{xx} w$$ \hspace{1cm} (+ smaller terms)  \hspace{1cm} (1.20)

Since the quantity in Eqn. (1.20) is quite close to unity one may write

$$\frac{1}{\left| \frac{\partial \bar{R}}{\partial x} \right|} = 1 - (\frac{\partial u}{\partial x} - Z_{xx} w)$$  \hspace{1cm} (1.21)
and thus
\[ \bar{e}_1' = \frac{\partial \bar{R}}{\partial x} - \bar{e}_2' + \left( \frac{\partial y}{\partial x} - Z_{xy} w \right) \bar{e}_2 + \left( \frac{\partial w}{\partial x} + Z_{xx} u + Z_{xy} v \right) \bar{e}_n \]  
(1.22)

Repeating exactly the same procedure in the y direction gives
\[ \bar{e}_2' = \left( \frac{\partial u}{\partial y} - Z_{xy} w \right) \bar{e}_1 + \bar{e}_2' + \left( \frac{\partial w}{\partial y} + Z_{xy} u + Z_{yy} v \right) \bar{e}_n \]  
(1.23)

The normal to the strained surface can be found by taking the vector product of the two tangent vectors:
\[ \bar{e}_n' = \bar{e}_1' \times \bar{e}_2' = \left( \frac{\partial w}{\partial x} + Z_{xx} u + Z_{xy} v \right) \bar{e}_1 - \left( \frac{\partial w}{\partial y} + Z_{xy} u + Z_{yy} v \right) \bar{e}_2 + \bar{e}_n \]  
(1.24)

Although some of the strains could be found from Eqns. (1.22)–(1.24), we will only use them in setting up some more general relations.

Consider two neighboring points P and Q on the middle surface such that the arc P\(\bar{Q}\) is in the direction of \(\bar{e}_1\) (Figure 2).

\[ \bar{r}_a = \bar{r}_p + \frac{\partial \bar{r}}{\partial x} dx \]  
(1.25)

Further let P* and Q* be points on the normals at P and Q, respectively, and at the same distance \(\int\) from the middle surface. Then P* and Q* lie on a so-called parallel surface and the arc ds\(_x\) (\(\int\)) between them is originally
Figure 2
Displacement of $d\bar{x}_j(\mathbf{f})$ into $d\bar{x}_j(\mathbf{f})$

$P$ moves $u, v, w$ along $\bar{e}_1, \bar{e}_2, \bar{e}_3$ into $P'$. $P^*$ keeps at the same distance $\mathbf{f}$ from wherever $P$ moves but along the new normal $\bar{e}_n$ ($P^*$). The same relations hold for $Q, Q^*$, etc., except both the displacements and the unit vectors along which they occur are changed by $\frac{\partial}{\partial x}$. dx.
The length of the arc is

\[ ds = \sqrt{d\tilde{s}_x(\xi) \cdot d\tilde{s}_x(\xi)} = (1 - \xi z_{xx}) \, d\xi \quad (1.27) \]

After deformation \( P^* \) moves into \( P^*' \)

\[ \bar{R}_p(\xi) = \bar{R}_p + \xi \tilde{e}_n \quad (1.28) \]

and the new arc \( d\tilde{s}_x(\xi) \) is, just as before

\[ d\tilde{s}_x(\xi) = \frac{\partial}{\partial \xi} \left( \bar{R}_p + \xi \tilde{e}_n \right) \, d\xi \quad (1.29) \]

Now, \( R_p = \bar{r} + u\tilde{e}_1 + v\tilde{e}_2 + w\tilde{e}_n \) and with the use of Eqn. (1.24) for \( \tilde{e}_n \)

we obtain

\[ d\tilde{s}_x(\xi) = \frac{\partial}{\partial \xi} \left\{ \bar{r} + \left[ u - \xi \left( \frac{\partial w}{\partial x} + z_{xx} u + z_{xy} v \right) \right] \tilde{e}_1 + \left[ v - \xi \left( \frac{\partial w}{\partial y} + z_{xy} u + z_{yy} v \right) \right] \tilde{e}_2 + (w + \xi) \tilde{e}_n \right\} \, d\xi \quad (1.30) \]

The differentiation in Eqn. (1.30) may be performed by applying

Eqns. (1.12). The result is

\[ d\tilde{s}_x(\xi) = \left\{ \left[ (1 + \frac{\partial u}{\partial x} - z_{xx} w) + \xi \left( z_{xx} \frac{\partial^2 w}{\partial x^2} - z_{xx} \frac{\partial u}{\partial x} - z_{xy} \frac{\partial v}{\partial x} \right) \right] \tilde{e}_1 + \left[ \left( \frac{\partial v}{\partial x} - z_{xy} w \right) + \xi \left( z_{xy} \frac{\partial w}{\partial x} - z_{xy} \frac{\partial u}{\partial x} - z_{yy} \frac{\partial v}{\partial x} \right) \right] \tilde{e}_2 \right. + \left. \left[ \left( \frac{\partial w}{\partial x} + z_{xx} u + z_{xy} v \right) + \xi \left( -z_{xx} \frac{\partial w}{\partial x} - z_{xy} \frac{\partial w}{\partial y} \right) \right] \tilde{e}_n \right\} \, d\xi \quad (1.31) \]
The change of the arc length due to the displacement is

\[
\Delta x(t) = d\vec{S}_x(t) - d\vec{S}_x(t) = \left\{ \left( \frac{du}{dx} - z_{xx} w \right) + \xi \left( -\frac{\partial^2 w}{\partial x^2} - z_{xx} \frac{du}{dx} - z_{xy} \frac{dv}{dx} \right) \right\} \hat{e}_1
\]

\[+ \left[ \left( \frac{\partial v}{\partial x} - z_{xy} w \right) + \xi \left( -\frac{\partial^2 w}{\partial x \partial y} - z_{xy} \frac{du}{dx} - z_{yy} \frac{dv}{dx} \right) \right] \hat{e}_2 \]  

(1.32)

\[+ \left[ \left( \frac{\partial w}{\partial x} + z_{xx} u + z_{xy} v \right) + \xi \left( -z_{xx} \frac{du}{dx} - z_{xy} \frac{dw}{dx} \right) \right] \hat{e}_n \right\} dx
\]

The linear strain in the x direction is defined as the change of the length of \(d\vec{S}\) in the x direction divided by the original length. Thus,

\[
\varepsilon_{xx}(t) = \frac{\Delta x(t)}{|d\vec{S}_x(t)|} \hat{e}_1 = \left( \frac{du}{dx} - z_{xx} w \right) + \xi \left( -\frac{\partial^2 w}{\partial x^2} - z_{xx} \frac{du}{dx} - z_{xy} \frac{dv}{dx} \right)
\]

\[= \varepsilon_{xx} + \xi K_{xx} \]  

(1.33)

where

\[\varepsilon_{xx} = \frac{du}{dx} - z_{xx} w\]

\[K_{xx} = -\frac{\partial^2 w}{\partial x^2} - z_{xx} \frac{du}{dx} - z_{xy} \frac{dv}{dx}\]

An analogous procedure in the y direction would give

\[
\Delta y(t) = d\vec{S}_y(t) - d\vec{S}_y(t) = \left\{ \left( \frac{du}{dy} - z_{xy} w \right) + \xi \left( -\frac{\partial^2 w}{\partial x \partial y} - z_{xy} \frac{dv}{dy} - z_{xx} \frac{du}{dy} \right) \right\} \hat{e}_1
\]

\[+ \left[ \left( \frac{\partial v}{\partial y} - z_{yy} w \right) + \xi \left( -\frac{\partial^2 w}{\partial y^2} - z_{xy} \frac{du}{dy} - z_{yy} \frac{dv}{dy} \right) \right] \hat{e}_2 \]

\[+ \left[ \left( \frac{\partial w}{\partial y} + z_{xy} u + z_{yy} v \right) + \xi \left( -z_{xy} \frac{dw}{dy} - z_{yy} \frac{dw}{dy} \right) \right] \hat{e}_n \right\} dy
\]

(1.34)
\[
\varepsilon_{yy}(t) = \frac{\Delta_y(t)}{|dS_y(t)|} \mathbf{e}_2 = (\frac{\partial v}{\partial y} - z_{yy}w) + \xi (\frac{\partial^2 w}{\partial y^2} - z_{xy} \frac{\partial u}{\partial y} - z_{yy} \frac{\partial v}{\partial y}) \\
= \varepsilon_{yy} + \xi \kappa_{yy}
\]

where
\[
\varepsilon_{yy} = \frac{\partial v}{\partial y} - z_{yy}w \\
\kappa_{yy} = -\frac{\partial^2 w}{\partial y^2} - z_{xy} \frac{\partial u}{\partial y} - z_{yy} \frac{\partial v}{\partial y}
\]

Finally to obtain an expression for the shear strain we find the unit vectors in the strained directions of the arcs \(dS_x(\mathbf{f})\) and \(dS_y(\mathbf{f})\) and form their scalar product. We get
\[
\tilde{e}_1(t) = \tilde{e}_1 + \left[\left(\frac{\partial v}{\partial x} - z_{xy}w\right) + \xi \left(-\frac{\partial^2 w}{\partial x \partial y} - z_{xy} \frac{\partial u}{\partial x} - z_{yy} \frac{\partial v}{\partial x}\right)\right] \mathbf{e}_2
\]
\[
+ \left[\left(-\frac{\partial w}{\partial x} - z_{xx}u - z_{xy}v\right) + \xi \left(-z_{xx} \frac{\partial w}{\partial x} - z_{xy} \frac{\partial w}{\partial y}\right)\right] \mathbf{e}_n
\]

\[
\tilde{e}_2(t) = \left[\left(\frac{\partial u}{\partial y} - z_{xy}w\right) + \xi \left(-\frac{\partial^2 w}{\partial x \partial y} - z_{xy} \frac{\partial u}{\partial y} - z_{yy} \frac{\partial v}{\partial y}\right)\right] \mathbf{e}_1 + \mathbf{e}_2
\]
\[
+ \left[\left(-\frac{\partial w}{\partial y} - z_{xy}u - z_{yy}v\right) + \xi \left(-z_{xy} \frac{\partial w}{\partial x} - z_{yy} \frac{\partial w}{\partial y}\right)\right] \mathbf{e}_n
\]
and hence
\[ \varepsilon_{xy}(t) = \ddot{e}_1(t) + \dot{e}_2(t) = \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z_{xy} w \right) \]
\[ + \dot{z} \left[ -2 \frac{\partial^2 w}{\partial x \partial y} - z_{xx} \frac{\partial u}{\partial y} - z_{xy} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - z_{yy} \frac{\partial v}{\partial x} \right] \]
\[ = \varepsilon_{xy} + \dot{z} \kappa_{xy} \quad (1.37) \]

where
\[ \varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z_{xy} w \]
\[ \kappa_{xy} = -2 \frac{\partial^2 w}{\partial x \partial y} - z_{xx} \frac{\partial u}{\partial y} - z_{xy} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - z_{yy} \frac{\partial v}{\partial x} \]

Equations (1.33), (1.35) and (1.37) give all strains in a plane parallel to the middle surface and at the distance \( f \) from it as a function of the displacement of the middle surface. According to assumption (b) in Art. 0.1 any other strains that may arise are negligible and cause negligible stresses, hence all significant stresses in the shell should be derivable from these three equations.

The derived strain-displacement relations lend themselves to very obvious physical interpretation. When \( f = 0 \) we obtain the strains of the middle surface
\[ \varepsilon_{xx} = \frac{\partial u}{\partial x} - z_{xx} w \]
\[ \varepsilon_{yy} = \frac{\partial v}{\partial y} - z_{yy} w \]
\[ \varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z_{xy} w \quad (1.38) \]
If the quantities multiplying \( \xi \) - i.e., all \( \xi \)'s - were zero then the strains \( (1.38) \) would be valid everywhere across the thickness of the shell. Hence, there would also be a uniform stress distribution without any bending. This so-called membrane state of stress is very similar to the state of "generalized plane stress" of elasticity theory and we notice that the formulae \( (1.38) \) are different from the corresponding plane stress formulae by the addition of a term depending on \( w \). An almost trivial example may give some insight into how those terms arise.

Consider a ring and take a fiber \( A-B \) along an arc in it (Figure 3) with central angle \( \theta \) and radius \( R \). Let us use the notation \( 1/R = z_{ss} \). The length of the arc is

\[
\text{ds} = Rd\theta \tag{1.39}
\]

If now a displacement \( w \) occurs along the normal to the arc, which is the \( R \) direction, the new arc length is

\[
\text{ds}' = (R-w)d\theta \tag{1.40}
\]

and therefore, the unit strain

\[
\varepsilon = \frac{\text{ds}' - \text{ds}}{\text{ds}} = \frac{(R-w)d\theta - Rd\theta}{Rd\theta} = -\frac{w}{R} = -z_{ss}w \tag{1.41}
\]

This is the same kind of term that occurs in the expressions of the membrane strains.

Let us now investigate the terms in Eqns. \( (1.33), (1.35) \) and \( (1.37) \) that depend on \( \xi \):
Figure 3
Linear strain due to the normal displacement of a circular fiber
\[ K_{xx} = -\frac{\partial^2 w}{\partial x^2} - Z_{xx} \frac{\partial u}{\partial x} - Z_{xy} \frac{\partial v}{\partial x} \]
\[ K_{yy} = -\frac{\partial^2 w}{\partial y^2} - Z_{xy} \frac{\partial u}{\partial y} - Z_{yy} \frac{\partial v}{\partial y} \]
\[ K_{xy} = -2 \frac{\partial^2 w}{\partial x \partial y} - Z_{xx} \frac{\partial u}{\partial y} - Z_{xy} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - Z_{yy} \frac{\partial v}{\partial x} \]

(1.42)

If the middle surface had no strains - i.e., all \( \epsilon \)'s were zero - then the stresses at the middle surface would also be zero. The strains, and the stresses would be proportionate to \( f \), the distance from the stress-free, neutral, middle surface. This means that we would have tensions on one side and equal compressions on the other - the resultant of these would be a zero force component accompanied by a non-zero moment. These are then the strains that give rise to the bending in the shell. Indeed, if we isolate the first term of say \( K_{xx} \), we notice that it is identical to what the elementary treatment of pure bending of beams gives for the curvature of an originally straight bar after deflection: 6

\[ \frac{1}{\phi} = -\frac{\partial^2 w}{\partial x^2} \]

(1.43)

Since a shell is curved in the unstrained state, the \( \kappa \)'s give not the curvature of the shell but the "changes of curvature" (which is their accepted name) with \( K_{xy} \) often referred to as the twist of the surface.

6 In elementary texts one would usually find that the transverse deflection is denoted by \( y \) rather than by \( w \).
We shall again refer to the simplified example of a ring to understand better the way the terms depending on \( u \) and \( v \) arise in the \( k \)'s. To illustrate, for example, the meaning of \(-z_{xx} \frac{\partial u}{\partial x}\) in \( k_{xx} \) consider an arc of a circle originally \( dx \) long (Figure 4).

Let the arc undergo some strain in the tangential direction only.

If the unit strain is \( \frac{\partial u}{\partial x} \) the elongation of the arc is \( \frac{\partial u}{\partial x} \cdot dx \). If the end moves parallel to itself, \( \frac{d}{dx} \) the new elongated arc will still define the original central angle but with a different radius, \( R' \). This angle, \( d\theta \), may be expressed both from the strained and unstrained arcs:

\[
\frac{dx}{R'} = \frac{dx + \frac{\partial u}{\partial x} \cdot dx}{R'}
\]

(1.44)

Dividing Eqn. (1.44) by \( dx \) and introducing \( z_{ss} = \frac{1}{R}, z_{ss}' = \frac{1}{R'} \):

\[
z_{ss} - z_{ss}' = k 
\]

we get

\[
k = \frac{1}{R} - \frac{1}{R'} = \frac{1}{R} \frac{\partial u}{\partial x} \left( \frac{1}{R} - \frac{1}{R'} \right) \frac{\partial u}{\partial x} - \frac{1}{R} \frac{\partial u}{\partial x} = -z_{ss} \frac{\partial u}{\partial x} - k \frac{\partial u}{\partial x}
\]

(1.45)

Finally, neglecting \( \frac{\partial u}{\partial x} \) in comparison with unity in Eqn. (1.45) we may obtain

\[
k = -z_{ss} \frac{\partial u}{\partial x}
\]

(1.46)

---

7 This procedure is, rigorously speaking, not correct since if the end stays parallel to itself there must also be some displacement along the normal. However, the magnitude of the normal displacement is of the order of magnitude of the square of the in-plane strain and is thus negligible in our example.
Figure 4
Change of curvature due to tangential strain of a circular fiber
Eqn. (1.46) is now exactly in the same form in which this term arises in the expressions for the changes of curvature. It is now seen that this and like terms are due to changes in curvature that accompany changes of arc-length without change of the subtended angle.

The formulae (1.45) for the changes of curvature contain some terms that depend on the normal deflection \( w \) and some terms that depend on the membrane strains. In the theory of shallow shells, all terms in the changes of curvature other than the second derivatives of \( w \), are neglected. Several authors start out with order of magnitude considerations that enable them to dispose of these during the derivation so that they do not occur in the final formulae; some authors will carry them until the changes of curvature are found and will neglect them then. Two different explanations of this, frequently found in the literature will now be briefly discussed.

One of the sources often referred to in this respect is Donnell's work (26). He writes:

"(The above mentioned approximation)... is applicable in all thin-wall problems in which the deformation consists of a large number of waves in the circumferential direction."

Mushtari's reasoning is differently worded (Ref. 9 as quoted in Ref. 2) but it really means the same thing. He requires that the displacements of the middle surface be "rapidly changing functions" so that with every differentiation they will increase.

---

Donnell is dealing with cylindrical shells for which the curvature in the non-circumferential direction is zero and thus the questionable terms drop out by themselves.
Another argument simply observes that one expects more in-plane rigidity, hence smaller \( u \) and \( v \) than \( w \). In addition to the in-plane displacements being already small, in Eqns. (1.42) they are multiplied by the curvatures and this makes them negligible.

Many cases are readily visualized where any of the above arguments seem very reasonable. We submit, however, that their validity is affected not only by the shape of the shell, but also its loading and ways of support. For example, if the in-plane loading is significant - as might easily be the case, if a shell although in itself shallow is supported at a steep slope - and especially if the shell is not supported on all sides, one would attach less weight to the above arguments.

Therefore, none of the foregoing arguments seem to the writer so general that the question would thereby be definitely settled. It is proposed that one of the points to be checked in trying to assess the limitations of the theory of shallow shells be the question of errors in changes of curvature. Unfortunately, as will presently be explained, this checking is possible only numerically in each actually solved case.

If the terms under discussion were retained, they would result in the third partial derivatives of \( u \) and \( v \) appearing in the equilibrium equation (1.79 in Art. 1.4) which would make the introduction of the usual stress function impossible. Thus, the shallow shell equations, in the form we now know them and in which they can be solved with comparative ease, would
not exist and thus a basis of comparison - an "exact" solution - is not available. It is suggested that in cases other than well restrained shells with predominantly normal loading the solution, once obtained from shallow shell theory, be substituted back into the expressions of curvature-changes and that the resulting corrections be used as a measure of the applicability of shallow shell theory.

With these reservations, we shall replace Eqns. (1.42) by Eqns. (1.47):

\[ K_{xx} = -\frac{\partial^2 w}{\partial x^2} \]
\[ K_{yy} = -\frac{\partial^2 w}{\partial y^2} \]
\[ K_{xy} = -2 \frac{\partial^2 w}{\partial x \partial y} \]

These expressions are approximate beyond the approximations of linear theory and of Love.

1.3 Relations Between Strains and Stress Resultants

Through the equations of the previous article the strains at any point in the cross-section of the shell may be found if the displacements of the middle surface are fully specified. Using Hooke's law, we may find all stresses from the strains and hence the stresses, too, may be expressed in terms of the same displacements.
This enables us to replace the stresses at different points of the cross-section by their resultants over the thickness of the shell. This possibility is in no way connected to the shallowness of the shell; it is a consequence of assumptions (b) and (c) in Art. 0.1.

On the basis of our discussion in the previous article, the stresses due to the membrane strains - the so-called membrane stresses - have a zero moment resultant and the stresses due to the changes of curvature - the bending stresses - have no force resultant.

An element of the shell is shown in Figure 5. It measures $dx_1$ and $dy_1$ in the $x_1$ and $y_1$ directions at the middle surface, however, due to the curvature of the surface the normal sections which form its boundaries have different widths, $db_x$ or $db_y$, at any other value of $f$.

The curvatures of the middle surface being $z_{xx}$ and $z_{yy}$ the radii of curvature are:

\[
R_x = \frac{1}{z_{xx}} \quad \text{and} \quad R_y = \frac{1}{z_{yy}}
\]

and therefore,

\[
\frac{db_x}{R_x + f} = \frac{dx_1}{R_x}
\]

from which

\[
db_x = \left(1 + \frac{f}{R_x}\right)dx_1 = \left(1 + \frac{f}{z_{xx}}\right)dx_1
\]

\[
db_y = \left(1 + \frac{f}{R_y}\right)dx_1 = \left(1 + \frac{f}{z_{yy}}\right)dx_1
\]
Figure 5
An element of the shell bounded by normal sections
According to assumption (d) in Art. 0.1 the shell is thin, i.e.

\[ h = \delta L \]  

(1.51)

where \( \delta \) is a small number of the order of \( (cL) \) or smaller.

Therefore, we may write

\[ 1 + \xi z_{xx} \leq 1 + hc = 1 + \delta cL \approx 1 \]  

(1.52)

Thus, using \( dh_x = dx \) and \( dh_y = dy \) we can now evaluate the stress resultants on the shell element.\(^9\)

Let \( N_{x_1x_1} \) and \( N_{y_1y_1} \) be the resultant normal forces and \( N_{x_1y_1} \) the shear force in the \( x_1y_1 \) plane each per unit length. Then

\[ dx_1 N_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{x_1x_1} dx_1 dz = \sigma_{x_1x_1} \int_{-\frac{h}{2}}^{\frac{h}{2}} dz = \sigma_{x_1x_1} h dx_1 \]  

(1.53)

or

\[ N_1 = \sigma_{x_1x_1} h \]

But from Hooke's law we know that

\[ \sigma_{x_1x_1} = \frac{E}{1 - \nu^2} \left( \epsilon_{x_1x_1} + \nu \epsilon_{y_1y_1} \right) \]  

(1.54)

where \( E \) is Young's modulus and \( \nu \) is Poisson's ratio. Thus, we obtain

\[ N_{x_1x_1} = \frac{Eh}{1 - \nu^2} \left( \epsilon_{x_1x_1} + \nu \epsilon_{y_1y_1} \right) \]  

(1.55)

---

\(^9\) Neglecting the small quantity in Eqn. (1.52) physically corresponds to taking the centroid of the section as if it were at the middle surface.
Similarly

\[ N_{y_1y_1} = \frac{Eh}{1-\nu^2} \left( \epsilon_{y_1y_1} + \nu \epsilon_{x_1x_1} \right) \tag{1.56} \]

The same procedure may be used for \( N_{x_1y_1} \) except the relevant stress-strain relation is

\[ \tau_{x,y} = \frac{E}{2(1+\nu)} \epsilon_{x_1y_1} = \frac{E}{1-\nu^2} \cdot \frac{1-\nu}{2} \epsilon_{x_1y_1}, \tag{1.57} \]

and this leads to

\[ N_{x_1y_1} = \frac{Eh}{1-\nu^2} \cdot \frac{1-\nu}{2} \epsilon_{x_1y_1}, \tag{1.58} \]

As remarked previously, the membrane strains - being uniform across the thickness of the shell - generate no moment. Thus, for the bending moments we need to consider only that part of the strain which varies with \( x \), i.e., the changes of curvature. Thus,

\[ M_{x_1x_1} \, dx_1 = dx_1 \int_{-\frac{h}{2}}^{\frac{h}{2}} \int \frac{Eh}{1-\nu^2} \left( \kappa_{x_1x_1} + \nu \kappa_{y_1y_1} \right) \, dt \tag{1.59} \]

and hence

\[ M_{x_1x_1} = \frac{Eh^3}{12(1-\nu^2)} \left( \kappa_{x_1x_1} + \nu \kappa_{y_1y_1} \right) \tag{1.60} \]

similarly

\[ M_{y_1y_1} = \frac{Eh^3}{12(1-\nu^2)} \left( \kappa_{y_1y_1} + \nu \kappa_{x_1x_1} \right) \tag{1.61} \]

For the twisting moment, again referring to Eqn. (1.57), we obtain

\[ M_{x_1y_1} = \frac{E}{2(1+\nu)} \int_{-\frac{h}{2}}^{\frac{h}{2}} \kappa_{x_1y_1} \, dt = \frac{Eh^3}{12(1-\nu^2)} \cdot \frac{1-\nu}{2} \kappa_{x_1y_1} \tag{1.62} \]
It should be pointed out that in accordance with our remarks in Art. 1.1 the components of the above stress resultants along the x-y axes equal the corresponding $x_1$-$y_1$ components.

We shall close this article by giving a list of all strain-displacement and stress strain relations derived in Arts. 1.2 and 1.3.

Membrane strains in terms of the displacements of the middle surface:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} - z_{xx} w$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} - z_{yy} w$$

$$\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z_{xy} w$$

(1.63)

Changes of curvature in terms of the displacements of the middle surface:

$$K_{xx} = -\frac{\partial^2 w}{\partial x^2}$$

$$K_{yy} = -\frac{\partial^2 w}{\partial y^2}$$

$$K_{xy} = -2 \frac{\partial^2 w}{\partial x \partial y}$$

(1.64)

Membrane forces in terms of the membrane strains:

$$N_{xx} = S(\varepsilon_{xx} + \nu \varepsilon_{yy})$$

$$N_{yy} = S(\varepsilon_{yy} + \nu \varepsilon_{xx})$$

$$N_{xy} = S \frac{1-\nu}{2} \varepsilon_{xy}$$

(1.65)

where

$$S = \frac{E h}{1-\nu^2}$$
Moments in terms of changes of curvature:

\[ M_{xx} = D(K_{xx} + \nu K_{yy}) \]

\[ M_{yy} = D(K_{yy} + \nu K_{xx}) \]  \hspace{1cm} (1.66)

\[ M_{xy} = D \frac{1-\nu}{2} K_{xy} \]

where

\[ D = \frac{Eh}{12(1-\nu^2)} \]

1.4 Equilibrium of a Shell Element

Figure 6 shows an element of a shell in the coordinate system used and all forces and moments acting on it. For clarity, these are shown only on the near sides. The right hand rule is used for the moment vectors.

All quantities are shown positive. They conform to the sign convention usually adopted in works on the theory of elasticity in that they correspond to stresses that are positive when they are in the direction of the positive axes on a face whose outer normal is in the direction of a positive axis. They are in agreement with the formulae in Art. 1.3.

In addition to the membrane forces and bending and twisting moments introduced previously, Figure 6 also shows the transverse shearing forces \((Q_{x1} \text{ and } Q_{y1})\) and the components of the load per unit area, \(X_1, Y_1\) and \(Z_1\). All of these components are shown acting along the local tangents \(x_1\) and \(y_1\) and perpendicular to them. In the present work only
Figure 6
Forces and moments acting on an element of the shell
such in-plane loads will be admitted that possess a potential function $V$ for which

$$X_i = -\frac{\partial V}{\partial x_i}$$

$$Y_i = -\frac{\partial V}{\partial y_i}$$

and it will be assumed that $V$ is known.

We shall first write moment equations with respect to two edges of the shell-element. After dividing by $dx_1dy_1$ these give:

$$\frac{\partial M_{x_1}}{\partial x_1} + \frac{\partial M_{x_1y_1}}{\partial y_1} - Q_{x_1} = 0$$

$$\frac{\partial M_{y_1}}{\partial y_1} + \frac{\partial M_{x_1y_1}}{\partial x_1} - Q_{y_1} = 0$$

Eqns. (1.64) and (1.66) may now be used to express the moments and the transverse shears in terms of the displacements:

$$M_{x_1} = -D \left( \frac{\partial^2 w}{\partial x_1^2} + \nu \frac{\partial^2 w}{\partial y_1^2} \right)$$

$$M_{y_1} = -D \left( \frac{\partial^2 w}{\partial y_1^2} + \nu \frac{\partial^2 w}{\partial x_1^2} \right)$$

$$M_{x_1y_1} = -D (1-\nu) \frac{\partial^2 w}{\partial x_1 \partial y_1}$$

so that

$$Q_{x_1} = -D \left( \frac{\partial^3 w}{\partial x_1^3} + \frac{\partial^3 w}{\partial x_1 \partial y_1^2} \right) = -D \frac{\partial}{\partial x_1} \nabla^2 w$$

$$Q_{y_1} = -D \left( \frac{\partial^3 w}{\partial x_1^2 \partial y_1} + \frac{\partial^3 w}{\partial y_1^3} \right) = -D \frac{\partial}{\partial y_1} \nabla^2 w$$
where \( \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2} \) is the Laplace operator.

Next we write equilibrium equations for the force components in the \( x_1, y_1 \) and \( z_1 \) directions respectively. After dividing by \( dx_1 dy_1 \) these become:

\[
\frac{\partial N_{x_1}}{\partial x_1} + \frac{\partial N_{x_1 y_1}}{\partial y_1} - Z_{xx} Q_{x_1} - Z_{xy} Q_{y_1} + X_1 = 0
\]

\[
\frac{\partial N_{y_1}}{\partial y_1} + \frac{\partial N_{x_1 y_1}}{\partial x_1} - Z_{yy} Q_{y_1} - Z_{xy} Q_{x_1} + Y_1 = 0
\] (1.71)

\[
\frac{\partial Q_{x_1}}{\partial x_1} + \frac{\partial Q_{y_1}}{\partial y_1} + Z_{xx} N_{x_1} + 2Z_{xy} N_{x_1 y_1} + Z_{yy} N_{y_1} + Z_z = 0
\]

A stress function \( \phi \) is now introduced defined by the following equations:

\[
N_{x_1} = \frac{\partial^2 \phi}{\partial y_1^2} - Z_{xx} D \nabla^2 w + V
\]

\[
N_{y_1} = \frac{\partial^2 \phi}{\partial x_1^2} - Z_{yy} D \nabla^2 w + V
\] (1.72)

\[
N_{x_1 y_1} = -\frac{\partial^2 \phi}{\partial x_1 \partial y_1} - Z_{xy} D \nabla^2 w
\]

It can be verified by direct substitution that the first two of Eqns. (1.71) are identically satisfied by Eqns. (1.72) and that the third one, with Eqns. (1.70) being substituted for the transverse shears becomes:

\[
D \nabla^4 w - \nabla^2 \phi + D \left[ Z_{xx}^2 + 2Z_{xy}^2 + Z_{yy}^2 \right] \nabla^2 w = (Z_{xx} + Z_{yy}) V + Z_z
\] (1.73)
where $\nabla^2$ and $\nabla^4$ are the Laplace and the biharmonic operator, respectively, and

$$
\nabla_k^2 = \frac{\partial^2}{\partial x_1^2} - 2Z_{xy} \frac{\partial^2}{\partial x_1 \partial y_1} + \frac{\partial^2}{\partial y_1^2}
$$

(1.74)

Recalling now that for shallow surfaces $\frac{\partial}{\partial x_1} = \frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y_1} = \frac{\partial}{\partial y}$ and also that all forces along any of the local axes are approximately equal their components along the fixed axes, we may re-write both our final results and the intermediate definitions in the $x$-$y$ system.

Thus, we have from Eqns. (1.67), (1.69), (1.70), (1.72), (1.73) and (1.74) respectively:

$$
\begin{align*}
\chi &= -\frac{\partial V}{\partial x} \\
\gamma &= -\frac{\partial V}{\partial y}
\end{align*}
$$

(1.75)

$$
\begin{align*}
M_x &= -D \left( \frac{\partial^2 W}{\partial x^2} + \nu \frac{\partial^2 W}{\partial y^2} \right) \\
M_y &= -D \left( \frac{\partial^2 W}{\partial y^2} + \nu \frac{\partial^2 W}{\partial x^2} \right) \\
M_{xy} &= -D(1-\nu) \frac{\partial^2 W}{\partial x \partial y}
\end{align*}
$$

(1.76)

Had we started in the $x$-$y$ system some terms containing the slopes would have entered into each of the Eqns. (1.71) and to eliminate those we would have had to neglect the slopes themselves rather than their squares.
\[ Q_x = -D \frac{\partial}{\partial x} \nabla^2 w \]
\[ Q_y = -D \frac{\partial}{\partial y} \nabla^2 w \]  
(1.77)

\[ N_x = \frac{\partial^2 \varphi}{\partial y^2} - z_{xx} D \nabla^2 w + V \]
\[ N_y = \frac{\partial^2 \varphi}{\partial x^2} - z_{yy} D \nabla^2 w + V \]  
(1.78)

\[ N_{xy} = -\frac{\partial^2 \varphi}{\partial x \partial y} - z_{xy} D \nabla^2 w \]

\[ D \nabla^4 w - \nabla_k^2 \varphi + D [z_{xx}^2 + 2z_{xy}^2 + z_{yy}^2] \nabla^2 w = (z_{xx} + z_{yy})V + Z \]  
(1.79)

where
\[ \nabla^4 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^3 \partial y} + \frac{\partial^4}{\partial y^4} \]
\[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \]  
(1.80)

and
\[ \nabla_k^2 = z_{yy} \frac{\partial^2}{\partial x^2} - 2z_{xy} \frac{\partial^2}{\partial x \partial y} + z_{xx} \frac{\partial^2}{\partial y^2} \]
For the time being, Eqn. (1.79) will be accepted as the equation whose fulfillment, when supplemented by Eqns. (1.75) to (1.78) and (1.80), is a necessary and sufficient condition of equilibrium.

1.5 Compatibility of Strains

The strain-displacement relations (1.63) reveal that the three membrane strains are not entirely independent; that they should be derivable from continuous displacements imposes certain differentiability requirements on them.

Expressing first the derivatives of \( u \) and \( v \) from Eqns. (1.63) we get

\[
\frac{\partial u}{\partial x} = \varepsilon_{xx} + Z_{xx} w \\
\frac{\partial v}{\partial y} = \varepsilon_{yy} + Z_{yy} w \\
\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \varepsilon_{xy} + 2Z_{xy} w
\]  

(1.81)

Closely following now the procedure used in plane elasticity, we differentiate the first of Eqns. (1.81) twice with respect to \( y \), the second twice with respect to \( x \) and the third with respect to \( x \) and \( y \) and finally subtract the first two from the third. All terms containing \( u \) and \( v \) cancel out and we get

\[
-\nabla^2 w - \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} - \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = 0
\]  

(1.82)
The Eqns. (1.65) may now be used to introduce the membrane forces into Eqn. (1.82) which may, in turn, be expressed in terms of \( \varphi \) through Eqns. (1.78). This leads to the equation of compatibility in terms of the normal deflection \( w \) and the stress function \( \varphi \):

\[
Eh \beta_n^2 w + \beta_n^2 \varphi - D \left[ \beta_n^2 - \nu \beta_n^2 \right] \varphi^2 w = - (1 - \nu) \nabla^2 \nabla
\]

where

\[
\beta_n^2 \equiv Z_{xx} \frac{\partial^2}{\partial x^2} + 2Z_{xy} \frac{\partial^2}{\partial x \partial y} + Z_{yy} \frac{\partial^2}{\partial y^2}
\]

Eqns. (1.79) and (1.83) are the shallow shell equations. It will be noticed that both of them contain terms - shown in square brackets - not found elsewhere to the best of the writer's knowledge. These are due to defining \( \varphi \) in such a manner - Eqns. (1.72) - that there was no need to neglect the transverse shear in the equilibrium equations, which is the procedure used by other authors.

In the next article the conditions will be found for which consistent application of our approximations will lead us to neglect these terms in square brackets. Then the equations will take their conventional form and thus \( \varphi \) and \( w \) will be unchanged. However, the membrane stresses obtained from such a solution will be different due to the different way in which \( \varphi \) is related to the membrane stresses.
1.6 Simplification of the Equilibrium and Compatibility Equations

For simplicity the following argument is presented with respect to the homogeneous equations. Were the non-homogeneous terms included the expressions would become even lengthier without any change in the gist of the argument.

We re-write the two equations in question (1.79) and (1.83) with their right hand side set to zero:

\[
D \psi^4 w - \psi_k^2 \psi + D [ z_{xx}^2 + 2 z_{xy}^2 + z_{yy}^2 ] \nabla^2 w = 0
\]

\[
E h \nabla_k^2 w + \psi^4 \psi - D [ \psi_k^2 - \psi_{n}^2 ] \nabla^2 w = 0
\] (1.84)

The terms in square brackets are present because the stress function, defined by Eqns. (1.78), is not the usual Pucher type (4) but has been expanded to include \( w \). It is intended to show that the solution of the above equations does not change considerably by neglecting these terms.

Notice that if \( w \) is a rapidly changing function and the shell thickness \( h \) is small, then the first and last terms of the second equation are of the same order of magnitude and therefore no valid reason exists to eliminate either of them just on this basis.

Anticipating to some extent what will be discussed in Art. 2.2, we shall assume that Eqns. (1.84) have a solution in the form

\[
w = A e^{\lambda x} e^{\mu y}
\]

\[
\psi = B e^{\lambda x} e^{\mu y}
\] (1.85)
Here \( \lambda \) will be some suitable complex quantity with magnitude

\[
|\lambda| = \frac{n\pi}{L_x} \tag{1.86}
\]

where \( n \) is an integer and \( L_x \) is the length of the shell in the \( x \) direction.

When (1.85) is substituted into (1.84), a homogeneous system of two algebraic equations in the two unknowns \( A \) and \( B \) results which will possess a non-trivial solution only if \( \mu \) is such that the determinant of the system vanishes. This condition is the one that determines \( \mu \). When the determinant is expanded it gives an equation of degree 4 in \( \mu^2 \) and is of the form:

\[
\sum_{i=0}^{4} \frac{\mu^{2i}}{L^{i-2}} \left( c_i + d_i \frac{Z_{\alpha\beta}Z_{\beta\gamma}L^4}{h^2} + \left[ f_i Z_{\alpha\beta}Z_{\gamma\delta}L^2 \right] \right) = 0 \tag{1.87}
\]

where \( c_i, d_i \) and \( f_i \) are algebraic expressions of order of magnitude unity or zero and \( z_{\alpha\beta} \) and \( z_{\gamma\delta} \) are the curvatures and the term in square brackets contains all terms derivable from the part which is in square brackets in Eqns. (1.84).

\[\text{For example for } i = 4 \text{ we would have}
\mu^6 \left( \frac{1}{12 (1-\nu)} + 0 + 0 \right)
\]

and for \( i = 2 \)

\[
\mu^4 \left( \frac{n^4 \pi^4}{4} + (1-\nu^2) \frac{Z_{xx}^2 L^4}{h^2} + \frac{n^2 \pi^2}{6} (\nu Z_{xx} Z_{yy} + 2 Z_{xy} Z_{xx}^2 + Z_{yy}^2 + \nu Z_{yy}^2) L^2 \right)
\]

All the other terms are of the same structure. That the expression is not symmetric in \( x \) and \( y \) is due to the operator \( \nabla_x^2 - \nu \nabla_y^2 \).
As defined in Art. 1.1, for a shallow shell (cL) - the supremum of \((z_{\alpha\beta} x)\) or \((z_{\alpha\beta} y)\) in the region of interest - is small and therefore \((z_{\alpha\beta} z \cdot L^2) \ll 1\). We are dealing with thin shells such that \(h = \delta L\) where \(\delta\) is of the same order of smallness as (cL) or smaller. Then the first two terms in Eqn. (1.87) are of order unity or greater because:

\[
\frac{Z_{\alpha\beta} Z_{\gamma\delta} L^4}{h^2} = \frac{Z_{\alpha\beta} Z_{\gamma\delta} L^2}{\delta^2} \geq O(1)
\]  

(1.88)

but the third term is of the order of \((cL)^2\) and may be neglected. It is thus seen that the characteristic equation of a shallow, thin shell does not change significantly when Eqns. (1.73) and (1.79) are replaced by:

\[
D v^2 w - v_k^2 \varphi = (Z_{xx} + Z_{yy}) V \cdot Z
\]

\[
E h v_k^2 w + v_k^4 \varphi = -(1-\nu) v_k^2 \varphi
\]

(1.89)

These are the conventional equations of the theory of shallow shells.

It is seen that the introduction of \(w\) in the stress function did not change the form of the equations. It does, however, affect the stresses obtainable from the same equations. This should mean an especially significant improvement in problems where in some regions the transverse shears are larger than the membrane forces, e.g., in the vicinity of a transversely loaded free edge.

\[12\]

In the sense used here "does not change significantly" means that for the terms now being discussed the same order of magnitude considerations hold which were the basis of neglecting some other terms in the derivation of the same equations. Thus, retaining these terms would not improve the accuracy of the result.
In Art. 1.1 it was shown that to any shell with maximum curvature there belongs a region R of its tangent plane with characteristic measure L such that the shell is shallow within R. This places an upper bound on L, i.e., if the maximum quantity we are willing to neglect in comparison with unity is $\epsilon^2$ than

$$L_{\text{max}} = \frac{\epsilon}{C}$$  \hspace{1cm} (1.90)

The quantity $(cL)$ measures the shallowness of the shell;  $\gamma$, as introduced above, measures its "thinness". Eqn. (1.88) may then be interpreted to mean that the condition under which the terms in the square brackets in Eqn. (1.84) are negligible is that the shell be at least as thin as it is shallow, i.e., that

$$\frac{h}{L} = \delta \approx cL$$

This sets a lower bound on L, since we have

$$\frac{h}{L_{\text{min}}} = \epsilon$$  \hspace{1cm} (1.91)

Eqns. (1.90) and (1.91), when read in conjunction, mean that L must satisfy the following inequality:

$$\frac{h}{\epsilon} \leq L \leq \frac{\epsilon}{C}$$  \hspace{1cm} (1.92)

For example, if we were willing to tolerate an error of, say, 4 per cent, then

$$\epsilon^2 = 0.04 \quad \text{or} \quad \epsilon = 0.20$$
and this means that the shell would have to have at least a radius of curvature five times its span and could not be thicker than $1/25$ of the same radius of curvature.

If $L$ violates the left inequality in (1.92) then Eqns. (1.83) may not legitimately be replaced by Eqns. (1.89); if $L$ violates the right inequality the shell is not shallow and the present theory is not applicable to it.

1.7 Transformation to Oblique Coordinates

We shall now transform Eqns. (1.89) from the rectangular $x$-$y$ coordinate system to the oblique $\xi$-$\eta$ system (Figure 7). This transformation should be thought of as follows:

The shell itself and the functions $w$ and $q$ are all surfaces over the coordinate plane, at least in the mathematical sense. The shallow shell equations, as well as the stresses and strains derivable from them, involve only the curvatures of these surfaces or their derivatives. These are properties of the surfaces, independent of the coordinate system in which the surface is described. Thus, while the operations needed to find the wanted characteristic value depend on the coordinate system, the value itself is unchanged.

Regardless of the limitations currently being discussed, the shell would have to be thin enough for Love's first approximation to be valid.
A simplified example shall serve to illustrate the point. Assume

\[ w = V = 0, \] in which case the first of Eqns. (1.78) reads

\[ N_x = \frac{\partial^2 \phi}{\partial y^2} \]  \hspace{1cm} (1.93)

This means that the membrane force in the x direction equals the second derivative in the direction perpendicular to it. When \( \phi \) is given as a function of \( f \) and \( \eta \), the validity of Eqn. (1.93) is not affected but to find the second derivative with respect to \( y \), one would have to use the chain rule:

\[ N_x = \frac{\partial^2 \phi}{\partial y^2} \left[ \left( \frac{\partial}{\partial \xi} \cdot \frac{\partial f}{\partial y} + \frac{\partial}{\partial \eta} \cdot \frac{\partial f}{\partial y} \right)^2 \right] \]  \hspace{1cm} (1.94)

The definitions of normal and shear stress will be left unchanged by this transformation, i.e., the stress components are still perpendicular to each other. If the section on which they occur happens to be parallel to one of the axes then the shear stress will be parallel to that axis, too, but the normal stress will be perpendicular to it and hence not parallel to the other axis. Let, for example, the \( n_f \) and \( n_\eta \) directions be perpendicular to \( f \) and \( \eta \), respectively; then a section parallel to \( f \) has the stresses \( \sigma_{n_f, f} \) and \( \tau_{n_\eta, f} \) on it (Figure 7).

In what follows we will, rather mechanically, evaluate the derivatives in Eqn. (1.94) and the differential operators that occur in this work. It will be convenient to introduce the notation:
Figure 7
Relation of the x-y and $\xi-\eta$ coordinate systems and stresses on an oblique element
\[
\sin \omega = \alpha \\
\cos \omega = \beta
\]

(1.95)

Using these, in Figure 7 it is seen that

\[
x = \frac{\xi}{\alpha} + \beta \eta
\]

(1.96)

or the inverse of these

\[
\xi = \frac{1}{\alpha} (\alpha x - \beta y)
\]

\[
\eta = \frac{1}{\alpha} y
\]

(1.97)

Then

\[
\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi}
\]

(1.98)

We may now apply Eqns. (1.98) repeatedly to find all the second and fourth order partial derivatives needed. Occasionally making use of the

relation \( \alpha^2 + \beta^2 = \sin^2 \omega + \cos^2 \omega = 1 \) we get the following final results:

\[
\nabla^2 = \frac{1}{\alpha^2} \left( \frac{\partial^2}{\partial \xi^2} - 2\beta \frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2} \right)
\]

\[
\nabla_k^2 = \frac{1}{\alpha^2} \left( Z \frac{\partial^2}{\partial \xi^2} - 2Z \frac{\partial^2}{\partial \xi \partial \eta} + Z \frac{\partial^2}{\partial \eta^2} \right)
\]

(1.99)

\[
\nabla^4 = \frac{1}{\alpha^4} \left( \frac{\partial^4}{\partial \xi^4} - 4\beta \frac{\partial^4}{\partial \xi^3 \partial \eta} + (2 + 4\beta^2) \frac{\partial^4}{\partial \xi^2 \partial \eta^2} - 4\beta \frac{\partial^4}{\partial \xi \partial \eta^3} + \frac{\partial^4}{\partial \eta^4} \right)
\]
The equation of the shell surface is now expressed in the form

\[ z = \frac{1}{2} \left( z_{\xi\xi} \xi^2 + 2 z_{\xi\eta} \xi \eta + z_{\eta\eta} \eta^2 \right) \]  
(1.100)

and the connection between the old and the new curvatures is

\[ z_{xx} = z_{\xi\xi} \]
\[ z_{xy} = \frac{1}{\alpha} \left( -\beta z_{\xi\xi} + z_{\xi\eta} \right) \]  
(1.101)
\[ z_{yy} = \frac{1}{\alpha^2} \left( \beta^2 z_{\xi\xi} - 2 \beta z_{\xi\eta} + z_{\eta\eta} \right) \]

or, inversely

\[ z_{\xi\xi} = z_{xx} \]
\[ z_{\xi\eta} = \beta z_{xx} + \alpha z_{xy} \]  
(1.102)
\[ z_{\eta\eta} = \beta^2 z_{xx} - 2 \alpha \beta z_{xy} + \alpha^2 z_{yy} \]

As long as it is understood that the operators in the equation have their form as given in this article, the form of the shallow shell equations (Eqns. 1.89) is unchanged.

Occasionally the need arises to find directional derivatives in the \( \xi - \eta \) system. Let \( \theta \) be the angle from the \( +\xi \) axis to the \( +t \) direction, measured counterclockwise. The application of the chain rule gives

\[ \frac{\partial}{\partial t} = \frac{1}{\alpha} \left[ (\alpha \cos \theta - \beta \sin \theta) \frac{\partial}{\partial \xi} + \sin \theta \frac{\partial}{\partial \eta} \right] \]  
(1.103)
2. SOLUTION OF THE SHALLOW SHELL EQUATIONS

2.1 Remarks on the Existence and Uniqueness of the Solution

The author was unable to find in the literature known to him either any rigorous proof of the existence of the solution to Eqns. (1.89) or any reference to what the necessary boundary conditions might be to guarantee uniqueness.

While realizing that it does not constitute a mathematically acceptable procedure, the author looks upon the solution to be presented and upon the procedure which will be used to formulate and satisfy the boundary conditions as a constructional proof. Probably the best way to formulate this statement would be to say that the solution obtained is, indeed, the solution of some shallow shell problem and engineering judgement indicates that the problem solved and the problem that was to solved are acceptably close.

Nevertheless, when applying an approximate method, it would be comforting to have some assurance that the formulation of the problem is adequate to guarantee the existence of a unique solution to which one would want to converge.

2.2 The Homogeneous Equation

The homogeneous system obtainable from Eqns. (1.89) is

\[ \begin{align*}
D \nabla^4 w - \varphi_k^2 \varphi &= 0 \\
Eh(\varphi_k^2 w + \nabla^4 \varphi) &= 0
\end{align*} \]  

(2.1)
This is a system of two linear partial differential equations with constant coefficients. Its solution is usually obtained by the substitution (1.85) as described in Art. 1.6. However, this procedure leads to an algebraic equation with complex coefficients; it is usually more convenient to work with polynomials with real coefficients only.

The solution will therefore be sought in the form:

\[ w = (A \cos \lambda x + B \sin \lambda x) e^{\mu y} \]

\[ \phi = (C \cos \lambda x + F \sin \lambda x) e^{\mu y} \]

where \( \lambda \) will be assigned suitable values of the kind \( \lambda = \frac{n \pi}{L_x} \) and \( \mu \) will be found from the condition that not all of the coefficients \( A, B, C, F \), should vanish. When (2.2) are substituted into (2.1) and the coefficients of like terms are separately set equal to zero, the following system of equations results:

\[
\begin{bmatrix}
Db_{11} & Db_{12} & b_{13} & b_{14} \\
-Db_{12} & Db_{11} & -b_{24} & b_{23} \\
-Eh_{b13} & -Eh_{b14} & b_{34} & b_{33} \\
Eh_{b14} & -Eh_{b13} & -b_{44} & b_{43}
\end{bmatrix}
\begin{bmatrix}
A \\
B \\
C \\
F
\end{bmatrix}
= 0
\]

(2.3)
where
\begin{align*}
 b_{11} &= \lambda^4 - d\lambda^2\mu^2 + \mu^4 \\
 b_{12} &= 4\beta(\lambda^2\mu - \lambda\mu^2) \\
 b_{13} &= \alpha^2(z_{\eta\eta}^2 - z_{jj}^2\mu^2) \\
 b_{14} &= 2\alpha^2 z_{\eta\eta}^2 \lambda\mu
\end{align*}

This system must have a zero determinant if the solution is to be non-trivial. Expanding the determinant yields the following algebraic equation with real coefficients:

\[ \sum_{i=0}^{8} a_{2i} \mu^{2i} = 0 \tag{2.4} \]

where
\begin{align*}
 a_{16} &= \frac{D}{Eh} \\
 a_{14} &= \frac{D}{Eh} (-8 + 16\beta^2)\lambda^2 \\
 a_{12} &= \frac{D}{Eh} (28 - 96\beta^2 + 96\beta^4)\lambda^4 + 2 z_{jj}^2 \alpha^4 \\
 a_{10} &= \frac{D}{Eh} (-56 + 240\beta^2 - 384\beta^4 + 256\beta^6)\lambda^6 + \alpha^4 (64\beta z_{jj}^2 z_{\eta\eta} - 8 z_{jj}^2) \\
 &\quad \quad -4 z_{\eta\eta} z_{\eta\eta} - 8 z_{jj}^2 z_{jj} - 48\beta^2 z_{jj}^2 \lambda^2 \lambda^2
\end{align*}
\[ a_8 = \frac{D}{E_h} \left( \text{cont'd.} \right) \]
and

$$a_2 = \frac{D}{Eh} (-8 + 16 \beta^2) \lambda^{10} + (64 \beta z_{1f} z_{H} - 8 z_{H}^2 - 48 \beta^2 z_{H}^2$$

$$-4 z_{1f} z_{H} - 8 z_{1f}^2) \alpha^4 \lambda^{10} + \frac{Eh}{D} (8 z_{1f}^2 z_{H}^2 - 4 z_{1f} z_{H}^2) \alpha^8 \lambda^6$$

$$a_0 = \frac{D}{Eh} \lambda^{10} + 2 z_{H}^2 \alpha^4 \lambda^{10} + \frac{Eh}{D} z_{H}^4 \alpha^8 \lambda^8$$

(2.4)

Eqn. (2.4) is of degree 8 in $\mu^2$. Since the equation has real coefficients its roots will be either real or they will occur in complex conjugate pairs. This is important because it allows us to extract real eigenfunctions from the complex eigenvalues.

Assume now that the eigenvalues of the matrix in Eqn. (2.3) have been found. Let $\mu_1$ be a root of (2.4) hence an eigenvalue of (2.3).

The system (2.3) with $\mu = \mu_1$, is a set of four homogeneous algebraic equations, linear in the four unknowns, A, B, C, and F; since its determinant is zero, it possesses a non-trivial solution. The array (A, B, C, F) unique within an arbitrary constant multiplier, will be referred to hereinafter as the eigenvector belonging to $\mu_1$ and will be denoted by $\gamma(\mu_1)$.

If $\mu_1$ is a real root $\gamma(\mu_1)$ is also real and the associated eigenfunctions are also real. This case is quite routine and poses no special problems.
When $\mu_1$ is complex, the situation is somewhat more involved. The procedure followed for real roots would lead to complex-valued eigenfunctions and is therefore, ruled out. In order that a real-valued solution can be obtained one must recall the following.

a) If $\mu_1$ is a complex eigenvalue, $\bar{\mu}_1$ is also an eigenvalue.

b) Let the matrix on the left side of Eqn. (2.3) be denoted by $[M(\mu)]$. Since each term in $M$ is a polynomial with real coefficients in $\mu$, it follows that:

$$[M(\mu)] = \overline{[\overline{M(\mu)}]}$$  \hspace{1cm} (2.5)

c) Let $\gamma$ be an eigenvector of a singular complex matrix $M$, such that $\gamma \neq 0$ and

$$[M] \gamma = 0$$  \hspace{1cm} (2.6)

In Appendix A we prove that then the eigenvector associated with $\bar{M}$ is $\bar{\gamma}$, i.e., that Eqn. (2.6) implies that

$$[\bar{M}] \bar{\gamma} = 0$$  \hspace{1cm} (2.7)

We shall now introduce the following notations:

$$\gamma(\mu_1) = \begin{pmatrix} a_1 + i a_2 \\ b_1 + i b_2 \\ c_1 + i c_2 \\ f_1 + i f_2 \end{pmatrix}$$

$$\mu_1 = \rho_1 + i \sigma_1$$  \hspace{1cm} (2.8)
where \( i = \sqrt{-1} \) and all other quantities on the right are real.

It is a consequence of the statements (a), (b) and (c) above that if the eigenfunctions belonging to \( \mu \) are

\[
\begin{align*}
\psi^* &= (A \cos \lambda \xi + B \sin \lambda \xi) e^{\mu \eta} \\
\Phi^* &= (C \cos \lambda \xi + F \sin \lambda \xi) e^{\mu \eta}
\end{align*}
\] (2.9)

then the eigenfunctions belonging to \( \mu \) are

\[
\begin{align*}
\psi'^* &= (\overline{A} \cos \lambda \xi + \overline{B} \sin \lambda \xi) e^{\overline{\mu} \eta} \\
\Phi'^* &= (\overline{C} \cos \lambda \xi + \overline{F} \sin \lambda \xi) e^{\overline{\mu} \eta}
\end{align*}
\] (2.10)

and then the following two solutions are linearly independent and real valued:

\[
\begin{align*}
\psi_1 &= \psi^* + \psi'^* = (a_1 \cos \lambda \xi \cos \theta \eta - a_2 \cos \lambda \xi \sin \theta \eta \\
&\quad + b_1 \sin \lambda \xi \cos \theta \eta - b_2 \sin \lambda \xi \sin \theta \eta) e^{\mu \eta} \\
\phi_1 &= \phi^* + \phi'^* = (c_1 \cos \lambda \xi \cos \theta \eta - c_2 \cos \lambda \xi \sin \theta \eta \\
&\quad + f_1 \sin \lambda \xi \cos \theta \eta - f_2 \sin \lambda \xi \sin \theta \eta) e^{\mu \eta}
\end{align*}
\] (2.11)

and

\[
\begin{align*}
\psi_2 &= \psi^* - \psi'^* = (a_2 \cos \lambda \xi \cos \theta \eta + a_1 \cos \lambda \xi \sin \theta \eta \\
&\quad + b_2 \sin \lambda \xi \cos \theta \eta + b_1 \sin \lambda \xi \sin \theta \eta) e^{\mu \eta} \\
\phi_2 &= \phi^* - \phi'^* = (c_2 \cos \lambda \xi \cos \theta \eta + c_1 \cos \lambda \xi \sin \theta \eta \\
&\quad + f_2 \sin \lambda \xi \cos \theta \eta + f_1 \sin \lambda \xi \sin \theta \eta) e^{\mu \eta}
\end{align*}
\] (2.12)
Among the 16 roots of Eqn. (2.4) some may be multiple roots. If, say $\mu_k$ has multiplicity $p$ then the associated eigenfunctions are

$$w = [(A_1 \cos \lambda f + B_1 \sin \lambda f) + (A_2 \cos \lambda f + B_2 \sin \lambda f) \eta + ...$$
$$+ (A_p \cos \lambda f + B_p \sin \lambda f) \eta^{p-1}] e^{\mu_k \eta}$$

$$\varphi = [(C_1 \cos \lambda f + F_1 \sin \lambda f) + (C_2 \cos \lambda f + F_2 \sin \lambda f) \eta + ...$$
$$+ (C_p \cos \lambda f + F_p \sin \lambda f) \eta^{p-1}] e^{\mu_k \eta}$$

In Eqn. (2.13) the (4p) quantities $A_1, B_1, ..., F_p$ are not independent of each other. This question is not fully explored in the literature known to the author and was therefore subjected to some investigation. The results of this are given in Appendix B.

2.3 Particular Solutions

In the present work we shall confine ourselves to non-singular loads, i.e., to loads that are described by functions expandable in a power series or a double trigonometric series in $f$ and $\eta$.

This is one instance where it is felt that allowances have to be made for the limitations of the computer. In particular, certain special cases could be solved by more direct methods than what we will suggest. These are discussed in what follows but will not be utilized in our solution. It is felt that we will lose some elegance by the strong-arm method we feel compelled to choose.
Singular solutions of selected special cases are found in the literature (29). It would be extremely difficult to incorporate them in a general computer program; moreover, it is felt that for practical purposes, a singularity, such as for example a concentrated force, can be handled with sufficient accuracy by a truncated Fourier series spreading it out over a small, but finite, area.

Another procedure available in the literature (30) is limited to shells on shear diaphragms where the solution of the plate problem, having the same boundary conditions and loading as the shell, is known. The particular solution is then assumed as the sum of the plate solution plus some correction terms and it can be shown that these correction terms converge faster than if the entire solution were expanded in one series. The boundaries must rest on shear diaphragms so that the appropriate trigonometric series will satisfy the boundary conditions automatically. Obviously, if the proper conditions exist, this method can be very advantageous, but it is not the intention of the present work to impose such strict restrictions on the methods to be selected.

The cases where the load is represented by power series and where it is represented by trigonometric series will now be discussed separately.
Let the vertical load $Z$ and the potential $V$ associated with the horizontal loads $X$ and $Y$ be such that substitution into the right hand side of Eqns. (1.89) results in two polynomials in $\xi$ and $\eta$ - one for each equation - with $q$ denoting the highest exponent of $\xi$ and $r$ denoting the highest exponent of $\eta$. Then quantities $a_{ij}$, $b_{ij}$ can be found such that

$$w_p = \sum_{i=0}^{q+4} \sum_{j=0}^{r+4} a_{ij} \xi^i \eta^j$$

$$\varphi_p = \sum_{i=0}^{q+4} \sum_{j=0}^{r+4} b_{ij} \xi^i \eta^j$$

(2.14)

are solutions of Eqns. (1.89) by simply substituting (2.14) into (1.89) and equating coefficients of like powers. Actually the problem that arises is not to find a solution but rather to decide on a suitable set out of the possible solutions, because the procedure leads to fewer equations for the $a$-s and $b$-s than there are unknowns. While any solution is equally acceptable from the strictly mathematical point of view the convergence of the overall result is greatly improved by an appropriate choice. In general, one should attempt to find the particular solution that comes closest to satisfying the boundary conditions. It should be pointed out that one may even add linear and constant terms to $w$ where there is no corresponding loading.  

For example, if the shell is supported on elastic foundations, the average deflection of the supports may be calculated and included as a term in the particular solution for $w$. This will improve convergence.
Next consider loads in terms of a trigonometric series. Assume that for a particular term of this loading the equations to be solved take the following form:

\[ D \nabla^4 w - \nabla_k^2 \varphi = A_1 \cos \gamma \cos \delta \eta + B_1 \cos \gamma \sin \delta \eta + C_1 \sin \gamma \cos \delta \eta + F_1 \sin \gamma \sin \delta \eta \]

\[ E h \nabla_k^2 w - \nabla^4 \varphi = A_2 \cos \gamma \cos \delta \eta + B_2 \cos \gamma \sin \delta \eta + C_2 \sin \gamma \cos \delta \eta + F_2 \sin \gamma \sin \delta \eta \]  \hspace{1cm} (2.15)

It should be noted that by selecting \( \gamma \) or \( \delta \) to be zero the above expression includes, as a special case, loads that vary in one direction only.

The standard procedure now consists of assuming a solution in the form of the right hand side of Eqns. (2.15) with \( A, B, \ldots \) replaced by undetermined coefficients, then substituting this solution into Eqns. (2.15) and equating coefficients of like terms on the two sides. At first sight this seems to lead to a system of eight simultaneous equations - four different kinds of terms in each of two equations - with eight unknowns. It can readily be shown that four of these equations uncouple from the other four. To do this we observe that the operators in Eqns. (2.15)

involve only second and fourth partial derivatives. Then two cases arise:
a) If the order of differentiation is even with respect to one variable it is also even with respect to the other. This will replace any of the terms of the kind \( \cos \gamma \int \cos \eta \) by identical terms (except, of course, for constant multipliers).

b) If the order of differentiation is odd with respect to one variable, it is also odd with respect to the other. This will replace cosine by sine and vice versa; thus both the \( \gamma \) and the \( \eta \) factors in each term change character, i.e.,

\[
(\cos \cos) \text{ turns into } (\sin \sin) \text{ and vice versa}
\]

\[
(\cos \sin) \text{ turns into } (\sin \cos) \text{ and vice versa.}
\]

Therefore, in neither case do these two kinds of terms mix. Now assume

\[
w = A_1^{*} \cos \gamma \int \cos \eta + B_1^{*} \cos \gamma \int \sin \eta + C_1^{*} \sin \gamma \int \cos \eta + F_1^{*} \sin \gamma \int \sin \eta
\]

\[
\varphi = A_2^{*} \cos \gamma \int \cos \eta + B_2^{*} \cos \gamma \int \sin \eta + C_2^{*} \sin \gamma \int \cos \eta + F_2^{*} \sin \gamma \int \sin \eta
\]

This procedure gives the following systems of equations

\[
[A^+] \begin{bmatrix} A_1^{*} \\ F_1^{*} \\ A_2^{*} \\ F_2^{*} \end{bmatrix} = \begin{bmatrix} A_1 \\ F_1 \\ A_2 \\ F_2 \end{bmatrix} \quad \text{and} \quad [A^-] \begin{bmatrix} B_1^{*} \\ C_1^{*} \\ B_2^{*} \\ C_2^{*} \end{bmatrix} = \begin{bmatrix} B_1 \\ C_1 \\ B_2 \\ C_2 \end{bmatrix}
\]
where the matrices \([A^+]\) and \([A^-]\) are given by

\[
\begin{bmatrix}
DP_0 & DP_1 & P_0 & P_1 \\
DP_1 & DP_0 & P_0 & P_1 \\
-EhP_2 & -EhP_3 & P_0 & P_1 \\
-EhP_3 & -EhP_2 & P_0 & P_1
\end{bmatrix}
\] (2.18)

and

\[
\begin{align*}
P_0 &= \frac{1}{\alpha^4} \left[ \gamma^4 + (2 + 4\beta^2) \gamma^2 \delta^2 + \delta^4 \right] \\
P_1 &= \pm \frac{4\beta}{\alpha^4} (\gamma^3 \delta + \gamma \delta^3) \\
P_2 &= \frac{1}{\alpha^2} (Z_{\eta\eta} \gamma^2 + Z_{\eta\eta} \delta^2) \\
P_3 &= \pm \frac{2}{\alpha^2} Z_{\eta\eta} \gamma \delta
\end{align*}
\] (2.19)

and for \(P_1\) and \(P_3\) the + sign should be selected for \([A^+]\) and the - sign for \([A^-]\).

In case the matrix (2.18) is singular the functions (2.16) are solutions to the homogeneous equations and solution in the assumed form does
not exist. As is known from the theory of linear differential equations with constant coefficients, the solution may then be obtained by repeating the above procedure but assuming each constant in (2.16) to be replaced by a polynomial in either $f$ or $\eta$ with undetermined coefficients.

2.4 In-Plane Displacements of a Shallow Shell

Assuming that $w$ and $\phi$ have been found, one can find all stresses and strains by differentiation, using Eqns. (1.64)-(1.66) and (1.76) - (1.78).

The situation is more complicated if $u$ and $v$, the displacements in the "plane" of the shell are needed.

From Eqns. (1.63) the derivatives of $u$ and $v$ may be obtained.

Expressing the membrane strains by Eqns. (1.65) and substituting them into (1.63) gives these derivatives in terms of the membrane forces which, in turn, may be expressed by Eqn. (1.78) as derivatives of the stress function. This purely algebraic manipulation results in:

\[
\frac{\partial u}{\partial x} = \frac{1}{Eh} \left[ \frac{\partial^2 \phi}{\partial y^2} - \nu \frac{\partial^2 \phi}{\partial x^2} + (1-\nu)V - D (z_{xx} - \nu z_{yy}) \nabla^2 w \right] + z_{xx} w
\]

\[
\frac{\partial v}{\partial y} = \frac{1}{Eh} \left[ \frac{\partial^2 \phi}{\partial x^2} - \nu \frac{\partial^2 \phi}{\partial y^2} + (1-\nu)V - D (z_{yy} - \nu z_{yy}) \nabla^2 w \right] + z_{yy} w
\]

\[
\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\frac{2(1+\nu)}{Eh} \left[ \frac{\partial^2 \phi}{\partial x \partial y} + D z_{xy} \nabla^2 w \right] + 2z_{xy} w
\]

(2.20)
The first two of Eqns. (2.20) may be integrated directly:

\[
\begin{align*}
    u &= \frac{1}{Eh} \left[ \int \frac{\partial^2 \phi}{\partial y^2} \, dx - \nu \frac{\partial \phi}{\partial x} + (1 - \nu) \int V \, dx - D(z_{xx} - \nu z_{yy}) \int \nabla^2 w \, dx \right] \\
    &\quad + z_{xx} \int w \, dx - \frac{f_1(y)}{Eh} \\
    v &= \frac{1}{Eh} \left[ \int \frac{\partial^2 \phi}{\partial x^2} \, dy - \nu \frac{\partial \phi}{\partial y} + (1 - \nu) \int V \, dy - D(z_{yy} - \nu z_{xx}) \int \nabla^2 w \, dy \right] \\
    &\quad + z_{yy} \int w \, dy - \frac{f_2(x)}{Eh}
\end{align*}
\]  

where \( f_1 \) is a function of \( x \) only and \( f_2 \) is a function of \( y \) only and so far both of these are undetermined. They can be found from the last of Eqns. (2.20).

When Eqns. (2.21) are differentiated with respect to \( y \) and \( x \), respectively, and the result substituted into the shear strain equation, after some obvious re-arrangement one obtains:
It is clear that the functions \( f_1 \) and \( f_2 \) will exist only if the right side "falls apart" into two parts, one a function of \( x \) only, the other a function of \( y \) only. That this indeed happens will now be shown.

On the right side of Eqn. (2.22) the lengthy expression consists of three kinds of terms: some are integrated with respect to \( x \), some others with respect to \( y \) and some others are not integrated at all. We shall transform them all into terms of the form:

\[
\iint (\ldots) \, dx \, dy
\]
by differentiating and integrating with respect to whatever variable is needed to achieve this. In the course of this process, terms that are contained in the expression on the right hand side and which are functions of one of the variables only, or are constant, may be lost. It is understood that these would be absorbed in the constants of integration implied by the indefinite integral.

This process gives:

\[ \frac{df_1}{dx} + \frac{df_2}{dy} = \int \int \left[ Eh \nabla^2 w + \nabla^4 \varphi - D(\nabla_k^2 - \nu \nabla_n^2) \nabla^2 w + (1-\nu) \nabla^2 V \right] dx dy \]

(2.23)

The integral in Eqn. (2.23) is, according to Eqn. (1.83), equal to zero and hence we have

\[ \frac{df_1}{dx} + \frac{df_2}{dy} = \int \int 0 \, dx dy = g_1(x) + g_2(y) \]

(2.24)

with constants, if any, being absorbed in either of the g-s. Q.E.D.

We will now proceed to show that for any term of the complementary solution in the form (2.2) we have

\[ g_1 = g_2 = 0 \]

(2.25)

implying

\[ f_1 = ax + b \]

\[ f_2 = -ay + c \]

(2.26)

which corresponds to the rigid body motions of the shell.
To establish this result, we first note that $a_0$ in Eqn. (2.4) is a positive definite quantity since with the exception of the shell thickness and elastic constants - which are never negative - all other quantities appear in even powers. Therefore, $\mu = 0$ is not a possible solution of Eqn. (2.4) and the eigenfunctions consist of terms which are products of $\cos \lambda x$ or $\sin \lambda x$ with $e^{ny}$. Repeated differentiation or integration - which are the operations on the right hand side of Eqn. (2.23) - will change the coefficients of these terms but will not affect their basic nature; i.e., it will neither lead to the vanishing of $x$ or $y$ from them nor will it lead to $\cos^2$ or $\sin^2$ terms with the possibility of the dropping out of one of the variables from some combination of them. Thus, the only possibility which allows no mixed $x$-$y$ term is if the coefficients of all terms actually vanish. This proves the validity of Eqns. (2.25) and (2.26) for any $\omega$ and $\varphi$ that are complementary solutions and are of the form (2.2).

A similar argument for terms of the particular solution would not be valid. For terms of the power series type Table 1 shows all the possible combinations which lead to a non-vanishing right hand side for Eqn. (2.23) and hence contribute to Eqns. (2.21).

The entries in Table 1, subjected to some restriction, are such that when those restrictions are violated the contribution of that particular term is a constant. This may be absorbed in the rigid body motion of the shell and hence need not be considered separately. All other power series
<table>
<thead>
<tr>
<th>( w )</th>
<th>( \varphi )</th>
<th>( V )</th>
<th>( \frac{df_1}{dx} )</th>
<th>( \frac{df_2}{dy} )</th>
<th>Restrictions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( Axy^n )</td>
<td></td>
<td></td>
<td>( \frac{AEnz_{yy}y^{n+1}}{n+1} )</td>
<td></td>
</tr>
<tr>
<td>2.</td>
<td>( Ay^n )</td>
<td></td>
<td></td>
<td>( 2AEnz_{xy}y^n )</td>
<td>( n \neq 0 )</td>
</tr>
<tr>
<td>3.</td>
<td>( Ax^l )</td>
<td></td>
<td>( 2AEnz_{xy}x^l )</td>
<td></td>
<td>( l \neq 0 )</td>
</tr>
<tr>
<td>4.</td>
<td>( Ax^l y )</td>
<td></td>
<td>( \frac{AEnz_{xx}x^{l+1}}{l+1} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.</td>
<td></td>
<td>( Bx^3 y^m )</td>
<td></td>
<td>( -\frac{6A}{m+1}y^{m+1} )</td>
<td></td>
</tr>
<tr>
<td>6.</td>
<td></td>
<td>( Bxy^m )</td>
<td></td>
<td>( -2mBy^{m-1} )</td>
<td>( m \neq 1 )</td>
</tr>
<tr>
<td>7.</td>
<td></td>
<td>( Bx^k y )</td>
<td></td>
<td>( -2kBx^{k-1} )</td>
<td></td>
</tr>
<tr>
<td>8.</td>
<td></td>
<td>( Bx^k y^3 )</td>
<td></td>
<td>( -\frac{6B}{k+1}x^{k+1} )</td>
<td></td>
</tr>
<tr>
<td>9.</td>
<td></td>
<td></td>
<td></td>
<td>( -(1-v)C_x y^{q+1} )</td>
<td></td>
</tr>
<tr>
<td>10.</td>
<td></td>
<td></td>
<td>( Cx^p y )</td>
<td>( -(1-v)C_x x^{p+1} )</td>
<td></td>
</tr>
</tbody>
</table>

Table 1
Non-vanishing terms of the power-series type entering the right side of Eqn. (2.23)
terms of the particular solution must cancel out each other because none of them separate into functions of one variable only.

The argument presented for the terms of the homogeneous solution can be repeated for those trigonometric terms of the particular solution which contain both \(x\) and \(y\) and therefore these do not contribute to Eqn. (2.25). When a term contains only one variable, substitution into Eqn. (2.25) reveals that neither \(\varphi\) nor \(V\) contribute anything while if

\[
W = H \cos yx + J \sin yx + K \cos dy + L \sin dy \tag{2.27}
\]

one obtains

\[
\frac{df}{dx} = -2Eh z_{xy} (H \cos yx + J \sin yx) - 2D z_{xy} y^2 (1 + \nu)(\cos yx + \sin yx) \tag{2.28}
\]

\[
\frac{df}{dy} = -2Eh z_{xy} (K \cos dy + L \sin dy) - 2D z_{xy} d^2 (1 + \nu)(\cos dy + \sin dy)
\]

In each of the two Eqns. (2.28) the second term stems from those terms of Eqn. (1.83) which were neglected in Art. 1.5 and it can therefore be argued that for consistency they should also be neglected here. Indeed, a comparison of the order of magnitude of the first and second terms shows that they are in the relation as unity is to \(h^2 y^2\). For most applications \(y\) - whose dimension is \(1/\text{length}\) - will have a magnitude of the order \(1/L\) and hence

\[
h^2 y^2 \approx \left(\frac{h}{L}\right)^2 \ll 1 \tag{2.29}
\]
which means that without recourse to the arguments presented in Art. 1.5 they may be eliminated from Eqns. (2.28). Our only reason in carrying them so far was to point out that these terms will make a significant contribution to the in-plane displacement in the case, admittedly rather unlikely in practice, when the load, and hence $w$, are extremely rapidly changing functions.

Having thus taken care of the constants of integration - except for the rigid body motion which is to be found from the boundary conditions - we may now proceed to list the other terms appearing in Eqns. (2.21).

The differentiations and integrations may be performed in a very routine manner. We neglect, as explained above, the terms containing $\nabla^2 w$ and thus for a typical term of the complementary solution we obtain the following

If a term of $w$ is

$$w = K \left[ (A_i \cos \lambda \cos \phi \sin \theta + B_i \cos \lambda \sin \theta + \epsilon C_i - \epsilon F_i) \sin \lambda \cos \theta y + (\epsilon C_i + \epsilon F_i) \right]$$

$$+ F_i \sin \lambda \sin \theta y \right] e^{\theta y}$$

then

$$u_w = \frac{K z_{xx}}{\lambda} \left[ (-C_i \cos \lambda \cos \phi \sin \theta - F_i \cos \lambda \sin \theta + A_i \sin \lambda \cos \theta y$$

$$+ B_i \sin \lambda \sin \theta y) e^{\theta y} \right]$$

$$v_w = \frac{K z_{yy}}{\kappa^2 + \epsilon^2} \left[ (\epsilon A_i - \epsilon B_i) \cos \lambda \cos \phi \sin \theta + (\epsilon A_i + \epsilon B_i) \cos \lambda \sin \phi \sin \theta$$

$$+ (\epsilon C_i - \epsilon F_i) \sin \lambda \cos \phi \sin \theta + (\epsilon C_i + \epsilon F_i) \sin \lambda \sin \phi \sin \theta \right] e^{\theta y}$$
Similarly, if

\[
\phi = K \left[ (A_2 \cos \lambda x \cos \beta y + B_2 \cos \lambda x \sin \beta y + C_2 \sin \lambda x \cos \beta y \\
+ F_2 \sin \lambda x \sin \beta y) e^{\psi y} \right]
\]

(2.32)

then

\[
u_\phi = \frac{K}{Eh} \left\{ \left[ \lambda (\phi^2 - \delta^2 - \nu) C_2 + 2 \lambda \phi \delta F_2 \right] \cos \lambda x \cos \beta y \\
+ \left[ \lambda (\phi^2 - \delta^2 - \nu) F_2 - 2 \lambda \phi \delta C_2 \right] \cos \lambda x \sin \beta y \\
+ \left[ \lambda (-\phi^2 + \delta^2 + \nu) A_2 - 2 \lambda \phi \delta B_2 \right] \sin \lambda x \cos \beta y \\
+ \left[ \lambda (-\phi^2 + \delta^2 + \nu) B_2 + 2 \lambda \phi \delta A_2 \right] \sin \lambda x \sin \beta y \right\} e^{\psi y}
\]

\[
u_\psi = \frac{K}{Eh} \left\{ \left[ - \frac{\lambda^2}{\phi^2 + \delta^2} (\rho A_2 - 6B_2) - \nu (\rho A_2 + 6B_2) \right] \cos \lambda x \cos \beta y \\
+ \left[ - \frac{\lambda^2}{\phi^2 + \delta^2} (6A_2 - \rho B_2) + \nu (6A_2 - \rho B_2) \right] \cos \lambda x \sin \beta y \\
+ \left[ - \frac{\lambda^2}{\phi^2 + \delta^2} (\rho C_2 - 6F_2) - \nu (\rho C_2 + 6F_2) \right] \sin \lambda x \cos \beta y \\
+ \left[ - \frac{\lambda^2}{\phi^2 + \delta^2} (6C_2 + \rho F_2) + \nu (6C_2 - \rho F_2) \right] \sin \lambda x \sin \beta y \right\} e^{\psi y}
\]

(2.33)
In the above \( K \) is the arbitrary constant yet to be found and \( A_1, A_2, \ldots F_2 \) are the known components of the eigenvector entering the complementary solution. The functions \( w \) and \( \varphi \) actually consist of a sum of terms of this type - the same summation should be carried out for the displacements \( u \) and \( v \).

E.g., the complementary solution could be written in the form

\[
    w = \sum K_i w_i
    \quad \tag{2.34}
\]

\[
    \varphi = \sum K_i \varphi_i
\]

and the part of \( u \) and \( v \) corresponding to the complementary solution is

\[
    u_{\text{compl.}} = \sum K_i (u_{w_i} + u_{\varphi_i})
    \quad \tag{2.35}
\]

\[
    v_{\text{compl.}} = \sum K_i (v_{w_i} + v_{\varphi_i})
\]

The other terms appearing in Eqns. (2.21) - i.e., those coming from the particular solution or from \( V \) - require no further discussion.

With the aid of Table 1 and Eqns. (2.28), (2.31) and (2.33) the functions \( f_1 \) and \( f_2 \), and hence from Eqns. (2.24) \( u \) and \( v \) may be determined except for the rigid body motion of the shell and the so-far arbitrary constants entering into the complementary solution. To each value of \( \lambda \) chosen in the expansion of the complementary solution, there will belong 16 arbitrary constants; three more are introduced to describe the rigid body motion - the solution presented on the foregoing pages allows all properties of the shell to be
expressed as functions of these constants. We are now in a position to formulate the boundary conditions.
3. BOUNDARY CONDITIONS

3.1 Physical Description of the Supporting Structure

A major stumbling block encountered in producing a mathematical model for shells which is solvable, yet close to the actual structure, is the formulation of realistic boundary conditions.

About the only boundary condition frequently encountered in structural practice which is easy to deal with in analysis is the free boundary. It has no restraints on any of the displacements and all of the stresses are known along the boundary – hence, it is what came to be called a "boundary condition of the first kind."

Unfortunately, shells do not usually have all their boundaries free and many shells have no free boundary at all. For the convenience of the analyst the concept of the shear diaphragm was introduced, which is supposed to have infinite stiffness in its own plane and no stiffness out of its plane. There are two major objections to this:

a) Even when the ratio of the actual stiffnesses in and out of the plane of the supporting structure is such that, for practical purposes, it satisfies the requirements set for a shear diaphragm, it will certainly restrain tangential motion along the boundary and this effect is usually ignored.

b) It is deemed an undesirable circumstance when the demands of ease of analysis have a governing influence on the layout of the structure; yet the use of shear diaphragms seems to be motivated exactly by such considerations.
In the present work, an attempt will be made to consider, in its
generality, the problem of a shell supported by edge beams which, in turn,
are resting on columns. The edge beams are elastic members whose deforma-
tions are governed by ordinary differential equations, exact within the
Bernoulli–Navier hypothesis and the limitations of small deflection theory.
The consistent formulation in mathematical terms of the interdependence of
the shell and its edge beams will be the subject of the subsequent articles;
in the present article the overall structural concept will be considered.

The edge of the shell will always be considered supported by an edge
beam whose flexural, torsional and axial stiffness is known everywhere. Any
of these quantities may vary along the beam. A free edge will be considered
an edge beam with zero stiffness.

In addition, the edge may also be supported by isolated springs.
This is quite an unlikely occurrence, but it is needed so that a shear dia-
phragm may be adequately handled. This aspect will become clear when the
method of satisfying the boundary conditions is discussed.

The edge beam is assumed to be governed by the differential equations
of a straight member although it may actually be slightly curved. This feature
may have a rather adverse influence on the results if the axial compression
along the edge beam is large. However, to include the effect of the axial
force on the deformation of the edge beam would have rendered the boundary
conditions non-linear. This would change the character of the whole problem
and render it quite unwieldy. It is suggested that if a situation is encountered where the curvature of the edge beam seems to be of concern one may reach an acceptable solution by assigning fictitious reduced values to the stiffness of the edge beam so that the actual deformations - to be calculated by hand on the basis of the edge beam loads supplied by a first approximation - are approximated, within reason, by the fictitious ones.

It is further assumed that the shell is connected to the edge beam on the line of the shear centers of the latter, i.e., only the rotations of the shell cause torsion in the beam. The character of the problem would not be changed if this restriction were lifted; it would lead to more involved but still linear boundary equations.

The equations expressing the effect of the edge beams on the shell, irrespective of the way the edge beams themselves are supported and connected to each other, will be referred to as "shell boundary equations." They will be satisfied approximately only (see Art. 3.4).

The edge beams rest on columns. The columns have a specified stiffness such that they resist displacement of the point of support in any direction. However, the rotations of the edge beams are not restrained by the columns - the support is assumed to be through a ball-and-socket type connection.
In keeping with our remarks in Art. 2.3 singularities are not considered. Therefore, the columns have a finite width along the edge beam.

The number of columns is not limited in principle. Thus, the shell may be supported in a statically indeterminate manner.

There may or may not be columns in the corners of the shell. If there are not the two intersecting edge beams support each other in that point, or rather one of them supports the other. This support prevails not only vertically, but also for in-plane forces. The forces transmitted here are important because they enter the differential equations of the edge beams as boundary conditions.

The equations expressing the effect of the supporting columns and intersecting edge beams on the edge beams will be referred to as "edge beam boundary equations." They will be satisfied exactly.

The boundary conditions, as outlined above, include many important applications. There are others, however, which may, and in fact do, occur in practice and which cannot be treated as special cases of an edge beam. These are the cases when the shell is bounded by a structure whose elastic behavior is governed by partial, rather than ordinary, differential equations. Such is the case for example, when two shells meet at an angle. If a satisfactory solution were found to this problem, it would become possible to

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The kind of connection between a column and a beam passing continuously over it that has been described is frequently implied, although not explicitly stated, in structural practice. It corresponds, for example, to a continuous floor beam, whose moments are calculated on spans taken as center-to-center of columns, but whose shears are calculated at the face of the column.
consider non-shallow shells as a series of shallow panels and thereby extend the range of applicability of the shallow shell equations. Incorporating the present procedure into an iteration scheme may possibly lead to a solution. It is felt this would be an important area for further research.

3.2 Formulation of the Shell Boundary Equations

In this and the following article we will use a coordinate system such that the axis $t$ is taken along the boundary and the axis $n$ is the inner normal to the boundary (Figure 8). Thus, $t$ is always parallel to one of the coordinate axes $\xi$ or $\eta$ but $n$ is parallel to the other one only in the case of a rectangular shell.

In the method of point matching the boundary equations are set up in several isolated boundary points. Therefore, in the discussion that follows each equation refers to a point $P_k$ along the boundary and it is understood that similar equations are written for each of the other points included in the point matching. The constants that occur in these equations may vary from point to point; thus variable stiffness edge beams may be considered without additional trouble.

As is well known, four boundary conditions are to be satisfied at every point of a shell boundary. Each of these incorporates a statement of static equilibrium as well as kinematic compatibility. For brevity these equations will be referred to by the force to which they correspond. They will now be considered one by one in detail.
Figure 8
The $t$-$n$ coordinate system in the boundary point $P_k$
a) Transverse shear boundary equation.

The displacement corresponding to transverse shear is the normal deflection \( w \). This boundary condition expresses the condition of static equilibrium that the contribution of the shell to the vertical loads on the edge beam equals the transverse shear in the shell at the boundary. It also expresses the compatibility condition that the normal deflection of the boundary of the shell and that of the edge beam are equal.

As mentioned before, the possibility of an external spring will also be considered. When we use the term "transverse shear" in connection with the boundary condition, we mean not only the force \( Q \) as shown in Figure 6 and defined by Eqn. (1.77) but also the effect of the twisting moment at the boundary. Detailed descriptions of this may be found in numerous books on plate or shell theory (for example, Ref. (2) contains a very clear treatment of the problem) and here we will simply state that the transverse force \( V_n \), to be balanced by external forces at a boundary is given by

\[
V_n = Q_n + \frac{\partial M_{nt}}{\partial t} = -D \left[ \frac{\partial^3 w}{\partial n^3} + (2 - \nu) \frac{\partial^3 w}{\partial n \partial t^2} \right] \quad (3.1)
\]

The vertical deflection of the edge beam being equal, by reasons of continuity, to the normal deflections of the shell we will denote both by \( w \). If the total vertical external load on the beam is denoted by \( p^* \) the equation to be satisfied by the beam is

\[
EI_v \frac{\partial^4 w}{\partial t^4} = p^* \quad (3.2)
\]
where \((EI_v)\) is the flexural stiffness of the beam in the vertical plane and both \(w\) and \(p^*\) are positive in the direction of the \(+z\) axis, i.e., upwards.

The load \(p^*\) contains the following:

1. from the shell: \(-V_n\)
2. external edgeload, if any: \(p\)
3. from an external spring with spring coefficient \(k_1\): \(-k_1w\)

The negative sign for items (1) and (2) arises from the fact that for positive transverse shear in the shell or positive normal deflection, respectively, the load on the edge beam is downward.

Thus,

\[ p^* = -V_n + p - k_1w \]  

(3.3)

and with this Eqn. (3.2) becomes

\[ k_1w + EI_v \frac{\partial^4 w}{\partial t^4} + V_n = p \]  

(3.4)

To make it possible to include a special case, to be discussed presently, in the same equation we will introduce a constant multiplier, \(\varphi\), in Eqn. (3.4) in front of \(V_n\). Thus, the final form of the shell boundary equation for transverse shear is

\[ k_1w + EI_v \frac{\partial^4 w}{\partial t^4} + \varphi V_n = p \]  

(3.5)

When \(\varphi = 1\) this is the general equation for elastic support. It also includes a number of special cases as follows:
Free boundary with edgeload:

\[ k_i = EI_y = 0, \quad q = 1, \quad V_n = p \quad (3.6) \]

Shear diaphragm:

\[ k_i = 1, \quad EI_y = q = p = 0, \quad w = 0 \quad (3.7) \]

Forced deflection \( \delta \) along the boundary (this could be used to obtain influence coefficients):

\[ k_i = 1, \quad EI_y = q = 0, \quad p = \delta, \quad w = \delta \quad (3.8) \]

It is now clear that \( q \) usually takes the value unity and it had to be introduced so that in certain special cases, when \( V_n \) is not required in the equation, it may be set to zero.

Eqn. (3.5) with the special interpretations expressed by Eqns. (3.6) - (3.8), fully describe the behavior of the edge beam and shell boundary in the vertical plane.

b) Bending moment boundary equation.

The displacement corresponding to bending moment along the boundary is the rotation in the \( n \) direction \( \frac{\partial w}{\partial n} \). From the point of view of the equilibrium of the edge beam, the shell bending moments cause change of the torque in the beam and for structural continuity the twist angle of the beam has to equal the rotation of the shell. Thus, in writing the differential equation for
the torsion of the edge beam \( \frac{\partial w}{\partial n} \) may be used for the twist angle. The possibility of having a rotational spring will again be included.

Let \((GJ)\) be the torsional rigidity of the edge beam and \(T^*\) be the total torque acting, taken positive when its sense of rotation coincides with \(+\frac{\partial w}{\partial n}\), then the equation to be satisfied by the beam is

\[
GJ \frac{\partial \left( \frac{\partial w}{\partial n} \right)}{\partial t} = T^*
\]

(3.9)

The torque \(T^*\) contains the following:

1) from the shell: \(-M_{nn}\)
2) External edge load, if any: \(T\)
3) from an external spring with torsional spring constants \(k_2\): \(-\frac{k_2}{2} \frac{\partial w}{\partial n}\)

The negative sign for items (1) and (3) arises from the fact that for \(+\frac{\partial w}{\partial n}\) the bending moment in the shell or the moment in the spring cause torques in the direction of \(-\frac{\partial w}{\partial n}\) in the beam.

Thus,

\[
T^* = -M_{nn} + T - k_2 \frac{\partial w}{\partial n}
\]

(3.10)

Introducing in Eqn. (3.10) the constant \(\varphi\) just as we did in the transverse shear equation and substituting it into Eqn. (3.9) we obtain

\[
k_2 \frac{\partial w}{\partial n} + GJ \frac{\partial^2 w}{\partial n \partial t} + \varphi M_{nn} = T
\]

(3.11)
When $q = 1$ this is the general equation for the boundary of the shell being supported elastically against rotation. Closely corresponding to the previously discussed case of Eqn. (3.5) this, too, includes the following special cases.

Free boundary with edge moment:

$$k_2 = GJ = 0, \quad \varphi = 1, \quad M_{mn} = M$$ (3.12)

Fully restrained edge:

$$k_2 = 1, \quad GJ = \varphi = T = 0, \quad \frac{\partial w}{\partial n} = 0$$ (3.13)

Forced rotation $\theta$ along the boundary (to obtain influence coefficients):

$$k_2 = 1, \quad GJ = \varphi = 0, \quad T = \theta, \quad \frac{\partial w}{\partial n} = \theta$$ (3.14)

Thus Eqn. (3.11) fully describes the rotational behavior of the edge beam and shell boundary in the $n$ direction.

c) Membrane normal force boundary equation.

The displacement corresponding to the membrane normal force $N_{nn}$ is in-plane motion along $n$. We will denote this by $u_n$ (even though in some cases it may be equal to what was earlier denoted by $v$), positive in the $+n$ direction.

It is readily seen that this condition and that for the transverse shear are very much alike. Indeed, if we introduce the following changes into the description on Pages 87-89:
vertical to be replaced by horizontal

$w$ to be replaced by $u_n$

$V_n$ to be replaced by $N_{nn}$

the two become identical. Avoiding the repetitious argument we will just list the relevant definitions and relations.

Let $k_3$ be the constant of a horizontal spring, if any; $EI_h$ be the flexural stiffness of the edge beam in the horizontal plane; $\varphi$ be a constant whose value is either 0 or 1; $q$ be the external horizontal load along the edge beam, positive when in the $+n$ direction. Then the general form of the shell boundary condition for membrane normal force is

$$k_3 u_n + EI_h \frac{\partial^4 u_n}{\partial t^4} + \varphi N_{nn} = q \quad (3.15)$$

Special cases:

Free boundary with edge load:

$$k_3 = EI_h = 0, \quad \varphi = 1, \quad N_{nn} = q \quad (3.16)$$

Full restraint:

$$k_3 = 1, \quad EI_h = \varphi = q = 0, \quad u_n = 0 \quad (3.17)$$

Forced displacement:

$$k_3 = 1, \quad EI_h = \varphi = 0, \quad q = u_n^*, \quad u_n = u_n^* \quad (3.18)$$
d) Membrane shear force boundary equation.

The displacement corresponding to membrane shear force is in-plane displacement along the \( t \) axis. This displacement will be denoted \( u_t \) and will be taken positive when it is in the direction of the \(+t\) axis.

From the point of view of the boundary conditions the effect of the membrane shear force and the twisting moment are to be combined in a manner similar to that we referred to when transverse shear was discussed. Without going into details, we introduce the quantity

\[
H_n = N_{nt} - z_{tt} M_{nt} \tag{3.19}
\]

and in the ensuing discussion— for lack of a generally accepted term—we will refer to \( H_n \) as the "effective boundary membrane shear."

The contribution of the shell to the axial load in the edge beam equals the effective boundary membrane shear. For structural continuity the tangential displacement of the shell boundary and that of the edge beam must be equal.

The beam must satisfy the equation

\[
EA \frac{\partial (\partial u_t)}{\partial t} = P^* \tag{3.20}
\]

where \( A \) is the cross sectional area of the beam and \( P^* \) is the total axial load acting in the \( t \) direction which contains the contribution of the shell, the effect of any external spring and possible edge load.
If \( k_4 \) is the spring constant and \( P \) is the edge load, the general elastic restraint is expressed by

\[
k_4 u_t + EA \frac{\partial^2 u_t}{\partial t^2} + \varphi H_n = P
\]

(3.21)

where the constant \( \varphi \) has the value unity. Eqn. (3.20) includes, as special cases, the following:

Free boundary with edge load:

\[
k_4 = 0, \quad EA = 0, \quad \varphi = 1, \quad H_n = P
\]

(3.22)

Full restraint:

\[
k_4 = 1, \quad EA = \varphi = P = 0, \quad u_t = 0
\]

(3.23)

Forced displacement:

\[
k_4 = 1, \quad EA = \varphi = 0, \quad P = u_t^*, \quad u_t = u_t^*
\]

(3.24)

Eqns. (3.5), (3.11), (3.15) and (3.21) are the expressions describing the four boundary conditions to be satisfied in any point on the boundary of a shell stiffened with an edge beam.

3.3 Formulation of the Edge Beam Boundary Equations

The various displacements of the edge beams are equal to corresponding displacements of the shell at the boundary. Since the shell displacements are compatible - by virtue of the second of Eqns. (1.89) - this automatically assures the compatibility of the edge beam displacements.
The same cannot be said for the equilibrium of the edge beams. The differential equations, which we called shell boundary equations and which were set up in the previous section, will assure the equilibrium of the edge beams only when the proper constants of integration are used.

Among these constants there may be some displacement-like quantities but, as was explained above, those do not lead to new equations since they are included in the shell boundary equations. It is readily seen that whenever $\psi = 0$ in any of the shell boundary equations (Eqns. 3.5, etc.) then exactly this is the case. For example, consider a point where the shell is fully fixed against rotation in a plane normal to $t$. This condition would be expressed by Eqn. (3.13) for the shell, however, the same equation would be used to express this condition for the edge beam. Those constants of integration of the differential equations that describe the behavior of the edge beams which are force-like quantities do not appear in the shell boundary equations. For example, Eqn. (3.4) governs the variation of vertical shear in the edge beam and brings it in accord with the load it receives from the shell; but we have, so far, no information that would allow us to determine the actual value of the shear, rather than its variation, in any cross section.

We shall now proceed to find all forces that act on the edge beams and express them in terms of known constants and the same functions that describe the behavior of the shell.
The edge beams receive loads

a) directly from the shell

b) from the supporting columns

c) from the external springs

d) from the neighboring edge beams at the corners

e) external loads on the edge beam

Each of these items will be considered separately.

a) the load transmitted to an edge beam from the shell consists of the integral, along the edge beam, of the quantities \(-V_n, -M_{nn}, -N_{nn}, -H_n\). Expressions for all these, in terms of \(\phi, w\) and the elastic and geometric constants of the shell, have already been given. In general, the resultant of all of these for a given edge beam will be a force and a moment. Assume that this resultant has been found and we shall denote it by \(S\).

b) To find the resultant of the forces coming from the columns we shall first find the force exerted by one column, say the \(i\)-th, along the edge beam. For definiteness and without loss of generality, it will be assumed that the edge beam is located at a boundary where the \(+t\) and \(+f\) directions coincide. The application to any other possibility - i.e., when \(+t = -f\) or \(+t = +\eta\) - is obvious.

Let the center of the column have the coordinates \(f_i, \eta_i\) and the width of the column in the \(f\) direction be \(b_i\). The total vertical component of the column reaction includes the difference between the shears
in the edge beam at the two faces of the column and the transverse shear over the width $b_i$. Thus, we have

$$C_{i,z} = EI_v \left[ \frac{\partial^3 w(\frac{b_i}{2}, \eta_i)}{\partial \xi^3} - \frac{\partial^3 w(\frac{b_i}{2}, \eta_i)}{\partial \eta^3} \right]$$

$$+ \int_{\xi=b_i/2}^{\xi=b_i/2} \mathbf{v}_n \, d\xi$$

(3.25)

Let the force causing unit displacements of the column in the $\xi$ and $\eta$ directions have the components $h_{\xi,i}$ and $h_{\eta,i}$ in the $\xi$ and $\eta$ directions, respectively. Then

$$C_{i,\xi} = -h_{\xi,i} \quad u(\xi, \eta)$$

$$C_{i,\eta} = -h_{\eta,i} \quad v(\xi, \eta)$$

(3.26)

The resultant of all column forces may now be obtained by summing vectorially all forces $C_i$ from $i=1$ to $i=n$ where $n$ is the number of columns supporting the beam. Let this force be denoted by $C$.

c) The forces coming from an external spring are found in two different ways, depending on the value of $\varphi$. When $\varphi \neq 0$ we have a true spring and the force on the edge beam is found from the load displacement relations of a linear spring.
\[ F_j = -k_j d_j \quad (j = 1, 2, 3, 4) \] (3.27)

where for \( j = 1 \): \( d_1 = w \) and \( F_1 \) is a vertical force

for \( j = 2 \): \( d_2 = \frac{\partial w}{\partial n} \) and \( F_2 \) is a moment whose vector is along the \( t \) axis

for \( j = 3 \): \( d_3 = u_n \) and \( F_3 \) is a force in the \( n \) direction

for \( j = 4 \): \( d_4 = u_t \) and \( F_4 \) is a force in the \( t \) direction

It was already pointed out that when \( \phi = 0 \) there is no true spring but rather a rigid external restraint is imposed on the boundary. This simply means that this restraint takes directly the forces transmitted from the shell, i.e., its effect on the edge beam is equal to the integral, over the length of such support, of the quantities \( V_n, M_{nn}, N_{nn}, H_n \). These then, effectively, cancel out a part of what was considered in Sec. (a); whether one deletes them there, or includes them here, has no bearing on the final result. Our decision to include these forces in \( S \) with a negative sign and here with a positive sign is motivated by considerations of ease of programming.

Assume then that the resultant of all of the spring forces has been found and denote it by \( F \).

d) To find the forces transmitted from one edge beam to the other, we first consider the stress resultants in one edge beam, say the \((m-1)\)-th. Let the vertical and horizontal bending stiffness, the torsional rigidity and the axial rigidity of the beam be \((EI_v)_{m-1}', (EI_h)_{m-1}', (GJ)_{m-1}', (EA)_{m-1}'\)
respectively. Then at any point, and, in particular, at the corner of the
shell where the \((m-1)\)-th beam meets the \(m\)-th one, the stress-resultants
in the \((m-1)\)-th beam are:

- **vertical shear**
  \[
  (EI_v)_{m-1} \frac{\partial^3 w}{\partial t^3}
  \]

- **vertical bending moment**
  \[
  (EI_v)_{m-1} \frac{\partial^2 w}{\partial t^2}
  \]

- **horizontal shear**
  \[
  (EI_h)_{m-1} \frac{\partial^3 u_n}{\partial t^2}
  \]

- **horizontal bending moment**
  \[
  (EI_h)_{m-1} \frac{\partial^2 u_n}{\partial t^2}
  \]

- **twisting moment**
  \[
  (GJ)_{m-1} \frac{\partial^2 w}{\partial t \partial n}
  \]

- **axial force**
  \[
  (EA)_{m-1} \frac{\partial u_k}{\partial t}
  \]

In Figure 9 we show edge beams \((m-1)\), \(m\) and \((m+1)\). As a con-
sequence of the way the orientation of the \(n\)-\(t\) system is taken, it is seen
that the stress resultants, as listed above, act from beam \((m-1)\) on beam \(m\).
To obtain the forces acting on beam \(m\) at the other end, not only are the sub-
scripts to be changed from \((m-1)\) to \((m+1)\) but also all quantities are to be
taken with a negative sign.

Assume that these have been found and denote their resultant by
\(B_{m-1}\) and \(B_{m+1}\) respectively.
Figure 9
Adjacent edge beams and their local coordinate systems
e) The external loads on the edge beam are all given. Let their resultant be denoted by \( L \).

We shall further introduce the symbol \( \equiv \) to mean "statistically equivalent." The equilibrium of the \( m \)-th edge beam is then expressed by

\[
(S, C, F, B_{m-1}, B_{m+1}, L) \equiv 0
\]

(3.28)

This statement of static equilibrium is equivalent to six scalar equations. These should be written for each of the four edge beams. Thus, the system of edge beam boundary conditions consists of 24 equations.

The equilibrium of the whole shell is implied by these equations and need not be considered separately. This becomes obvious if the four Eqns. (3.28) - one for each edge beam - are considered together. The \( B \)-s then cancel out and \( S \), the integral of all boundary forces along the edge, may be replaced by the total load on the shell. What is then left is a statement of the equilibrium of the whole structure.

3.4 The Extended Point Matching Method With Constraints

In Ref. (24) Hulbert gives a very good and up-to-date historical account of the development and applications of the so-called point matching method to boundary value problems. In the present work, we will not repeat the interesting sidelights mentioned by Hulbert but will give a full explanation of the method itself. Furthermore, an extension of the method will be
introduced to include restraints.

Let the functions $f_1, f_2, \ldots, f_r$ satisfy a set of $r$ simultaneous linear partial differential equations in some region $R$ and on the boundary $B$ of $R$. Assume that these functions contain $n$ arbitrary constants, $A_1, A_2, \ldots, A_n$ to be used to satisfy the boundary conditions. The boundary conditions will be stated in the form:

$$g_i(f_1, f_2, \ldots, f_r) \bigg|_{on \ B} = 0 \quad (i = 1, 2, \ldots, j)$$

(3.29)

Numerous important cases are known when a set of $A_i$-s may be found for which Eqns. (3.29) will be satisfied exactly. When this is so some of the conditions are - at least on some part of $B$ - identically satisfied by the $f$-s for any $A$ and those that are not are expandable in terms of the $f$-s on $B$. If this expansion gives exactly the right number of equations for the $A$-s the problem has an exact solution in terms of the $f$-s, even though that solution may be an infinite series. For example, the so-called Levy solution of plates is a case like this.

In a general case, however, it may be impossible to generate solutions that will solve a sufficient number of the boundary conditions identically and, further, the number of equations to be satisfied by the $A$-s

16

Frequently $n$ will be infinity.
may exceed the number of $A$-s available. Then, although the $f$-s satisfy the
differential equations, there exists no exact solution of the boundary value
problem in terms of the $f$-s.

One possible approach in such cases is to alter some physical
boundary conditions so that they will match the mathematically desirable
solution. This, for example, is the usual procedure in the membrane theory
of shells: some boundary values may be specified but some others have to be
accepted as results of the solution rather than conditions that may be imposed
upon the solution. Coining a term, these may be called boundary requirements,
as opposed to the boundary conditions. The usability of these results depends
largely on how closely the boundary requirements may be met and how signifi-
cant is the effect of not meeting them exactly.

In the point matching method all boundary conditions are retained
but the boundary on which they are satisfied contains only a finite number of
discreet points of $B$. The number of points, at which the solution "matches"
the boundary conditions, is suitably selected so that there are exactly the
required number of equations for determining the $A$-s.

Thus, Eqn. (3.29) would be replaced by

$$g_i(f_1, f_2, \ldots, f_r) \bigg|_{at \ P_k \ on \ B} = 0 \quad (i = 1, 2, \ldots, j)$$

(3.30)
where the number of points selected - i.e., the range of the subscript $k$ - would be such that the total number of algebraic equations of the form (3.30) would be exactly $n$, which is the number of $A$-s.

When the $A$-s so obtained are substituted into the boundary conditions some expressions other than $g_i$ result, whose value is not everywhere zero on the boundary. Let $g_i^*$ be this approximation of $g_i$ then we may accept, as a measure of the error of our approximation, the value $\Delta$ defined by

\[ \Delta = \sum_{i=1}^{J} \int_{\mathcal{B}} (g_i^*)^2 \, ds \] (3.31)

where $s$ is the arc length along $\mathcal{B}$ and the integration extends over the whole boundary.

The magnitude of $\Delta$ obviously depends not only on $f_1, f_2, \ldots, f_r$ but also on the location of the points $P_k$ at which we chose to make the solution match the boundary conditions exactly. One would wish to find the choice of $P_k$-s which, for a given $n$, will minimize $\Delta$.

(The following example, although it does not sound highly scientific, is very illustrative of the procedure.

Given a rigid wire frame which forms a closed space curve, say a warped rectangle - this corresponds to the boundary $\mathcal{B}$. There is also a semi-rigid celluloid sheet, which cannot be stretched at will but can still be subjected to some warping - this corresponds to the different functions describable by the set of $f$-s in the region when the $A$-s are changed. It is desired to fit the celluloid sheet over the wire frame.

If, by sheer luck, the shape of the frame is such that the sheet can be twisted into a shape where it will fit over the frame everywhere, we may use glue to fasten it all around and obtain an "exact" solution. On the
other hand, having spent some time in trying to do this, we may get convinced that with our frame and our sheet it cannot be done. One would then be tempted to take a stapler, force the sheet to meet the frame in as many points as possible, and staple it to the frame at these points. After a number of points have been stapled to the frame the sheet will become rigid and we have reached a position where we cannot apply any more staples. This is point matching.

Just how closely the sheet fits in between the staples is, of course, the measure of how good an approximation of the glued case we attained. This depends on the given frame and sheet, over which we have little control - but it also depends on how lucky we were in selecting the points at which we applied our staples. Obviously, if somebody else used the same number of staples at different locations a different surface would result which may fit better or worse than the original.

Indeed, one may argue that it is more important to minimize the "error" $\Delta$ than to match the boundary conditions exactly in a set of points selected in an arbitrary manner. Thus $\Delta$ could be expressed as a function of the $A$-s and then minimized with respect to each $A$. The set of $A$-s which makes all these derivatives vanish simultaneously is accepted as the best solution obtainable from the set of functions with the number of arbitrary constants chosen.

When the integral in Eqn. (3.31) is evaluated by the trapezoid rule the procedure really corresponds to writing the boundary equations at a large number of uniformly spaced points, say $m(m > n)$, and satisfying these $m$ equations in $n$ unknowns in the least square sense. The algebraic procedure for this was formulated by Hulbert in a very systematic and clear way.

---

17 I.e., it will not admit of any more inextensional deformation.
fashion and he called this the "extended point matching technique."

(To return to our wire frame and plastic sheet: We may decide that rather than using a limited number of staples and achieving perfect fit where they are applied but having no control over the intermediate points we will do without the stiff steel staples and take a narrow rubber tape, glue one longitudinal edge to the frame and the other one to the celluloid sheet. Then the rubber tape will pull the sheet as close to the frame as it possibly can. At some parts the sheet will pass above the frame and at others it will pass below, where two such opposite parts meet it will actually touch the frame. This is the best overall fit we can obtain with the given sheet for the given frame. This picture corresponds to evaluating $\Delta$ by the continuous integral. Using the trapezoid rule, or, which is the same, writing a large number of equations at isolated points and satisfying them in the least square sense, could be thought of as replacing the rubber tape by a large number of rubber bands, each tied to the wire frame with one end and the celluloid sheet with the other. Unlike with the staples, now there is no limit on the number of rubber bands that may be used – since they are elastic they will not hold the sheet rigidly and, in any case, in the limit they would only be equivalent to the continuous rubber tape).

In our problem, in addition to the shell boundary conditions, which are in the form (3.29) but which will only be satisfied in the least square sense, we also have the system of edge beam boundary equations which we would like to satisfy exactly. Let these latter equations be of the form

$$h_i(f_1, f_2, \ldots, f_r) = 0 \quad (i = 1, 2, \ldots, p) \quad (3.32)$$

Our problem, then, is to minimize $\Delta$ subject to the restraints (3.32).

(To exploit our illustration fully, we may attach special importance to some points where we would like to have contact between the frame and the sheet. We would place staples at these points and retain the rubber bands everywhere else. We would thus obtain the best possible fit consistent with the requirement that at some specified points we want exact match.)
Having given a description of what we wish to do we shall now develop the algebra it involves.

A typical boundary equation written for some point on the boundary has the form

\[ c_{ij} A_j = s_i \quad (j = 1, 2, \ldots, n) \quad (3.33) \]

where we have adopted the summation convention for repeated subscripts and where \( c_{ij} \) and \( s_i \) are known algebraic expressions depending on the particular boundary condition and expressible in terms of the parameters of the problem and the functions appearing in \( w \) and \( \varphi \) and where \( A_1, A_2, \ldots, A_n \) are so far undetermined constants. Let us write \( m \) boundary equations, each of the form (3.33) with \( i \) varying from 1 to \( m \), where \( m \) is greater than \( n \) but is otherwise arbitrary.

In addition to the boundary equations (3.33) we have \( p \) equations of the form

\[ q_{kj} A_j = t_k \quad (k = 1, 2, \ldots, p) \quad (3.34) \]

which we insist on satisfying exactly.

If \( p = n \), the number of unknowns equals the number of Eqns. (3.34) and we have no control over Eqn. (3.33). Hence, we shall always select \( n \) considerably larger than \( p \).

If our solution were exact then, according to Eqn. (3.30), we should have \( v_i = 0 \) for all \( i \), where
\[ v_i = c_{ij} A_j - s_i \]  

(3.35)

However, our solution is not exact and therefore, in general, the \( v_i \)-s will not vanish.

It is now wanted to find the \( A \)-s such that

\[ \Delta = \sum_{i=1}^{m} v_i^2 \]  

(3.36)

is a minimum while Eqns. (3.34) are exactly satisfied.

We shall introduce the following matrix notation\(^{18}\)

\[
\begin{align*}
C_{mxn} &= [c_{ij}] \\
Q_{pxn} &= [q_{kij}] \\
A_n &= [A_i] \\
T_p &= [t_k] \\
S_m &= [s_j]
\end{align*}
\]

and then

\[ \Delta = \sum_{i=1}^{m} v_i^2 = (C_{mxn} A_n - S_m)^T (C_{mxn} A_n - S_m) \]  

(3.37)

We will introduce a column vector \( L_p \), whose components are unknown Lagrange multipliers and shall seek the minimum of

\[ \Delta^* = \Delta + 2 (Q_{pxn} A_n - T_p)^T L \]  

(3.38)

\(^{18}\) The first subscript is the number of rows in the matrix, the second the number of columns. Matrices with one subscript only denote column-vectors.
Substituting $\Delta$ from Eqn. (3.37) and differentiating with respect to $A$ we obtain

$$\frac{\partial \Delta}{\partial A} = 0 = 2(C^TC A - C^TS) + 2 Q^T L$$ \hspace{1cm} (3.39)

We may rewrite Eqn. (3.39) thus

$$\begin{bmatrix} C^TC & Q^T \\ \end{bmatrix}_{[n \times (n+p)]} \begin{bmatrix} A \\ L \end{bmatrix}_{(n+p)} = (C^TS)_n$$ \hspace{1cm} (3.40)

These are $n$ equations in $(n+p)$ unknowns, the unknowns being the $n$ components of $A$ and the $p$ components of $L$. We attach the system of Eqns. (3.34) to this

$$\begin{bmatrix} C^C & Q^T \\ Q & 0 \end{bmatrix} \begin{bmatrix} A \\ L \end{bmatrix} = \begin{bmatrix} C^TS \\ T \end{bmatrix}$$ \hspace{1cm} (3.41)

Eqn. (3.41) is the scheme for solution of what we will term the "extended point matching method with restraints." The $L$ part of its solution vector may be discarded, however, the $A$ part is such that the restraints (3.34) will be satisfied exactly and the sum of the square of the residuals (3.35) will be minimized.

Applying this procedure to the case in question, we recall that in Art. 2.2 we found that to each $\lambda$ selected there will be 16 arbitrary constants in the complementary solution. In Art. 2.4 we saw that three more arbitrary
constants appear in the expressions for the in-plane motion of the shell.

Assume that it was decided to truncate the solution after $b$ different $\lambda$-'s then all boundary conditions may be expressed in terms of the undetermined constants $A_1, A_2, \ldots, A_n$, where $n = 16b + 3$.

The number of equations to be satisfied exactly is six for each of the four edge beams, i.e., $p = 24$. In each boundary point of the shell there will be four shell boundary equations to satisfy; if one should use, say, 10 points on each side one has $m = 10 \times 4 \times 4 = 160$.

We have now put forward all information necessary to find the quantities that appear in Eqn. 3.41 and thus find the undetermined constants in the general solution of the shallow shell equations. The solution so found will have the following features:

a) Eqns. (1.89) will be exactly satisfied everywhere within the region of interest.

b) The shell, with its edge beams will be in equilibrium and will have moved in accordance with the elastic properties of the supporting columns.

c) The boundary conditions at the joint between the shell and its edge beams will be satisfied at a number of isolated points but will not be satisfied between these points. However, the solution obtained will approach the best solution in the selected form to the extent that the trapezoid rule approaches the integral when applied to evaluate the deviation of the actual boundary conditions from the theoretical ones.
4. COMPUTATIONAL SEQUENCE

In the previous chapters a complete solution has been presented to the problem formulated in Art. 0.4. The solution has been presented in sufficient detail that it can form the basis of a computer program and that such program can be written by somebody engaged in numerical work without the need for him to be familiar with shell theory.

Indeed, such a program is currently under preparation but it does not form an integral part of the present work, and the numerical results obtained will be reported elsewhere. To facilitate the writing of this program the computational sequence to be followed is given below and the pertinent equations repeated so that reference to the previous chapters is not necessary. The equations, however, are quoted with their original number and all symbols have the same meaning here as they have in the part of the text that the equations are taken from.

Assign the values $n = 1, 2, \ldots$ in the expression $\lambda_n = \frac{n \pi}{L} \frac{L}{\alpha}$ and truncate this sequence at some arbitrary, suitable limit, say $b$. To each $\lambda$ find 16 values of $\mu$ by solving Eqn. (2.4).
\[
\sum_{i=0}^{8} a_{xi} \mu^{2i} = 0
\]

(2.4)

where

\[
a_{xx} = \frac{D}{Eh}
\]

\[
a_{xy} = \frac{D}{Eh} (-8 + 16 \beta^2) \lambda^2
\]

\[
a_{yx} = \frac{D}{Eh} (-56 + 240 \beta^2 - 384 \beta^4 + 256 \beta^6) \lambda^6 + \alpha^4 (64 \beta z_{ff} z_{\eta} - 8 z_{j1}^2
\]

\[-4 z_{ff} z_{\eta} - 8 z_{j1}^2 - 48 \beta^2 z_{j1}^2) \lambda^2
\]

\[
a_{yy} = \frac{D}{Eh} (70 - 320 \beta^2 + 576 \beta^4 - 512 \beta^6 + 256 \beta^8) \lambda^8 + (-64 \beta z_{ff} z_{\eta}
\]

\[-192 \beta z_{ff} z_{\eta} - 256 \beta^2 z_{ff} z_{\eta} + 2 \eta + 16 z_{ff} z_{\eta}
\]

\[+96 \beta^2 z_{ff} z_{\eta} + 32 z_{j1}^2 + 192 \beta^2 z_{j1}^2 + 12 z_{ff}^2 + 96 \beta^2 z_{j1}^2
\]

\[+32 \beta^4 z_{ff}^2) \alpha^4 + \frac{Eh}{D} \alpha^2 z_{ff}^2
\]
and

\[ a_s = \frac{D}{2\eta h} (-56 + 240\beta^2 - 384\beta^4 + 256\beta^6)\lambda^4 + (192\beta z_f z_{\eta}) \]
\[ + 192\beta z_f z_{\eta} + 256\beta^3 z_f z_{\eta} + 256\beta^3 z_f z_{\eta} - 24 z_f z_{\eta} \]
\[ - 192\beta^4 z_f z_{\eta} - 64\beta^4 z_f z_{\eta} - 48 z_{f \eta}^2 - 384\beta^4 z_{f \eta}^2 - 8 z_{f \eta}^2 \]
\[ - 128\beta^4 z_{f \eta}^2 - 48\beta^2 z_{f \eta}^2 - 8 z_{f \eta}^2 - 18z_{f \eta}^2)\alpha^4\lambda^4 \]
\[ + \frac{2\eta h}{D} (8 z_{f \eta}^2 z_{f \eta}^2 - 4 z_{f \eta}^2 z_{\eta}^2)\alpha^4\lambda^4 \]

\[ a_q = \frac{D}{2\eta h} (28 - 96\beta^2 + 96\beta^4)\lambda^4 + (-192\beta z_f z_{\eta} - 64\beta z_{f \eta}^2 z_{f \eta} \]
\[ - 256\beta^3 z_f z_{\eta} + 12 z_{f \eta}^2 + 96\beta^2 z_{f \eta}^2 + 32\beta^2 z_{f \eta}^2 + 96\beta^2 z_f z_{\eta} \]
\[ + 16 z_{f \eta}^2 + 32 z_{f \eta}^2 + 192\beta^2 z_{f \eta}^2 + 2 z_{f \eta}^2)\alpha^4\lambda^4 + \frac{2\eta h}{D} (16z_{f \eta}^4 \]
\[ - 16 z_{f \eta}^2 z_{f \eta}^2 + 6 z_{f \eta}^2 z_{\eta}^2)\alpha^4\lambda^4 \]

\[ a_2 = \frac{D}{2\eta h} (-8 + 16\beta^2)\lambda^4 + (64\beta z_f z_{\eta} - 8 z_{\eta}^2 - 48\beta z_{f \eta}^2 \]
\[ - 4 z_{f \eta}^2 z_{\eta}^2)\alpha^4\lambda^4 + \frac{2\eta h}{D} (8 z_{f \eta}^2 z_{f \eta}^2 - 4 z_{f \eta}^2 z_{\eta}^2)\alpha^4\lambda^4 \]

\[ a_0 = \frac{D}{2\eta h} \lambda^{16} + 2 z_{f \eta}^2 \alpha^4\lambda^{12} + \frac{2\eta h}{D} z_{f \eta}^2 \alpha^4\lambda^{12} \]

\[ (2.4) \]

(cont'd.)
For each non-multiple root $\mu$ thus obtained solve Eqn. (2.3) for the eigenvector $\gamma(\mu) = (A, B, C, F)$.

\[
\begin{bmatrix}
Db_{11} & Db_{12} & b_{13} & b_{14} \\
-Db_{12} & Db_{11} & -b_{13} & b_{13} \\
-Ehb_{13} & -Ehb_{14} & b_{11} & b_{12} \\
Ehb_{14} & -Ehb_{13} & -b_{11} & b_{15}
\end{bmatrix}
\begin{bmatrix}
A \\
B \\
C \\
F
\end{bmatrix}
= 0
\quad (2.3)
\]

where

\[
\begin{align*}
b_{11} &= \lambda^4 - d^2 \mu^2 + \mu^4 \\
b_{12} &= 4\beta (\lambda^2 \mu - \lambda \mu^2) \\
b_{13} &= \alpha^2 (z_{11} \lambda^2 - z_{11} \mu^2) \\
b_{14} &= 2 \alpha^2 z_{11} \lambda \mu
\end{align*}
\]

In general both $\mu$ and the eigenvector will be complex. Thus, for some root $\mu_1$ we may write

\[
\mu_1 = \rho_1 + i \sigma_1 
\quad (2.8)
\]

and then the complementary solution consists of the sum of terms as those given by Eqns. (2.11) and (2.12) for each $(\lambda, \mu_1)$ pair in the arbitrarily truncated series and each having an arbitrary constant multiplier.
\[ w_1 = w^* + w^{**} = (a_1 \cos \lambda \cos \delta \eta - a_2 \cos \lambda \sin \delta \eta \]
\[ + b_1 \sin \lambda \cos \delta \eta - b_2 \sin \lambda \sin \delta \eta)e^{i \eta} \quad (2.11) \]

\[ \phi_1 = \phi^* + \phi^{**} = (c_1 \cos \lambda \cos \delta \eta - c_2 \cos \lambda \sin \delta \eta \]
\[ + f_1 \sin \lambda \cos \delta \eta - f_2 \sin \lambda \sin \delta \eta)e^{i \eta} \]

and

\[ w_2 = w^* - w^{**} = (a_2 \cos \lambda \cos \delta \eta + a_1 \cos \lambda \sin \delta \eta \]
\[ + b_2 \sin \lambda \cos \delta \eta + b_1 \sin \lambda \sin \delta \eta)e^{i \eta} \quad (2.12) \]

\[ \phi_2 = \phi^* - \phi^{**} = (c_2 \cos \lambda \cos \delta \eta + c_1 \cos \lambda \sin \delta \eta \]
\[ + f_2 \sin \lambda \cos \delta \eta + f_1 \sin \lambda \sin \delta \eta)e^{i \eta} \]

To find the particular solution express \( Z \) and \( V \) in the form of a power series and a trigonometric series, both in \( x \) and \( \eta \). Substitute these into the right hand side of Eqns. (1.89). Assume that the power series part resulting from this substitution has \( q \) and \( r \) as the highest exponents of \( x \) and \( \eta \). Then the particular solution corresponding to this part is found by assuming

\[ w_p = \sum_{i=0}^{q+4} \sum_{j=0}^{r+4} a_{ij} x^i \eta^j \]
\[ \phi_p = \sum_{i=0}^{q+4} \sum_{j=0}^{r+4} b_{ij} x^i \eta^j \quad (2.14) \]

substituting these on the left hand side of Eqns. (1.89) and equating coefficients.
of equal powers. This procedure yields fewer equations for the determination of the unknowns \((a_{ij} \text{ and } b_{ij})\) than their number; some of these coefficients can therefore be set arbitrarily.

Let the trigonometric series part consist of terms each of which is of the form

\[
D \nabla^4 w - \nabla_k^2 \varphi = A_1 \cos \gamma \cos \delta \eta + B_1 \cos \gamma \sin \delta \eta + C_1 \sin \gamma \cos \delta \eta + F_1 \sin \gamma \sin \delta \eta \\
Eh \nabla_k^2 w - \nabla^4 \varphi = A_2 \cos \gamma \cos \delta \eta + B_2 \cos \gamma \sin \delta \eta + C_2 \sin \gamma \cos \delta \eta + F_2 \sin \gamma \sin \delta \eta
\] (2.15)

Then a particular solution is

\[
w = A_1^* \cos \gamma \cos \delta \eta + B_1^* \cos \gamma \sin \delta \eta + C_1^* \sin \gamma \cos \delta \eta + F_1^* \sin \gamma \sin \delta \eta \\
\varphi = A_2^* \cos \gamma \cos \delta \eta + B_2^* \cos \gamma \sin \delta \eta + C_2^* \sin \gamma \cos \delta \eta + F_2^* \sin \gamma \sin \delta \eta
\] (2.16)

where the coefficients are found from the following equations

\[
\begin{bmatrix}
A_1^* \\
F_1^* \\
A_2^* \\
F_2^*
\end{bmatrix} =
\begin{bmatrix}
A_1 \\
F_1 \\
A_2 \\
F_2
\end{bmatrix} \quad \text{and} \quad
\begin{bmatrix}
B_1^* \\
C_1^* \\
B_2^* \\
C_2^*
\end{bmatrix} =
\begin{bmatrix}
B_1 \\
C_1 \\
B_2 \\
C_2
\end{bmatrix}
\] (2.17)

where the matrices \([A^+]\) and \([A^-]\) are given by
\[
\begin{pmatrix}
DP_0 & DP_1 & P_2 & P_3 \\
DP_1 & DP_0 & P_2 & P_3 \\
-E\hbar P_2 & -E\hbar P_3 & P_0 & P_1 \\
-E\hbar P_3 & -E\hbar P_2 & P_1 & P_0
\end{pmatrix}
\]

(2.18)

and

\[
P_0 = \frac{1}{\alpha^4} \left[ \gamma^4 + (2 + 4\beta^2)\gamma^2 \sigma^2 + \sigma^4 \right]
\]

\[
P_1 = \pm \frac{4\beta}{\alpha^4} (\gamma^3 \sigma + \gamma \sigma^3)
\]

(2.19)

\[
P_2 = \frac{1}{\alpha^2} (z_{\eta\eta} \gamma^2 + z_{\eta\sigma} \sigma^2)
\]

\[
P_3 = \pm \frac{2}{\alpha^2} z_{\eta\sigma} \gamma \sigma
\]

and for \(P_1\) and \(P_3\) the + sign should be selected for \([A^+]\) and the - sign for \([A^-]\).

The in-plane displacements \(u\) and \(v\) are found in four parts:

a) To each term of the complementary solution there belong the following displacements:

For

\[
\begin{align*}
\mathbf{w} &= K \left[ (A_x \cos \lambda x \cos \beta y + B_y \cos \lambda x \sin \beta y + C_4 \sin \lambda x \cos \beta y \\
&+ f_2 \sin \lambda x \sin \beta y) e^{i \gamma} \right]
\end{align*}
\]

(2.30)

one gets
\[ u_w = \frac{K z_{xy}}{\lambda} \left[ (-C_1 \cos \lambda_x \cos \lambda_y - F_1 \cos \lambda_x \sin \lambda_y + A_1 \sin \lambda_x \cos \lambda_y \\
+ B_1 \sin \lambda_x \sin \lambda_y) e^{\phi y} \right] \]  
(2.31)

\[ v_w = \frac{K z_{yy}}{\phi^2 + \sigma^2} \left[ (\phi A_1 - \phi B_1) \cos \lambda_x \cos \lambda_y + (\phi A_1 + \phi B_1) \cos \lambda_x \sin \lambda_y \\
+ (\phi C_1 - \phi F_1) \sin \lambda_x \cos \lambda_y + (\phi C_1 + \phi F_1) \sin \lambda_x \sin \lambda_y \right] e^{\phi y} \]

and for \[ \phi = K \left[ (A_2 \cos \lambda_x \cos \lambda_y + B_2 \cos \lambda_x \sin \lambda_y + C_2 \sin \lambda_x \cos \lambda_y \\
+ F_2 \sin \lambda_x \sin \lambda_y) e^{\phi y} \right] \]  
(2.32)

one gets

\[ u_\phi = \frac{K}{Eh} \left\{ \left[ \lambda (\phi^2 - \sigma^2 - \nu) \right] C_2 + 2 \lambda \phi \phi F_2 \right\} \cos \lambda_x \cos \lambda_y \\
+ \left\{ \left[ \lambda (\phi^2 - \sigma^2 - \nu) \right] F_2 - 2 \lambda \phi \phi C_2 \right\} \cos \lambda_x \sin \lambda_y \\
+ \left\{ \lambda (-\phi^2 + \sigma^2 + \nu) A_2 - 2 \lambda \phi \phi B_2 \right\} \sin \lambda_x \cos \lambda_y \\
+ \left\{ \lambda (-\phi^2 + \sigma^2 + \nu) B_2 + 2 \lambda \phi \phi A_2 \right\} \sin \lambda_x \sin \lambda_y \right\} e^{\phi y} \]  
(2.33)

\[ v_\phi = \frac{K}{Eh} \left\{ \left[ -\frac{\lambda^2}{\phi^2 + \sigma^2} (\phi A_2 - \phi B_2) - \nu (\phi A_2 + \phi B_2) \right] \cos \lambda_x \cos \lambda_y \\
+ \left[ -\frac{\lambda^2}{\phi^2 + \sigma^2} (6A_2 + \phi B_2) + \nu (6A_2 - \phi B_2) \right] \cos \lambda_x \sin \lambda_y \\
+ \left[ -\frac{\lambda^2}{\phi^2 + \sigma^2} (\phi C_2 - \phi F_2) - \nu (\phi C_2 + \phi F_2) \right] \sin \lambda_x \cos \lambda_y \\
+ \left[ -\frac{\lambda^2}{\phi^2 + \sigma^2} (6C_2 + \phi F_2) + \nu (6C_2 - \phi F_2) \right] \sin \lambda_x \sin \lambda_y \right\} e^{\phi y} \]
b) The contribution of the in-plane load is

\[ u = \frac{1-\nu}{Eh} \int V \, dx \]

\[ v = \frac{1-\nu}{Eh} \int V \, dy \]  \hspace{1cm} (4.1)

c) In addition there are two additive functions that enter as constants of integration. They are

\[ u = -\frac{\dot{f}_2(y)}{Eh} \]

\[ v = -\frac{\dot{f}_1(x)}{Eh} \]  \hspace{1cm} (4.2)

and the derivatives of \( f_1 \) and \( f_2 \) may be found from Table 1 on Page 75.

d) Finally the rigid body motions of the shell are given by

\[ f_1 = ax + b \]

\[ f_2 = -ay + c \]  \hspace{1cm} (2.26)

The sum of all these gives the in-plane motions. This sum contains the same arbitrary constants as the complementary solution and 3 more unknowns which enter as the components of the rigid body motion. Denote the total number of unknowns by \( n \) and let them be grouped as components of the column vector \( A \).
To find these unknowns we first form the matrices $C_{m \times n}$ and $Q_{24 \times n}$.

The $C$ matrix is formed by selecting a suitable number of points on the shell boundary and writing four equations in each of these points:

\[
k_{w}w + EI_{v} \frac{\partial w}{\partial t^{4}} + qV_{n} = P \quad (3.5)
\]

\[
k_{s} \frac{\partial w}{\partial n} + GJ \frac{\partial^{2} w}{\partial n \partial t} + qM_{mn} = T \quad (3.11)
\]

\[
k_{3}u_{n} + EI_{h} \frac{\partial^{4} u_{n}}{\partial t^{4}} + qN_{mn} = Q \quad (3.15)
\]

\[
k_{4}u_{t} + EA \frac{\partial^{4} u_{t}}{\partial t^{2}} + qH_{n} = P \quad (3.21)
\]

Each of these equations can be written in terms of the $n$ unknowns and if $m$ is four times the number of selected boundary points an $m \times n$ matrix results. Let the constant terms of the above equations form the column vector $S$.

The $Q$ matrix consists of the 6 static equilibrium equations for each of the 4 edge beams. This part is described in Art. 3.3 and is essentially without any derivations that could be omitted here. It does not, therefore, defeat the purpose of the present chapter if the reader is referred to the indicated part for the formulation of $Q$. 

The unknown vector $A$ is then found by solving

$$
\begin{bmatrix}
C^T & a^T \\
Q & 0
\end{bmatrix}
\begin{bmatrix}
A \\
L
\end{bmatrix} =
\begin{bmatrix}
C^T S \\
T
\end{bmatrix}
$$

(3.41)

where $L$ is a vector of Lagrange multipliers and may be discarded after the solution has been obtained.

This finally determines the stress function $\phi$ and the displacements $u, v, w$. With these known the bending moments, transverse shears and membrane stresses may be found from

$$
M_x = -D \left( \frac{\partial^2 w}{\partial x^2} + v \frac{\partial^2 w}{\partial y^2} \right)
$$

$$
M_y = -D \left( \frac{\partial^2 w}{\partial y^2} + v \frac{\partial^2 w}{\partial x^2} \right)
$$

$$
M_{xy} = -D (1-v) \frac{\partial^2 w}{\partial x \partial y}
$$

$$
Q_x = -D \frac{\partial}{\partial x} \nabla^2 w
$$

$$
Q_y = -D \frac{\partial}{\partial y} \nabla^2 w
$$

$$
N_x = \frac{\partial^2 \phi}{\partial y^2} - Z_{xx} D \nabla^2 w + V
$$

$$
N_y = \frac{\partial^2 \phi}{\partial x^2} - Z_{yy} D \nabla^2 w + V
$$

$$
N_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} - Z_{xy} D \nabla^2 w
$$

(1.76)
If the shell has a non-rectangular shape an oblique coordinate system \( \xi - \eta \) is used in all preceding equations. In this system derivatives in the x-y directions are found by

\[
\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x} \cdot \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \cdot \frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi}
\]

and

\[
\frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y} \cdot \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y} \cdot \frac{\partial}{\partial \eta} = \frac{1}{\alpha} (-\beta \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta})
\]

(1.98)

the differential operators entering the equations are

\[
\nabla^2 = \frac{1}{\alpha^2} \left( \frac{\partial^2}{\partial \xi^2} - 2\beta \frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2} \right)
\]

\[
\nabla_x^2 = \frac{1}{\alpha^2} \left( Z \eta \frac{\partial^2}{\partial \xi^2} - 2Z \xi \frac{\partial^2}{\partial \xi \partial \eta} + Z \frac{\partial^2}{\partial \eta^2} \right)
\]

(1.99)

\[
\nabla^4 = \frac{1}{\alpha^4} \left( \frac{\partial^4}{\partial \xi^4} - 4\beta \frac{\partial^4}{\partial \xi^3 \partial \eta} + (2+4\beta^2) \frac{\partial^4}{\partial \xi^2 \partial \eta^2} - 4\beta \frac{\partial^4}{\partial \xi \partial \eta^3} + \frac{\partial^4}{\partial \eta^4} \right)
\]

and the directional derivative in some direction \( t \) at an angle \( \theta \) from the \( + \xi \) axis is

\[
\frac{\partial}{\partial t} = \frac{1}{\alpha} \left( (\alpha \cos \theta - \beta \sin \theta) \frac{\partial}{\partial \xi} + \sin \theta \frac{\partial}{\partial \eta} \right)
\]

(1.103)
5. REMARKS ON THE COMPUTER PROGRAM

While the majority of our remarks will be pertinent whatever type of high speed electronic computer is used, some of them will refer to problems one would be likely to encounter in the use of IBM computers, in particular the IBM 7094.

The problem naturally lends itself to division into several parts.

These will now be taken up one by one.

5.1 Solution of the Homogeneous Equation

The first major operation is to solve the characteristic polynomial (2.4). The coefficients of this polynomial contain anywhere from the 0th to the 16th powers of $\lambda$. Since $\lambda$ will generally be a small number, say of the order of $10^{-2}$, the coefficients will show a variation of many orders of magnitude. Therefore, in using a computer program to find roots of Eqn. (2.4), overflows or underflows are likely to occur, unless this condition is born in mind when the program is written.

Let $\mu^*$ denote the eigenvalues of the matrix in Eqn. (2.3) and $\mu^{**}$ the roots of Eqn. (2.4). Ideally these two are equal but when (2.4) is obtained by numerical expansion from (2.3) using say n significant digits and then solved by some iterative technique presumably using the same degree of precision then one will generally find that the $\mu^{**}$'s are not
exact eigenvalues. This problem has been investigated exhaustively (27) and it is found that by raising the number of digits carried in all intermediate operations by say $k$, the number of correct digits in the result will also be raised by $k$. Thus, double precision arithmetic will, in most practical cases, yield correct single precision roots.

It is necessary that the eigenvalues be obtained in conjugate complex pairs. Therefore, the Bairstow technique (28), which finds quadratic factors and thus supplies exact conjugates, is preferred over algorithms that find one root at a time.

As regards multiple roots, the program will have to search for these before it sets up the eigenfunctions. Exact multiples will, of course, very rarely be produced by any iterative procedure and therefore a tolerance should be set up within the program for accepting two roots as being identical roots. For example, if $\mu_i = a_k + ib_k$ and $\mu_j = a_j + ib_j$ are two roots one would accept them as identical and replace them by their algebraic mean if

$$(1 - \epsilon) < \frac{a_k}{a_j} < (1 + \epsilon)$$

and

$$(1 - \epsilon) < \frac{b_k}{b_j} < (1 + \epsilon)$$

where $\epsilon$ is some small number, say $10^{-4}$. If, as a result of $\epsilon$ being too small, some roots that should really be treated as if they were multiple roots will be treated unique this will lead to nothing more serious
than poor conditioning with respect to these two roots; since it is suggested that many terms be included in the solution the detrimental effect of such an error will probably be negligible.

It is felt that it should be sufficient to provide for double roots in the program with an error exit for roots with higher multiplicity. The chances of this ever occurring are very slim.

Assume that \( \mu_1 \) is a complex eigenvalue and \( \gamma(\mu_2) = \gamma_1 + i\gamma_2 \) is the associated eigenvector, also complex. The eigenvector is then obtained by substituting \( \mu_1 \) into the characteristic matrix and solving an equation of the form:

\[
\begin{bmatrix} M(\mu_1) \end{bmatrix} \gamma(\mu_1) = 0
\]

(5.2)

where \( [M(\mu_1)] \) is a complex singular matrix. If a complex matrix inverter subroutine is not available, one may obtain the components of \( \gamma \) through real operations only as follows:

Separate the real and imaginary parts of \( [M(\mu_1)] \) and \( \gamma(\mu_1) \):

\[
\begin{bmatrix} M(\mu_1) \end{bmatrix} = [m_1] + i[m_2]
\]

(5.3)

\[
\gamma(\mu_1) = \gamma_1 + i\gamma_2
\]

where \( m_1 \) and \( m_2 \) are real 4 x 4 matrices and \( \gamma_1 \) and \( \gamma_2 \) are real column vectors with 4 components each.
We may then write:

\[ [M(\mu_1)] y(\mu_1) = \left[ \begin{bmatrix} m_1 \\ -m_2 \end{bmatrix} + i \begin{bmatrix} m_2 \\ m_1 \end{bmatrix} \right] \cdot (y_1 + i y_2) \]

\[ = \begin{bmatrix} m_1 \\ -m_2 \end{bmatrix} (y_1, y_2) + i \begin{bmatrix} m_2 \\ m_1 \end{bmatrix} (y_1, y_2) = 0 \quad (5.4) \]

where \((y_1, y_2)\) stands for the column vector with 8 components obtained by simply writing \(y_1\) after \(y_2\).

Separate the real and imaginary parts of Eqn. (5.4) and it is then clearly equivalent to the matrix equation:

\[ \begin{bmatrix} \tilde{M} \end{bmatrix} (y_1, y_2) = 0 \quad (5.5) \]

where \([\tilde{M}]\) is an 8 x 8 matrix made up of \(m_1\) and \(m_2\) as its four quadrants as follows:

\[ \begin{bmatrix} \tilde{M} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} m_1 \\ -m_2 \end{bmatrix} \\ \begin{bmatrix} m_2 \\ m_1 \end{bmatrix} \end{bmatrix} \quad (5.6) \]

Eqn. (5.5) is pure real and hence may be handled in the conventional manner.

One final remark may be in order regarding the actual technique of solving Eqn. (5.5). If \(\mu_1\) is a single root of Eqn. (2.4) we have a theorem that guarantees that \([\tilde{M}]\) has at least one 7 x 7 minor which is non-singular. This does not mean that some of the minors could not have a vanishing determinant. The difficulty in determining if a 7 x 7 minor is singular or not stems from the round-off errors of the numerical solution.
which may cause a determinant that should vanish theoretically to appear to have a non-zero value. Although from physical considerations one tends to feel that no component of the eigenvector should be zero and hence no minor should be singular, an extra precaution against extreme ill-conditioning may be introduced by finding the minor whose determinant has the greatest absolute value and using it to find the eigenvector.

5.2 Particular Solutions

This will prove to be the shortest and least complicated part of the program.

It is suggested that power series terms of the particular solution be read by the program as input values, rather than being generated by it. The reason for this is that, as has been pointed out in Section 2.3, these are far from being unique and the judgements involved in the selection of the most suitable one is not the kind that a computer can efficiently make.

The handling of trigonometric series terms is quite routine. It is suggested that no special arrangements, other than an error exit, be provided for the special and unlikely case when the load happens to be a solution of the homogeneous equations.
5.3 Boundary Conditions

The work to be done here consists of four major parts from the standpoint of the computer.

a) In addition to information received from the preceding parts of the program, we need here the elastic constants and loads that appear in the boundary equations. The point matching method has the flexibility to provide for the coefficients of these equations to vary from point to point and therefore this should be allowed for. At the same time, in most applications these coefficients will have a constant value, at least along one side of the shell. If this is the case, it would be a waste of both key-punching and input time to read in the same number over and over again. It is, therefore, suggested that a switch be set by the program to make it possible to read multiple values of only those coefficients that actually vary.

b) When the boundary equations are formulated both \( w \) and \( \varphi \) will have to be differentiated many times in different directions. All these derivatives may be expressed in terms of derivatives in the \( \xi \) and \( \eta \) directions (See Eqns. 1.98). It is, therefore, suggested that a subroutine be set up to give these derivatives for any point, orientation and function. This subroutine will be entered something like 50,000 times in an average program, therefore every effort must be made to make it very efficient.

c) It is very unlikely that there will be enough room in core for the coefficient matrix \( C \) of Eqn. (3.33). It will, therefore, be necessary to
write it out on some auxiliary storage facility, say a tape. It is then necessary that this matrix be pre-multiplied by its own transpose. This is not quite straight-forward since it cannot be brought into core all at once. Furthermore, it is possible that the matrix $[C^T C]$ which is smaller than $C$, is still too big and must be stored on another tape. We will give an outline of the procedure that may be used.

Assume that there is enough room in core to accommodate $(k+1)$ rows of $C$ and partition $C$ as follows:

$$
[C]_{m \times n} = \begin{bmatrix}
\vdots \\
. & . & C_i & . & . \\
\vdots \\
. & . & C_r & . & . \\
\vdots \\
\end{bmatrix}_{k \leq k}
$$

(5.7)

Then

$$
C^T C = \sum_{i=1}^{r} C_i^T C_i
$$

(5.8)

where each partial product has the same dimensions as the final sum, i.e., $n \times n$ and hence by hypothesis, neither can be stored in core.

We still have available room for 1 row and we shall establish a temporary location into which we will read some row, say the $j$th of the part of $[C^T C]$ already generated. We then add to this the $j$th row of the current
partial product \( C_1^T C_1 \) which is available since \( C_1 \) is in core. Letting \( j \) vary from 1 to \( n \) and repeating this \( r+1 \) times - the first pass is needed to initialize \( C^T C \) - gives the desired product.

d) The solution of Eqn. (3.41) will involve the inversion of a matrix probably well over a 100 x 100. It is assumed that an appropriate matrix inverter subroutine, operating from auxiliary storage, will be available in the machine library.

5.4 Evaluation of Results

The solution, once obtained, will be far too bulky for hand calculation of any result that is needed. Therefore, one should anticipate all quantities that may be of interest and write a subroutine for their evaluation. For similar reasons, the program should punch all of its previous results on cards so that this last part may be entered directly, if needed, with the results of a previous run.

The derivative subroutine, mentioned above, will prove indispensable here, too. A major problem is that the same quantities will be used in evaluating different things but there will not be enough room for their temporary storage. For example, the second derivatives of \( w \) are needed, implicitly or explicitly, for the membrane forces, membrane strains, bending moments, bending strains, overall stresses, evaluation of the approximation in reducing the expression for the change of curvature and
evaluation of the approximation involved in neglecting the square brackets in Eqns. (1.84). Yet, for clarity, one does not wish to print all of these quantities for one point and then again all for the next; one would rather have the membrane forces for the whole shell, then the bending moments for the whole shell, and so on. Thus it will be necessary to evaluate the second derivatives of w as well as a number of other quantities, repeatedly. About all that is problematical in this final part of the program is to find the most preferable order in which to work so as to reduce repetitions as much as possible. This is closely tied in with most efficient use of the storage available since repetitions are primarily due to lack of temporary storage space.

In this respect, it will not be without interest how the tensor quantities that appear can be handled with a minimum amount of temporary storage and repetitions. For definiteness we will use stresses as an illustration here but obviously the same procedure could be used for strains, moments, membrane forces, curvatures.

Assume then that x-y are axes perpendicular to each other and

that $\sigma_{xx}, \sigma_{yy}, \tau_{xy}$ have been found for a number of points and stored in the locations A, B, and C respectively. We wish to print the principal stresses $\sigma_1$ and $\sigma_2$, the angle $\alpha$ between x and the major principal direction and the principal shear stress $\tau_{max}$.
The following procedure uses no additional storage and no reference to anything else but the values already found. Admittedly, it gives the result in an unusual order.

We start with

\[ A = \sigma_{xx} \]
\[ B = \sigma_{yy} \]
\[ C = \tau_{xy} \]

We do some rearranging

\[ \frac{\sigma_{xx} + \sigma_{yy}}{2} = \frac{A + B}{2} \rightarrow A \]
\[ \frac{\sigma_{xx} - \sigma_{yy}}{2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \sigma_{yy} = A - B \rightarrow B \]

Now we find \( \alpha \), store in, and print from, \( C \):

\[ \alpha = \frac{1}{2} \arctan \frac{2 \tau_{xy}}{\sigma_{yy} - \sigma_{xx}} = \frac{1}{2} \arctan \left( \frac{-C}{B} \right) \rightarrow C \]

Next \( \tau_{\text{max}} \) is stored in, and printed from, \( C \):

\[ \tau_{\text{max}} = \frac{\sigma_x - \sigma_y}{2 \cos 2\alpha} = \frac{B}{\cos 2C} \rightarrow C \]

Finally \( \sigma_1 \) and \( \sigma_2 \) are found and printed thus

\[ \sigma_1 = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \tau_{\text{max}} = A + C \rightarrow B \]
\[ \sigma_2 = (\sigma_{xx} + \sigma_{yy}) - \sigma_1 = 2A - B \rightarrow A \]
APPENDIX A

Let $A$ be a complex singular matrix with eigenvector $\xi$ so that

$$\begin{align*}
A\xi &= 0 \\
(A.1)
\end{align*}$$

Prove that then

$$\bar{A}\bar{\xi} = 0$$

(A.2)

Let the real and imaginary parts of $A$ be $A_1$ and $A_2$ and of $\xi$ be $\xi_1$ and $\xi_2$. Then Eqn. (A.1) may be written

$$\begin{align*}
A_1\xi_1 + iA_2\xi_2 + i(A_1\xi_1 + A_2\xi_2) &= 0 \\
(A.3)
\end{align*}$$

and Eqn. (A.2) may be written

$$\begin{align*}
\bar{A}\bar{\xi} &= (A_1 - iA_2)(\xi_1 - i\xi_2) - (A_1\xi_1 - A_2\xi_2) - i(A_1\xi_2 + A_2\xi_1) \\
(A.4)
\end{align*}$$

By virtue of Eqn. (A.3) both real and imaginary parts of the right hand side of Eqn. (A.4) vanish. Q.E.D.
APPENDIX B

Given a system of two partial differential equations in the unknown functions \( X(s, t) \) and \( Y(s, t) \) in the form

\[
\Phi_1(X) + \Gamma_1(Y) = 0
\]

\[
\Phi_2(X) + \Gamma_2(Y) = 0
\]

(B.1)

where the operators \( \Phi_1, \ldots, \Gamma_2 \) are all linear partial differential operators with constant coefficients. We shall investigate the solution of this system under some special conditions to be stated later.

Assume a separable solution in the form

\[
X = e^{As} \cdot x(t)
\]

(B.2)

\[
Y = e^{As} \cdot y(t)
\]

Substitution of (B.2) into (B.1) reduces the system to an ordinary one of the form:

\[
F_r(x) + G_r(y) = 0 \quad (r = 1, 2)
\]

(B.3)

where the operators \( F_r \) and \( G_r \) are of the form

\[
F_r(\cdot) = \sum_{i=0}^{q_{r,r}} f_{r,i} D^i(\cdot)
\]

(B.4)

\[
G_r(\cdot) = \sum_{i=0}^{q_{r,r}} g_{r,i} D^i(\cdot)
\]

with the quantities \( f_{r,i} \) and \( g_{r,i} \) constant.
We define the following algebraic operators

\[ \varphi_{r}(\lambda) = \sum_{i=0}^{q_{r}} f_{r,i} \lambda^{i} \quad (B.5) \]

\[ \varphi_{r}(\lambda) = \sum_{i=0}^{q_{r}} g_{r,i} \lambda^{i} \]

Assume now that the equation

\[ P(\lambda) = \varphi_{1}(\lambda) \varphi_{2}(\lambda) - \varphi_{2}(\lambda) \varphi_{1}(\lambda) = 0 \quad (B.6) \]

has the \( p \)-ple root \( \lambda = \lambda_{1} \), i.e., that

\[ P(\lambda_{1}) = \frac{dP(\lambda_{1})}{d\lambda} = \ldots \frac{d^{(p-1)}P(\lambda_{1})}{d\lambda^{p-1}} = 0 \quad (B.7) \]

We wish to show that then

\[ x = \sum_{k=0}^{p-1} A_{k} t^{k} e^{\lambda_{1}t} \quad (B.8) \]

\[ y = \sum_{k=0}^{p-1} B_{k} t^{k} e^{\lambda_{1}t} \]

are solutions of Eqns. \( (B.3) \) with the \( B_{k} \) being arbitrary and the \( A_{k} \) expressible in terms of the \( B_{k} \) or vice versa.
First consider the operator $F_r(\ )$ operating on a typical term in (B.8):

$$F_r(A_k t^k e^{\lambda_1 t}) = \sum_{i=0}^{q_{1,r}} f_{r,i} \frac{d^i}{dt^i} (A_k t^k e^{\lambda_1 t}) =$$

$$= \sum_{i=0}^{q_{1,r}} f_{r,i} \sum_{j=0}^{k} \binom{i}{j} \lambda_1^{i-j} \frac{k!}{(k-j)!} A_k t^{k-j} e^{\lambda_1 t} =$$

$$= e^{\lambda_1 t} \sum_{j=0}^{k} A_k \sum_{i=0}^{q_{1,r}} \binom{i}{j} \frac{i!}{j!(i-j)!} \frac{k!}{(k-j)!} \lambda_1^{i-j} =$$

$$= e^{\lambda_1 t} \sum_{j=0}^{k} A_k \binom{k}{j} \varphi_r^{(j)}(\lambda_1) t^{k-j} \quad (B.9)$$

where

$$\varphi_r^{(j)}(\lambda_1) = \sum_{i=0}^{q_{1,r}} f_{r,i} \frac{i!}{(i-j)!} \lambda_1^{i-j} \quad (B.10)$$

Note that when $j=0$ is substituted into Eqn. (B.10) it reduces to $\varphi_r^{(0)}$ as defined by Eqn. (B.5) and thus the two definitions are consistent. Also, notice that by definition we have

$$\frac{d^j \varphi_r^{(a)}(\lambda)}{d\lambda^j} = \sum_{i=0}^{q_{1,r}} f_{r,i} \frac{i!}{(i-j)!} \lambda_1^{i-j} = \varphi_r^{(j)}(\lambda) \quad (B.11)$$
In an entirely analogous manner we define

\[ \chi_r^{(j)}(\lambda) = \sum_{i=0}^{q_2,r} g_{r,i} \frac{i!}{(i-j)!} \lambda_{i-j} \]  

(B.12)

and thus, by Eqn. (B.9) we have

\[ G_r(B_k t^k e^{\lambda t}) = e^{\lambda t} \sum_{j=0}^{k} B_k \binom{k}{j} \chi_r^{(j)}(\lambda_i) t^{k-j} \]  

(B.13)

Using Eqn. (B.9) and (B.13) we may now substitute the assumed solution (B.8) into (B.3):

\[ e^{\lambda t} \sum_{k=0}^{p-1} \sum_{j=0}^{k} \binom{k}{j} \left[ A_k \varphi_r^{(j)}(\lambda_i) + B_k \delta_r^{(j)}(\lambda_i) \right] t^{k-j} = 0 \]  

(B.14)

The term \( e^{\lambda t} \) may obviously be divided out of Eqn. (B.14). We are then left with a polynomial of degree \( (p-1) \) in \( t \) – it will vanish for all values of \( t \) only if the coefficient of each power of \( t \) separately vanishes.

We replace \( (k-j) \) by \( j \) and use the identity \( \binom{k}{j} = \binom{k}{k-j} \) in Eqn. (B.14) and then equate each coefficient separately to zero. We obtain

\[ \sum_{k=j}^{p-1} \binom{k}{j} \left[ A_k \varphi_r^{(k-j)}(\lambda_i) + B_k \delta_r^{(k-j)}(\lambda_i) \right] = 0 \]  

(B.15)

\( (j=0,1,\ldots,p-1) \)

The meaning of Eqns. (B.15) can be appreciated better if they are written out in detail. We list them in the order of decreasing \( j \) for both values of \( r \) and, for simplicity, dropping the argument \( \lambda_i \) from \( \varphi \) and \( \delta \):
\[
\begin{bmatrix}
\begin{pmatrix}
(p-1)
\end{pmatrix} \varphi_1^{(0)} & \begin{pmatrix}
(p-1)
\end{pmatrix} \chi_1^{(0)} & 0 & 0 \\
\begin{pmatrix}
(p-1)
\end{pmatrix} \varphi_2^{(0)} & \begin{pmatrix}
(p-1)
\end{pmatrix} \chi_2^{(0)} & 0 & 0 \\
\begin{pmatrix}
(p-2)
\end{pmatrix} \varphi_1^{(1)} & \begin{pmatrix}
(p-2)
\end{pmatrix} \chi_1^{(1)} & \begin{pmatrix}
(p-2)
\end{pmatrix} \varphi_1^{(0)} & \begin{pmatrix}
(p-2)
\end{pmatrix} \chi_1^{(0)} & 0 \\
\begin{pmatrix}
(p-2)
\end{pmatrix} \varphi_2^{(1)} & \begin{pmatrix}
(p-2)
\end{pmatrix} \chi_2^{(1)} & \begin{pmatrix}
(p-2)
\end{pmatrix} \varphi_2^{(0)} & \begin{pmatrix}
(p-2)
\end{pmatrix} \chi_2^{(0)} & 0 \\
\begin{pmatrix}
(p-3)
\end{pmatrix} \varphi_1^{(2)} & \begin{pmatrix}
(p-3)
\end{pmatrix} \chi_1^{(2)} & \begin{pmatrix}
(p-3)
\end{pmatrix} \varphi_1^{(1)} & \begin{pmatrix}
(p-3)
\end{pmatrix} \chi_1^{(1)} & \begin{pmatrix}
(p-3)
\end{pmatrix} \varphi_1^{(0)} & \begin{pmatrix}
(p-3)
\end{pmatrix} \chi_1^{(0)} & 0 \\
\begin{pmatrix}
(p-3)
\end{pmatrix} \varphi_2^{(2)} & \begin{pmatrix}
(p-3)
\end{pmatrix} \chi_2^{(2)} & \begin{pmatrix}
(p-3)
\end{pmatrix} \varphi_2^{(1)} & \begin{pmatrix}
(p-3)
\end{pmatrix} \chi_2^{(1)} & \begin{pmatrix}
(p-3)
\end{pmatrix} \varphi_2^{(0)} & \begin{pmatrix}
(p-3)
\end{pmatrix} \chi_2^{(0)} & 0 \\
\vdots & & & & & \\
\begin{pmatrix}
0
\end{pmatrix} \varphi^{(p-1)} & \begin{pmatrix}
0
\end{pmatrix} \chi^{(p-1)} & \begin{pmatrix}
0
\end{pmatrix} \varphi^{(p-2)} & \begin{pmatrix}
0
\end{pmatrix} \chi^{(p-2)} & 0 & 0 \\
\begin{pmatrix}
0
\end{pmatrix} \varphi_1^{(p-1)} & \begin{pmatrix}
0
\end{pmatrix} \chi_1^{(p-1)} & \begin{pmatrix}
0
\end{pmatrix} \varphi_1^{(p-2)} & \begin{pmatrix}
0
\end{pmatrix} \chi_1^{(p-2)} & 0 & 0 \\
\begin{pmatrix}
0
\end{pmatrix} \varphi_2^{(p-1)} & \begin{pmatrix}
0
\end{pmatrix} \chi_2^{(p-1)} & \begin{pmatrix}
0
\end{pmatrix} \varphi_2^{(p-2)} & \begin{pmatrix}
0
\end{pmatrix} \chi_2^{(p-2)} & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
A_{p-1} \\
B_{p-1} \\
\vdots \\
A_{o} \\
B_{o} \\
\end{bmatrix} = 0
\]
When the determinant of the system (B.16) is broken up into 2 x 2 minors it is seen that along the main diagonal we have the same
minor repeated p times, namely
\[ \begin{bmatrix}
\varphi_{1}^{(0)} & \varphi_{1}^{(0)} \\
\varphi_{2}^{(0)} & \varphi_{2}^{(0)}
\end{bmatrix}
= \varphi_{1}^{(0)}\varphi_{2}^{(0)} - \varphi_{2}^{(0)}\varphi_{1}^{(0)} = P(\lambda_1) \]  
(B.17)

Since the part above this "diagonal of 2 x 2-s" has zero in every position the value of the determinant of system (B.16) is
\[ \Delta = [P(\lambda_1)]^p = 0 \]  
(B.18)
by Eqn. (B.7). Thus, there exists a non-trivial solution.

In what follows we shall assume that not all of the quantities \( \varphi_1^{(0)}(\lambda_1), \varphi_2^{(0)}(\lambda_1), \varphi_1^{(0)}(\lambda_2), \varphi_2^{(0)}(\lambda_2) \) vanish. If they should all vanish the problem becomes trivial because this means that the original equations uncouple and the A-s are independent of the B-s. If not all of them vanish we may, by re-labeling some of the quantities involved, arrange it so that \( \varphi_1^{(0)} \neq 0 \).

Solution of the system of equations (B.16) leads to a recurrence formula. It can be more readily discerned if one starts with solving for \( A_{p-1} \) in terms of \( B_{p-1} \).
The first two equations form a homogeneous system whose determinant is $P(\lambda)$ and thus the two equations are compatible. The solution is

$$A_{p-1} = -\frac{\Phi^{(o)}_r}{\Phi^{(o)}_r} B_{p-1} \quad (r=1,2) \quad (B.19)$$

and the result is unchanged whether $r$ is taken 1 or 2. In Eqn. (B.19) either $A_{p-1}$ or $B_{p-1}$ is arbitrary.

The second two equations, after rearrangement, look like this

$$\Phi^{(o)}_1 A_{p-2} + \chi^{(o)}_1 B_{p-2} = -\left(\begin{array}{c} p-1 \\ p-2 \end{array} \right) \left(\Phi^{(i)}_1 A_{p-1} + \chi^{(i)}_1 B_{p-1} \right) \quad (B.20)$$

$$\Phi^{(o)}_2 A_{p-2} + \chi^{(o)}_2 B_{p-2} = -\left(\begin{array}{c} p-1 \\ p-2 \end{array} \right) \left(\Phi^{(i)}_2 A_{p-1} + \chi^{(i)}_2 B_{p-1} \right)$$

This is an inhomogeneous system, unless $A_{p-1} = B_{p-1} = 0$. We want to avoid having to take this solution but since the rank of the matrix of the coefficients on the left side is 1 - due to Eqn. (B.6) - a solution will exist only if the rank of the augmented matrix is also 1. This is clearly the case if the determinant (B.21) vanishes

$$\begin{vmatrix} \Phi^{(o)}_1 & \left(\Phi^{(i)}_1 A_{p-1} + \chi^{(i)}_1 B_{p-1} \right) \\ \Phi^{(o)}_2 & \left(\Phi^{(i)}_2 A_{p-1} + \chi^{(i)}_2 B_{p-1} \right) \end{vmatrix} \quad (B.21)$$
Substituting Eqn. (B.19) with \( r = 2 \) in the top row and \( r = 1 \) in the bottom row and using Eqn. (B.11) to differentiate the functions \( \varphi \) and \( \gamma \) (B.21) may be reduced to

\[
\begin{bmatrix}
\varphi_1^{(0)} \gamma_2^{(0)} + \varphi_1^{(1)} \gamma_2^{(1)} - (\varphi_1^{(0)} \gamma_1^{(0)} + \varphi_2^{(0)} \gamma_1^{(0)})
\end{bmatrix} B_{p-1} = \frac{dP(\lambda_i)}{d\lambda} B_{p-1} = 0
\]

(B.22)

Eqn. (B.21) is true because - as a consequence of \( \lambda_i \) being a multiple root - in Eqn. (B.7) we established that the derivative multiplying \( B_{p-1} \) vanishes.

Thus, the inhomogeneous system (B.20), even though its matrix is singular, possesses a solution. Expressing \( A_{p-2} \) from either of the two equations we get

\[
A_{p-2} = -\frac{1}{\varphi_1^{(0)}} \left[ (p-1) \left( \frac{\varphi_1^{(0)}}{\varphi_1^{(1)}} - \frac{\varphi_1^{(1)}}{\varphi_1^{(0)}} \right) B_{p-1} + \frac{\varphi_1^{(0)}}{\varphi_1^{(1)}} B_{p-2} \right]
\]

\[
= -\frac{p-1}{p-2} \frac{d}{d\lambda} \left( \frac{\gamma_1^{(0)}}{\varphi_1^{(0)}} \right) B_{p-1} - \frac{\gamma_1^{(0)}}{\varphi_1^{(0)}} B_{p-2}
\]

(B.23)

Carrying this procedure further it can be shown that the condition that the equations from which we wish to solve for \( A_{p-j} \) possess a solution without selecting \( A_{p-1} = B_{p-1} = 0 \) is always that

\[
\frac{d^{(i-1)} P(\lambda_i)}{d\lambda^{i-1}} = 0
\]

(B.24)
which is satisfied by Eqn. (B.7) for all \( j \) such that \( 1 \leq j \leq p \).

The solution itself can be obtained in exactly the same manner as (B.23) was obtained. The procedure gives

\[
A_{p-j} = - \sum_{k=p-j}^{p-1} \binom{k}{p-j} \frac{d^{(k-p+j)} \left( \frac{\varphi f^{(a)}}{\delta f^{(a)}} \right)}{d \lambda^{(k-p+j)}} \bigg|_{\lambda = \lambda_1} B_k \tag{B.25}
\]

Thus, it is seen that each \( A \) is a function of not only the \( B \) with the same subscript, but all other \( B \)-s bearing a higher subscript.
BIBLIOGRAPHY

(This list includes only those items to which actual reference is made in the text. A rather exhaustive bibliography of the field has been compiled by Nash (16), (17).)


I, Peter E. Korda, was born on December 5, 1931 in Budapest, Hungary where I also received my basic education. I received the degree Dipl. Eng. from the Technical University of Budapest in 1954 with a major in Structural Engineering.

From 1954 to 1956 I worked for a structural design office in Budapest, mostly on reinforced concrete industrial buildings. In the fall of 1956, I fled from Hungary for political reasons and went to London, England. I spent about a half of a year there working for a structural consulting office.

From 1957 till 1960 I worked for the Dominion Bridge Company in Montreal, Canada. I held various positions in their structural design department. My last position before I left was that of Assistant Project Designer.

In 1960 I came to The Ohio State University to pursue graduate studies. I first started in the Department of Civil Engineering, later transferred to Engineering Mechanics. I have held various research and teaching positions at the University, and am currently an instructor in the Department of Engineering Mechanics.