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For \( k = 2 \), \( D_{\mathcal{F}}(k) = D(k) \) and for \( k = 3 \), \( \Theta_{\mathcal{F}} \) is known to be \( \leq \frac{5}{2} \). Hardy's paper [8] in 1916 also included a proof that

\[
\Theta_{\mathcal{F}} \geq \frac{(k-1)/2}{k}
\]

for each \( k \geq 2 \). Littlewood [16] showed in 1912 that the Riemann conjecture that the \( \zeta \) function has no zeros for \( \Re(s) > \frac{1}{2} \) implies \( \Theta_{\mathcal{F}} \leq \frac{1}{2} \) for each \( k \geq 2 \). This led Hardy to conjecture that \( \Theta_{\mathcal{F}} = \frac{(k-1)/2}{k} \). This conjecture is still unsettled.

In 1956, Erdős and Fuchs [4] made a striking generalization of Hardy's theorem (7). They showed that if

\[
\alpha, \alpha_2, \ldots, \alpha_m, \ldots
\]

is an infinite sequence of integers such that

\[ 0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_m \leq \ldots \]

and if \( f(m) \) denotes the number of solutions of

\[ \alpha_i + \alpha_j = m \]

then

\[
\tau(m) = \sum_{m \leq x} f(m) = \alpha x + \beta x^\alpha + \gamma (x^{1/4} (\log x)^{1/2})
\]

\( c > 0, 0 < \alpha < 1 \) cannot hold. This result yields a slightly weaker result than (7) if we let \( \{\alpha_m\} \) be the sequence of squares. In an unpublished paper, Tull has shown that if the term \( \alpha x \) in (10) is replaced with a more general term, a similar theorem may still be
Theorem 1 [19, page 43]: $\sigma_\varphi(\mathcal{X})$ is a convex function on $\mathcal{X}$.

Theorem 2 [19, page 43]: $\sigma_\varphi(\mathcal{X})$ is a continuous function on $\mathcal{X}$.

The proofs of Theorems 1 and 2 occupy the rest of this section.

Lemma 1 [19, page 33]: For $\lambda + \rho > 0$, $0 < \rho < 1$, $\sigma_\varphi(\lambda + \rho)$ is a convex function of $\varphi$. Thus, $\sigma_\varphi(\lambda + \rho)$ is a continuous function of $\varphi$ for $\lambda + \rho > 0$, $0 < \varphi < 1$.

Proof: Let $\lambda + \rho > 0$, $0 < \varphi$, $0 < \varphi_2 < 1$. From Hölder's Inequality and (5.12)

$$
\left( \int_0^\infty |\theta_{\lambda}(u)|^{\rho \varphi} du \right)^{\rho} \leq \left( \int_0^\infty |\theta_{\lambda}(u)|^{\rho \varphi} du \right)^{\rho} \cdot \left( \int_0^\infty |\phi_{\lambda}(u)|^{\rho \varphi_2} du \right)^{\rho \varphi_2}
$$

$$
= \Theta\left( e^{\omega \rho} \frac{\rho \varphi - \rho}{\rho \varphi_2 - \rho} \sigma_{\varphi_1}(\lambda + \rho) + \frac{\rho \varphi - \rho}{\rho \varphi_2 - \rho} \sigma_{\varphi_2}(\lambda + \rho) + \varepsilon^2 \right)
$$

and the lemma follows from another application of (5.12).

Lemma 2 [19, page 36]: Let $0 < \varphi_1, \varphi_2 < 1$, $\frac{\varphi_1 - \varphi_2}{2} < \lambda < \frac{\varphi_1 - \varphi_2}{2}$.

Proof: Let $0 < \varphi_1, \varphi_2 < 1$, $\varphi = (\varphi_1 + \varphi_2)/2$, $\varphi = (\varphi_1 - \varphi_2)/2$, $\lambda < \varphi + \frac{\varphi_2}{2}$.

$\lambda - \lambda > 0$, $\lambda + \lambda > 0$, and let $\delta$ be a sufficiently small positive number.

From Theorem 3.1, with $\gamma > \nu$, since $\lambda - \varphi + \rho = \lambda - \varphi$.
by Lemma 7.2 with \( \lambda = \lambda - \bar{\sigma} - 1 \). We note that we used here only that \( \bar{\sigma} < \lambda < \bar{\sigma} + 1 \).

From (1) and Theorem 3.1,

\[
(2) = \frac{\Gamma(\lambda+\lambda-\bar{\sigma}+1)}{\Gamma(\lambda+\bar{\sigma}+1) \Gamma(\lambda-\bar{\sigma})} \int_0^\mu B_{\lambda-\bar{\sigma}}(\xi) (\xi-\eta)^{\lambda-\bar{\sigma}-1} d\eta.
\]

In a manner similar to that in which we derived (2), we obtain

\[
(3) = \frac{\Gamma(\lambda+\lambda-\bar{\sigma}+1)}{\Gamma(\lambda-\bar{\sigma}) \Gamma(2\lambda-2\bar{\sigma})} \int_0^\mu B_{\lambda-\bar{\sigma}-1}(\xi) (\xi-\eta)^{2\lambda-2-\bar{\sigma}-1} d\eta.
\]

We will also need the following estimate. Applying
Hölder's Inequality with $\kappa = (\lambda - \bar{\nu}) (1 - \delta)$ and $\gamma \leq m < \zeta$,

$$\int_u^n (n - y)^{\lambda - \bar{\nu} - 2} \, dn \leq \left( \int_u^n (n - y)^{\frac{(\lambda - \bar{\nu} - 2)}{1 - \delta}} \, dn \right)^{-\frac{1}{\delta}}$$

(4)

Now, for $u < \nu < \zeta$, applying Theorem 3.1 to $B_{x-P}$ and to $B_{x-P}(u)$ with $\kappa - \nu = (\lambda - \bar{\nu}) + (\lambda - \bar{\nu})$ and splitting the integral for the former, we have

(5) $B_{x-P}(\omega) = B_{x-P}(u) + \frac{\Gamma(\nu + \mu)}{\Gamma(\lambda - \nu + \mu) \Gamma(\lambda - \bar{\nu})} \int_u^n B_{x-P}(y) \int_u^y (y - x)^{\lambda - \bar{\nu} - 2} \, dx \, dy$

$$+ \frac{\Gamma(\nu + \mu)}{\Gamma(\lambda - \nu + \mu) \Gamma(\lambda - \bar{\nu})} \int_u^n B_{x-P}(y) \int_y^\zeta (y - x)^{\lambda - \bar{\nu} - 1} \, dx \, dy.$$

Integrating (5), from Lemma 3.1, (4), and (3), we have uniformly for $\gamma \geq u$

$$\int_u^n B_{x-P}(\omega) (\omega - \nu)^{\lambda - \bar{\nu} - 1} \, d\omega = \frac{B_{x-P}(u)}{\lambda - \bar{\nu}} (\gamma - u)^{\lambda - \bar{\nu}}$$

$$+ O\left( \int_u^n \int \left[ B_{x-P}(y) \int_u^y (y - x)^{\lambda - \bar{\nu} - 2} \, dx \, dy \right] (\gamma - \nu)^{\lambda - \bar{\nu} - 1} \, d\omega \right)$$

$$+ \frac{\Gamma(\nu + \mu)}{\Gamma(\lambda - \nu + \mu) \Gamma(\lambda - \bar{\nu})} \int_u^n B_{x-P}(y) \int_y^\zeta (y - x)^{\lambda - \bar{\nu} - 1} (\gamma - \nu)^{\lambda - \bar{\nu} - 1} \, dx \, dy \, d\omega \, dy.$$
\[ = \frac{\Theta_{\lambda-\delta}(\mu)}{\lambda-\delta} (\zeta-u)^{\lambda-\delta} \]

+ \( O(\zeta^2/\frac{\zeta-u}{\lambda-\delta}) \int_0^1 \Theta_{\lambda-\delta}(\varphi) \left( \frac{\zeta-u}{\lambda-\delta} \right)^{\lambda-\delta} \zeta^{\lambda-1} (\zeta-u) d\varphi \) d\zeta

+ \frac{\Gamma(\lambda-\delta)}{\Gamma(\lambda-\delta-1) \Gamma(2\lambda-2\delta)} \int_0^\infty \Theta_{\lambda-\delta}(\varphi) (\zeta-u)^{\lambda-2\delta-1} d\varphi

(6) 

\[ = \frac{\Theta_{\lambda-\delta}(\mu)}{\lambda-\delta} (\zeta-u)^{\lambda-\delta} \]

+ \( O((\zeta-u)^{\lambda-\delta} (u-w)^{\lambda-\delta} (u-w)^{\lambda-\delta}) \int_0^1 \Theta_{\lambda-\delta}(\varphi) \left( \frac{\zeta-u}{\lambda-\delta} \right)^{\lambda-\delta} (\zeta-u) d\varphi \) d\zeta

From our assumptions, we have, uniformly for \( \zeta \geq u \),

\[ (\zeta-u)^{\lambda-\delta} \int_0^\infty \Theta_{\lambda-\delta}(\varphi) (\zeta-u)^{\lambda-\delta} (u-w)^{\lambda-\delta} d\varphi \]

\[ = O((\zeta-u)^{\lambda-\delta} \int_0^1 \Theta_{\lambda-\delta}(\varphi) \left( \frac{\zeta-u}{\lambda-\delta} \right)^{\lambda-\delta} (u-w)^{\lambda-\delta} d\varphi) \]

(7) 

and similarly,

\[ (\zeta-u)^{2\lambda-2\delta} \int_0^\infty \Theta_{\lambda-\delta}(\varphi) (\zeta-u)^{\lambda-\delta} (u-w)^{\lambda-\delta} d\varphi \]

\[ = O((\zeta-u)^{\lambda-\delta} \int_0^1 \Theta_{\lambda-\delta}(\varphi) \left( \frac{\zeta-u}{\lambda-\delta} \right)^{\lambda-\delta} (u-w)^{\lambda-\delta} d\varphi) \]

(8)
Substituting (2) into (6) and applying (7) and (8), we get

\[ (9) \quad B_{\lambda-\rho}(u) = \Theta\left( \int_0^u |B_{\lambda-\rho}(y)| (\lambda-\rho)^{-1} \, dy \right) \]

\[ + \Theta\left( \int_0^u |B_{\lambda-\rho}(y)| (\lambda-\rho)^{-1} \, dy \right) \cdot \frac{u}{\lambda-\rho} \]

If we call the two integrals appearing here \( J_1(u) \) and \( J_2(u) \), we see that \( J_1(u) \) and \( J_2(u) \) are less than \( \infty \), and as in the proof of (5.4), we have that \( J_1(u), J_2(u) > 0 \) for \( u = u_0 > 0 \). For such \( u \), choose \( \gamma = u \) so that

\[ \gamma - u = \left( \frac{J_2(u)}{J_1(u)} \right)^{1/(\lambda - 2a)} > 0. \]

From (9) we get

\[ (10) \quad B_{\lambda-\rho}(u) \]

\[ = \Theta\left( \int_0^u |B_{\lambda-\rho}(y)| (\lambda-\rho)^{-1} \, dy \right) \cdot \frac{u}{\lambda-\rho} \]

\[ \text{for } u = u_0. \]

Further, from Hölder's Inequality with \( \alpha = \frac{\gamma}{2} \), we get

\[ \int_0^u |B_{\lambda-\rho}(u)|^{1/\gamma} \, du = \Theta(1) \]

\[ + \Theta\left( \int_0^u \left( \int_0^u |B_{\lambda-\rho}(y)| (\lambda-\rho)^{-1} \, dy \right)^{\gamma/2} \, du \right) \]

\[ \left( \int_0^u \left( \int_0^u |B_{\lambda-\rho}(y)| (\lambda-\rho)^{-1} \, dy \right)^{\gamma/2} \, du \right)^{\gamma/\gamma} \]

and thus, from Lemma 3.2 and (5.12)
Hence, from (5.12)

\[(11) \quad \sigma_{\frac{c_1}{l+\delta}} (x) \leq \frac{1+\delta}{2} \sigma_{\frac{c_1}{l+\delta}} (x-\delta - \frac{c_1}{l+\delta}) + \frac{1-\delta}{2} \sigma_{\frac{c_1}{l+\delta}} (x+\delta - \frac{c_1}{l+\delta})\]

for \(0 < c_1, c_2 < 1, \quad \frac{c_2-c_1}{2} < \lambda < \frac{c_2-c_1}{2} + \frac{1}{2}\), and \(\delta\) sufficiently small and \(> 0\).

By Lemma 1,

\[\sigma_{\frac{c_1}{l+\delta}} (x-\lambda - \frac{\delta c_1}{l+\delta}) = \sigma_{\frac{c_1}{l+\delta}} (x-\lambda - \frac{c_1}{l+\delta})\]

and

\[\sigma_{\frac{c_2}{l+\delta}} (x+\lambda + \frac{\delta c_2}{l+\delta}) = \sigma_{\frac{c_2}{l+\delta}} (x+\lambda - \frac{c_2}{l+\delta})\]

are continuous functions of \(c_1/(l+\delta)\), and \(c_2/(l+\delta)\) respectively, for \(0/(l+\delta), c_2/(l+\delta) \in (0,1)\) and \(\delta\) sufficiently small. Thus, they are continuous functions of \(\delta\) for \(\delta > 0\) and \(\delta\) sufficiently small. Thus, taking the limit of (11) as \(\delta \to 0^+\), we have our lemma for \((c_2-c_1)/2 - \lambda < (c_2-c_1 + 1)/2\).
If $(r_2 - r_1)/2 = \lambda$, then Lemma 2 reduces to Lemma 1.

Lemma 3 [19, page 37]: $\sigma\epsilon(\alpha)$ is convex on the interior of $\mathcal{H}$.  

Proof: Let $(\rho, \alpha), (\rho', \alpha')$ be two points in the interior of $\mathcal{H}$. Set $\psi(\alpha) = \sigma_{\rho + u} (\alpha - \alpha') (\alpha + u (\alpha' - \alpha))$ for $u \in (\alpha, \beta)$ where $(\alpha, \beta)$ denotes an interval such that $u \in (\alpha, \beta)$ implies $(\rho + u (\alpha' - \alpha), \alpha + u (\alpha' - \alpha))$ is in the interior of $\mathcal{H}$.

Suppose that for each choice of $\alpha, \rho, \alpha', \rho', \alpha$ and $\beta$ as above,

$$\psi(\alpha \epsilon u_{1} + (1 - \alpha \epsilon) u_{2}) \leq \alpha \epsilon \psi(u_{1}) + (1 - \alpha \epsilon) \psi(u_{2})$$

holds for $\alpha \epsilon u_{1} \epsilon u_{2} \epsilon \beta$, $0 \leq \alpha \epsilon \leq 1$. Then it follows that $\sigma\epsilon(\alpha)$ is convex on the interior of $\mathcal{H}$.

If $|u_{1} - u_{2}|$ is sufficiently small, then Lemma 2 implies

$$\psi\left(\frac{u_{1} + u_{2}}{2}\right) = \sigma_{\rho + u_{2}} \left(\frac{\alpha_{1} + \alpha_{2}}{2}\right) \leq \frac{\sigma_{\rho}(u_{1}) + \sigma_{\rho}(u_{2})}{2} = \frac{\psi(u_{1}) + \psi(u_{2})}{2}.$$  

Now, since $\sigma\epsilon(\alpha)$ is bounded by $\alpha_{1}$, it follows from [10, Theorem 111] that $\psi(\alpha)$ is continuous in $\alpha \epsilon u \epsilon \beta$, and so, with [10, Theorem 110], the restriction on the $u$ values in (13) may be removed. Thus (12) holds from continuity and the lemma follows.
Lemma 4 [19, page 41]: Let \( 0 < c_1, p_2 < 1 \), \( 0 < \omega < 1 \), 
\( \sigma \in (-\infty) \cup (-\omega) \), \( \omega < p_2 \). Then

\[
\sigma c_1 + (1 - \omega) p_2 \left( \sigma c_1 + (1 - \omega) \omega \right) \leq \sigma c_1 (1 - c_1) + (1 - \omega) \sigma c_2 (\omega).
\]

Proof: We list first the hypotheses under which we shall prove the theorem and some of the immediate consequences thereof.

(14) \( 0 < c_1, p_2 < 1 \), 
\( 0 < \omega < 1 \), 
\( \sigma \in (-\infty) \cup (-\omega) \), \( \omega < p_2 \), 
where \( \sigma \) is an integer \( \geq M \), 
\( \log \frac{1}{j} = m j < u < m j + 1 \), \( \log (e^{u - 1}) = f(u) \), 
\( u < \tau \), \( j > 0 \) and sufficiently small,

\( \rho = \sigma c_1 + (1 - \omega) p_2 \), \( \lambda' = \sigma c_1 (1 - c_1) + (1 - \omega) \omega \), \( \lambda = \lambda' + e - c_2 \).

From these assumptions we easily see that

\( 0 > \lambda - p_2 > -1 \)
\( 1 > \lambda = \sigma c_1 (1 + \lambda - p_2) > 0 \)
\( 0 > \lambda - p = -\omega + (1 - \omega) (\lambda - c_2) > -1 \)
\( \lambda > \lambda' > 0 \).

Now, for \( u + \omega < \tau \), we have

\[
\mathcal{H}_{\lambda, \rho}(\omega) = \mathcal{H}_{\lambda, \rho}(\omega) + \int_{\omega}^{\infty} \mathcal{H}(u, \omega) \, du - \int_{\omega}^{\infty} \mathcal{H}(u, \omega) \, du,
\]
Thus,

\[ B_{\frac{\lambda^p}{\lambda}}(u) = B_{\frac{\lambda^p}{\lambda}}(u) + \int_{u}^{\nu} (\nu - \eta) \frac{\lambda^p}{\lambda} \, d\eta + \int_{0}^{\epsilon} (\eta - u) \frac{\lambda^p}{\lambda} \, d\eta + \int_{\epsilon}^{u} (\nu - u) \frac{\lambda^p}{\lambda} \, d\eta, \quad (\nu > u). \]

Thus,

\[
(15) \int_{u}^{\nu} B_{\frac{\lambda^p}{\lambda}}(\nu) (\nu - \nu)^{-1} \, d\nu = B_{\frac{\lambda^p}{\lambda}}(u) \int_{u}^{\nu} (\nu - \nu)^{-1} \, d\nu + \int_{u}^{\nu} \left( \int_{u}^{\nu} (\nu - \eta) \frac{\lambda^p}{\lambda} \, d\eta \right) (\nu - \nu)^{-1} \, d\nu
\]

\[
+ \Theta \left( \int_{u}^{\nu} \left( \int_{0}^{\epsilon} (\eta - u) \frac{\lambda^p}{\lambda} \, d\eta \right) \frac{\lambda^p}{\lambda} \, d\nu \right) (\nu - \nu)^{-1} \, d\nu
\]

\[ = E_1 + E_2 + E_3 + E_4. \]

First,

\[ E_1 = B_{\frac{\lambda^p}{\lambda}}(u) (\nu - u)^{-1}. \]

Consider \( E_2 \). Here again we may change the order of integration since the discontinuities of the integrand are of measure zero with respect to the product measure, and by Lemma 3.1,
\[ E_2 = \int_\sigma^\tau \int_n^{(\tau - n)^{1 - \alpha}} (-n)^{1 - \alpha} d\sigma d\Theta(n) \]

\[ = \frac{\Gamma((-n)^{1 - \alpha}) \Gamma(\alpha)}{\Gamma((-n)^{1 - \alpha} + \alpha)} \int_n^{\tau} (-n)^{-\alpha \tau + \alpha - 1} d\Theta(n) \]

\[ = \frac{\Gamma((-n)^{1 - \alpha}) \Gamma(\alpha)}{\Gamma((-n)^{1 - \alpha} + \alpha)} \int_n^{\tau} (-n)^{-\alpha \tau + \alpha - 1} d\Theta(n) . \]

Next, we consider \( E_3 \). Since \( \int_u^{(\tau - n)(\tau - n)} (-n)^{-\alpha} dy \)

is a bounded continuous function in the region of integration, we may change the order of the outside two integrals and

\[ E_3 = O\left( \int_0^{(\tau - n)^{1 - \alpha}} \left( \int_n^{\tau} (-n)^{-\alpha \tau + \alpha - 1} d\sigma \right) d\Theta(n) \right) . \]

By Hölder's Inequality with \( \alpha = 1 + \tau - \epsilon - \delta > 0 \), we have

\[ \int_n^{\tau} (-n)^{-\alpha \tau + \alpha - 1} d\sigma \leq \left( \int_n^{\tau} (-n)^{1 - \alpha} d\sigma \right)^{\tau - \alpha} \left( \int_n^{\tau} (-n)^{1 - \alpha} d\sigma \right)^{1 - \alpha} \]

\[ = O\left( (\tau - n)^{\alpha} (\tau - n)^{1 - \alpha} + (1 - \alpha) \right) \]

\[ = O\left( (\tau - n)^{1 + \tau - \epsilon - \delta} (u - n)^{\delta - 1} \right) . \]
established by the same type of proof. His result involves slowly oscillating functions. A positive real valued continuous function \( L(x) \), defined for \( x > x_0 \), is called slowly oscillating in case

\[
\lim_{x \to \infty} \frac{L(cx)}{L(x)} = 1
\]

for each \( c > 0 \). \( L(x) \) is characterized by the form

\[
L(x) = a \exp \int_{x_0}^{x} e^{-t} \, dt
\]

where \( a > 0 \), \( a(x) \to 1 \) and \( s(x) \to 0 \) as \( x \to \infty \) \([13]\).

Tull considered \( L(x) \) of the special form

\[
L(x) = a \exp \int_{x_0}^{x} e^{-t} s(t) \, dt
\]

He further required that \( L \) have two derivatives and that \( L''(x) \) be monotone for \( x > x_1 \). This class of functions, although restricted, includes all powers of \( \log x \) and \( \log \log x \) and many other common functions. The theorem mentioned is that

\[
\forall(x) = xL(x) + d \times x + O(x^{\gamma - \delta}) \quad 0 < \alpha < 1, \delta > 0
\]
cannot hold.

The question of whether or not a theorem concerning Dirichlet series similar to that of Erdös and Fuchs is true remained open for several years until Richert \([20]\) published in 1961 a proof of the following theorem:

Let \( k \) be an integer greater than 1 and
Hence,

$$E_3 = \Theta((\tau-u)^{\lambda'-p-\delta} \sum_{\l}^\infty (\tau-\eta)^{-\delta-1} \sum_{\l}^\infty (\tau-\eta)^{-\delta-1} d\eta \ dV(0, \eta))$$

$$= \Theta((\tau-u)^{\lambda'-p-\delta+\lambda} \sum_{\l}^\infty (\tau-\eta)^{-\delta-1} dV(0, \eta))$$

$$= \Theta((\tau-u)^{\lambda'-p-\delta-\lambda} \sum_{\l}^\infty (\tau-\eta)^{-\delta-1} dV(0, \eta)).$$

Finally,

$$E_4 = \Theta((\tau-u)^{\lambda} \sum_{\l}^\infty (\tau-\eta)^{-\delta} dV(0, \eta)).$$

Now, applying these calculations to (15), we have

$$(16) \sum_{\l}^\infty \Theta(x-\eta)(\tau-\eta)^{-\delta-1} d\eta = \Theta^{\lambda'-p}(u)(\tau-u)^{\lambda}/\lambda$$

$$+ \frac{\Gamma(\lambda'-p+1)}{\Gamma(\lambda'-p+1)} \sum_{\l}^\infty (\tau-\eta)^{-\delta-1} d\eta(\tau-\eta)^{-\delta-1} dV(0, \eta)$$

$$+ \Theta((\tau-u)^{\lambda'-p-\delta} \sum_{\l}^\infty (\tau-\eta)^{-\delta-1} dV(0, \eta))$$

$$+ \Theta((\tau-u)^{\lambda} \sum_{\l}^\infty (\tau-\eta)^{-\delta} dV(0, \eta)).$$

Now, from (7.2),

$$(17) \sum_{\l}^\infty (\tau-\eta)^{-\delta-1} d\eta(\tau-\eta)^{-\delta-1} dV(0, \eta) = \Theta^{\lambda'-p}(u)(\tau)$$

$$- \frac{(\tau-u)^{\lambda'-p+1}}{\Gamma(\lambda'-p+1)} \sum_{\l}^\infty \Theta(x-\eta)(\tau-\eta)^{\lambda'-p} d\eta(\tau-\eta)^{-\delta-1} dV(0, \eta).$$

We also have uniformly for $\tau \geq u$,,
Similarly,

\begin{equation}
(19) \quad (\tau - u)^{\lambda - e_2 + 1} \int_0^\mu B_{\lambda - e_2}(\nu) (\tau - \nu)^{-1} (u - \nu)^{\lambda - e_2 + 1} d\nu
= O((\tau - u)^{\delta} \int_0^\mu B_{\lambda - e_2}(\nu) (u - \nu)^{\delta - 1} d\nu),
\end{equation}

We note again that in (2) we used only that 
\( \bar{\delta} \leq \lambda - \bar{e} + 1 \), and then from (2) with \( \bar{e} = e_2 = \bar{e} \)
\( x = \bar{x} \), \( \lambda = \lambda' \), we obtain

\[ \frac{\Gamma(\lambda' - \lambda' - e + 1)}{\Gamma(\lambda' - e + 1) \Gamma(\lambda)} \int_0^\nu B_{\lambda' - e}(\nu) (\tau - \nu)^{-1} d\nu \]

\[ = B_{\lambda' - e}(\bar{\nu}) - \frac{(\tau - u)^{\lambda}}{\Gamma(\lambda) \Gamma(\lambda - \lambda')} \int_0^\mu B_{\lambda' - e - \lambda}(\nu) (\tau - \nu)^{-1} (u - \nu)^{-\lambda} d\nu. \]

or

\begin{equation}
(20) \quad \frac{\Gamma(\lambda' - e_2 + 1)}{\Gamma(\lambda' - e_2 + 1) \Gamma(\lambda)} \int_0^\nu B_{\lambda' - e_2}(\nu) (\tau - \nu)^{-1} d\nu
= B_{\lambda - e_2}(\bar{\nu}) - \frac{(\tau - u)^{\lambda}}{\Gamma(\lambda) \Gamma(\lambda - \lambda')} \int_0^\mu B_{\lambda - e_2}(\nu) (\tau - \nu)^{-1} (u - \nu)^{-\lambda} d\nu. \]

We have, from (16) and (17),
Solving (20) for
\[ \int_u^x \Theta_{x'} (\omega) (\omega - u)^{\lambda - 1} \, d\omega = \int_u^x \Theta_{x'} (\omega) (\omega - u)^{\lambda - 1} \, d\omega \]
and substituting in (21), we obtain
\[ \Theta_{x'} (\omega) (\omega - u)^{\lambda - 1} = \Theta((\omega - u)^{1 + \lambda - p} \int_0^{\frac{\omega}{\lambda}} (\omega - \eta)^{\frac{\lambda - 1}{\lambda}} \, d\nu \Theta (\omega, \nu)) \]
\[ + \Theta((\omega - u)^{1 + \lambda - p} \int_0^{\frac{\omega}{\lambda}} (\omega - \eta)^{\frac{\lambda - 1}{\lambda}} \, d\nu \Theta (\omega, \nu)) \]
\[ + \Theta((\omega - u)^{1 + \lambda - p} \int_0^{\frac{\omega}{\lambda}} (\omega - \eta)^{\frac{\lambda - 1}{\lambda}} \, d\nu \Theta (\omega, \nu)), \]
In the last term we used estimates (18) and (19). Thus,
\[ \Theta_{x'} (\omega) = \Theta((\omega - u)^{1 + \lambda - p} \int_0^{\frac{\omega}{\lambda}} (\omega - \eta)^{\frac{\lambda - 1}{\lambda}} \, d\nu \Theta (\omega, \nu)) \]
\[ + \Theta((\omega - u)^{-1} \int_0^{\frac{\omega}{\lambda}} (\omega - \eta)^{\frac{\lambda - 1}{\lambda}} \, d\nu \Theta (\omega, \nu)) \]
\[ + \Theta((\omega - u)^{-1} \int_0^{\frac{\omega}{\lambda}} (\omega - \eta)^{\frac{\lambda - 1}{\lambda}} \, d\nu \Theta (\omega, \nu)) \]
\[ + \Theta((\omega - u)^{-1} \int_0^{\frac{\omega}{\lambda}} (\omega - \eta)^{\frac{\lambda - 1}{\lambda}} \, d\nu \Theta (\omega, \nu)). \]
We call the first two integrals appearing here $J_1(u)$ and $J_2(u)$, respectively. Then, $J_1(u)$, $J_2(u)$ are $<\infty$, and $J_1(u)$, $J_2(u) > 0$ for $j \geq j_0 \geq M$. We choose $\gamma > u$ so that $(\gamma - u) : (J_2(u) / J_1(u)) \sum_{k=0}^{+\infty} r_k > 0$, that is, so that the first two "$O$" terms above are equal. We then obtain for $j_0 \leq j \leq e^u < j + 1$

$$\Theta_{\lambda - \phi}(u)$$

$$= \Theta \left( \int_{(u-n)^{+\infty}}^{(u-n)^{\lambda - \phi}} \left( \int_{(u-n)^{\lambda - \phi}}^{(u-n)^{+\infty}} \left( \int_{(u-n)^{\lambda - \phi}}^{(u-n)^{+\infty}} \right) \right) \right)$$

Therefore, from (2.5.4)

$$\Theta_{\lambda - \phi}(u)$$

$$= \Theta \left( \int_{(u-n)^{+\infty}}^{(u-n)^{\lambda - \phi}} \left( \int_{(u-n)^{\lambda - \phi}}^{(u-n)^{+\infty}} \left( \int_{(u-n)^{\lambda - \phi}}^{(u-n)^{+\infty}} \right) \right) \right)$$

We denote the first "$O$" term by

$$\left\{ J_1^{\lambda - \phi} + J_2^{\lambda - \phi} \right\} J_3^{\lambda - \phi}.$$
Using (2.5.4) again and integrating, we have

\[
\int_{m+1}^{m+2} |B_{d-\rho}(u)| \frac{1}{e} \, du
\]

\[
= \mathcal{O}(\int_{m+1}^{m+2} \frac{|f(u)|}{|u-m+\frac{1}{2}|} \, du)
\]

\[
+ \mathcal{O}( \int_{m+1}^{m+2} (u - m)^{\frac{\sigma}{2}} \, du)
\]

\[
+ \mathcal{O}(1_{m+1}^{\frac{\sigma}{2}} \int_{m+1}^{m+2} (u - m)^{\frac{\sigma}{2}} \, du)
\]

\[
= E_1 + E_2 + E_3.
\]

First,

\[
E_2 = \mathcal{O}( (a_2)^{\frac{1}{\sigma}} (m+1 - m)^{\frac{\sigma}{2}}) = \mathcal{O}( (a_2)^{\frac{1}{\sigma}} (m)^{\frac{\sigma}{2}}).
\]

Considering \( E_2 \),

\[
\int_{m+1}^{m+2} (u - m)^{\frac{\sigma}{2}} \, du = \int_{m+1}^{m+2} (u - m)^{\frac{\sigma}{2}} \, du
\]

\[
= \mathcal{O}(e^{u(1+\epsilon\sigma)} \int_{m+1}^{m+2} (u - m)^{\frac{\sigma}{2}} \, du)
\]

\[
= \mathcal{O}(e^{u(1+\epsilon\sigma)} (u - m)^{\frac{\sigma}{2}})
\]

\[
= \mathcal{O}(e^{u(-\frac{1}{2} + \epsilon\sigma)}).
\]
Therefore,

\[ E_2 = O \left( S_m^{m+1} \left( \int_{u=0}^u (u-n)^{L_\alpha} \, d(e^nL(n))^{1/2} \, du \right) \right) 
= O \left( S_m^{m+1} e^{\int_{u=0}^u \left( 1 - \frac{\mu}{2} + \frac{1}{2} \epsilon \right) \, du \right) \right). \]

We now apply these estimates to (22), sum over all \( j \in \mathcal{E}^w \) and apply Hölter's Inequality with

\[ \alpha = \frac{\mu}{2} \left( \frac{x}{n-x} \right) = \frac{c}{1+c-n} \left( \frac{\mu(1+x-x_2)+\delta}{1-x-x_2} \right) \leq 1. \]

to obtain:

\[ \sum_{j=0}^\infty \left| \theta_{n+1} (u) \right|^{1/2} du = O(1) \]

\[ + \Theta \left( \left( \int_{u=0}^x \frac{\log (e^{u/\alpha})}{x^{1-x_2}} + \int_{u=0}^x \frac{x-x_2}{x-x_2} \int_{u=0}^x \frac{x-x_2}{x-x_2} du \right) \left( \int_{u=0}^x \frac{\log (e^{u/\alpha})}{x^{1-x_2}} + \int_{u=0}^x \frac{x-x_2}{x-x_2} du \right) \right) \]

\[ + \Theta \left( \sum_{j=0}^\infty \left( \int_{u=0}^x \frac{e^{u/\alpha}}{x^{1-x_2}} + \int_{u=0}^x \frac{x-x_2}{x-x_2} du \right) \right) \]

\[ + \Theta \left( \int_{u=0}^x e^{u/\alpha} \int_{u=0}^x \frac{x-x_2}{x-x_2} du \right) \]

\[ = O(1) + E_1^* + E_2^* + E_3^* \]

The first factor of \( E_1^* \), is, by (2.5.4),

\[ \Theta \left( \left( \int_{u=0}^x \frac{\log (e^{u/\alpha})}{x^{1-x_2}} + \int_{u=0}^x \frac{x-x_2}{x-x_2} du \right) \right). \]

Now,
\[
\mathcal{F}_1 = \sum_{n=1}^{\ell(u)} \delta(e^n) \mathcal{F}_n = \sum_{n=1}^{\ell(u)} e^n \mathcal{F}_n (1 + \delta(e^n) (u - n)^{\delta - 1} \, dn \\
= \Theta \left( \frac{u^{a_1 \delta}}{a_1} \int_0^{\ell(u)} (u - n)^{\delta - 1} \, dn \right) \\
= \Theta \left( e^{u (c_1 + \varepsilon)} \right),
\]
and thus,

\[
(24) \quad \int_0^{\ell(u+1)} \mathcal{F}_1^{1/\alpha} \, du = \Theta \left( e^{u (c_1 + \varepsilon)} \right).
\]

Also,

\[
\sum_{m \in e^{-1}} (u - mm)^{\delta - 1} = \Theta \left( \int_0^{\ell(u+1)} (u - n)^{\delta - 1} \, dn \right) \\
= \Theta \left( e^{u (c_1 + \varepsilon)} \right).
\]

Thus, by Hölder's Inequality with \( \alpha = c_1 \),

\[
\left( \sum_{m \in e^{-1}} l_{am} (u - mm)^{\delta - 1} \right)^{1/\alpha} \leq \left( \sum_{m \in e^{-1}} l_{am}^{1/\alpha} (u - mm)^{\delta - 1} \right)^{1/\alpha} \left( \sum_{m \in e^{-1}} (u - mm)^{\delta - 1} \right)^{1/\alpha} \\
= \Theta \left( e^{u (c_1 + \varepsilon)} \right) \sum_{m \in e^{-1}} l_{am}^{1/\alpha} (u - mm)^{\delta - 1} 
\]

Then, by Theorem 6.2,

\[
(25) \quad \int_0^{\ell(u+1)} \mathcal{F}_1^{1/\alpha} \, du = \Theta \left( e^{u (c_1 + \varepsilon)} \right) \sum_{m \in e^{-1}} l_{am}^{1/\alpha} \int_{mm+1}^{\ell(u+1)} (u - mm)^{\delta - 1} \, du \\
= \Theta \left( e^{u (c_1 + \varepsilon) + \frac{a_1 \Delta}{c_1} + \frac{c_2 \Delta}{c_1} \varepsilon} \right).
\]
Now, since \( \alpha_0 / \epsilon_1 \geq 1 \), the right sides of (24) and (25) are both

\[
O\left( e^{\omega \frac{1}{\epsilon_1} \frac{1}{1 + \omega / \epsilon_1} + \frac{\alpha_0}{\epsilon_1} + \frac{\epsilon}{\epsilon_1} )} \right).
\]

Therefore, the first factor of \( \epsilon_2^k \) is

\[
O\left( e^{\omega \frac{1}{\epsilon_1} \frac{1}{1 + \omega / \epsilon_1} + \frac{\epsilon}{\epsilon_1} )} \right).
\]

From Lemma 5.2 and (5.12), we have that

\[
\sum_{0}^{\mu} \int \frac{\log(e^{\omega t})}{| \beta \epsilon_0(\omega)| (\omega - \nu)^{\frac{1}{2} - 1} \omega} \frac{1 + X^\epsilon \rho - \delta}{1 + X^\epsilon \rho - \delta} \, d \omega = O\left( e^{\omega \frac{1}{\epsilon_1} \frac{1}{1 + \omega / \epsilon_1} + \frac{\epsilon}{\epsilon_1} )} \right)
\]

Therefore,

\[
\epsilon_2^k = O\left( e^{\omega \frac{1}{\epsilon_1} \frac{1}{1 + \omega / \epsilon_1} + \frac{X^\epsilon \rho - \delta}{1 + X^\epsilon \rho - \delta} \sigma (\omega + \frac{\epsilon(\epsilon_2 - e_2)}{1 + X^\epsilon \rho - \delta} \right) + \epsilon)^2.
\]

For \( \epsilon_2^k \), we use partial summation, and from Theorem 6.2,

\[
\epsilon_2^k = \sum_{\lambda} \epsilon^{\lambda^k / \kappa_0} e^{-\omega / \kappa_0}
\]

\[
= O\left( \sum_{\lambda} \epsilon^{\lambda^k / \kappa_0} (\epsilon^{\lambda^k / \kappa_0} (\epsilon^{\lambda^k / \kappa_0} + \epsilon)) + O(e^{\omega (\epsilon_2 \kappa_0 - e^{-\omega / \kappa_0} + \epsilon)})
\]

\[
= O\left( \sum_{\lambda} \epsilon^{\lambda^k / \kappa_0} \epsilon^{\lambda^k / \kappa_0} (\epsilon^{\lambda^k / \kappa_0} + \epsilon) + O(e^{\omega (\epsilon_2 \kappa_0 - e^{-\omega / \kappa_0} + \epsilon)})
\]

\[
= O\left( \sum_{\lambda} \epsilon^{\lambda^k / \kappa_0} \epsilon^{\lambda^k / \kappa_0} (\epsilon^{\lambda^k / \kappa_0} + \epsilon) + O(e^{\omega (\epsilon_2 \kappa_0 - e^{-\omega / \kappa_0} + \epsilon)})
\]

\[
= O\left( \sum_{\lambda} \epsilon^{\lambda^k / \kappa_0} \epsilon^{\lambda^k / \kappa_0} (\epsilon^{\lambda^k / \kappa_0} + \epsilon) + O(e^{\omega (\epsilon_2 \kappa_0 - e^{-\omega / \kappa_0} + \epsilon)})
\]

\[
= O\left( \sum_{\lambda} \epsilon^{\lambda^k / \kappa_0} \epsilon^{\lambda^k / \kappa_0} (\epsilon^{\lambda^k / \kappa_0} + \epsilon) + O(e^{\omega (\epsilon_2 \kappa_0 - e^{-\omega / \kappa_0} + \epsilon)})
\]

\[
= O\left( \sum_{\lambda} \epsilon^{\lambda^k / \kappa_0} \epsilon^{\lambda^k / \kappa_0} (\epsilon^{\lambda^k / \kappa_0} + \epsilon) + O(e^{\omega (\epsilon_2 \kappa_0 - e^{-\omega / \kappa_0} + \epsilon)})
\]
Thus,
\[ \mathcal{E}_2^x = \mathcal{O}(e^{\omega \frac{\alpha e - \mu}{\varepsilon} + \varepsilon \varepsilon}) \]
since \( \alpha e \geq \varepsilon \geq \mu \).

Now,
\[ \mathcal{E}_3^x = \mathcal{O}(e^{\omega \frac{\alpha e - \mu}{\varepsilon} + \varepsilon \varepsilon}) \]
and since \( \alpha e / \varepsilon \geq 1 \),
\[ \mathcal{E}_2^x + \mathcal{E}_3^x = \mathcal{O}(e^{\omega \frac{\alpha e - \mu}{\varepsilon} + \varepsilon \varepsilon}). \]

Now, using our estimates on \( \mathcal{E}_1^x \) and \( \mathcal{E}_2^x + \mathcal{E}_3^x \), and applying (2.5.4), we obtain

\[
(26) \quad \left( \int_0^\infty \eta_{k \alpha e} (\omega) \eta_{\alpha e} \right)^6
\]
\[ = \mathcal{O}(e^{\omega \left\{ \frac{\alpha e - \mu}{\varepsilon} - \frac{1 + \frac{\alpha e - \mu}{\varepsilon}}{1 + \frac{\mu}{\varepsilon}} \sigma_b (\mu - c_2 b + \varepsilon b) \right\} + \varepsilon \varepsilon}) \]
\[ + \mathcal{O}(e^{\omega \left\{ \alpha e - \mu + \varepsilon \varepsilon \right\}}) \]
where
\[ b = \frac{(\mu'b - c \delta \frac{\alpha e - \mu}{\varepsilon}}{1 + \mu' - c - \delta}. \]
By (5.12)

\[
(27) \sigma_p(x) \leq \max \left( \frac{\lambda}{1+x-c_2} (\alpha p_1+\gamma) + \frac{1+x-c_2}{1+x-c_2} \sigma_b(x-c_2+b), \alpha e-k' \right).
\]

Since, by Lemma 1, we have the continuity of \( \sigma_b(x-c_2+b) \) as a function of \( b \), we may take the limit of (27) as \( \delta \to 0 \). From our assumptions,

\[
np = \frac{\lambda}{1+x-c_2}, \quad (1-np) = \frac{1+x-c_2}{1+x-c_2}
\]

so

\[
\sigma_p(x') \leq \max \left( np(\alpha p_1+\gamma) + (1-np) \sigma_{c_2}(x'), \alpha e-x' \right).
\]

\( \alpha e-x' \) is convex, so by Theorem 7.1

\[
\alpha e-x' \leq np \alpha p_1 + (1-np) \alpha e - x'
\]

\[
\leq np \alpha p_1 + (1-np)(\sigma_{c_2}(x')+x') - x'
\]

\[
\leq \alpha(\sigma_{p_1}+1-p_1) + (1-np) \sigma_{c_2}(x).
\]

Therefore, for \( \sigma(\alpha p_1+1-p_1) \leq \sigma(\alpha p_1+1-p_1) + (1-np) \sigma_{c_2}(x) \)

\[
\sigma_{p_1}(x) \leq \sigma(x) + (1-np) \sigma_{c_2}(x) + \sigma_{p_1}(x) + (1-np) \sigma_{c_2}(x)
\]

for \( \sigma(\alpha p_1+1-p_1) \leq \sigma(x) + (1-np) \sigma_{c_2}(x) + \sigma_{p_1}(x) + (1-np) \sigma_{c_2}(x) \) as asserted.

**Lemma 5 [19, page 41]: On \( \mathbb{H} \), \( \sigma(x) \) is a decreasing function of \( x \), and \( \sigma(x)+x \) is an increasing function of \( x \). Thus, for \( x, x \geq 0, \sigma \leq p \leq 1 \).**
be a sequence of complex numbers such that

$$a_1, a_2, \ldots, a_m, \ldots$$

is a sequence of complex numbers such that

$$\sum_{m \leq x} a_m = x + \mathcal{O}(x^{k-1} + \varepsilon)$$

for each $\varepsilon > 0$. Then

$$\sum_{m_1 \ldots m_k \leq x} a_{m_1} \ldots a_{m_k} = x P_k(\log x) + \mathcal{O}(x^{k-1} - \delta)$$

cannot hold for any $\delta > 0$, where $P_k(\log x)$ is a polynomial in $\log x$ of degree $k-1$ with leading coefficient greater than 0. Again this theorem may be specialized to obtain a weak form of (6) or to obtain (9).

We see that this result is analogous to (10) for $k=2$, but we have added the additional hypothesis (11).

Tull suggested that I investigate the possibility of replacing the main terms of (11) and (12) with $x L_1(x)$ and $x L_k(x)$, where $L_1(x)$ and $L_k(x)$ are slowly oscillating functions. As a result of one of his theorems on Dirichlet multiplication [24],

$$\sum_{m \leq x} a_m = x L_1(x) + \mathcal{O}(x^{k-1} + \varepsilon)$$

for each $\varepsilon > 0$, where $L_1$ is a special slowly oscillating
Proof: \( \sigma_e(x) \) is decreasing on \( H \) from (5.13) and Definitions 5.2 and 5.3.

Let \( 0 < x_1 < x < e < 1 \). From Lemma 4 with 
\( p_1 = p_2 = \rho \), 
\( p_3 = (x - x_1) / (1 + x - e) \) 
using Theorem 7.1,

\[
\sigma_e(x) \leq \frac{(x - x_1)(\sigma_e(x) + x + 1 - p) + (1 + x - e) \sigma_e(x_1)}{1 + x - e} = \sigma_e(x) + x - x_1.
\]

Thus,

\[
(28) \quad \sigma_e(x_1) + x_1 \leq \sigma_e(x) + x
\]
or equivalently

\[
(29) \quad -1 \leq \frac{\sigma_e(x) - \sigma_e(x_1)}{x - x_1}
\]
for \( 0 < x_1 < x < e < 1 \).

From Lemma 3, we have that \( \sigma_e(x) \) is a convex function of \( x \) , for fixed \( e \), \( 0 < e < 1 \), \( x > 0 \). Therefore, from (2.5.21), the difference quotient, (29) is an increasing function of \( x \) and also of \( x_1 \), \( x_1 < x \). Hence, the restriction \( x < e \) may be dropped. Thus, (28) is proved for \( 0 < e < 1 \), \( x \geq x_1 > 0 \).

Now, if \( x > x_1 > 0 \), let \( \max (1 - x_1, 0) < e < 1 \) and \( \delta > 0 \). Then, from (5.15), (5.16) and (28) as proven so far,
\[ \sigma_1(x_i) + \delta \leq \sigma_0(x_i + \epsilon) - \sigma_0(x_i) + \delta \]

Since \( 1-\epsilon \) and \( \delta \) are arbitrary small numbers greater than 0, (28) holds for \( x_i \geq 0, \epsilon = 1 \). By (28) with \( 0 < \epsilon \leq 1 \), \( 0 < x_i \leq x \),

\[ \sigma_0(x_i + \epsilon) + \epsilon \leq \sigma_0(x + \epsilon) + \epsilon. \]

Taking the limit as \( \epsilon \to 0 \), we have

\[ \sigma_0(x_i) + \epsilon \to \sigma_0(x) + \epsilon. \]

To establish (28) in case \( x_i = 0 \), we take the limit as \( x_i \to 0 \).

Thus, (28) holds for \( 0 < \epsilon \leq 1 \), \( 0 < x_i \leq x \), and the Lipschitz conditions follow since

\[ 0 \leq \sigma_0(x) - \sigma_0(x_i) \leq x - x_i. \]

Lemma 6 [19, page 42]: Let \( 0 \leq \epsilon \leq 1 \) and \( x + \epsilon \geq 0 \). Then \( \sigma_0(x + \epsilon) \) is a decreasing function of \( \epsilon \), and \( \sigma_0(x + \epsilon) + \epsilon \) is an increasing function of \( \epsilon \).
Proof: (5.16) and Definition 5.2 imply that

\[ \sigma_{\epsilon}(x+\epsilon) \leq \sigma_{\epsilon}(x+\epsilon) \quad \text{for } x+\epsilon > 0, \, 0 \leq \epsilon, \, \delta \leq 1. \]

Lemma 5 implies \( \sigma_{\epsilon}(x) \) is a continuous function of \( x \) for \( 0 \leq \epsilon \leq 1, \, x \geq 0 \), so that (30) also holds for \( x+\epsilon \geq 0, \, 0 \leq \epsilon, \, \delta \leq 1 \).

(5.14), (5.14), Definition 5.2 and the continuity of \( \sigma_{\epsilon}(x) \) in \( x \) yield

\[ \sigma_{\epsilon}(x) \leq \sigma_{\epsilon}(x) \]

for \( x \geq 0, \, 0 \leq \epsilon, \, \delta \leq 1 \). Thus, Lemma 5 implies

\[ \sigma_{\epsilon}(x+\epsilon) + \epsilon \leq \sigma_{\epsilon}(x+\epsilon) + \epsilon \leq \sigma_{\epsilon}(x+\epsilon) + \epsilon \]

for \( x+\epsilon \geq 0, \, 0 \leq \epsilon, \, \delta \leq 1 \).

Lemma 7 [19, page 42]: On \( X \), \( \sigma_{\epsilon}(x) \) is an increasing function of \( \epsilon \), and \( \sigma_{\epsilon}(x)-\epsilon \) is a decreasing function of \( \epsilon \). Thus, for \( x \geq 0, \, 0 \leq \epsilon, \, \delta \leq 1 \),

\[ |\sigma_{\epsilon}(x)-\sigma_{\epsilon}(x)| \leq |\epsilon, \, \delta|. \]

Proof: \( \sigma_{\epsilon}(x) \) is an increasing function of \( \epsilon \) from (31). That \( \sigma_{\epsilon}(x)-\epsilon \) is decreasing follows from Lemmas 5 and 6, since, if \( 0 \leq \epsilon, \, \delta \leq 1, \, x \geq 0 \),

\[ \sigma_{\epsilon}(x) \leq \sigma_{\epsilon}(x+\epsilon), \, \delta \leq \sigma_{\epsilon}(x)-\epsilon. \]
Also,

\[ 0 \leq \sigma_{e}(x) - \sigma_{e}(x') \leq P - P, \]

so the Lipschitz condition follows, and this completes the proof of the lemma.

Now, from Lemmas 5 and 7,

\[ |\sigma_{e_{2}}(x_{2}) - \sigma_{e_{1}}(x_{1})| \leq |x_{2} - x_{1}| + |P - P| \]

for \( x_{1} \geq 0, \ 0 \leq P_{1} \leq 1, \ P_{2} \leq 1 \). This proves Theorem 2, and Theorem 2 together with Lemma 3 proves Theorem 1.

3.9. Extensions of Some Theorems

In this section we extend Lemma 8.4 and Theorem 7.1.

Theorem 1 [19, page 43]: Let \( x \geq 0, \ 0 \leq P_{1}, P_{2} \leq 1, \ 0 \leq \omega \leq 1 \) and \( \omega(1 - P_{1}) \leq (1 - \omega)x \). Then

\[ \sigma_{e_{1} + (1 - \omega)e_{2}}(\omega(1 - P_{1}) + (1 - \omega)x) \leq \sigma_{e_{1} + (1 - \omega)e_{2} + (1 - \omega)x}. \]

Proof: For \( 0 \leq P_{1}, P_{2} \leq 1, \ 0 \leq \omega \leq 1, \ \omega(1 - P_{1})/(1 - \omega) < x < e_{2} \), Theorem 1 reduces to Lemma 8.4. We wish to remove the restriction \( x < e_{2} \). To do this, we note that

\( \omega(1 - P_{1})/(1 - \omega) < x \) is equivalent to \( \omega < x/(1 - P_{1} + x) \);

assume \( x \geq e_{2} > 0 \); and let

\[ \Phi(u) = \begin{cases} \sigma_{u, e_{1} + (1 - u)e_{2}}(u(1 - e_{1}) + (1 - u)x) & \text{for } 0 \leq u < x/(1 - P_{1} + x) \, \text{; } \\ e_{1} + 1 - P_{1} & \text{for } u = 1 \end{cases} \]
We need to show that

\[ \varphi(\alpha + (1-\alpha) u) \leq \alpha \varphi(1) + (1-\alpha) \varphi(u) \]

for \( 0 < \alpha + (1-\alpha) u \leq \frac{1}{1+\varepsilon} \) in the special case \( u = 0 \). Now, since \( u(\varepsilon, 1) + (1-u) \alpha \to 0 \) as \( u \to \frac{1}{1+\varepsilon} \), and \( u + (1-u) \rho_2 \geq \min(\rho_1, \rho_2) > 0 \) for \( 0 < u < 1 \),

\[ \chi' = u(\varepsilon, 1) + (1-u) \alpha \leq u + (1-u) \rho_2 = \rho_2' \]

for \( u_1 \) such that \( 0 < \frac{1}{1+\varepsilon} - \delta < u_1 \leq \frac{1}{1+\varepsilon} \), where \( \delta > 0 \) and sufficiently small. Fix some \( u_1 \) satisfying this condition. Then, by Lemma 8.4, (1) holds for \( u = u_1 \) since \( \alpha + (1-\alpha) u_1 \leq \frac{1}{1+\varepsilon} \) is equivalent to \( \alpha \leq \frac{1}{1+\varepsilon} \).

Now, consider \( u_2 \) such that \( u_1 < u_2 < \frac{1}{1+\varepsilon} \).

Let \( \rho_1 = (u_2 - u_1)(1-u_1) \). Then \( \rho_1 + (1-\rho_1) u_1 = u_2 \), and by (1) with \( u = u_1 \),

\[ \varphi(u_2) \leq \rho_1 \varphi(1) + (1-\rho_1) \varphi(u_1) \]

Let \( \rho_2 = \frac{\rho_1}{u_2} \). Then \( 1-\rho_2 = (u_2 - u_1)/u_2 \), and by Theorem 8.1,

\[ \varphi(u_1) \leq \rho_2 \varphi(u_2) + (1-\rho_2) \varphi(0) \]

Thus,
\[ \psi(u_2) \leq \alpha_1 \psi(1) + (1-\alpha_1) \psi_2 \psi(u_2) + (1-\alpha_1)(1-\alpha_2) \psi(0), \]

and so

\[ \psi(u_2) \leq \left( \alpha_1 / (1-(1-\alpha_1)\alpha_2) \right) \psi(1) + \left( (1-\alpha_1)(1-\alpha_2) / (1-(1-\alpha_1)\alpha_2) \right) \psi(0). \]

With some simple algebra this yields

\[ (2) \quad \psi(u_2) \leq u_2 \psi(1) + (1-u_2) \psi(0). \]

Consider \( 0 < u_3 < u_1 \). Choose \( u_2 \) as above so that \( u_1 < u_2 < \mathcal{X}/(1-\alpha_1+\mathcal{X}) \). Then (2) holds. Let \( \alpha_1 = u_3/u_2 \). Then \( 1-\alpha_1 = (u_2-u_3)/u_2 \), and as above, Theorem 8.1 implies

\[ \psi(u_3) \leq \alpha_1 \psi(u_2) + (1-\alpha_1) \psi(0). \]

Thus,

\[ \psi(u_3) \leq u_2 \alpha_1 \psi(1) + \left( (1-u_2) \alpha_1 + (1-\alpha_1) \right) \psi(0). \]

Hence, (1) holds for all \( u \geq u_1 \) such that \( 0 \leq u \leq \mathcal{X}/(1-\alpha_1+\mathcal{X}) \), but \( u_1 \) is arbitrary in \( (\mathcal{X}/(1-\alpha_1+\mathcal{X}) - \delta, \mathcal{X}/(1-\alpha_1+\mathcal{X}) \) , so (1) holds for all \( u \) such that \( 0 < u < \mathcal{X}/(1-\alpha_1+\mathcal{X}) \).

The boundary cases are now handled by the continuity of \( \mathcal{C}_0(\mathcal{X}) \) and \( \alpha_\mathcal{X} \).
Theorem 2 [19, page 44]: Let $X \geq 0$, $0 \leq \epsilon \leq 1$. Then

$$\alpha \epsilon \leq \sigma_{\epsilon}(X) + X.$$  

Proof: Theorem 7.1, together with the continuity of $\sigma_{\epsilon}(X)$ and $\alpha \epsilon$ yield

$$\alpha \epsilon \leq \sigma_{\epsilon}(\alpha)$$  

for $0 \leq \epsilon \leq 1$. Now, since by Lemma 8.5, $\sigma_{\epsilon}(X) + X$ is an increasing function of $X$, the theorem follows.
4. SOME APPLICATIONS OF STRONG RIESZ SUMMABILITY TO THE CARLSON $\varphi$ FUNCTION

4.1. Theorems

In this chapter we shall combine our results on the functions $\psi_1(\varphi)$, $\psi_2$, and $\varphi(\chi)$ in a manner similar to that of Richert's [19, chapter 8].

We first make some estimates concerning the integral

$$\frac{1}{2\pi i} \int_{\delta-iT}^{\delta+iT} e^{ys} s^{-(\chi-\delta+1)} ds$$

as $T \to \infty$. Here $\int_{\delta-iT}^{\delta+iT}$ indicates the integral over the straight line from $\delta-iT$ to $\delta+iT$, and $\arg s$ is 0 for $s$ real and positive. The "$C$" constants in the following do not depend on $\gamma$ or $T$.

Let $\delta > 0$, $\chi-\delta > 1$, $\gamma < 0$. For $T > \min (\delta, 1)$, applying Cauchy's theorem to the integral over the rectangle with corners $\delta \pm iT$, $\tau \pm iT$, we get

$$\frac{1}{2\pi i} \int_{\delta-iT}^{\delta+iT} e^{ys} s^{-(\chi-\delta+1)} ds = \mathcal{O}(\sum_{\delta} e^{y\sigma} (\sigma+iT)^{-(\chi-\delta+1)} d\sigma) + \mathcal{O}(\sum_{\tau} e^{y\tau} (\tau+iT)^{-(\chi-\delta+1)} d\tau)$$

$$= \mathcal{O}(T^{-(\chi-\delta+1)} \sum_{\delta} e^{y\sigma} d\sigma) + \mathcal{O}(e^{yT} T^{-(\chi-\delta+1)} \sum_{\tau} d\tau).$$
Thus,

\[ (2) \frac{1}{2\pi i} \int_{\delta-iT}^{\delta+iT} e^{ys} s^{-(\omega-\rho+1)} ds = O(e^{\delta T} T^{-(\omega-\rho)}) + O(e^{\delta T} T^{-(\omega-\rho)}) \]

for \( \delta > 0 \), \( \omega-\rho > -1 \), \( y < 0 \), \( T > \delta \). Further, from (1), we see that if \( \delta > 0 \), \( \omega-\rho > 0 \), \( y < 0 \), \( T > \delta \),

\[ \frac{1}{2\pi i} \int_{\delta-iT}^{\delta+iT} e^{ys} s^{-(\omega-\rho+1)} ds = O(e^{\delta T} T^{-(\omega-\rho)}) + O(e^{\delta T} T^{-(\omega-\rho)}) \]

Now, if \( \delta > 0 \), \( \omega-\rho > -1 \), \( y > 0 \), for

\[ T > \min(\delta, 1) \]

we first make a substitution and then apply Cauchy's theorem over the contour formed from the rectangle with corners \( y \delta \pm iyT \), \( -yT \mp iyT \), and the Hankel contour \( -L_i \), where \( -L_i \) is the path along the negative real axis from \( -T \) to \( -\epsilon \), \( 0 < \epsilon \leq \delta \), around the circle \( |s| = \epsilon \) to the negative real axis, and back to \( -yT \). On the first part of the path, \( \omega \) is \( \Pi \), and on the last part, \( -\Pi \). We thus obtain

\[ \frac{1}{2\pi i} \int_{\delta-iT}^{\delta+iT} e^{ys} s^{-(\omega-\rho+1)} ds = \frac{\omega y^{\omega-\rho}}{2\pi i} \int_{yT-i\epsilon}^{yT+i\epsilon} e^{s} s^{-(\omega-\rho+1)} ds \]
Now, let \( L_i' \) be the extension of the contour \( L_i \), obtained by letting \(-\infty < -T < -\infty\). It is well known [26, page 245] that

\[
\frac{1}{2\pi i} \int_{L_i'} e^s s^{-(\nu+1)} ds = \frac{1}{\Gamma(\nu+1)}.
\]

Now,

\[
\frac{1}{2\pi i} \int_{L_i'-L_i} e^s s^{-(\nu+1)} ds = \Theta(s^\infty e^{\sigma}) \int_{-\infty}^{\infty} e^{\sigma} \Gamma(\nu+1) ds
\]

Thus,

\[
\frac{1}{2\pi i} \int_{L_i'-L_i} e^{sT} s^{-(\nu+1)} ds = \frac{e^{-\nu T}}{\Gamma(\nu+1)}
\]

\[
+ \Theta(s^\infty e^{\sigma}) \int_{-\infty}^{\infty} e^{\sigma} \Gamma(\nu+1) ds
\]

\[
+ \Theta(s^\infty e^{\sigma}) \int_{-\infty}^{\infty} e^{\sigma} \Gamma(\nu+1) ds
\]

\[
+ \Theta(s^\infty e^{\sigma}) \int_{-\infty}^{\infty} e^{\sigma} \Gamma(\nu+1) ds
\]
function, implies

\[ (14) \sum_{m_i \cdots m_k} a_{m_i} \cdots a_{m_k} = x L_{\infty}(x) + O(x^\theta) \]

where \( L_{\infty}(x) \) is a slowly oscillating function and \( \theta = \frac{2\pi \sqrt{1+4e}}{2\pi} \).

Thus, it is of interest to find a lower bound for the \( \omega \) satisfying (14). We were able to establish such a theorem only with some further restrictions on \( L, \) and \( L_\infty \).

In order to establish (12), Richert used results from a recent paper [19] in which he developed the theory of strong Riesz summability and applied it to Dirichlet series. We found that Richert's paper [20] contains everything necessary to prove that the \( \omega \) in (14) must be \( \geq \frac{1}{4} \) when \( L_4(x) \) is a polynomial in \( \log x \). The method used to deduce the theorem in this way depends heavily on the fact that the function \( Z_\infty(s) \) which is the analytic continuation of the function \( \sum_{m=1}^{\infty} a_m m^{-s} \), \( \Re s > 1 \), has a pole at \( s = 1 \) when \( L_4(x) \) is a polynomial in \( \log x \). In order to prove that \( \omega \geq \frac{1}{4} \) in a more general case, we first made further restrictions on \( L_4(x) \) and \( L_{\infty}(x) \) (see (2.2.2) and (2.2.3)) which, although they no longer guarantee that \( Z_\infty(s) \) has a pole at \( s = 1 \), do guarantee an analytic continuation of \( Z_\infty(s) \) into a region larger than \( \Re s > 1 \). We then developed the theory of strong Riesz summability for certain Stieltjes integrals. Nearly all of the methods used by Richert are still
Thus,

\[
(4) \quad \frac{1}{2\pi i} \int_{\delta-iT}^{\delta+iT} e^{ys} s^{-\nu} ds = \frac{y^{\nu}}{\Gamma(\nu+1)}
\]

\[
+ \mathcal{O}(y^{-\nu} e^{\delta y} T^{-\nu+1}) + \mathcal{O}(e^{-\delta T} T^{-\nu-1})
\]

for \( \delta > 0, \ \nu > -1, \ y > 0, \ T > \delta \).

(4) implies

\[
\lim_{T \to \infty} \frac{1}{2\pi i} \int_{\delta-iT}^{\delta+iT} e^{ys} s^{-\nu} ds = \frac{y^{\nu}}{\Gamma(\nu+1)}
\]

for \( y > 0, \ \nu > -1 \), and in particular for \( \nu > 0 \), so

\[
\frac{1}{2\pi i} \int_{\delta-iT}^{\delta+iT} e^{ys} s^{-\nu} ds = y^{\nu}/\Gamma(\nu+1) + \mathcal{O}(e^{\delta y} T^{-\nu})
\]

\[
= y^{\nu}/\Gamma(\nu+1) + \mathcal{O}(e^{\delta y} T^{-\nu})
\]
for $\delta > 0$, $\chi - \rho > 0$, $\gamma > 0$, $\tau > \delta$.

We have now proved the following lemma:

**Lemma 1:** Let $\delta > 0$, $\tau > \delta$, $\gamma > 0$, $\chi - \rho > -1$.

Then

$$\frac{1}{2\pi i} \int_{\gamma - i\tau}^{\gamma + i\tau} e^{zs} s^{-(\chi - \rho + 1)} ds = Q(z) + E_1,$$

where

$$Q(z) = \begin{cases} \frac{\gamma^{\chi - \rho}/\Gamma(\chi - \rho + 1)}{s^{\chi - \rho + 1}} & \gamma > 0 \\ 0 & \gamma \leq 0 \end{cases}$$

and

(4) \quad $E_1 = O(1_{\gamma > -1} e^{\chi \delta T - (\chi - \rho)}) + O(1_{\chi - \rho > 0} e^{1/4\tau T - (\chi - \rho)}).$

Further, if $\chi - \rho > 0$, then

(5) \quad $E_1 = O(1_{\chi - \rho > 0} e^{\chi \delta T - (\chi - \rho)}).$

**Lemma 2** [19, page 55]: Let $0 < \rho < 1$, $e^w \neq e^{i\omega}$, $\sigma_o > s$, and $\chi > \chi_{\rho}(\sigma_o)$. Then

(6) \quad $\Theta_{\chi - \rho}(\omega) = \frac{\Gamma(\chi - \rho + 1)}{2\pi i} \int_{\sigma_o - i\infty}^{\sigma_o + i\infty} e^{ws} \frac{\psi(s)}{s^{\chi - \rho + 1}} ds$

where

$$\int_{\sigma_o - i\infty}^{\sigma_o + i\infty} = \lim_{T \to \infty} \int_{\sigma_o - iT}^{\sigma_o + iT},$$

(7) \quad $\int_{-\infty}^{\infty} |\psi(\sigma_o + it)|^{1/2e} |\sigma_o + it|^{-\frac{\chi - \rho + 1}{1-2e}} dt < \infty.$
Proof: First, we may assume for the purposes of this proof that \( a_{n+1} \) contains the discontinuity of \( L \) at \( \omega + 1 \).

For \( \delta > \alpha_1 + \varepsilon > \alpha_1 \), \( T \gg \delta \), by Lemma 1,

\[
\frac{\Gamma(\alpha_1+1)}{2\pi i} \left\{ \sum_{\gamma_1} e^{i\gamma_1 T} \left[ \psi(s) + 2 - (\alpha_1+1) \right] \right\} \prod_{\nu \neq 1} \frac{1}{\nu - \delta} = \sum_{m=1}^{\infty} a_m \frac{\Gamma(\alpha_1+1)}{2\pi i} \left\{ \sum_{\gamma_1} e^{i\gamma_1 T} \left[ \psi(s) + 2 - (\alpha_1+1) \right] \right\} \prod_{\nu \neq 1} \frac{1}{\nu - \delta} \\
- \frac{\Gamma(\alpha_1+1)}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} e^{\omega T} \frac{e^{-\omega T}}{\omega - \alpha_1 + 1} d\omega
\]

\[
= \sum_{m=1}^{\infty} a_m (\omega - m\omega)^{\alpha_1} \\
+ O((\omega - \log[\log e\omega]) + \frac{1}{\log[e\omega] + \omega}) (\int_0^\infty e^{(\omega+1)u} e^{(\omega-u)\omega} du) T^{-(\alpha_1+1)} \\
+ O((\int_0^\infty e^{(\omega+1)u} e^{-(\omega-u)\omega} du) T^{-(\alpha_1+1)} \\
- E_2
\]

\[
= \sum_{m=1}^{\infty} a_m (\omega - m\omega)^{\alpha_1} \\
+ O(T^{-(\alpha_1+1)}) \\
+ O(T^{-(\alpha_1+1)}) \\
- E_2
\]
We now consider $E_2$.

$$\int_{0}^{\omega+1} e^{-su} d(e^u E(u)) = \int_{0}^{\omega+1} e^{-su} e^u E(u) (1+\delta(e^u)) \, du.$$ 

Let $f(u) = e^u E(u) (1+\delta(e^u))$. Then

$$\int_{0}^{\omega+1} e^{-su} d(e^u E(u)) = \left[ f(u) e^{-su} s^{-1} \right]_{0}^{\omega+1} - \int_{0}^{\omega+1} e^{-su} s^{-1} f'(u) \, du$$

$$= f(\omega+1) e^{(\omega+1)s} s^{-1} - \int_{0}^{\omega+1} e^{-us} s^{-1} f'(u) \, du.$$ 

Thus, by Lemma 1,

$$\frac{\Gamma(\chi+\rho+1)}{2\pi i} \int_{\delta-iT}^{\delta+iT} e^{ws} s^{-(\chi+\rho+1)} \int_{0}^{\omega+1} e^{-us} d(e^u E(u)) \, ds$$

$$= \frac{\Gamma(\chi+\rho+1)}{2\pi i} f(\omega+1) \int_{\delta-iT}^{\delta+iT} e^{-s} s^{-(\chi+\rho+2)} \, ds$$

$$- \frac{\Gamma(\chi+\rho+1)}{2\pi i} \int_{0}^{\omega+1} f'(u) \int_{\delta-iT}^{\delta+iT} e^{(u-w)s} s^{-(\chi+\rho+2)} \, ds \, du$$

$$= \Theta(T^{-(\chi+\rho+1)} + e^{-T} T^{-(\chi+\rho)})$$

$$+(\int_{0}^{\omega} f'(u) (w-u)^{-\chi+\rho+1} \, du) \frac{1}{\chi+\rho+1} + \Theta(\int_{0}^{\omega} e^{(w-u)s} T^{-(\chi+\rho+1)} \, du)$$

$$= \frac{1}{\chi+\rho+1} \int_{0}^{\omega} f'(u) (w-u)^{-\chi+\rho+1} \, du + \Theta(T^{-(\chi+\rho+1)})$$

$$= \int_{0}^{\omega} (w-u)^{-\chi+\rho} d(e^u E(u)) + \Theta(T^{-(\chi+\rho+1)}).$$

Also,

$$\frac{\Gamma(\chi+\rho+1)}{2\pi i} \int_{\delta-iT}^{\delta+iT} e^{ws} s^{-(\chi+\rho+1)} \int_{\omega+1}^{\infty} e^{-us} d(e^u E(u)) \, ds$$
Therefore,

\[
\frac{\Gamma(\kappa+\rho+1)}{2\pi i} \int_{\sigma-iT}^{\sigma+iT} e^{\omega s} s^{-(\kappa+\rho+1)} \psi(s) \, ds = B_{\kappa-\rho}(\omega) + \mathcal{O}(T^{-\rho})
\]

We now wish to move the path of integration to the line \( \sigma = \sigma_0 \). If \( \sigma_0 > \kappa \), we choose \( \delta = \sigma_0 \). If \( \sigma_0 \leq \kappa \), we integrate around the rectangle with corners \( \delta \pm i\tau \), \( \sigma_0 \pm i\tau \). Since \( \sigma_0 > \sigma \geq 0 \), the integrand is analytic in this rectangle, and on the horizontal paths, the integral is

\[
\mathcal{O}(T^{-\rho(\sigma)-\kappa+\rho-1+\varepsilon})
\]

as \( T \to \infty \), for each \( \varepsilon > 0 \), by Lemma 2.5.3. Since

\( -\rho(\sigma)+\varepsilon \) is an increasing function of \( \varepsilon \) by (2.5.13), our assumptions yield

\( -\rho(\sigma)+\varepsilon \leq -\rho(\sigma)+\varepsilon \leq \kappa+\varepsilon \).

Thus,

\[
\frac{\Gamma(\kappa-p+1)}{2\pi i} \int_{\sigma_0-iT}^{\sigma_0+iT} e^{\omega s} s^{-(\kappa-\rho+1)} \psi(s) \, ds
\]

\[
= B_{\kappa-\rho}(\omega) + \mathcal{O}(T^{-(\kappa-\rho+1)}) + \mathcal{O}(T^{-\rho_{2}})
\]
So
\[
\lim_{T \to \infty} \frac{1}{2\pi i} \int_{\gamma_0 - iT}^{\gamma_0 + iT} e^{\omega s} s^{-(\lambda - \sigma + i)} \psi(s) \, ds = \Theta_{\lambda - \sigma} (\omega).
\]
This proves (6).

Now, \( \psi(s) s^{-(\lambda - \sigma + i)} \) is analytic on the vertical line \( \sigma = \sigma_0 \), and by the definition of \( \psi_{\lambda - \sigma} (\sigma_0, \gamma) \),
\[
J(T) = \int_{-T}^{T} |\psi(\sigma_0 + it)|^{\gamma - \sigma} \, dt = O(T^{\sigma_0 \gamma})
\]
as \( T \to \infty \), for each \( \epsilon > 0 \). We have
\[
\int_{-T}^{T} |\psi(\sigma_0 + it)|^{\gamma - \sigma} \, dt = O(1) + O(T^{\gamma - \sigma(1 - \frac{\lambda - \sigma + i}{1 - \sigma}}) = O(1) + O(T^{\gamma - \sigma(1 - \frac{\lambda - \sigma + i}{1 - \sigma}})
\]
By assumption \( \lambda > -\nu_{\lambda - \sigma} (\sigma_0) \), so
\[
\int_{-T}^{T} |\psi(\sigma_0 + it)|^{\gamma - \sigma} \, dt = O(1),
\]
and this proves (7).

Lemma 3 [19, page 56]: Let \( 0 < \sigma = \nu_2 \), \( \sigma_0 > S \), and \( \lambda > -\nu_{\lambda - \sigma} (\sigma_0) \). Then
\[
\nu_2 (\lambda) \leq \sigma_0.
\]
Proof: Set \( f(t) = \psi(\sigma_0 + it) (\sigma_0 + it)^{-(\lambda - \sigma + i)} \) and
\[
F(\sigma, T) = \frac{1}{2\pi i} \int_{-T}^{T} e^{it\sigma} f(t) \, dt.
\]
From (7), \( f(t) \) belongs to \( L_{\lambda - \sigma} (-\infty, \infty) \). From [23, Theorem 74] for each \( T > 0 \), \( F(\sigma, T) \in L_{\nu_2} (-\infty, \infty) \).
as a function of $n_r$, and there exists $F(n_r) \in L^1(\infty, \infty)$ such that

$$P(n_r) = \lim_{r \to \infty} P(n_r, r) \quad (\text{mean } k_0)$$

and

$$\sum_{n=1}^{\infty} |P(n_r)|^{k_0} n_r \leq \frac{1}{(2\pi)^{1/r-1}} \left\{ \int_{-\infty}^{\infty} |f(t)|^{k-\sigma} dt \right\}^{1/k_0} \leq \infty.$$

On the other hand, for $e^{\nu_0} \neq [e^{\nu_0}]$, from (6) $\lim_{r \to \infty} P(n_r, r)$ exists, and thus by [7, Chapter XI, Theorems 24 and 21] for almost every $n_r$,

$$P(n_r) = \frac{1}{(2\pi)^{1/r} P(n_r \rho)} \theta_{n_r}(\nu_r) e^{-i\sigma_0 \nu_r}.$$

As a consequence, we obtain

$$\left( \int_{\nu_0}^{\omega} \theta_{n_r}(u) |^{k_0} du \right)^{1/k_0} \leq e^{\sigma_0 (e^{\omega / \nu_0})^{k_0}} e^{-\frac{\sigma_0 u}{e} \omega}$$

$$= \Theta(e^{\sigma_0 \omega} \left( \int_{\nu_0}^{\omega} |F(n_r)|^{k_0} d\nu_r \right)^{1/k_0} \omega)$$

Thus, from (3.5.12), $\sigma_0(\nu) \leq \sigma_0$ and the lemma is proved.

**Lemma 4 [19, page 57]:** Let $\lambda > 0$, $\lambda_2 \leq \rho < 1$ and $\sigma \geq \sigma_0(\nu)$. Then

$$\neg \nu_1 \leq \nu_0(\sigma) \leq \lambda.$$

**Proof:** Let $\sigma_0(\nu) < \sigma'' < \sigma' < \sigma$. By (3.5.12)

$$\left( \int_{\nu_0}^{\omega} \theta_{n_r}(u) |^{k_0} du \right)^{1/k_0} = \Theta(e^{\omega \sigma''})$$
and thus by (3.5.6),

\[
\left( \int_0^\infty \mathcal{B}_{\chi, \rho}(u)^\nu e^{-\sigma u} du \right)^\rho = \Theta(1).
\]

Now, from Hölder’s Inequality with \( \alpha = \rho \),

\[
\int_0^\infty \mathcal{B}_{\chi, \rho}(u) e^{-\sigma u} du \leq \left( \int_0^\infty \mathcal{B}_{\chi, \rho}(u)^\nu e^{-\sigma u} du \right)^\rho \left( \int_0^\infty e^{-\frac{\sigma u}{\rho-\rho}} du \right)^{1-\rho}
\]

\[
= \Theta(1).
\]

Thus, \( \int_0^\infty \mathcal{B}_{\chi, \rho}(u) e^{-\sigma u} du \) is analytic for \( \sigma > \sigma_0(w) \).

We assert that

\[
(8) \quad \psi(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathcal{B}_{\chi, \rho}(u) e^{us} du
\]

for \( \sigma > \sigma_0(w) \), where the analytic continuation for \( \psi(s) \) into this region was established in Section 3.5.

It is sufficient to show that (8) holds for \( s \) real and greater than \( \alpha_i \), for then by the uniqueness of an analytic function, it holds in the larger region. For \( \sigma > \alpha_i \geq \sigma_0(w) \),

\[
\int_0^\infty \mathcal{B}_{\chi, \rho}(u) e^{us} du = \int_0^\infty e^{us} \int_0^\infty (u-\tau)^{\chi-\rho} d\mathcal{B}(\tau) \, du
\]

\[
= \int_0^\infty e^{-su} \int_0^\infty e^{s(\tau-u)} (\tau-u)^{\chi-\rho} d\mathcal{B}(\tau) \, du
\]

\[
= \int_0^\infty e^{-su} \int_0^\infty e^{-\rho s (\tau-u)} \tau^{\chi-\rho} \, d\mathcal{B}(\tau) \, du
\]

\[
= \Gamma(\chi-\rho) \int_0^\infty e^{-su} \tau^{\chi-\rho} \, d\mathcal{B}(\tau).
\]
The interchange of integration is justified by the measurability of the integrand and the fact that

\[ \int_0^\infty e^{-s \nu} d\nu \theta(s, \nu) < \infty \]

for \( s > \alpha \).

Now, for \( \sigma > \sigma_0(x) \) we set

\[ F(\nu) = \begin{cases} \left( \frac{\sqrt{2\pi}}{1(\nu - \sigma + 1)} \right) B_{\nu-\sigma}(u) e^{-\nu \sigma} \text{ for } \nu > 0 \quad \text{for } \nu \leq 0 \end{cases} \]

\[ f(t, \nu) = \frac{1}{\sqrt{2\pi}} \int_{-\nu}^{\nu} e^{-\nu (\sigma - t)} F(\nu) d\nu \]

\[ = \frac{1}{\Gamma(\nu - \sigma + 1)} \int_{\nu}^{\nu} B_{\nu-\sigma}(u) e^{-u (\sigma - t)} d\nu. \]

We have already noticed that by (3.5.6), \( F(\nu) \) belongs to \( L_{\nu}(-\infty, \infty) \). Since \( \frac{1}{2} \leq \nu < 1 \), there exists as before by [23, Theorem 74] a \( f(t) \in L_{\nu}(-\infty, \infty) \) such that

\[ f(t) = \lim_{\nu \to \infty} f(t, \nu) \quad \text{(mean \( L_{\nu} \))} \]

and

\[ \int_{-\infty}^{\infty} \left| f(t) \right|^{\nu_{\sigma}} dt \leq \frac{1}{(2\pi)^{\frac{\lambda}{2}} \Gamma_{\nu_{\sigma}}^{-1}} \left\{ \int_{-\infty}^{\infty} \left| F(\nu) \right|^{\nu_{\sigma}} d\nu \right\}^{\frac{\lambda}{\nu_{\sigma}}} \]

On the other hand, from above, \( \lim_{\nu \to \infty} f(t, \nu) \) also exists, so by [7, Chapter XI, Theorems 24 and 21], almost everywhere

\[ f(t) = \Psi(\sigma + i\tau) (\sigma + i\tau)^{-\nu} \sigma_{\nu_{\sigma}}. \]
Thus, it follows that
\[
\int_T |\psi(\sigma+i\tau)|^{\frac{1}{1-\sigma}} \, d\tau \leq |\sigma+i\tau|^{\frac{1}{1-\sigma}} \int_T |\psi(t)|^{\frac{1}{1-\sigma}} \, dt
\]
\[
= 0 \left( t \frac{1}{1-\sigma} \right).
\]

The lemma now follows from the definition of \( -\psi(\sigma) \).

Theorem 1 [19, page 58]: Let \( 0 \leq \sigma \leq \frac{1}{4} \). Then

(9) \( \sigma(\chi) \leq \sigma \)

for \( \sigma > S \), \( \chi \geq -\psi_0(\sigma) \)

(10) \( -\psi_0(\sigma(\chi)) \geq \chi \)

for \( \chi \geq 0 \), in case \( \sigma(\chi) > S \);

and

(11) \( \sigma(0) \leq \psi_0 \).

Proof: (9) follows from Lemma 3 and the continuity of \( \sigma(\chi) \) and \( -\psi_0(\sigma) \).

(10) follows from (9) and the continuity of \( -\psi(\sigma) \).

If \( \sigma > \psi_0 \), then \( -\psi_0(\sigma) = 0 \). Thus, (9) implies \( \sigma(0) \leq \sigma \). This implies (11).
applicable in the more general theory. The results are all of the same form as those he stated, and the proofs, with a few exceptions, are the same. We shall give these results in Chapters 3 and 4.

In Chapter 2 a transformation is performed which, with these results, allows a proof of the desired theorem, Theorem 2.3.1, by the same method Richert used to prove a lemma to (12) [20, Satz 3].
Theorem 2 [19, page 59]: Let \( \mathcal{N} \leq r \leq 1 \). Then

(12) \[ -\mathcal{N}_r (\sigma) \leq \mathcal{N} \]

for \( \sigma > \mathcal{N} \), \( \sigma \geq \mathcal{N}_r (\mathcal{N}) \), \( \mathcal{N} \geq 0 \);

(13) \[ \mathcal{N}_r (\mathcal{N}_r (\sigma)) \geq \sigma \]

for \( \mathcal{N} < \sigma < \mathcal{N}_r \); and

(14) \[ \mathcal{N}_r \leq \mathcal{N}_r (\mathcal{N}) \]

Proof: We may extend Lemma 4 to (12) by the continuity of \( \mathcal{N}_r (\mathcal{N}) \) and \( \mathcal{N}_r (\mathcal{N}) \).

(13) follows from the contrapositive of (12) and continuity, and (14) follows directly from (12).

Theorem 3 [19, page 60]: Let \( \mathcal{N} \geq 0 \), \( 0 \leq r \leq 1 \). Then

\[ \lim_{\mathcal{N} \to \infty} \mathcal{N}_r (\mathcal{N}) = \mathcal{N} \]

Proof: By the remark following (3.5.12) and the continuity of \( \mathcal{N}_r (\mathcal{N}) \),

\[ \mathcal{N}_r (\mathcal{N}) \geq \mathcal{N} \]

for \( 0 \leq r \leq 1 \), \( \mathcal{N} \geq 0 \).
For $0 \leq \sigma \leq 1$, since, from Lemma 3.8.5, $\sigma_\sigma(\lambda)$ is a decreasing function of $\lambda$, the limit
\[
\lim_{\lambda \to \infty} \sigma_\sigma(\lambda) = \sigma_\sigma
\]
exists and

(15) $\sigma_\sigma(\lambda) \geq \sigma_\sigma \geq S$

for $\lambda \geq 0$. Now, if $\sigma_\sigma > S$, then there is a $\sigma$ such that $S < \sigma < \sigma_\sigma$, and since, from Lemma 3.8.6, $\sigma_\sigma(\lambda+\sigma)$ is a decreasing function of $\sigma$,
\[
\sigma_\sigma(\lambda+\sigma) \leq \sigma_\sigma(\lambda)
\]
By Theorem 1,
\[
\sigma_\sigma(\lambda) \leq \sigma
\]
for sufficiently large $\lambda$, so
\[
\sigma_\sigma(\lambda+\sigma) \leq \sigma < \sigma_\sigma
\]
for sufficiently large $\lambda$, which contradicts (15). Therefore, $\sigma_\sigma = S$. 

Theorem 4 [19, page 61]: Let \( \sigma > S \), \( \kappa \geq 0 \), \( 0 \leq \rho \leq \frac{1}{2} \), and
\[
M_\rho (\sigma, \kappa) = \text{Max} (\neg \rho (\sigma) - \kappa, 0),
\]
Then,
\[
(16) \quad \sigma (\kappa) \leq \alpha (1 - \rho) + \frac{\alpha (1 - \rho) - \sigma}{1 + \kappa - \rho + M_\rho (\sigma, \kappa)} \leq \sigma + M_\rho (\sigma, \kappa).
\]
Also,
\[
(17) \quad \alpha (1 - \rho) \leq \gamma (1 - \rho).
\]

Proof: From Theorem 3.9.2
\[
\alpha (1 - \rho) \leq \sigma (\kappa) + \kappa
\]
for \( \kappa \geq 0 \), \( 0 \leq \rho \leq \frac{1}{2} \), and hence, from (9)
\[
(18) \quad \alpha (1 - \rho) \leq \sigma (\neg \rho (\sigma)) + \neg \rho (\sigma) \leq \sigma + \neg \rho (\sigma).
\]
Thus, (17) is clear.

For \( M_\rho (\sigma, \kappa) = 0 \), (16) is identical to (9).
Otherwise, \( M_\rho (\sigma, \kappa) = \neg \rho (\sigma) - \kappa > 0 \). We set
\[
\kappa' = \neg \rho (\sigma) \quad \text{and} \quad \kappa'' = M_\rho (\sigma, \kappa) / (1 + \kappa - \rho + M_\rho (\sigma, \kappa)).
\]
It follows that
\[
\omega (1 - \rho) = \frac{(1 - \rho) (\neg \rho (\sigma) - \kappa)}{1 + \kappa - \rho + \neg \rho (\sigma) - \kappa} = \frac{\neg \rho (\sigma) (1 - \rho) - (1 - \rho) \kappa}{1 - \rho + \neg \rho (\sigma)}
\]
and

\[(1-\alpha \varepsilon) \chi' = -\sqrt{1-\varepsilon}(\sigma) \frac{1+\varepsilon-\varepsilon}{1-\varepsilon + v_{1-\varepsilon}(\sigma)} = \frac{-\sqrt{1-\varepsilon}(\sigma)(1-\varepsilon) + \chi_1 v_{1-\varepsilon}(\sigma)}{1-\varepsilon + v_{1-\varepsilon}(\sigma)} \].

Thus, \( \alpha \varepsilon (1-\varepsilon) \leq (1-\alpha \varepsilon) \chi' \), and Theorem 3.9.1 implies

\[\sigma_0 (\alpha \varepsilon (\varepsilon-1) + (1-\alpha \varepsilon) \chi') \leq \sigma_0 (\alpha \varepsilon + 1-\varepsilon) + (1-\alpha \varepsilon) \sigma_0 (\chi')\]

and since \( \sigma_0 (-\sqrt{1-\varepsilon}(\sigma)) \leq \sigma \),

\[(19) \quad \sigma_0 (\chi') \leq (\alpha \varepsilon + 1-\varepsilon) \left(1 - \frac{1+\varepsilon-\varepsilon}{1+\varepsilon-\varepsilon + M_0(\sigma, \chi')}\right) + \sigma_0 \left(\frac{1+\varepsilon-\varepsilon}{1+\varepsilon-\varepsilon + M_0(\sigma, \chi')}\right).\]

The first inequality of (16) follows after simplification.

Using (18), the right side of (19) is less than or equal to

\[\sigma + \sqrt{1-\varepsilon}(\sigma) + (1-\varepsilon) \frac{\sqrt{1-\varepsilon}(\sigma) + 1-\varepsilon}{1-\varepsilon + \sqrt{1-\varepsilon}(\sigma)} \leq \sigma + \sqrt{1-\varepsilon}(\sigma) - \chi.\]

This completes the proof of Theorem 4.

Theorem 5 [19, page 61]: Let \( \sigma > S \). Then

\[\psi_2 \leq \alpha \gamma_2 + \gamma_2 - \frac{\alpha \gamma_2 + \gamma_2 - \sigma}{1+\gamma_2 v_2(\sigma)} \leq \sigma + \sqrt{1-\varepsilon}(\sigma).\]

Proof: In (16) let \( \chi = 0 \), \( \varepsilon = \gamma_2 \) and use (11) with \( \varepsilon = \gamma_2 \).
Theorem 6 [19, page 67]: Let $0 \leq \rho, \leq \rho \leq 1$, $\sigma > S$, $\rho \geq \frac{1}{2}$.

Then

(20) $\nabla_{\rho} (\sigma + \rho - \rho) \leq \nabla_{\rho} (\sigma)$

and

(21) $\nabla_{\rho} + \rho \leq \nabla_{\rho} + \sigma$.

Proof: (21) is an immediate consequence of (20). Since, by Theorem 2.5.1, $\nabla_{\rho} (\sigma)$ is a convex function on the line segment from $(\rho, \sigma + \rho)$ to $(\sigma, \sigma)$,

$\nabla_{\rho} (\sigma + \rho - \rho) \leq \frac{\rho - \rho}{\rho} \nabla_{\rho} (\sigma + \rho) + \frac{\rho}{\rho} \nabla_{\rho} (\sigma)$.

Thus, in order to prove (20) we need only show that

(22) $\nabla_{\rho} (\sigma + \rho) \leq \nabla_{\rho} (\sigma)$

for $\sigma > S$, $\frac{1}{2} \leq \rho \leq 1$.

If $\sigma + \rho > \alpha$, then (22) holds, since the left side is $0$. Hence, we may assume $S < \sigma \leq \alpha + \rho$.

Now, \( \lim_{\lambda \to \infty} \sigma_{0}(\lambda) = S \)

and \( \alpha \leq \sigma_{0}(\lambda) + \lambda \)

for $\lambda > 0$ by Theorem 3.9.1.

Hence, by the continuity of $\sigma_{0}(\lambda)$, there exists a $\lambda_{0} > 0$ such that $\sigma_{0}(\lambda_{0}) = \sigma$.

Thus,

\( \sigma_{0}(\lambda_{0}) + \rho \leq \alpha \leq \sigma_{0}(\lambda_{0}) + \lambda_{0} \)

and \( \rho \leq \lambda_{0} \).
Applying the convexity of $\mathcal{V}_0(\sigma)$,

$$-\mathcal{V}_0(\sigma_1(x_0) + \rho) \leq \frac{x_0 - \rho}{x_0} \mathcal{V}_0(\sigma_1(x_0)) + \frac{\rho}{x_0} \mathcal{V}_0(\sigma_1(x_0) + x_0),$$

and since $\sigma_1(x_0) + x_0 \geq x$, implies $\mathcal{V}_0(\sigma_1(x_0) + x_0) = 0$,

$$\mathcal{V}_0(\sigma + \rho) = \mathcal{V}_0(\sigma_1(x_0) + \rho) \leq \frac{x_0 - \rho}{x_0} \mathcal{V}_0(\sigma_1(x_0)).$$

Now, (12) implies

$$-\mathcal{V}_0(\sigma + \rho) \leq x_0 - \rho.$$

On the other hand, since by Lemma 3.8.6, $\sigma_0(x + \rho)$ is a decreasing function of $\rho$,

$$\sigma \geq \sigma_1(x_0) \leq \sigma_1 - \rho (x_0 - \rho),$$

By (2.5.11), $-\mathcal{V}_0(\sigma)$ is a decreasing function of $\sigma$ and thus, (10) implies

$$-\mathcal{V}_0(\sigma) = \mathcal{V}_0(\sigma_1(x)) \geq \mathcal{V}_0(\sigma_1 - \rho (x_0 - \rho)) \geq x_0 - \rho.$$

This proves (22), and hence, (20) is proved.

4.2. Conclusion

We have now established all the results we need to prove Theorem 2.3.1. (2.7.1) follows from (2.6.5) and (4.1.21). (2.7.2) follows from Theorem 4.1.5, and (2.7.3) follows from (4.1.17).
LIST OF REFERENCES


[16] J. E. Littlewood, "Quelques consequences de l'hypothese que la fonction $\zeta(s+it)$ de Riemann n's pas de Zeros dans le demi plan $\sigma \geq \frac{1}{2}$," Comptes rendus de l'Academie des sciences, Paris, 154, 263-266, 1912.


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2. A GENERALIZATION OF A THEOREM
BY H. E. RICHERT

2.1. Introduction

This chapter contains a statement of the main theorem of this thesis, Theorem 2.3.1; a development of the analytic continuation of the generating function with which we will be concerned; a development of the theory of the Carlson \( \sqrt{\cdot} \) function which, although it is more general than that given by Richert [19, page 47], is in every respect the same; and the proof of Theorem 2.3.1, assuming the results of Chapters 3 and 4.

2.2. Hypotheses

Throughout this work we shall assume the following conditions:

\[(1) \quad a_1, a_2, \ldots, a_m, \ldots\]

is a sequence of complex numbers.

We define, for \( k = 1, 2, 3, \ldots \)

\[A_k(x) = \sum_{m_1 \cdots m_k < x} a_{m_1} \cdots a_{m_k}\]

\[a_k, m = A_k(m+1) - A_k(m).\]
Let \( a_{N_k}, N_k \) be the first non-zero element in the sequence
\[
\left\{ a_k, \downarrow \infty \right\}_{\downarrow \infty}.
\]

(2) For each integer \( k \geq 1 \), we assume
\[
L_k(x) = \begin{cases} 
    c_k \frac{\sum_{\downarrow 1}^{\uparrow x} \delta_k(t) \, dt}{g_{k+1}}, & x > N_{k+1} \\
    0, & 1 \leq x \leq N_{k+1}
\end{cases}
\]
where \( c_k > 0 \) and \( \delta_k(t) \) is bounded and Lebesgue integrable on each interval \([N_{k+1}, x] \) and \( \delta_k(t) \to 0 \) as \( t \to \infty \).

It is easily verified that \( L_k(x) \) is a slowly oscillating function [13], that is, \( L_k \) is continuous and positive valued, and
\[
\lim_{x \to \infty} \frac{L_k(x)}{L_k(x)} = 1
\]
for every \( C > 0 \). It is also easily shown that
\[
L_k(x) = O(x^\epsilon), \quad \{L_k(x)\}^{\frac{1}{\epsilon}} = O(x^\epsilon)
\]
for each \( \epsilon > 0 \).

(3) We assume \( L_k(x) = L_k(e^x) \) has an analytic continuation, \( L_k(z) \), into some region containing the region
\[
|\arg z| \leq \phi_k, \quad |z| \geq \log(N_{k+1})
\]
Further, we assume

\[ L_K(z) = O(L_K(|z|)) \]

as \(|z| \to \infty\), uniformly for \(|\arg z| \leq \phi_K\).

Thus, \( x L_K(x) \) has derivatives of all orders for \( x > N_K + 1 \), and

\[ \frac{d}{dx} (x L_K(x)) = L_K(x) (1 + c_k \delta_K(x)) \]

almost everywhere, and we may assume \( \delta \) is such that equality holds everywhere. It follows that \( x L_K(x) \) is monotone increasing for \( x \geq M_K \), for some \( M_K \geq N_K \).

In all the proofs we shall assume \( c_K = 1 \), but this is no restriction.

\(4\) We assume, for \( K = 1, 2, 3, \ldots \)

\[ A_K(x) = \sum_{m=1}^{\infty} a_{K,m} = x L_K(x) + O(x^{\theta_K}) \]

where \( 0 \leq \theta_K < 1 \). We let \( Q_K \) be the greatest lower bound of such \( \theta_K \). Thus \( 0 \leq Q_K < 1 \).

2.3. The Main Theorem

Under the hypotheses of Section 1, we have:

Theorem 1: Let \( K \) be an integer greater than 2. Then

\[ Q_1 \leq (K-1)/(2K) \] \text{implies} \[ Q_K \geq (K-1)/(2K) \]

and \( Q_1 < 1/2 \) \text{implies} \[ Q_2 \geq 1/4 \].
The proof of Theorem 1 is given in Section 7. First, we shall develop the language and some of the theorems needed for that proof.

2.4. An Analytic Continuation

In the next three sections, we fix \( \kappa \geq 1 \) and delete the subscript \( \kappa \) from all quantities.

We define

\[
B(x) = A(x) - x \ln(x), \quad \Omega(x) = B(e^x), \quad \Omega(x) = A(x).
\]

Let \( s = \sigma + it \), \( \sigma \) and \( t \) real. Then the integral

\[
(1) \quad \gamma(s) = \int_1^\infty x^{-s} \, d\, B(x) = \int_0^\infty e^{-sx} \, d\, \Omega(x)
\]

converges for \( \sigma > Q \), and

\[
(2) \quad \gamma(s) = s \int_0^\infty e^{-sx} \, d\, \Omega(x)
\]

whenever the integral (1) converges and \( \sigma > 0 \). Also, (2) converges absolutely for \( \sigma > Q \) [27, pages 35-41].

Similarly,

\[
(3) \quad Z(s) = \int_1^\infty x^{-s} \, d\, A(x) = s \int_0^\infty A(x) \, e^{-sx} \, dx
\]

and
(4) \( W(s) = \int_{0}^{\infty} x^{-s} \, L(x) \, dx = s \int_{0}^{\infty} (e^x \, L(e^x)) \, e^{-xs} \, dx \)

for \( \sigma > 1 \), since \( L(x) = O(x^\epsilon) \) for each \( \epsilon > 0 \).

These integrals are all analytic functions of \( s \) in their regions of convergence [27, page 57].

Now, for \( \sigma > 1 \),

\[
\mathcal{Z}(s) = s \int_{0}^{\infty} a(x) \, e^{-xs} \, dx
\]

\[
= s \int_{0}^{\infty} b(x) \, e^{-xs} \, dx + s \int_{0}^{\infty} e^x \, L(x) \, e^{-xs} \, dx
\]

\[
= \mathcal{Y}(s) + W(s).
\]

We need an analytic continuation for

\[
W(s) s^{-1} = \int_{\log(N+1)}^{\infty} L(x) \, e^{-x(s-1)} \, dx.
\]

For each \( R > \log(N+1) \), let \( \mathcal{C}_R \) be the boundary, in the positive sense, of the region

\[
R \geq |z| \geq \log(N+1) \quad , \quad 0 \leq \arg{z} \leq \phi.
\]

Applying the Cauchy theorem to \( \mathcal{C}_R \), we have, for \( \sigma > 1 \),
A GENERALIZATION OF AN $\Omega$ RESULT
IN MULTIPLICATIVE NUMBER THEORY

DISSERTATION
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BY
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*****

The Ohio State University
1962

Approved by

Adviser
Department of Mathematics
\[
\int_{\log(N+n)}^{R} L(x) e^{-x(\sigma-1)} \, dx
\]

\[
= -iR \int_{0}^{\phi} L(R e^{i\theta}) e^{-R(\sigma-1)e^{i\theta}} e^{i\theta} \, d\theta
\]
\[
+ i \log(N+n) \int_{0}^{\phi} L(e^{i\theta} \log(N+n)) e^{-(\log(N+n))(\sigma-1)e^{i\theta}} e^{i\theta} \, d\theta
\]
\[
+ \int_{\log(N+n)}^{R} L(x e^{i\phi}) e^{-x(\sigma-1)} e^{i\phi} e^{i\phi} \, dx
\]

\[
= I_1 + I_2 + I_3 + I_4.
\]

Now, for \( \pi/2 > \phi > \epsilon > 0 \), \( \cos \phi > 0 \).

Let \( 0 < \epsilon < (\sigma-1)(\cos \phi)/2 \), then

\[
I_1 = O(R e^{\epsilon R} e^{-R(\sigma-1) \cos \phi}) = O(R e^{-\epsilon R}).
\]

Thus, letting \( R \to \infty \),

(5) \( W(\sigma) e^{-\sigma} = \sum_{\log(N+n)}^{\infty} L(x e^{i\phi}) e^{-x(\sigma-1)} e^{i\phi} e^{i\phi} \, dx \)

\[
+ i \log(N+n) \int_{0}^{\phi} L(e^{i\theta} \log(N+n)) e^{-(\log(N+n))(\sigma-1)e^{i\theta}} e^{i\theta} e^{i\theta} \, d\theta
\]

\[
= g_1(\sigma) + g_2(\sigma)
\]

for \( \sigma > 1 \).

A similar equation holds if we replace \( \phi \) by \( -\phi \).
We now consider the region of convergence of
\[ \int_{\log(s+1)}^{\infty} L(x e^{i\phi}) e^{-e^{i\phi}x(s-1)} e^{i\phi} \, dx. \]

This integral converges uniformly and absolutely in the half plane
\[ \text{RE} (s-1) e^{i\phi} \geq \varepsilon \]
for \( \varepsilon > 0 \).

This is the region
\[ (\cos \phi)(\sigma-1) - t \sin \phi \geq \varepsilon \]
or
\[ t \leq (\cot \phi) \sigma - \cot \phi - \varepsilon / \sin \phi. \] (6)

If we replace \( \phi \) by \(-\phi\), we obtain the region
\[ t \geq -(\cot \phi) \sigma + \cot \phi + \varepsilon / \sin \phi. \] (7)

Considering the right side of (5) with \( \sigma \) replaced by \( s \), both with \( \phi \) and \(-\phi\), we have two functions that agree with \( W(s) s^{-\frac{1}{2}} \) for \( s = \sigma > 1 \) and that are each analytic in certain regions, (6) and (7) respectively. Therefore, we have an analytic continuation for \( W(s) s^{-\frac{1}{2}} \) into a region, \( \mathcal{I} \), which is defined as follows:
\[ -\phi - \pi / 2 \leq \text{Arg} \left( s-1 \right) < \phi + \pi / 2. \]
We thus have an analytic continuation for $Z(s)$ into the intersection of the half plane $\sigma > \sigma^*$ with $\mathcal{D}$.

We now investigate the behavior of $W(s) s^{-1}$ on a vertical line segment $\sigma = \sigma^*$, $t > T_0 > 0$. We may choose $T_0$ such that the segment lies in the region of convergence of

$$q_i(s) = \int_{\log(N+1)}^{\infty} \frac{e^{-ix}}{x} e^{- i \phi} e^{i (s-1) \phi} e^{- i \phi} \, dx$$

$$= \Theta \left( \int_{0}^{\infty} e^{-x (\cos \phi + t \sin \phi - \varepsilon)} \, dx \right)$$

$$= \Theta \left( \frac{1}{i (\sigma-1) \cos \phi + t \sin \phi - \varepsilon} \right)$$

$$= \Theta (t^{-1})$$

as $t \to \infty$, uniformly for $\alpha \leq \sigma$, $t \geq T_0(a)$ for each real $a$.

Now,

$$q_{\tau}(s) = -i \log(N+1) \int_{0}^{\Phi} \frac{e^{-i \phi}}{\log(N+1)} e^{-i\phi} e^{i (s-1) \phi} e^{- i \phi} \, d\theta$$

$$= \Theta \left( \int_{0}^{\Phi} e^{-\log(N+1) (\cos \theta + t \sin \theta)} \, d\theta \right)$$

$$= \Theta \left( \int_{0}^{\Phi} e^{-kt \sin \theta} \, d\theta \right), \quad k > 0$$
\[ = \Theta \left( \int_0^\delta e^{-\lambda t} \sin \delta \, d\delta \right) + \Theta \left( \int_\delta^\infty e^{-\lambda t} \sin \delta \, d\delta \right) \]
\[ = \Theta \left( \frac{1 + e^{-\lambda (\delta(1-\epsilon)) t}}{\lambda (1-\epsilon) t} \right) + \Theta \left( e^{-\lambda t} \sin \delta \right) \]
\[ = \Theta (t^{-1}) \]

as \( t \to \infty \), uniformly in the region \( \alpha \leq \sigma \), \( t \geq T_0(\alpha) \), for each real \( \alpha \).

Therefore,

\[ W(s) s^{-1} = \Theta (|g_0(s)|) + \Theta (|g_2(s)|) \]

(8) \[ W(s) s^{-1} = \Theta (|t|^{-1}) \]

uniformly for \( \alpha \leq \sigma \), \( t \geq T_0(\alpha) \), for each real \( \alpha \).

A similar argument replacing \( -\phi \) by \( \phi \) yields (8) uniformly for \( \alpha \leq \sigma \), \( |t| \geq T_0(\alpha) \), for each real \( \alpha \).

It is clear from (2) that

(9) \[ \Psi(s) = \Theta (|t|) \]

as \( |t| \to \infty \), uniformly for \( Q + \epsilon \leq \sigma \leq Q + K \), for each \( \epsilon \) and \( K \) such that \( 0 < \epsilon < K \).
Finally, we show there is a $\sigma^* \geq 0$ such that

$\sigma \geq \sigma^*$ implies that

$$|Y(s)| \geq K(\sigma) > 0.$$ (10)

First,

$$\int_1^{\infty} x^{-\sigma} dV_B(1,x) \leq \int_1^{\infty} x^{-\sigma} dV_A(1,x) + \int_1^{\infty} x^{-\sigma} dV(u,L(u))(1,x).$$

It is well known [11, Theorem 9] that $\int_1^{\infty} x^{-\sigma} dV_A(1,x)$ converges for $\sigma > 2$, and from our assumptions on $L(u)$,

$$\int_1^{\infty} x^{-\sigma} dV(u,L(u))(1,x)$$

converges for $\sigma > 1$.

Let $\alpha_1$ be the abscissa of convergence of

$$\int_1^{\infty} x^{-\sigma} dV_A(1,x).$$

Then $1 \leq \alpha_1 \leq 2$, and $\alpha_1$ is the abscissa of the absolute convergence of both

$$\int_1^{\infty} x^{-\sigma} dA(x) \quad \text{and} \quad \int_1^{\infty} x^{-\sigma} dB(x).$$

We have that

$$|Y(s)| = |\int_1^{\infty} x^{-\sigma} dA(x)| = |\int_1^{\infty} x^{-\sigma} dA(x)|
\geq |N^{-6} a_N| - |\int_{N+1}^{\infty} x^{-\sigma} dB(x)|.$$
Now, 
\[
\int_{n+1}^{\infty} x^{-5} \, dB(x) = (n+1)^{-5} \int_{n+1}^{\infty} \left( \frac{x}{n+1} \right)^{-5} \, dB(x).
\]

When \( \sigma > \alpha \), this integral converges absolutely, and 
\[
\left| \int_{n+1}^{\infty} (x/(n+1))^{-5} \, dB(x) \right| \leq \int_{n+1}^{\infty} (x/(n+1))^{-\sigma} \, dB(x).
\]

This last integral is a decreasing function of \( \sigma \). Thus, 
\[
\left| \int_{n+1}^{\infty} x^{-5} \, dB(x) \right| \leq (n+1)^{-\sigma} K_2
\]

for \( \sigma > \alpha \), where \( K_2 \) is some positive constant.

Therefore, 
\[
\left| \int_{1}^{\infty} x^{-5} \, dB(x) \right| \geq \left( |a_n| - K_2 \left( \frac{n+1}{n} \right)^{-\sigma} \right) (n)^{-\sigma}
\]

for \( \sigma > \alpha \).

Hence, 

(11) \[
\left| \int_{1}^{\infty} x^{-5} \, dB(x) \right| \geq \left( |a_n|/2 \right) n^{-\sigma} > 0
\]

for \( \sigma > \sigma^* > \alpha_1 \geq 1 \).

A similar argument shows that there is a \( \sigma^* \) such that

(12) \[
|Z(s)| \geq \left( |a_n|/2 \right) n^{-\sigma} > 0
\]

for \( \sigma \geq \sigma^* \).
2.5. The Carlson Function

We shall give here the proofs of some properties of the Carlson function [19, page 47].

Definition 1: Let \( P(Y) = \inf \{ x \mid x \geq 0, \text{ and there exists a } \xi \text{ such that } Y(s) \text{ is analytic and } O(|t|^f) \text{ uniformly for } \sigma \geq x, |t| \geq T_0(x) \text{ for some } T_0(x) > 0 \} \).

In the same way, we define \( P(Z) \).

We define \( S = S(Y) \) in a similar way except that we require \( Y \) to be analytic in the entire region \( \sigma \geq x \). We do not define \( S(Z) \).

Definition 2: Let \( \sigma > P(Y) \), \( 0 < \varepsilon \leq 1 \). Then the Carlson function, \( \mathcal{V}_\rho(\sigma) \), is defined by

\[
\mathcal{V}_\rho(\sigma, Y) = \mathcal{V}_\rho(\sigma) = \inf \left\{ \xi \left( \frac{1}{\pi} \int_{-\pi}^{\pi} |Y(\sigma + it)|^{\rho} dt \right)^{\frac{1}{\rho}} \right\} = O(T^f)
\]

where \( S_{T_0}^{T} \) means \( S_{T_0}^{T} = S_{T_0}^{T_0} + S_{T_0}^{T} \) with \( T_0 \geq T_0(Y, \sigma) \).

\( \mathcal{V}_\rho(\sigma, Z) \) is defined in the same way.

The analytic continuation established for \( \mathcal{W}(s) \) together with (4.8) show that \( P(Y) = P(Z) = P \), and from (4.9),

\[
0 \leq P \leq S \leq Q \leq 1.
\]

We have defined the Carlson function for functions of type \( Y \) and \( Z \). We will state and prove the theorems in terms of \( Y \), but they also hold for \( Z \).
We wish to show that Definition 2 is independent of the choice of $T_0$. We first see that $Y(\sigma+it)$ is not identically 0 for $t \geq T_0$, for this would imply

$Y(s) \equiv 0$, which contradicts (4.10) (4.12) for $Z$).

Thus,

$$\int_{-T}^{T} |Y(\sigma+it)|^{\frac{1}{2}} dt > K, > 0$$

for $T \geq T_1, > T_0$.

Let $K_2$ be a real number, then

$$\left( \frac{1}{T} \int_{-T}^{T} |Y(\sigma+it)|^{\frac{1}{2}} dt \right)^{\frac{1}{2}} + \frac{K_2}{T}$$

$$\leq \left( \frac{1}{T} \int_{-T}^{T} |Y(\sigma+it)|^{\frac{1}{2}} dt \right)^{\frac{1}{2}} + \left( \frac{|K_2|}{T} \right)^{\frac{1}{2}}$$

$$= O\left( \left( \frac{1}{T} \int_{-T}^{T} |Y(\sigma+it)|^{\frac{1}{2}} dt \right)^{\frac{1}{2}} \right)$$

as $T \to \infty$.

This completes the proof that Definition 2 is independent of the choice of $T_0$ if we let

$$K_2 = \pm \left( \int_{-T_0}^{T_0} \int_{-T_0}^{T_0} |Y(\sigma+it)|^{\frac{1}{2}} dt \right).$$

We used here: (with $\rho = \frac{1}{2}$, $q = 1$)

Jensen's Inequality [10, Theorems 19 and 167]:

If $0 < \rho < q$, then

(1) $\left( \sum |a_j|^q \right)^{\frac{\rho}{q}} \leq \left( \sum |a_j|^\rho \right)^{\frac{1}{\rho}}$.
We shall also use:

Hölder's Inequality [10, Theorems 11 and 188]:

If \( \alpha > 0 \), \( \beta > 0 \) and \( \alpha + \beta = 1 \), and \( \mu \) is a positive measure, then

\[
(2) \quad \int |f^r g^s| \, d\mu \leq \left( \int |f|^{\alpha} \, d\mu \right)^{\alpha} \left( \int |g|^{\beta} \, d\mu \right)^{\beta},
\]

and as a special case we have

\[
(3) \quad \sum |d_n e_m| \leq \left( \sum |d_n|^{\alpha} \right)^{\alpha} \left( \sum |e_m|^{\beta} \right)^{\beta}.
\]

As a special case of this and Jensen's Inequality, we have

\[
(4) \quad \left( \sum_{j=1}^{K} |d_j| \right)^{\frac{1}{\alpha}} \leq K^{\frac{\alpha - 1}{\alpha}} \left( \sum_{j=1}^{K} |d_j| \right)^{\frac{1}{\alpha} - \phi},
\]

with \( \phi = 1 \) if \( 0 < \alpha < 1 \), \( \phi = \frac{1}{\alpha} \) if \( \alpha > 1 \).

Minkowski's Inequality [10, Theorem 198]:

If \( 0 < \rho < 1 \), then

\[
(5) \quad \left( \int |f + g|^{\rho} \, d\mu \right)^{\rho} \leq \left( \int |f|^{\rho} \, d\mu \right)^{\rho} + \left( \int |g|^{\rho} \, d\mu \right)^{\rho}.
\]

For \( \sigma > \rho = P(Y) \), \( 0 < \rho < \rho \leq 1 \), let \( \alpha = \rho / \sigma \),

\( \beta = 1 - \rho \), \( \beta^{-1} = (\rho - 1) / \sigma^{-1} \). Then \( \alpha, \beta > 0 \) and \( \alpha + \beta = 1 \),
and applying Hölder's Inequality, we have
\[
\left(\frac{1}{T} \int_T^\infty |y^\sigma(t)|^{\frac{1}{\sigma}} \, dt\right)^\sigma \leq \left(\frac{1}{T} \int_T^\infty |y^\sigma(t)|^{\frac{1}{\sigma'}} \, dt\right)^{\sigma'} \left(\frac{1}{T} \int_T^\infty dt\right)^{\sigma-\sigma'} = O\left(T^{-\psi_0(\sigma) + \epsilon}\right).
\]

Therefore,

\[
\psi_\rho(\sigma) \leq \psi_\rho(\sigma)
\]

for \(\sigma > P\), \(0 < \epsilon < \rho \leq 1\).

**Definition 3:** For \(\sigma > P\), we let \(\psi_0(\sigma) = \lim_{\rho \to 0} \psi_\rho(\sigma)\).

This limit exists because of (6).

**Definition 4:** A real valued function \(f(x,y)\) is said to be convex on a convex region \(\mathcal{F}\) of the \((x,y)\)-plane in case \((x_1,y_1), (x_2,y_2) \in \mathcal{F}\) and \(0 \leq \sigma \leq 1\) imply

\[
f(\sigma x_1 + (1-\sigma)x_2, \sigma y_1 + (1-\sigma)y_2) \leq \sigma f(x_1,y_1) + (1-\sigma)f(x_2,y_2).
\]

**Theorem 1:** Let \(G\) be the region of the \((\sigma, \rho)\)-plane defined by \(\sigma > P\), \(0 \leq \rho \leq 1\). Then on \(G\), \(\psi_\rho(\sigma)\) is

\(\psi_\rho(\sigma)\) a continuous function,

\(\psi_\rho(\sigma)\) a convex function,

\(\psi_\rho(\sigma)\) a decreasing function of \(\rho\),
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(11) a decreasing function of $\sigma$, 
(12) non-negative, 
(13) 0 for $\sigma > \alpha$, 

and 

(14) $\nu_\sigma(\sigma) + \varepsilon$ is an increasing function of $\sigma$. 

The proof of Theorem 1 occupies the rest of this section.

Lemma 1: For each $\sigma > \rho$, $\nu_\sigma(\sigma)$ is a convex function of $\rho$ for $0 < \rho \leq 1$.

Proof: Let $0 < \rho_1 < \rho_2 \leq 1$, $\sigma > \rho$, $0 < \rho_0 < 1$ and $\rho = \rho_0 \rho_1 + (1 - \rho_0) \rho_2$. We apply Hölder's Inequality (2) with $\alpha = \frac{\rho_1}{\rho}$, $\beta = 1 - \alpha = \frac{(1 - \rho_0) \rho_2}{\rho}$ to obtain

$$
\left(\frac{1}{\alpha} \int_0^\tau |Y(\sigma + \rho t)|^{\alpha \rho_1} dt\right)^\frac{1}{\alpha} \leq \left(\frac{1}{\alpha} \int_0^\tau |Y(\sigma + \rho_1 t)|^{\alpha \rho_1} dt\right)^\frac{1}{\alpha} \left(\frac{1}{\beta} \int_0^\tau |Y(\sigma + \rho_2 t)|^{\beta \rho_2} dt\right)^\frac{1}{\beta}.
$$

This proves Lemma 1.

Lemma 2: $\nu_\sigma(\sigma)$ is convex on $G_1: \sigma > \rho$, $0 < \rho \leq 1$.

Proof: It is sufficient to show that (7) holds for

$\sigma_2 > \sigma_1 > \rho$, $0 < \rho_1, \rho_2 \leq 1$ and $0 < \rho_0 < 1$. Set $\sigma = \rho_0 \sigma_1 + (1 - \rho_0) \sigma_2$, $\rho = \rho_0 \rho_1 + (1 - \rho_0) \rho_2$ and $s = \sigma + it$ with $T_0(\sigma) = T_0 = \frac{4}{t^2} |t| \leq \tau$. Let $I$ be the rectangle
with corners

$$\sigma_j + i \text{ sgn} (\tau) \tau \to, \quad \sigma_j + 2i\tau, \quad j = 1, 2.$$  

Integrating in a positive direction with \( \gamma > 0 \), independent of \( z = x + iy \), by the Cauchy integral formula

$$\gamma(z) = \frac{1}{2\pi i} \int_C \frac{e^{(z-s)^2}}{(z-s)} \gamma(s) \, ds,$$

Since we are to the right of \( P \), on both horizontal paths we have

$$\sigma_1 - \sigma \leq \text{Re}(z-s) \leq \sigma_2 - \sigma$$  

$$\text{Re}((z-s)^2) = (x-\sigma)^2 - (y-t)^2$$  

$$\leq K - (\frac{A}{\lambda})^2$$

uniformly in \( t \), and by (4.9)

$$\gamma(z) = \Theta(12t) = \Theta(e^{t\lambda}).$$

Therefore, the integrals on these paths are

$$\Theta\left( (\gamma^{\sigma_1-\sigma} + \gamma^{\sigma_2-\sigma}) e^{-t\lambda/2} \right)$$

uniformly in \( \gamma \).

On the left vertical path, we have

$$\text{Re}(z-s) = \sigma - \sigma, \quad e^{(z-s)^2} = \Theta(e^{-(\gamma-t)^2})$$;
on the right vertical path,

\[ RL(z-s) = \sigma_2 - \sigma, \quad e^{(z-s)^2} = O(e^{-|y-t|^2}). \]

Thus,

\[ \psi(s+it) = O\left( \sum_{n=1}^{\infty} e^{-n|y|}\int_{-\infty}^{\infty} e^{-|y|^2} \psi(\sigma_2 + i y) dy + e^{\frac{|y|^2}{2}} \right). \]

We may replace \( \int_{-\infty}^{\infty} \psi(\sigma_2 + iy) dy \) by \( \int_{-\infty}^{\infty} \psi(\sigma_2 + iy) dy \). We then choose \( \tau \) so as to balance the two terms on the right of (15). If we are to have the following relation,

\[ \gamma^{\sigma_i-\sigma} \beta_1(t) = \gamma^{\sigma_2-\sigma} \beta_2(t) \quad \text{or} \quad \beta_1(t), \beta_2(t) > 0, \]

then

\[ \gamma^{\sigma_2-\sigma_1} = \beta_1(t)/\beta_2(t) \]

or

\[ \gamma = \left( \frac{\beta_1(t)}{\beta_2(t)} \right)^{\frac{1}{\sigma_1-\sigma_2}} > 0. \]

In this case we have

\[ \gamma^{\sigma_1-\sigma} \beta_1(t) = \beta_2(t) \left( \frac{\sigma_1-\sigma}{\sigma_2-\sigma_1} \right) (\beta_1(t))^{\frac{\sigma_1-\sigma}{\sigma_2-\sigma_1}} = \beta_2(t)^{1-\alpha'} \beta_1(t)^{\alpha'} \]

since

\[ \frac{\sigma_1-\sigma}{\sigma_2-\sigma_1} = -\frac{\sigma_1-\sigma_1(1-\alpha')\sigma_2}{\sigma_2-\sigma_1} = 1-\alpha'. \]
It is now clear that for the proper choice of \( \gamma > 0 \),

\[
Y(\sigma+it) = \Theta \left( \left\{ \int_{-1+i\epsilon}^{1+i\epsilon} e^{-\left(\gamma - \epsilon \right)^2} |Y(\sigma+i\gamma)| dy + e^{-t/2} \right\}^{\frac{1}{\gamma}} \right)
\]

for \( |t| > 4T_0 \), \( \sigma_1 < \sigma < \sigma_2 \).

We apply Hölder's Inequality (2) first with

\[\alpha = \rho, \quad \beta = 1 - \rho, \quad \text{and then with} \quad \beta = 1 - \rho_2, \quad \alpha = \rho_2\]

to obtain

\[
Y(\sigma+it) = \Theta \left( \left\{ \int_{-1+i\epsilon}^{1+i\epsilon} e^{-\left(\gamma - \epsilon \right)^2} |Y(\sigma+i\gamma)| \right\}^{\alpha} \left\{ \int_{-1+i\epsilon}^{1+i\epsilon} e^{-\left(\gamma - \epsilon \right)^2} dy \right\}^{\beta} \right)^{\frac{1}{\alpha}}
\]

\[
\times \left\{ \left\{ \int_{-1+i\epsilon}^{1+i\epsilon} e^{-\left(\gamma - \epsilon \right)^2} |Y(\sigma_2+i\gamma)| \right\}^{\alpha_2} \left\{ \int_{-1+i\epsilon}^{1+i\epsilon} e^{-\left(\gamma - \epsilon \right)^2} dy \right\}^{\beta} \right\}^{\frac{1}{\alpha_2}}
\]

(\text{It should be noted that if} \( \rho \), \text{or} \( \rho_2 = 1 \), \text{Hölder's Inequality does not apply, but the equation is true anyway.})

We see that

\[
\int_{-1+i\epsilon}^{1+i\epsilon} e^{-\left(\gamma - \epsilon \right)^2} dy = \Theta \left( \int_{-3+i\epsilon}^{3+i\epsilon} e^{-u^2} du \right) = \Theta(1)
\]

so

\[
Y(\sigma+it) = \Theta \left( \left\{ \int_{-1+i\epsilon}^{1+i\epsilon} e^{-\left(\gamma - \epsilon \right)^2} |Y(\sigma+i\gamma)| \right\}^{\frac{1}{\rho}} \left\{ \int_{-1+i\epsilon}^{1+i\epsilon} e^{-\left(\gamma - \epsilon \right)^2} dy \right\}^{\frac{1}{1-\rho}} \right)^{\frac{1}{\rho}}
\]

\[
\times \left\{ \left\{ \int_{-1+i\epsilon}^{1+i\epsilon} e^{-\left(\gamma - \epsilon \right)^2} |Y(\sigma_2+i\gamma)| \right\}^{\frac{1}{\rho_2}} \left\{ \int_{-1+i\epsilon}^{1+i\epsilon} e^{-\left(\gamma - \epsilon \right)^2} dy \right\}^{\frac{1}{1-\rho_2}} \right\}^{\frac{1}{\rho_2}}
\]
Integrating with $T_{0}^{*} = 4 T_{0} (\sigma_{1})$ , we have

$$F_{0} (\tau) \equiv \int_{-T}^{T} \left[ y (\sigma + i \tau) \right]^{1/2} d\tau$$

$$= \Theta \left( \int_{-T}^{T} \left\{ \int_{-2}^{2} e^{-y (y - t)^{2}} \left[ y (\sigma + i t) \right]^{1/2} dy \right\}^{2} + e^{-\frac{t^{2}}{2}} \right)^{\frac{1}{2}}$$

$$\times \left( \int_{-2}^{2} e^{-2(y - t)^{2}} \left[ y (\sigma + i t) \right]^{1/2} dy \right)^{2} + e^{-\frac{t^{2}}{2}} \right)^{\frac{1}{2}}$$

Again applying Hölder's Inequality (2) with $\alpha = \frac{\alpha 0}{\tau}$, $\beta = 1 - \alpha = 1 - \frac{\alpha 0}{\tau} = \left( 1 - \frac{\alpha 0}{\tau} \right) \frac{1}{\tau}$, we have

$$F_{0} (\tau) = \Theta \left( \int_{-T}^{T} \left\{ \int_{-2}^{2} e^{-y (y - t)^{2}} \left[ y (\sigma + i t) \right]^{1/2} dy \right\}^{2} + e^{-\frac{t^{2}}{2}} \right)^{\frac{1}{2}}$$

$$\times \left( \int_{-2}^{2} e^{-2(y - t)^{2}} \left[ y (\sigma + i t) \right]^{1/2} dy \right)^{2} + e^{-\frac{t^{2}}{2}} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}$$

We can, of course, replace $\pm 2 t$ by $\pm 2 T$ as the limits of the inner integral to obtain a weaker estimate, and then from (4) we get

$$F_{0} (\tau) = \Theta \left( \int_{-T}^{T} \left\{ \int_{-2}^{2} e^{-y (y - t)^{2}} \left[ y (\sigma + i t) \right]^{1/2} dy \right\}^{2} + e^{-\frac{t^{2}}{2}} \right)^{\frac{1}{2}}$$

$$\times \left( \int_{-2}^{2} e^{-2(y - t)^{2}} \left[ y (\sigma + i t) \right]^{1/2} dy \right)^{2} + e^{-\frac{t^{2}}{2}} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}$$

$$= \Theta \left( \int_{-T}^{T} \left\{ \int_{-2}^{2} e^{-y (y - t)^{2}} dt \right\}^{2} + e^{-\frac{t^{2}}{2}} \right)^{\frac{1}{2}}$$

$$\times \left( \int_{-2}^{2} e^{-2(y - t)^{2}} dt \right)^{2} + e^{-\frac{t^{2}}{2}} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}$$

$$= \Theta \left( \int_{-T}^{T} \left\{ \int_{-2}^{2} e^{-y (y - t)^{2}} dt \right\}^{2} + e^{-\frac{t^{2}}{2}} \right)^{\frac{1}{2}}$$

$$\times \left( \int_{-2}^{2} e^{-2(y - t)^{2}} dt \right)^{2} + e^{-\frac{t^{2}}{2}} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}$$
Hence, from (16) we get
\[
\left( \frac{F_\rho(T)}{T} \right)^\rho = \Theta \left( \frac{\varepsilon}{T} \sum_{k=1}^{T} |Y(\sigma + iy)|^{\mu k} \, dy + \frac{\varepsilon}{T} \right)
\]
since 
\[
\sum_{t} \left[ Y(\sigma + iy) \right]^{\mu k} \, dy > K > 0
\]
for 
\[
T \geq T_1, \quad \rho = 1, 2
\]
we have
\[
\left( \frac{F_\rho(T)}{T} \right)^\rho = \Theta \left( T^{\mu k} \left[ Y(\sigma + iy) \right]^{\mu k} \, dy + \frac{\varepsilon}{T} \right)
\]
This proves Lemma 2.

We can now prove (12) and (13). Consider \( Y(\sigma) \) on a vertical line to the right of the abscissa of absolute convergence \( \alpha \). There \( Y(\sigma) \) is uniformly bounded in \( t \), and therefore,
\[
(18) \quad Y_\sigma(t) \leq \delta
\]
for \( \sigma > \alpha \), and \( 0 \leq \rho \leq 1 \).

From (4.11), for \( \sigma > \sigma^* \), \( |Y(\sigma)| \geq \left( \frac{1}{2} \right) N^{\sigma > \alpha} \) and hence, \( Y_\sigma(\sigma) = 0 \) for \( \sigma > \sigma^* \).

We point out that this last paragraph also holds for \( Z(\tau) \), using (4.12) instead of (4.11).
For $P < \sigma_1 \leq \sigma$, $0 < \rho \leq 1$, $\sigma_2 > \max(\sigma, \sigma^*)$ and $\psi = (\sigma_2 - \sigma)/(\sigma_2 - \sigma_1)$, we may apply Lemma 2 to obtain

$$\psi_0(\sigma) \leq \frac{\sigma - \sigma_1}{\sigma_2 - \sigma_1} \psi(\sigma_1) + \frac{\sigma_2 - \sigma}{\sigma_2 - \sigma_1} \psi(\sigma_2)$$

(19) \[ \psi_0(\sigma) \leq \frac{\sigma - \sigma_1}{\sigma_2 - \sigma_1} \psi(\sigma_1) . \]

Therefore, (19) also holds for $\rho = 0$. If we also assume $\sigma > \sigma^*$, we obtain

$$0 \leq \frac{\sigma_2 - \sigma}{\sigma_2 - \sigma_1} \psi(\sigma_1)$$

or

$$0 \leq \psi_0(\sigma) .$$

(12) together with (18) imply (13).

(10) follows from (6) and the definition of (11), when $0 < \rho \leq 1$, is an immediate consequence of convexity and the fact that $\psi_0(\sigma) = 0$ for $\sigma > \alpha$. (11) then follows from the definition of $\psi_0(\sigma)$.

Lemma 3: If $\sigma > P$, then

$$\psi_0(\sigma) = \mu(\sigma) = \inf \{ \xi | Y(\sigma + \xi) = 0 \}$$

where $\mu(\sigma)$ is the Lindelöf $\mu$ function for $Y$ (or $Z$), [11, page 14].
Proof: Recall (17): For \( P \leq \sigma, \leq \delta < \sigma \), \( \delta < \rho < 1 \),
\( \rho < \rho_1, \rho_2 \leq 1 \),
\[
  \Psi(\sigma+iT) = \Theta\left( \left( \sum_{2t}^{T} e^{-\left(y-T\right)^2} |\Psi(\sigma+iy)|^{\rho} dy \right)^{\frac{1}{\rho}} + e^{-\frac{T^2}{2}} \right)^{\frac{1}{2}}
\]

In this estimate we choose \( \delta = \delta - \rho \),
\( \sigma_1 = \sigma - \delta \),
\( \sigma_2 = \sigma + \delta \), \( \rho_1 = \rho_2 = \rho > 0 \), \( \rho_2 = \nu \frac{1}{2} \) and \( T = |t| \); then
\[
  \Psi(\sigma+iT) = \Theta\left( \left( \sum_{2t}^{T} e^{-\left(y-T\right)^2} |\Psi(\sigma+iy)|^{\rho} dy \right)^{\frac{1}{\rho}} + e^{-\frac{T^2}{2}} \right)^{\frac{1}{2}}
\]

\[
  = \Theta\left( T \frac{\nu_1(\sigma-\delta) + \nu_1(\sigma+\delta)}{2} + \varepsilon \right).
\]

And so for \( P \leq \sigma - \delta < \sigma \), \( \delta < \rho < 1 \), we have

(20) \[ \mu(\sigma) = (\nu_1(\sigma-\delta) + \nu_1(\sigma+\delta))/2 + \varepsilon \]

(21) Remark: It is easy to see, by geometric considerations, that if \( \Psi \) is a convex function in the interval \( (a, b) \), then for each \( c \) in \( (a, b) \), the difference quotient,

\[
  \Psi_c(x) = \frac{\Psi(x) - \Psi(c)}{x - c}
\]

is a non-decreasing function of \( x \) on \( (a, c) \cup (c, b) \).

Hence, \( \Psi_c(x) \) has a right and a left limit as \( x \) tends
that is, $\psi$ has a right and a left derivative everywhere in $(a, b)$ and thus is continuous on $(a, b)$ [1, page 49].

Now, we have shown that for fixed $\sigma > 0$, $\psi(\sigma)$ is a convex function of $\sigma$, and thus, $\psi(\sigma)$ is a continuous function of $\sigma$. We let $\delta \to 0$ in (20) to obtain

\begin{equation}
(21) \quad \mu(\sigma) \leq \psi(\sigma) + \rho
\end{equation}

for $\sigma > P$, $0 < \rho < 1$.

Thus,

$$\mu(\sigma) \leq \psi(\sigma).$$

We also have

$$\left( \frac{1}{1} \int_0^T \psi(\sigma + it)^{\frac{1}{\rho}} dt \right)^\rho = O \left( \left( \frac{1}{1} \int_0^T \frac{\mu(\sigma) + \rho}{t} dt \right)^\rho \right) = O \left( T^{\mu(\sigma) + \rho} \right)$$

so

$$\psi(\sigma) \leq \mu(\sigma)$$

for $\sigma > P$, $0 < \rho < 1$, and, hence, for $\sigma = 0$.

This proves Lemma 3.

We now prove (14). From (21),

$$\psi(\sigma) = \mu(\sigma) \leq \psi(\sigma) + \rho$$
for \( \sigma > P \), \( 0 < P \leq 1 \), so (14) is proved for \( \epsilon = 0 \).

By convexity,

\[
\frac{\nabla \rho_1(\sigma)}{\epsilon_2} \leq \frac{(\rho_2 - \rho_1) \nabla \rho_2(\sigma) + (\rho_2 - \rho_1) \nabla \rho_1(\sigma)}{\epsilon_2}
\]

for \( \sigma > P \), \( 0 < P \leq \sigma < P \leq 1 \). Taking the limit as \( \rho_1 \to 0 \), we have

\[
\frac{\nabla \rho_1(\sigma)}{\epsilon_2} \leq \frac{(\rho_2 - \rho_1) \nabla \rho_2(\sigma) + \rho_1 \nabla \rho_2(\sigma) + \rho_2 \nabla \rho_1(\sigma)}{\epsilon_2}
\]

\[
\leq \nabla \rho_2(\sigma) + (\rho_2 - \rho_1).
\]

This proves (14).

We can now prove (8), that \( \nabla \rho(\sigma) \) is continuous on \( G \). Given \((\rho_0, \sigma_0)\) in \( G \), \( \epsilon > 0 \), let

\[
\rho_1 = \max \{ 0, \rho_0 - \epsilon/4 \} \quad \text{and} \quad \rho_2 = \min \{ 1, \rho_0 + \epsilon/4 \}.
\]

Since \( \nabla \rho_2(\sigma) \) is a continuous function of \( \sigma \), we may choose a \( \delta > 0 \) such that \( |\sigma - \sigma_0| < \delta \) implies that \( \sigma > P \) and

\[
|\nabla \rho_2(\sigma) - \nabla \rho_2(\sigma_0)| < \epsilon/4.
\]

Let \( \sigma_1 = \sigma_0 - \delta \), \( \sigma_2 = \sigma_0 + \delta \). Since \( \nabla \rho(\sigma) \) is a decreasing function in each variable, for each \((\rho, \sigma)\) in the rectangle with corners \((\rho_i, \sigma_i)\), \( i, k = 1, 2 \).
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\[ | \nu_\sigma (\sigma) - \nu_{\rho_0} (\sigma_0) | \leq \nu_\rho (\sigma, \gamma) - \nu_{\rho_2} (\sigma_2) \]
\[ \leq \nu_{\rho_2} (\sigma_2) - \nu_{\rho_2} (\sigma_0) + \rho_2 - \rho_1 \]

by (14). Thus,
\[ | \nu_\sigma (\sigma) - \nu_{\rho_0} (\sigma_0) | \leq \nu_{\rho_2} (\sigma_2) - \nu_{\rho_2} (\sigma_0) + \nu_{\rho_2} (\sigma_0) - \nu_{\rho_2} (\sigma_2) + \rho_2 - \rho_1 \]
\[ \leq \epsilon/4 + \epsilon/4 + \epsilon/2 = \epsilon. \]

This proves (8).

(9) now follows from Lemma 2 and the continuity of \(-\nu_{\rho_0} (\sigma)\).

2.6. The \( \gamma \) Function

We first remark that, though the theorems of Section 5 are only true for \( \gamma \) and \( \mathbb{Z} \), the definition of \(-\nu_\rho (\sigma)\) also applies to \( \mathcal{W} \) for \( \sigma > 0 \), and \(-\nu_\rho (\sigma, \mathcal{W})\) due to (4.8).

Thus, by Minkowski's Inequality (5.5) for \( 0 < \rho \leq 1 \), \( \sigma > \rho \),
\[ \nu_\rho (\sigma, \mathcal{W}) \leq \nu_\rho (\sigma, \gamma) \leq \nu_\rho (\sigma, \mathcal{W}) \]
\[ \leq \max \left( \nu_\rho (\sigma, \gamma), 0 \right) \]
\[ \leq \nu_\rho (\sigma, \gamma) \]
and

$$\nu_\rho(\sigma, \gamma) \leq \max (\nu_\rho(\sigma, Z), \nu_\rho(\sigma, W))$$

$$\leq \nu_\rho(\sigma, Z).$$

Therefore,

$$\nu_\rho(\sigma, Z) = \nu_\rho(\sigma, \gamma)$$

for $$\sigma > \rho$$, $$0 \leq \rho \leq 1$$.

Henceforth we will make no distinction between $$\nu_\rho(\sigma, Z)$$ and $$\nu_\rho(\sigma, \gamma)$$.

**Definition 1:** For $$0 \leq \rho \leq 1$$, let

$$\gamma_\rho(\gamma) = \gamma_\rho(Z) = \inf \{ \sigma > S = S(\gamma) | \nu_\rho(\sigma, \gamma) = 0 \}.$$

It is more natural to allow $$\sigma$$ in this definition to be $$\sigma > P(\gamma)$$, but we choose this convention for notational convenience.

From (5.13), (5.12) and (5.11), for $$0 \leq \rho \leq 1$$,

(1) $$S \leq \gamma_\rho \leq \alpha_1$$

and

(2) $$\nu_\rho(\sigma) = 0$$

for $$\sigma > \gamma_\rho.$$
Theorem 1: We have for \( 0 \leq \rho \leq 1 \), \( \gamma_\rho \) is a

(3) continuous function,

(4) convex function,

(5) decreasing function,

and for \( 0 < \rho_1 \leq \rho_2 \leq 1 \), \( \sigma > \gamma_{\rho_2} \),

(6) \( -\gamma_{\rho_2}(\sigma) \leq \rho_2 - \rho_1 \).

Proof: To prove (5) we choose \( \sigma > \gamma_{\rho_1} \), then

\[
0 = -\gamma_{\rho_1}(\sigma) = -\gamma_{\rho_2}(\sigma)
\]

by (5,10). Thus, \( \sigma > \gamma_{\rho_2} \).

(6) follows from (5,14) since \( \sigma > \gamma_{\rho_1} \) implies

\[
-\gamma_{\rho_1}(\sigma) + \rho_1 \leq -\gamma_{\rho_2}(\sigma) + \rho_2 = \rho_2.
\]

\( \gamma_\rho \) is convex since (5,9) with \( \sigma_1 > \gamma_{\rho_1}, \sigma_2 > \gamma_{\rho_2} \),

\( 0 \leq \sigma \leq 1 \), implies

\[
\gamma_{\rho_1}(\sigma_1) + (1-\rho_1)\rho_2 \leq \rho_1 - \gamma_{\rho_1}(\sigma_1) + (1-\rho_1) - \gamma_{\rho_2}(\sigma_2) = 0
\]

and thus,

\[
\gamma_{\rho_1}(\sigma_1) + (1-\rho_1)\rho_2 \leq \rho_1 - \sigma_1 + (1-\rho_1) \sigma_2
\]

so

\[
\gamma_{\rho_1}(\sigma_1) + (1-\rho_1)\rho_2 \leq \rho_1 \gamma_{\rho_1} + (1-\rho_1) \gamma_{\rho_2}.
\]

The convexity of \( \gamma_\rho \) implies the continuity of \( \gamma_\rho \)

for \( 0 < \rho < 1 \). Thus, to prove (4), we need only show
(7) \( \lim_{\rho \to 0^+} Y_\rho = Y_0 \)

and

(8) \( \lim_{\rho \to 1^-} Y_\rho = Y_1 \).

From (5)

\[ \lim_{\rho \to 0^+} Y_\rho \leq Y_0 \leq \alpha < \infty. \]

For \( \sigma > \lim_{\rho \to 0^+} Y_\rho \), there exists a \( \rho > 0 \) such that \( 0 < \rho < \rho_1 \) implies \( \sigma > Y_\rho \) and, hence, that

\[ -\nu_\rho(\sigma) = 0. \]

Therefore, \( -\nu_0(\sigma) = 0 \) and \( Y_0 \leq \sigma \).

This implies (7).

From (5)

\[ \lim_{\rho \to 1^-} Y_\rho \geq Y_1. \]

Now,

\[ Y_{\rho_1 \cdot \rho + (1 - \rho_1) \rho_2} = \rho \cdot Y_{\rho_1} + (1 - \rho) Y_{\rho_2}, \]

with \( \rho_1 = 0 \), \( \rho_2 = 1 \) and \( 0 < \rho \leq 1 \), implies

\[ Y_{(1 - \rho) \cdot \rho + \rho \cdot \rho} \leq \rho \cdot Y_0 + (1 - \rho) Y_1. \]

Hence,

\[ \lim_{\rho \to 1^-} Y_\rho \leq Y_1. \]
and (8) is proved.

We now state the convexity of $\gamma_\rho(\sigma)$ in another form.

**Theorem 2:** For $0 \leq \rho \leq 1$, $\rho \sigma < \sigma < \delta_\rho$, 
$$\gamma_\rho(\sigma) \leq \frac{\delta_\rho - \sigma}{\delta_\rho - \sigma_i} \gamma_\rho(\sigma_i).$$

**Proof:** By the convexity of $\gamma_\rho(\sigma)$, if $\sigma_2 \geq \delta_\rho$, 
$$\gamma_\rho(\sigma) \leq \frac{\sigma_2 - \sigma}{\sigma_2 - \sigma_i} \gamma_\rho(\sigma_1) + \frac{\sigma - \sigma_i}{\sigma_2 - \sigma_i} \gamma_\rho(\sigma_2) = \frac{\delta_\rho - \sigma}{\sigma_2 - \sigma_i} \gamma_\rho(\sigma).$$

Let $\sigma_2 \to \delta_\rho$, then Theorem 2 follows.

Finally we give a proof found in [6].

**Theorem 3:** $\gamma_{\rho_2}(\sigma) \leq \frac{1}{2}$, for $\sigma > Q$.

**Proof:** Let $\sigma > Q$, then 
$$\gamma(s) = s \int_0^\infty e^{-su} \Theta(u) \, du,$$
$$\overline{\gamma(s)} = \overline{s} \int_0^\infty e^{-\overline{s}u} \overline{\Theta(u)} \, du,$$

and both integrals are uniformly absolutely convergent for $-T \leq t \leq T$, $\sigma \geq Q + \varepsilon$. Also, $\Theta(u) = \Theta(e^{u(Q+\frac{1}{2})})$.

Thus,

$$J(T) = \int_{-T}^{T} \left| \gamma(\sigma + it) \right|^2 \, dt$$
$$= \int_0^\infty \int_0^\infty \Theta(u) \overline{\Theta(u')} e^{-\sigma(u+u')} \int_{-T}^{T} e^{-ti(u-u')} \, dt \, du \, du'.$$
Now, for $T > 1$, 

$$J(T) = \Theta \left( \int_0^\infty \int_0^\infty e^{-\frac{1}{2}(u+\nu)} \left| \int_T^T t^2 e^{-t(u-\nu)} dt \right| \ dv \ du \right)$$

$$+ \Theta \left( \int_0^\infty \int_0^\infty e^{-\frac{1}{2}(u+\nu)} T \ dv \ du \right)$$

$$= E_1 + \Theta(T).$$

For $T > 1$, 

$$\left| \int_T^T t^2 e^{-it(u-\nu)} dt \right| = \Theta(T^3),$$

and for $u \neq \nu$

$$\left| \int_T^T t^2 e^{-it(u-\nu)} dt \right| = \left| \int_T^T \frac{e^{-it(u-\nu)} t^2}{\nu - \nu} \ dt + \frac{2}{\nu^2} \int_T^T t e^{-it(u-\nu)} dt \right|$$

$$= \Theta \left( \frac{T^2}{|u-\nu|} \right).$$

Thus, we have

$$E_1 = \Theta \left(T^2 \int_0^\infty \int_0^\infty e^{-\frac{1}{2}(u+\nu)} \min(T, \frac{1}{|u-\nu|}) \ dv \ du \right)$$

$$= \Theta \left(T^2 \int_0^\infty e^{-\frac{1}{2}u} \left( \int_0^\infty e^{-\frac{1}{2}\nu} \min(T, \frac{1}{|u-\nu|}) \ dv \right) du \right).$$

But

$$\int_0^\infty e^{-\frac{1}{2}u} \min(T, \frac{1}{|u-\nu|}) \ dv$$

$$= \Theta(\int_0^\infty e^{-\frac{1}{2}u} \ dv) + \Theta(\int_{\frac{1}{|u-\nu|}}^\infty \min(T, \frac{1}{|u-\nu|}) \ dv)$$
\[ = \Theta(1) + \Theta\left( S_0^{\sqrt{T}} \int \omega \right) + \Theta\left( S_0^{\sqrt{T}} \frac{d\omega}{\omega} \right) \]

\[ = \Theta(T^{1/2}) \]

for \( \delta > 0 \). Thus,

\[ E_1 = \Theta(T^{2+\delta}), \]

Thus, the theorem follows by the definition of \( -\vartheta_{y_2}(\sigma) \).

2.7. Proof of Theorem 3.1

We now assemble what more we need to prove Theorem 3.1.

3.1. The following results will be proved in Chapters 3 and 4.

For \( k \geq 1 \),

(1) \( \delta_{y_2}(Z_k) \leq \chi_c(Z_k) \leq \delta_{y_2}(Z_k) + y_2 - \sigma \)

for \( 0 < \sigma \leq y_2 \). See (4.1.21).

(2) \( \delta_{y_2}(Z_k) \leq \sigma + \vartheta_{y_2}(\sigma, Z_k) \)

for \( \sigma > S_k \). See Theorem 4.1.5.
(3) If $\nabla\gamma_2(\sigma, Z_i) = 0$ for a $\sigma > s_i$, then

$$\sum_{m \leq x} |a_m|^2 = \Theta(x^{2\sigma+\varepsilon})$$

for each $\varepsilon > 0$. See (4.1.17).

Proof of Theorem 3.1: First, we note that $Z_i^{\mathbb{R}} = (Z_i)^\mathbb{R}$ [11, page 61]. Thus, by the definition of $\nabla\varphi(\sigma)$,

$$\nabla\gamma_1(\sigma, Z_i) = \nabla\gamma_1(\sigma, Z_i^{\mathbb{R}}) = \nabla\varphi(\sigma, Z_i^{\mathbb{R}}).$$

Now,

$$\begin{align*}
(A_1(x))^2 &= (x L_1(x) + \Theta(x^{0_1}))^2 = x^2 L_1^2(x) + \Theta(x^{1+\alpha}, L_1(x)).
\end{align*}$$

Since $L_1(x) = \Theta(x^{\varepsilon_1})$ and $L_1^2(x) = \Theta(x^{4\varepsilon_1})$ for each $\varepsilon > 0$, there exists an $x_o$ and a $K(x_o) > 0$ such that

$$\begin{align*}
K(x^{1-\varepsilon}) &\leq |A_1(x)|^2(x) \leq \sum_{m \leq x} |a_m|^2
\end{align*}$$

for $x \geq x_o$. We used here (5.4) with $\alpha = \gamma_2$.

We assume now that $k \geq 3$. From our assumptions,

$$S(\gamma_1) = S_1 \leq Q \leq (k-1)/(2k).$$

We suppose that $Q(k \leq (k-1)/(2k)$.

If $\nabla\gamma_2(\sigma, Z_i) = 0$ for some $\sigma$ where $s_i < \sigma < \gamma_2$, then, from (3),
\[ \sum_{m=0}^{\infty} |a_m|^2 = O(x^{\sigma + \xi}) \]

for each \( \xi > 0 \). This contradicts (5). Therefore,

\[ \gamma_{\frac{1}{2}}(Z_i) \geq \frac{1}{2} \]

From (1),

\[ \gamma_{\frac{1}{2}}(Z_i) \leq \gamma_{\frac{1}{2}}(Z_i) \leq \gamma_{\frac{1}{2}}(Z_i) + \frac{1}{2} - \frac{1}{2} \]

Since \( -\psi_\sigma(\sigma_i) \) is a decreasing function of \( \sigma \), (2) implies

\[ \gamma_{\frac{1}{2}}(Z_i) \leq \sigma_i + \psi_{\frac{1}{2}}(\sigma_i, Z_i) \leq \sigma_i + \psi_{\frac{1}{2}}(\sigma_i, Z_i) = \sigma_i + \psi_{\frac{1}{2}}(\sigma_i, Z_i) \]

for \( \sigma_i > (k-1)/(2+k) \).

Now, let \( Q_k < \sigma < (k-1)/(2+k) < \sigma_i < \gamma_{\frac{1}{2}}(Z_i) \leq \gamma_i(Z_i) \).

Then, by Theorem 6.2,

\[ -\psi_i(\sigma, Z_k) \leq \frac{\gamma_i(Z_k) - \sigma_i}{\gamma_i(Z_k) - \sigma} - \psi_i(\sigma, Z_k) \]

Since \( -\psi_i(\sigma) \leq -\psi_{\frac{1}{2}}(\sigma) \),

\[ -\psi_i(\sigma, Z_k) \leq \frac{\delta_{\frac{1}{2}}(Z_i) - \sigma_i}{\delta_{\frac{1}{2}}(Z_i) - \sigma} - \psi_{\frac{1}{2}}(\sigma, Z_k) \]

The following inequality may be easily verified.

If \( a \leq x \leq b, \ c \leq y \leq c \), then \( \frac{a-c}{a-d} \leq \frac{b-c}{b-d} \).
Thus, from Theorem 6.3,

\[
\frac{\gamma_{\nu_2}(Z_i) - \sigma_i}{\gamma_{\nu_2}(Z_i) - \sigma} \leq \frac{\gamma_{\nu_2}(Z_i) + \frac{1}{2} - \frac{1}{2} - \sigma_i}{\gamma_{\nu_2}(Z_i) + \frac{1}{2} - \frac{1}{2} - \sigma}
\]

for \( Q_{2k} < \sigma < (k-1)/(2k) \), \( a_2 < \nu_2 < \gamma_{\nu_2}(Z_i) \).

We set \( \sigma_i = \gamma_{\nu_2}(Z_i) - \frac{1}{2} + \varepsilon \), \( 0 < \varepsilon < \frac{1}{2} \). Then

\[
\gamma_{\nu_2}(Z_i) \leq \frac{1}{2} \left( \frac{\gamma_{\nu_2}(Z_i) + \frac{1}{2} - \frac{1}{2} - \sigma_i}{\gamma_{\nu_2}(Z_i) + \frac{1}{2} - \frac{1}{2} - \sigma} \right) + \gamma_{\nu_2}(Z_i) - \frac{1}{2} + \varepsilon .
\]

Taking the limit as \( \varepsilon \to 0 \), we get

\[
1 \leq \frac{(k-1)/(2k)}{\gamma_{\nu_2}(Z_i) + \frac{1}{2} - \frac{1}{2} - \sigma}
\]

for \( Q_{2k} < \sigma < (k-1)/(2k) \), so by (6),

\[
\gamma_{\nu_2} \leq \gamma_{\nu_2}(Z_i) \leq \sigma + \frac{1}{2} \to \gamma_2 .
\]

This contradiction implies \( Q_{2k} \geq (k-1)/(2k) \), \( k \geq 3 \).

Now, in case \( k=2 \), we suppose \( Q_1 < \gamma_2 - \delta \), \( \delta > 0 \), and \( Q_2 < \gamma_4 \). We then obtain (8) and (9) for \( \sigma_i > \gamma_2 - \delta \). (9) becomes

\[
\frac{\gamma_{\nu_2}(Z_i) - \sigma_i}{\gamma_{\nu_2}(Z_i) - \sigma} \leq \frac{1}{4} \frac{\gamma_{\nu_2}(Z_i) - \sigma_i}{\gamma_{\nu_2}(Z_i) - \sigma}
\]

for \( Q_2 < \sigma < \gamma_2 - \delta < \sigma_i < \gamma_{\nu_2}(Z_i) \).
1. INTRODUCTION

The problem with which this dissertation is concerned arose naturally from several previous investigations, the first of which was carried out by Dirichlet in 1849 [3]. Let $n$ be a positive integer and $d(n)$ be the number of positive divisors of $n$. Let $\theta_2$ be the greatest lower bound of the numbers $\theta$ such that

$$
\sum_{m \leq x} d(m) = x \log x + (1 - \theta) x + O(x^{\theta})
$$

as $x \to \infty$, where $\gamma$ is Euler's constant. We adopt the standard meanings for the $O$, $\sigma$, and $\sim$ symbols [22, page 1]. Dirichlet considered the problem of finding $\theta_2$. He was able to show by elementary means, that is, he did not use the theory of complex functions, that (1) holds with $\theta = 1/2$. He stated that he felt this could be improved, but he made no further conjecture.

Voronoi [25] showed in 1903, by elementary means, the error term in (1) is $O(x^{1/2} \log x)$.

As in the case of many other problems in multiplicative number theory, this problem is closely related to the properties of the Riemann $\zeta$ function. The Riemann $\zeta$ function is defined as the analytic continuation, into the
Hence, \( \frac{1}{2} - \sigma \leq \delta_{\frac{1}{2}}(Z) - \sigma \leq \frac{1}{4} \).

Thus \( \sigma \geq \frac{1}{4} \) and \( Q_2 \geq \frac{1}{4} \).

This completes the proof of Theorem 3.1.
3. STRONG RIESZ SUMMABILITY

3.1. Introduction

In this chapter we give certain results in the theory of strong Riesz summability of the type of integral with which we were concerned in Chapter 2. These results are all analogous to results for series given by Richert [19] and Glatfeld [5].

3.2. Assumptions

We assume that \( C(u) \) is a function of bounded variation on every interval \([-\infty, x] \), \( x > 0 \), and that \( C(0) = 0 \).

Then,

\[
C(u) = M(u) + J(u)
\]

where \( M(u) \) is continuous, and \( J(u) \) is a step function.

We assume the variation of \( M(u) \) on the interval \( [\omega_1, \omega_2] \), \( \forall \omega_1, \omega_2 \) satisfies a Lipschitz condition,

\[
\forall M(\omega_1, \omega_2) \leq K_1 (\omega_2 - \omega_1)
\]

where the \( K_1 \) is uniform for \( \omega_1, \omega_2 \) in the bounded interval \( [0, K_2] \), for each \( K_2 > 0 \). We further
assume that \( J(\omega) \) is left continuous and has at most a
denumerable number of discontinuities,

\[
0 = m_1 < m_2 < \ldots < m_m < \ldots
\]

with no finite cluster point.

3.3. Riesz Means

We define

\[
C(\omega) = \int_{0}^{\infty} (\omega - u)^x \; d\mathcal{L}(u).
\]

This integral exists in the Riemann sense for \( x > 0 \).

For \( x > -1 \), \( \omega \neq m_j \), we form the measure \( m(\mathcal{L}(\omega)) \),
using the function \( \mathcal{L}(\omega) \), in the case \( \omega = m_k \)
for some \( k \), we form \( m(\mathcal{L}(\omega)) \) equal to \( \mathcal{L}(\omega) \) for \( u \leq \omega \), and equal to \( \mathcal{L}(\omega) \) for \( u \geq \omega \).

Now, \( \mathcal{L}(\omega) \) is left continuous at \( \omega \), and hence,
\( \mathcal{L}(\omega) \) is continuous at \( \omega \).

Also, \( (\omega - u)^x \) is continuous on \( [0, \omega) \). Thus,
\( (\omega - u)^x \) is \( m(\mathcal{L}(\omega)) \) measurable. For \( m_j < \omega \neq m_{j+1} \),
\( 0 \leq \omega \leq K_2 \), \( -1 \leq x \leq 0 \)

\[
\left| \int_{0}^{\omega} (\omega - u)^x \; d\mathcal{L}(u) \right| \leq K_1 \int_{m_j}^{\omega} (\omega - u)^x \; du + \int_{0}^{m_j} (\omega - u)^x \; d\mathcal{L}(u) \leq K_1 (\omega - m_j)^{x+1} + (\omega - m_j)^x \mathcal{L}(0, K_2) \leq \infty.
\]
Thus, \((\omega - u)^\chi\) is \(m(C^\omega)\) integrable for \(\chi > -1\). We then write

\[
\int_0^\omega (\omega - u)^\chi \, d\mu(u) \quad \text{for} \quad \int_0^\omega (\omega - u)^\chi \, d\mu(C^\omega(u)).
\]

We now investigate the continuity of \(C^\chi(\omega)\).

If \(0 < \omega_2 < \omega_1 < \omega \leq K_2\), \(\chi > -1\), \(\neq 0\), then

\[
f(\omega, \omega_1) = |\int_0^{\omega_1} (\omega - u)^\chi \, d\mu(u) - \int_0^\omega (\omega - u)^\chi \, d\mu(u)|
\]

\[
= |\int_0^{\omega_2} (\omega - u)^\chi - (\omega_1 - u)^\chi \, d\mu(u) + \int_{\omega_2}^{\omega_1} (\omega - u)^\chi \, d\mu(u) - \int_{\omega_2}^{\omega_1} (\omega_1 - u)^\chi \, d\mu(u)|. 
\]

Thus,

\[
f(\omega, \omega_1) = \Theta \left( \int_0^{\omega_2} (\omega - u)^\chi \, d\nu \, dV(C(0, u)) \right)
\]

\[
+ \Theta \left( \int_{\omega_2}^{\omega_1} (\omega - u)^\chi \, dV(C(0, u)) \right)
\]

\[
+ \Theta \left( \int_{\omega_2}^{\omega_1} (\omega_1 - u)^\chi \, dV(C(0, u)) \right).
\]

Now, if \(\chi > 0\),

\[
f(\omega, \omega_1) = \Theta \left( (\omega - \omega_1) \xi(\omega_1 - \omega_2)^{\chi - 1} \omega^{\chi - 1/2} V(C(0, K_2)) \right)
\]

\[
+ \Theta \left( (\omega - \omega_2)^\chi \, V(C(0, K_2)) \right)
\]

\[
+ \Theta \left( (\omega_1 - \omega_2)^\chi \, V(C(0, K_2)) \right).
\]
If \( \lambda > 1 \), set
\[
\omega_2 = \omega_1 - (\omega - \omega_1)^{1/(2 \lambda - 1)}.
\]

If \( \lambda = 1 \), set
\[
\omega_2 = \omega_1 - (\omega - \omega_1).
\]

Then \( \omega_2 < \omega_1 \), \( \omega_2 \to \omega_1 \), \( f(\omega, w) \to 0 \), as \( \omega \to \omega^- \)
or as \( \omega \to \omega^+ \).

Since, for \( \lambda > 0 \), it is easily seen that
\( C_{\lambda}(u) \to 0 \) as \( u \to 0 \), we have shown \( C_{\lambda}(\omega) \) is continuous for \( \lambda > 0 \), \( \omega \geq 0 \).

If \( -1 < \lambda < 0 \), let \( m_{\frac{1}{\lambda}} < \omega_2 < \omega_1 < \omega \leq m_{\frac{1}{\lambda}+1} \), then

\[
f(\omega, w) = \Theta((\omega - \omega_1)(\omega_1 - \omega_2)^{-1} \forall (0, k_2)) + \Theta(\int_{\omega_2}^{\omega_1} (\omega - u)^{x} \, dV M(o, u)) + \Theta(\int_{\omega_2}^{\omega_1} \omega_1 - u)^{x} \, dV M(o, u)) = \Theta((\omega - \omega_1)(\omega_1 - \omega_2)^{-1}) + \Theta((\omega - \omega_2)^{x+1}) + \Theta((\omega_1 - \omega_2)^{x+1}).
\]
Set \( \omega_2 = \omega_1 - (\omega - \omega_1)^{1/2}\). Then \( \omega_2 < \omega_1 \),
\( \omega_2 \to \omega_1 \), \( \int (\omega, \omega_1) \to 0 \) as \( \omega_1 \to \omega^- \) or \( \omega \to \omega^+ \).

This shows \( C_\chi(\omega) \), \(-1 < \chi < 0\), is left continuous everywhere and continuous for \( \omega \neq m_\chi \).

For \( \chi > 0 \), integrating by parts, we have

\[
C_\chi(\omega) = \chi \int_0^\omega C(\omega) (\omega - \omega')^{\chi - 1} \, d\omega.
\]

We now derive a more general formula of this type.

Theorem 1: For \( \chi > -1 \) and \( \chi > 0 \), we have

\[
C_{\chi+\lambda}(\omega) = \frac{\Gamma(\chi+\lambda+1)}{\Gamma(\chi+\lambda)} \int_0^\omega C_\chi(\omega') (\omega - \omega')^{\chi - 1} \, d\omega.
\]

In order to prove this, we need the following lemma.

Lemma 1: Let \( \omega > \tau \geq 0 \), \( \chi > 0 \) and \( \chi > 0 \). Then

\[
\int_\tau^\omega (\omega - \tau)^{\lambda - 1} (\omega - u)^{\chi - 1} \, du = \frac{\Gamma(\lambda) \Gamma(\chi)}{\Gamma(\chi+\lambda)} (\omega - \tau)^{\chi+\lambda - 1}.
\]

Proof of Lemma 1: From [22, page 56], we have

\[
\frac{\Gamma(\lambda) \Gamma(\chi)}{\Gamma(\chi+\lambda)} = \int_0^1 (\lambda - 1)^{\chi - 1} (\chi - 1)^{\lambda - 1} \, d\lambda
\]

for \( \chi > 0 \), \( \chi > 0 \).

Let \( \nu = (\omega - \tau)/(\omega - \tau) \), \( x = \chi \), \( y = \chi \). Then

\[
\frac{\Gamma(\lambda) \Gamma(\chi)}{\Gamma(\chi+\lambda)} = \int_{\tau}^\omega \left( \frac{\omega - \tau}{\omega - \tau} \right)^{\chi - 1} \left( \frac{\omega - u}{\omega - \tau} \right)^{\chi - 1} \, du
\]

\[
= (\omega - \tau)^{\chi - \lambda + 1} \int_{\tau}^\omega (\omega - \tau)^{\chi - 1} (\omega - u)^{\chi - 1} \, du.
\]

This proves Lemma 1.
Proof of Theorem 1: The integral

\[ \int_0^\infty C_\lambda(u) (w-u)^{\lambda-1} \, du \]

exists for \( \lambda > -1 \), \( x > 0 \). To see this, let \( \lambda \geq 0 \).

Then \( C_\lambda(u) \) is bounded in \([0, \omega]\). If \( 0 > \lambda > -1 \), then by left continuity, \( C_\lambda(\omega) \) is bounded in \([\omega, \omega]\) for some \( \omega > 0 \). On the interval \([0, \omega]\), \((w-u)^{\lambda-1}\) is bounded, and by (2), \( C_\lambda(u) \) is integrable.

Thus,

\[ \int_0^\omega C_\lambda(u) (w-u)^{\lambda-1} \, du = \int_0^\omega (w-u)^{\lambda-1} \left( \int_0^u (u-v)^{\lambda} \, dC^{\omega}(v) \right) \, du \]

(4)

Now, interchanging the order of integration, we have

(5) \[ \int_0^\omega C_\lambda(u) (w-u)^{\lambda-1} \, du = \int_0^\omega \int_0^u (w-u)^{\lambda-1} (u-v)^{\lambda} \, dC^{\omega}(v) \, du \, dC^{\omega}(u). \]

Thus, by Lemma 1,

\[ \int_0^\omega C_\lambda(u) (w-u)^{\lambda-1} \, du = \frac{\Gamma(\lambda) \Gamma(\lambda+1)}{\Gamma(\lambda+\lambda+1)} \int_0^\omega (w-u)^{\lambda+\lambda} \, dC^{\omega}(u) \]

\[ \int_0^\omega C_\lambda(u) (w-u)^{\lambda-1} \, du = \frac{\Gamma(\lambda) \Gamma(\lambda+1)}{\Gamma(\lambda+\lambda+1)} \int_0^\omega (w-u)^{\lambda+\lambda} \, dC^{\omega}(u) \]

\[ = \frac{\Gamma(\lambda) \Gamma(\lambda+1)}{\Gamma(\lambda+\lambda+1)} C^{\lambda+\lambda}(\omega). \]
We need only justify the interchange in (5). Let

\[ f^*(u, \nu) = (\omega - u)^{\lambda - 1} (u - \nu)^\beta, \]

and for \( 0 < u, \nu < \omega, \)

\[ f(u, \nu) = \begin{cases} f^*(u, \nu), & u > \nu^2 \\ 0, & u \leq \nu \end{cases}, \]

\[ f(u, \omega) = 0 \]

for \( u = \omega. \)

The right side of (4) is

\[ \int_0^\omega \int_0^\omega f(u, \nu) \ d\nu C^\omega(\omega)du. \]

We work with the integral

\[ \int_0^\omega \int_0^\omega f(u, \nu) \ d\nu C^\omega(\omega, \nu) \ d\omega. \]

If we can show that \( f(u, \nu) \) is integrable with respect to the product measure

\[ m^x = m(V C^\omega(\omega, \nu)) \times m(\omega) \]

on the rectangle \( R_\omega = [0, \omega] \times [0, \omega] \), then it will be

\[ m^x = m(N_\omega C^\omega(\omega, \nu)) \times m(\omega) \]

and

\[ m^x = m(P_\omega C^\omega(\omega, \nu)) \times m(\omega) \]

integrable, \( \gamma = 1, 2 \), where \( N_\omega C^\omega(\omega, \nu) \) and \( P_\omega C^\omega(\omega, \nu) \)
are the negative and positive variations, respectively, of $\mathcal{C}_t^\omega$ on $[0,\omega]$ for $t=1$, and of $\Delta_m^\omega$ for $j=2$. Then, $f(u,\nu)$ will be

$$m = m(\mathcal{C}_t^\omega(\nu)) \times m(\omega)$$

integrable, and Fubini's theorem [17, page 143] tells us that all the integrals involved exist and that we may interchange the order of integration.

To show that $f(u,\nu)$ is $m^\#$ measurable, we will show that it is continuous on $\mathcal{R}_\omega$ except at a set of $m^\#$ measure zero. To show this is sufficient, we show that for each real number $\alpha$, the set of points where $f \geq \alpha$ is $m^\#$ measurable. We cover the discontinuities of $f$ with an open (with respect to $\mathcal{R}_\omega$) set $S_{\gamma m}$ of measure $< \gamma m$. Then the set of points, not in $S_{\gamma m}$, where $f \geq \alpha$ is closed and hence $m^\#$ measurable. The union of such sets, for all $\gamma$, is again an $m^\#$ measurable set, say $F$, and the set of points where $f \geq \alpha$ is $F_\alpha = F \cup F^\#$ where $F^\#$ is some set of points contained in $\bigcap m S_{\gamma m}$. Therefore, $F^\#$ is of $m^\#$ measure zero, and $F_\alpha$ is an $m^\#$ measurable set.

It is easy to see that $f(u,\nu)$ is discontinuous at most for $u=\omega$, or $u=\nu$. Suppose the $m(\mathcal{V}^\omega(\nu))$ measure of $m_\delta^\#$, for $0 < m_\delta^\# < \omega$ is $\Lambda(\delta)$. Choose $\varepsilon > 0$. Then there exists a $\delta' > 0$ such that
for all $\omega$ such that $0 \leq m_{\omega} \leq \omega$. Now, choose $\delta'$ such that $0 < \delta \leq \delta'$ and $\delta \leq \varepsilon/(2K_3 \varepsilon)$.

$m_{\omega} \leq m_{\omega}^{\delta}$, Then the total $m^*$ measure of the rectangles

$[m_{\omega}^{\delta} - \delta, m_{\omega}^{\delta} + \delta] \times [m_{\omega}^{\delta} - \delta, m_{\omega}^{\delta} + \delta]$

with $m_{\omega} \leq \omega$ is less than $\varepsilon$.

Now, consider the portions of the line $u = \omega$ not in these rectangles. For $\omega$ in $[m_{\omega}^{\delta} + \delta/2, m_{\omega}^{\delta+1} - \delta/2]$,

$\forall C^u(\omega, \omega) = \forall M^u(\omega, \omega) \leq K_1 (\omega - \omega)$. 

It is thus clear that these segments are of $m^*$ measure zero.

Finally, we can enclose the segment $u = \omega$ in a rectangle whose $m^*$ measure is $\leq \forall C^u(0, \omega) \times \varepsilon_1$, for arbitrary $\varepsilon_1 > 0$, hence in a rectangle whose $m^*$ measure is less than $\varepsilon$.

Thus, the discontinuities of $\omega$ may be enclosed in a set of $m^*$ measure $< 2\varepsilon$, for each $\varepsilon > 0$, and they form a set of $m^*$ measure zero.

Therefore, $\omega$ is $m^*$ measurable on $\mathcal{R}_\omega$. 

complex plane with $1+0i$ deleted, of the function defined for $\Re s > 1$ by

$$
(2) \quad \zeta(s) = \sum_{m=1}^{\infty} m^{-s} = \sum_{m=1}^{\infty} e^{-s \log m} .
$$

This series is one of a class of complex series called ordinary Dirichlet series which are of the form

$$
(3) \quad \sum_{m=1}^{\infty} a_m m^{-s}
$$

where $\{a_m\}$ is an arbitrary complex sequence. A Dirichlet series converges in a half plane $\Re s > \sigma_0$, $-\infty < \sigma_0 \leq \infty$ [11].

We note that the series (2) converges absolutely for $\Re s > 1$ so that if we square the series (2), we may arrange the terms as we choose, and the resulting series converges absolutely for $\Re s > 1$. Thus,

$$
(4) \quad \left\{ \zeta(s) \right\}^2 = \left( \sum_{m=1}^{\infty} m^{-s} \right) \left( \sum_{m=1}^{\infty} m^{-s} \right)
$$

$$
= \sum_{m=1}^{\infty} \left( \sum_{m=1}^{\infty} \frac{k^{-s}}{k} \right)
$$

$$
= \sum_{m=1}^{\infty} d(k) k^{-s} .
$$

Using the properties of the $\zeta$ function, Landau [14] was able to show in 1912, in a way much shorter than Voronoi's, that $\Theta_2 \leq \frac{1}{3}$. Since that time, as the $\zeta$
Now, for each $K_4 > 0$, by Fubini's theorem

$$
\int_{R^n} \min (f, K_4) \, dm^* = \int_0^\infty \int_0^\infty \min (f, K_4) \, d\nu (0, u) \, du
$$

Thus, by the bounded convergence theorem $f$ is $m^*$ integrable, and

$$
\int_{R^n} f \, dm^* \leq \int_0^\infty \int_0^\infty \nu (0, u) \, du < \infty.
$$

3.4. Strong Riesz Summability

We now introduce the concept of strong Riesz summability and prove some basic theorems.

**Definition 1:** Let $\lambda$ and $\rho$ be real numbers such that $0 < \lambda$, $0 < \rho \leq 1$. We say that $C(x)$ is strongly Riesz summable to sum $\rho$, if

$$
\lim_{\omega \to \infty} C(x) = \rho \left| R, \lambda \right|^{1/\rho}
$$

in case

$$
\left( \int_0^\infty \left| C(x) - \rho \, u^{\lambda-\rho} \right|^{1/\rho} \, du \right)^\rho = \epsilon (\omega^x)
$$

as $\omega \to \infty$. 
Theorem 1 [5]: Let $C(\omega)$ and $D(\omega)$ satisfy the conditions of Section 2, and let $\alpha_1$ and $\alpha_2$ be real. If

$$\lim_{\omega \to \infty} C(\omega) = c \quad |R, \chi|^{\gamma},$$

and

$$\lim_{\omega \to \infty} D(\omega) = d \quad |R, \chi|^{\gamma},$$

then

$$\lim_{\omega \to \infty} (\alpha_1 C(\omega) + \alpha_2 D(\omega)) = \alpha_1 c + \alpha_2 d \quad |R, \chi|^{\gamma}.$$ 

Proof: Clearly $E(\omega) = \alpha_1 C(\omega) + \alpha_2 D(\omega)$ satisfies the conditions of Section 2. We have

$$E_\chi(\omega) = \int_0^\omega (\omega - u)^\chi \ d(\alpha_1 C(\omega) + \alpha_2 D(\omega))$$

$$= \alpha_1 \int_0^\omega (\omega - u)^\chi \ dC(\omega) + \alpha_2 \int_0^\omega (\omega - u)^\chi \ dD(\omega)$$

$$= \alpha_1 C_\chi(\omega) + \alpha_2 D_\chi(\omega).$$

Now, from Minkowski's Inequality

$$\left( \int_0^\omega |E_{\chi, \rho}(\omega) - (\alpha_1 c + \alpha_2 d) u^{\chi - \rho} |^{\gamma} du \right)^{\frac{1}{\gamma}}$$

$$= \Theta \left( \left( \int_0^\omega |a_1 C_{\chi, \rho}(\omega) - a_1 c u^{\chi - \rho} |^{\gamma} du \right)^{\frac{1}{\gamma}} \right)$$

$$+ \Theta \left( \left( \int_0^\omega |a_2 D_{\chi, \rho}(\omega) - a_2 d u^{\chi - \rho} |^{\gamma} du \right)^{\frac{1}{\gamma}} \right)$$

$$= \Theta \left( \omega^{\chi} \right).$$
This proves Theorem 1.

Suppose that

\[ \lim_{\omega \to \infty} C(\omega) = \mu \quad \text{and} \quad \lim_{\omega \to \infty} C(\omega) = \mu' \quad \text{for} \quad \mathbb{R}, \mathbb{N}^{1/\alpha}. \]

Then from Minkowski's Inequality

\[ |\mu - \mu'| \omega^\alpha = \left( \frac{\mu}{\mu'} \int_0^\omega \left( C_{\mu'}(u) - \mu u\mathbb{N}^{1/\alpha} - \mu' u\mathbb{N}^{1/\alpha}\right) \frac{du}{\mu'} \right)^{1/\alpha} \]

\[ \leq \left( \frac{\mu}{\mu'} \int_0^\omega \left( C_{\mu'}(u) - \mu u\mathbb{N}^{1/\alpha}\right) \frac{du}{\mu'} \right)^{1/\alpha} \]

\[ + \left( \frac{\mu}{\mu'} \int_0^\omega \left( C_{\mu'}(u) - \mu' u\mathbb{N}^{1/\alpha}\right) \frac{du}{\mu'} \right)^{1/\alpha} \]

\[ = \Theta(\omega^\alpha). \]

Thus,

(1) \[ \mu - \mu' = 0 \]

and limits \[ |\mathbb{R}, \mathbb{N}^{1/\alpha} \] are unique.

Consider, for a real number \( \mu \),

\[ \mathcal{C}(\omega) = \left\{ \mu, \omega > 0 \right\} \cup \left\{ 0, \omega = 0 \right\}. \]

Then, for \( \omega > 0 \),

\[ \mathcal{C}\mathbb{N}(\omega) = \int_0^\omega (\omega-u)^\alpha \mathcal{C}(u) \]

\[ = \mu \omega^\alpha. \]
Thus,

\[ \lim_{\omega \to \infty} C(\omega) = \mu \mid R, \lambda \mid^{\frac{1}{\rho}} \]

for each \( \lambda > 0 \) and each \( 0 < \rho \leq 1 \).

**Theorem 2 [5]:** Let \( \lambda > \lambda_1 > 0 \), \( 0 < \rho \leq 1 \). Then

\[ \lim_{\omega \to \infty} C(\omega) = \mu \mid R, \lambda \mid^{\frac{1}{\rho}} \]

implies

\[ \lim_{\omega \to \infty} C(\omega) = \mu \mid R, \lambda \mid^{\frac{1}{\rho}}. \]

**Proof:** It is clear from previous remarks that we may assume \( \phi = 0 \). From Theorem 3.1, with \( \lambda = \lambda - \lambda_1 > 0 \),

\[ |C_{\lambda-\rho}(\omega)|^{\frac{1}{\rho}} = |C_{\lambda+\xi-\rho}(\omega)|^{\frac{1}{\rho}} \]

\[ = O\left( \int_0^w C_{\lambda-\rho}(\omega)(\omega-u)^{\lambda-1} du \right)^{\frac{1}{\rho}}. \]

From Hölder's Inequality with \( \alpha = \rho \), \( \beta = 1 - \rho \),

\[ \phi(\omega) = C_{\lambda-\rho}(\omega)(\omega-u)^{\lambda-1} = (\omega-u)^{(1-\rho)(\lambda-1)}, \]

we have

\[ |C_{\lambda-\rho}(\omega)|^{\frac{1}{\rho}} \]

\[ = O\left( \int_0^w |C_{\lambda-\rho}(\omega)|^{\frac{1}{\rho}} (\omega-u)^{\lambda-1} du \right)^{\frac{1}{\rho}} \]

Therefore,
\[ S_0^\omega |C_{x-\varepsilon}(x)|^{\nu_0} \, dx \]

\[ = \Theta \left( (\omega^{\lambda} + \lambda + \lambda) \right) \]

An interchange of order of integration may be made here on the same grounds as before because the discontinuities of the integrand in the square \([0, \omega] \times [0, \omega]\) lie on the diagonal, edge and a finite number of horizontal lines, which are all of measure zero with respect to the product measure. Thus,

\[ S_0^\omega |C_{x-\varepsilon}(x)|^{\nu_0} \, dx \]

\[ = \Theta \left( (\omega^{\lambda + \lambda} + \lambda + \lambda) \right) \]

This proves Theorem 2.
Theorem 3 [5]: Let $X+P > 0$, $0 < e_i = e_i < 1$. If
\[ \lim_{\omega \to \infty} C(\omega) = P \quad \text{for } \omega \in \mathbb{R}, X+P \leq \omega, \]
then
\[ \lim_{\omega \to \infty} C(\omega) = P \quad \text{for } \omega \in \mathbb{R}, X+P \leq \omega. \]

Proof: Again we may assume $e = 0$. We let $\gamma = \gamma e + \tau$, thus $\frac{c}{e} + \gamma, \tau = 1$. Without loss of generality we may assume $\gamma > 0$, $\gamma / e < 1$. Therefore, from Hölder's Inequality, with $\alpha = \gamma e$, $\beta = e, \tau$, we have
\[
\int_0^\infty \left| C_{X+P-e}(\omega) \right|^\gamma d\omega \leq \left( \int_0^\infty \left| C_{X+P-e}(\omega) \right|^{\gamma e} d\omega \right)^{\gamma e} \left( \int_0^\infty d\omega \right)^{\tau e}.
\]
\[
= \Theta \left( \omega \left( \frac{X+P}{e} + \tau e \right) \right)
\]
\[
= \Theta \left( \omega \left( \frac{X+P}{e} \right) \right).
\]

This proves Theorem 3.

We now need the following lemma.

Lemma 1: Let $f(x) > 0$ for $x > c$ and $\int_c^x f(t) dt \to \infty$ as $x \to \infty$. If $f(x) = o(\phi(x))$ as $x \to \infty$, then
\[
\int_c^x f(t) dt = o \left( \int_c^x \phi(t) dt \right).
\]

Proof: Given $\varepsilon > 0$, there exists a $K > 0$ such that
Thus,

$$| S^x_c \phi dt | \leq \varepsilon S^x_k \phi dt \leq \varepsilon S^x_c \phi dt$$

for $x \geq x_0$, $x_0$ sufficiently large. This proves Lemma 1.

Theorem 4 [5]: Let $\lambda > 0$, $0 < p \leq 1$. If

$$\lim_{\omega \to \infty} C(\omega) = \beta |R, \lambda|^{\frac{1}{\beta}}$$

and $\lambda = 0$, $0 < \rho_1 \leq \rho < 1$

or $\lambda > 0$, $0 < \rho_1 \leq \rho = 1$

then

$$\lim_{\omega \to \infty} C(\omega) = \beta |R, \lambda + \lambda|^{\frac{1}{\beta}}$$

Proof: We may assume $\beta = 0$. For $\lambda = 0$, $\rho < \rho_1$, from Theorem 3.1,

$$C_{\lambda, \rho}(\omega) = O \left( |S^\omega \lambda \rho \omega \rho \omega^{-\rho} \omega^{-\rho-1} \, d\omega| \right).$$

Hölder's Inequality, with $\alpha = \rho$, $\beta = 1 - \rho$, implies
Thus, from Lemma 1

\[
\left( \int_0^\omega |C_{\chi_1}(\tau)|^{\gamma_1} d\tau \right)^{\gamma_1} = \Theta \left( \left( \int_0^\omega \tau^{\chi_1-1} d\tau \right)^{\gamma_1} \right) = \Theta (\omega^{\chi_1}).
\]

In the case \( \chi > 0 \), \( \rho = 1 \), we may, by Theorem 2, assume \( \rho < 1 \). From Theorem 3.1,

\[
|C_{\chi+\lambda-\rho,\chi}(\tau)|^{\gamma_1} = \Theta \left( \left( \int_0^\tau |C_{\chi+1}(\omega) |^{\gamma_1} d\omega \right)^{\gamma_1} \right).
\]

Applying Hölder's Inequality with \( \chi = \rho \), \( \beta = 1 - \rho \),

\[
|C_{\chi+\lambda-\rho,\chi}(\tau)|^{\gamma_1} = \Theta \left( \left( \int_0^\tau |C_{\chi+1}(\omega)|^{\gamma_1} d\omega \right)^{\gamma_1} \right) \left( \int_0^\tau |C_{\chi+1}(\omega)|^{\chi \gamma_1} d\omega \right)^{(1-\chi)\gamma_1}
\]

\[
= \Theta \left( \left( \int_0^\tau |C_{\chi+1}(\omega)|^{\gamma_1} d\omega \right)^{\gamma_1} \right) \left( \int_0^\tau |C_{\chi+1}(\omega)|^{\gamma_1} d\omega \right)^{(1-\chi)\gamma_1}
\]

\[
= \Theta \left( \left( \int_0^\tau |C_{\chi+1}(\omega)|^{\chi \gamma_1} d\omega \right)^{(1-\chi)\gamma_1} \right).
\]
Therefore,

\[
\left( \int_0^\omega \left| c_{x+\gamma-\rho}(\tau) \right|^{\frac{1}{\gamma}} d\tau \right) = \Theta \left( \int_0^\omega c_{x-\eta}(u) \left| c_{x-\eta}(u) \right|^{\frac{\gamma}{\gamma-1}} \left( x-u \right)^{-\frac{\gamma}{\gamma-1}} d\tau du \right)
\]

\[
= \Theta \left( \omega^{x/(\gamma-1)} \int_0^\omega \left| c_{x-\eta}(u) \right| \left( x-u \right)^{-\frac{\gamma}{\gamma-1}} du \right)
\]

where the inversion is justified as before. Thus,

\[
\lim_{\omega \to \infty} c(\omega) = \infty \quad \text{for} \quad \omega \in \mathbb{R}, x \neq \gamma
\]

implies

\[
\left( \int_0^\omega \left| c_{x+\gamma-\rho}(\tau) \right|^{\frac{1}{\gamma}} d\tau \right) = \Theta \left( \omega^{x-ne/(\gamma-1)} \right)
\]

This completes the proof of Theorem 4.

**Theorem 5:** If \( \lim_{\omega \to \infty} c(\omega) = \infty \quad \text{for} \quad \omega \in \mathbb{R}, x \neq \gamma \), with \( x > 0 \),

\( 0 < \rho < 1 \), then

\[
\left( \int_0^\omega \left| c_{x-\rho}(u) \right|^{\frac{1}{\gamma}} du \right)^\rho = \Theta (\omega^x).
\]

**Proof:** From Minkowski's Inequality

\[
\left( \int_0^\omega \left| c_{x-\rho}(u) \right|^{\frac{1}{\gamma}} du \right)^\rho \leq \left( \int_0^\omega c_{x-\rho}(u) - \rho u^{x-\rho} \left| \gamma \right| du \right) + \left( \rho \int_0^\omega u^{x-1} du \right)^\rho
\]

\[
= \Theta (\omega^x) + \Theta (\omega^x)
\]

\[
= \Theta (\omega^x).
\]
Finally, we remark that \( |R, \lambda|^{1/\rho} \) is a regular method of summability for \( \lambda - \rho \geq 0 \). To see this, suppose

\[
\lim_{\omega \to \infty} C(\omega) = \rho, \quad 0 < \rho \leq 1,
\]

then

\[
\left( \int_{-\infty}^{\infty} |C(\omega) - \rho|^{1/\rho} d\omega \right)^\rho = \Theta(\omega),
\]

by Lemma 1, so

\[
\lim_{\omega \to \infty} C(\omega) = \rho \quad |R, \rho|^{1/\rho}.
\]

Therefore, by Theorem 2,

\[
\lim_{\omega \to \infty} C(\omega) = \rho \quad |R, \lambda|^{1/\rho}
\]

for \( \lambda - \rho \geq 0 \).

Moreover, if

\[
\lim_{\omega \to \infty} C(\omega) = d \quad |R, \lambda|^{1/\rho}
\]

with \( \lambda - \rho \geq 0 \), then \( d = \rho \) by Theorem 2.

3.5. Strong Riesz Summability
of Certain Stieltjes Integrals

We now apply the theory of strong Riesz summability to Dirichlet series and integrals of the type (2.4.1). We recall that dropping the subscript \( \lambda \),

function has been more thoroughly investigated, the upper bound on $\Theta_2$ has been lowered several times. The best estimate thus far is $1^{3/40}$ by Hua in 1949 [12].

In 1916, Hardy [8] proved that

\[ (5) \sum_{m \leq x} d(m) = x \log x + (1 - \gamma) x + \sigma (x^{1/4} (\log x)^{1/4} \log \log x) \]

cannot hold. In fact, he proved that there exists a $K > 0$ such that the error term in (5) is

\[ > K x^{1/4} (\log x)^{1/4} \log \log x \]

for a certain sequence of $x$ values tending to $\infty$ and

\[ - K x^{1/4} (\log x)^{1/4} \log \log x \]

for another sequence of $x$ values tending to $\infty$. This is a little stronger than (5), and he called it an $\Omega$ result, written

\[ (6) \sum_{m \leq x} d(m) = x \log x + (1 - \gamma) x + \Omega (x^{1/4} (\log x)^{1/4} \log \log x) \]

No improvement on this lower bound has been made since Hardy's result. This dissertation is concerned with a more general but slightly weaker form of this theorem.

The geometrical interpretation of $D(x) = \sum_{m \leq x} d(m)$ is the number of points in the first quadrant with integral coordinates between a hyperbola and the axes. A similar problem exists for the number of points $A(x)$ in the first
For given complex $S$, we set

\[ A(x) = \sum_{m \leq x} a_m, \]

\[ B(x) = A(x) - xL(x), \quad x \geq 1, \]

\[ B(\omega) = B(e^\omega), \quad \omega \geq 0. \]

To show that $B(\omega)$ satisfies the conditions of Section 2, we write

\[ B(\omega) = -e^\omega L(\omega) + A(\omega), \]

\[ M_{B^*}(\omega) = \begin{cases} -e^\omega L(\omega) + (N+1) L(N+1^+), & \omega > \log(N+1) \\ 0, & \omega \leq \log(N+1) \end{cases} \]

where

\[ \lim_{x \to x_0^+} L(x) = L(x_0^+) \quad \text{and} \quad N \text{ is from (2.2.1), and} \]

\[ J_{B^*}(\omega) = \begin{cases} A(\omega) - (N+1) L(N+1^+), & \omega > \log(N+1) \\ A(\omega), & \omega \leq \log(N+1) \end{cases} \]
Then \( J_0(\omega) \) is a step function with discontinuities
\[ m_+ = \log \gamma, \quad \text{and} \]
\[ |M_\theta(\omega) - M_\theta(\omega_1)| \leq K_2 (\omega_2 - \omega_1) \]
for \( 0 \leq \omega_1 \leq \omega_2 \leq K_2 \), by the mean value theorem. Thus,
\[ \forall M_\theta(\omega_1, \omega_2) \leq K_1 (\omega_2 - \omega_1). \]

In addition,
\[ c(\omega) = \int_0^\omega e^{-us} dM_\theta(u) \]
\[ = \int_0^\omega e^{-us} dM_\theta(u) + \int_0^\omega e^{-us} dJ_\theta(u) \]
\[ = M_c(\omega) + J_c(\omega). \]

For \( 0 \leq \omega_1 \leq \omega_2 \leq K_2 \),
\[ \forall M_c(\omega_1, \omega_2) \leq \int_{\omega_1}^{\omega_2} e^{-u\sigma} d\forall M_\theta(u) \]
\[ \leq K_1 (\omega_2 - \omega_1) (e^{-\omega_1\sigma} + e^{-\omega_2\sigma}) \]
\[ \leq K_1 (\omega_2 - \omega_1). \]

Also, \( J_c(\omega) \) is a step function with discontinuities at
\[ m_+ = \log \gamma. \] Hence, \( C(\omega) \) satisfies the conditions of Section 2.
The following important lemma is taken from [2, page 53].

**Lemma 1:** Let \( \lambda > 0 \), \( 0 < p \leq 1 \) and \( h(\lambda) = -[\lambda^2] \). Then for \( \omega > 0 \), \( h = h(\lambda - p) \), \( s \) complex

\[
C_{\lambda-p}(\omega) = B_{\lambda-p}(\omega) e^{-\omega s} + \frac{\Gamma(\lambda-p+1)}{h! \Gamma(\lambda-p)} \int_0^\omega B_h(\nu) (e^{-\nu s} - e^{\nu s}) (\omega - \nu)^{-\lambda-p-1} d\nu
\]

and

\[
B_{\lambda-p}(\omega) = C_{\lambda-p}(\omega) e^{\omega s} + \frac{\Gamma(\lambda-p+1)}{h! \Gamma(\lambda-p)} \int_0^\omega C_h(\nu) (e^{\nu s} - e^{-\nu s}) (\omega - \nu)^{-\lambda-p-1} d\nu
\]

Proof: If \( s = 0 \), \( C_{\lambda-p}(\omega) = B_{\lambda-p}(\omega) \) and the lemma is trivial. Thus, we assume \( s \neq 0 \). We first prove the first equation.

For \( s \neq 0 \), \( \epsilon > 0 \), \( \lambda > -1 \), from [27, page 12]

\[
C_\lambda(\omega) = \int_0^\omega (\omega - \tau)^\lambda dC(\tau) + \int_{\omega-\epsilon}^\omega (\omega - \tau)^\lambda dC(\tau)
\]

\[
= \int_0^\omega (\omega - \tau)^\lambda e^{-\tau s} d\Theta(\tau) + \int_{\omega-\epsilon}^\omega (\omega - \tau)^\lambda dC(\tau)
\]

Thus, from the conditions on \( C \) and \( \Theta \),

\[
C_\lambda(\omega) = \int_0^\omega (\omega - \tau)^\lambda e^{-\tau s} d\Theta(\tau) + O(\epsilon \cdot \epsilon \lambda)
\]

as \( \epsilon \to 0^+ \); hence,
\[ C_{\chi}(\omega) = \int_0^\omega (\omega - \tau)^{\chi} e^{-\tau s} \, dB(\tau) \]
\[ = e^{-\omega s} \int_0^\omega (\omega - \tau)^{\chi} \, dB(\tau) + \int_0^\omega (\omega - \tau)^{\chi} (e^{-\tau s} - e^{-\omega s}) \, dB(\tau). \]

\[ (1) \quad = e^{-\omega s} \mathcal{B}_N(\omega) + \int_0^\omega (\omega - \tau)^{\chi} (e^{-\tau s} - e^{-\omega s}) \, dB(\tau), \]

We now make some preparations in order to integrate this last integral by parts.

We consider

\[ H(\tau) = (\omega - \tau)^{\chi} (e^{-\tau s} - e^{-\omega s}) \]

for \( 0 \leq \tau \leq \omega \).

\[ |H(\tau)| = |(\omega - \tau)^{\chi}| |\int_0^\omega e^{-\tau s} \, d\mu| \]

\[ \leq (\omega - \tau)^{\chi+1} (\omega - \tau) (e^{-\omega s} + 1) = \Theta((\omega - \tau)^{\chi+1}) \]

which \( \to 0 \) as \( \tau \to \omega \) for \( \chi > -1 \). Thus, \( H(\tau) \) has a removable discontinuity at \( \tau = \omega \). We therefore let

\[ H(\omega) = 0. \]

Also,

\[ H'(\tau) = \chi (\omega - \tau)^{\chi-1} (e^{-\tau s} - e^{-\omega s}) - \frac{s}{\omega - \tau} (\omega - \tau)^{\chi} e^{-\tau s} \]

\[ = \Theta((\omega - \tau)^{\chi}) \]

as \( \tau \to \infty \). Thus, for \( \chi > -1 \), \( H'(\tau) \) is integrable.
over \([0, \omega]\), and since \(H'(\tau)\) is continuous on \([0, \omega]\), \(H(\tau)\) is absolutely continuous on \([0, \omega]\).

We now consider for \(0 \leq \tau < \omega\),

\[
H^{(m)}(\tau) = (-1)^m \sum_{k=0}^{m} \frac{P_k(\lambda_i)}{P_{k-m+1}(\lambda)} (e^{-\tau} - e^{-\omega}) (\omega - \tau)^{\lambda-m} + \sum_{k=1}^{m} \sum_{\nu=1}^{\infty} \frac{P_k(\lambda_i)}{P_{k-m+1}(\lambda)} e^{-\nu\tau} (\omega - \tau)^{\lambda-m+k}.
\]

From (2), we see that

\[
H^{(m)}(\tau) = o((\omega - \tau)^{\lambda-m+1})
\]

as \(\tau \to \omega\). Thus, \(H^{(m)}(\tau)\) is integrable over \([0, \omega]\), and continuous over \([0, \omega]\) for \(m < \lambda+2\). Hence, \(H^{(m)}(\tau)\) is absolutely continuous on \([0, \omega]\) for all \(m < \lambda+1\).

If \(k\) is a non-negative integer,

\[
\int_0^\tau \theta_k(t) \, dt = \int_0^\tau \int_0^t (t-u)^k \, d\theta(u) \, dt
\]

\[
= \int_0^\tau \int_u^\tau (t-u)^k \, dt \, d\theta(u)
\]

\[
= \frac{1}{k+1} \int_0^\tau (\tau-u)^{k+1} \, d\theta(u)
\]

\[
= \frac{1}{k+1} \theta_{k+1}(\tau).
\]

Now, since \(\lambda(\lambda) < \lambda+1\), \(H^{(m)}(\tau)\) is absolutely continuous for \(m < \lambda(\lambda)\), and since \(\theta_{m}(\tau)\) is integrable for each \(m > 0\), we may integrate
\[ S_0^\omega H(\tau) \, d\mathcal{B}(\tau) \]

by parts \( h(\lambda) + 1 \) times to obtain

\[ S_0^\omega H(\tau) \, d\mathcal{B}(\tau) = \frac{(\cdot )^{h(\lambda)+1}}{h(\lambda)} \int_0^\omega H(h(\lambda)+1) \, \mathcal{B}_{h(\lambda)}(\tau) \, d\tau. \]

Thus,

\[ c_{x}(\omega) = e^{-\omega_S} \mathcal{B}_{\mathcal{N}}(\omega) \frac{\Gamma(x+1)}{\Gamma(h+1) \Gamma(x-h)} \int_0^\omega \mathcal{B}_{h}(\tau) (e^{-\tau_S} - e^{-\omega_S}) (\omega - \tau)^{x-h-1} \, d\tau \]

\[ + \frac{1}{\Gamma(h+1)} \sum_{k=1}^{h+1} s^k \frac{\Gamma(x+1)}{\Gamma(x+h+h)} \int_0^\omega \mathcal{B}_{h}(\tau) e^{-\tau_S} (\omega - \tau)^{x-h+h+1} \, d\tau. \]

Substituting \( x-p \) for \( x \), we have proved our assertion.

From [27, page 12],

\[ \mathcal{B}(\omega) = \int_0^\omega e^{\mu_S} e^{-\omega_S} \, d\mathcal{B}(\omega) = \int_0^\omega e^{\mu_S} \, d\mathcal{C}(\omega), \]

We may therefore interchange the roles of \( \mathcal{B} \) and \( \mathcal{C} \) if we replace \( S \) by \(-S\), and we obtain the second equation of Theorem 1.

We shall find useful the following lemma:

Lemma 2: Let \( \delta > 0 \), \( 0 < p \leq 1 \) and \( 0 \leq \xi(\omega) \leq L(0, \omega) \), then

\[ (\int_0^\omega \left( \int_0^\omega f(\omega) \, (\omega - \nu)^{\delta-1} \, d\nu \right)^{\frac{\nu}{\sigma}} \, d\omega) = o\left( \omega^\delta \left( \int_0^\omega f(\omega)^{\frac{\nu}{\sigma}} \, d\nu \right)^{\frac{\nu}{\sigma}} \right). \]
Proof: If \(0 \leq p \leq 1\), we have
\[
\int_0^\infty \left( \int_0^u f(v) (u-v)^{s-1} \, dv \right)^p \, du
\]
\[
= \Theta \left( \int_0^\infty f(v) (u-v)^{s-1} \, dv \right) \left( \int_0^u (u-v)^{s-1} \, dv \right)^{\frac{1}{p}} \, du
\]
from Hölder's Inequality if \(0 < p < 1\), trivially if \(p = 1\).

Interchanging the order of integration,
\[
\int_0^u \left( \int_0^\infty f(v) (u-v)^{s-1} \, dv \right)^p \, du
\]
\[
= \Theta \left( \int_0^\infty f(v) (u-v)^{s-1} \, dv \right) \int_0^u (u-v)^{s-1} \, dv \, du
\]
\[
= \Theta \left( \int_0^\infty f(v) (u-v)^{s-1} \, dv \right) \int_0^u (u-v)^{s-1} \, dv
\]
\[
= \Theta \left( \int_0^\infty f(v) \, dv \right).
\]

If
\[
\gamma(s) = \int_0^\infty e^{-us} \, d\mathbb{B}(u)
\]
for \(s\) sufficiently large, then we say \(\gamma(s)\) is \(1R, \chi^{1/p}\)
summable at the point \(s\) in case there exists a \(p\) such that
\[
\lim_{\omega \to \infty} \int_0^u e^{-us} \, d\mathbb{B}(u) = p \, 1R, \chi^{1/p},
\]
that is,
\[
\lim_{\omega \to \infty} \gamma(s) = p \, 1R, \chi^{1/p}.
\]
We now proceed to establish an abscissa of \( |R, \lambda|^{\gamma} \) summability.

Lemma 3 [19, page 26]: Let \( \lambda > 0 \), \( 0 < \gamma \leq 1 \). Suppose \( \gamma(s) \) is \( |R, \lambda|^{\gamma} \) summable at the point \( s = \sigma + it \) where \( \sigma > 0 \). Then for each \( \sigma > \sigma' \)

\[
\left( \int_{0}^{\infty} |B_{\gamma}(u)|^{\gamma} \, du \right)^{\gamma} = O \left( e^{\sigma^{\gamma}} \right).
\]

Proof: Applying Lemma 1 with \( s = s' \), \( \omega = u \),

\[
B_{\gamma}(u) = C_{\gamma}(u) \, e^{\omega s'}' + \frac{\Gamma(\lambda - p + 1)}{h!} \int_{0}^{\infty} C_{\gamma}(u)(e^{\omega s'} - e^{u s'}) (u - v)^{\lambda - p - 1} dv
\]

Now, since \( \sigma' > 0 \) for \( \lambda \neq u \),

\[
|e^{\omega s'} - e^{u s'}| = |s| \int_{s'}^{u} e^{y s'} \, dy \leq |s| \ e^{\omega s'} (u - s')
\]

Therefore, all the integrals on the right side of (3) are

\[
O \left( e^{\omega s'} \sum_{h=1}^{n} \int_{0}^{\infty} |C_{\gamma}(u)| (u - v)^{\lambda - p - h + 1} \, dv \right).
\]

Thus, by (2.5.4)

\[
\left( \int_{0}^{\infty} |B_{\gamma}(u)|^{\gamma} \, du \right)^{\gamma}
\]

\[
= O \left( \left( \int_{0}^{\infty} |C_{\gamma}(u)|^{\gamma} \, du \right)^{\gamma} \right) + \Theta \left( e^{\omega s'} \sum_{h=1}^{n} \left( \int_{0}^{\infty} |C_{\gamma}(u)| (u - v)^{\lambda - p - h + 1} \, dv \right)^{\gamma} \, du \right)^{\gamma}.
\]
By Theorem 4.5, the first "O" term is \( \Theta(e^{\omega \sigma} \omega^x) \).

In the second "O" term, we apply Lemma 2 with 
\[ f(\nu) = \left| c_h(\nu) \right| \]
and then apply the following inequality which follows from Theorems 4.2 and 4.5:
\[
\left( \int_0^\omega \left| c_h(\nu) \right|^{\nu e \lambda} d\nu \right)^e = \Theta(\omega^{x+e}).
\]

Thus,
\[
\left( \int_0^\omega \beta_{\nu-\rho}(\nu) \right)^e = \Theta(e^{\omega \sigma} \omega^x) + \Theta(e^{\omega \sigma} \omega^{x+\nu+1})
\]
\[
= \Theta(e^{\omega \sigma})
\]
for each \( \sigma > \sigma' \).

Lemma 4 [19, page 26]: Let \( \lambda > 0 \), \( 0 < \rho \leq 1 \), and suppose
\[
\left( \int_0^\omega \beta_{\nu-\rho}(\nu) \right)^e = \Theta(x^{\sigma'}).
\]

Then for each \( \sigma > \sigma' \), \( \gamma(s) \) is \( I, \lambda \nabla \nu \) summable. Moreover, there exists an analytic continuation for \( \gamma(s) \) into the region \( \sigma > \sigma' \) given by the strong Riesz limit.

Proof: We recall from (2.2.1) and (2.2.2) that \( a_n \) is the first non-zero element in A and \( \lambda(x) = 0 \) for \( x \leq N+1 \).

Thus, for \( N < e^\omega < N+1 \),
\[
\beta_{\nu-\rho}(\nu) = \int_0^\nu (\omega - \nu)^{\lambda-\rho} d\beta(\nu)
\]
\[
= a_n (\omega - \omega_{N+1})^{\lambda-\rho}.
\]
We then have for $\varepsilon^\omega > N+1$

\[
\left( \int_0^\omega \| B_{x-\varepsilon}(u) \|_{\varepsilon^\omega} \, du \right)^\varepsilon \geq \left( \int_{\log N}^{\log(N+1)} |a_N| (u - \log N)^{\frac{h}{2} - \varepsilon} \, du \right)^\varepsilon
\]

\[
= \frac{|a_N|^{\varepsilon^0}}{\varepsilon^0} \left( \log(N+1)^{\frac{h}{2} - \varepsilon} \right)^\omega > 0.
\]

Therefore,

\[(4) \quad \sigma' \geq 0.\]

If $h = h(x-\varepsilon) > x-\varepsilon$, by Theorem 3.1,

\[
\| B_h(u) \|_{\varepsilon^0} = O \left( \left( \int_0^\omega \| B_{x-\varepsilon}(u) \| (u - \varepsilon)^{h-\varepsilon+1} \, du \right)^\varepsilon \right).
\]

Therefore, from Lemma 2,

\[(5) \quad \left( \int_0^\omega \| B_h(u) \|_{\varepsilon^0} \, du \right)^\varepsilon
\]

\[
= O \left( \left( \int_0^\omega \left( \int_0^u \| B_{x-\varepsilon}(u) \| (u - \varepsilon)^{h-\varepsilon+1} \, du \right)^\varepsilon \right)^\varepsilon \right)
\]

\[
= O \left( \omega^{h-\varepsilon^e} \left( \int_0^\omega \| B_{x-\varepsilon}(u) \|_{\varepsilon^0} \, du \right)^\varepsilon \right)
\]

\[
= O \left( \omega^{h-\varepsilon^e} e^{\omega \sigma'} \right).
\]

(5) also holds if $h = x-\varepsilon$, since this is our assumption.

Let $I$ be the region

\[I = \left\{ s \mid \sigma' + \varepsilon \leq \sigma \leq \varepsilon, |t| \leq t \right\} \]
quadrant with integral coordinates under the circle with radius $\sqrt{x}$ centered at the origin.

In 1906, Sierpinski [21] used Voronoi's method to obtain the estimate $R(x) = \Theta (x^{\sqrt{2}} \log x)$ where $R(x)$ is defined by

$$A(x) = \pi x + \sqrt{x} + R(x).$$

Results on Dirichlet's divisor problem have been matched by estimates on $R(x)$, and in most cases the same methods are applicable. The relation between $D(x)$ and $A(x)$ is made clearer by considering the following generating functions:

Let

$$F(z) = \sum_{m=0}^{\infty} A(m) z^m$$

where

$$A(m) = \begin{cases} 1 & \text{for } m \text{ a square} \\ 0 & \text{otherwise} \end{cases}.$$ 

$F(z)$ converges absolutely for $|z| < 1$, so

$$\left\{ F(z) \right\}^2 = \left( \sum_{m=0}^{\infty} A(m) z^m \right) \left( \sum_{m=0}^{\infty} A(m) z^m \right)$$

$$= \sum_{k=0}^{\infty} \left( \sum_{m_1, m_2} A(m_1) A(m_2) \right) z^k$$

$$= \sum_{k=0}^{\infty} f(k) z^k$$

converges absolutely for $|z| < 1$. Here $f(k)$ is the
where $\varepsilon$, $k$, $\Gamma$, are fixed positive numbers with 
$\sigma^2 + 4 \varepsilon < k$. If $s \leq 1$,

$$
(6) \quad \int_{-\omega}^{\omega} |B_{k,s}(\omega)|^{\frac{3}{2}} e^{-a \sqrt{s}} d\omega
$$

$$
= \left\{ \left( \int_{-\omega}^{\omega} |B_{k,s}(\omega)|^{\frac{3}{2}} d\omega \right)^{\frac{3}{2}} + \frac{\omega}{\varepsilon} \left( \int_{-\omega}^{\omega} |B_{k,s}(\omega)|^{\frac{3}{2}} d\omega \right)^{\frac{3}{2}} \right\} e^{-a \sqrt{s}} d\omega
$$

$$
= \Theta(1) + \Theta(\int_{-\omega}^{\omega} e^{-\frac{(s-\varepsilon)}{s}} d\omega) = \Theta(1)
$$
as $\omega \to \infty$, uniformly for $s \leq 1$.

Also,

$$
(7) \quad \left( \int_{-\omega}^{\omega} |B_{k,s}(\omega)|^{\frac{3}{2}} e^{-a \sqrt{s}} d\omega \right) = \Theta(1)
$$
as $\omega \to \infty$, uniformly for $s \leq 1$, holds by a similar argument.

Now,

$$
\int_{0}^{\infty} |B_{k,s}(\omega)| e^{-\sqrt{s}} d\omega
$$
is uniformly absolutely convergent for $s \leq 1$ due to (7), if $\rho = 1$. If $\rho < 1$, then Hölder's Inequality, with

$\alpha = \rho$, $\beta = 1 - \rho$, implies

$$
\int_{0}^{\omega} |B_{k,s}(\omega)| e^{-\sqrt{s}} d\omega \leq \left( \int_{0}^{\omega} |B_{k,s}(\omega)|^{\frac{3}{2}} d\omega \right)^{\frac{3}{2}} \left( \int_{0}^{\omega} e^{-\sqrt{s}} d\omega \right)^{\frac{1}{2}}
$$

$$
= \Theta(1)
$$
as $\omega \to \infty$, uniformly for $s \leq 1$. 
Now, let
\[ \zeta = \frac{s^{h+1}}{h!} \int_0^\infty B_h(\nu) e^{-\alpha \nu} d\nu. \]

If we can show
\[ \lim_{\omega \to -\infty} \zeta(\omega) = \rho \mid R, X \mid \gamma \]
for \( \sigma > \sigma' \), then we will have an analytic continuation for \( \gamma(s) \) into the region \( \sigma > \sigma' \), because if \( C \) is \( \mid R, X \mid \gamma \) summable and convergent, it is summable to its ordinary limit.

Let \( \sigma > \sigma' \), and \( \delta \) be such that \( 0 < \delta < \sigma - \sigma' \).

Then, we have that
\[
\frac{s^{h+1}}{h!} \int_0^\mu B_h(\nu) e^{-\alpha \nu} (u-\nu)^{X-p} d\nu - \rho u^{X-p} \\
= \frac{s^{h+1}}{h!} \int_0^\mu B_h(\nu) e^{-\alpha \nu} ((e-x) \int_0^\nu (u-y)^{X-p-1} dy) d\nu + o(u^{X-p}).
\]

Thus, from Lemma 1,
\[
C_{X-p}(u) - \rho u^{X-p} = O \left( \mid B_{X-p}(u) \mid e^{-u\sigma} \right) \\
+ O \left( \int_0^\mu \mid B_h(\nu) \mid (e^{-\alpha \nu} - e^{-u\nu}) (u-\nu)^{X-p-h+1} d\nu \right) \\
+ O \left( \int_\frac{h}{h} \mid B_h(\nu) \mid e^{-\alpha \nu} (u-\nu)^{X-e-h+\gamma-1} d\nu \right)
\]
+ O\left( \int_0^u B_h(\omega) \left| e^{-\omega \sigma} (\rho-x) \int_0^\omega \left( u-y \right)^{\lambda-\rho-1} dy \right| d\omega \right) \\
+ o\left( u^{\lambda-\rho} \right)
\]
\]
\]
\]
= E_1 + E_2 + E_3 + E_4 + E_5,

To simplify $E_2$ we note that for $\omega > 0$,

$\omega \leq \nu \leq u$,

\[
| e^{-\omega \nu} - e^{-\nu \omega} | = \left| \int_\omega^\nu e^{-y \nu} dy \right| \leq \int_\omega^\nu e^{-\omega \nu} (u-\nu) \, d\nu,
\]

and

\[
| e^{-\nu \omega} - e^{-\nu \nu} | \leq e^{-2 \omega \nu}.
\]

Therefore, for $1 \leq \nu \leq u$,

\[
| e^{-\nu \omega} - e^{-\nu \nu} | = O(e^{-\omega \nu} (u-\nu)^{\lambda-\rho}),
\]

for each $\lambda, \rho \in [0, 1]$ , but for purposes of the following estimate we choose

$\nu - (\lambda-\rho) < \nu \leq \min (1, \nu+\rho)$.

Thus,

$E_2 = O\left( \int_0^u B_h(\omega) e^{-\omega \sigma} (u-\omega)^{\lambda-\rho-\lambda+\nu-1} d\omega \right)$.

Considering $E_4$ , we choose $\omega$ such that

$\max (0, \rho-x) \leq \omega \leq \rho$.
We then choose $\varphi$ such that

$$\max(0, x-\varepsilon) < \varphi < x - \varepsilon + \varepsilon.$$ 

For $1 \leq \nu < u$,

$$\int_0^\nu (u-y)^{x-\varepsilon-1} \, dy = \int_0^\nu (u-y)^{x-\varepsilon-\varphi-1} \, dy \\
\leq u^\varphi \int_0^\nu (u-y)^{x-\varepsilon-\varphi-1} \, dy$$

(9) $$= O(u^\varphi (u-\nu)^{x-\varepsilon-\varphi-1 \nu}),$$

and also

(10) $$\int_0^\nu (u-y)^{x-\varepsilon-1} \, dy = O(u^\varphi (u-\nu)^{x-\varepsilon-\varphi})$$

since $\varphi > \max(0, x-\varepsilon)$.

Therefore, since $0 < \nu < 1$, for $1 \leq \nu < u$,

(11) $$\int_0^\nu (u-y)^{x-\varepsilon-1} \, dy = O(u^\varphi (u-\nu)^{x-\varepsilon-\varphi+\nu-1 \nu^{-\nu}})$$

uniformly in $\nu$. This may be seen by comparing $\nu$ and $u-\nu$. If $\nu = u-\nu$, then $(\frac{\nu}{u-\nu})^{-\nu} \geq 1$ and multiplying (10) by this factor gives (11). If $\nu \leq u-\nu$, multiplying (9) by $(\frac{u-\nu}{\nu})^\varphi$ yields (11).

Finally, we remark $\nu^{-\nu} = O(e^{a\nu})$. 


Now,

$$E_4 = \Theta\left( u^\varphi \int_0^u \Omega_{B_h(\omega)} | e^{\nu(-\sigma+\delta)} (u-\nu)^{\chi-\varphi+\nu-1} d\nu \right).$$

Thus,

$$C_{X-\varphi}(u) = \Theta(1 \Omega_{X-\varphi}(u) | e^{-\mu \varphi}) + \Theta\left( \int_0^u \Omega_{B_h(\omega)} | e^{-\nu \sigma} (u-\nu)^{X-\varphi-h+\nu-1} d\nu \right).$$

Hence, applying (2.5.4) twice, we have

$$\left( \int_0^u \Omega_{B_h(\omega)} | e^{-\varphi \sigma} (u-\nu)^{X-\varphi-h+\nu-1} d\nu \right)^\varphi$$

$$= \Theta\left( \left( \int_0^u \Omega_{B_h(\omega)} | e^{-\frac{\varphi \sigma}{\varphi} \varphi} (u-\nu)^{X-\varphi-h+\nu-1} d\nu \right)^\varphi \right).$$

Hence, applying (2.5.4) twice, we have

$$\left( \int_0^u \Omega_{B_h(\omega)} | e^{-\varphi \sigma} (u-\nu)^{X-\varphi-h+\nu-1} d\nu \right)^\varphi$$

$$= \Theta\left( \left( \int_0^u \Omega_{B_h(\omega)} | e^{-\frac{\varphi \sigma}{\varphi} \varphi} (u-\nu)^{X-\varphi-h+\nu-1} d\nu \right)^\varphi \right).$$
Now, from Lemma 2, (6) and (7),
\[
(\int_0^\infty \vert A_{x-p}(u) - e^{x-p} u^{x-p} \vert^{1/\nu} du)^\nu = O(1) + O(\omega^{x-p-n+\alpha_1}) + O\left(\frac{\epsilon^{2}}{\nu^{2}}\right)\omega^{x-p-n} + O(\omega^{x-p+\nu}) + o(\omega^{x})
\]
\[
= o'(\omega^{x}).
\]
Therefore, \(Y(s)\) is \(|R, X|^{1/\nu}\) summable for each \(s\) with \(\sigma > \sigma'\).

**Theorem 1** [19, page 29]: Let \(X > 0, 0 < \rho < 1\). If \(Y(s)\) is \(|R, X|^{1/\nu}\) summable at the point \(s'\) with \(\sigma' > 0\), then \(Y(s)\) is \(|R, X|^{1/\nu}\) summable for each \(s\) with \(\sigma > \sigma'\).

**Proof:** The theorem follows immediately from Lemmas 3 and 4.

**Definition 1:** For \(X > 0, 0 < \rho < 1\),
\[
\sigma_{0}(X) = \inf \left\{ \xi > 0 \mid Y(s) \text{ is } |R, X|^{1/\nu} \text{ summable for } \sigma \geq \xi \right\}.
\]
\(\sigma_{0}(X)\) is called the abscissa of \(|R, X|^{1/\nu}\) summability for \(Y(s)\) and from Theorem 1, \(\sigma > \sigma_{0}(X)\) implies \(Y(s)\) is \(|R, X|^{1/\nu}\) summable, and \(0 < \sigma < \sigma_{0}(X)\) implies \(Y(s)\) is not \(|R, X|^{1/\nu}\) summable.

Also, if \(\xi > 0, X > 0, 0 < \rho < 1\), then Lemmas 1 and 2 imply
\[
(\int_0^\infty |A_{x-p}(u)|^{1/\nu} du)^\nu = O\left(e^{\omega(\sigma_{0}(X) + \varepsilon)}\right),
\]
and conversely, if
\[(S_{\omega} | B_{\nu-C}(\omega) |^{\nu} d\nu) = \Theta(e^{\omega r}),\]
then \(r < \sigma_{p}(\lambda)\). Thus,

(12) \(\sigma_{p}(\lambda) = \inf \left\{ \xi \mid (S_{\omega} | B_{\nu-C}(\omega) |^{\nu} d\nu) = \Theta(e^{\omega \xi}) \right\}^{p},\)

and hence, from Lemma 4, \(Y(s)\) has an analytic continuation into the region \(\Re(s) > \sigma_{p}(\lambda), \lambda > 0, 0 < p < 1\).

From Theorems 4.2, 4.4, 4.4, and 4.3, respectively, we get

(13) \(\sigma_{p}(\lambda) \leq \sigma_{p}(\lambda_{1})\)
for \(\lambda = \lambda_{1} > 0, 0 < p < 1\).

(14) \(\sigma_{p}(\lambda) \leq \sigma_{p}(\lambda)\)
for \(\lambda > 0, 0 < p_{1} < p < 1\).

(15) \(\sigma_{p}(\lambda + \delta) \leq \sigma_{p}(\lambda)\)
for \(\lambda > 0, 0 < p < 1, \delta > 0\).

(16) \(\sigma_{p}(\lambda + \rho) \leq \sigma_{p}(\lambda + \rho)\)
for \(\lambda + \rho > 0, 0 < \rho, \leq p < 1\).

Because of (16), we can make the following definition:

Definition 2: For \(\lambda > 0\),

\[\sigma_{p}(\lambda) = \lim_{\rho \to 0+} \sigma_{p}(\lambda + \rho)\]
Since $\sigma_p(X)$ is regular for $X - p > 0$,
$\sigma_0(X) = \sigma_1$.

And Lemma 4 implies
$S \leq \sigma_0(X)$.

(13) implies $\sigma_p(X + \rho) \leq \sigma_p(X, + \rho)$ for $X + \rho > 0$,
$0 < \rho \leq 1$, and thus,

(17)
$\sigma_p(X) \leq \sigma_p(X,)$

for $X > \rho_1 > 0$, $0 \leq \rho \leq 1$.

Definition 3: For $0 \leq \rho \leq 1$,

$\sigma_p(\alpha) = \lim_{\eta \to 0^+} \sigma_p(\eta) \leq \infty$.

Thus, given $\Psi(s)$, we can define $\sigma_p(X)$ in a region $H = \{ X \in \mathbb{C} \mid \Re X \geq \alpha, 0 \leq \rho \leq \frac{1}{2} \}$.

3.6. Generalized Absolute Convergence

We now give some results concerning a generalized concept of absolute convergence of a Dirichlet series. This development may be found in [19, page 21].

Definition 1: Let $0 < \rho \leq 1$ and $Z(s) = \sum \frac{a_m}{m^s}$. By $\alpha_\rho$ we mean the abscissa of convergence of

$\sum \frac{1}{m^{\rho \alpha}} m^{-\frac{s}{\rho}}$.

Let $0 < \alpha < 1$. We apply Jensen's Inequality (2.5.1), with $q = \frac{1}{\rho}$, $p = \frac{1}{\rho}$. Thus, $p < q$, and
for each real \( \sigma \) and integer \( k \),

\[
\left( \sum_{i=1}^{\kappa} |a_m|^{\kappa_i} m^{-\sigma_0} \right)^{\rho_i} \leq \left( \sum_{i=1}^{\kappa} |a_m|^{\kappa_i} m^{-\sigma_0} \right)^k.
\]

Therefore, \( \alpha_0 \geq \alpha_0 \).

Let \( 0 < \epsilon \leq \epsilon \leq 1 \), \( \sigma > \sigma_0 + \epsilon \), \( \sigma > \sigma_1 > \epsilon \).

We apply Hölder's Inequality (2.5.3), with \( e_m = m^{\frac{\sigma - \sigma_0}{\epsilon}} \),

\[
d_m = |a_m|^{\kappa_i} m^{-\frac{\sigma - \sigma_0 + \epsilon}{\epsilon}}, \quad \alpha = \frac{\epsilon}{\epsilon}, \quad \beta = 1 - \alpha = (\epsilon - \epsilon)/\epsilon,
\]

and we obtain

\[
\left( \sum_{i=1}^{\kappa} |a_m|^{\kappa_i} m^{-\frac{\sigma - \sigma_0}{\epsilon}} \right)^{\beta} \leq \left( \sum_{i=1}^{\kappa} |a_m|^{\kappa_i} m^{-\frac{\sigma - \sigma_0 + \epsilon}{\epsilon}} \right) \left( \sum_{i=1}^{\kappa} m^{-\frac{\epsilon \sigma - \sigma_0}{\epsilon}} \right)^{\alpha - \beta}.
\]

Thus, \( \sigma \geq \sigma_0 - \epsilon \), which implies \( \sigma_1 - \epsilon \geq \sigma_0 - \epsilon \).

Definition 2: \( \alpha_0 = \lim_{\epsilon \to 0^+} \alpha_0 \).

It is easily seen that \( \alpha_1 \geq \alpha_0 \geq \alpha_1 \). We now have shown:

Theorem 1: Let \( 0 < \epsilon \leq \epsilon \leq 1 \). Then \( \alpha_0 \leq \alpha_0 \)

and \( \alpha_0 - \epsilon \leq \alpha_0 - \epsilon \).

If \( \sigma > \sigma_0 \), then there is a \( \epsilon > 0 \) such that \( \sigma > \sigma_0 \). Thus,

\[
\sum_{i=1}^{\kappa} |a_m|^{\kappa_i} m^{-\sigma_0}
\]

converges, and \( a_m = O(m^\sigma) \). If \( a_m = O(m^\sigma) \) holds for \( \sigma < \sigma_0 \), then for sufficiently small \( \epsilon > 0 \) and for \( \sigma' = \sigma + 2 \epsilon \leq \sigma_0 \), we have

\[
|a_m|^{\kappa_i} m^{-\sigma_0} = O(m^{-2}).
\]
Then
\[ \sum_{m=1}^{\infty} |a_m|^{1/\varepsilon} m^{-5/\varepsilon} \]
converges, but \( \sigma' < \alpha_0 < \alpha_9 \). This contradicts the definition of \( \alpha_9 \).

**Theorem 2**: If \( \alpha < \rho \leq 1 \) and \( \alpha_0 > 0 \), then

\[ \alpha_9 = \inf \xi \xi |(\sum_{m=1}^{\infty} |a_m|^{1/\varepsilon})^{\rho} = O(x^{\xi}) \].

Also,
\[ \alpha_0 = \inf \xi |a_m| = O(m^{\xi}) \].

**Proof**: The first assertion follows from above. The last follows from the well-known fact that a Dirichlet series \( \sum_{m=1}^{\infty} a_m m^{-s} \) converges for \( \sigma > \xi > 0 \) if and only if
\[ \sum_{m=1}^{\infty} a_m = O(x^{\xi + \varepsilon}) \]
for each \( \varepsilon > 0 \).

For the \( \Lambda(x) \) with which we deal, \( \alpha_9 \geq 1 \), so from Theorem 1, \( \alpha_9 - \rho \geq \alpha_9 - 1 \geq 0 \). Thus \( \alpha_9 \geq \rho \) for \( 0 \leq \rho \leq 1 \) and hence,
\[ (\sum_{m=1}^{\infty} |a_m|^{1/\varepsilon})^{\rho} = O(x^{\alpha_9 + \varepsilon}) \]
for \( 0 \leq \rho \leq 1 \), \( \varepsilon > 0 \).

Also, since \( \alpha_9 \leq 2 \), we have \( \alpha_9 \leq 2 \) for \( 0 \leq \rho \leq 1 \).

**Theorem 3**: \( \alpha_9 \) is a convex function of \( \rho \) for \( 0 \leq \rho \leq 1 \).

**Proof**: We assume that \( 0 < \alpha_9 < 1 \), \( 0 < \rho_1 < \rho_2 < 1 \). Set \( \rho = \rho_1 \rho_2 + (1-\rho_1) \rho_2 \). We apply H"{o}lder's Inequality with
number of representations of $k$ as the sum of two squares. Thus,

$$A(x) = \sum_{0 \leq k \leq x} f(k)$$

At present, as with the divisor problem, it is known that

$$R(x) = \mathcal{O}(x^{\frac{12}{k^2} + \varepsilon})$$

for each $\varepsilon > 0$ and that

$$(7) \quad R(x) = \Omega(x^{\frac{k}{2}} (\log x)^{\frac{k}{2}})$$

Piltz [18] obtained in 1881 an elementary result concerning a generalization of Dirichlet's divisor problem. Let $d_k(m)$ be the number of ordered factorizations of $m$ into $k$ factors and

$$D_k(x) = \sum_{m \leq x} d_k(m), \quad k \geq 2$$

Piltz showed that

$$(8) \quad D_k(x) = x P_k(\log x) + \mathcal{O}(x^\Theta)$$

with $\Theta = 1 - \frac{1}{2} k$ where $P_k(\log x)$ is a polynomial in $\log x$ of degree $k - 1$. Hardy and Littlewood [9] showed in 1922 that for each $k \geq 4$ the greatest lower bound $\Theta_k$ of the exponents $\Theta$ in (8) is $\leq (k - 1)/(k + 2)$.
\[ \alpha = \rho, \quad \beta = 1 - \alpha = \rho (1 - \rho^2) / \rho \quad . \] Then
\[
\left( \sum_{m=1}^{\infty} l a_m^{\rho/\rho} \right)^\rho = \left( \sum_{m=1}^{\infty} l a_m^{\rho/\rho} \right)^\rho \leq \left( \sum_{m=1}^{\infty} l a_m^{\rho/\rho} \right)^{\rho}, \left( \sum_{m=1}^{\infty} l a_m^{\rho/\rho} \right)^{1-\rho} \rho^2 \\
= O( \chi^{\rho} \alpha \epsilon_1 + (1 - \rho^2) \alpha \epsilon_2 + \epsilon ) \]

Thus,
\[ \alpha \epsilon_1 \leq n^\sigma \alpha \epsilon_1 + (1 - n^\sigma) \alpha \epsilon_2 \]

Also,
\[ \alpha \rho + (1 - n^\sigma) \epsilon_2 \leq \alpha \rho \rho + (1 - n^\sigma) \epsilon_2 \leq \rho \alpha \rho + (1 - n^\sigma) \alpha \epsilon_2 \]

and taking the limit as \( \rho \to 0 \),
\[ \alpha \rho + (1 - n^\sigma) \epsilon_2 \leq \rho \alpha \rho + (1 - n^\sigma) \alpha \epsilon_2 \]

Theorem 4: \( \alpha \epsilon_1 \) is continuous for \( \alpha \leq \rho \leq 1 \).
Proof: The continuity in \( \alpha \leq \rho < 1 \) follows from convexity.
Continuity at \( \rho = 0 \) follows from the definition of \( \alpha \),
and at \( \rho = 1 \) from \( \alpha \rho - \rho \geq \alpha - 1 \), which implies
\[ 1 - \rho \geq \alpha - \alpha \rho \geq 0 \]

3.7. Relations between \( \alpha \epsilon_1 \) and \( \sigma_\rho(x) \).
We return to the properties of \( \sigma_\rho(x) \) and show
Lemma 1 [19, page 30]: \( \sigma_\rho(x) \leq x, \quad x^\infty \quad \text{for} \quad x \geq 0 \),
\[ 0 \leq \rho \leq 1 \]
Proof: For $0 < \rho < 1$, $x > 0$, from Hölder's Inequality
\[
(S_o \mid B_{x \cdot \rho}(u) \mid \kappa^p d u)^{\rho} \leq (S_o \mid S_o^u (u - \tau) \kappa^p d V \beta(o, \tau) \mid \kappa^p d u)^{\rho}
\]
\[
\leq (S_o \mid S_o^u (u - \tau) \kappa^p d V \beta(o, \tau) \mid \kappa^p d u)^{\rho}
\]
This holds also for $\rho = 1$, trivially.

Now, for $u \leq \omega$, $\kappa = e^u$,
\[
S_o^u d V \beta(o, u) = \Theta(S_o \mid d V \mu(o, u) + S_o^\omega d V A(o, \tau))
\]
\[
= \Theta(S_o \mid L(\tau) (1 + \delta(\tau)) d \tau + \kappa^{\alpha_1 + \varepsilon})
\]
\[
= \Theta(\kappa^{\alpha_1 + \varepsilon} + \kappa^{\alpha_1 + \varepsilon}) = \Theta(\kappa^{\alpha_1 + \varepsilon})
\]
\[
= \Theta(e^{u(\alpha_1 + \varepsilon)}).
\]
We used here that $\tau L(\tau)$ is monotone for $\tau \geq M$.

Hence,
\[
(S_o \mid B_{x \cdot \rho}(u) \mid \kappa^p d u)^{\rho} = \Theta(e^{u(\alpha_1 + \varepsilon)})(S_o \mid S_o^u (u - \tau) \kappa^p d V \beta(o, \tau) d u)^{\rho}).
\]
As before, we may interchange the order of integration to obtain
\[
(S_o \mid B_{x \cdot \rho}(u) \mid \kappa^p d u)^{\rho} = \Theta(e^{u(\tau + \varepsilon)})(S_o^{\omega} \mid S_o^u (u - \tau) \kappa^p d V \beta(o, \tau))^{\rho})
\]
\[
= \Theta(e^{u(\tau + \varepsilon)})(S_o^{\omega} \mid S_o^u (u - \tau) \kappa^p d V \beta(o, \tau))^{\rho})
\]
\[
\left( \int_0^w \delta_{u-e(u)} \, du \right)' = \Theta \left( e^{w(\alpha+\epsilon)} \omega^{\alpha} e^{\omega \phi(\alpha+\epsilon)} \right)
\]

\[
= \Theta \left( e^{\omega (\alpha+2\epsilon)} \right).
\]

Now, (5.12) implies the lemma for \( \alpha \neq 1 \), \( \lambda > 0 \).

We take limits to establish the cases \( \alpha = 0 \), or \( \lambda = 0 \).

**Lemma 2 [26, page 4]:** For \( 0 \leq \eta < u < \gamma \), \(-1 < \lambda < 0\),

\[
\frac{(\gamma-u)^{m+1}}{\Gamma(-\lambda) \Gamma(\lambda+1)} \int_0^u (\eta-u) (\gamma-u)^{m} (u-\eta)^{-m-1} \, d\eta = (\gamma-\eta)^{m}.
\]

**Proof:** Let

\[
\tau = \frac{y \gamma (u-\eta) + \eta (\gamma-u)}{y (u-\eta) + (\gamma-u)}.
\]

Then

\[
\frac{d\tau}{\gamma-\tau} = \frac{(u-\eta) \, dy}{y (u-\eta) + (\gamma-u)}.
\]

\[
\tau - \eta = \frac{y (\gamma-\eta) (u-\eta)}{y (u-\eta) + (\gamma-u)}.
\]

and

\[
u - \tau = \frac{(u-y) (\gamma-u) (u-\eta)}{y (u-\eta) + (\gamma-u)}.
\]

Thus,
\[
\int_{\eta}^{\mu} (u-\eta)^{x-1} (u-\nu)^{-x-1} \, d\nu
\]

\[=
\left( \int_{0}^{1} y^x (1-y)^{-x-1} \, dy \right) (\mu-\eta)^x (u-\eta)^{x+1} (\mu-\nu)^{-x-1} (u-\nu)^{-x-1}
\]

\[=
\frac{(\mu-\eta)^x (\mu-\nu)^{-x-1}}{\Gamma(-x) \Gamma(x+1)}
\]

We shall prove the following important theorem.

Theorem 1 [19, page 32]: For \(0 < x < r \leq 1\),

\[\alpha \leq \sigma_0 (x) + x\]

and

\[\sigma_0 (x) > 0\]

Proof: \(\sigma_0 (x) > 0\) follows from the first part of the theorem and the fact that \(\alpha \geq \varepsilon\).

Let \(0 < x < q \leq 1\), \(\varepsilon > 0\), \(j\) be an integer greater than \(N+1\). We denote \(l_{N+1}\) by \(m_j\), and take \(m_{j-1} < u \leq m_j < \nu \leq m_{j+1}\). Then by Lemma 2,

\[
\int_{0}^{\mu} (y-N)^x \, d\theta (y) = \int_{0}^{\mu} \left( \frac{(y-N)^x}{\Gamma(x+1)} \right) \int_{0}^{\mu} (y-N)^x (y-N)^{-x-1} \, dy \, d\theta (y)
\]

\[=
\frac{(\mu-N)^x}{\Gamma(x+1)} \int_{0}^{\mu} (y-N)^{-x-1} \int_{0}^{\mu} (y-N)^x \, d\theta (y) \, dy
\]

where the interchange of order of integration is justified as before. Thus,

(1) \[\int_{0}^{\mu} (y-N)^x \, d\theta (y) = \frac{(\mu-N)^x}{\Gamma(x+1)} \int_{0}^{\mu} \theta (y) (y-N)^{-x-1} \, dy\]
Therefore,

\[
(2) \int_0^\infty (\nu - \nu)^{\lambda + \theta} \, d\vartheta(\nu) = \mathcal{B}_{2,0}(\nu) - \frac{(\nu - \nu)^{\lambda + \theta + 1}}{\Gamma(\lambda + \theta + 1) \Gamma(\lambda + \theta + 1)} \int_0^\mu \frac{e^{\nu \vartheta}}{(\nu - \nu)^{\lambda + \theta + 1}} \, d\vartheta(\nu)
\]

\[
= O\left(1 \mathcal{B}_{2,0}(\nu)\right)
\]

\[
+ O((\nu - \nu)^{\lambda + \theta + 1} \int_0^{m_{\lambda + \theta - 2}} \vartheta(\nu) |\mathcal{B}_{2,0}(\nu)| (\nu - \nu)^{-\lambda + \theta + 1} \, d\vartheta(\nu))
\]

\[
+ O((\nu - \nu)^{\lambda + \theta} \int_{m_{\lambda + \theta}}^\mu \vartheta(\nu) |\mathcal{B}_{2,0}(\nu)| (\nu - \nu)^{-\lambda + \theta + 1} \, d\vartheta(\nu))
\]

where, as in the following, the "O" constants depend only on \(\lambda\), \(\mu\), \(\nu\), \(\theta\) and \(\varepsilon\).

Now,

\[
\int_0^\nu (\nu - \nu)^{\lambda + \theta} \, d\vartheta(\nu) = (\nu - m_\lambda)^{\lambda + \theta} \alpha_\lambda - \int_0^\nu (\nu - \nu)^{\lambda + \theta} \, d\left(e^\nu \vartheta(\nu)\right).
\]

From (2.2.2) and (2.2.3),

\[
(3) \int_0^\nu (\nu - \nu)^{\lambda + \theta} \, d\left(e^\nu \vartheta(\nu)\right) = \int_0^\nu (\nu - \nu)^{\lambda + \theta} e^\nu \vartheta(\nu) (1 + \delta(\varepsilon^{\nu})) \, d\vartheta(\nu)
\]

\[
= O\left(i \mu(i) \int_0^\nu (\nu - \nu)^{\lambda + \theta} \, d\vartheta(\nu)\right)
\]

\[
= O\left(i \mu(i) (\nu - \nu)^{\lambda + \theta + 1}\right).
\]

Thus, from (1), (2) and (3), we have
Using (2.5.4) and Hölder's Inequality (2.5.3), we obtain

\[
|a_{\phi}|^{\nu_{\phi}} (\xi - m_{\phi})^{\delta_{\phi}} = \mathcal{O}(1) \beta_{0}(\xi)^{\nu_{0}}
\]

\[
+ \mathcal{O}\left((\xi - u)^{\nu_{0}} \int_{0}^{m_{\phi}^{-1}} |\beta_{0}(u)| (u - \nu)^{-\nu_{0} + p - 2} d\nu\right)
\]

\[
+ \mathcal{O}\left(\int_{0}^{u} |\beta_{0}(u)| (u - \nu)^{-\nu_{0} + p - 1} d\nu\right)
\]

\[
+ \Theta\left(\mathcal{H}(\xi) (\xi - u)^{\nu_{0} + 1}\right).
\]

We shall denote (5) by \( I = E_{1} + E_{2} + E_{3} + E_{4} \).

Our procedure will now be to integrate both sides of (5); first \( \int_{m_{\phi}}^{m_{\phi}+\epsilon} d\xi \), and then \( \int_{m_{\phi}+\epsilon}^{m_{\phi}+\epsilon + 1} d\xi \). We will handle each term separately for convenience.

\[
I = \sum_{m_{\phi}}^{m_{\phi}+\epsilon} I d\xi = |a_{\phi}|^{\nu_{\phi}} \frac{\phi}{\delta_{\phi}} (\xi - m_{\phi})^{\frac{\log(1 + \epsilon)}{\epsilon}}
\]

\[
= |a_{\phi}|^{\nu_{\phi}} \frac{\phi}{\delta_{\phi}} \left(\log(1 + \epsilon)^{\nu_{0}}\right).
\]
\[
S_{m_j}^{\tau} I^* d\nu = \frac{e}{\tau} (a \delta)^{1/\nu} \left( \log (1 + i^{-1/\nu}) \right)^{1/\nu} \log \left( \frac{\delta}{x^j} \right).
\]

\[
E_1^* = S_{m_j}^{\log(1+\frac{1}{x})} E_1 d\nu = O(S_{m_j}^{\tau} E_1 d\nu) = O(S_{m_j}^{\tau + 1} (\beta_{\nu\rho}(\nu)|^{1/\nu} d\nu).
\]

\[
S_{m_j}^{\tau} E_1^* d\nu = O((m_j-m_j-\tau) S_{m_j}^{\tau + 1} (\beta_{\nu\rho}(\nu)|^{1/\nu} d\nu),
\]

\[
E_2^* = S_{m_j}^{\log(1+\frac{1}{x})} E_2 d\nu = O(S_{m_j}^{\tau} E_2 d\nu)
\]

\[
= O(S_{m_j}^{\tau + 1} (\beta_{\nu\rho}(\nu)|^{1/\nu} d\nu).
\]

\[
S_{m_j}^{\tau} E_2^* d\nu = O((m_j-\nu-1) S_{m_j}^{\tau + 2} (\beta_{\nu\rho}(\nu)|^{1/\nu} d\nu),
\]

\[
E_3^* = S_{m_j}^{\log(1+\frac{1}{x})} E_3 d\nu = O(S_{m_j}^{\tau + 1} E_3 d\nu).
\]
\[ E_3^* = O((m d_{i+1} - u)^\frac{\nu e}{c} (u - m d_{i+1})\frac{\nu e}{c} + \nu e - \frac{\nu e}{c} \int_{m d_{i+2}}^u \beta_{i+1} \nu (\nu e) \nu (u - \nu e)^{-\frac{\nu e}{c} + 1} d\nu) \]

\[ = O\left( \int_{m d_{i+2}}^u \beta_{m d_{i+2}} \nu (\nu e) \nu (u - \nu e)^{-\frac{\nu e}{c} + 1} d\nu \right) \]

\[ \sum_{m d_{i+1}}^{m d_{i+2}} E_3^* d\nu = O\left( \int_{m d_{i+1}}^{m d_{i+2}} \beta_{m d_{i+2}} \nu (\nu e) \nu (u - \nu e)^{-\frac{\nu e}{c} + 1} d\nu \right) \]

\[ = \sum_{m d_{i+1}}^{m d_{i+2}} E_4^* d\nu = O\left( \int_{m d_{i+1}}^{m d_{i+2}} \beta_{m d_{i+2}} \nu (\nu e) \nu (u - \nu e)^{-\frac{\nu e}{c} + 1} d\nu \right) \]

\[ E_4^* = \sum_{m d_{i+1}}^{m d_{i+2}} E_4^* d\nu = O\left( \int_{m d_{i+1}}^{m d_{i+2}} \beta_{m d_{i+2}} \nu (\nu e) \nu (u - \nu e)^{-\frac{\nu e}{c} + 1} d\nu \right) \]

\[ = O\left( \int_{m d_{i+1}}^{m d_{i+2}} \beta_{m d_{i+2}} \nu (\nu e) \nu (u - \nu e)^{-\frac{\nu e}{c} + 1} d\nu \right) \]

\[ = O\left( \int_{m d_{i+1}}^{m d_{i+2}} \beta_{m d_{i+2}} \nu (\nu e) \nu (u - \nu e)^{-\frac{\nu e}{c} + 1} d\nu \right) \]

\[ = O\left( \int_{m d_{i+1}}^{m d_{i+2}} \beta_{m d_{i+2}} \nu (\nu e) \nu (u - \nu e)^{-\frac{\nu e}{c} + 1} d\nu \right) \]
Now, from these estimates, we have

\[
(6) \quad |a_\xi| \leq \Theta \left( e^{(1+\varepsilon) \frac{\gamma}{\nu}} \int_{m_\xi}^{m_{\xi+1}} |B_{\lambda-\sigma}(\nu)| \frac{\nu}{\nu + 2} d\nu \right)
\]

\[
+ \Theta \left( e^{(1+\varepsilon) \frac{\gamma}{\nu}} - \frac{\nu}{\nu + 2} \int_{m_\xi}^{m_{\xi+1}} |B_{\lambda-\sigma}(\nu)| \frac{\nu}{\nu + 2} \left( m_{\xi} - \nu \right)^{\nu - 2} d\nu \right)
\]

\[
+ \Theta \left( e^{(1+\varepsilon) \frac{\gamma}{\nu}} \int_{m_\xi}^{m_{\xi+1}} |B_{\lambda-\sigma}(\nu)| \frac{\nu}{\nu + 2} d\nu \right)
\]

\[
+ \Theta \left( \frac{\nu}{\nu + 2} \int_{m_\xi}^{m_{\xi+1}} |B_{\lambda-\sigma}(\nu)| \frac{\nu}{\nu + 2} \left( m_{\xi} - \nu \right)^{\nu - 2} d\nu \right)
\]

We observe that

\[
\sum_{e^{\gamma_1} + \frac{1}{2} \leq \nu \leq e^{\gamma_2}} e^{\frac{\gamma}{\nu}} + e^{\frac{\gamma}{\nu} - 2} \left( m_{\xi} - \nu \right)^{\nu - 2}
\]

\[
\leq e^{\frac{\gamma}{\nu}} \left( e^{\nu \left( \frac{e - 2}{4} \right)} \left( \log(e^{\nu} + 1) - \nu \right)^{\nu - 2} \right)
\]

\[
+ e^{\frac{\gamma}{\nu}} \int_{\log(e^{\nu} + 1)}^{\nu} e^{\nu \left( \frac{e - 2}{4} \right)} \left( \nu - \nu \right)^{\nu - 2} d\nu
\]

\[
= \Theta \left( e^{\nu \frac{\gamma}{\nu}} \right).
\]

Finally, we sum (6) over $\xi$ and

\[
\sum_{\xi} e^{\nu \frac{\gamma}{\nu}} |a_\xi| \leq \Theta \left( 1 \right)
\]

\[
+ \Theta \left( e^{\frac{\nu \gamma}{\nu} + (1+\varepsilon) \frac{\log(e^{\nu} + 1)}{\nu}} \int_{\nu - \sigma}^{\nu} |B_{\lambda-\sigma}(\nu)| \frac{\nu}{\nu + 2} d\nu \right)
\]

\[
+ \Theta \left( \frac{\log(e^{\nu})}{\nu} \int_{\nu - \sigma}^{\nu} |B_{\lambda-\sigma}(\nu)| \sum_{e^{\gamma_1} + \frac{1}{2} \leq \nu} e^{\frac{\nu \gamma}{\nu} + (1+\varepsilon) \frac{\log(e^{\nu} + 1)}{\nu}} \left( m_{\xi} - \nu \right)^{\nu - 2} d\nu \right)
\]
+ \Theta \left( e^{\omega (\xi + 1) + \sigma_o + \delta^2} \right)
+ \Theta \left( e^{\omega (\xi + 1) + 1} \right).

Thus, for each \( \delta > 0 \),

\[
\left( \frac{\xi}{\delta} e^{\lambda \eta / \gamma} \right)^{\sigma_o} = \Theta \left( e^{\omega \xi (\xi + 1) + \sigma_o \eta + \delta^2} \right) + \Theta \left( e^{\omega \xi (\xi + 1) + \sigma_o \eta (\frac{\eta}{\gamma} - 1)^2} \right).
\]

Hence,

\[
\alpha_o \leq \max \left( \eta (\xi + 1) + \sigma_o \eta, \xi - \alpha \right)
\]

where \( \alpha > 0 \). But \( \alpha_o \geq \xi > \xi - \alpha \).

Therefore,

\[
\alpha_o \leq \eta (\xi + 1) + \sigma_o \eta.
\]

Since \( \xi \) is an arbitrary positive number, we have

\[
\alpha_o \leq \eta + \sigma_o \eta.
\]

This proves Theorem 1.

3.8. The Convexity of \( \sigma_o (\eta) \)

This section is concerned with the convexity and the continuity of \( \sigma_o (\eta) \) on the region

\[
\mathcal{H} = \{(\xi, \eta) \mid \eta \geq 0, 0 \leq \xi \leq \bar{\xi} \}.
\]