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A GENERAL THEORY OF THIN
ELASTIC SHELLS FOR ISOTROPIC AND ORTHOTROPIC
MATERIALS

DISSERTATION

Presented in Partial Fulfillment of the Requirements
for the Degree Doctor of Philosophy in the
Graduate School of The Ohio State University

By

Thomas Joseph Kozik, B. S., M. S.

The Ohio State University
1962

Approved by

[Signature]
Adviser
Department of Engineering Mechanics
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<td>average of the difference of the middle surface curvatures ( L = \frac{k_1 k_2}{2} )</td>
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1 Only the basic symbols are listed here. Additional, more specialized symbols are defined in the specific sections in which they are used in the text.
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<td>( u, v )</td>
<td>middle surface displacements tangent to that surface and in the directions of the principal curvilinear coordinates ( \beta = \text{constant} ) and ( \alpha = \text{constant} ), respectively</td>
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<td>$\nu$</td>
<td>Poisson's ratio</td>
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<td>middle surface normal stresses</td>
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<td>$\sigma_\alpha$, $\sigma_\beta$, $\sigma_\gamma$</td>
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<tr>
<td>$\sigma_{\alpha\beta}$</td>
<td>shearing stress acting on an equidistant surface</td>
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<td>$\chi$</td>
<td>rotation of an element of area lying on an equidistant surface</td>
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INTRODUCTION

In the last decade, a great need has arisen for structures with a high ratio of load capacity to weight. At first, attempts to satisfy this condition were directed solely toward metallurgical improvement. The result was the development of new construction materials which possessed high strength to weight ratios. However, these attempts to solve the basic problem resulted in the improvement of the materials composing the structure without affecting the structure itself. The weight saving effected by the use of new materials was adequate for a time. However, the demand for lightness continued until it was obvious that in order to continue decreasing the weight without sacrificing load carrying capacity, not only would the material have to be improved, but the shape and type of structure as well.

One type of structure which heretofore has been relatively neglected is the thin elastic shell. Save for some limited applications, this member has been considered one of academic interest alone. However, the shell is known to possess certain advantages over other types of structures, the chief one being
that it is able, in many cases, to redistribute the stresses from a given load throughout its interior without the presence of bending so that the peak stress in the shell will be smaller than the corresponding stress in another structure. Thus in many applications, a sizable weight reduction is possible by the use of a shell instead of other types of structures because a lesser amount of material is necessary to prevent failure from a given loading. The one great disadvantage of shells is the difficulty in quantitatively analyzing these members.

H. Aron, using the Kirchoff hypothesis, made the first successful attempt at deriving the shell equations. This work, however, contained some errors which were detected by A. E. H. Love and published by him in a paper with the corresponding corrections. Later, Love incorporated this paper in his book which, in part, dealt extensively with the problem of thin shell analysis.

Love's work on thin shells was not devoid of deficiencies, especially in the matter of small term analysis. However, in spite of its shortcomings, the comprehensiveness of the study

---

made his work a standard reference for future investigators. Moreover, it set the pattern of derivation and solution of thin shell equations for many years.

The one great problem encountered in shell analysis has always been the reduction of the number of equations to be solved. This condition is especially true if the equations are expressed in terms of stress resultants. Increased engineering interest in the shell as a practical structure is causing a great deal of effort to be expended for ways to reduce, simplify, and give physical meaning to the shell equations. In this way, hopefully, shell theory may be placed within the grasp of the ordinary practicing engineer rather than being relegated to the domain of the applied mathematician. Works by such authors as Timoshenko², Flügge³, and Novozhilov⁴ are notable for their attempts to accomplish this aim.


The demand for lightness of structure is leading to a refinement of construction even in shells. In those instances in which the directions of the major loads are completely known, a weight savings sometimes may be effected by utilizing materials which manifest their greatest strength in only those directions and regions of highest stress. These materials may either intrinsically possess the directional strength properties or they may be deliberately constructed to obtain them. However, whether by accident or by plan, the fact that directional properties exist means that the material can no longer be considered isotropic and hence the shell analysis becomes more involved.

This dissertation is concerned with two analyses of shells. The first is for isotropic materials. The second is for a specific form of anisotropic materials, namely those possessing curvilinear orthotropy.

The derivation of the equations for isotropic shells to be presented is suggested by the work of V. Z. Vlasov.\(^5\) It differs from other derivations in that the equations of equilibrium and hence the final shell equations are not expressed in terms of

stress resultants, but rather in terms of the three components of the middle surface displacement.

The principle advantage of deriving and stating shell equations in terms of displacements rather than stress resultants is that it reduces the number of equations to be studied. Since there are only the three unknown displacement components in the equations, then ultimately there can be only three independent equations. In contrast, shell equations expressed in terms of stress resultants ultimately reduce to five in number.

Another advantage of the shell displacement equations is the ease with which they can be simplified. Since there are only the three variables to contend with, a direct term-by-term comparison will usually suffice in establishing which terms are negligible. Still another advantage of the equations is the fact that various groups of terms in the equation can be given physical interpretation and hence studied as separate phenomena.

The analysis of a curvilinearly orthotropic shell is not as restrictive as it might appear. Engineering practice indicates that this type of anisotropy is in fact the most frequently encountered. The analysis of this type of shell presented here is unique in that no comparable work appears in the literature. Though various particular cases have been solved, as yet there
has been no published work which formulates the equations of the general orthotropic shell in terms of the middle surface displacement components.

The complete set of equations governing an orthotropic cylinder are given as an application of the derived general orthotropic shell theory. These cylinder equations are in turn applied to the problem of the determination of influence coefficients for axisymmetric end loads.
CHAPTER I

THE ISOTROPIC SHELL

1. Definitions and Assumptions

A shell may be defined as any structure whose geometry may be completely characterized by a surface lying midway between the inner and outer surfaces of the structure. This characterizing surface is defined as the middle surface.

In dealing with shells, it is advantageous to classify them into one or more of the following categories. That is, shells may be considered either thick or thin, elastic or plastic, isotropic or anisotropic, and whether the shell deflections are large or small. This dissertation will be solely concerned with thin elastic shells which manifest small displacements and further, the present chapter will be concerned with isotropic shells.

The two fundamental assumptions used in the derivation of thin shell equations are accredited to Kirchhoff, who first formulated them in regard to flat plates, and to G. Aron and
A. E. H. Love who extended them to thin shells. These assumptions may be reduced to the two following statements:

i) Straight lines initially perpendicular to the middle surface of the shell remain straight and perpendicular to that surface after deformation.

ii) Stresses normal to planes parallel to the middle surface may be neglected in comparison with the other stresses.

These assumptions are nothing more than generalizations of the basic assumption used in deriving the beam bending formula, namely that plane sections remain plane after deformation. As might then be expected, all the inherent inaccuracies of simple beam theory will be carried over to shell theory. Thus the effects of the transverse shearing stresses on the deformation of the shell are excluded and the shell equations must not be expected to yield accurate results in regions of high shear loads. Further, neglecting the normal stresses normal to the middle surface of the shell and hence idealizing the problem to that of plane stress, one disregards the effects of these stresses on the shortening and lengthening of the normal fibers. Again, thin shell theory might not be expected to yield accurate results in regions of large normal loads. Fortunately, as in the case of beams, the majority of problems of interest are such that the
the effects of the above inaccuracies are negligible. In fact, various investigators\textsuperscript{1} studied the effects of the Kirchoff hypotheses on the accuracy of shell solutions and found that in most situations, the errors are of order $\delta k$, where $\delta$ is the thickness of the shell and $k$ is a representative middle surface curvature. Their study also establishes a unique limit to the refinement that is possible in developing shell theory based on the Kirchoff hypotheses. This fact becomes extremely important in simplifying the shell equations in that terms whose effects are of order $(\delta k)$ or smaller in comparison with unity may be discarded arbitrarily.

As is true of most of the problems formulated in either elasticity or strength of materials, linear differential equations are desired as the shell equations. Since the present work is concerned with small displacement shell theory, it will be assumed that squares, cubes, and products of displacements and their derivatives may be discarded. Thus, linear differential equations in the displacements will result.

\textsuperscript{1}See Love (1), Novozhilov and Finkelstein (2), Donnel (3).
2. **Surface or Curvilinear Coordinates**

Any given surface, \( F(x, y, z) = c \), may be adequately described by means of two parameters, \( \alpha \) and \( \beta \), such that

\[
\begin{align*}
x &= x(\alpha, \beta) \\
y &= y(\alpha, \beta) \\
z &= z(\alpha, \beta)
\end{align*}
\]

These parameters form a set of coordinate lines on the surface and are called the curvilinear coordinates of that surface. It is obvious that an infinite set of such coordinates may be drawn.

Not all such possible sets of coordinate lines are convenient to use and it is found that the most convenient of all the possible sets are coordinate lines coinciding with the lines of principal curvature of the surface. Relative to such a set, simplifications in mathematical expressions result which would not be possible if some other set of coordinates were used. Further, since lines of principal curvature are orthogonal, the curvilinear coordinates coinciding with these lines will also be orthogonal to each other. To differentiate that set of coordinates coinciding with lines of principal curvature from any other set that may be drawn on the surface, the term principal curvilinear coordinates, or principal coordinates will be used.
A line of length $ds$ lying on the surface may be expressed in terms of the principal coordinates as

$$ds^2 = A^2 \, da^2 + B^2 \, d\beta^2$$

where the coefficients $A$ and $B$ are the Lamé parameters. If in particular, the length of line coincides with a $\beta = \text{constant}$ coordinate line, the length becomes

$$ds = A \, da$$

while if it coincides with an $a = \text{constant}$ coordinate line, then

$$ds = B \, d\beta$$.

The above two equations may be considered the definitions of the parameters $A$ and $B$ in that $ds$ is usually easily expressible in terms of $da$ and $d\beta$.

Letting $k_1$ and $k_2$ represent the respective curvatures of the surface along the principal coordinate lines $\beta = \text{constant}$ and $a = \text{constant}$, the relation between the Lamé parameters and the curvatures must satisfy the following two conditions:

1) Condition of Codazzi

$$\frac{\partial}{\partial a} (k_2 B) = k_1 \frac{\partial B}{\partial a}$$

$$\frac{\partial}{\partial \beta} (k_1 A) = k_2 \frac{\partial A}{\partial \beta}$$

(2.1)
ii) Condition of Gauss

\[ \frac{\partial}{\partial a} \left( \frac{1}{A} \frac{\partial B}{\partial a} \right) + \frac{\partial}{\partial \beta} \left( \frac{1}{B} \frac{\partial A}{\partial \beta} \right) = -ABk_1k_2 \]  

(2.2)

This concludes the section on surfaces. It should not in any way be construed that the theory of surfaces may be presented in such a brief form, but rather the above information is normally sufficient for the development of shell theory. For further information on surfaces, references are included in the Bibliography.

3. Shell Surfaces

The thickness of a thin shell can be considered as being made up of a series of equidistant surfaces. On any one of the surfaces, a set of principal coordinates lines can be drawn and these coordinates lines will be equidistant to a set that would be drawn on any other surface making up the shell. Let there be such a set of coordinates on the middle surface of the shell and let these be designated as \( \alpha, \beta \) coordinates. Further let \( \gamma \) be a coordinate measuring distance perpendicular to the middle surface. The positive direction of \( \gamma \) may be determined at any point by drawing an orthonormal triad of vectors \( \hat{e}_\alpha, \hat{e}_\beta, \hat{e}_\gamma \) such that \( \hat{e}_\alpha \) and \( \hat{e}_\beta \) lie in the positive increasing directions of \( \alpha \) and \( \beta \). The direction of \( \hat{e}_n \) and hence that of positive \( \gamma \) will
be determined from the vector equation $\mathbf{e}_\gamma = \mathbf{e}_a \times \mathbf{e}_\beta$. These unit vectors are shown in Figure 1. Normally, $\alpha$, $\beta$, and $\gamma$ are so chosen that positive $\gamma$ is directed from the center of curvature associated with $k_1$ to the surface itself.

Consider now some equidistant surface lying a distance $\gamma$ from the middle surface. Since the surfaces are equidistant from each other, the principal coordinates of the middle surface are applicable to this surface, the two sets of coordinates being equidistant from each other. An element of length $ds$ on this equidistant surface can then be written as

$$ds^2 = \frac{da^2}{h_1^2} + \frac{d\beta^2}{h_2^2}.$$

For increments of length parallel to the coordinates $\alpha$ and $\beta$,

$$ds = \frac{da}{h_1},$$

$$ds = \frac{d\beta}{h_2}.$$

The reciprocal of the quantities $h_1$ and $h_2$ are the Lame' parameters of the parallel surface and the last two equations their definition. These quantities can be expressed in terms of Lame' parameters, $A$ and $B$, of the middle surface and the distance between the two surfaces by expressing the lengths $ds$
Figure 1. Orthonormal triad of vectors on the shell middle surface.
in the equations defining $h_1$ and $h_2$ in terms of middle surface functions. Thus

$$h_1 = \frac{1}{A(1 + k_1 \gamma)} \quad (3.1)$$

$$h_2 = \frac{1}{B(1 + k_2 \gamma)}$$

where $k_1$ and $k_2$ are the curvatures of the middle surface along the $\alpha$ and $\beta$ coordinate lines.

If a coordinate $\gamma'$, normal to an equidistant surface were constructed at some point of that surface, then it would coincide with the coordinate $\gamma$ constructed at the same corresponding point of the middle surface, and hence

$$d\gamma' = d\gamma.$$  

The equation of Gauss can be written for an equidistant surface as follows:

$$\frac{\partial}{\partial \alpha} \left( h_2 \frac{\partial h_2^{-1}}{\partial \alpha} \right) + \frac{\partial}{\partial \beta} \left( h_2 \frac{\partial h_2^{-1}}{\partial \beta} \right) = -k_1 k_2 AB \quad (3.2)$$

where $k_1$ and $k_2$ are curvatures of the middle surface. Other relations which are useful for differentiation off the middle surface are

$$h_1 \frac{\partial h_1^{-1}}{\partial \alpha} = \frac{1}{A} \frac{\partial B}{\partial \alpha} \quad (3.3)$$

$$h_2 \frac{\partial h_2^{-1}}{\partial \beta} = \frac{1}{B} \frac{\partial A}{\partial \beta}$$
This last set of equations may easily be verified by sub-
stitutions for $h_1$ and $h_2$ and the use of the condition of Codazzi
and the equation of Gauss.

4. Equations of Strain and Displacement

Every single point in the shell can be considered lying
on some surface, parallel to, and displaced a distance $\gamma$ from
the middle surface. The strains of any point then can be ex-
pressed in terms of the parameters of the surface in which
it lies.

Let $u_a$, $u_\beta$, and $u_\gamma$ be the displacements of a point par-
allel to the three coordinates of the parallel surface in which
the point lies. From the general theory of elasticity the ex-
pressions for the three normal strains $e_a$, $e_\beta$, $e_\gamma$ and the
shearing strains $e_{a\beta}$, $e_{\beta\gamma}$, $e_{\gamma a}$ may be written as

$$
e_a = h_1 \frac{\partial u_a}{\partial a} + h_1 h_2 \frac{\partial h_1}{\partial \beta} u_\beta + h_1 \frac{\partial h_1}{\partial \gamma} u_\gamma$$

$$e_\beta = h_2 \frac{\partial u_\beta}{\partial \beta} + h_1 h_2 \frac{\partial h_2}{\partial \beta} u_a + h_2 \frac{\partial h_2}{\partial \gamma} u_\gamma$$

See Love (1) or Novozhilov (4) for derivation of these equations.
The Kirchoff hypothesis enables the displacements \( u_a \) and \( u_\beta \) to be expressed in terms of the corresponding displacements \( u, v, \) and \( w \) of the middle surface \(^3\). Thus

\[
\begin{align*}
\frac{u_a}{h_a} &= (1 + k_1 \gamma) u - \gamma \frac{\partial w}{\partial a} \\
\frac{u_\beta}{h_\beta} &= (1 + k_2 \gamma) v - \gamma \frac{\partial w}{\partial \beta}
\end{align*}
\]

It should be noted that \( u_\gamma \) is not expressible in terms of \( u, v, \) and \( w \) from the Kirchoff hypothesis and, in fact, will be found during the course of derivation of the shell equations.

5. **Shell Equations**

Utilizing a method first proposed by Vlasov (5) but incorrectly carried out by him, the shell equations in the present work will be exhibited in terms of the displacements \( u, v, \) and \( w \) of the middle surface.

\(^3\)See Love (1) or Novozhilov (4) for the development of these expressions.
Consider an element of the shell located a distance $\gamma$ above and parallel to the middle surface of the shell. On the middle surface of the shell let there be a set of principal curvilinear coordinates $\alpha$ and $\beta$ and a normal coordinate $\gamma$. On the parallel surface then, there can be drawn a set of principal curvilinear coordinates parallel to those on the middle surface and having the same scale value. Furthermore, a normal coordinate $\gamma$ to the parallel surface will also be a normal coordinate to the middle surface of the shell such that $d\gamma = d\gamma'$. The sides of the element will be normal to and assumed to coincide with the $\alpha$ and $\beta$ coordinate lines drawn on the parallel surface. Let the thickness of the element be $d\gamma$. The element is shown in Figure 2.

In order to insure the equilibrium of the element, force summations in three perpendicular directions must each yield zero value. The three directions chosen for the element will be in the directions of the tangent lines to the $\alpha$, $\beta$, $\gamma$ coordinate lines of parallel surface. These tangent directions are indicated by the unit vectors $\hat{e}_\alpha$, $\hat{e}_\beta$, and $\hat{e}_\gamma$ whose directions coincide with the directions of the increasing coordinates $\beta$ and $\gamma$. 
Figure 2. Element located at a distance $\gamma$ above the middle surface.
In establishing the equilibrium equations of the element, consideration should be given not only to the dimensional changes of the element but also to the fact that because of the curvature of the element, forces acting on the faces of the element may have components in directions other than the ones that they are nominally acting.

In order to maintain a simplicity and clarity of representation, two figures are included. Figure 3 represents the stresses on those faces of the element which can be seen, the stresses on the back faces being opposite but differing by a differential amount. The quantities $p_a$, $p_\beta$, and $p_\gamma$ are the body forces acting on the element, i.e., forces per unit volume. Figure 4 portrays the dimensions of the element.

Summing forces in the $e_a$, $e_\beta$, and $e_\gamma$ directions, the following three equations of equilibrium in terms of the stresses result:

$$\frac{\partial}{\partial a} \left( \frac{\sigma_a}{h_2} \right) - \frac{\partial h^{-1}}{\partial a} \sigma_\beta + h_1 \frac{\partial}{\partial \beta} \left( -\frac{\sigma_\beta}{h_2} \right) + h_1 \frac{\partial}{\partial \gamma} \left( \frac{\sigma_\gamma}{h_2 h_1 h_2} \right) + \frac{p_a}{h_1 h_2} = 0 \quad (5.1)$$

---

The actual derivation is not included in that it parallels that of the stress resultants. See Love (1), p. 534, or Novozhilov (4) p. 33.
Figure 3. General stress condition on an element.
Figure 4. Dimensions of the element.
The stress strain relations for the case of plane stress, $\sigma_y = 0$, can be written as

$$\sigma_a = \frac{E}{(1 - \nu^2)} (e_a + \nu e_\beta)$$

$$\sigma_\beta = \frac{E}{(1 - \nu^2)} e_\beta + \nu e_a$$

$$\sigma_{a\beta} = \frac{E}{2(1 + \nu)} e_{a\beta}$$

and

$$e_\gamma = \frac{-\nu}{(1 - \nu)} (e_a + e_\beta)$$

where $E$ is Young's modulus and $\nu$ is Poisson's ratio. The stress strain relations may yet be presented in another way. Thus,

$$\sigma_a = (\tau + 2\mu)\Delta - 2\mu (e_\beta + e_\gamma)$$

$$\sigma_\beta = (\tau + 2\mu)\Delta - 2\mu (e_a + e_\gamma)$$

$$\sigma_{a\beta} = \mu e_{a\beta}$$
where \( \tau \) and \( \mu \) are the coefficients of elasticity of Lame and are related to \( E \) and \( \nu \) by the relation

\[
\tau = \frac{E \nu}{(1 + \nu)(1 - 2\nu)}
\]

\[
\mu = \frac{E}{2(1 + \nu)}
\]

(5.6)

The quantity \( \Delta \) is the volume dilatation or change in volume per unit volume and is related to the strains by the relation

\[
\Delta = e_a^\alpha + e_\beta^\beta + e_\gamma^\gamma
\]

(5.7)

In the present work it will be more convenient to use the stress strain relations in the form of (5.5).

Substituting the stress strain relations into the equilibrium equations, expressing the strains in terms of the displacement equations (4.1), rearranging the ensuing equations and using the relations (3.1), (3.2), and (3.3), the following equations result:

\[
(\tau + 2\mu) \frac{1}{h_2^2} \frac{\partial \Delta}{\partial \alpha} - 2\mu \frac{1}{h_1^2} \frac{\partial \chi}{\partial \beta} + 2\mu ABK u_a - 2\mu \frac{1}{h_2^2} \frac{\partial u}{\partial \alpha} + h_1 \frac{\partial}{\partial \gamma} \left( \frac{\sigma_{\gamma \gamma}}{h_2^2 h_2} \right) + \frac{p_a}{h_1 h_2} = 0
\]

(5.8)
The symbols in the last three equations are the same as previously defined with the following additions.

\[ H = \frac{k_1 + k_2}{2} \]

\[ K = k_1 k_2 \]  

\[ 2 \chi = h_1 h_2 \left[ \frac{\partial}{\partial \alpha} \left( \frac{u_2}{h_2} \right) \frac{\partial}{\partial \beta} \left( \frac{u_1}{h_1} \right) \right] \]

The quantity \( \chi \) is called the normal rotation and represents the rotation of the element in a tangent plane to the parallel surface. To aid in visualizing this parameter, Figure 5 represents a rectangular lamina whose adjacent sides are bound by the unit vectors \( \vec{e}_\alpha \) and \( \vec{e}_\beta \). The line \( a - a \) represents the diagonal of
Figure 5. Rotation of a lamina on an equidistant surface.
this lamina. After deformation of the shell the adjacent sides of the lamina are no longer orthogonal but are still bound by the unit vectors \( \hat{e}_\alpha \) and \( \hat{e}_\beta \), the vectors also assuming some new orientation with respect to each other. The angles \( \theta_1 \), \( \theta_2 \), and \( \theta_3 \) represent the orientations of the deformed vector positions to the respective undeformed positions. The line \( a' - a' \) still represents the diagonal of the lamina which now has been deformed to a parallelogram. The angle \( \theta_3 \) is quite small and thus the plane of the vectors \( \hat{e}_\alpha \) and \( \hat{e}_\beta \) may still be considered the plane of the vectors \( \hat{e}_\alpha \) and \( \hat{e}_\beta \). From geometry then

\[ 2 \chi = \theta_1 - \theta_2 \]

Substitution for the angles \( \theta_1 \) and \( \theta_2 \) leads to equation (5.8).

It might be pointed out that the shearing strain \( \varepsilon_{\alpha \beta} \) is the sum of the angles \( \theta_1 \) and \( \theta_2 \). Hence the rotation, though it in itself is not a shearing strain, barring the case of rigid body motions is of the same or smaller order of magnitude as the shearing strain.

---

5 The determinations of these angles and, in fact, the quantitative expressions for the rotations of the unit vectors are completely worked out by Novozhilov (4), p. 10.
Equations (5.8), (5.9), and (5.10) represent the equilibrium equations in terms of the displacements for an element located within the shell. If each of the equations were multiplied by the term \((\text{d}a\text{d}\beta\text{d}y)\), then each of the equations would represent respectively, the resultant force on the element in the \(\bar{e}_\alpha\), \(\bar{e}_\beta\), and \(\bar{e}_\gamma\) directions. Integrating each of the equations with respect to \(\gamma\) over the thickness of the shell \(\delta\) would then represent resultant forces in the direction of the tangents to the \(\alpha\), \(\beta\), and \(\gamma\) coordinate directions of the middle surface of an element of finite height \(\delta\). The length and width dimensions of the element could be characterized by its dimensions on the middle surface, namely \(A\alpha\) and \(B\beta\). Further, equations (5.8) and (5.9) could be multiplied by \((\gamma\text{d}\gamma\text{d}a\text{d}\beta)\) the effect being that of taking moments about the tangents to \(\alpha\) and \(\beta\) coordinates of the middle surface of the resultant forces in the \(\bar{e}_\beta\) and \(\bar{e}_\alpha\) directions of the element respectively. (Note that moment contributions from the resultant force on the \(\bar{e}_\gamma\) direction are neglected in that their effects, within the accuracy limitations of \((\delta k)\), lead to no new information about the shell.) Integrating these two moment conditions over the thickness of the shell would then result in the resultant moment condition on the same element as described on the force integration.
The displacements $u_\alpha$ and $u_\beta$ occurring in the equilibrium equations are expressible on terms of the displacements $u$, $v$, and $w$ of the middle surface. Thus, if integrations of the five equilibrium equations are carried out, then neglecting for the moment consideration of the displacement $u_\gamma$, five equations would result involving the five unknowns $u$, $v$, $w$, $\tau_{\alpha\gamma}$, and $\tau_{\beta\gamma}$. Following a fairly standard procedure, the shearing stresses $\tau_{\alpha\gamma}$ and $\tau_{\beta\gamma}$ could be eliminated, thus yielding three equations in terms of the three displacements of the middle surface.

With the idea of integration of the three equilibrium equations in mind, close inspection of the three equations (5.8), (5.9), and (5.10) reveals partly why the equations were put in that form. Save for the terms $\Delta$, $\chi$, and $u_\gamma$, each of the terms in the three equations can be expressed as some finite polynomial in the variable $\gamma$. Thus integration of those terms with respect to $\gamma$ can proceed without any particular difficulty.

Consider now the dilatation $\Delta$ and the rotation $\chi$. These quantities if expressed in terms of their definitions (5.7) and (5.11) would lead to difficulties in the integrations with respect to $\gamma$. However, these terms might be expanded in power series of the form

$$\Delta = \Delta_0 + \sum_{i=1}^{\infty} \Delta_i \gamma^i.$$
\[ \chi = \chi_o + \sum_{i=1}^{\infty} \chi_i \gamma^i \]

where it should be noted that \( \Delta_i \) and \( \chi_i \) are functions of \( \alpha \) and \( \beta \).

The dilatation represents the sum of the three strains \( e_{\alpha} \), \( e_{\beta} \), and \( e_{\gamma} \), and the rotation has involved in its expression the same terms as the shearing strain \( e_{\alpha\beta} \). Hence, as might be expected, the order of accuracy of \( \Delta \) and \( \chi \) will be the same as that of the strains. However, the strains are defined in terms of the displacements \( u_{\alpha}, u_{\beta}, \) and \( u_{\gamma} \), which in turn can be expressed in terms of the displacements of the middle surface by the Kirchoff hypothesis whose inherent errors are of the order \( k\delta \) as previously discussed. The net effect is that the strains, if the Kirchoff hypothesis is to be used in conjunction with them, may be considered linear functions of the variable \( \gamma \) without incurring any greater error than that inherently present. On this basis, the series for the dilatation and the rotation may be truncated so that

\[ \Delta = \Delta_o + \Delta_1 \gamma \quad (5.12) \]

\[ \chi = \chi_o + \chi_1 \gamma \]
In order to evaluate the dilatation \( \Delta \), the strain \( e_y \) must be known. For the assumed case of plane stress, this strain is given as

\[
e_y = \frac{-\nu}{(1-\nu)} (e_a + e_\beta)
\]

By adding and subtracting \( e_y \) it may also be written as

\[
e_y = \frac{-\nu}{(1-2\nu)} \Delta
\]

or, in terms of Lamé parameters of elasticity

\[
e_y = \frac{-\tau}{2\mu} \Delta
\]  

(5.13)

Thus the strain \( e_y \) is directly proportional to the dilatation. Expressing the dilatation by (5.12), the strain may be written as a function of \( y \), that is

\[
e_y = \frac{-\tau}{2\mu} (\Delta_0 + \Delta_1 y)
\]  

(5.14)

Utilizing the definition of \( e_y \) in (4.1), the displacement \( u_y \) may be found by integration. Thus,

\[
u_y = \frac{-\tau}{2\mu} (\Delta_0 + \frac{\Delta_1 y^2}{2}) + w
\]  

(5.15)

where the symbols have the same meaning as before.
Consider now evaluating the coefficients of the truncated series for the rotation \( \chi \) and the dilatation \( \Delta \). Substituting the expressions for \( h_2, h_1, u_\alpha \), and \( u_\beta \) into the equation (5.11) for \( 2\chi \).

\[
\frac{1}{AB (1 + k_1 \gamma)(1 + k_2 \gamma)} \left\{ \frac{\partial}{\partial a} \left[ B (1 + k_2 \gamma)^2 \nu - \gamma (1 + k_2 \gamma) \frac{\partial w}{\partial \beta} \right] - \frac{\partial}{\partial \beta} \left[ A (1 + k_1 \gamma)^2 u - \gamma (1 + k_1 \gamma) \frac{\partial w}{\partial a} \right] = 2 \chi \right. 
\]

The denominator \((1 + k_1 \gamma)(1 + k_2 \gamma)\) may be expanded in series form. Thus,

\[
\frac{1}{(1 + k_1 \gamma)(1 + k_2 \gamma)} = 1 - (k_1 + k_2) \gamma
\]

where the series has been truncated at the first power of \( \gamma \). Expanding the expression and retaining terms up to and including the first power of \( \gamma \)

\[
2 \chi = \frac{1}{AB} \left[ \frac{\partial}{\partial a} (B \nu) - \frac{\partial}{\partial \beta} (A u) \right] - (k_1 - k_2) \left[ \frac{A}{B} \frac{\partial}{\partial \beta} \left( \frac{u}{A} \right) + \frac{B}{A} \frac{\partial}{\partial a} \left( \frac{\nu}{B} \right) \right] \gamma
\]

Thus equating coefficients of the truncated series for \( \chi \)

\[
\chi_0 = \frac{1}{2AB} \left[ \frac{\partial}{\partial a} (B \nu) - \frac{\partial}{\partial \beta} (A u) \right]
\]

\[
\chi_1 = -\frac{(k_1 - k_2)}{2} \left[ \frac{A}{B} \frac{\partial}{\partial \beta} \left( \frac{u}{A} \right) + \frac{B}{A} \frac{\partial}{\partial a} \left( \frac{\nu}{B} \right) \right]
\]

(5.16)
The dilatation $\Delta$ is defined as

$$\Delta = \left( e_\alpha + e_\beta + e_\gamma \right)$$

Substituting in the expressions for the strains, it may similarly be put in the form

$$\Delta = h_1 h_2 \left[ \frac{\partial}{\partial \alpha} \left( \frac{u_a}{h_2} \right) + \frac{\partial}{\partial \beta} \left( \frac{u_\beta}{h_1} \right) + \frac{\partial}{\partial \gamma} \left( \frac{u_\gamma}{h_1 h_2} \right) \right]$$  \hspace{1cm} (5.17)

Substituting for $u_a$, $u_\beta$, and $u_\gamma$ with equations (4.2) and (5.12), expanding and retaining terms up to and including the first power of $\gamma$,

$$\Delta = h_1 h_2 \left\{ \left[ \frac{\partial}{\partial \alpha} (Bu) + \frac{\partial}{\partial \beta} (Av) + 2HABw - \frac{\tau}{2\mu} AB\Delta_o \right] \right\}$$

$$+ \left[ 2 \frac{\partial}{\partial \alpha} (HBu) + 2 \frac{\partial}{\partial \beta} (HAv) - \frac{\partial}{\partial \alpha} \left( \frac{B}{A} \frac{\partial w}{\partial \alpha} \right) - \frac{\partial}{\partial \beta} \left( \frac{A}{B} \frac{\partial w}{\partial \beta} \right) \right.$$  

$$- \frac{\tau}{\mu} AB \left( 2H\Delta_o + \Delta \right) + 2 ABKw) \right\} \times 1$$

Expanding for the coefficient $h_1 h_2$ as before, this term can also be put in the form

$$h_1 h_2 = \frac{1}{AB} \left( 1 - 2 H\gamma \right)$$
where higher power terms in $\gamma$ have been neglected. Substituting the expression for $h_1 h_2$, the final expansion for the dilatation becomes

$$
\Delta = \left\{ \frac{1}{AB} \left[ \frac{\partial}{\partial a} (Bu) + \frac{\partial}{\partial \beta} (Av) + 2 HABw \right] - \frac{\tau}{2\mu} \Delta_0 \right\}
$$

$$
\left( \frac{2}{AB} \left[ \frac{\partial}{\partial a} (HBu) + \frac{\partial}{\partial \beta} (HAv) \right] - \frac{1}{AB} \left[ \frac{\partial}{\partial a} \left( \frac{B}{A} \frac{\partial w}{\partial a} \right) + \frac{\partial}{\partial \beta} \left( \frac{A}{B} \frac{\partial w}{\partial \beta} \right) \right]
\right)

- \frac{2H}{AB} \left[ \frac{\partial}{\partial a} (Bu) + \frac{\partial}{\partial \beta} (Av) \right] + 2 (K - 2H^2)w - \frac{\tau}{\mu} H \Delta_0 - \frac{\tau}{2\mu} \Delta_1 \right) \gamma
$$

Thus assuming $\Delta$ to be given by the truncated series (5.12), and equating coefficients of $\gamma$, two equations result which must be solved simultaneously for $\Delta_0$ and $\Delta_1$. The results of the solution are

$$
\Delta_0 = \frac{2\mu}{(\tau + 2\mu)AB} \left[ \frac{\partial}{\partial a} (Bu) + \frac{\partial}{\partial \beta} (Av) \right] + \frac{4\mu}{(\tau + 2\mu)} Hw \tag{5.18}
$$

$$
\Delta_1 = \frac{4\mu}{(\tau + 2\mu)AB} \left[ \frac{\partial}{\partial a} (HBu) + \frac{\partial}{\partial \beta} (HAv) \right] - \frac{2\mu}{(\tau + 2\mu)AB}
$$

$$
\left[ \frac{\partial}{\partial a} \left( \frac{B}{A} \frac{\partial w}{\partial a} \right) + \frac{\partial}{\partial \beta} \left( \frac{A}{B} \frac{\partial w}{\partial \beta} \right) \right] + \frac{8\mu^2}{(\tau + 2\mu)^2} \frac{H}{AB} \left[ \frac{\partial}{\partial a} (Bu) + \frac{\partial}{\partial \beta} (Av) \right]
$$

$$
+ \frac{4\mu}{(\tau + 2\mu)} \left[ K - \frac{4(\tau + \mu)}{(\tau + 2\mu)} H^2 \right] w
$$
In the intermediate steps of the derivation of the shell equation it will be convenient not to express $\Delta_o$, $\Delta_1$, $\chi_o$, $\chi_1$ and $u, v$ by their respective values until the final simplifications are made.

Consider now the derivation of the shell equations in terms of the displacement $u$, $v$, and $w$. Following the procedure previously outlined with regard to the integration of the displacement equations, the following five equations result from the equations (5.8), (5.9), and (5.10).

\[
(\tau + 2\mu)B \left( \frac{\partial \Delta_o}{\partial a} + \frac{k_2 \delta^2}{12} \frac{\partial \Delta_1}{\partial a} \right) - 2\mu A \left( \frac{\partial \chi_o}{\partial \beta} + \frac{k_1 \delta^2}{12} \frac{\partial \chi_1}{\partial \beta} \right) + 2\mu ABKu
\]

\[
- \frac{2\mu}{\delta} \int_{-\delta/2}^{\delta/2} \frac{\partial}{\partial \gamma} \left( \frac{1}{h_2} \frac{\partial u}{\partial a} \right) d\gamma + \frac{ABk_1}{\delta} Q_1 + \frac{ABK}{\delta} = 0
\]

\[
(\tau + 2\mu)A \left( \frac{\partial \Delta_o}{\partial \beta} + \frac{k_1 \delta^2}{12} \frac{\partial \Delta_1}{\partial \beta} \right) + 2B \left( \frac{\partial \chi_o}{\partial a} + \frac{k_2 \delta^2}{12} \frac{\partial \chi_1}{\partial a} \right) + 2\mu ABKv
\]

\[
- \frac{2\mu}{\delta} \int_{-\delta/2}^{\delta/2} \frac{\partial}{\partial \gamma} \left( \frac{1}{h_1} \frac{\partial u}{\partial \beta} \right) d\gamma + \frac{ABk_2}{\delta} Q_2 + \frac{ABY}{\delta} = 0
\]

\[
-2(\tau + 2\mu)B \left( H \Delta_o + \frac{K A_1 \delta^2}{12} \right) + 2\mu \left( \frac{\partial}{\partial a} (Bk_2 u) + \frac{\partial}{\partial \beta} (Ak_1 v) \right) + \frac{4\mu ABK}{\delta}
\]

\[
\int_{-\delta/2}^{\delta/2} u \gamma d\gamma + \frac{4\mu AB}{\delta} \int_{-\delta/2}^{\delta/2} \left( H + K \gamma \right) \frac{\partial u}{\partial \gamma} d\gamma + \frac{\partial}{\partial a} \left( \frac{BO_1}{\delta} \right) + \frac{\partial}{\partial \beta} \left( \frac{AO_2}{\delta} \right) + \frac{AB}{\delta} Z = 0
\]

(5.19)
\[(\tau + 2\mu) \frac{B \delta^3}{12} \left( \frac{\partial \Delta}{\partial a} + k_2 \frac{\partial \Delta}{\partial a} \right) - 2\mu \frac{A \delta^3}{12} \left( \frac{\partial \chi_1}{\partial \beta} + k_1 \frac{\partial \chi_1}{\partial \beta} \right) + 2\mu \frac{ABK \delta^3}{12} \]

\[\left( k_1 u - \frac{1}{A} \frac{\partial w}{\partial a} \right) - 2\mu \int_{-\delta/2}^{\delta/2} \gamma \frac{\partial}{\partial \gamma} \left( \frac{1}{h_2} \frac{\partial u}{\partial \gamma} \right) d\gamma - ABQ_1 = 0\]

\[(\tau + 2\mu) \frac{A \delta^3}{12} \left( \frac{\partial \Delta}{\partial \beta} + k_1 \frac{\partial \Delta}{\partial \beta} \right) + 2\mu \frac{B \delta^3}{12} \left( \frac{\partial \chi_1}{\partial a} + k_2 \frac{\partial \chi_1}{\partial a} \right) + 2\mu \frac{ABK \delta^3}{12} \]

\[\left( k_2 v - \frac{1}{B} \frac{\partial w}{\partial \beta} \right) - 2\mu \int_{-\delta/2}^{\delta/2} \gamma \frac{\partial}{\partial \gamma} \left( \frac{1}{h_1} \frac{\partial u}{\partial \gamma} \right) d\gamma - ABQ_2 = 0\]

(5.19 contd)

where

\[Q_1 = \frac{1}{B} \int_{-\delta/2}^{\delta/2} \frac{\sigma a \gamma}{h_2} d\gamma\]

\[Q_2 = \frac{1}{A} \int_{-\delta/2}^{\delta/2} \frac{\gamma \beta \gamma}{h_1} d\gamma\]

\[X = \frac{1}{AB} \left[ \frac{\sigma a \gamma}{h_1 h_2} \right]_{-\delta/2}^{\delta/2} + \int_{-\delta/2}^{\delta/2} \frac{\rho a}{h_1 h_2} d\gamma \]
The quantities $Q_1$ and $Q_2$ are the stress resultants of the shearing stresses, $\sigma_{a\gamma}$ and $\sigma_{b\gamma}$, while the quantities $X$, $Y$, $Z$ represent the external loading on the shell per unit of area of the middle surface. These external loadings are composed of the surface loadings and the body loadings on the shell and are assumed to be known quantities.

The five equations of (5.19) are in terms of the three displacements of the middle surface, $u$, $v$, and $w$ and the two stress resultants $Q_1$ and $Q_2$. Elimination of the unknowns $Q_1$ and $Q_2$ will then yield three equations in terms of the desired three unknowns.

Solving the last two equations of (5.19) for $Q_1$ and $Q_2$ the following expressions result.
These expressions for \( Q_1 \) and \( Q_2 \) may now be substituted into the first three of the equations (5.19) and the result will be the desired three equations.

Because of certain simplifications that will be made, it will be more convenient to handle each of the substitutions separately. Further, the symmetry of the subscript in the first two equations and in the expressions for \( N_1 \) and \( N_2 \) is such that only one of these equations need be operated on, the second following by a mere interchange of subscripts.
Substituting for \( Q_1 \) into the first of equations of \((5.19)\) and regrouping terms

\[
(\tau + 2\mu) B \frac{\partial \Delta}{\partial a} + 2\mu A \frac{\partial \chi_0}{\partial \beta} + 2\mu ABKu + \frac{ABX}{\delta} + (\tau + 2\mu) \frac{Bk_1 \delta^2}{12}
\]

\[
\left( \frac{\partial \Delta}{\partial a} + k \frac{\partial \Delta}{\partial \alpha} \right) - 2\mu \frac{A\kappa \delta^2}{12} \left( \frac{\partial \chi_1}{\partial \beta} + k \frac{\partial \chi_0}{\partial \beta} \right) + 2\mu \frac{ABKk_1 \delta^2}{12}
\]

\[
\left( k_1 u - \frac{1}{A} \frac{\partial w}{\partial a} \right) - \frac{2\mu}{\delta} \int_{-\delta/2}^{\delta/2} \frac{\partial}{\partial y} \left( \frac{1}{h^2} \frac{\partial u}{\partial a} \right) dy - \frac{2\mu}{\delta} k_1
\]

\[
\int_{-\delta/2}^{\delta/2} \frac{\partial}{\partial y} \left( \frac{1}{h^2} \frac{\partial u}{\partial a} \right) dy = 0
\]

Evaluating the integrals

\[
\int_{-\delta/2}^{\delta/2} \frac{\partial}{\partial y} \left( \frac{1}{h^2} \frac{\partial u}{\partial a} \right) dy = \frac{\tau B}{2\mu} \frac{\partial \Delta}{\partial a} + Bk_2 \delta \frac{\partial w}{\partial a} - \frac{Bk_2 \delta^3}{16\mu} \frac{\partial \Delta_1}{\partial a}
\]

\[
\int_{-\delta/2}^{\delta/2} \frac{\partial}{\partial y} \left( \frac{1}{h^2} \frac{\partial u}{\partial a} \right) dy = - \frac{\tau B \delta^3}{24\mu} \frac{\partial \Delta_1}{\partial a} - \frac{\tau Bk_2 \delta^3}{12\mu} \frac{\partial \Delta_0}{\partial a}
\]
Substituting the value of the integrals into the above equation and combining terms:

\[ 2(\gamma + \mu)B \left[ 1 + \frac{(\tau + 2\mu)}{24(\tau + \mu)} K\delta^2 \right] \frac{\partial \Delta}{\partial \alpha} - 2\mu A \left( 1 + \frac{k_1^2 \delta^2}{12} \right) \frac{\partial \chi_0}{\partial \beta} \]

\[ + 2\mu ABK \left( 1 + \frac{k_1^2 \delta^2}{12} \right) \frac{\partial w}{\partial \alpha} + \frac{ABX}{\delta} \]

\[ - \mu \frac{Ak_1 \delta^2}{6} \frac{\partial \chi_1}{\partial \beta} + \left[ \frac{(\tau + \mu)}{6} k_1 + \frac{\tau}{8} k_2 \right] \delta^2 \frac{\partial \Delta_1}{\partial \alpha} = 0 \]

The Kirchhoff hypothesis introduces an error of the order \( \delta k \) into the shell equations and hence it would be superfluous to attempt a greater accuracy of expression of any term of the shell equations.

For this reason, the following approximation will be made:

\[ (1 + \delta^2 k^2) \approx 1 \]

where \( k \) represents one or the other of the curvatures and \( k^2 \) may represent the product of the curvatures \( k_1 k_2 \).

On the basis of the above approximation, the first equilibrium expression may be simplified to:

\[ 2(\gamma + \mu)B \frac{\partial \Delta}{\partial \alpha} - 2\mu A \frac{\partial \chi_0}{\partial \beta} + 2\mu ABKu - 2\mu BK_2 \frac{\partial w}{\partial \alpha} + \frac{ABX}{\delta} \]

\[ - \mu \frac{Ak_1 \delta^2}{6} \frac{\partial \chi_1}{\partial \beta} + \left[ \frac{(\tau + \mu)}{6} k_1 + \frac{\tau}{8} k_2 \right] \delta^2 \frac{\partial \Delta_1}{\partial \alpha} = 0 \]
Examination of the term $\lambda_0$ as expressed by (5.16) shows that it may also be put in the form

$$-\frac{B}{A} \frac{\partial}{\partial \alpha} \left( \frac{\nu}{B} \right) + \frac{A}{B} \frac{\partial}{\partial \beta} \left( \frac{\mu}{A} \right) + \frac{2}{A} \frac{\partial v}{\partial a} - \frac{2}{B} \frac{\partial u}{\partial \beta}$$

Save for the coefficient $(k_1 - k_2)/2$ and the signs, the first two terms of the expression for $\lambda_0$ are the same as that of the expression for $\lambda_1$ in (5.16). Hence the expression for $\lambda_1$ might be discarded by a term by term comparison with the expression for $\lambda_0$.

A more appealing reason for discarding the term $\lambda_1$ is based on the physical interpretation of this term. The two quantities $\lambda_0$ and $\lambda_1$ represent components of the truncated series expression for the rotation $\chi$. The value $\lambda_0$ represents the rotation on the middle surface and the value $\lambda_1$ represents a perturbation on this value due to movement away from the middle surface and hence would be expected to be smaller than $\lambda_0$. Thus multiplying $\lambda_1$ by a term of order $\delta^2 k^2$ would justify discarding $\lambda_1$. The same physical reasoning can be used as justification for discarding the term $\Delta_1$. 
Making the simplifications outlined above, the first, and hence by an interchange of subscripts the second equations of equilibrium assume the form

\[
2(\tau + \mu) B \frac{\partial \Delta}{\partial a} - 2\mu A \frac{\partial \chi}{\partial \beta} + 2\mu ABKu - 2\mu BK \frac{\partial w}{\partial a} + \frac{ABX}{\delta} = 0
\]

(5.22)

\[
2(\tau + \mu) A \frac{\partial \Delta}{\partial \beta} + 2\mu B \frac{\partial \chi}{\partial a} + 2\mu ABKv - 2\mu Ak \frac{\partial w}{\partial \beta} + \frac{ABY}{\delta} = 0
\]

Substituting for \( \Delta \) and \( \chi \), the final form of the first two equilibrium equations becomes

\[
\frac{4\mu(\tau + \mu)}{(\tau + 2\mu)} \frac{1}{A} \left[ \frac{\partial^2}{\partial a^2} (Bu) + \frac{\partial^2}{\partial a^2} (Av) \right] + \frac{8\mu(\tau + \mu)}{(\tau + 2\mu)} \frac{B}{A} \frac{\partial}{\partial a} (Hw)

- \frac{\mu}{B} \left[ \frac{\partial^2}{\partial a \partial \beta} (Bv) - \frac{\partial^2}{\partial \beta^2} (Au) \right] + 2\mu ABKu - 2\mu BK \frac{\partial w}{\partial a} + \frac{ABX}{\delta} = 0
\]

(5.23)

\[
\frac{4\mu(\tau + \mu)}{(\tau + 2\mu)} \frac{1}{B} \left[ \frac{\partial^2}{\partial \beta^2} (Av) + \frac{\partial^2}{\partial a \partial \beta} (Bu) \right] + \frac{8\mu(\tau + \mu)}{(\tau + 2\mu)} \frac{A}{B} \frac{\partial}{\partial \beta} (Hw)

- \frac{\mu}{A} \left[ \frac{\partial^2}{\partial a \partial \beta} (Au) - \frac{\partial^2}{\partial \beta^2} (Bv) \right] + 2\mu ABKv - 2\mu Ak \frac{\partial w}{\partial \beta} + \frac{ABY}{\delta} = 0
\]

(5.24)
Substituting the expressions for \( Q_1 \) and \( Q_2 \) into the third of the equations of the system (5.19), the equation takes the form

\[
-2(\tau+2\mu)AB \left( H\Delta_o + \frac{\partial}{\partial \alpha} \right) + 2\mu \left[ \frac{\partial}{\partial a} \left( \frac{\partial \Delta_1}{\partial a} \right) + \frac{\partial}{\partial \beta} \left( \frac{\partial \Delta_1}{\partial \beta} \right) \right] \\
+ \frac{ABZ}{\delta} + (\tau+2\mu) \left[ \frac{\partial}{\partial a} \left( \frac{B}{A} \frac{\partial \Delta_1}{\partial a} \right) + \frac{\partial}{\partial \beta} \left( \frac{A}{B} \frac{\partial \Delta_1}{\partial \beta} \right) \right] \\
+ (\tau+2\mu) \frac{\delta^2}{12} \left[ \frac{\partial}{\partial a} \left( \frac{B}{A} k_1 \frac{\partial \Delta_1}{\partial a} \right) + \frac{\partial}{\partial \beta} \left( \frac{A}{B} k_1 \frac{\partial \Delta_1}{\partial \beta} \right) \right] - \frac{\mu \delta^2}{6} \\
\left[ \frac{\partial}{\partial a} \left( k_1 \frac{\partial \chi_o}{\partial \beta} \right) + \frac{\partial}{\partial \beta} \left( k_2 \frac{\partial \chi_o}{\partial a} \right) \right] + \frac{\mu \delta^2}{6} \left[ \frac{\partial}{\partial a} \left( Bk_1 u_1 \right) - \frac{\partial}{\partial \beta} \left( Bk_2 v_2 \right) \right] \\
- \frac{\mu \delta^2}{6} \left[ \frac{\partial}{\partial a} \left( \frac{KB}{A} \frac{\partial w}{\partial a} \right) + \frac{\partial}{\partial \beta} \left( \frac{KA}{B} \frac{\partial w}{\partial \beta} \right) \right] + \frac{4\mu ABK}{\delta} \int_{-\delta/2}^{\delta/2} u_d\gamma \\
+ \frac{4\mu AB}{\delta} \int_{-\delta/2}^{\delta/2} \left( H + K \gamma \right) \frac{\partial u}{\partial \gamma} d\gamma - \frac{2\mu}{\delta} \frac{\partial}{\partial a} \left[ \frac{1}{A} \int_{-\delta/2}^{\delta/2} \gamma \frac{\partial}{\partial \gamma} \left( \frac{1}{h_2} \frac{\partial u}{\partial \beta} \right) d\gamma \right] \\
- \frac{2\mu}{\delta} \frac{\partial}{\partial \beta} \left[ \frac{1}{B} \int_{-\delta/2}^{\delta/2} \gamma \frac{\partial}{\partial \gamma} \left( \frac{1}{h_1} \frac{\partial u}{\partial \beta} \right) d\gamma \right] = 0
\]

Evaluating the integrals

\[
\int_{-\delta/2}^{\delta/2} u_d\gamma = \frac{\tau \delta^3 \Delta_1}{48\mu} + w_\delta \\
\int_{-\delta/2}^{\delta/2} \left( H + K \gamma \right) d\gamma = -\frac{\tau}{2\mu} \delta H\Delta_o - \frac{\tau}{24\mu} \delta^3 K\Delta_1
\]
The remaining two integrals have been evaluated when considering the first two equilibrium equations.

Combining terms, the third equilibrium equation becomes

\[
2\mu \left[ \frac{\partial}{\partial a} (Bk_2 u) + \frac{\partial}{\partial \beta} (Ak_1 v) \right] + \frac{\mu \delta^2}{6} \left[ \frac{\partial}{\partial a} (BKk_1 u) - \frac{\partial}{\partial \beta} (BKk_2 v) \right] \\
- 4 (\tau + \mu) ABH\Delta_o + \left( \frac{\tau + \mu}{6} \right) \delta^2 \left[ \frac{\partial}{\partial a} \left( \frac{B}{A} \frac{\partial \Delta}{\partial a} \right) + \frac{\partial}{\partial \beta} \left( \frac{A}{B} \frac{\partial \Delta}{\partial \beta} \right) \right] \\
+ \frac{(3\tau + 2\mu)}{12} \delta^2 \left[ \frac{\partial}{\partial a} \left( \frac{B}{A} \frac{\partial \Delta}{\partial a} \right) + \frac{\partial}{\partial \beta} \left( \frac{A}{B} \frac{\partial \Delta}{\partial \beta} \right) \right] - \frac{(5\tau + 4\mu)}{12} ABKk^2 \Delta_1 \\
+ 4\mu ABKw - \frac{\mu \delta^2}{6} \left[ \frac{\partial}{\partial a} \left( k_1 \frac{\partial \chi}{\partial a} \right) + \frac{\partial}{\partial \beta} \left( k_2 \frac{\partial \chi}{\partial \beta} \right) \right] \\
- \frac{\mu \delta^2}{6} \left[ \frac{\partial}{\partial a} \left( \frac{KB}{A} \frac{\partial w}{\partial a} \right) + \frac{\partial}{\partial \beta} \left( \frac{KA}{B} \frac{\partial w}{\partial \beta} \right) \right] + \frac{ABZ}{\delta} = 0
\]

Utilizing the criteria stated for simplifying the first two equilibrium equations, the only term which can be discarded by a direct comparison with other terms in the above equation is the underlined term.

An assumption that is inherent in any derivation of thin elastic shells subjected to small deformations is that the derivations of \( u \) and \( v \) are of the same or higher order of magnitude than the displacements \( u \) and \( v \). The justification for this assumption is a practical one based on the solution of many problems.
Utilizing this assumption and carrying out an order of term comparison, further simplifications may be made in the above equation. The expression for $\Delta_1$ may be written so as to exclude any of the displacement terms $u$ and $v$. Further, the expressions for $\Delta_0$ and $\chi_0$ when preceded by the coefficient $\delta^2$ may be discarded.

Substituting for $\Delta_0$, $\Delta_1$, $\chi_0$ and simplifying in accordance with what was stated in the preceding paragraph, the third equation of equilibrium may be written as

$$
2\mu \left[ \frac{\partial}{\partial \alpha} (Bk_2 u) + \frac{\partial}{\partial \beta} (Ak_1 v) \right] - \frac{8\mu (\tau + \mu)}{(\tau + 2\mu)} H \left[ \frac{\partial}{\partial \alpha} (Bv) + \frac{\partial}{\partial \beta} (A v) \right]
$$

$$
+ 4\mu AB \left[ K - \frac{4(\tau + \mu)}{(\tau + 2\mu)} H^2 \right] w - \frac{\mu (\tau + \mu)}{3(\tau + 2\mu)} AB\delta^2 \nabla^2 e w
$$

$$
+ \frac{2\mu (\tau + \mu)}{3(\tau + 2\mu)} AB\delta^2 \nabla^2 e \left[ K - \frac{4(\tau + \mu)H^2}{(\tau + 2\mu)} \right] w + \frac{\mu(3\tau + 2\mu)}{3(\tau + 2\mu)} \delta^2 (H\nabla^2 e - L\nabla^2 h) w
$$

$$
+ \frac{\mu(5\tau + 4\mu)}{6(\tau + 2\mu)} ABK\delta^2 \nabla^2 e w - \frac{\mu^2}{6} \left[ \frac{\partial}{\partial \alpha} \left( \frac{KB}{A} \frac{\partial w}{\partial \alpha} \right) + \frac{\partial}{\partial \beta} \left( \frac{KA}{B} \frac{\partial w}{\partial \beta} \right) \right] + \frac{ABZ}{8} = 0
$$

In the above equation, the elliptic differential operator $\nabla^2_e$ and the hyperbolic differential operator $\nabla^2_h$ are introduced and are defined by the equations

$$
\nabla^2_e = \frac{1}{AB} \left[ \frac{\partial}{\partial \alpha} \left( \frac{B}{A} \frac{\partial}{\partial \alpha} \right) + \frac{\partial}{\partial \beta} \left( \frac{A}{B} \frac{\partial}{\partial \beta} \right) \right]
$$

$$
\nabla^2_h = \frac{1}{AB} \left[ B^2 \frac{\partial}{\partial \alpha} \left( \frac{1}{AB} \frac{\partial}{\partial \alpha} \right) - A^2 \left( \frac{1}{AB} \frac{\partial}{\partial \beta} \right) \right]
$$

(5.25)
The mixed operator \( (H \nabla_e^2 - L \nabla_h^2) \) is defined by the equation

\[
H \nabla_e^2 - L \nabla_h^2 = \frac{1}{AB} \left[ \frac{\partial}{\partial \alpha} \left( \frac{B}{A} k_2 \frac{\partial}{\partial \alpha} \right) + \frac{\partial}{\partial \beta} \left( \frac{A}{B} k_1 \frac{\partial}{\partial \beta} \right) \right]
\]

where

\[
L = \frac{k_1 - k_2}{2}
\]

The sum of the changes of curvature of an inextensional middle surface can be shown\(^6\) to be the Laplacian of the displacement \( w \), i.e., \( \nabla_e^2 w \). In the case of a stretchable middle surface, the sum of the middle surface curvature changes may be written as the Laplacian of \( w \) plus terms depending on the tangential displacements \( u \) and \( v \). However, the effects of the \( u \) and \( v \) terms are small such that \( \nabla_e^2 w \) will be of the same order of magnitude as the curvature change sum and thus also of the same order of magnitude as a curvature change.\(^7\) In thin shell theory, the changes in curvature when prefixed by the square of the thickness of the shell are usually discarded in comparison to the strains and displacements, the reason being that practice indicates the effect of such terms to be much smaller than order \( \delta k \). This assumption will be carried to the present derivation.

---

\(^6\) See Love (1), p. 524 for the expressions for curvature change.

\(^7\) See discussion at end of chapter for further clarification.
Inspection of the third equation reveals that all terms involving the Laplacian of $w$ or a function of $w$ may be discarded since each of those terms can be shown to be of the same or smaller order than $\delta^2 \nabla^2 w$, and on the basis of what was stated above, this term may be discarded. Hence, the third equilibrium equation becomes

$$2\mu \left[ \frac{\partial}{\partial \alpha} \left( B k_2 u \right) + \frac{\partial}{\partial \beta} \left( A k_1 v \right) \right] - \frac{8\mu(\tau + \mu)}{(\tau + 2\mu)} H \left[ \frac{\partial}{\partial \alpha} \left( B u \right) + \frac{\partial}{\partial \beta} \left( A v \right) \right]$$

$$(5.26)$$

$$+ 4\mu AB \left[ K - \frac{4(\tau + \mu)}{(\tau + 2\mu)} H^2 \right] w - \frac{\mu(\tau + \mu)}{3(\tau + 2\mu)} AB \delta^2 \nabla^2 e w + \frac{ABZ}{\delta} = 0$$

Thus the three equations (5.23), (5.24), and (5.26) completely define the deformed middle surface. Summarizing these equations, they are as follows:

$$\frac{4\mu(\tau + \mu)}{(\tau + 2\mu)} \frac{1}{A} \left[ \frac{\partial^2}{\partial \alpha^2} (B u) + \frac{\partial^2}{\partial \alpha \partial \beta} (A v) \right] + \frac{8\mu(\tau + \mu)}{(\tau + 2\mu)} B \frac{\partial (Hw)}{\partial \alpha}$$

$$(5.27)$$

$$- \frac{\mu}{B} \left[ \frac{\partial^2 (B v)}{\partial \alpha \partial \beta} - \frac{\partial^2 (A u)}{\partial \beta^2} \right] + 2\mu AB Ku - 2\mu B k_2 \frac{\partial w}{\partial \alpha} + \frac{ABX}{\delta} = 0$$

$$\frac{4\mu(\tau + \mu)}{(\tau + 2\mu)} \frac{1}{B} \left[ \frac{\partial^2 (A v)}{\partial \beta^2} + \frac{\partial^2 (B u)}{\partial \alpha \partial \beta} \right] + \frac{8\mu(\tau + \mu)}{(\tau + 2\mu)} A \frac{\partial (Hw)}{\partial \beta}$$

$$- \frac{\mu}{A} \left[ \frac{\partial^2 (A u)}{\partial \alpha \partial \beta} - \frac{\partial^2 (B v)}{\partial \alpha^2} \right] + 2\mu AB Kv - 2\mu A k_1 \frac{\partial w}{\partial \beta} + \frac{ABY}{\delta} = 0$$
48

\[ 2\mu \left( \frac{\partial}{\partial a} (Bk_2u) + \frac{\partial}{\partial \beta} (Ak_1v) \right) - \frac{\partial \mu(\tau + \mu)}{(\tau + 2\mu)} H \left[ \frac{\partial}{\partial a} (Bu) + \frac{\partial}{\partial \beta} (Av) \right] \]

+ \[ 4\mu AB \left[ K - \frac{4(\tau + \mu)}{(\tau + 2\mu)} H^2 \right] w - \frac{\mu(\tau + \mu)}{3(\tau + 2\mu)} AB \delta^2 \nabla^4 w + \frac{ABZ}{\delta} = 0 \]

(5.27 contd)

In terms of the elastic constant \( E \) and \( \nu \), the coefficients of the above equations take on a simpler form. Thus, substituting for the Lame constants of elasticity the equation (5.6), the system (5.27) becomes

\[ \frac{1}{A} \left[ \frac{\partial^2 (Bu)}{\partial a^2} + \frac{\partial^2 (Av)}{\partial a \partial \beta} \right] + 2B \frac{\partial (Hw)}{\partial a} - \frac{(1 - \nu)}{2B} \left[ \frac{\partial^2 (Bu)}{\partial a \partial \beta} - \frac{\partial^2 (Av)}{\partial \beta^2} \right] \]

+ (1 - \nu) ABKu - (1 - \nu) Bk_2 \frac{\partial w}{\partial a} + \frac{(1 - \nu^2) ABX}{E \delta} = 0

\[ \frac{1}{A} \left[ \frac{\partial^2 (Av)}{\partial \beta^2} + \frac{\partial^2 (Bu)}{\partial a \partial \beta} \right] + 2 \frac{\partial (Hw)}{\partial \beta} - \frac{(1 - \nu)}{2A} \left[ \frac{\partial^2 (Av)}{\partial a \partial \beta} - \frac{\partial^2 (Bu)}{\partial a^2} \right] \]

+ (1 - \nu) ABK\nu - (1 - \nu) Ak_1 \frac{\partial w}{\partial \beta} + \frac{(1 - \nu^2) ABY}{E \delta} = 0

(1 - \nu) \left[ \frac{\partial}{\partial a} (Bk_2u) + \frac{\partial}{\partial \beta} (Ak_1v) \right] - 2H \left[ \frac{\partial}{\partial a} (Bu) + \frac{\partial}{\partial \beta} (Av) \right]

+ 2AB[(1 - \nu)K - 2H^2]w - AB \frac{\delta^2}{12} \nabla^4 w + \frac{(1 - \nu^2) ABZ}{E \delta} = 0 \]

Vlasov (5) proposes yet a third form of the above equations by introducing the auxiliary variables \( \phi(a, \beta) \) and \( \psi(a, \beta) \) defined by the equations

\[ u = \frac{1}{A} \frac{\partial \phi}{\partial a} + \frac{1}{B} \frac{\partial \psi}{\partial \beta} \]

\[ v = \frac{1}{B} \frac{\partial \phi}{\partial \beta} - \frac{1}{A} \frac{\partial \psi}{\partial a} \]
Substituting these relations into (5.28) and operating on the equations, the following set of equations result.

\[
\nabla_e^4 \phi + 2 \nabla_e^2 (H w) - (1 - \nu) (H \nabla_e^2 - L \nabla_h^2) w = - \frac{(1 - \nu^2)}{E \delta} \frac{1}{AB} \left[ \frac{\partial}{\partial a} (BX) + \frac{\partial}{\partial \beta} (AY) \right]
\]

\[
\frac{(1 - \nu)}{2} \nabla_e^4 \psi + \frac{(1 - \nu)}{AB} \left[ \frac{\partial}{\partial a} (k_1 \frac{\partial w}{\partial \beta}) - \frac{\partial}{\partial \beta} (k_2 \frac{\partial w}{\partial a}) \right] = - \frac{(1 - \nu^2)}{E \delta} \frac{1}{AB} \left[ \frac{\partial}{\partial \beta} (AX) - \frac{\partial}{\partial a} (BY) \right]
\]

\[
- 2H \nabla_e^2 \phi + (1 - \nu) (H \nabla_e^2 - L \nabla_h^2) \phi + (1 - \nu) \frac{1}{AB} \left[ \frac{\partial}{\partial a} (k_1 \frac{\partial \psi}{\partial \beta}) - \frac{\partial}{\partial \beta} (k_2 \frac{\partial \psi}{\partial a}) \right] = 0
\]

\[
- \frac{\delta^2}{12} \nabla_e^4 w - 2 [2H^2 - (1 - \nu)K] = - \frac{(1 - \nu^2)}{E \delta} Z
\]

6. **Stress Resultants, Strains, and Stresses**

The derived shell equations are in terms of the displacements of the middle surface. In order to completely effect a solution, the boundary conditions must also be stated explicitly in terms of these displacements, a situation which cannot always be guaranteed. In fact, frequently the boundary conditions are given in terms of edge loadings which in turn are expressible in terms of
stress resultants and thus it becomes of utmost concern to relate the displacements to the stress resultants.

Problems also arise where the stress and strains within the shell become of prime concern. In order to examine these quantities it is necessary to know the relation between them and the displacement solutions of the shell equations.

The purpose of this section is to disclose the relations between stress, strains, stress resultants and the displacements of the middle surface.

a) **Strains**

Consider the four strains, $e_a$, $e_\beta$, $e_\gamma$, and $e_{a\beta}$ as defined by (4.1)

$$
e_a = h_1 \frac{\partial u_a}{\partial a} + h_1 h_2 \frac{\partial h_1}{\partial \beta} u_\beta + h_1 \frac{\partial h_1}{\partial \gamma} u_\gamma$$

$$e_\beta = h_2 \frac{\partial u_\beta}{\partial \beta} + h_1 h_2 \frac{\partial h_2}{\partial a} u_a + h_2 \frac{\partial h_2}{\partial \gamma} u_\gamma$$

$$e_\gamma = \frac{\partial u_\gamma}{\partial \gamma}$$

$$e_{a\beta} = \frac{h_1}{h_2} \frac{\partial}{\partial a} (h_2 u_\beta) + \frac{h_2}{h_1} \frac{\partial}{\partial \beta} (h_1 u_a)$$
Because of the symmetry of definition in the expressions for $e_a$ and $e_\beta$, only one need be determined and the other will follow by an interchange of subscripts. The expression for $e_\gamma$ had been determined by equation (5.13) and hence need not be derived, but rather put in a more convenient form.

Substituting the expressions for $u_a$, $u_\beta$ as given by equation (4.3) and $u_\gamma$ as given by (5.15), into the expression for $e_a$, the result becomes after a regrouping of terms

$$
e_a = h_1 h_2 \left( \frac{\partial u}{\partial a} + \frac{\partial w}{\partial \beta} + ABk_1 w \right) + \left[ k_1 (\frac{\partial u}{\partial a} + \frac{\partial A}{\partial \beta} + ABk_1 w) \right]

+ AB \left( -\frac{1}{A} \frac{\partial}{\partial a} \left( \frac{1}{A} \frac{\partial w}{\partial a} - k_1 u \right) - \frac{1}{AB} \frac{\partial A}{\partial \beta} \left( \frac{1}{B} \frac{\partial w}{\partial \beta} - k_2 v \right) \right)

- \frac{\tau}{2\mu} ABk_1 \Delta_o \gamma$$

Defining $e_1$, $e_\gamma$, $e_3$, $e_12$ as the strains $e_a$, $e_\beta$, $e_a\beta$ respectively on the middle surface, that is,

$$e_1 = \frac{1}{A} \frac{\partial u}{\partial a} + \frac{1}{AB} \frac{\partial A}{\partial \beta} v + k_1 w$$

$$e_2 = \frac{1}{B} \frac{\partial v}{\partial \beta} + \frac{1}{AB} \frac{\partial B}{\partial a} u + k_2 w$$

$$e_3 = -\frac{\tau}{2\mu} \Delta_o$$

$$e_12 = \frac{B}{A} \frac{\partial}{\partial a} \left( \frac{v}{B} \right) + \frac{A}{B} \frac{\partial}{\partial \beta} \left( \frac{u}{A} \right)$$
Then the expression for $e_\alpha$ may be written as

$$e_\alpha = h_1 h_2 AB \{ e_1 + [k_2 e_1 + k_1 e_3 - \frac{1}{A} \frac{\partial}{\partial \alpha} \left( \frac{1}{A} \frac{\partial w}{\partial \alpha} - k_1 u \right) + \frac{1}{B} \frac{\partial}{\partial \beta} \left( \frac{1}{B} \frac{\partial w}{\partial \beta} - k_2 v \right)] \gamma \}$$

Let $K_1$ and $K_2$ represent the respective changes in curvatures in the lines $\beta = constant$ and $\alpha = constant$ on the middle surface. Designating as $k_1'$ and $k_2'$ the curvatures of the deformed principal coordinate axes, then

$$K_1 = k_1' - k_1$$

$$K_2 = k_2' - k_2$$

(6.2)

However, $K_1$ and $K_2$ may be expressed in terms of the deformation of the middle surface as

$$K_1 = -\frac{1}{A} \frac{\partial}{\partial \alpha} \left( \frac{1}{A} \frac{\partial w}{\partial \alpha} - k_1 u \right) - \frac{1}{AB} \frac{\partial A}{\partial \beta} \left( \frac{1}{A} \frac{\partial w}{\partial \beta} - k_2 v \right)$$

$$K_2 = -\frac{1}{B} \frac{\partial}{\partial \beta} \left( \frac{1}{B} \frac{\partial w}{\partial \beta} - k_2 v \right) - \frac{1}{AB} \frac{\partial B}{\partial \alpha} \left( \frac{1}{A} \frac{\partial w}{\partial \alpha} - k_1 u \right)$$

(6.3)

Inspection of the expression for $e_\alpha$ indicates it may then be written as

$$e_\alpha = h_1 h_2 AB \{ e_1 + [k_2 e_1 + k_1 e_3 + K_1'] \gamma \}$$

---

Expanding \( h_1 h_2 \) in series form and truncating the series at the first power of \( \gamma \)

\[
h_1 h_2 = \frac{1}{AB} (1 - 2H \gamma)
\]

Substituting this expression into the expression for \( e_a \) and retaining only a linear expression in \( \gamma \), the result becomes

\[
e_a = e_1 + (K_1 - k_1 e_1 + k_1 e_3) \gamma
\]  (6.4)

By analogy

\[
e_\beta = e_2 + (K_2 - k_2 e_2 + k_2 e_3) \gamma
\]  (6.5)

Equation (6.4) and (6.5) are the resultant expressions for the strains. However, the strain \( e_3 \) has yet to be evaluated. Now

\[
e_3 = -\frac{\tau}{2\mu} \Delta_o
\]

However, \( \Delta_o \) represents the volume dilatation on the middle surface and thus

\[
e_3 = -\frac{\tau}{2\mu} (e_1 + e_2 + e_3)
\]

Solving for \( e_3 \)

\[
e_3 = -\frac{\tau}{(\tau + 2\mu)} (e_1 + e_2)
\]  (6.6)
Thus the expression for $e_a$ and $e_\tau$ assume the final form

$$
e_a = e_1 + \frac{2(\tau + \mu)}{(\tau + 2\mu)} k_1 e_1 - \frac{\tau}{(\tau + 2\mu)} k_1 e_2 \gamma$$

$$
e_\beta = e_2 + \frac{2(\tau + \mu)}{(\tau + 2\mu)} k_2 e_2 - \frac{\tau}{(\tau + 2\mu)} k_2 e_1 \gamma \quad (6.7)$$

Rather than expand the above expressions in terms of $u$, $v$, and $w$, it will be more convenient for the work that follows to leave the strain expressions in the above form.

Substituting the expressions for $h_1$, $h_2$, $u_a$ and $u_\beta$ into the equation for the shear strain $e_{a\beta}$, the result after rearranging and discarding $\gamma^2$ terms becomes

$$
e_{a\beta} = AB h_1 h_2 \left( \frac{1}{A} \frac{\partial v}{\partial a} + \frac{1}{B} \frac{\partial u}{\partial \beta} - \frac{v}{AB} \frac{\partial B}{\partial a} - \frac{u}{AB} \frac{\partial A}{\partial \beta} \right)$$

$$+ \left[ 2k_1 \left( \frac{1}{B} \frac{\partial u}{\partial \beta} - \frac{1}{AB} \frac{\partial A}{\partial \beta} u \right) + 2k_2 \left( \frac{1}{A} \frac{\partial v}{\partial a} - \frac{1}{AB} \frac{\partial B}{\partial a} v \right) 
- \frac{2}{AB} \left( \frac{\partial^2 w}{\partial a \partial \beta} - \frac{1}{B} \frac{\partial B}{\partial a} \frac{\partial w}{\partial \beta} - \frac{1}{A} \frac{\partial A}{\partial \beta} \frac{\partial w}{\partial a} \right) \right] \gamma$$

Let $\tau$ represent the twist of the middle surface. That is, let it represent the ratio of the relative angular change of two opposite sides of an element drawn on the middle surface to the distance between them. Its analytical definition is given as

$$
\tau = - \frac{1}{AB} \left( \frac{\partial^2 w}{\partial a \partial \beta} - \frac{1}{A} \frac{\partial A}{\partial \beta} \frac{\partial w}{\partial a} - \frac{1}{B} \frac{\partial B}{\partial a} \frac{\partial w}{\partial \beta} \right)
+ k_1 \left( \frac{1}{B} \frac{\partial u}{\partial \beta} - \frac{1}{AB} \frac{\partial A}{\partial \beta} u \right) + k_2 \left( \frac{1}{A} \frac{\partial v}{\partial a} - \frac{1}{AB} \frac{\partial B}{\partial a} v \right) \quad (6.8)
$$

---

Inspection of the expression for \( e_{a\beta} \) reveals that the first term is the shearing strain on the middle surface \( e_{12} \) while the second term is the twist of the middle surface. Thus

\[
e_{a\beta} = ABh_1 h_2 [e_{12} + \tau \gamma]
\]

Expanding and truncating \( h_1 h_2 \) as before, the final expression for \( e_{a\beta} \) becomes

\[
e_{a\beta} = e_{12} + 2(\tau - He_{12}) \gamma
\]

(6.9)

The expression for \( e_{\gamma} \) has been given by (5.18) as

\[
e_{\gamma} = -\frac{\tau}{2\mu} \Delta
\]

where \( \Delta \) is the dilatation at some point inside the shell. Thus

\[
\Delta = e_a + e_\beta + e_{\gamma}
\]

and hence the expression for \( e_{\gamma} \) may be written as

\[
e_{\gamma} = -\frac{\tau}{(\tau + 2\mu)} (e_a + e_\beta)
\]

Substituting (6.7) for \( e_a \) and \( e_\beta \), the result becomes

\[
e_{\gamma} = -\frac{\tau}{(\tau + 2\mu)} \{ (e_1 + e_2) + [(K_1 + K_2) - \frac{1}{(\tau + 2\mu)} (\tau k_2 + 2(\tau + \mu)k_1)e_1
\]

\[\quad - \frac{1}{(\tau + 2\mu)} (\tau k_1 + 2(\tau + \mu)k_2)e_2] \gamma \}
\]

(6.10)
An immediate consequence of the above expression is that the component of the dilatation $\Delta_1$ may be given a physical interpretation. Thus from the expression

$$e_\gamma = -\frac{\tau}{2\mu} (\Delta_0 + \Delta_1 \gamma)$$

it follows that

$$\Delta_1 = \frac{2\mu}{(\tau + 2\mu)} \left[ (K_1 + K_2) - \frac{1}{(\tau + 2\mu)} (\tau k_2 + 2(\tau + \mu)k_1) e_1 \right.
$$

$$- \left. \frac{1}{(\tau + 2\mu)} (\tau k_1 + 2(\tau + \mu)k_2) e_2 \right]$$

(6.11)

In his book, Novozhilov\(^{10}\) points out that the expressions for the stress resultants may be simplified without exceeding the basic error of the Kirchoff hypothesis. Essentially the simplification consists of discarding the strain terms in comparison with the change of curvature and the twist of the shell. Following this procedure in the present work, the expressions for the strains simplify considerably and become

$$e_\alpha = e_1 + K_1 \gamma$$

$$e_\beta = e_2 + K_2 \gamma$$

$$e_\gamma = -\frac{\tau}{(\tau + 2\mu)} \left[ (e_1 + e_2) + (K_1 + K_2) \gamma \right]$$

$$e_{\alpha\beta} = e_{12} + 2\tau \gamma$$

\(^{10}\)See Novozhilov (4), pp. 51-54.
Similarly, the dilatation component $\Delta_1$ becomes

$$\Delta_1 = \frac{2\mu}{(\tau + 2\mu)} (K_1 + K_2) \quad (6.13)$$

b) **Stress**

The stresses $\sigma_a$, $\sigma_\beta$, and $\sigma_\alpha\beta$ have been defined as

$$\sigma_a = (\tau + 2\mu)\Delta - 2\mu (e_\beta + e_\gamma)$$

$$\sigma_\beta = (\tau + 2\mu)\Delta - 2\mu (e_a + e_\gamma)$$

$$\sigma_{\alpha\beta} = \mu e_{12}$$

Substituting the truncated series expression for $\Delta$ and the strain expressions (6.13), the above equation reduces to

$$\sigma_a = \frac{1}{(\tau + 2\mu)} \left\{ [4\mu(\tau + \mu)e_1 + 2\mu\tau e_2] + [4\mu(\tau + \mu)K_1 + 2\mu\tau K_2] \right\} \gamma$$

$$\sigma_\beta = \frac{1}{(\tau + 2\mu)} \left\{ [2\mu\tau e_1 + 4\mu(\tau + \mu)e_2] + [2\mu\tau K_1 + 4\mu(\tau + \mu)K_2] \right\} \gamma$$

$$\sigma_{\alpha\beta} = \mu e_{12} + 2\mu\tau \gamma \quad (6.14)$$

c) **Stress Resultants**

The stress resultants or forces and moments per unit length of the middle surface are defined as follows:

$$T_1 = \frac{1}{B} \int_{-\delta/2}^{\delta/2} \frac{\sigma_a}{h_2} \, d\gamma$$

$$T_2 = \frac{1}{A} \int_{-\delta/2}^{\delta/2} \frac{\sigma_\beta}{h_1} \, d\gamma \quad (6.15)$$
\[ T_{12} = \frac{1}{B} \int_{-\delta/2}^{\delta/2} \frac{\sigma_{a\beta}}{h_2} \, d\gamma \]

\[ T_{21} = \frac{1}{A} \int_{-\delta/2}^{\delta/2} \frac{\sigma_{a\beta}}{h_1} \, d\gamma \]

\[ Q_1 = \frac{1}{B} \int_{-\delta/2}^{\delta/2} \frac{\sigma_{a\gamma}}{h_2} \, d\gamma \]

\[ Q_2 = \frac{1}{A} \int_{-\delta/2}^{\delta/2} \frac{\sigma_{\beta\gamma}}{h_1} \, d\gamma \]

\[ M_1 = \frac{1}{B} \int_{-\delta/2}^{\delta/2} \frac{\sigma_{a\gamma}}{h_2} \gamma d\gamma \]

\[ M_2 = \frac{1}{A} \int_{-\delta/2}^{\delta/2} \frac{\sigma_{\beta\gamma}}{h_1} \gamma d\gamma \]

\[ M_{12} = \frac{1}{B} \int_{-\delta/2}^{\delta/2} \frac{\sigma_{a\beta}}{h_2} \gamma d\gamma \]

\[ M_{21} = \frac{1}{A} \int_{-\delta/2}^{\delta/2} \frac{\sigma_{a\beta}}{h_1} \gamma d\gamma \]

(6.15 contd)
The directions of these resultants which are consistent with the assumed directions of displacements are shown in Figure 6 and Figure 7. Note that force functions X, Y, Z given by (5.20) are not shown.

Substituting the expressions for stresses as given by (6.12) and the expressions for $h_1$ and $h_2$, the above relations become upon integration

\[
T_1 = \frac{\delta}{(\tau + 2\mu)} [4\mu (\tau + \mu)e_1 + 2\mu \tau e_2]
\]

(6.16)

\[
T_2 = \frac{\delta}{(\tau + 2\mu)} [2\mu \tau e_1 + 4\mu (\tau + \mu)e_2]
\]

\[
T_{12} = T_{21} = \mu e_{12}
\]

\[
M_1 = \frac{\delta^3}{12(\tau + 2\mu)} [4\mu (\tau + \mu)K_1 + 2\mu \tau K_2]
\]

\[
M_2 = \frac{\delta^3}{12(\tau + 2\mu)} [2\mu \tau K_1 + 4\mu (\tau + \mu)K_2]
\]

\[
M_{12} = M_{21} = \frac{\mu \delta^3}{6} \tau
\]

In the evaluation of the above expressions, use was made of a statement made earlier, namely that simplification within the limits of accuracy of shell theory is possible by neglecting the strain components in comparison with the changes in curvature.
Figure 6. Stress resultant forces acting on an element of the middle surface.
Figure 7. Stress resultant moments acting on an element of the middle surface.
An explicit relation has not been given for \( Q_1 \) and \( Q_2 \).

The visual procedure in either shells or plates is to solve for these quantities from the equilibrium equations. Thus in the present case, referring to equation (5.21) and substituting for \( u_2 \), the following relations are obtained.

\[
Q_1 = \frac{(\tau + \mu) \delta^3}{6A} \frac{\partial \Delta}{\partial \alpha} + \frac{(3\tau + 2\mu)}{12A} k_2 \delta^3 \frac{\partial \Delta}{\partial \beta} - \frac{2\mu \delta^3}{12B} \left(\frac{\partial \chi_1}{\partial \beta} + k_1 \frac{\partial \chi_0}{\partial \beta}\right)
\]

\[
+ \frac{\mu K \delta^3}{6} \left(k_1 u - \frac{1}{A} \frac{\partial w}{\partial \alpha}\right)
\]

\[
Q_2 = \frac{(\tau + \mu) \delta^3}{6B} \frac{\partial \Delta}{\partial \beta} + \frac{(3\tau + 2\mu)}{12B} k_1 \delta^3 \frac{\partial \Delta}{\partial \alpha} + \frac{2\mu \delta^3}{12A} \left(\frac{\partial \chi_1}{\partial \alpha} + k_2 \frac{\partial \chi_0}{\partial \alpha}\right)
\]

\[
+ \frac{\mu K \delta^3}{6} \left(k_2 v - \frac{1}{B} \frac{\partial w}{\partial \beta}\right)
\]

The physical interpretation for \( \chi \) the rotation had been previously given. Further, the two components of the truncated series for \( \chi \) may also be given a physical interpretation. From equation (5.16), the terms for \( \chi_1 \) may be rearranged so that

\[
\chi_1 = \left(\frac{k_2 - k_1}{2}\right) e_{12}
\]

Thus the one component of \( \chi \) is directly dependent on the shearing deformation of the middle surface. The other component \( \chi_0 \) is merely the rotation of an element on the middle surface.
Substituting the expression for $\Delta_1$ as given by (6.13) and the expressions for $\Delta_0$ and $\chi_1$, the equations for $Q_1$ and $Q_2$ may be considerably simplified. Thus following a procedure previously used, the strains may be disregarded in comparison with the changes in curvature. Further the displacements $u$, $v$, and $w$ will be of the same order as the strains and their expressions may too be discarded. Now the rotation $\chi_0$ need not be a small term. However, if the shell is restrained against rigid body motion as a whole, the case normally encountered, then the term $\chi_0$ must be small and from its physical nature be of the same order of magnitude as the shearing strain $e_{12}$ and thus also discarded.

Hence the equations for $Q_1$ and $Q_2$ simplify into the following two

$$Q_1 = \frac{\mu(\tau + \mu)}{3(\tau + 2\mu)} \frac{\delta^3}{A} \frac{\partial}{\partial \alpha} (K_1 + K_2)$$

$$Q_2 = \frac{\mu(\tau + \mu)}{3(\tau + 2\mu)} \frac{\delta^3}{B} \frac{\partial}{\partial \beta} (K_1 + K_2)$$

(6.19)

d) **Kirchoff Boundary Conditions**

One of the consequences of using the Kirchoff hypothesis in either plates or shells is that the edge conditions cannot be

---

The word edge is here assumed to mean an edge of a shell coinciding with one of the principal coordinate lines. For a more general discussion of edges which do not necessarily coincide with principal coordinate lines see Love (1).
identically satisfied. In particular, the number of boundary conditions available will exceed the number of constants available upon integration of the differential equations governing the deformed shell.

In order to reduce the number of boundary conditions, certain of the stress resultants are combined together to define a new effective stress resultant which now must be satisfied on a boundary. The net effect of such a transformation is that the stresses and displacements thus evaluated will be in error; however, the error will be negligible save in the immediate vicinity of the edge.

On an edge $\alpha = \text{constant}$, two effective values of shear may be given. Thus

$$T_{12}^{\text{eff}} = T_{12} + k_2 M_{12}$$

$$Q_1^{\text{eff}} = Q_1 + \frac{1}{B} \frac{\partial M_{12}}{\partial \beta}$$

Similarly, on the edge $\beta = \text{constant}$

$$T_{21}^{\text{eff}} = T_{21} + k_1 M_{21}$$

$$Q_2^{\text{eff}} = Q_2 + \frac{1}{A} \frac{\partial M_{21}}{\partial \alpha}$$

12 The deviation of these conditions is standard and may be found in any of the publications on general shell theory listed in the Bibliography.
Substituting for the various stress resultants

\[ T_{12 \text{ eff}} = \mu \left( e_{12} + \frac{k_2 \delta^3}{6} \tau \right) \]  

(6.22)

\[ Q_{1 \text{ eff}} = \mu \delta^3 \left[ \frac{(\tau + \mu)}{3(\tau + 2\mu)} \frac{1}{A} \frac{\partial}{\partial \alpha} (K_1 + K_2) + \frac{1}{B} \frac{\partial \tau}{\partial \beta} \right] \]

\[ T_{21 \text{ eff}} = \mu \left( e_{12} + \frac{k_2 \delta^3}{6} \tau \right) \]

(6.23)

\[ Q_{2 \text{ eff}} = \mu \delta^3 \left[ \frac{(\tau + \mu)}{3(\tau + 2\mu)} \frac{1}{B} \frac{\partial}{\partial \beta} (K_1 + K_2) + \frac{1}{A} \frac{\partial \tau}{\partial \alpha} \right] \]

d) Summary

The stresses, strains, and stress resultants have been expressed in terms of Lame's constants of elasticity. These same functions when expressed in terms of Young's Modulus, \( E \), and Poisson's ratio, become

\[ e_a = e_1 + K_1 \gamma \]

\[ e_\beta = e_2 + K_2 \gamma \]

\[ e_{\alpha\beta} = e_{12} + 2\tau \gamma \]

\[ e_\gamma = -\frac{\nu}{(1 - \nu)} [(e_1 + e_2) + (K_1 + K_2) \gamma] \]

\[ \sigma_a = \frac{E}{(1 - \nu^2)} [(e_1 + \nu e_2) + (K_1 + \nu K_2) \gamma] \]

\[ \sigma_\beta = \frac{E}{(1 - \nu^2)} [(e_2 + \nu e_1) + (K_2 + \nu K_1) \gamma] \]
\[ \sigma_{a \beta} = \frac{E}{2(1 + \nu)} \left( e_{12} + 2\tau \gamma \right) \]

\[ T_1 = \frac{E \delta}{(1 - \nu^2)} \left( e_1 + \nu e_2 \right) \]

\[ T_2 = \frac{E \delta}{(1 - \nu^2)} \left( e_2 + \nu e_1 \right) \]

\[ M_1 = \frac{E \delta^3}{12(1 - \nu^2)} \left( K_1 + \nu K_2 \right) \]

\[ M_2 = \frac{E \delta^3}{12(1 - \nu^2)} \left( K_2 + \nu K_1 \right) \]

\[ T_{12 \text{ eff}} = \frac{E}{2(1 + \nu)} \left( e_{12} + \frac{k_2 \delta^3}{6} \right) \]

\[ T_{21 \text{ eff}} = \frac{E}{2(1 + \nu)} \left( e_{12} + \frac{k_1 \delta^3}{6} \right) \]

\[ Q_{1 \text{ eff}} = \frac{\delta^3}{12(1 - \nu^2)} E \left[ \frac{1}{A} \frac{\partial}{\partial \alpha} (K_1 + K_2) + \frac{6(1 - \nu)}{B} \frac{\partial \tau}{\partial \beta} \right] \]

\[ Q_{2 \text{ eff}} = \frac{\delta^3 E}{12(1 - \nu^2)} \left[ \frac{1}{B} \frac{\partial}{\partial \beta} (K_1 + K_2) + \frac{6(1 - \nu)}{A} \frac{\partial \tau}{\partial \alpha} \right] \]
7. **Concluding Remarks**

As has been mentioned earlier, the derivation of the shell equations is based on a method proposed by V. Z. Vlasov (5). However, in his derivation, proper concern is not placed on the expression for the normal displacement of the shell $u_\gamma$. Reasoning that the Kirchoff hypothesis yields linear variations in the tangential displacements $u_a$ and $u_\beta$, Vlasov assumes a linear variation in $u_\gamma$. In particular he expresses $u_\gamma$ in the form

$$u_\gamma = w + w* \gamma$$

where $w*$ is a function of $a$ and $\beta$ only and its form is to be determined in the derivation. Restricting his derivation to thin shells, he then discards $w*$ as being negligible and is left with the result that the normal displacement is a constant through the thickness of the shell and equal to that of the middle surface. The strain $e_\gamma$ is thus identically zero and hence the problem solved is one of plane strain.

The error in Vlasov's development lies in the fact that the shell problem, like the plate problem, is assumed to be one of plane stress. Hence by Hooke's Law, once the two strains $e_a$ and $e_\beta$ are specified, the third strain $e_\gamma$ cannot be chosen arbitrarily but must be in elastic accord with the two specified strains. Thus
in the shell problem solved, the strain $e^\gamma$ turns out to be a linear function of $\gamma$ while the displacement $u^\gamma$ is a parabolic function of that variable and no further simplifications are permitted in the exponents of $\gamma$ for the solution of the equations of the deformed shell. It might also be noted that expressing the strain $e^\gamma$ and the displacement $u^\gamma$ as just mentioned leads to expected parabolic variations in $\gamma$ of the transverse shearing strains $e_{a\gamma}$ and $e_{\beta\gamma}$ whereas Vlasov's expressions lead to linear variations in those strains. Further, equations (5.29) degenerate to the plate equation for $k_1 = k_2 = 0$ while the corresponding equations of Vlasov, when left in terms of the Lamé constants $\lambda$ and $\mu$ do not.

The derivation presented by Vlasov is most remarkable in that two errors tend to cancel each other so that his final differential equations for the middle surface displacements when expressed with the elastic constants $E$ and $\nu$ are correct. The first error, as previously noted, is in his expression for the displacement $u^\gamma$. The second error lies in an incorrect expression for the relation between the Lamé constants $\lambda$ and $\mu$ and the elastic constants $E$ and $\nu$. In particular, Vlasov states the relation as

$$\tau = \frac{E\nu}{(1 - \nu^2)}; \quad \mu = \frac{E}{2(1 + \nu)}$$
Comparing these expressions with the true relations as given by (5.6) reveals that the above expression for $\lambda$ is incorrect.

The problem of a correct expression for the normal strain $e_\gamma$ and displacement $u_\gamma$ seems to be unique to the displacement derivation of the shell equations. When the shell equations are expressed in terms of stress resultants, an assumption of plane stress in conjunction with the assumption of $e_\gamma = 0$ and $u_\gamma = w$ is admissable. Thus in the present work, the expressions for the stress resultants could have been derived in an alternative way by solving for the normal strains $e_\alpha$, $e_\beta$, $e_\alpha\beta$ as if $u_\gamma = w$ and the results found for the stress resultants would be identical to equations (6.21).

Analyzing the original expressions for the strain and the dilatation component $\Delta_1$ as given by equations (6.7) and (6.11), respectively, and noting how they enter the equilibrium equations stated in terms of displacements as well as the ensuing simplifications that are then made, it must be concluded that the shell displacement equations (5.28) or (5.29) could just as easily have been derived by using the more simple strain and dilatation expressions as given by (6.12) and (6.13). One must then conclude that the shell equations in terms of displacements and the shell equations in terms of stress resultants are completely compatible.

\[^{13}\text{See Novozhilov (4), p. 52.}\]
within the limits of accuracy dictated by the Kirchoff hypothesis. Thus the stress resultants found from the displacement shell equations will check those found from the stress resultant shell equations and vice versa.

A natural question might arise at this point, namely: If the shell displacement equations can be derived by using the simplified strain and dilatation expressions who not use them in the derivation? The answer to this question lies in the chapters that follow dealing with the orthotropic case where the methods used in deriving the orthotropic displacement equations parallels those used for the isotropic case. The simplifications that are possible in the isotropic expressions for strain and dilatation arise from the fact that certain groups of terms may be given a direct physical interpretation. In the orthotropic case, because of the complexity of expression, such interpretations may be difficult to find and, hence, for the purpose of simplifying the orthotropic equations, an expanded term by term comparison with the isotropic case will prove more advantageous.

In the derivation of the shell equations particular difficulty arises in analyzing the order of magnitude of the changes in curvatures of the middle surface, $K_1$ and $K_2$. Thus, these terms are assumed to be large enough so that the strains $e_1$, $e_2$, and $e_{12}$ may
be discarded in comparison with them, yet small enough so that these terms when prefixed by square of the thickness of the shell may be discarded in comparison with the strains.

That the changes in curvature are large enough to neglect comparing strains is not disputed since various analysts have substantiated that result. However, the second difficulty has as yet not been resolved. Almost universally, authors on shell theory have neglected the term of the form $\delta^2 K$ in the derivation of the shell equations stating that its inclusion leads to negligible effects. This author has investigated the effects of this term for the case of a cylindrical shell and indeed found that its effects are extremely small, smaller than order $\delta k$. However, it still remains to show that terms of the form $\delta^2 K$ will be negligible for all shells.

As amplification may be made in the definitions for $K_1$ and $K_2$ as given by (6.3) by noting how these quantities enter in the derivation of the displacement shell equations and into the expressions for the stress resultants. In all these equations, the tangential displacement terms $u$ and $v$ may be omitted in the definitions of $K_1$ and $K_2$ without violating the form or accuracy of the equations.

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15 See Kozik (6).
The net effect of such a definition would be to make the normal
displacement as given in the curvature expressions a sole func-
tion of the moment stress resultants $M_1$ and $M_2$. Thus barring
the case where $w$ is either a constant or a linear function of $\alpha$
and $\beta$, if $w$ is identically zero, the moment stress resultants
are zero and vice versa. Hence, $w$ would be a direct indicator
of the presence and magnitude of bending moments. Note however
that such an analogous conclusion cannot be made in regard to the
tangential displacement $u$ and $v$ and the force stress resultants
$T_1$ and $T_2$. In this case $T_1$ and $T_2$ are directly dependent upon
the strains $e_1$ and $e_2$ which in turn are expressed by (6.1) in
terms of $u$, $v$, and the normal displacement $w$.

As has been pointed out in the derivation, the simplified
expressions for the strain (6.12) are completely in accord with
the accuracy of shell solutions. A more detailed set of expres-
sions yields results whose effects are small and doubtful since
they fall within the error boundaries of the Kirchoff hypothesis.
Some authors, reasoning that the expression for $e_{\alpha}$ is the same
as that obtained for thin curved beams, fall prey to the temptation
of using the more complicated expression for $e_{\alpha}$ hoping to parallel
the strain expression of medium thickness curved beams and thus

---

assure themselves of solutions to shell of medium thickness. As pointed out, such is not the case. The Kirchoff hypothesis places a limit on the accuracy of strain expressions and thus the shell solutions must of necessity always be thin shell solutions. For this reason, any shell problem other than the thin shell case must be treated as a three dimensional elasticity problem.

The form of the strain, stress and stress resultant expressions as given in the summary equations (6.21) leads to one important conclusion. That is, that these functions may be found by superpositioning. Thus the strains $e_1$, $e_2$ and the corresponding stress resultants may be found independently of the change in curvature $K$ or torsion $\tau$ expressions and hence the corresponding moments. In particular, the shell may be analyzed as a thin membrane in order to determine the strains of the middle surface and then analyzed as a structure with an inextensional middle surface to determine the quantities $K$ and $\tau$.

Such a breakdown of the problem leads to great simplifications in the solution of shells especially if the shell equations are expressed in terms of stress resultants. In fact, most authors deriving the stress resultant shell equations include a special section on membrane solutions of the shell problem, pointing out that except for regions in the vicinity of concentrated or distributed
applied moments, the shell will behave as a membrane. Further, Novozhilov (4) uses the membrane solution as a starting point for finding the general shell solution. In particular, he is able to show that the membrane solution represents the particular solution to the general shell differential equation.

From what has been earlier stated in regard to the displacements, it is seen that in general, the middle surface displacements cannot be combined by superpositioning. Thus the displacement \( w \) enters the \( K_\lambda \) and the \( e \) expressions. However, since \( K_\lambda \) and for that matter \( \tau \), can be simplified such that the tangential displacements \( u \) and \( v \) do not appear in them, then the tangential displacements \( u \) and \( v \) may be found from the membrane solution of the shell.
REFERENCES


CHAPTER II
THE ORTHOTROPIC SHELL

8. **Introductory Remarks**

A material is said to be anisotropic, or possess anisotropy, if at a given point in that material the elastic properties are directionally dependent. Isotropy, then, is the special case of anisotropy where the elastic properties at a point are the same in all directions.

The elastic properties of a material are quantitatively expressible in terms of elastic constants. The latter quantities are constants which relate the stress to the strain. In the case of an isotropic material, these quantities are usually the modulus of elasticity $E$ and Poisson's ratio $\nu$. Thus anisotropy is a property which manifests itself through the stress-strain relations.

Much of what has been said in regard to the isotropic shell in the previous chapter is equally applicable to the anisotropic shell. Since anisotropy is a condition which dictates the relation between stress and strain, any definitions or conclusions reached based solely on a consideration of strains and displacements or stresses and stress resultants will be valid no matter what the elastic
properties may be. Thus the equilibrium equations of an element of a shell derived solely in terms of stresses, equation (5.1), (5.2), and (5.3), will be valid for anisotropic as well as isotropic materials. Similarly, the definition of the strains and stress resultants as given by equations (4.1) and (6.15) will carry over to the anisotropic shell.

The Kirchoff hypothesis is concerned only with deformations and the only criteria governing its validity is the relative thinness of a shell. Hence no matter what the elastic properties may be, if a shell is thin enough the hypotheses as elucidated in the previous chapter may be applied with the result that the tangential displacements, \( u_a \) and \( u_\beta \), will be given by equation (4.2) of Chapter I.

Up to this point, the above discussion is applicable to any anisotropic shell. The work to be presented in this chapter will deal with the orthotropic shell, a particular form of anisotropy. The derivation of the equations will be patterned very closely after that of the isotropic shell. Thus the end result is to present a set of equations in terms of the displacements of the middle surface. Unlike the isotropic case, the orthotropic shell equations

\[ \text{The definition of the orthotropic shell will be reserved for the following section.} \]
will not be reduced to a system of three equations in three unknowns. Because of the number of terms involved and their complexity of expression, the orthotropic equations will be brought to a state of development equivalent to equation (5.19) of the previous chapter. Any further reduction and simplifications will entail the choosing of a specific shell shape.

The specification of a point in a shell is purely a function of the shape of the undeformed shell and the coordinate system used. Thus in deriving the orthotropic equations, principal curvilinear coordinates $\alpha$ and $\beta$, may again be used. In fact, all that has been stated in Sections 3 and 4 of the previous chapter are directly applicable to the present case.

9. Orthotropic Stress Strain Relations

a) General Considerations

The most general form of Hooke's law at a point in a body is

$$e_x = a_{11} \sigma_x + a_{12} \sigma_y + a_{13} \sigma_z + a_{14} \gamma_{xz} + a_{15} \gamma_{zx} + a_{16} \gamma_{xy}$$

$$e_y = a_{21} \sigma_x + a_{22} \sigma_y + a_{23} \sigma_z + a_{24} \gamma_{xz} + a_{25} \gamma_{zx} + a_{26} \gamma_{xy}$$

$$e_{xy} = a_{61} \sigma_x + a_{62} \sigma_y + a_{63} \sigma_z + a_{64} \gamma_{xz} + a_{65} \gamma_{zx} + a_{66} \gamma_{xy}$$
The constants $a_{ij}$ are the elastic constants of the material and in general are 36 in number. However, these constants are also the coefficients of a homogeneous quadratic strain energy function and are therefore connected by relations which ensure the existence of that function.\(^2\) These relations are of the form

$$a_{ij} = a_{ji}$$

Thus the number of independent constants is reduced from 36 to 21.

If there exists a plane passing through the point in a body such that the elastic properties of the body are symmetric with respect to that plane, then that point is said to possess one plane of elastic symmetry. A clearer idea of this symmetry can be obtained in the following manner. Let there be drawn a radius vector from the point in the body under consideration such that the point is always the origin of the vector and the length of the vector is proportional to the modulus of elasticity, $E$. Then as the vector rotates about the point, the length of the vector will either increase or decrease depending on the value of $E$ in the particular direction it is pointing. The tip of the vector describes a geometric closed surface and if this surface possesses a plane of symmetry passing through the origin of the vector describing it, the point in question

is said to possess a plane of elastic symmetry. As a consequence of the above, any two directions symmetric with respect to a plane of symmetry will possess equivalent elastic properties, and, further, the number of independent elastic constants at the point will be reduced to 13.  

If the point possesses three mutually perpendicular planes of elastic symmetry, the body is said to be orthotropic at that point in its elastic properties and the number of independent elastic constants is nine in number. Assuming now that the \( x, y, z \) axes are normal to the planes of symmetry, the stress strain relations may then be written as

\[
\begin{align*}
e_x & = a_{11} \sigma_x + a_{12} \sigma_y + a_{13} \sigma_z \\
e_y & = a_{12} \sigma_x + a_{22} \sigma_y + a_{23} \sigma_z \\
e_z & = a_{13} \sigma_x + a_{23} \sigma_y + a_{33} \sigma_z
\end{align*}
\]

\(^3\) For a further discussion of planes of symmetry and consequent reduction of elastic constants, see Lekhnitski (2), p. 14.

\(^4\) Directions normal to planes of symmetry are called principal directions.

\(^5\) See Lekhnitski (2), p. 15.
The above elastic constants may also be expressed in terms of the moduli of elasticity, Poisson's ratio $\nu$ and the modulus of rigidity $G$ in the three principal directions. Thus

\[
a_{11} = \frac{1}{E_1}, \quad a_{12} = \frac{-\nu_{12}}{E_1}, \quad a_{13} = \frac{-\nu_{13}}{E_1}, \quad a_{44} = \frac{1}{G_{23}}
\]

\[
a_{22} = \frac{1}{E_2}, \quad a_{23} = \frac{-\nu_{23}}{E_2}, \quad a_{55} = \frac{1}{G_{13}}
\]

\[
a_{33} = \frac{1}{E_3}, \quad a_{23} = \frac{-\nu_{32}}{E_3}, \quad a_{66} = \frac{1}{G_{12}}
\]

where $G_{ij}$ is the modulus of rigidity. Because of the equality of the coefficients $a_{ij} = a_{ji}$ it also follows that

\[
\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}
\]

\[
\frac{\nu_{13}}{E_1} = \frac{\nu_{31}}{E_3}
\]

\[
\frac{\nu_{23}}{E_2} = \frac{\nu_{32}}{E_3}
\]
b) **An Application to an Orthotropic Shell**

As had been mentioned earlier, the shell being considered in the present analysis is assumed to be orthotropic. The orthotropy of the shell is such that if an equidistant surface is drawn through any point of the shell and if the principal curvilinear coordinates $\alpha$ and $\beta$ are drawn on that surface together with the normal $\gamma$ to the surface, then the normals to the three planes of elastic symmetry of the point will be tangent to the $\alpha$, $\beta$, and $\gamma$ coordinates passing through that point. This type of orthotropy is frequently called curvilinear orthotropy.

Since the thin shell is considered a plane stress case, i.e., $\sigma_z = 0$, the stress strain relations become

\[
\begin{align*}
\epsilon_\alpha &= a_{11} \sigma \alpha + a_{12} \sigma \beta \\
\epsilon_\beta &= a_{12} \sigma \alpha + a_{22} \sigma \beta \\
\epsilon_\gamma &= a_{13} \sigma \alpha + a_{23} \sigma \beta \\
\epsilon_{\alpha\beta} &= a_{66} \sigma_{\alpha\beta}
\end{align*}
\]

(9.1)

---

6 The term "equidistant surface" has been defined in Chapter I, Section 3.
Solving the above equations for the stresses $\sigma_\alpha$ and $\sigma_\beta$

$$\sigma_\alpha = \frac{a_{22}e_\alpha - a_{12}e_\beta}{a_{11}a_{22} - a_{12}^2}$$

$$\sigma_\beta = \frac{a_{11}e_\beta - a_{12}e_\alpha}{a_{11}a_{22} - a_{12}^2}$$ (9.2)

The strain $e_\gamma$ will be an unknown as in the isotropic case, and it also may be evaluated in terms of $e_\alpha$ and $e_\beta$.

$$e_\gamma = \frac{(a_{22}a_{13} - a_{12}a_{23})e_\alpha + (a_{11}a_{23} - a_{12}a_{13})e_\beta}{a_{11}a_{22} - a_{12}^2}$$ (9.3)

In the isotropic case, the stress strain relations are exhibited using Lamé's parameter of elasticity and the volume dilatation. In the orthotropic case, a similar form of relationship is desired. However, in the orthotropic case, there does not exist a definition of Lamé's parameters. Consider then deriving a set of coefficients, $\tau_\alpha$, $\tau_\beta$, $\mu_\alpha$, $\mu_\beta$, $\mu_{\alpha\beta}$ such that these coefficients degenerate to the Lamé parameters in the limiting case of isotropy. In particular, the coefficients are to be such that the stress strain relations may be written in the form

$$\sigma_\alpha = (\tau_\alpha + 2\mu_\alpha)\Delta - 2\mu_\alpha (e_\beta + e_\gamma)$$

$$\sigma_\beta = (\tau_\beta + 2\mu_\beta)\Delta - 2\mu_\beta (e_\alpha + e_\gamma)$$ (9.4)

$$\sigma_{\alpha\beta} = 2\mu_{\alpha\beta} e_{\alpha\beta}$$
where $\Delta$ is the volume dilatation and is again defined as

$$\Delta = e_\alpha + e_\beta + e_\gamma.$$ 

Substituting the expressions for the stresses, equation (2.2) and equation (2.3) for the strain $e_\gamma$ into the equations (2.4) and equating the coefficient of the strains $e_\alpha$ and $e_\beta$, the following definitions for the elastic coefficients $\tau_\alpha$, $\tau_\beta$, $\mu_\alpha$, $\mu_\beta$, and $\mu_{\alpha\beta}$ result.

$$\tau_\alpha = \frac{-a_{12}}{(a_{11}a_{22} - a_{12}^2) + (a_{11}a_{23} - a_{12}a_{23})}$$

$$\tau_\beta = \frac{-a_{12}}{(a_{11}a_{22} - a_{12}^2) + (a_{22}a_{13} - a_{12}a_{23})}$$

$$2\mu_\alpha = \frac{a_{22} + a_{12} + a_{23}}{(a_{11}a_{22} - a_{12}^2) + (a_{11}a_{23} - a_{12}a_{23})}$$

$$2\mu_\beta = \frac{a_{11} + a_{12} + a_{13}}{(a_{11}a_{22} - a_{12}^2) + (a_{22}a_{13} - a_{12}a_{23})}$$

$$\mu_{\alpha\beta} = \frac{1}{a_{66}}$$

As pointed out, the above coefficients are not Lamé's coefficients since the latter exists only in the isotropic case. However, the above coefficients do degenerate to those of Lamé in the limiting case of isotropy as may be verified by a simple check. For the sake
of brevity of nomenclature, the above coefficients will be called the Lamé orthotropic coefficients of elasticity in the remainder of this work.

10. Orthotropic Shell Derivation

The derivation of the orthotropic shell equations will be completely analogous to that of the isotropic case. In the derivation, the form of the equations will be maintained such that upon inspection the equations will degenerate to those derived for the isotropic shell. In this manner, not only will algebraic errors be minimized in that a check will exist at each step of the development, but more importantly, those terms which are peculiar only to orthotropy will be apparent.

Choosing the same element as in the isotropic case, the equations of equilibrium in terms of stresses may again be written as

\[
\frac{\partial}{\partial a} \left( \frac{a}{h_2^2} \right) - \frac{\partial h_{1}^{-1}}{\partial a} \sigma_{\beta} + h_{1} \frac{\partial}{\partial \beta} \left( \frac{a \beta}{h_2^2} \right) + h_{1} \frac{\partial}{\partial \gamma} \left( \frac{a \gamma}{h_1 h_2} \right) + \frac{p_{a}}{h_1 h_2} = 0
\]

\[
\frac{\partial}{\partial \beta} \left( \frac{a}{h_1^2} \right) - \frac{\partial h_{1}^{-1}}{\partial \beta} \sigma_{\alpha} + h_{2} \frac{\partial}{\partial a} \left( \frac{a \alpha}{h_2^2} \right) + h_{2} \frac{\partial}{\partial \gamma} \left( \frac{a \epsilon_{\gamma}}{h_2^2 h_1} \right) + \frac{p_{\beta}}{h_1 h_2} = 0
\]

\[
-\frac{1}{h_2} \frac{\partial h_{1}^{-1}}{\partial \gamma} \sigma_{\alpha} - \frac{1}{h_1} \frac{\partial h_{2}^{-1}}{\partial \gamma} \sigma_{\beta} + \frac{\partial}{\partial a} \left( \frac{a \gamma}{h_2^2} \right) + \frac{\partial}{\partial \beta} \left( \frac{a \gamma}{h_1^2} \right) + \frac{\partial}{\partial \gamma} \left( \frac{a \gamma}{h_1 h_2} \right) + \frac{p_{\gamma}}{h_1 h_2} = 0
\]
Substituting the stress-strain relations, equation (9.4),
and the expressions for strain, equation (4.1) of Chapter I, into
the equilibrium equations and rearranging, the following three
equations result.

\[
\begin{align*}
\frac{(\tau_a + 2\mu_a)}{h_2} \frac{\partial \Delta}{\partial a} - \frac{2\mu_a}{h_1} \frac{\partial \chi}{\partial \beta} + 2\mu_{ABKu} a - 2\mu_a \frac{\partial}{\partial \gamma} \left( \frac{1}{h_2} \frac{\partial u\gamma}{\partial a} \right) \\
+ h_1 \frac{\partial}{\partial \gamma} \left( \frac{\alpha_\gamma}{h_2} \right) + \frac{p_a}{h_1 h_2} + \left[ (\tau_a + 2\mu_a) - (\tau_\beta + 2\mu_\beta) \right] \Delta \frac{\partial h^{-1}}{\partial a} \\
+ 2(\mu_a - \mu_{a\beta}) \frac{\partial}{\partial \beta} \left( h_2 \frac{\partial h^{-1}}{\partial \beta} \right) u_a - 2(\mu_a - \mu_{\beta}) h_1 \frac{\partial u_a}{\partial a} \frac{\partial h^{-1}}{\partial a} \\
- 2(\mu_a - \mu_{a\beta}) \frac{\partial}{\partial a} \left( u_\gamma \frac{\partial h^{-1}}{\partial \gamma} \right) + 2(\mu_\beta - \mu_{a\beta}) h_1 h_2 \frac{\partial u_\gamma}{\partial a} \frac{\partial h^{-1}}{\partial \beta} \frac{\partial h^{-1}}{\partial \beta} u_\beta \\
- 2(\mu_a - \mu_\beta) \frac{\partial}{\partial \gamma} \left( u_\gamma \frac{\partial h^{-1}}{\partial \gamma} \right) = 0
\end{align*}
\] (10.1)
\[
\begin{align*}
\frac{\tau_\beta + 2\mu_\beta}{h_1} &\quad \frac{\partial \Delta}{\partial \beta} + \frac{2\mu_\alpha}{h_2} \frac{\partial \chi}{\partial \alpha} + 2\mu_\beta ABKu_\beta - 2\mu_\beta \frac{\partial}{\partial \gamma} \left( \frac{1}{h_1} \frac{\partial u_\gamma}{\partial \beta} \right) \\
+ h_2 &\quad \frac{\partial}{\partial \gamma} \left( \frac{\sigma_{\beta \gamma}^2}{h_1^2} \right) + \frac{\partial \beta}{h_1 h_2} + \left[ \left( \tau_\alpha + 2\mu_\alpha \right) - \left( \tau_\alpha + 2\mu_\alpha \right) \right] \Delta \frac{\partial h_{1,1}^{-1}}{\partial \beta} \\
+ 2(\mu_\beta - \mu_\alpha) &\quad \frac{\partial}{\partial \alpha} \left( \frac{\partial h_2}{\partial \alpha} \right) u_\beta - 2(\mu_\beta - \mu_\alpha) h_2 \frac{\partial h_1^{-1}}{\partial \beta} \frac{\partial u_\beta}{\partial \beta} \\
- 2(\mu_\beta - \mu_\alpha) &\quad \frac{\partial}{\partial \gamma} \left( \frac{\partial h_{1,1}^{-1}}{\partial \beta} \right) u_\alpha \\
- 2(\mu_\beta - \mu_\alpha) &\quad \frac{\partial}{\partial \gamma} \left( \frac{\partial h_1^{-1}}{\partial \beta} \right) u_\alpha = 0 \\
\end{align*}
\]

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\[
\begin{align*}
-2 \left[ \frac{\tau_\alpha + 2\mu_\alpha}{\tau_\beta + 2\mu_\beta} \right] (H + Ky) AB \Delta + 2\mu_\alpha \frac{\partial}{\partial \alpha} (Bk_2 u_\alpha) \\
+ 2\mu_\beta \frac{\partial}{\partial \beta} (Ak_1 u_\beta) + \frac{4(\mu_\alpha + \mu_\beta)}{2} ABKu_\gamma + \frac{4(\mu_\alpha + \mu_\beta)}{2} AB(H + Ky) \frac{\partial u_\gamma}{\partial \gamma} \\
+ \frac{\partial}{\partial \alpha} \left( \frac{\sigma_{\gamma \alpha}^2}{h_2} \right) + \frac{\partial}{\partial \beta} \left( \frac{\sigma_{\gamma \beta}^2}{h_1} \right) + \frac{\partial}{\partial \gamma} \left( \frac{\sigma_{\gamma \gamma}^2}{h_1 h_2} \right) \\
- \left[ \left( \tau_\alpha + 2\mu_\alpha \right) - \left( \tau_\beta + 2\mu_\beta \right) \right] ABL \Delta - 2(\mu_\alpha - \mu_\beta) Bk_2 \frac{\partial u_\alpha}{\partial \alpha} \\
- 2(\mu_\beta - \mu_\alpha) Ak_1 \frac{\partial u_\beta}{\partial \beta} + 2(\mu_\alpha - \mu_\beta) ABL \frac{\partial u_\gamma}{\partial \gamma} = 0 \\
\end{align*}
\]

(10.2)
In the above three equations, the symbols used have the same definitions as in the isotropic case. Thus the dilatation $\Delta$ is given as

$$\Delta = e_a + e_\beta + e_\gamma$$

and the normal rotation $\chi$ as

$$2\chi = h_1 h_2 \left[ \frac{\partial}{\partial a} \left( \frac{u_3}{h_2} \right) - \frac{\partial}{\partial \beta} \left( \frac{u_a}{h_1} \right) \right]$$

Again, as in the isotropic case, it will be convenient to represent the dilatation and the rotation by means of truncated power series. In particular,

$$\chi = \chi_o + \chi_1 \gamma$$

$$\Delta = \Delta_o + \Delta_1 \gamma$$

Since $\chi$ is defined in terms of the tangential displacements $u_a$ and $u_\beta$, its expression is unchanged from the isotropic case. Thus from equation (5.16) of the previous chapter

$$\chi_o = \frac{1}{2AB} \left[ \frac{\partial}{\partial a} (Bv) - \frac{\partial}{\partial \beta} (Au) \right]$$

$$\chi_1 = - \frac{(k_1 - k_2)}{2} \left[ \frac{A}{B} \frac{\partial}{\partial \beta} \left( \frac{u}{A} \right) + \frac{B}{A} \frac{\partial}{\partial a} \left( \frac{\gamma}{B} \right) \right]$$

The dilatation $\Delta$ as well as the strain $e_\gamma$, though defined in terms of the strains and the displacement $u_\gamma$, respectively, are not independent of the elastic properties. The plane stress assumption
inherent in thin shells results in the strain $e_\gamma$ being a function of the normal strains $e_\alpha$, $e_\beta$ as well as the elastic constants of the material.

In the isotropic case, the ultimate dependence of the functions $\Delta$, $e_\gamma$ and $u_\gamma$ on the elastic properties and the middle surface displacements $u$, $v$, and $w$ is fairly simple to obtain. However, because of the more complicated stress-strain relations, such is not the case for the orthotropic shell. To maintain a continuity of thought through the derivation, these three unknown functions will not be developed until last.

The three equations (3.1), (3.2), and (3.3) represent the equilibrium equation in terms of displacements of an element of a shell situated as in the isotropic case. In order to obtain equations in terms of the middle surface displacements, the procedure used in the isotropic case will be followed. That is, five equations of equilibrium will be obtained by integration in terms of the middle surface displacements $u$, $v$, and $w$ as well as the transverse shearing forces $Q_1$ and $Q_2$. 
Substituting the displacement expressions derived from the Kirchhoff hypothesis

\[ u_a = (1 + k_1 \gamma)u - \frac{Y}{A} \frac{\partial w}{\partial a} \]

\[ u_b = (1 + k_2 \gamma)v - \frac{Y}{B} \frac{\partial w}{\partial b} \]

and the truncated series

\[ \Delta = \Delta_0 + \Delta_1 \gamma \]

\[ \chi = \chi_0 + \chi_1 \gamma \]

into equations (3.1), (3.2), and (3.3) and proceeding as outlined in the preceding paragraph, the following equations result.

\[ B(\tau_a + 2\mu_a) \left( \frac{\partial \Delta_0}{\partial a} + \frac{k_2 \delta^2}{12} \frac{\partial \Delta_1}{\partial a} \right) + 2\mu_a ABKu - 2\mu_a A(\Delta_0 \frac{\partial \chi_0}{\partial b} + \frac{k_1 \delta^2}{12} \frac{\partial \chi_1}{\partial b} + \frac{\partial \Delta_0}{\partial a}) \]

\[ \frac{2\mu_a}{\delta} \int_{-\delta/2}^{\delta/2} \frac{\partial}{\partial \gamma} \left( \frac{1}{h_2} \frac{\partial y}{\partial a} + \frac{ABk_1 N_1}{\delta} + \frac{ABX}{\delta} + 2(\mu_a - \mu_\beta) \right) \frac{\partial u}{\partial a} - \frac{1}{A} \frac{\partial B}{\partial a} \frac{\partial u}{\partial a} - 2(\mu_a - \mu_\beta) \frac{\partial^2 v}{\partial a \partial b} + 2(\mu_a - \mu_\beta) \]

\[ \frac{1}{AB} \frac{\partial A}{\partial b} \frac{\partial B}{\partial a} v - \frac{2(\mu_a - \mu_\beta)}{\delta} \int_{-\delta/2}^{\delta/2} \frac{\partial}{\partial \gamma} \left( u \frac{\partial h^{-1}}{\partial a} \right) \frac{\partial v}{\partial a} d\gamma = 0 \]  \hspace{1cm} (10.4)
\[
A(\tau + 2\mu_\beta)(\frac{\partial \Delta}{\partial \beta} + \frac{k_1\delta^2}{12} \, \frac{\partial \Delta}{\partial \beta} + 2\mu_\beta \, ABKv + 2\mu_a \beta B(\frac{\partial \chi_o}{\partial a} + \frac{k_1\delta^2}{12} \, \frac{\partial \chi_1}{\partial a})
\]

\[-\frac{2\mu_\beta}{\delta} \int_{-\delta/2}^{\delta/2} \frac{\partial}{\partial \gamma} \left( \frac{1}{h_1} \, \frac{\partial u}{\partial \beta} \right) d\gamma + \frac{ABk_2N_2}{\delta} + \frac{ABY}{\delta} + 2(\mu_\beta - \mu_a \beta)
\]

\[-\frac{\partial}{\partial a} \left( \frac{1}{A} \, \frac{\partial B}{\partial a} \right) \nu + \left[ (\tau + 2\mu_\beta) - (\tau + 2\mu_a) \right] \frac{\partial A}{\partial a} \left( \Delta^o + \frac{k_1\delta^2}{12} \, \Delta_1 \right)
\]

\[-2(\mu_\beta - \mu_a ) \frac{1}{B} \frac{\partial A}{\partial B} \frac{\partial v}{\partial \beta} - 2(\mu_\beta - \mu_a \beta) \frac{\partial^2 u}{\partial a \partial \beta} + 2(\mu_a - \mu_a \beta)
\]

\[
\frac{1}{AB} \frac{\partial B}{\partial a} \frac{\partial A}{\partial \beta} u - 2 \frac{(\mu_\beta - \mu_a )}{\delta} \int_{-\delta/2}^{\delta/2} \frac{\partial}{\partial \gamma} \left( u \frac{\partial h_1^{-1}}{\partial \beta} \right) d\gamma = 0 \quad (10.5)
\]

\[-2[(\tau + 2\mu_a) + (\tau + 2\mu_\beta)] AB(H \Delta^o + \frac{k_1\delta^2}{12} \, \Delta_1) + 2\mu_a \frac{\partial}{\partial a} (Bk_2u)
\]

\[+ 2\mu_\beta \frac{\partial}{\partial \beta} (Ak_1v) + 4(\frac{\mu_a + \mu_\beta}{2}) \frac{ABK}{\delta} \int_{-\delta/2}^{\delta/2} u \gamma d\gamma + 2 \frac{(\mu_a + \mu_\beta)}{\delta}
\]

\[
\int_{-\delta/2}^{\delta/2} (H + KY) \frac{\partial u}{\partial \gamma} d\gamma + \frac{\partial}{\partial a} \left( \frac{BQ_1}{\delta} \right) + \frac{\partial}{\partial \beta} \left( \frac{A\Omega_2}{\delta} \right) + \frac{ABZ}{\delta}
\]

\[-[\tau + 2\mu_a] - (\tau + 2\mu_\beta) ABL\Delta^o - 2(\mu_a - \mu_\beta) Bk_2 \frac{\partial u}{\partial a}
\]

\[+ 2 \frac{(\mu_a - \mu_\beta)}{\delta} ABL \int_{-\delta/2}^{\delta/2} \frac{\partial u}{\partial \gamma} d\gamma = 0 \quad (10.6)
\]
\[ \begin{align*}
&\left(\tau_a + 2\mu_a\right)B \left(\frac{\partial \Delta_1}{\partial \alpha} + k_2 \frac{\partial \Delta}{\partial \alpha}\right) + 2\mu_a \ ABK \left(k_1 u - \frac{1}{A} \frac{\partial w}{\partial \alpha}\right) \\
&- 2\mu_{a\beta} \left(\frac{\partial \chi_1}{\partial \beta} + k_1 \frac{\partial \chi}{\partial \beta}\right) - \left(\frac{24\mu_a}{\delta^2} \int_{-\delta}^{\delta} \gamma \frac{\partial}{\partial \gamma} \left(\frac{1}{h_2^2} \frac{\partial w}{\partial \alpha}\right) \ dy - 12 \frac{ABN_1}{\delta^3}\right) \\
&+ \left[\left(\tau_a + 2\mu_a\right) - \left(\tau_\beta + 2\mu_\beta\right)\right] \frac{\partial B}{\partial \alpha} \left(\Delta_1 + k_2 \Delta\right) + 2\mu_a - \mu_\beta \\
&- 2 \frac{\partial}{\partial \beta} \left(\frac{1}{B} \frac{\partial A}{\partial \beta}\right) \left(k_1 u - \frac{1}{A} \frac{\partial w}{\partial \alpha}\right) \\
&- 2 \left(\frac{\mu_a - \mu_\beta}{A} \right) \frac{\partial B}{\partial \alpha} \frac{\partial}{\partial \alpha} \left(k_1 u - \frac{1}{A} \frac{\partial w}{\partial \alpha}\right) - 2 \left(\mu_a - \mu_{a\beta}\right) \frac{\partial^2}{\partial a \partial \beta} \left(k_2 v - \frac{1}{B} \frac{\partial w}{\partial \beta}\right) \\
&+ 2 \left(\frac{\mu_\beta - \mu_{a\beta}}{AB} \right) \frac{\partial A}{\partial \beta} \frac{\partial B}{\partial \alpha} \left(k_2 v - \frac{1}{B} \frac{\partial w}{\partial \beta}\right) - 24 \frac{\mu_a - \mu_\beta}{\delta^3} \int_{-\delta}^{\delta} \gamma \frac{\partial}{\partial \gamma} \left(u^2 \frac{\partial w}{\partial \alpha}\right) \ dy \\
&+ \left[\left(\tau_a + 2\mu_a\right) - \left(\tau_\beta + 2\mu_\beta\right)\right] B \frac{\partial k_2}{\partial \alpha} \Delta_1 = 0 \\
&\text{(10.7)}
\end{align*} \]
\[(\tau_\beta + 2\mu_\beta) A \left( \frac{\partial A}{\partial \beta} + k_1 \frac{\partial o}{\partial \beta} \right) + 2\mu_\beta \text{ABK} \left( k_y - \frac{1}{B} \frac{\partial w}{\partial \beta} \right) + 2\mu_{a\beta} B \left( \frac{\partial \chi_1}{\partial a} + k_2 \frac{\partial \chi_0}{\partial a} \right) \]

\[- \frac{24\mu_\beta}{\delta^3} \int_{\delta 2}^{\delta 2} \frac{\partial}{\partial \gamma} \left( \frac{1}{h_1} \frac{\partial u_y}{\partial \beta} \right) \mathrm{d}\gamma - 12 \frac{\text{ABN}^2}{\delta^3} + \left[ (\tau_\beta + 2\mu_\beta) - (\tau_a + 2\mu_a) \right] \]

\[\frac{\partial A}{\partial \beta} (\Delta_1 + k_1 \Delta_o) + 2(\mu_\beta - \mu_{a\beta}) \frac{\partial}{\partial a} \left( \frac{1}{A} \frac{\partial B}{\partial \beta} \right) (k_y - \frac{1}{B} \frac{\partial w}{\partial \beta}) - 2 \frac{(\mu_\beta - \mu_{a\beta})}{B} \]

\[\frac{\partial A}{\partial \beta} \frac{\partial}{\partial \beta} (k_y - \frac{1}{B} \frac{\partial w}{\partial \beta}) - 2(\mu_\beta - \mu_{a\beta}) \frac{\partial}{\partial \alpha \partial \beta} (k_1 u - \frac{1}{A} \frac{\partial w}{\partial \alpha}) + 2 \frac{(\mu_\beta - \mu_{a\beta})}{AB} \frac{\partial B}{\partial \alpha} \frac{\partial A}{\partial \beta} (k_1 u - \frac{1}{A} \frac{\partial w}{\partial \alpha}) \]

\[- \frac{24}{\delta^3} \int_{\delta 2}^{\delta 2} \gamma \frac{\partial}{\partial \gamma} (u_y) \left( \frac{\partial h_1}{\partial \beta} \right) \mathrm{d}\gamma + \left[ (\tau_\beta + 2\mu_\beta) - (\tau_a + 2\mu_a) \right] \frac{\partial A}{\partial \beta} = 0 \]

(10.8)

Consider now determining the analytical expressions for \(\Delta, e_y, \) and \(u_y.\)

**Displacement** \(u_y\) and **strain** \(e_y\)

From the condition of plane stress, the expression for \(e_y\) as found in equation (2.3) is given as

\[e_y = \frac{(a_{22}a_{13} - a_{12}a_{23})e_a}{(a_{11}a_{22} - a_{12}^2)} + \frac{(a_{11}a_{23} - a_{12}a_{13})e_\beta}{(a_{11}a_{22} - a_{12}^2)} \]
or more conveniently

\[ e_\gamma = C_a e_a + C_\beta e_\beta \]

However, \( e_\gamma \) may also be written in terms of the volume dilatation by adding and subtracting \( C_a e_\gamma \) and \( C_\beta e_\beta \). Thus

\[ e_\gamma = \frac{C_a}{(1 + C_a)} \Delta + \frac{(C_\beta - C_a)}{(1 + C_a)} e_\beta \]

It must be apparent that \( e_\gamma \) may just as easily be expressed as

\[ e_\gamma = \frac{C_\beta}{(1 + C_\beta)} \Delta + \frac{(C_a - C_\beta)}{(1 + C_\beta)} e_a \]

In the isotropic case, the expression for the strain \( e_\gamma \) is symmetrical in the subscripts \( a \) and \( \beta \). In order that a similar situation exist for the orthotropic case, the two expressions for \( e_\gamma \) will be added. The resulting expressions for the strain then becomes

\[ e_\gamma = \frac{1}{2} \left[ \frac{C_a}{(1 + C_a)} + \frac{C_\beta}{(1 + C_\beta)} \right] \Delta + \frac{1}{2} \left[ \frac{(C_\beta - C_a)}{(1 + C_a)} e_\beta + \frac{(C_a - C_\beta)}{(1 + C_\beta)} e_a \right] \]

The constants in the above expression can be expressed in terms of the orthotropic Lame' coefficients. However, all attempts to do so have led to relations which are so complicated as to be useless. For simplicity, therefore, it seems to be preferable
to introduce two new (but not independent) elastic constants $\eta_\alpha$ and $\eta_\beta$ defined as follows.

$$\eta_\alpha = \frac{C_\alpha}{(1 + C_\alpha)}$$

$$\eta_\beta = \frac{C_\beta}{(1 + C_\beta)}$$

(10.9)

Thus the expression for $e_\gamma$ may be written as

$$e_\gamma = \frac{(\eta_\alpha + \eta_\beta)}{2} \Delta + \frac{\eta_\alpha - \eta_\beta}{2(1 - \eta_\beta)} e_\beta + \frac{\eta_\beta - \eta_\alpha}{2(1 - \eta_\alpha)} e_\alpha$$

(10.10)

In order to solve for the strain $e_\gamma$, the expression for $u_\gamma$ must be first determined. Substituting the expressions for the strains into equation (10.10)

$$e_\alpha = h_1 \frac{\partial u_\alpha}{\partial \alpha} + h_1 h_2 \frac{\partial h^{-1}}{\partial \beta} u_\beta + h_1 \frac{\partial h^{-1}}{\partial \gamma} u_\gamma$$

$$e_\beta = h_2 \frac{\partial u_\beta}{\partial \beta} + h_1 h_2 \frac{\partial h^{-1}}{\partial \alpha} u_\alpha + h_2 \frac{\partial h^{-1}}{\partial \gamma} u_\gamma$$

$$e_\gamma = \frac{\partial u_\gamma}{\partial \gamma}$$
and rearranging, a first order differential equation results.

\[
\frac{1}{h_1 h_2} \frac{\partial u}{\partial \gamma} = - \left[ \frac{\eta_a - \eta_a}{2(1 - \eta_a)} \frac{\partial h_2^{-1}}{\partial \gamma} + \frac{\eta_a - \eta_a}{2(1 - \eta_a)} \frac{\partial h_2^{-1}}{\partial \gamma} \right] u_\gamma \\
+ \frac{(\eta_a + \eta_a)}{2} \frac{\Delta}{h_1 h_2} + \frac{(\eta_a - \eta_a)}{2(1 - \eta_a)} \left[ \frac{1}{h_1} \frac{\partial u_a}{\partial \gamma} + \frac{\partial h_2^{-1}}{\partial \gamma} \right] \\
+ \frac{(\eta_a - \eta_a)}{2(1 - \eta_a)} \left[ \frac{1}{h_2} \frac{\partial u_a}{\partial \gamma} + \frac{\partial h_2^{-1}}{\partial \gamma} \right]
\]  

(10.11)  

The resultant solution to equation (10.11) will consist of the homogeneous and particular solutions. However, before such a solution can be found, the functions \( \Delta, h_1, h_2, u_a, \) and \( u_\beta \) must be expressed as functions of the variable \( \gamma \).

Expressions for \( u_a \) and \( u_\beta \), which result from the Kirchoff hypothesis, have previously been stated. The expression for the dilatation \( \Delta \) will again be the truncated power series

\[
\Delta = \Delta_o + \Delta_1 \gamma
\]

while in Chapter I the expressions for \( h_1 \) and \( h_2 \) have been shown to be

\[
h_1 = \frac{1}{A(1 + k_1 \gamma)}
\]

\[
h_2 = \frac{1}{B(1 + k_2 \gamma)}
\]
Substitution of all these expressions into equation (10.11)

leads to the following linear differential equation

\[
(1 + 2K_v + K_v^2) \frac{\partial u}{\partial y} - \frac{1}{2} \left\{ \left[ \frac{(\eta - \eta_a)k_2}{(1 - \eta_a)} + \frac{(\eta - \eta)k_1}{(1 - \eta)} \right] + \left[ \frac{(\eta - \eta_a)}{(1 - \eta_a)} k_1 k_2 + \frac{(\eta - \eta)}{(1 - \eta)} k_1 k_2 \right] \gamma \right\} \gamma = F(a, \beta, \gamma) \tag{10.12}
\]

where

\[
F(\gamma) = \frac{(\eta + \eta_a)}{2} \Delta_o + \frac{(\eta - \eta_a)}{2(1 - \eta)} \frac{1}{B} \frac{\partial v}{\partial \beta} + \frac{(\eta - \eta)}{2(1 - \eta_a)} \frac{1}{A} \frac{\partial u}{\partial \alpha} + \frac{(\eta - \eta_a)}{2(1 - \eta_a)} \frac{1}{B} \frac{\partial (k_2 v)}{\partial (k_2 \gamma)} \\
\frac{1}{AB} \frac{\partial B}{\partial \alpha} u + \frac{(\eta - \eta_a)}{2(1 - \eta_a)} \frac{1}{A} \frac{\partial A}{\partial \beta} v + \left[ (\eta_a + \eta) \Delta_o + \frac{(\eta - \eta)}{2(1 - \eta_a)} \right] \frac{(\eta + \eta_a)}{2} \frac{1}{B} \frac{\partial (k_2 v)}{\partial (k_2 \gamma)} \\
+ \frac{(\eta - \eta_a)}{2(1 - \eta_a)} \frac{1}{A} \frac{\partial \gamma}{\partial \alpha} (k_1 u) - \frac{(\eta - \eta_a)}{2(1 - \eta_a)} \frac{1}{B} \frac{\partial \gamma}{\partial \beta} \frac{1}{B} \frac{\partial w}{\partial \delta} - \frac{(\eta - \eta)}{2(1 - \eta_a)} \frac{1}{A} \frac{\partial \gamma}{\partial \sigma} \frac{1}{A} \frac{\partial w}{\partial \delta} \\
+ \frac{(\eta - \eta_a)}{2(1 - \eta_a)} \frac{k_1}{B} \frac{\partial v}{\partial \beta} + \frac{(\eta - \eta_a)}{2(1 - \eta_a)} \frac{k_2}{A} \frac{\partial u}{\partial \alpha} + \frac{(\eta - \eta)}{2(1 - \eta_a)} \frac{1}{A} \frac{\partial \gamma}{\partial \alpha} k_1 u + \frac{(\eta - \eta)}{2(1 - \eta_a)} \frac{1}{B} \frac{\partial \gamma}{\partial \beta} \frac{1}{B} \frac{\partial w}{\partial \delta} \\
\frac{1}{AB} \frac{\partial A}{\partial \beta} k_2 v + \frac{(\eta - \eta)}{2(1 - \eta_a)} \frac{1}{A B} \frac{\partial A}{\partial \sigma} \frac{1}{A B} \frac{\partial \gamma}{\partial \sigma} \frac{1}{A B} \frac{\partial w}{\partial \delta} \\
\left[ \frac{\eta + \eta_a}{2} \Delta_1 + \frac{\eta_a + \eta}{2(1 - \eta_a)} \frac{k_1}{B} \frac{\partial \gamma}{\partial \beta} (k_2 v) + \frac{(\eta - \eta_a)}{2(1 - \eta_a)} \frac{1}{A} \frac{\partial \gamma}{\partial \alpha} \frac{1}{A} \frac{\partial w}{\partial \delta} \right] \gamma + \\
\frac{(\eta - \eta_a)}{2(1 - \eta_a)} \frac{k_1}{B} \frac{\partial \gamma}{\partial \beta} \frac{1}{B} \frac{\partial w}{\partial \delta} - \frac{(\eta - \eta_a)}{2(1 - \eta_a)} \frac{k_2}{A} \frac{\partial \gamma}{\partial \alpha} \frac{1}{A} \frac{\partial w}{\partial \delta} + \frac{(\eta - \eta)}{2(1 - \eta_a)} \frac{1}{A B} \frac{\partial \gamma}{\partial \alpha} \frac{1}{A B} \frac{\partial w}{\partial \delta} \frac{1}{2} \\
+ \frac{(\eta - \eta_a)}{2(1 - \eta_a)} \frac{1}{A B} \frac{\partial \gamma}{\partial \beta} \frac{1}{A B} \frac{\partial \gamma}{\partial \alpha} \frac{1}{A B} \frac{\partial \gamma}{\partial \delta} \frac{1}{2} \frac{\partial \gamma}{\partial \gamma} \gamma + \\
\left[ \frac{\eta + \eta_a}{2} \Delta_1 \right] \gamma^3
\]
For the purpose of obtaining a solution, the differential equation for $u_\gamma$ may be written symbolically as

\[ (1 + 2H\gamma + K\gamma^2) \frac{\partial u_\gamma}{\partial \gamma} - (b_o + b_1 \gamma) u_\gamma = F_o + F_1 \gamma + F_2 \gamma^2 + F_3 \gamma^3 \]  

(10.14)

where $F_i = F_i(a, \beta)$

The solution to the above equation may be obtained in a straightforward manner. Thus

\[ u_\gamma = A e^{p(\gamma)} + e^{p(\gamma)} \int \frac{F_o + F_1 \gamma + F_2 \gamma^2 + F_3 \gamma^3}{1 + 2H\gamma + K\gamma^2} e^{-p(\gamma)} \, d\gamma \]  

(10.15)

where $A$ is some undetermined constant and

\[ p(\gamma) = \int \frac{(b_o + b_1 \gamma)}{(1 + 2H\gamma + K\gamma^2)} \, d\gamma \]

In order to obtain a polynomial form of solution for $u_\gamma$, the various functions involved in its expression will be expanded in truncated power series. Thus the denominator of the integrand of the expression for $p(\gamma)$ may be written as

\[ \frac{1}{(1 + 2H\gamma + K\gamma^2)^2} = 1 - 2H\gamma - (4H^2 + 3K)\gamma^2 \]
Hence the expression for $p(\gamma)$ becomes

$$p(\gamma) = b_0 \gamma - (H b_0 + \frac{b_1}{2}) \gamma^2$$

where the expression has been truncated to include only second order terms of $\gamma$.

The expressions for the exponentials may now be evaluated and are given below in their truncated form.

$$e^{p(\gamma)} = 1 + b_0 \gamma - (b_0 H + \frac{b_1}{2} - \frac{b_0^2}{2}) \gamma^2$$

$$e^{-p(\gamma)} = 1 - b_0 \gamma + (b_0 H + \frac{b_1}{2} + \frac{b_0^2}{2}) \gamma^2$$

With the series substitutions indicated above and the boundary condition that $u_\gamma = w$ at $\gamma = 0$, a polynomial form of equation (10.15) is readily obtained. Thus truncating this polynomial to order two in $\gamma$, the following solution for the displacement $u_\gamma$ results.
\[ u \gamma = w + \left( \frac{(\eta_\alpha + \eta_\beta)}{2} \Delta + \frac{(\eta_\beta - \eta_\alpha)}{2(1-\eta_\beta)} \frac{1}{B} \frac{\partial v}{\partial \beta} + \frac{(\eta_\alpha - \eta_\beta)}{2(1-\eta_\alpha)} \frac{1}{A} \frac{\partial u}{\partial \alpha} + \frac{(\eta_\alpha - \eta_\beta)}{2(1-\eta_\alpha)} \frac{1}{AB} \frac{\partial A}{\partial \alpha} \frac{\partial A}{\partial \alpha} \right) \gamma \\
+ \left[ \frac{1}{AB} \frac{\partial A}{\partial \beta} + \frac{(\eta_\beta - \eta_\alpha)}{2(1-\eta_\beta)} \frac{1}{AB} \frac{\partial B}{\partial \alpha} + \frac{(\eta_\alpha - \eta_\beta)}{2(1-\eta_\alpha)} \frac{1}{A} \frac{\partial \gamma}{\partial \alpha} + \frac{(\eta_\alpha - \eta_\beta)}{2(1-\eta_\alpha)} \frac{1}{B} \frac{\partial \gamma}{\partial \beta} \right] \gamma^2 \\
+ \left[ (\eta_\alpha + \eta_\beta) \Delta + \frac{(\eta_\beta - \eta_\alpha)}{4(1-\eta_\beta)^2} \frac{(\eta_\alpha + \eta_\beta)}{2} k_2 \Delta + \frac{(\eta_\alpha - \eta_\beta)}{4(1-\eta_\alpha)} \frac{(\eta_\alpha + \eta_\beta)}{2} k_1 \Delta \right] \\
- (\eta_\alpha + \eta_\beta) \Delta + \frac{(\eta_\beta - \eta_\alpha)}{2(1-\eta_\beta)} \frac{1}{B} \frac{\partial v}{\partial \beta} + \frac{(\eta_\alpha - \eta_\beta)}{2(1-\eta_\alpha)^2} \frac{1}{A} \frac{\partial u}{\partial \alpha} \left( k_1 u \right) \\
+ \frac{(\eta_\beta - \eta_\alpha)}{2(1-\eta_\beta)} \frac{1}{B} \frac{\partial v}{\partial \beta} + \frac{(\eta_\alpha - \eta_\beta)}{2(1-\eta_\alpha)^2} \frac{1}{A} \frac{\partial u}{\partial \alpha} \left( k_2 v \right) \\
+ \frac{(\eta_\beta - \eta_\alpha)}{1-\eta_\beta} \frac{1}{AB} \frac{\partial u}{\partial \alpha} k_1 + \frac{(\eta_\beta - \eta_\alpha)^2}{4(1-\eta_\beta)^2} \frac{1}{B} \frac{\partial \gamma}{\partial \beta} + \frac{(\eta_\beta - \eta_\alpha)(\eta_\alpha - \eta_\beta)}{4(1-\eta_\beta)(1-\eta_\alpha)} \frac{1}{B} \frac{\partial \gamma}{\partial \alpha} k_1 \gamma \\
+ \frac{(\eta_\beta - \eta_\alpha)^2}{4(1-\eta_\beta)^2} \frac{1}{AB} \frac{\partial u}{\partial \alpha} k_2 + \frac{(\eta_\beta - \eta_\alpha)(\eta_\alpha - \eta_\beta)}{4(1-\eta_\beta)(1-\eta_\alpha)} \frac{1}{AB} \frac{\partial \gamma}{\partial \alpha} k_1 \gamma + \frac{(\eta_\beta - \eta_\alpha)(\eta_\alpha - \eta_\beta)}{4(1-\eta_\beta)(1-\eta_\alpha)} \frac{1}{AB} \frac{\partial \gamma}{\partial \alpha} k_2 \gamma \\
+ \frac{1}{AB} \frac{\partial A}{\partial \beta} k_2 \gamma + \frac{(\eta_\beta - \eta_\alpha)^2}{4(1-\eta_\alpha)^2} \frac{1}{AB} \frac{\partial A}{\partial \beta} k_1 \gamma \right] \gamma^2 \\
- \frac{(\eta_\beta - \eta_\alpha)}{1-\eta_\alpha} \frac{1}{AB} \frac{\partial B}{\partial \alpha} \frac{\partial \gamma}{\partial \alpha} Hu \]
Examination of the last coefficient of $\gamma^2$ in the preceding equation indicates that each term in that coefficient may be shown to be of order $\delta k$, or smaller than the remaining terms in the expression for $u_\gamma$. Since the basic error in shell theory employing the Kirchoff hypotheses is of order $\delta k$, then the last coefficient of $\gamma^2$ may be discarded. Hence the simplified equation for the displacement becomes

$$u_\gamma = w + \left[ \frac{(\eta_\alpha + \eta_\beta)}{2} \Delta_0 + \frac{(\eta_\beta - \eta_\alpha)}{2(1-\eta_\alpha)} B \frac{\partial v}{\partial \beta} + \frac{(\eta_\alpha - \eta_\beta)}{2(1-\eta_\alpha)} A \frac{\partial u}{\partial a} + \frac{(\eta_\alpha - \eta_\beta)}{2(1-\eta_\alpha)} \right] \gamma^2$$

$$- \frac{(\eta_\beta - \eta_\alpha)}{2(1-\eta_\alpha)} \frac{1}{AB} \frac{\partial B}{\partial a} u + \frac{(\eta_\alpha - \eta_\beta)}{2(1-\eta_\alpha)} \frac{1}{AB} \frac{\partial A}{\partial a} v + \frac{(\eta_\alpha - \eta_\beta)}{2(1-\eta_\alpha)} k_2 w + \frac{(\eta_\alpha - \eta_\beta)}{2(1-\eta_\alpha)} k_1 w \right] \gamma$$

$$+ \left[ \frac{(\eta_\alpha + \eta_\beta)}{2} \Delta_1 - \frac{(\eta_\beta - \eta_\alpha)}{2(1-\eta_\alpha)} \frac{1}{B} \frac{\partial}{\partial \beta} \left( \frac{1}{B} \frac{\partial w}{\partial \beta} \right) - \frac{(\eta_\alpha - \eta_\beta)}{2(1-\eta_\alpha)} \frac{1}{B} \frac{\partial A}{\partial a} \right]$$

$$- \frac{(\eta_\beta - \eta_\alpha)}{2(1-\eta_\alpha)} \frac{1}{AB} \frac{\partial B}{\partial a} \frac{\partial w}{\partial a} - \frac{(\eta_\alpha - \eta_\beta)}{2(1-\eta_\alpha)} \frac{1}{AB} \frac{\partial A}{\partial \beta} \frac{\partial w}{\partial \beta} \right] \frac{\gamma^2}{2} \right]$$

(10.16)
The strain $e_\gamma$ may now be found by differentiating the above expression. Thus

$$e_\gamma = \left[ \frac{(\eta_a + \eta_\beta)}{2} \right] \Delta_0 + \frac{(\eta_\beta - \eta_a)}{2(l-\eta_\beta)} \frac{1}{B} \frac{\partial v}{\partial \beta} + \frac{(\eta_a - \eta_\beta)}{2(l-\eta_a)} \frac{1}{A} \frac{\partial u}{\partial a} + \frac{(\eta_\beta - \eta_a)}{2(l-\eta_\beta)} \frac{1}{AB} \frac{\delta B}{\delta a} u$$

$$+ \frac{(\eta_\beta - \eta_a)}{2(l-\eta_\beta)} \frac{1}{AB} \frac{\partial A}{\partial \beta} v + \frac{(\eta_a - \eta_\beta)}{2(l-\eta_a)} k_2 w + \frac{(\eta_a - \eta_\beta)}{2(l-\eta_a)} k_1 w'] + \left[ \frac{(\eta_a + \eta_\beta)}{2} \Delta_1 \right]$$

$$- \frac{(\eta_\beta - \eta_a)}{2(l-\eta_\beta)} \frac{1}{B} \frac{\partial}{\partial \beta} \left( \frac{1}{B} \frac{\partial w}{\partial \beta} \right) - \frac{(\eta_a - \eta_\beta)}{2(l-\eta_a)} \frac{1}{A} \frac{\partial}{\partial a} \left( \frac{1}{A} \frac{\partial w}{\partial a} \right)$$

$$- \frac{(\eta_\beta - \eta_a)}{2(l-\eta_\beta)} \frac{1}{A} \frac{\partial B}{\partial a} \frac{\partial w}{\partial a} - \frac{(\eta_a - \eta_\beta)}{2(l-\eta_a)} \frac{1}{AB} \frac{\partial A}{\partial \beta} \frac{\partial w}{\partial \beta} \right] \gamma \quad (10.17)$$

### Volume Dilatation $\Delta$

The volume dilatation $\Delta$ is defined as

$$\Delta = e_a + e_\beta + e_\gamma$$

Since the strains are defined in terms of the displacements, the dilatation may also be written as

$$\Delta = h_1 h_2 \left[ \frac{\partial}{\partial a} \left( \frac{u_a}{h_2} \right) + \frac{\partial}{\partial \beta} \left( \frac{u_\beta}{h_1} \right) + \frac{\partial}{\partial \gamma} \left( \frac{u_\gamma}{h_1 h_2} \right) \right]$$

Substituting the expression for $u_a$ and $u_\beta$ as determined from the Kirchoff hypothesis and the expression for $u_\gamma$ as determined from equation (10.16) into the above expression for $\Delta_1$ and further representing the product $h_1 h_2$ by the truncated series

$$h_1 h_2 = \frac{1}{AB} \left( 1 - 2H_\gamma + K_\gamma^2 \right)$$
the expression for \( \Delta \) after rearrangement becomes

\[
\Delta = \frac{1}{AB} \left[ \frac{\partial}{\partial \alpha} (Bu) + \frac{\partial}{\partial \beta} (Av) \right] + \frac{(\eta_\alpha + \eta_\beta)}{2} \Delta_o + 2Hw + \Phi_1
\]

\[
+ \left\{ \frac{2}{AB} \left[ \frac{\partial}{\partial \alpha} (BH_u) + \frac{\partial}{\partial \beta} (AH_v) \right] - \frac{2H}{AB} \left[ \frac{\partial}{\partial \alpha} (Bu) + \frac{\partial}{\partial \beta} (Av) \right] \right\}
\]

\[
- \nabla_e^2 w + (\eta_\alpha + \eta_\beta)H \Delta_o + \frac{(\eta_\alpha + \eta_\beta)}{2} \Delta_1 + 2(K - 2H^2)w
\]

\[+ 2 \Phi_2 + 2H \Phi_1 \} \gamma
\]

In the above expression

\[
\nabla_e^2 = \frac{1}{AB} \left[ \frac{\partial}{\partial \alpha} \left( \frac{B}{A} \frac{\partial}{\partial \alpha} \right) + \frac{\partial}{\partial \beta} \left( \frac{A}{B} \frac{\partial}{\partial \beta} \right) \right]
\]

\[
\Phi_1 = \frac{(\eta_\beta - \eta_\alpha)}{2(1-\eta_\beta)} \frac{1}{B} \frac{\partial}{\partial \beta} v + \frac{(\eta_\alpha - \eta_\beta)}{2(1-\eta_\alpha)} \frac{1}{A} \frac{\partial}{\partial \alpha} u + \frac{(\eta_\beta - \eta_\alpha)}{2(1-\eta_\beta)} \frac{1}{AB} \frac{\partial}{\partial \alpha} u
\]

\[
+ \frac{(\eta_\alpha - \eta_\beta)}{2(1-\eta_\alpha)} \frac{1}{AB} \frac{\partial}{\partial \beta} v + \frac{(\eta_\beta - \eta_\alpha)}{2(1-\eta_\beta)} \frac{1}{AB} \frac{\partial}{\partial \beta} v + \frac{(\eta_\alpha - \eta_\beta)}{2(1-\eta_\alpha)} \frac{1}{AB} \frac{\partial}{\partial \alpha} u
\]

\[
\Phi_2 = \frac{(\eta_\beta - \eta_\alpha)}{4(1-\eta_\beta)} \frac{1}{B} \frac{\partial}{\partial \beta} \left( \frac{1}{B} \frac{\partial}{\partial \beta} w \right) - \frac{(\eta_\alpha - \eta_\beta)}{4(1-\eta_\alpha)} \frac{1}{A} \frac{\partial}{\partial \alpha} \left( \frac{1}{A} \frac{\partial}{\partial \alpha} w \right)
\]

\[
- \frac{(\eta_\beta - \eta_\alpha)}{4(1-\eta_\beta)} \frac{1}{A} \frac{\partial}{\partial \alpha} \frac{\partial}{\partial \alpha} w - \frac{(\eta_\alpha - \eta_\beta)}{4(1-\eta_\alpha)} \frac{1}{AB} \frac{\partial}{\partial \beta} \frac{\partial}{\partial \beta} w
\]

\[= \frac{(\eta_\beta - \eta_\alpha)}{4(1-\eta_\beta)} \frac{1}{A} \frac{\partial}{\partial \alpha} \frac{\partial}{\partial \alpha} w - \frac{(\eta_\alpha - \eta_\beta)}{4(1-\eta_\alpha)} \frac{1}{AB} \frac{\partial}{\partial \beta} \frac{\partial}{\partial \beta} w
\]

\[(10.18)\]
The dilatation $\Delta$ is expressed as a truncated series

$$\Delta = \Delta_0 + \Delta_1 \gamma$$

Equating coefficients in the above expression for $\Delta$ leads to the following solutions for the components $\Delta_0$ and $\Delta_1$.

$$\Delta_0 = \frac{\mathcal{J}}{AB} \left[ \frac{\partial}{\partial \alpha} (Bu) + \frac{\partial}{\partial \beta} (Av) \right] + 2 \mathcal{J} H w + \mathcal{J} \Phi_1 \quad (10.19)$$

$$\Delta_1 = \frac{2}{AB} \left[ \frac{\partial}{\partial \alpha} (BHu) + \frac{\partial}{\partial \beta} (AHv) \right] + \mathcal{J} \left( \frac{\mathcal{J}(\eta_a + \eta_\beta)}{AB} - 2 \right) H$$

$$\left[ \frac{\partial}{\partial \alpha} (Bu) + \frac{\partial}{\partial \beta} (Av) \right]$$

$$- \mathcal{J} \nabla_e^2 w + 2 \mathcal{J} \left( K - \left[ 2 - \mathcal{J}(\eta_a + \eta_\beta) \right] H^2 \right) w + 2 \mathcal{J} \Phi_2$$

$$+ \mathcal{J} \left[ 2 + (\eta_a + \eta_\beta) \mathcal{J} \right] H \Phi_1 \quad (10.20)$$

The constant $\mathcal{J}$ is defined by the equation

$$\mathcal{J} = \frac{2}{2 - (\eta_a + \eta_\beta)} \quad (10.21)$$
11. **Concluding Remarks**

The five equilibrium equations (10.4) to (10.8), together with the expressions for the displacement $u$, equation (10.14), the strain $e$, equation (10.15), and the dilatation components, equations (10.17) and (10.18), allow a solution for any orthotropic thin shell. As is seen by inspection of these equations, combining them as was done in the preceding chapter dealing with the isotropic case will lead to extremely long and complicated expressions. In fact this has been attempted on an experimental basis and it was concluded that by far the simplest approach is to apply the component equations to each particular case under study.

The simplifications to be made in the application of the orthotropic equations will follow those made for the isotropic case. Since the Kirchoff hypothesis is in force, terms of order $\delta k$ in comparison to unity may be discarded.

The orthotropy of the shell is defined such that the elastic constants relative to corresponding planes of elastic symmetry at any two points of the shell are identical. However, the normals to two of these planes are tangent to the principal curvilinear coordinates, $\alpha$ and $\beta$, lying on any parallel surface, and the third normal coincides with the $\gamma$ axis, the normal to the parallel surface. Since
the normal for any parallel surface is also the normal to the shell, the net effect is that planes of elastic symmetry are parallel for points lying along a normal to the shell. However, because of the curvature of the principal coordinate lines, planes of elastic symmetry are not parallel for any two points lying on an equi-distant surface.

A material is said to be homogeneous if at any two points within that material the planes of elastic symmetry are parallel to each other. Thus, it is obvious that an orthotropic shell is not a homogeneous body since the planes of elastic symmetry will not, in general, be parallel to each other. Inspection of the expression for the dilatation $\Delta$ and the displacement $u_\gamma$ reveals terms which are concerned with the derivatives of the Lamé parameters, $A$ and $B$, as well as the curvatures $k_1$ and $k_2$. These terms represent the rotations of the planes of elastic symmetry along the coordinate lines $\alpha$ and $\beta$ and, as is to be expected, vanish in the case of isotropy.
REFERENCES


CHAPTER III
THE ORTHOTROPIC CYLINDRICAL SHELL

12. Displacement Equations

The general orthotropic equations derived in Chapter II will be applied to the case of a right circular cylindrical thin shell of radius $a$ and thickness $\delta$.

For a cylindrical surface the set of orthogonal lines of principal curvatures are lines parallel to the generatrix of the cylinder and the circle formed by the intersection of the cylinder and a plane perpendicular to its axis. Further, the curvatures of the surface along the lines of principal curvature are constant.

The lines of principal curvatures of the cylindrical surface will also be the principal curvilinear coordinate lines of the middle surface of the cylindrical shell. Let the lines $\beta = \text{constant}$ coincide with the curvature lines parallel to the generatrix and hence the lines $\alpha = \text{constant}$ will coincide with the circular circumferential lines. The coordinate axis $\gamma$ will be normal to the shell middle surface and positive when directed outward.
from the axis of the shell. Figure 8 shows the shell with its dimensions and a set of curvilinear coordinate axes drawn on its middle surface.

Consider now a set of coordinates \( x \) and \( \theta \) describing a point on the middle surface of the shell. The coordinate \( x \), measured along the axis of the shell will locate a plane perpendicular to the axis of the shell and passing through the point in question. The coordinate \( \theta \) will measure the included angle between two radial lines in that plane, one of the lines going to the point in question, the other being arbitrarily chosen as the reference position. For points off the middle surface, a third coordinate \( z \) will be used. This coordinate is measured normal to the middle surface and directed from the normal projection of the point on that surface to the point itself. The positive direction of \( z \) will be outward from the axis of the shell. This set of coordinates is shown in Figure 9.

In order to determine the Lamé parameters \( A \) and \( B \) as well as the curvilinear coordinates \( a \) and \( \beta \), consider two line segments lying on the middle surface of the shell. Let one of the segments, \( ds_1 \), be parallel to the curvilinear coordinate line \( \beta = \text{constant} \) while
Figure 8. Curvilinear coordinate lines on the middle surface of a cylindrical shell.
Figure 9. Specification of a Point "P" Located within the Shell.
the other segment, \( ds_2 \), will be parallel to the curvilinear coordinate line \( a = \text{constant} \). From the definition of the Lamé parameters

\[
\begin{align*}
  ds_1 &= A \, da \\
  ds_2 &= B \, d\beta
\end{align*}
\]

However, in terms of the coordinates \( x \) and \( \theta \) it is obvious that

\[
\begin{align*}
  ds_1 &= dx \\
  ds_2 &= a \, d\theta
\end{align*}
\]

Letting

\[
\begin{align*}
  A &= 1 \\
  B &= 1
\end{align*}
\]

it then follows that

\[
\begin{align*}
  da &= dx \\
  d\beta &= a \, d\theta
\end{align*}
\]

Thus the principal curvilinear coordinate lines are defined in terms of the readily measured cylindrical coordinates. It is also apparent from the definition of the \( z \) and \( \gamma \) coordinates that

\[
d\gamma = dz
\]

As previously mentioned, the curvatures of the middle surface along the principal coordinate lines are constant. Since \( k_1 \) is defined to be curvature of the shell along the line \( \beta = \text{constant} \) while
\( k_2 \) is the curvature along the line \( a = \text{constant} \), then from the property of a cylindrical surface

\[
\begin{align*}
k_1 &= 0 \\
k_2 &= \frac{1}{a}
\end{align*}
\]

It follows that

\[
\begin{align*}
h_1 &= 1 \\
h_2 &= \frac{1}{1 + \frac{z}{a}}
\end{align*}
\]

On the basis of the preceding relationships the five equations, equation (10.4) to equation (10.8) of the previous chapter become after modification

\[
(\tau_a + 2\mu_a) \left( \frac{\partial \Delta_o}{\partial x} + \frac{2}{12a} \frac{\partial \Delta_1}{\partial x} \right) - \frac{2\mu_a}{a} \frac{\partial \chi_o}{\partial \theta} - \frac{2\mu_a}{\delta} \int_{-\delta/2}^{\delta/2} \frac{\partial}{\partial z} \left( \frac{1}{h_2} \frac{\partial u_z}{\partial x} \right) dz
\]

\[
+ \frac{X}{\delta} - 2 \frac{\mu_a - \mu_a^*}{a} \frac{\partial^2 u}{\partial x \partial \theta} = 0
\]

\( (\tau_\beta + 2\mu_\beta) \frac{\partial \Delta_o}{\partial \theta} + 2\mu_\beta \left( \frac{\partial \chi_o}{\partial x} + \frac{2}{12a} \frac{\partial \chi_1}{\partial x} \right) - \frac{2\mu_\beta}{a \delta} \int_{-\delta/2}^{\delta/2} \frac{\partial}{\partial z} \left( -\frac{\partial u_z}{\partial \theta} \right) dz
\]

\[
+ \frac{Q_\theta}{a \delta} + \frac{Y}{\delta} - 2 \frac{\mu_\beta - \mu_\beta^*}{a} \frac{\partial^2 u}{\partial x \partial \theta} = 0
\]
The dilatation components from equation (10.19) of Chapter II become after simplification

\[ \Delta_0 = \frac{1}{(1 - \eta)} \frac{\partial u}{\partial x} + \frac{1}{a} \frac{\partial v}{\partial \theta} + \frac{1}{(1 - \eta)} \frac{w}{a} \]  

(12.6)

\[ \Delta_1 = -\frac{1}{(1 - \eta)} \frac{1}{a} \frac{\partial^2 w}{\partial \theta^2} - \frac{1}{(1 - \eta)} \frac{\partial^2 w}{\partial x^2} + \frac{\eta}{a} \frac{\partial u}{\partial x} + \frac{\eta}{(1 - \eta)} \frac{1}{a} \frac{\partial v}{\partial \theta} + \frac{\eta}{(1 - \eta)} \left( \frac{\partial v}{\partial \theta} - \frac{w}{a} \right) \]  

(12.7)
The displacement equation for \( u_\gamma \) from equation (10.16) of Chapter II becomes

\[
u_z = w + \left( \frac{\eta_a^+ - \eta_\beta}{2} \right) \Delta_0 + w \frac{\eta_\beta - \eta_a}{2a(1-\eta_\beta)} + \frac{(\eta_\beta - \eta_a)}{2(1-\eta_\beta)} \frac{1}{a} \frac{\partial v}{\partial \theta}
\]

\[
+ \frac{(\eta_a - \eta_\beta)}{2(1-\eta_a)} \frac{\partial u}{\partial x} z + \left[ \frac{1}{2} \left( \frac{\eta_a + \eta_\beta}{2} \right) \Delta_1 - \frac{1}{2(1-\eta_a)} \frac{\partial^2 w}{\partial x^2} \right] z^2
\]

\[
- \frac{(\eta_\beta - \eta_a)}{2(1-\eta_\beta)} \frac{1}{a} \frac{\partial^2 w}{\partial \theta^2} z^2
\]

(12.8)

The rotation components \( \chi_o \) and \( \chi_1 \) have been shown to be independent of the elastic properties. Thus from equation (5.16) of Chapter I

\[
\chi_o = \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{1}{a} \frac{\partial u}{\partial \theta} \right)
\]

(12.9)

\[
\chi_1 = \frac{1}{2a} \left( \frac{\partial v}{\partial x} + \frac{1}{a} \frac{\partial u}{\partial \theta} \right)
\]

(12.10)

In order to maintain a simplicity of expression, the substitution of the dilatation and rotation components as well as the displacement \( u_\gamma \) will not be made until the last steps of the derivation. Proceeding now as in the isotropic derivation, the expressions for the transverse forces \( Q_1 \) and \( Q_2 \) will be found from equations (12.4) and (12.5) and then substituted into the first three equations of that set. However, inspection of equation (12.1) reveals that
that expression does not contain the stress resultant $Q_1$. Hence the substitutions need be made only in equations (12.2) and (12.3).

Making the substitutions indicated above, the following three equations result for the middle surface displacements $u$, $v$, and $w$.

\[
\begin{align*}
\left(\tau_a + 2\mu_a\right) & \left(\frac{\partial^2 o}{\partial x^2} + \frac{\delta^2}{12a} \frac{\partial \Delta_1}{\partial x}\right) - 2 \frac{\mu_a \beta}{a} \frac{\partial \chi}{\partial \theta} - 2 \frac{\mu_a}{\beta} \int_{-\delta/2}^{\delta/2} \frac{\partial}{\partial z} \left(\frac{1}{h_z} \frac{\partial u_z}{\partial x}\right) dz \\
+ \frac{Y}{\delta} & - 2 \frac{(\mu_a - \mu_a \beta)}{a} \frac{\partial^2 v}{\partial x \partial \theta} = 0
\end{align*}
\]  
\[\text{(12.11)}\]

\[
\begin{align*}
\left(\tau_\beta + 2\mu_\beta \right) & \frac{\partial \Delta}{\partial \theta} + \frac{\delta^2}{12a} \frac{\partial \Delta_1}{\partial \theta} + 2 \frac{\mu_a \beta}{a} \left(1 + \frac{\delta^2}{12a^2}\right) \frac{\partial \chi}{\partial x} \\
+ \frac{\mu_a \beta}{3a} \frac{\partial \chi}{\partial x} & - \frac{\delta^2}{a \beta} \int_{-\delta/2}^{\delta/2} \frac{\partial}{\partial z} \left(\frac{\partial u_z}{\partial \theta}\right) dz - \frac{2\mu_\beta}{a^2 \beta} \int_{-\delta/2}^{\delta/2} \frac{\partial^2}{\partial z^2} \left(\frac{\partial u_z}{\partial \theta}\right) dz \\
+ \frac{Y}{\delta} & - 2 \frac{(\mu_\beta - \mu_a \beta)}{a} \frac{\partial^2 u}{\partial x \partial \theta} + (\mu_\beta - \mu_a \beta) \frac{\delta^2}{6a^2} \frac{\partial^3 w}{\partial x^2 \partial \theta} = 0
\end{align*}
\]  
\[\text{(12.12)}\]
A direct substitution of the dilatation and rotation components as well as the displacement \( u_z \) into the above equations would lead to long and involved expressions. In order to show more clearly the various simplifications that will be made, it is felt that it will be more advantageous to simplify the various functions before substitution. As a further aid in the simplification process, each of the equations will be treated individually.

Before any substitutions or simplifications can be made in the first of the equations, equation (12.11), the integral involved in that equation must be evaluated. Thus substituting the expression for \( u_z \) and \( h_z \) and integrating

\[
-\frac{2\mu_a}{\delta} \int_{-\delta/2}^{\delta/2} \theta \left( \frac{1}{h_2} \frac{\partial u_z}{\partial x} \right) dx = -\mu_a (\eta_a + \eta_\beta) \frac{\partial \Delta_1}{\partial x} - 2\mu_a \left[ 1 + \frac{\eta_a - \eta_\beta}{2(1-\eta_\beta)} \right] \frac{1}{a} \frac{\partial w}{\partial x}
\]
In the isotropic case it is shown that terms of the type $\delta^2 K_1$ or $\delta^2 K_2$ lead to so called "tie rod" effects when appearing in the displacement "equation" of the shell. These terms are shown to smaller than order $\delta k$ in comparison with other terms appearing in the equation and hence may be discarded. For a cylinder, $K_1$ and $K_2$ are nothing more than the second derivatives of $w$ with respect to $x$ and $\theta$ respectively and thus these derivatives may be discarded.

Another simplification to be made is also based on isotropic results. In that derivation it is shown that the dilatation component $\Delta_1$ when preceded by a coefficient containing $\delta^2$ may be approximated by the expression for the change in curvature of the middle surface. Further, the curvature change may be approximated by the Laplacian of $w$ which for the coordinates chosen is given as

$$\frac{\partial^2 w}{\partial x^2} + \frac{1}{a^2} \frac{\partial^2 w}{\partial \theta^2}$$

Thus it must be concluded that all terms containing the displacement $u$ and $v$ as well as the term containing the displacement $w$ may be discarded in the expression of $\delta^2 \Delta_1$ as being of order smaller than $\delta k$ in comparison with the second order derivatives of $w$. 
The stress-strain relations should not have any effect on the order of magnitude of a term. Thus the above conclusions found for the isotropic case should carry over to the orthotropic case. As a consequence of this, the orthotropic expression for $\delta^2 \Delta_1$ may be simplified so that it too contains only the second derivatives of $w$.

If the shell is continuous in slope and deflection in the initial and deformed shape, then the smallness of the change in curvature must also imply that its first derivative with respect to either $x$ or $\theta$ must also be small and of the same order of magnitude or smaller than the change in curvature. Since the second derivatives of $w$ when multiplied by a coefficient containing $\delta^2$ are being discarded in comparison with the displacement $w$, it then follows that its third derivatives must also be discarded.

Examination of the terms containing $\Delta_1$ in equation (12.11) reveals two things. First, the coefficient of these terms contain the square of the thickness $\delta_1$ and secondly, $\Delta_1$ enters as a first derivative with respect to $x$. On the basis of what has been stated in the previous paragraphs, it must be concluded that as far as the first of the transformed equilibrium equations are concerned, the term $\Delta_1$ may be discarded.
The final form of equation (12.11) after substitutions for $\Delta_o$ and $x_o$ is given below. Note that the simplifications eliminate all terms containing $\delta^2$, a situation that was found to be true in the isotropic case.

\[
\left[ \frac{\tau_a + 2\mu_a}{(1-\eta_a)} - \frac{2\mu_a \eta_a}{(1-\eta_a)} \right] \frac{\partial^2 u}{\partial x^2} + \frac{\mu_{a\beta} \delta^2}{a^2} \frac{\partial^2 u}{\partial \theta^2} + \left[ \frac{\tau_a + 2\mu_a}{(1-\eta_\beta)} - \frac{2\mu_a \eta_\beta}{(1-\eta_\beta)} - \mu_{a\beta} - 2(\mu_a - \mu_{a\beta}) \right] \frac{1}{a} \frac{\partial^2 v}{\partial x \partial \theta} + \left[ \frac{\tau_a + 2\mu_a}{(1-\eta_\beta)} - \frac{2\mu_a \eta_\beta}{(1-\eta_\beta)} \right] \frac{1}{a} \frac{\partial w}{\partial x} + \frac{X}{\delta} = 0
\]

(12.14)

Consider now operating on equation (12.12). Evaluating each of the integrals involved in that equation separately, the following equation is obtained.

\[
- \frac{2\mu_\beta}{a \delta} \int_{-\delta/2}^{\delta/2} \frac{\partial}{\partial z} \left( \frac{\partial u_a}{\partial \theta} \right) dz = \mu_\beta \left( \eta_a + \eta_\beta \right) \frac{1}{a} \frac{\partial \Delta_o}{\partial \theta} - \frac{\mu_\beta (\eta_\beta - \eta_a)}{a} \frac{1}{2} \frac{\partial w}{\partial \theta}
\]

\[
- \frac{2\mu_\beta}{a \delta} \int_{-\delta/2}^{\delta/2} z \frac{\partial}{\partial z} \left( \frac{\partial u_a}{\partial \theta} \right) dz = - \frac{\mu_\beta (\eta_a + \eta_\beta)}{12 \left( \eta_a \right)} \frac{\delta^2}{a^2} \frac{\partial \Delta_a}{\partial \theta}
\]

\[
+ \frac{\mu_\beta (\eta_\beta - \eta_a)}{12 \left( \eta_\beta \right)} \frac{\delta}{a} \frac{3}{4 \delta} \frac{\partial w}{\partial \theta} + \frac{\mu_\beta (\eta_a - \eta_\beta)}{12 \left( 1 - \eta_a \right)} \frac{\delta^2}{a^2} \frac{\partial w}{\partial x \partial \theta} + \frac{\mu_\beta (\eta_a - \eta_\beta)}{12 \left( 1 - \eta_\beta \right)} \frac{\delta^2}{a^2} \frac{\partial^2 w}{\partial x \partial \theta}
\]
The simplification of the equation as well as the integrals will be based upon the same arguments as were used in the simplification of equation (12.11). The net result is that all terms which are preceded by a coefficient involving $\delta^2$ are discarded.

Thus combining terms, the second transformed equilibrium equation, equation (12.12) becomes

$$
\left[ \frac{\tau_\beta + 2\mu_\beta}{(1-\eta_\beta)} - \frac{2\mu_\beta \eta_\beta}{(1-\eta_\beta)} \right] \frac{1}{a} \frac{\partial^2 v}{\partial \theta^2} + \mu_\alpha \frac{\partial^2 v}{\partial x^2}
$$

$$
+ \left[ \frac{\tau_\beta + 2\mu_\beta}{(1-\eta_\alpha)} - \frac{2\mu_\beta \eta_\alpha}{(1-\eta_\alpha)} - \mu_\alpha - 2 (\mu_\beta - \mu_\alpha) \right] \frac{1}{a} \frac{\partial^2 u}{\partial x \partial \theta} \tag{12.15}
$$

$$
+ \left[ \frac{\tau_\beta + 2\mu_\beta}{(1-\eta_\beta)} - \frac{2\mu_\beta \eta_\beta}{(1-\eta_\beta)} \right] \frac{1}{a} \frac{\partial w}{\partial \theta} + \frac{Y}{\delta} = 0
$$

As in the first two transformed equilibrium equations, the simplification of the third equation of that set will start with an evaluation of the integrals.

$$
\frac{2\mu_\beta}{a} \delta \int_{-\delta/2}^{\delta/2} \frac{\partial u_z}{\partial z} \, dz = \mu_\beta \left( \frac{\eta + \eta_\beta}{a} \right) \Delta_0 + \frac{\mu_\beta (\eta - \eta_\beta)}{a^2 (1-\eta_\beta)^2} w
$$
The remaining integral is a repetition of the first of the above three integrals and hence need not be evaluated.

Many of the simplifications that can be made in equation (12.13) are of the same type that have been made for equations (12.11) and (12.12) Thus second and third order derivatives of \( w \), when having a coefficient involving \( \delta^2_1 \) may be neglected. Similarly, the expression for \( \Delta_1 \) when multiplied by a coefficient involving \( \delta^2_1 \) may be simplified so as to contain only the second order derivatives of \( w \).
Moreover, a term by term comparison of the expressions involving the tangential displacements \( u \) and \( v \) will result in the elimination of these expressions or their derivatives where the coefficients involve the square of the shell thickness \( \delta^2 \). Thus the net result of the simplification of the third equation will be the elimination of all terms containing the coefficient \( \delta^2 \) save those involving the fourth derivatives of the displacement \( w \). The equation may then be written as

\[
\left( \frac{\tau_\beta + 2\mu_\beta}{a} \delta_0 + \left( \frac{\tau_\alpha + 2\mu_\alpha}{a} \right) \frac{\delta^2}{12} \frac{\partial^2 \Delta_1}{\partial x^2} \right) + \left( \frac{\tau_\beta + 2\mu_\beta}{a} \right) \frac{\delta^2}{12a^2} \frac{\partial^2 \Delta_1}{\partial \theta^2} \\
+ 2\frac{\mu_\beta}{a} \frac{\partial u}{\partial x} + \left[ (\mu_\alpha - \mu_\alpha \beta)\left( \mu_\beta - \mu_\alpha \beta \right) \right] \frac{\delta^2}{6a^2} \frac{\partial^4 w}{\partial x^2 \partial \theta^2} \\
+ \frac{2\mu_\beta \eta_\beta}{(1 - \eta_\beta)} \frac{1}{a^2} \frac{\partial^2 w}{\partial x^2} + \frac{2\mu_\beta}{(1 - \eta_\beta)} \frac{1}{a} \frac{\partial v}{\partial \theta} + \frac{2\mu_\beta \eta_\alpha}{(1 - \eta_\alpha)} \frac{1}{a} \frac{\partial u}{\partial x} \\
+ \frac{\mu_\alpha \eta_\alpha}{6(1 - \eta_\alpha)} \frac{\delta^2}{a^2} \frac{\partial^4 w}{\partial x^2 \partial \theta^2} + \frac{\mu_\alpha \eta_\beta}{6(1 - \eta_\alpha)} \frac{\delta^2}{a^2} \frac{\partial^4 w}{\partial x^2 \partial \theta^2} \\
+ \frac{\mu_\beta \eta_\beta}{6(1 - \eta_\beta)} \frac{1}{4} \frac{\partial^4 w}{\partial \theta^4} + \frac{\Delta_1}{\delta} = 0
\]

Substituting \( \Delta_0 \), \( \chi_0 \), and \( \Delta_1 \) in accord with what has been stated in the previous paragraph and combining terms, the final form of equation (12.13) results.
Equations (12.14), (12.15), and (12.16) form a set of three equations in terms of the three unknown middle surface displacements, \( u \), \( v \), and \( w \). However, the coefficients of these equations are long and involved. For simplicity then, consider rewriting the equations as

\[
\begin{align*}
\frac{\delta^2}{12} \left[ -\frac{(\tau + 2\mu)}{(1-\eta_a)} + \frac{2\mu \eta_a}{(1-\eta_a)} \right] \frac{\partial^4 w}{\partial x^2} + \frac{\delta^2}{12} \left[ -\frac{(\tau + 2\mu)}{(1-\eta_\beta)} + \frac{2\mu \eta_\beta}{(1-\eta_\beta)} \right] \frac{\partial^4 w}{\partial x^2} \\
+ 2(\mu_a - \mu_a \beta) - \frac{(\tau + 2\mu)}{(1-\eta_a)} + \frac{2\mu \eta_a}{(1-\eta_a)} + 2(\mu_\beta - \mu_\beta \beta) \right] \frac{1}{a^2} \frac{\partial^2 w}{\partial x^2} \\
+ \frac{\delta^2}{12} \left[ -\frac{(\tau + 2\mu)}{(1-\eta_\beta)} + \frac{2\mu \eta_\beta}{(1-\eta_\beta)} \right] \frac{1}{a^4} \frac{\partial^4 w}{\partial \theta^4} + \left[ -\frac{(\tau + 2\mu)}{(1-\eta_\beta)} + \frac{2\mu \eta_\beta}{(1-\eta_\beta)} \right] \frac{w}{a^2} \\
+ \frac{Z}{\delta} = 0
\end{align*}
\]
The algebraically simplified coefficients $A_1, A_2, \ldots, B_1, \ldots$, $C_1, \ldots, C_6$ are given by the following expressions:

\[
A_1 = \frac{1}{(1 - \eta_a)} \left[ (\tau_a + 2\mu_a) - 2\mu_a \eta_a \right]
\]

\[
A_2 = \mu_{a\beta}
\]

\[
A_3 = \frac{1}{(1 - \eta_\beta)} \left[ (\tau_\beta + 2\mu_\beta) - 2\mu_\beta \eta_\beta \right] - \mu_{a\beta} - 2(\mu_\beta - \mu_{a\beta})
\]

\[
A_4 = \frac{1}{(1 - \eta_\beta)} \left[ (\tau_\beta + 2\mu_\beta) - 2\mu_\beta \right]
\]

\[
B_1 = \frac{1}{(1 - \eta_\beta)} \left[ (\tau_\beta + 2\mu_\beta) - 2\mu_\beta \eta_\beta \right]
\]

\[
B_2 = \mu_{a\beta}
\]

\[
B_3 = \frac{1}{(1 - \eta_a)} \left[ (\tau_\beta + 2\mu_\beta) - 2\mu_\beta \eta_a \right] - \mu_{a\beta} - 2(\mu_\beta - \mu_{a\beta})
\]

\[
B_4 = \frac{1}{(1 - \eta_\beta)} \left[ (\tau_\beta + 2\mu_\beta) - 2\mu_\beta \eta_\beta \right]
\]

\[
C_1 = -\frac{1}{(1 - \eta_a)} \left[ (\tau_a + 2\mu_a) - 2\mu_a \eta_a \right]
\]

\[
C_2 = -\frac{1}{(1 - \eta_\beta)} \left[ (\tau_\beta + 2\mu_\beta) - 2\mu_\beta \eta_\beta \right] + 2(\mu_\beta - \mu_{a\beta})
\]

\[
-\frac{1}{(1 - \eta_a)} \left[ (\tau_\beta + 2\mu_\beta) - 2\mu_\beta \eta_a \right] + 2(\mu_\beta - \mu_{a\beta})
\]
\[ C_3 = - \frac{1}{(1 - \eta_\beta)} \left[ (\tau_\beta + 2\mu_\beta) - 2\mu_\beta \eta_\beta \right] \]

\[ C_4 = - \frac{1}{(1 - \eta_\beta)} \left[ (\tau_\beta + 2\mu_\beta) - 2\mu_\beta \right] \]

\[ C_5 = - \frac{1}{(1 - \eta_\alpha)} \left[ (\tau_\beta + 2\mu_\beta) - 2\mu_\beta \right] \]

\[ C_6 = - \frac{1}{(1 - \eta_\beta)} \left[ (\tau_\beta + 2\mu_\beta) - 2\mu_\beta \eta_\beta \right] \]

An alternate and more convenient way of expressing the above constants is in terms of the elastic constants of the material. Substituting the definitions for the orthotropic Lamé parameters as well as the definitions of \( \eta_\alpha \) and \( \eta_\beta \), the result becomes as follows.

\[ A_1 = \frac{a_{22}}{(a_{11} a_{22} - a_{12})^2} \]

\[ A_2 = \frac{1}{a_{66}} \]

\[ A_3 = \frac{1}{a_{66}} \left( - \frac{a_{12}}{a_{11} a_{22} - a_{12}^2} \right) \]

\[ A_4 = - \frac{a_{12}}{(a_{11} a_{22} - a_{12})^2} \]

\[ B_1 = \frac{a_{11}}{(a_{11} a_{22} - a_{12})^2} \]
B_2 = \frac{1}{a_{66}}

B_3 = \frac{1}{a_{66}} - \frac{a_{12}}{(a_{12} a_{22} - a_{12}^2)}

B_4 = \frac{a_{11}}{(a_{11} a_{22} - a_{12}^2)}

C_1 = \frac{-a_{22}}{(a_{11} a_{22} - a_{12}^2)}

C_2 = -2 \left[ \frac{2}{a_{66}} - \frac{a_{12}}{(a_{12} a_{22} - a_{12}^2)} \right]

C_3 = -\frac{a_{11}}{(a_{11} a_{22} - a_{12}^2)}

C_4 = -\frac{a_{11}}{(a_{11} a_{22} - a_{12}^2)}

C_5 = \frac{a_{12}}{(a_{11} a_{22} - a_{12}^2)}

C_6 = -\frac{a_{11}}{(a_{11} a_{22} - a_{12}^2)}
13. **Stress and Strain Expressions**

The expressions for the strains in terms of displacements which are stated in Chapter I, equation (4.1), are valid for anisotropic as well as isotropic materials. For convenience, these equations are repeated below.

\[
e_a = h_1 \frac{\partial u_a}{\partial a} + h_1 h_2 \frac{\partial h_1}{\partial \beta} u_\beta + h_1 \frac{\partial h_1}{\partial \gamma} u_\gamma
\]

\[
e_\beta = h_2 \frac{\partial u_\beta}{\partial \beta} + h_1 h_2 \frac{\partial h_2}{\partial a} u_a + h_2 \frac{\partial h_2}{\partial \gamma} u_\gamma
\]

\[
e_{a\beta} = \frac{h_1}{h_2} \frac{\partial}{\partial a} \left( h_2 u_\beta \right) + \frac{h_2}{h_1} \frac{\partial}{\partial \beta} \left( h_1 u_a \right)
\]

For the particular case of the cylindrical orthotropic shell being dealt with these equations reduce to the following.

\[
e_x = \frac{\partial u_x}{\partial x}
\]

\[
e_\theta = \frac{1}{a(1 + a)} \frac{\partial u_\theta}{\partial \theta} + \frac{u_z}{a(1 + a)}
\]

\[
e_x\theta = (1 + \frac{z}{a}) \frac{\partial}{\partial x} \left[ \frac{u_\theta}{(1 + z/a)} \right] + \frac{1}{a(1 + z/a)} \frac{\partial u_x}{\partial \theta}
\]

Now the Kirchhoff assumptions dictate that the tangential displacements, \( u_x \) and \( u_\theta \), be given by the following expressions.

\[
u_x = u - z \frac{\partial w}{\partial x}
\]

\[
u_\theta = (1 + \frac{Z}{a}) v - \frac{z}{a} \frac{\partial w}{\partial \theta}
\]
Substituting these relations as well as the expression for 
\( u_z \) into the strain expressions and expressing \( h_z \) by the truncated 
series 
\[
\frac{1}{(1 + \frac{Z}{a})} = 1 - \frac{z}{a}
\]
the expressions for the strains become

\[
e_x = \frac{\partial u}{\partial x} - z \frac{\partial^2 w}{\partial x^2} \tag{13.1}
\]

\[
e_\theta = \left[ \frac{w}{a} + \frac{1}{a} \frac{\partial v}{\partial \theta} \right] + \left[ \frac{\eta}{(1 - \eta)} \frac{1}{a} \frac{\partial v}{\partial \theta} - \frac{(1 - 2\eta)}{(1 - \eta)} \frac{w}{a} \right] + \left[ \frac{\eta_a}{(1 - \eta_a)} \frac{\partial u}{\partial x} - \frac{1}{a} \frac{\partial^2 w}{\partial \theta^2} \right] \frac{z}{a} \tag{13.2}
\]

\[
e_{x\theta} = \left( \frac{\partial v}{\partial x} + \frac{1}{a} \frac{\partial u}{\partial \theta} \right) + \left( \frac{\partial v}{\partial x} - \frac{1}{a} \frac{\partial u}{\partial \theta} - 2 \frac{\partial^2 w}{\partial x \partial \theta} \right) \frac{z}{a} \tag{13.3}
\]

The above expressions for the strain allow their calculation
once the displacements \( u, v, \) and \( w \) are known. For the purpose
of analyzing qualitatively the strains as well as the stress resultants to be derived from them, an alternate set of strain equations
will be more convenient to use.

Let \( e, e_z, \) and \( e_3 \) represent the middle surface normal strains
in the \( x, \theta, \) and \( z \) directions respectively. Further, let \( e_{12} \) repre-
sent the shear strain \( e_{x\theta} \) acting at the middle surface. Then
The most convenient way of expressing $e_3$ is from the component of the dilatation $\Delta_o$, where

$$\Delta_o = e_1 + e_2 + e_3$$

Substituting and rearranging

$$e_3 = \frac{\eta_a}{(1 - \eta_a)} e_1 + \frac{\eta_3}{(1 - \eta_3)} e_2 \quad (13.4)$$

It will also be convenient to have expressions for the change in curvature and twist of the middle surface. From equation (6.2) the expressions for the changes in curvature are

$$K_1 = -\frac{\partial^2 w}{\partial x^2} \quad (13.5)$$

$$K_2 = -\frac{1}{a^2} \frac{\partial^2 w}{\partial \theta^2} \quad (13.6)$$

The expression for the twist is given by equation (6.8). Thus

$$\tau = -\frac{1}{a} \frac{\partial^2 w}{\partial x \partial \theta} \quad (13.7)$$
Utilizing the above parameter, the alternate expressions for the strains become

\[ e_x = e_1 + K_1 z \]

\[ e_\theta = e_2 + K_2 z + \left[ \frac{\eta_\beta}{(1 - \eta_\beta)} \right] \frac{1}{a} \frac{\partial v}{\partial \theta} + \frac{(1 - 2 \eta_\beta)}{(1 - \eta_\beta)} \frac{w}{a} \]

\[ + \frac{\eta_a}{(1 - \eta_a)} \frac{\partial u}{\partial x} \frac{z}{a} \]

\[ e_{x\theta} = e_{12} + 2 \tau z + \left( \frac{1}{a} \frac{\partial v}{\partial x} - \frac{1}{z} \frac{\partial u}{\partial \theta} \right) z \]

It is pointed out in the isotropic development that when strains are being compared to either curvature changes or twist of the middle surface, the strains may be neglected since they are of order $6k$ compared to the latter quantities. There is no reason to doubt that the same situation is not true in the orthotropic case.

Further, the last expressions for the strains contain displacement terms or their derivatives as well as the middle surface strains. Inspection of these displacement terms show that they are component parts of the middle surface strains and hence will be of the same order or smaller than these quantities and, thus, they too
may be discarded. Thus the resultant strain expressions may be written as

\[ e_x = e_1 + K_1 z \]  
\[ e_\theta = e_2 + K_2 z \]  
\[ e_{x\theta} = e_{12} + 2\tau z \]  

(13.8)  
(13.9)  
(13.10)

As a consequence of the above, the strain expressions in terms of the displacement may be simplified to the following.

\[ e_x = \frac{\partial u}{\partial x} - z \frac{\partial^2 w}{\partial x^2} \]  
\[ e_\theta = \frac{1}{a} (w + \frac{\partial v}{\partial \theta}) - \frac{1}{a} \frac{\partial^2 w}{\partial \theta^2} z \]  
\[ e_{x\theta} = \frac{\partial v}{\partial x} + \frac{1}{a} \frac{\partial u}{\partial \theta} - \frac{2}{a} \frac{\partial^2 w}{\partial x \partial \theta} z \]  

(13.11)  
(13.12)  
(13.13)

The stress-strain relations are given by equation (9.4). For the particular case of the orthotropic cylinder these equations become

\[ \sigma_x = (\tau_\alpha + 2\mu_\alpha) \Delta - 2\mu_\alpha (e_\theta + e_z) \]  
\[ \sigma_\theta = (\tau_\beta + 2\mu_\beta) \Delta - 2\mu_\beta (e_x + e_z) \]  
\[ \sigma_{x\theta} = \mu_\alpha \beta e_{x\theta} \]
As was pointed out previously, the expression for the dilatation component $\Delta_1$ may be simplified so that it only contains the second derivatives of $w$. Further, the second derivatives of $w$ may be expressed in terms of the curvature changes so that the expression (12.7) for $\Delta_1$ may ultimately be simplified to

$$\Delta_1 = \frac{1}{(1 - \eta_a)} K_1 + \frac{1}{(1 - \eta_\beta)} K_2$$  \hspace{1cm} (13.14)

Similarly, the expression for the component $\Delta_o$ may also be written as

$$\Delta_o = \frac{e_1}{(1 - \eta_a)} + \frac{e_2}{(1 - \eta_\beta)}$$  \hspace{1cm} (13.15)

The expression for $e_z$ is found by differentiating expression (12.8). Thus

$$e_z = \left[ \frac{(\eta_a + \eta_\beta)}{2} \Delta_o + \frac{(\eta_\beta - \eta_a)}{2(1 - \eta_\beta)} \frac{1}{a} \frac{\partial v}{\partial \theta} + \frac{(\eta_a - \eta_\beta)}{2(1 - \eta_a)} \frac{\partial u}{\partial x} \right] + \frac{(\eta_\beta - \eta_a)}{2(1 - \eta_\beta)} \frac{w}{a} + \left[ \frac{(\eta_a + \eta_\beta)}{2} \Delta_1 - \frac{(\eta_\beta - \eta_a)}{2(1 - \eta_a)} \frac{\partial^2 w}{\partial x^2} - \frac{(\eta_a - \eta_\beta)}{2(1 - \eta_\beta)} \frac{1}{a^2} \frac{\partial^2 w}{\partial \theta^2} \right] z$$

The first bracketed term in the above expression is nothing more than $e_3$ while the second bracketed term may be expressed by curvature changes. Thus the expression for $e_z$ may be written as

$$e_z = \frac{\eta_a}{(1 - \eta_a)} e_1 + \frac{\eta_\beta}{(1 - \eta_\beta)} e_2 + \frac{\eta_a}{(1 - \eta_a)} K_1 z + \frac{\eta_\beta}{(1 - \eta_\beta)} K_2 z$$  \hspace{1cm} (13.16)
Substituting the expression of the strains \( e_x, e_\theta, \) and \( e_z \) into the stress strain relations, the following equations result.

\[
\sigma_x = \frac{1}{(1 - \eta_a)} \left[ (\tau_a + 2\mu_a) - 2\mu_a \eta_a \right] e_1 + \frac{1}{(1 - \eta_\beta)} \left[ (\tau_\beta + 2\mu_\beta) - 2\mu_\beta \eta_\beta \right] e_2 \\
+ \frac{1}{(1 - \eta_a)} \left[ (\tau_a + 2\mu_a) - 2\mu_a \eta_a \right] K_1 z + \frac{1}{(1 - \eta_\beta)} \left[ (\tau_\beta + 2\mu_\beta) - 2\mu_\beta \eta_\beta \right] K_2 z
\]

\( (13.17) \)

\[
\sigma_\theta = \frac{1}{(1 - \eta_a)} \left[ (\tau_\beta + 2\mu_\beta) - 2\mu_\beta \right] e_1 + \frac{1}{(1 - \eta_\beta)} \left[ (\tau_\beta + 2\mu_\beta) - 2\mu_\beta \eta_\beta \right] e_2 \\
+ \frac{1}{(1 - \eta_a)} \left[ (\tau_\beta + 2\mu_\beta) - 2\mu_\beta \right] K_1 z + \frac{1}{(1 - \eta_\beta)} \left[ (\tau_\beta + 2\mu_\beta) - 2\mu_\beta \eta_\beta \right] K_2 z
\]

\( (13.18) \)

\[
\sigma_{x\theta} = \mu_{a\beta} e_{12} + 2\mu_{a\beta} \tau z
\]

\( (13.19) \)

For convenience of expression, the stresses may be put in the form

\[
\sigma_x = M_1 e_1 + M_2 e_2 + [M_1 K_1 + M_2 K_2] z
\]

\( (13.20) \)

\[
\sigma_\theta = N_1 e_2 + N_2 e_1 + [N_1 K_2 + N_2 K_1] z
\]

\( (13.21) \)

\[
\sigma_{x\theta} = P e_{12} + (2 P \tau) z
\]

\( (13.22) \)
The constants \( M_1, M_2, \ldots P \) are defined by the equations

\[
M_1 = \frac{1}{(1 - \eta_a)} \left[ \tau_a + 2\mu_a - 2\mu_a \eta_a \right]
\]

\[
M_2 = \frac{1}{(1 - \eta_\beta)} \left[ (\tau_\beta + 2\mu_\beta - 2\mu_\beta \eta_\beta) \right]
\]

\[
N_1 = \frac{1}{(1 - \eta_\beta)} \left[ (\tau_\beta + 2\mu_\beta - 2\mu_\beta \eta_\beta) \right]
\]

\[
N_2 = \frac{1}{(1 - \eta_a)} \left[ (\tau_a + 2\mu_a - 2\mu_a \eta_a) \right]
\]

\[
P = \mu_{a\beta}
\]

The above constant may also be expressed in terms of the elastic constants. Thus

\[
M_1 = \frac{a_{22}}{(a_{11} a_{22} - a_{12}^2)}
\]

\[
M_2 = -\frac{a_{12}}{(a_{11} a_{22} - a_{12}^2)}
\]

\[
N_1 = \frac{a_{11}}{(a_{11} a_{22} - a_{12}^2)}
\]

\[
N_2 = -\frac{a_{12}}{(a_{11} a_{22} - a_{12}^2)}
\]

\[
P = \frac{1}{a_{66}}
\]
14. Stress Resultants

The definition of the stress resultants in terms of stresses for the isotropic shell is given by equation (6.15) of Chapter I. As has been pointed out these definitions hold equally well for an anisotropic shell and, hence, may be applied directly to the particular case being analyzed.

Substituting the values for $A$, $B$, $h_1$ and $h_2$ for a cylindrical shell into equations (6.15), the stress resultants become

\[ T_x = \int_{-\delta/2}^{\delta/2} \left(1 + \frac{z}{a}\right) \sigma_x \, dz \]

\[ T_\theta = \int_{-\delta/2}^{\delta/2} \sigma_\theta \, dz \]

\[ T_{x\theta} = \int_{-\delta/2}^{\delta/2} \sigma_{x\theta} \, d\theta \]

\[ M_x = \int_{-\delta/2}^{\delta/2} z \left(1 + \frac{z}{a}\right) \sigma_x \, dz \]

\[ M_\theta = \int_{-\delta/2}^{\delta/2} z \sigma_\theta \, dz \]

\[ M_{x\theta} = \int_{-\delta/2}^{\delta/2} z \sigma_{x\theta} \, dz \]

(14.1)
In the above equations a certain liberty is taken in tacitly assuming that \( T_{x\theta} = T_{\theta x} \) and \( M_{x\theta} = M_{\theta x} \). The justification for this step is based on the results of the isotropic derivation. In that case it is shown that those terms which cause the inequalities of the shear and twisting moment stress resultant are of order \( \delta k \) in comparison with other terms and hence may be neglected.

Since the same order of accuracy prevails in the orthotropic and isotropic analyses, it could be expected that the isotropic results in regard to the stress resultants being discussed would carry over to the present derivation.

Substitution of equations (13.20), (13.21), and (13.22) into the stress resultant expressions leads to the following set of equations.

\[
\begin{align*}
T_x &= \delta (M_1 e_1 + M_2 e_2) \\
T_{\theta} &= \delta (N_1 e_2 + N_2 e_1) \\
T_{x\theta} &= 2\delta P e_{12} \\
M_x &= \frac{\delta^3}{12} [M_1 K_1 + M_2 K_2] \\
M_{\theta} &= \frac{\delta^3}{12} [N_1 K_2 + N_2 K_1] \\
M_{x\theta} &= \frac{\delta^3 P \tau}{6}
\end{align*}
\] (14.2)
The expressions for the transverse shear stress resultants $Q_x$ and $Q_\theta$ are found by solving equations (12.4) and (12.5) for these quantities. Thus simplifying these expressions by combining terms and neglecting terms containing the tangential displacements $u$ and $v$ the following equations result.

$$Q_x = (\tau_a + 2\mu_a) \frac{\delta}{12} \frac{\partial}{\partial x} \Delta_1 + \frac{2\mu_a \eta_a}{(1 - \eta_a)} \frac{\delta}{12} \frac{\partial^3 w}{\partial x^3}$$

$$+ 2 \frac{\mu_a - \mu_\beta}{12} \frac{\delta}{a^2} \frac{\partial}{\partial \theta} \frac{\partial^3 w}{\partial x \partial \theta^2} + 2 \frac{\mu_\beta \eta_\beta}{(1 - \eta_\beta)} \frac{\delta}{12a^2} \frac{\partial^3 w}{\partial x^2 \partial \theta^2}$$

$$Q_\theta = (\tau_\beta + 2\mu_\beta) \frac{\delta}{12a} \frac{\partial}{\partial \theta} \Delta_1 + \frac{\mu_\beta \eta_\beta}{(1 - \eta_\beta)} \frac{\delta}{12a} \frac{\partial^3 w}{\partial \theta^3} + \frac{2\mu_\beta \eta_\beta}{(1 - \eta_\beta)} \frac{\delta}{12a} \frac{\partial^3 w}{\partial x^2 \partial \theta}$$

Substituting the expressions

$$\Delta_1 = \frac{1}{(1 - \eta_a)} K_1 + \frac{1}{(1 - \eta_\beta)} K_2$$

$$- \frac{\partial^2 w}{\partial x^2} = K_1$$

$$- \frac{1}{a^2} \frac{\partial^2 w}{\partial \theta^2} = K_2$$

$$- \frac{1}{a} \frac{\partial^2 w}{\partial x \partial \theta} = \tau$$

1The terms containing $u$ and $v$ can be shown to be of the same order of magnitude as the normal and shearing strains. Since these strains are of the order $\delta k$ compared to the curvature changes, they may be neglected.
The expressions for $Q_x$ and $Q_\theta$ may be written as

$$Q_x = \frac{1}{(1 - \eta_a)} \left[ (\tau_a + 2\mu_a) - 2\mu_a \eta_a \right] \frac{\delta^3 K_1}{12} \frac{\partial}{\partial x} + \frac{1}{(1 - \eta_\beta)}$$

$$\left[ (\tau_a + 2\mu_a) - 2\mu_a \right] \frac{\delta^3}{12} \frac{\partial K_2}{\partial x} + 2\mu_a \beta \frac{\delta^3}{12} \frac{\partial \tau}{\partial \theta} \tag{14.3}$$

$$Q_\theta = \frac{1}{(1 - \eta_\beta)} \left[ (\tau_\beta + 2\mu_\beta) - 2\mu_\beta \eta_\beta \right] \frac{\delta^3}{12} \frac{\partial K_2}{\partial \theta} + \frac{1}{(1 - \eta_a)}$$

$$\left[ (\tau_\beta + 2\mu_\beta) - 2\mu_\beta \right] \frac{\delta^2}{12a} \frac{\partial K_1}{\partial \theta} + 2\mu_a \beta \frac{\delta^3}{12} \frac{\partial \tau}{\partial x} \tag{14.4}$$

The above expression may also be written as

$$Q_x = M_1 \frac{\delta^3}{12} \frac{\partial K_1}{\partial x} + M_2 \frac{\delta^3}{12} \frac{\partial K_2}{\partial x} + 2P \frac{\delta^3}{12} \frac{\partial \tau}{\partial \theta} \tag{14.5}$$

$$Q_\theta = N_1 \frac{\delta^3}{12a} \frac{\partial K_2}{\partial \theta} + N_2 \frac{\delta^3}{12a} \frac{\partial K_1}{\partial \theta} + 2P \frac{\delta^3}{12} \frac{\partial \tau}{\partial x} \tag{14.6}$$

where the constants $M_1, M_2, N_1, N_2, P$ have been defined by equation (13.23).
15. Summary of Cylindrical Shell Equations

In terms of the elastic constants of the material, \( a_{ij} \), the displacement differential equations and the stress resultants may be written as follows.

\[
\frac{a_{22}}{(a_{11}a_{22} - a_{12}^2)} \frac{\partial^2 u}{\partial x^2} + \frac{1}{a_{66}} \frac{1}{a} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{a_{12}} \frac{\partial^2 u}{\partial \theta \partial \theta} = 0
\]

\[
+ \frac{a_{11}}{(a_{11}a_{22} - a_{12}^2)} \left( \frac{1}{a} \frac{\partial^2 v}{\partial x \partial \theta} - \frac{1}{a_{12}} \frac{\partial w}{\partial x} + \frac{X}{a} \right) = 0
\]

\[
- \frac{\partial^2 v}{\partial \theta^2} \left( \frac{a_{22}}{(a_{11}a_{22} - a_{12}^2)} \frac{\partial^4 w}{\partial x^4} + 2 \left( \frac{a_{12}}{(a_{11}a_{22} - a_{12}^2)} + \frac{2}{a_{66}} \right) \right)
\]

\[
\frac{1}{a^2} \frac{\partial^4 w}{\partial x^2 \partial \theta^2} + \frac{a_{11}}{(a_{11}a_{22} - a_{12}^2)} \left( \frac{1}{a} \frac{\partial^4 w}{\partial \theta^4} \right) - \frac{a_{11}}{(a_{11}a_{22} - a_{12}^2)} = 0
\]

\[
\frac{w}{a^2} + \frac{a_{12}}{(a_{11}a_{22} - a_{12}^2)} \frac{1}{a} \frac{\partial u}{\partial x} - \frac{a_{11}}{(a_{11}a_{22} - a_{12}^2)} \frac{1}{a} \frac{\partial v}{\partial \theta} + \frac{Z}{a} = 0
\]

(15.1)
\[ T_x = \frac{\delta}{(a_{11}a_{22} - a_{12}^2)} (a_{22}e_1 - a_{12}e_2) \]

\[ T_\theta = \frac{\delta}{(a_{11}a_{22} - a_{12}^2)} (a_{11}e_2 - a_{12}e_1) \]

\[ T_{x\theta} = \frac{2\delta}{a_{66}} e_{12} \]

\[ M_x = \frac{\delta^3}{12(a_{11}a_{22} - a_{12}^2)} (a_{22}K_1 - a_{12}K_2) \]

\[ M_\theta = \frac{\delta^3}{12(a_{11}a_{22} - a_{12}^2)} (a_{11}K_2 - a_{12}K_1) \quad (15.2) \]

\[ M_{x\theta} = \frac{\delta^3}{6a_{66}} \tau \]

\[ Q_x = \frac{\delta^3}{12(a_{11}a_{22} - a_{12}^2)} (a_{22} \frac{\partial K_1}{\partial x} - a_{12} \frac{\partial K_2}{\partial x}) + \frac{\delta^3}{6a_{66}} \frac{1}{a} \frac{\partial \tau}{\partial \theta} \]

\[ Q_\theta = \frac{\delta^3}{12(a_{11}a_{22} - a_{12}^2)} \frac{1}{a} (a_{11} \frac{\partial K_2}{\partial \theta} - a_{12} \frac{\partial K_1}{\partial \theta}) + \frac{\delta^3}{6a_{66}} \frac{\partial \tau}{\partial x} \]

The corresponding isotropic equations are

\[ \frac{1}{(1 - \nu^2)} \frac{\partial^2 u}{\partial x^2} + \frac{1}{2(1 + \nu)} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{2} \frac{1}{a^2} \frac{\partial^2 v}{\partial \theta^2} + \frac{1}{a^2} \frac{1}{\theta x \theta} \frac{\partial^2 v}{\partial \theta^2} + \frac{\nu}{(1 - \nu^2)} \frac{1}{a} \frac{\partial \omega}{\partial x} + \frac{X}{E \delta} = 0 \quad (15.3) \]
\[ \frac{1}{(1 - \nu^2)} \left( \frac{\delta^2 u}{\partial \theta^2} + \frac{1}{2(1 + \nu)} \frac{\delta^2 v}{\partial x^2} + \frac{1}{2(1 - \nu)} \frac{\delta^2 u}{\partial \theta \partial x} \right) \]

\[ + \frac{1}{(1 - \nu^2)} \frac{\delta w}{\partial \theta} + \frac{Y}{E\delta} = 0 \]

(15. 3 contd)

\[ - \frac{\delta^3}{12(1 - \nu^2)} \left[ \frac{\partial^4 w}{\partial x^4} + \frac{2}{a^2} \frac{\partial^4 w}{\partial x^2 \partial \theta^2} + \frac{1}{4} \frac{\partial^4 w}{\partial \theta^4} \right] - \frac{1}{(1 - \nu^2)} \frac{w}{a} \]

\[ - \frac{\nu}{(1 - \nu^2)} \frac{1}{a} \frac{\partial u}{\partial x} - \frac{1}{(1 - \nu^2)} \frac{1}{a} \frac{\partial v}{\partial \theta} + \frac{Z}{E\delta} = 0 \]

\[ T_x = \frac{\delta E}{(1 - \nu^2)} (e_1 + \nu e_2) \]

\[ T_\theta = \frac{\delta E}{(1 - \nu^2)} (e_2 + \nu e_1) \]

\[ T_{x \theta} = \frac{\delta E}{(1 + \nu)} \tau_{12} \]

(15. 4)

\[ M_x = \frac{E \delta^3}{12(1 - \nu^2)} (K_1 + \nu K_2) \]

\[ M_\theta = \frac{E \delta^3}{12(1 - \nu^2)} (K_2 + \nu K_1) \]

\[ M_{x \theta} = \frac{E \delta^3}{12(1 + \nu)} \tau \]
16. **Concluding Remarks**

The equations (15. 1) form a complete set for the determination of the middle surface displacements, i.e., v and w. Comparison of this set of equations with the corresponding isotropic ones reveals two things. First, the orthotropic equations do degenerate to those of the isotropic case, and second, no new additional terms arise in the equations because of the orthotropy of the material.

The only effect that orthotropy has on the displacement equations is to alter the constants.

The fact that the general form of the displacement equations is the same in the orthotropic as that of the isotropic case becomes of great importance. Excluding transformations of variables such as proposed by Vlasov in deriving equation (5. 29), any method that is applicable in solving the isotropic equations is equally applicable to the solution of the orthotropic equations. One such method which is extremely useful in engineering practice is the expansion of the displacements u, v, and w in the Fourier series.
The equations for the orthotropic stresses and strains, as the displacement equations, suffer only coefficient changes from their corresponding isotropic expressions. Examination of the orthotropic stress resultants when expressed in terms of middle surface strains, curvature changes, and torsion reveals that the stress resultants, with the exception of two, suffer only coefficient changes. However, the expressions for the transverse shearing forces \( Q_1 \) and \( Q_2 \), include the middle surface torsion, a term not found in the corresponding isotropic equations.

If the stresses, strains and stress resultants, even \( Q_1 \) and \( Q_2 \), were to be expressed in terms of the middle surface displacements, the same middle surface displacement functions found in the corresponding isotropic equations would result. Thus no new terms not found in the isotropic case would be present and the net effect of orthotropy would again be an alteration of the constant coefficients of these terms.

The cylindrical shell differential equations are stated in terms of middle surface displacements and hence the shell solution will be basically a displacement solution. Further, the expressions for the stresses, strains, and stress resultants are to be used in stating the boundary conditions to the differential displacement equation.
and hence ultimately must be expressed in displacement function form. Then from what has been stated in the preceding paragraphs in regard to the form of the orthotropic and isotropic differential equations as well as the expressions for the stress, strain and stress resultants, it must be concluded that for any cylindrical shell problem, the form of the resulting orthotropic displacement solution will be the same as that found in the corresponding isotropic case.

As a point of curiosity, the stress resultant expressions were found in yet another way. The stress-strain expressions were obtained directly from the generalized Hooke's law without recourse to the dilatation or Lamé's parameters, and then integrated. The resulting stress resultant expressions coincided exactly with those found in equation (14.2).

The implications of this coincidence of stress resultants is extremely important. It is obvious that an alternate method exists for solving any shell problem, isotropic or orthotropic, and that is by stating the differential shell equations in terms of stress resultants. However, the stress resultants are obtained in exactly the manner outlined above as an alternate method. Thus, congruence of the two expressions implies the following fact. The
expressions for the middle surface displacements that are found from the solutions of shell equations stated in terms of stress resultants will be exactly the same as those found from shell equations stated in terms of displacements.

The general orthotropic shell theory has included the material properties normal to the shell by means of the constants $a_{13}$ and $a_{23}$. Inspection of the shell equations (15. 1) reveals that for a cylinder these terms cancel out. Hence it must be concluded that an orthotropic cylindrical shell is independent of the elastic properties normal to its surface.
CHAPTER IV
APPLICATIONS

17. **Influence Coefficients for an Orthotropic Cylindrical Shell**

The results of Chapter III will now be used to determine the influence coefficients for a cylindrical orthotropic shell subjected to uniform moment and shear stress resultants along the cylinder's edge as shown in Figure 10. The shell is assumed to be semi-infinite.

Influence coefficients are that set of constants which arise in the determination of the edge displacement, \( w_o \), and slope, \( \frac{dw}{dx}_o \), by means of a linear combination of the edge shear and moment stress resultant, \( Q_x \) and \( M_x \) respectively. That is, if the edge displacements and slope may be put in the form

\[
\begin{align*}
    w_o &= \psi_{11} Q_{x_o} + \psi_{12} M_{x_o} \\
    \frac{dw}{dx}_o &= \psi_{21} Q_{x_o} + \psi_{22} M_{x_o}
\end{align*}
\]  

(17.1)

The quantities \( \psi_{ij} \) are defined as the influence coefficients.
Figure 10. Edge stress resultants $Q_{x_0}$ and $M_{x_0}$ acting on a semi-infinite cylindrical shell.
From the above discussion, it is obvious that the problem being dealt with is one possessing axial symmetry. Hence, as might be expected there can be no dependence on the angle \( \theta \) by any of the shell functions. Further, it is equally obvious that the tangential displacement \( v \) must be identically zero everywhere within the shell. Thus the shell equations (12.17) assume the simplified form.

\[
\begin{align*}
A_1 \frac{d^2 u}{dx^2} + A_4 \frac{dw}{dx} &= 0 \quad (17.2) \\
C_1 \frac{d^2}{dx^2} \left( \frac{4}{12} \frac{d^4 w}{dx^4} + \frac{C_4}{2} w + \frac{C_5}{a} \frac{du}{dx} \right) &= 0 \quad (17.3)
\end{align*}
\]

The boundary conditions to be above set of equations are in terms of stress resultants. Assuming now that the edge loadings \( Q_{x_0} \) and \( M_{x_0} \) are known, the following conditions result.

\[
\begin{align*}
T_x (x, \theta) &= 0 \quad (17.4) \\
Q_x (0, \theta) &= Q_{x_0} \\
M_x (0, \theta) &= M_{x_0}
\end{align*}
\]

From the definitions of the stress resultants, equations (14.2), the above boundary conditions may also be stated in terms of displacements. Simplifying the expressions so that there is no
dependency on the angle \( \theta \), equations (17.4) may be written as

\[
\delta \left[ M_1 \frac{du}{dx} + M_2 \frac{w}{a} \right] = 0 \quad (17.5)
\]

\[-\frac{\delta^3}{12} M_1 \frac{d^3 w}{dx^3} (x = 0) = Q_{x_o} \quad (17.6)\]

\[-\frac{\delta^3}{12} M_1 \frac{d^2 w}{dx^2} (x = 0) = M_{x_o} \quad (17.7)\]

Equation (17.5) yields a relation between \( w \) and \( \frac{du}{dx} \), that is

\[
\frac{du}{dx} = -\frac{M_2}{M_1} \frac{w}{a}
\]

When this relation is substituted into equation (17.3), a fourth order linear differential equation in \( w \) results. Combining terms, this equation may be written as

\[
\frac{d^4 w}{dx^4} + \frac{12}{a^2 b^2} \left( \frac{C_4}{C_1} - \frac{C_5}{C_1} \frac{M_2}{M_1} \right) w = 0
\]

Defining

\[
\beta^4 = \frac{3}{a^2 b^2} \left( \frac{C_4}{C_1} - \frac{C_5}{C_1} \frac{M_2}{M_1} \right) \quad (17.8)
\]

the differential equation is put in its final form.

\[
\frac{d^4 w}{dx^4} + 4 \beta^4 w = 0 \quad (17.9)
\]
The differential equation (17.9) is identical to the one obtained by Timoshenko\(^1\) for an axially symmetric loading on an isotropic cylindrical shell. The solution is relatively easy to obtain and is given as

\[ w = e^{-\beta x} (\bar{A} \cos \beta x + \bar{B} \sin \beta x) + e^{\beta x} (\bar{C} \cos \beta x + \bar{D} \sin \beta x) \]

where \(\bar{A}, \bar{B}, \bar{C},\) and \(\bar{D}\) are the constants of integration. Since the shell is semi-infinite and the external loadings are on the edge only, the displacement solution must be bounded for all values of \(x\). However, in order to ensure boundedness, the constants \(\bar{C}\) and \(\bar{D}\) must be identically zero. Thus the solution for the displacement \(w\) must be given as

\[ w = e^{-\beta x} (\bar{A} \cos \beta x + \bar{B} \sin \beta x) \]

The two constants of integration, \(\bar{A}\) and \(\bar{B}\), will be expressed in terms of the given edge loading. Thus, substituting the above expression for \(w\) into equations (17.6) and (17.7), the integration constants become

\[ \bar{A} = \frac{-6}{\beta^3 M_1} \left( \frac{Q}{\beta} + \frac{M}{x_0} \right) \]

\[ \bar{B} = \frac{6M}{\beta^3 M_1} \frac{x_0}{x_0} \]

\(^1\)See Timoshenko (1), p. 468. The solution to the differential equation is also given on that page.
The resulting solution for the displacement $w$ may then be written as

$$w(x) = -\frac{6}{\delta^3 \beta^2 M_1} e^{\beta x} \left[ \frac{Q_x}{\beta} + M_x \right] \cos \beta x - M_x \sin \beta x$$

(17.10)

At $x = 0$, let the deflection be $w_0$ and the corresponding slope $\frac{dw_0}{dx}$. Then from equation (17.10)

$$w_0 = -\frac{6}{\delta^3 \beta^2 M_1} \frac{Q_x}{\beta} - \frac{6 M_x}{\delta^3 \beta^2 M_1}$$

$$\frac{dw_0}{dx} = \frac{6 Q_x}{\delta^3 \beta^2 M_1} + \frac{12 M_x}{\delta^3 \beta M_1}$$

From the definition of the influence coefficients given by equation (17.1), it must be concluded that

$$\psi_{11} = -\frac{6}{\beta \delta^3 M_1}$$

$$\psi_{12} = -\frac{6}{\beta^2 \delta^2 M_1}$$

$$\psi_{21} = \frac{6}{\beta^2 \delta^2 M_1}$$

$$\psi_{22} = \frac{12}{\beta \delta^3 M_1}$$

(17.11)
The influence factor $\beta$ and the constant $M_1$ may be expanded in terms of the elastic constants of the material by use of equations (12.19) and (13.24). On the basis of these relationships,

$$\beta^4 = \frac{3}{a^2} \frac{(a_{11}a_{22} - a_{12}^2)}{a_{22}^2}$$  \hspace{1cm} (17.12)

$$M_1 = \frac{a_{22}}{(a_{11}a_{22} - a_{12}^2)}$$  \hspace{1cm} (17.13)

The corresponding isotropic values are given as

$$\beta^4 = \frac{3}{a^2} (1 - \nu^2)$$  \hspace{1cm} (17.14)

$$M_1 = \frac{1}{(1 - \nu^2)}$$  \hspace{1cm} (17.15)

To illustrate the difference between orthotropic and isotropic materials, several specific examples will be given for the calculation of the influence factor $\beta$ and the influence coefficients. In these examples, the material chosen will be wood. Since the material is inherently anisotropic, the isotropic values will be defined as the averages of the moduli of elasticity and Poisson's ratios. This procedure is in accord with the usual engineering practice for performing an isotropic analysis when confronted with an orthotropic material.
Example 1:

A plywood consisting of 3 or 5 layers of birch joined by a bakelite film has the following elastic properties.

\[ E_1 = 4.10 \times 10^4 \text{ psi} \quad \quad E_2 = 2.05 \times 10^4 \text{ psi} \]

\[ \nu_1 = 0.071 \quad \quad \nu_2 = 0.036 \]

\[ G = 0.23 \times 10^4 \text{ psi} \]

The average value of \( E \) and \( \nu \) are calculated as

\[ E_{av} = 3.08 \times 10^4 \text{ psi} \]

\[ \nu_{av} = 0.054 \]

Two sets of calculations will be presented for the orthotropic case. The first computation, indicated by the subscript 1, will be for the case of the direction of the largest value of \( E \) coinciding with the axis of the cylinder while the second calculation, indicated by the subscript 2, will be for the smallest \( E \) having a direct parallel to the cylinder axes.

The elastic constants for the first case are calculated as

\[ a_{11} = 24.4 \times 10^{-6} \quad \quad a_{22} = 48.8 \times 10^{-6} \]

\[ a_{12} = -1.735 \times 10^{-6} \quad \quad a_{21} = -1.735 \times 10^{-6} \]

\[ a_{66} = 435 \times 10^{-6} \]

\(^2\)Data is taken from Lekhnitski (2), p. 41.
while for the second case they are given as

\[ a_{11} = 48.8 \times 10^{-6} \quad a_{22} = 24.4 \times 10^{-6} \]
\[ a_{12} = -1.735 \times 10^{-6} \quad a_{21} = -1.735 \times 10^{-6} \]
\[ a_{66} = 435 \times 10^{-6} \]

On the basis of the above elastic properties, the following results for \( \beta \) are found.

\[ \beta_1 = 1.11 \sqrt{\frac{1}{a^2 \delta^2}} \]
\[ \beta_2 = 1.56 \sqrt{\frac{1}{a^2 \delta^2}} \]
\[ \beta_{av} = 1.32 \sqrt{\frac{1}{a^2 \delta^2}} \]

The values of the influence coefficients are given as

\[ \psi_{111} = -0.1066 \times 10^{-3} \left( \frac{a}{\delta^2} \right) \sqrt{a \delta} \]
\[ \psi_{121} = -0.1183 \times 10^{-3} \left( \frac{a}{\delta^2} \right) \]
\[ \psi_{211} = 0.1183 \times 10^{-3} \left( \frac{a}{\delta^2} \right) \]
\[ \psi_{221} = 0.2627 \times 10^{-3} \left( \sqrt{\frac{a \delta}{\delta^3}} \right) \]
\[
\begin{align*}
\psi_{11} &= -0.0763 \ (10)^{-3} \ (a/\delta^2)\sqrt{a\delta} \\
\psi_{12} &= -0.1196 \ (10)^{-3} \ (a/\delta^2) \\
\psi_{21} &= 0.1196 \ (10)^{-3} \ (a/\delta^2) \\
\psi_{22} &= 0.3735 \ (10)^{-3} \ (\sqrt{a\delta}/\delta^3) \\
\psi_{11}^{av} &= -0.0849 \ (10)^{-3} \ (a/\delta^2)\sqrt{a\delta} \\
\psi_{12}^{av} &= -0.1190 \ (10)^{-3} \ (a/\delta^2) \\
\psi_{21}^{av} &= 0.1190 \ (10)^{-3} \ (a/\delta^2) \\
\psi_{22}^{av} &= 0.2950 \ (10)^{-3} \ (\sqrt{a\delta}/\delta^3)
\end{align*}
\]

Example 3.2:

A plywood has the following properties

\[E_1 = 4.77 \ (10)^4 \text{ psi} \quad E_2 = 0.399 \ (10)^4 \text{ psi}\]
\[\nu_1 = 0.460 \quad \nu_2 = 0.0383\]
\[G = 0.409 \ (10)^4 \text{ psi}\]

The average values of \(E\) and \(\nu\) are calculated as

\[E^{av} = 2.59 \ (10)^4 \text{ psi}\]
\[\nu^{av} = 0.249\]

\[\text{Data taken from Lekhnitski (2), p. 40.}\]
As in example 1, two sets of calculations will be presented.

The calculations will be given the same subscript and meaning as in the previous example.

The elastic constants for the first case are calculated as

\[ a_{11} = 20.096 \times 10^{-5} \quad a_{22} = 251.25 \times 10^{-5} \]
\[ a_{12} = -0.964 \times 10^{-5} \quad a_{21} = -0.964 \times 10^{-5} \]
\[ a_{66} = 24.49 \times 10^{-5} \]

while for the second case, they are given as

\[ a_{11} = 251.25 \times 10^{-5} \quad a_{22} = 20.096 \times 10^{-5} \]
\[ a_{12} = -0.964 \times 10^{-5} \quad a_{21} = -0.964 \times 10^{-5} \]
\[ a_{66} = 24.49 \times 10^{-5} \]

The following results for \( \beta \) are calculated.

\[ \beta_1 = 0.696 \sqrt[4]{\frac{1}{a_{66}^2}} \]
\[ \beta_2 = 2.49 \sqrt[4]{\frac{1}{a_{66}^2}} \]
\[ \beta_{av} = 1.29 \sqrt[4]{\frac{1}{a_{66}^2}} \]
The corresponding values of the influence coefficients are calculated as follows.

\[
\begin{align*}
\psi_{11} & = -0.3661 \times 10^{-3} (a/\delta)^2 \sqrt{a\delta} \\
\psi_{12} & = -0.2549 \times 10^{-3} (a/\delta)^2 \\
\psi_{21} & = 0.2549 \times 10^{-3} (a/\delta)^2 \\
\psi_{22} & = 0.3545 \times 10^{-3} (\sqrt{a\delta}/\delta^3) \\
\psi_{11} & = -0.1026 \times 10^{-3} (a/\delta^2) \sqrt{a\delta} \\
\psi_{12} & = -0.2554 \times 10^{-3} (a/\delta^2) \\
\psi_{21} & = 0.2554 \times 10^{-3} (a/\delta^2) \\
\psi_{22} & = 1.2718 \times 10^{-3} (\sqrt{a\delta}/\delta^3) \\
\psi_{11} & = -0.3455 \times 10^{-3} (a/\delta^2) \sqrt{a\delta} \\
\psi_{12} & = -0.4459 \times 10^{-3} (a/\delta^2) \\
\psi_{21} & = 0.4459 \times 10^{-3} (a/\delta^2) \\
\psi_{22} & = 1.1502 \times 10^{-3} (\sqrt{a\delta}/\delta^3)
\end{align*}
\]
Example 3:

A piece of sitka spruce is made into a thin shell by peeling the wood circumferentially from the tree. Let $E_L$ represent the modulus of elasticity along the grain (parallel to the tree trunk), $E_R$ the modulus of elasticity in a direction perpendicular to the growth rings of the tree and $E_T$ the modulus of elasticity tangent to the growth rings. The elastic properties for this material are listed as $^4$

$$E_L = 17.27 \times 10^5 \text{ psi}$$
$$E_T = 0.743 \times 10^5 \text{ psi}$$
$$E_R = 1.347 \times 10^5 \text{ psi}$$

$$\nu_{LR} = 0.372$$
$$\nu_{LT} = 0.467$$
$$\nu_{RT} = 0.435$$

The radial direction of the shell will correspond to values with subscript R. As in the previous examples, two cases will be considered. The first case will be where the grain is parallel to the axis of the shell ($E_L = E_x$) while the second case will be where the grain runs in a circumferential direction ($E_T = E_x$).

As before, the subscript 1 will be used to denote the first and subscript 2 the second. The average values of $E$ and $\nu$ are calculated by considering $E_L$, $E_T$, and $\nu_{LT}$ and $\nu_{TL}$ alone. The average values are calculated as

$$E_{av} = 9.10 \times 10^5 \text{ psi}$$
$$\nu_{av} = 0.420$$

$^4$ Data for this wood is listed in (3), p. 79.
The elastic constants for the first case are computed as

\[ a_{11} = 0.579 \times 10^{-6} \quad a_{22} = 13.46 \times 10^{-6} \]

\[ a_{12} = -0.2704 \times 10^{-6} \quad a_{12} = -0.2704 \times 10^{-6} \]

while for the second case they are given as

\[ a_{11} = 13.46 \times 10^{-6} \quad a_{22} = 0.579 \times 10^{-6} \]

\[ a_{12} = -0.2704 \times 10^{-6} \quad a_{21} = -0.2704 \times 10^{-6} \]

The computed values of \( \beta \) are given as

\[ \beta_1 = 0.600 \sqrt{\frac{4}{a^2 \delta^2}} \]

\[ \beta_2 = 2.89 \sqrt{\frac{4}{a^2 \delta^2}} \]

\[ \beta_{av} = 1.26 \sqrt{\frac{4}{a^2 \delta^2}} \]

The corresponding values of the influence coefficients are as follows.

\[ \psi_{11} = -1.5932 \times 10^{-5} (a/\delta^2) \sqrt{a \delta} \]

\[ \psi_{12} = -0.9559 \times 10^{-5} (a/\delta^2) \]

\[ \psi_{21} = 0.9559 \times 10^{-5} (a/\delta^2) \]

\[ \psi_{22} = 1.1471 \times 10^{-5} (\sqrt{a \delta}/\delta^3) \]
\[
\begin{align*}
\psi_{11} & = -0.3315 \times 10^{-5} \left( \frac{a}{\delta^2} \right) \sqrt{a \delta} \\
\psi_{12} & = -0.9585 \times 10^{-5} \left( \frac{a}{\delta^2} \right) \\
\psi_{21} & = 0.9585 \times 10^{-5} \left( \frac{a}{\delta^2} \right) \\
\psi_{22} & = 5.5363 \times 10^{-5} \left( \frac{\sqrt{a \delta}}{\delta^3} \right) \\
\psi_{11}^{av} & = -0.2755 \times 10^{-5} \left( \frac{a}{\delta^2} \right) \sqrt{a \delta} \\
\psi_{12}^{av} & = -0.3466 \times 10^{-5} \left( \frac{a}{\delta^2} \right) \\
\psi_{21}^{av} & = 0.3466 \times 10^{-5} \left( \frac{a}{\delta^2} \right) \\
\psi_{22}^{av} & = 0.8720 \times 10^{-5} \left( \frac{\sqrt{a \delta}}{\delta^3} \right)
\end{align*}
\]

18. Concluding Remarks

The solutions for the clamping constants and the influence coefficients \( \psi_{ij} \) exhibit several peculiarities not in immediate evidence from an inspection of the equations alone. The peculiarities are sufficiently important that each of them warrants some discussion.

In the isotropic expressions for \( \beta \) and \( \psi_{ij} \), the only elastic parameter which enters the equations is Poisson's ratio \( \nu \). It might then be suspected that bounds on the possible variation of these two quantities could be obtained by substituting the maximum and minimum values of Poisson's ratio into the isotropic equations.
However, as evidenced by the results in the three examples solved, such is not the case. The actual variation of $\beta$ and $\psi_{ij}$ with the principal plane directions is much greater than might be predicted from the isotropic analysis mentioned. No explanation can be given for this effect save that it is a peculiarity of orthotropic shells.

Example 2 represents a case of mild orthotropy ($E^1/E^2 = 2$) while examples 2 and 3 are cases of large orthotropy ($E^1/E^2 \gg 10$). Examination of the computed values of $\beta$ for the three examples shows that the average values, found by averaging Poisson's ratio, falls between the maximum and minimum values of $\beta$ for a given material. This situation is not true for the influence coefficients $\psi_{ij}$. In example 1, the average values of $\psi_{ij}$, again computed by considering an average value of Poisson's ratio, lies between the maximum and minimum values for that material. However, as the orthotropic ratio $E_1/E_2$ is increased, as in examples 2 and 3, the average values of the influence coefficients no longer lie between the maximum and minimum values. Thus it must be concluded that when dealing with orthotropic materials, an averaging of the elastic properties and their use in isotropic equations does not necessarily lead to average values of the computed functions.
The variation of $\beta$ with the elastic constants in the orthotropic case is unexpected. In order to limit the propagation of an end disturbance along the axis of the shell, one would think that the material should be stiffened in that direction so as to make the effective value of $E_\chi$ larger than $E_{\phi}$. However, inspection of the results as well as the orthotropic expression for $\beta^4$ indicates that increasing $E_\chi$ alone will cause the disturbance to spread further along the length of the shell. In order to decrease the propagation, the tangential stiffness must be increased.

If one remembers that the shell being dealt with is subjected to axially symmetrical edge moments and shears, and that the only displacement of the shell is the radial displacement $w$ (neglecting the Poisson effect on axial stretch or contraction), the above results can be rationalized. Though the stiffness in the $\chi$ direction affect $\beta$, the stiffness in the $\theta$ direction acts as an elastic bandage about the cylinder. It is completely analogous to a variable pressure acting such as to restore the cylinder to its undeformed shape. As such, it might be expected that the $\theta$ stiffness would more markedly affect the propagation and also the magnitude of the displacements.
REFERENCES


CHAPTER V

CONCLUSIONS

Initially, it was hoped that the general orthotropic shell equations could be brought to the same state of development as those of the isotropic case. However, inspection of the intermediate derived expressions, especially those for the dilatation components and the displacement component $u_\gamma$ indicated that such an effort, though possible, would have rendered the resulting equations almost useless because of the complexity of expression. Hence, it was felt that presenting the set of five orthotropic equations with auxiliary expressions for $\Delta$ and $u_\gamma$ was the simplest method of handling the problem.

The relative ease with which the derived orthotropic equations can be applied to a specific problem is illustrated in the case of the cylinder. For that particular problem, the resulting three displacement equations are such that a Fourier series analysis can be made, since in each equation either odd or even derivatives of a displacement component with respect to a given variable appear.
Expressions for all but two of the stress resultants for the general orthotropic shell may be obtained without any consideration of the equilibrium equations. Substitution of the stress-strain relations and the displacements, as given by the Kirchoff hypothesis, into the definitions of the stress resultant suffices for the determination of these quantities.

The expressions for the transverse shear forces $Q_1$ and $Q_2$ are not directly calculated from their stress resultant definitions. In order to do so, the displacement $u_\gamma$ must be determined. The orthotropic development presented in this dissertation contains expressions for $u_\gamma$, and hence $Q_1$ and $Q_2$ may be calculated from definition. However, shell equations presented in terms of stress resultants do not include $u_\gamma$, and hence, $Q_1$ and $Q_2$ are calculated from equilibrium equations.

Excluding for the moment any consideration of $Q_1$ and $Q_2$, the stress resultants were indeed calculated as indicated above. The remarkable result was that these stress resultants, when expressed in terms of middle surface strains, changes of curvature or twist did not contain terms which were not common in their corresponding isotropic expressions. Thus the sole effect of orthotropy was merely to alter the constants.
Novozhilov's work (1) for isotropic shells possesses great advantages in the matter of equation solution. This work, presented in terms of stress resultants, utilizes asymptotic integration in solving the general shell equations. However, in order to develop the basic differential equation for this type of solution, a reduction in the order of the shell equations is mandatory. To effect such a reduction, complex stress resultants are introduced which ultimately by transformation reduce the shell equations to the one equation desired.

An attempt was made to reduce the general orthotropic shell equations expressed in terms of other resultants to an equation of the type utilized by Novozhilov for the asymptotic integration. However, after considerable effort, a complex transformation of the stress resultants could not be found for the reduction of the equations.

In general, it may be concluded that if a derivation for the orthotropic shell is to be patterned after any isotropic development, care should be exercised that the isotropic development is not dependent on any transformations. The experience to date indicates that a corresponding transformation for the orthotropic derivation will be difficult to make.
The derived orthotropic equations are applicable to any continuous material with constant orthotropic constants and a constant thickness $\delta$. However, in a great many engineering structures, the orthotropy is artificially created by either milling slots or adding longitudinal stillness to the shell. The net result is that the shell is discontinuous in its elastic properties.

To analyze this type of orthotropy, called constructional orthotropy, the shell is assumed to possess continuous orthotropic properties. The determination of these equivalent orthotropic constants is based on a consideration of the elongations and shear deformations of some representative section of the shell. Though the equivalent elastic constants due to the elongations and shearing of the shell are relatively easy to obtain, such is not the case when the equivalent Poisson's ratios are considered.

Orthotropic elasticity demands that the orthotropic constants $a_{ij}$ possess diagonal symmetry, that is, $a_{ij} = a_{ji}$. However, in considering structurally orthotropic shells, this condition can seldom be realized since the equivalent Poisson ratios $\nu_{21}$ and $\nu_{12}$, each of which are calculated separately will not obey the relation

$$\nu_{12} E_2 = \nu_{21} E_1$$

---

1 The quantities correspond to moduli of elasticity $E$ and rigidity $G$ in the isotropic case.
In order to circumvent this difficulty, one of two possible choices are usually made. The first is to calculate either \( \nu_{21} \) or \( \nu_{12} \) and then obtain the other from the above relationship. The second choice is to calculate both \( a_{21} \) and \( a_{12} \) and then average the two quantities thus obtaining a quantity \( \bar{a}_{12} \) which will satisfy the condition that \( \bar{a}_{12} = \bar{a}_{21} \).

Huffington (2) and Hoppmann (3) have made experimental studies of a stiffened cylindrical shell. Their results indicate that the above methods of defining equivalent orthotropic Poisson ratios can yield considerable error, especially in the lending stiffnesses. However, they offer no remedy to the problem.

Another problem encountered on structurally orthotropic shell is the possible dependency of the elastic constants on the coordinates \( a \) and \( \beta \). If, for example, stiffeners are placed along the principal coordinate lines emanating from the apex of a conical shell, a repeating section taken near the base will differ in size from that taken near the apex. Since the repeating section is to be used for the calculation of the equivalent orthotropic constants, it is obvious that these constants, when calculated for sections near the base will differ from those calculated near the apex. Thus, in this instance, the orthotropic constants will be dependent on the curvilinear coordinates.
Except for the cylindrical case, most structurally orthotropic shells will exhibit the variation of elastic properties with the curvilinear coordinates. In order to utilize an orthotropic analysis for the solution of such shells, either the elastic constants or thickness of the shell must be considered a function of the coordinates $a$ and $\beta$. By far, the most convenient choice would be in holding the elastic constants constant and varying the thickness of the shell.

From the discussion of the preceding paragraphs it is seen that the application of the derived orthotropic shell equations is limited in its use for structurally orthotropic materials. Except for the case of the cylinder, these equations must be applied to a material which is orthotropic in nature.

Toward the end of the work leading to this dissertation, a book by Ambartsumian (4) has appeared dealing with the anisotropic shell. Inspection of this book reveals that the equations of equilibrium are presented in terms of stress resultants and that the Kirchoff hypotheses are utilized in derivations. In order to develop a set of differential equations for the solution of the general anisotropic shell, the equilibrium equations, which are in terms of the stress resultants, are transformed to middle surface displacement equations by means of the stress-strain relations.
Ambartsumian's book is more general in scope in dealing with the anisotropic shell than the present work. In particular, the book deals with materials possessing but one plane of elastic symmetry which is tangent to the shell middle surface while the present work is confined to materials possessing three planes of elastic symmetry. Thus, Ambartsumian's work may be considered an extension to shells of Lekhnitski's work of anisotropic plates in that the latter work also assumes but one plane of elastic symmetry.

The problem of multi-layered laminated shells is becoming of increasing importance in engineering. In this type of a shell, each of the laminates possesses different elastic properties. To analyze this shell, an equivalent set of elastic properties are derived assuming the shell to be of one material. The equivalent elastic properties are anisotropic in nature and in particular can be shown to exhibit but one plane of elastic symmetry tangent to the shell middle surface.

Thus, from the preceding paragraph, it is obvious that a laminated shell cannot be analyzed by an orthotropic analysis and hence Ambartsumian's book can solve a class of problems which cannot be analyzed by the present work. However, the present work does develop the orthotropic equations in a unique manner which appears simpler than other derivations for orthotropic materials.
and it can be extended to materials with but one plane of elastic symmetry. Whether the extension of the present work to such materials will yield simpler expressions than those given by Ambartsumian is not known at the present time.

In the derivation of the shell equations in his book, Ambartsumian utilizes the Kirchoff hypotheses. Hence his analyses and solutions of the multi-layered shell problem have inherent in them the consequences of these hypotheses. However, in a recent survey of anisotropic shell analyses as applied to multi-layered shells (5), Ambartsumian points out that the shear deformations in such shells deform the normal of the shell middle surface to such an extent that the assumption of undeformable normals may lead to errors larger than $(\delta k)$. He further points out the need for some other assumptions in regard to normals in order to reasonably predict the behavior of the multi-layered shell.
REFERENCES


AUTOBIOGRAPHY

I, Thomas Joseph Kozik, was born in Jersey City, New Jersey, April 9, 1930. I received my secondary school training in Ferris High School, Jersey City, and my undergraduate education in aeronautical engineering at Rensselaer Polytechnic Institute, which granted me the Bachelor of Aeronautical Engineering degree in 1952. I received the Master of Science degree from The Ohio State University in 1957.

From 1952 to 1954 I was employed as a structural engineer by the Curtiss-Wright Corporation at Caldwell, New Jersey. From 1954 to 1956 I served on active duty in the U. S. Army Signal Corps at Fort Huachuka, Arizona.

In autumn of 1956 I was employed by The Ohio State University as a graduate assistant in the department of Engineering Mechanics while working on my Master of Science degree. Since 1957 I have been an instructor while completing the requirements for the Doctor of Philosophy degree.