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THE ATTITUDE MOTION AND STABILITY OF A SPINNING SATELLITE UNDER THE INFLUENCE OF THE EARTH'S GRAVITY GRADIENT TORQUE

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in the Graduate School of The Ohio State University

by


The Ohio State University
1962

Approved by

[signature]
Adviser
Department of Engineering Mechanics
To Kay, Mom, and Dad
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SYMBOLS

x, y, z  Body-fixed principal axes with origin at the mass center G of the satellite.

ξ, η, ζ  Node-system axes with origin at the mass center G and to be described more completely when used.

\( \mathbf{i} \)  Unit vector in the unperturbed orbit plane and in the direction of the forward motion of the satellite.

\( \mathbf{j} \)  Unit vector normal to the unperturbed orbit plane and such that \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) form a right-hand triad.

\( \mathbf{k} \)  Unit vector outward along the line joining the geocenter to the mass center G of the satellite. This line will be called the geocentric vertical.

A, B, C  Principal moments of inertia about the x, y, z axes.

I_v  A moment of inertia to be defined when used.

R_e  Radius vector of the earth at which the gravitational acceleration g is measured.

\( \mathbf{R}_G \)  Position vector measured from the geocenter to the mass center G of the satellite.

ϕ, θ, ψ  Euler angles to be defined when used.

w_x, w_y, w_z  Components of the absolute angular velocity vector \( \mathbf{w} \) of the satellite with respect to the principal body-fixed axes x, y, z.
SYMBOLS (cont'd)

$\omega_\xi$, $\omega_\eta$, $\omega_\zeta$ Components of the absolute angular velocity vector $\vec{\omega}$ of the satellite with respect to the node-system axes $\xi$, $\eta$, $\zeta$.

Subscript $^0$ denotes a steady motion quantity.

Subscript $^\delta$ denotes a deviational quantity from the steady motion.
INTRODUCTION

The attitude of a satellite is defined as the angular orientation of the satellite in space. This angular orientation is generally measured by the Euler angles which give the angular position of a set of axes fixed, or in some instances partially fixed, in the body such as the principal axes of inertia. The particular choice of Euler angles $\phi$, $\theta$, and $\psi$ used in this study is shown in Figure 4.

The problem of the attitude motion and stabilization of a satellite in orbit has become of increasing importance as the uses or applications of artificial satellites increase in scope as well as in precision. For example, the use of an artificial satellite for the accurate mapping of the surface of a celestial body would require not only an exact determination of the position of the satellite in its orbit but also the exact angular orientation or attitude of the satellite with respect to its orbit. The accuracy of the mapping would be greatly diminished if the angular position of the mapping equipment with respect to the celestial body was not known precisely. Another example in which
the attitude motion of the satellite would be of great importance would be in the area of manned space vehicles or stations. Not only would it be annoying and possibly discom forting for the crew to have their "surroundings," such as the star field or "horizon" continually change in orientation (125), but uncontrolled motions or disorientation of the spacecraft would naturally induce errors in observations or experiments conducted on board. These are but a few of the applications in which the attitude motion and the control of it are significant aspects of the design and analysis of proposed space vehicles.

Prior to the need for extremely accurate inertial guidance systems and prior to the coming of artificial satellites, the topic of attitude motion of a rigid body was largely confined to the study of motion of a body about a fixed point, for which there is a vast literature (44, 63, 64, 100, 124, 129, 140, 162, 164, 167, 180), and to the study of the most famous satellite in orbit, our moon (24, 105, 140). These problems were of interest to only classical dynamicists, a limited number of engineers working with gyroscopic instruments, and

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1 Numbers in parentheses refer to the Bibliography.
astronomers. However, in recent years, with the advent of missiles and satellites which require extremely accurate guidance and control, the field of attitude motion and stabilization has undergone renewed interest as well as an expansion into areas previously unexplored.

One facet of the problem of attitude motion and control is the problem of the change in the satellite's attitude due to perturbing torques. R. E. Roberson in a paper presented at the VIII International Astronautical Congress in Barcelona, 1957 (128), lists and discusses eleven possible sources of perturbing torques:

1. the earth's magnetic field
2. the earth's electric field
3. the sun's radiation pressure
4. the pressure of electromagnetic radiation from the satellite
5. the "atmospheric" drag
6. meteroid bombardment
7. cosmic ray bombardment
8. the gravitational fields of celestial bodies
9. the nonuniform rotation of reference coordinates
10. the moving parts within the satellite
11. the earth's gravitational field
More recently, another source causing change in the attitude of a satellite was discovered from observations made on Explorer I (122). The change in attitude, however, was not caused by a perturbing torque but rather by the dissipation of energy in the elastic parts of the satellite. In the case of Explorer I, the flexible antennas of the satellite provided an excellent source for energy dissipation. Papers by W. T. Thomson and G. S. Reiter (166) and L. Meirovitch (102) have only begun to analyze this phenomenon which is a whole new area in the problem of altitude motion and control of a space vehicle.

A search of the current literature revealed that many papers have been written on the attitude motion of a space vehicle and these have been listed in the Bibliography. Many of these papers dealt with the perturbing torques listed above and analyzed their effects on the attitude motion (8, 12, 26, 35, 43, 55, 66, 67, 77, 81, 82, 88, 109, 110, 112, 121, 129, 130, 131, 143, 146, 147, 148, 153, 165, 171, 174, 178, 179, 181).

It is the purpose of this dissertation to investigate analytically those aspects of the attitude motion and stabilization of a rigid, spinning satellite which result from the perturbing torque exerted by the earth's gravitational field. One of the objectives
of this paper is to demonstrate what some of the motions of a spinning satellite would be if it was not adequately controlled when acted on by the gravity gradient torque. Although the actual magnitude of the gravitational torque is small, it is significant because, unlike some of the other perturbing torques, it acts continuously and its cumulative effect over a long period could be serious.

This paper first presents a derivation of the resultant gravitational force, the gravity gradient torque, and the potential energy of the satellite. Following this, the equations of motion of the mass center and the general attitude equations of motion are derived. Two integrals of the equations of motion are also presented. However, due to the complexity of the equations of motion, various simplifying assumptions are made and the resulting special cases are studied. The first case studies the small oscillations about a steady precession of a spinning, unsymmetrical satellite. The second case touches on the small disturbances in the motion of an unsymmetrical spinning satellite. A study is then made of the behavioral motion and small oscillations of a spinning, symmetrical satellite.
Although the systems for the actual control of the attitude motion are also of great importance in the design of a space vehicle, this dissertation does not discuss or examine this particular phase of the problem. However, there is a wealth of material in the literature on the analysis, design, and construction of attitude control systems. A number of these papers are listed in the Bibliography (1, 2, 4, 14, 25, 32, 42, 54, 56, 60, 61, 68, 70, 72, 74, 80, 83, 84, 87, 91, 101, 114, 115, 116, 117, 118, 120, 137, 150, 152, 154, 158, 160, 163, 168, 172, 173, 176, 183).
THE RESULTANT GRAVITATIONAL FORCE, THE GRAVITY GRADIENT TORQUE, AND THE POTENTIAL ENERGY

The Resultant Gravitational Force

According to Newton's law of gravitation, the magnitude of the force of attraction between two particles is

\[ F = G \frac{m_1 m_2}{r^2} \quad (1) \]

where

- \( m_1 \) and \( m_2 \) are the masses of the particles;
- \( r \) is the distance between the particles;
- \( G \) is the universal constant of gravitation.

Application of this law to a particle on the earth's surface permits us to write

\[ G = \frac{g R_e^2}{M_e} \quad (2) \]

where

- \( g \) is the acceleration of gravity measured at \( R_e \), the radius of the earth;
- \( M_e \) is the mass of the earth.
If the earth is considered a sphere, the force of attraction of the earth on a particle \( m_i \) is

\[
\vec{F} = -\frac{gR_e^2m_i}{R_i^3} \vec{R}_i
\]  

(3)

where

\( \vec{R}_i \) is the position vector of \( m_i \) as measured from the center of the earth and is shown in Figure 1.

The resultant force exerted by the earth on a collection of particles or a finite body is

\[
\sum \vec{F}_i = -gR_e \sum \frac{m_i}{R_i^3} \vec{R}_i
\]  

(4)

As shown in Figure 1,

\[
\vec{R}_i = \vec{R}_G + \vec{r}_i = R_i \hat{K} + \vec{r}_i
\]  

(5)

where

\( \vec{R}_G \) is the position vector of the mass center \( G \) of the collection or body;

\( \vec{r}_i \) is the position vector of the particle \( m_i \) relative to the mass center \( G \);

\( \hat{K} \) is a unit vector outward from the geocenter along the line joining the geocenter to the mass center \( G \).

Thus,

\[
R_i^{-3} = \left[ (R_G \hat{K} + \vec{r}_i) \cdot (R_G \hat{K} + \vec{r}_i) \right]^{-\frac{3}{2}}.
\]  

(6)
Figure 1. Position Vectors $\vec{R}_i$, $\vec{R}_G$, $\vec{r}_i$
Earth Plane of the Unperturbed Orbit

\( \hat{I} \) -- Unit Vector in the Direction of Forward Motion
\( \hat{J} \) -- Unit Vector Normal to the Unperturbed Orbit Plane
\( \hat{K} \) -- Unit Vector along the Local Vertical

**Figure 2a.** The Unit Vectors \( \hat{I}, \hat{J}, \hat{K} \)

**Figure 2b.** The Unit Vectors \( \hat{i}, \hat{j}, \hat{k} \) and the Principal Axes \( x, y, z \)

**Figure 2.** The Unit Vectors \( \hat{i}, \hat{j}, \hat{k} \) and \( \hat{I}, \hat{J}, \hat{K} \)
Expanding the dot product,
\[ R_i^{-3} = R_G^{-3} \left[ 1 + 2 \left( \frac{r_i}{R_G} \right) \cdot \vec{K} + \left( \frac{r_i}{R_G} \right) \cdot \left( \frac{r_i}{R_G} \right) \right] \frac{3}{2}. \] (7)

Using the binomial expansion,
\[ R_i^{-3} = R_G^{-3} \left[ 1 - 3 \left( \frac{r_i}{R_G} \right) \cdot \vec{K} - \frac{15}{2} \left( \frac{r_i}{R_G} \right) \cdot \left( \frac{r_i}{R_G} \right)^2 \right]. \] (8)

The terms in \( \left( \frac{r_i}{R_G} \right)^3 \) and higher were neglected. For any reasonable satellite this is justified.

The resultant force can then be written
\[ \Sigma \vec{F}_1 = -\frac{g R^2}{R_G^3} \sum m_i \left[ 1 - 3 \left( \frac{r_i}{R_G} \right) \cdot \vec{K} - \frac{15}{2} \left( \frac{r_i}{R_G} \right) \cdot \left( \frac{r_i}{R_G} \right)^2 \right] \left[ R_G \vec{K} + \frac{r_i}{R_G} \right]. \] (9)

Expanding and using the fact that \( \sum m_i \frac{\vec{r_i}}{R_G} = 0 \), since this defines the position of the mass center \( G \), the resultant force is
\[ \Sigma \vec{F}_1 = -\frac{g R^2}{R_G^2} \sum m_i \vec{K} + \frac{3}{2} \frac{g R^2}{R_G^4} \sum m_i \frac{r_i^2}{R_G^2} \vec{K} \]
\[ -\frac{15}{2} \frac{g R^2}{R_G^4} \sum m_i \left( \frac{r_i}{R_G} \cdot \vec{K} \right)^2 \vec{K} + 3 \frac{g R^2}{R_G^4} \sum m_i \left( \frac{r_i}{R_G} \cdot \vec{K} \right) \frac{\vec{r_i}}{R_G}. \] (10)
For any vector space with bases \( \vec{i}, \vec{j}, \) and \( \vec{k} \) and \( \vec{I}, \vec{J}, \vec{K} \), we may express

\[
\vec{I} = c_{11} \vec{i} + c_{12} \vec{j} + c_{13} \vec{k} ; \\
\vec{J} = c_{21} \vec{i} + c_{22} \vec{j} + c_{23} \vec{k} ; \\
\vec{K} = c_{31} \vec{i} + c_{32} \vec{j} + c_{33} \vec{k} ;
\]

where

\[
c_{11} = \vec{i} \cdot \vec{I} , \quad c_{12} = \vec{j} \cdot \vec{I} , \quad c_{13} = \vec{k} \cdot \vec{I} ; \\
c_{21} = \vec{i} \cdot \vec{J} , \quad c_{22} = \vec{j} \cdot \vec{J} , \quad c_{23} = \vec{k} \cdot \vec{J} ; \\
c_{31} = \vec{i} \cdot \vec{K} , \quad c_{32} = \vec{j} \cdot \vec{K} , \quad c_{33} = \vec{k} \cdot \vec{K} .
\]

If \( \vec{i}, \vec{j}, \vec{k} \) are the unit vectors along the principal body-fixed axes \( x, y, z \), the coordinates of \( m_1 \) with respect to the principal axes may be written as

\[
x_1 = (\vec{r}_1 \cdot \vec{i}) , \quad y_1 = (\vec{r}_1 \cdot \vec{j}) , \quad z_1 = (\vec{r}_1 \cdot \vec{k}).
\]

The component of the force along the \( \vec{K} \) direction is

\[
\Sigma F_1 \cdot \vec{K} = -\frac{g R_e^2}{R G} \Sigma m_1 + \frac{3}{2} \frac{g R_e^2}{R G} \Sigma m_1 \vec{r}_1^2 \\
- \frac{9}{2} \frac{g R_e^2}{R G} \Sigma m_1 (\vec{r}_1 \cdot \vec{K})^2.
\]
Using equations (11), (12), (13) so that

\[( \vec{r}_1 \cdot \vec{K}) (\vec{r}_1 \cdot \vec{K}) = c_{31}^2 (\vec{r}_1 \cdot \vec{i})^2 + c_{32}^2 (\vec{r}_1 \cdot \vec{j})^2 + c_{33}^2 (\vec{r}_1 \cdot \vec{k})^2 + \]

[Cross product term \((\vec{r}_1 \cdot \vec{i})(\vec{r}_1 \cdot \vec{j})\), etc.];

\[( \vec{r}_1 \cdot \vec{K}) (\vec{r}_1 \cdot \vec{K}) = c_{31} x_1^2 + c_{32} y_1^2 + c_{33} z_1^2 + [\text{Cross product terms}] \]

and since \(x, y, z\) are principal axes,

\[\Sigma \vec{F} \cdot \vec{K} = -\frac{g R_e^2}{R_G^2} \Sigma m_i + \frac{3}{2} \frac{g R_e^2}{R_G^4} \Sigma m_i r_i^2 - \frac{9}{2} \frac{g R_e^2}{R_G^4} \Sigma [c_{31} m_i x_i^2 \]

\[+ c_{32} m_i y_i^2 + c_{33} m_i z_i^2]. \tag{16}\]

However,

\[\Sigma m_i r_i^2 = \Sigma m_i (x_i^2 + y_i^2 + z_i^2); \]

\[A = \Sigma m_i (y_i^2 + z_i^2); \quad B = \Sigma m_i (x_i^2 + z_i^2); \quad C = \Sigma m_i (x_i^2 + y_i^2); \tag{17}\]

\[A + B + C = \text{An invariant} = 2 \Sigma m_i r_i^2; \]

and thus,

\[\Sigma m_i x_i^2 = \frac{1}{2} (A + B + C) - A; \]

\[\Sigma m_i y_i^2 = \frac{1}{2} (A + B + C) - B; \tag{18}\]

\[\Sigma m_i z_i^2 = \frac{1}{2} (A + B + C) - C. \]
The resultant force in the \( \vec{K} \) direction can then be written

\[
\Sigma \vec{F}_1 \cdot \vec{K} = -\frac{gR_e^2}{R_G^2} \Sigma m_1 - \frac{3 gR_e^2}{4 R_G^2} [(A + B + C)

- 3 (c_{31} A + c_{32} B + c_{33} C)]
\]

(19)

where

\[ c_{31} = \vec{i} \cdot \vec{K}, \quad c_{32} = \vec{j} \cdot \vec{K}, \quad c_{33} = \vec{k} \cdot \vec{K}. \]

Similarly, the resultant force along an orthogonal \( \vec{J} \) direction is

\[
\Sigma \vec{F}_1 \cdot \vec{J} = \frac{3 gR_e^2}{R_G^4} \Sigma m_1 (\vec{r}_1 \cdot \vec{K})(\vec{r}_1 \cdot \vec{J})

= \frac{3 gR_e^2}{R_G^4} \left[ c_{31} c_{21} (C - A) + c_{32} c_{22} (C - B) \right],
\]

(20)

and the resultant force along the \( \vec{I} \) direction is

\[
\Sigma \vec{F}_1 \cdot \vec{I} = \frac{3 gR_e^2}{R_G^4} \Sigma m_1 (\vec{r}_1 \cdot \vec{K})(\vec{r}_1 \cdot \vec{I})

= \frac{3 gR_e^2}{R_G^4} \left[ c_{31} c_{11} (C - A) + c_{32} c_{12} (C - B) \right]
\]

(21)

where

\[ c_{21} = \vec{i} \cdot \vec{J}, \quad c_{22} = \vec{j} \cdot \vec{J}, \quad c_{11} = \vec{i} \cdot \vec{I}, \quad c_{12} = \vec{j} \cdot \vec{I}. \]

The advantage of expressing the resultant force and its components in this vector form is the relative ease with which the components along any three orthogonal axes such as \( \vec{I}, \vec{J}, \vec{K} \) may
then be found. All that is required is the transformation of the unit vectors \( \hat{1}, \hat{J}, \hat{K} \) into the unit vector \( \hat{i}, \hat{j}, \hat{k} \) or vice versa. This vector presentation was suggested in the paper by Nidey\(^2\) in which he used this technique to obtain the gravity gradient torque.

The Gravity Gradient Torque\(^3\)

The gravity gradient torque relative to the mass center \( G \) is

\[
\vec{M} = \sum_i \vec{r}_i \times \vec{F}_i
\]

(22)

where

\( \vec{r}_i \) is the position vector of the \( i \)th particle relative to the mass center \( G \);

\[
\vec{F}_i = -gR^2 \frac{m_i}{R^3_i} \vec{R}_i .
\]

Using equations (5) and (8),

\[
\vec{M} = -\frac{gR^2}{R^3_G} \sum_i \vec{r}_i \times \left( \frac{\vec{r}_i}{R_G} \cdot \vec{K} - \frac{15}{2} \left( \frac{\vec{r}_i}{R_G} \cdot \vec{K} \right)^2 \right) \left[ \vec{r}_i \times (R_G \vec{K} + \vec{r}_i) \right].
\]

(23)


\(^3\)Ibid.
Expanding, and neglecting terms of the order $R^{-4}$,

$$\mathbf{M} = \frac{3g R^2}{R_G^3} \Sigma m_i (\mathbf{r}_i \cdot \mathbf{K}) (\mathbf{r}_i \times \mathbf{K}) = 3 \lambda^2 \Sigma m_i (\mathbf{r}_i \cdot \mathbf{K}) (\mathbf{r}_i \times \mathbf{K}) \quad (24)$$

where

$$\lambda^2 = \frac{g R_e^2}{R_G^3}.$$  

If the orbit is circular, $\lambda$ is numerically equal to the geocentric angular velocity of the satellite.

The magnitude of the $\mathbf{I}$-component of the torque $\mathbf{M}$ is

$$\mathbf{M} \cdot \mathbf{I} = 3 \lambda^2 \Sigma m_i (\mathbf{r}_i \cdot \mathbf{K}) (\mathbf{r}_i \cdot \mathbf{J}) , \quad (25)$$

since $(\mathbf{r}_i \times \mathbf{K} \cdot \mathbf{I}) = (\mathbf{r}_i \cdot \mathbf{J})$.

If $\mathbf{r}_i = X_i \mathbf{I} + Y_i \mathbf{J} + Z_i \mathbf{K}$, then $\mathbf{r}_i \cdot \mathbf{K} = Z_i$ and $\mathbf{r}_i \cdot \mathbf{J} = Y_i$, and

$$\mathbf{M} \cdot \mathbf{I} = M_X = -3 \lambda^2 I_{ZY} \quad (26)$$

where

$$I_{ZY} = -\Sigma m_i Z_i Y_i .$$

However, this $X$-axis or direction is not a principal body axis.

Similarly, the $\mathbf{J}$-component of the moment $\mathbf{M}$ is

$$\mathbf{M} \cdot \mathbf{J} = M_Y = 3 \lambda^2 I_{XZ} \quad (27)$$

where

$$I_{XZ} = -\Sigma m_i X_i Z_i .$$
and the $K$-component is

$$\mathbf{M} \cdot \mathbf{K} = 3 \lambda^2 \sum m_i (\mathbf{r}_i \cdot \mathbf{K})(\mathbf{r}_i \times \mathbf{K} \cdot \mathbf{k}) = 0 \text{.} \quad (28)$$

Hence, the gravitational torque vector $\mathbf{M}$ is in a plane normal to the local vertical or $K$-axis. As an important consequence of this result, the angular momentum of the body with respect to the geocentric vertical is conserved.

In order to use Euler's equations and body-fixed principal axes $x$, $y$, $z$, the $i$, $j$, $k$ components of the gravitational torque $\mathbf{M}$ are required.

The $x$ or $i$-component of $\mathbf{M}$ is

$$M_x = \mathbf{M} \cdot \mathbf{i} = 3 \lambda^2 \sum m_i (\mathbf{r}_i \cdot \mathbf{K})(\mathbf{r}_i \times \mathbf{K} \cdot \mathbf{i}) \text{.} \quad (29)$$

However, since

$$\mathbf{K} = (\mathbf{i} \cdot \mathbf{K})\mathbf{i} + (\mathbf{j} \cdot \mathbf{K})\mathbf{j} + (\mathbf{k} \cdot \mathbf{K})\mathbf{k} \text{,} \quad (30)$$

$$(\mathbf{r}_i \cdot \mathbf{K})(\mathbf{r}_i \times \mathbf{K} \cdot \mathbf{i}) = [(\mathbf{i} \cdot \mathbf{K})(\mathbf{r}_i \cdot \mathbf{i}) + (\mathbf{j} \cdot \mathbf{K})(\mathbf{r}_i \cdot \mathbf{j}) + (\mathbf{k} \cdot \mathbf{K})(\mathbf{r}_i \cdot \mathbf{k})] \text{ times}$$

$$[\mathbf{i} \cdot \mathbf{k}](\mathbf{r}_i \cdot \mathbf{k}) + \mathbf{j} \times \mathbf{K} \text{ times} \quad (31)$$

Expanding,

$$(\mathbf{r}_i \cdot \mathbf{K})(\mathbf{r}_i \times \mathbf{K} \cdot \mathbf{i}) = -[(\mathbf{j} \cdot \mathbf{K})(\mathbf{k} \cdot \mathbf{K})(\mathbf{r}_i \cdot \mathbf{k}) \mathbf{j} \cdot \mathbf{k} + (\mathbf{j} \cdot \mathbf{K})(\mathbf{k} \cdot \mathbf{K})(\mathbf{r}_i \cdot \mathbf{j})] \text{ times}$$

$$+ \text{ cross-product terms in } (\mathbf{r}_i \cdot \mathbf{i})(\mathbf{r}_i \cdot \mathbf{k}) \text{, etc. } \text{.} \quad (32)$$
Since $x$, $y$, $z$ are principal axes, the cross product terms when summed over the body will be zero. Therefore,

$$M_x = \vec{M} \cdot \hat{i} = 3 \lambda^2 (\hat{j} \cdot \vec{K})(\hat{k} \cdot \vec{K}) \sum m_i \left[ -(\vec{r}_i \cdot \hat{k})^2 + (\vec{r}_i \cdot \hat{j})^2 \right]. \quad (33)$$

By adding and subtracting $(\vec{r}_i \cdot \hat{i})^2$ within the square brackets and since

$$B = \sum m_i (z_i^2 + x_i^2) = \sum m_i \left[ (\vec{r}_i \cdot \hat{k})^2 + (\vec{r}_i \cdot \hat{i})^2 \right] \quad \text{and} \quad (34)$$

$$C = \sum m_i (y_i^2 + x_i^2) = \sum m_i \left[ (\vec{r}_i \cdot \hat{j})^2 + (\vec{r}_i \cdot \hat{i})^2 \right], \quad \text{then} \quad (35)$$

$$M_x = \vec{M} \cdot \hat{i} = 3 \lambda^2 (\hat{j} \cdot \vec{K})(\hat{k} \cdot \vec{K}) \left[ C - B \right]. \quad (36)$$

Similarly, the $y$ and $z$ components of $\vec{M}$ are

$$M_y = \vec{M} \cdot \hat{j} = 3 \lambda^2 (\hat{i} \cdot \vec{K})(\hat{k} \cdot \vec{K}) \left[ A - C \right] \quad (37)$$

and

$$M_z = \vec{M} \cdot \hat{k} = 3 \lambda^2 (\hat{i} \cdot \vec{K})(\hat{j} \cdot \vec{K}) \left[ B - A \right]. \quad (38)$$

Therefore,

$$\vec{M} = 3 \lambda^2 (\hat{j} \cdot \vec{K})(\hat{k} \cdot \vec{K}) \left[ C - B \right] \hat{i} + 3 \lambda^2 (\hat{i} \cdot \vec{K})(\hat{k} \cdot \vec{K}) \left[ A - C \right] \hat{j} + 3 \lambda^2 (\hat{i} \cdot \vec{K})(\hat{j} \cdot \vec{K}) \left[ B - A \right] \hat{k}. \quad (39)$$

Observe that if $A = B = C$, $\vec{M} = 0$. 
Again, an advantage in using the vector presentation is the ease with which the components of the moment can be found in any set of orthogonal directions such as \( \mathbf{i}, \mathbf{j}, \mathbf{k} \). All that is required is the transformations of \( \mathbf{I}, \mathbf{J}, \mathbf{K} \) into \( \mathbf{i}, \mathbf{j}, \mathbf{k} \).

**The Potential Energy**

Considering the earth as a sphere, the potential at some point \( S \) which is a distance \( \rho \) from the geocenter is

\[
U = -\frac{GM_e}{\rho} = -\frac{gR_e^2}{\rho}
\]

where

- \( U \) is the potential per unit of mass at \( S \);
- \( G \) is the universal gravitational constant and is equal to \( \frac{gR_e^2}{M_e} \);
- \( M_e \) is the mass of the earth.

---

The potential of a finite body attracted by a spheriodal earth as developed by E. J. Routh is

\begin{equation}
U_{\text{Body}} = \sum_{i} \frac{g R_e^2 m_i}{\rho_i} ;
\end{equation}

\begin{equation}
U_{\text{Body}} = -g R_e^2 \frac{M}{R_G} - \frac{g R_e^2}{2R_G^3} [(A + B + C) - 3I_V] ;
\end{equation}

where

- \(\rho_i\) is the position vector of any particle \(m_i\) of the body as shown in Figure 3;

- \(M\) is the mass of the body;

- \(I_V\) is the mass moment of inertia with respect to the \(v\)-axis which is along the line going the geocenter to the mass center \(G\) of the body as shown in Figure 3.
Figure 3. The Potential of a Rigid Body Attracted by the Earth.
THE EQUATIONS OF MOTION

Description of the Euler Angles, The Coordinate Systems, and the Coordinate Transformations

The Euler Angles

Define the Euler angles as follows:

(i) A rotation \( \phi \) about the unit vector \( \vec{k} \);

(ii) A rotation \( \theta \) about the unit vector \( \vec{i} \);

(iii) A rotation \( \psi \) about the unit vector \( \vec{k} \). This set of Euler angles is shown in Figure 4.

The angular velocity \( \phi \) is sometimes referred to as the precession rate. The rotation \( \theta \) is often referred to as the angle of "dip" or "nod" and its rate of change \( \dot{\theta} \) is sometimes referred to as "nutation." The axis of the unit vector \( \vec{k} \) is called the "spin axis."

The Body-Fixed, Principal Axes

The coordinate axes \( x, y, z \) which correspond to the unit, orthogonal triad \( \vec{i}, \vec{j}, \vec{k} \) will be taken as body-fixed, principal axes. They are shown in Figure 5.

The Node System Axes

For the study of the motion of a symmetrical body, considerable simplification will be gained by using the coordinate
system \( \xi, \eta, \zeta \) which correspond to the unit triad \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) as shown in Figure 4. The coordinate axes \( \xi, \eta, \zeta \) will also be principal axes for a symmetrical body whose axis of symmetry will lie along \( \xi \). The \( \xi, \eta, \zeta \) coordinate system is called the "node system" axes since the axis \( \xi \) corresponding to \( \mathbf{i} \) is commonly called the "line of nodes." The node-system axes are shown in Figure 5.

The Coordinate Transformations

The transformation of the unit vectors \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) into the body-fixed, principal axes system \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) is given by

\[
\mathbf{i} = [\cos \phi \cos \psi - \sin \phi \cos \theta \sin \psi] \mathbf{i} + [- \cos \phi \sin \psi - \sin \phi \cos \theta \cos \psi] \mathbf{j} + [\sin \phi \sin \theta \mathbf{k} ;
\]

(43)

\[
\mathbf{j} = [\sin \phi \cos \psi + \cos \phi \cos \theta \sin \psi] \mathbf{i} + [- \sin \phi \sin \psi + \cos \phi \cos \theta \cos \psi] \mathbf{j} + [- \cos \phi \sin \theta \mathbf{k} ;
\]

(44)

\[
\mathbf{k} = [\sin \theta \sin \psi \mathbf{i} + [\sin \theta \cos \psi \mathbf{j} + [\cos \theta \mathbf{k} .
\]

(45)

The transformation of \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) into \( \mathbf{i}_\xi, \mathbf{j}_\eta, \mathbf{k}_\zeta \)

If the unit vectors along the node system axes \( \xi, \eta, \zeta \) are \( \mathbf{i}_\xi, \mathbf{j}_\eta, \mathbf{k}_\zeta \), then the transformation of \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) into \( \mathbf{i}_\xi, \mathbf{j}_\eta, \mathbf{k}_\zeta \) is given by

\[
\mathbf{i} = \cos \phi \mathbf{i}_\xi - \sin \phi \cos \theta \mathbf{j}_\eta + \sin \phi \sin \theta \mathbf{k}_\zeta ;
\]

(46)
Figure 4. The Euler Angles and the Coordinate Axes
Figure 5. The Node-System Axes $\xi, \eta$, and $\xi$ and the Body-Fixed Axes $x, y, z$. 
\[ \vec{J} = \sin \phi \vec{i}_\xi + \cos \phi \cos \theta \vec{j}_\eta - \cos \phi \sin \theta \vec{k}_\zeta ; \]  
\[ \vec{K} = \sin \theta \vec{j}_\eta + \cos \theta \vec{k}_\zeta . \]  

The transformation of \( \vec{i}, \vec{j}, \vec{k} \) into \( \vec{i}_\xi, \vec{j}_\eta, \vec{k}_\zeta \)

The transformations of the unit vectors \( \vec{i}, \vec{j}, \vec{k} \) into the node-system unit vectors \( \vec{i}_\xi, \vec{j}_\eta, \vec{k}_\zeta \) are given by

\[ \vec{i} = \cos \psi \vec{i}_\xi + \sin \psi \vec{j}_\eta ; \]  
\[ \vec{j} = -\sin \psi \vec{i}_\xi + \cos \psi \vec{j}_\eta ; \]  
\[ \vec{k} = \vec{k}_\zeta . \]
The Gravity Gradient Torque, the Potential Energy, 
and the Angular Velocity for the Body-Fixed Axes 
and the Node-System Axes

The Components of the Gravity Gradient 
Torque for Body-Fixed Axes

Using equations (43) - (44),

\[(\hat{i} \cdot \hat{K}) = \sin \theta \sin \psi, \quad (\hat{j} \cdot \hat{K}) = \sin \theta \cos \psi, \quad (\hat{k} \cdot \hat{K}) = \cos \theta\,.
\]

Thus,
\[
M_x = 3 \lambda^2 [\sin \theta \cos \psi \cos \theta] [C - B]; \quad (53)
\]
\[
M_y = 3 \lambda^2 [\cos \theta \sin \theta \sin \psi] [A - C]; \quad (54)
\]
\[
M_z = 3 \lambda^2 [\sin \theta \sin \psi \sin \theta \cos \psi] [B - A]. \quad (55)
\]

The Components of the Gravity Gradient 
Torque for the Node-System Axes

Since \( \vec{M} = M_x \hat{i} + M_y \hat{j} + M_z \hat{k} \), letting \( A = B \), and using 
equations (49) - (51) and (53) - (55), the components of the gravita-
tion torque along the node system axes are:

\[
M_\xi = 3 \lambda^2 [\sin \theta \cos \theta] [C - A]; \quad (56)
\]
\[
M_\eta = 0; \quad (57)
\]
\[
M_\zeta = 0. \quad (58)
\]
The Potential Energy of the Satellite - Body-Fixed Axes

If \( l, m, n \) are the direction cosines of the \( v \)-axis with respect to the body fixed, principal axes \( x, y, z \), then

\[
I_v = Al^2 + Bm^2 + Cn^2 \tag{59}
\]

where

\[
l = \vec{i} \cdot \vec{K} = \sin \theta \sin \psi ;
m = \vec{j} \cdot \vec{K} = \sin \theta \cos \psi ;
n = \vec{k} \cdot \vec{K} = \cos \theta. \tag{60}
\]

Thus, for body-fixed, principal axes,

\[
U = -gR_e^2 \frac{M}{R_G} - \frac{gR_e^2}{2R_G^3} \left[(A + B + C) - 3(A \sin^2 \theta \sin^2 \psi + B \sin^2 \theta \cos^2 \psi + C \cos^2 \theta)\right]. \tag{61}
\]

The Potential Energy of the Satellite - Node-System Axes

For a body of revolution, \( A = B \), the node-system axes, \( \xi, \eta, \zeta \), are principal axes.

The direction cosines of the \( v \)-axis with respect to \( \xi, \eta, \zeta \) are

\[
l = 0, \quad m = \sin \theta, \quad n = \cos \theta, \tag{62}
\]

so that \( I_v = A \sin^2 \theta + C \cos^2 \theta \) and

\[
U = -\frac{gR_e^2}{R_G} - \frac{gR_e^2}{2R_G^3} \left[(2A + C) - 3(A \sin^2 \theta + C \cos^2 \theta)\right]. \tag{63}
\]
**The Angular Velocity of the Satellite for Body-Fixed Axes**

The angular velocity of the satellite relative to the \( \vec{I}, \vec{J}, \vec{K} \) coordinate system is

\[
\frac{d\vec{n}}{dt} = \phi \vec{K} + \theta \vec{I} + \psi \vec{K} .
\] (64)

Using equation (48) and \( \vec{I}' = \vec{I}'' = \cos \psi \vec{i} - \sin \psi \vec{j} \),

\[
\frac{d\vec{n}}{dt} = [\phi \sin \theta \sin \psi + \theta \cos \psi] \vec{i}
+ [\phi \sin \theta \cos \psi - \theta \sin \psi] \vec{j}
+ [\phi \cos \theta + \psi] \vec{k} .
\] (65)

The angular velocity of the \( \vec{I}, \vec{J}, \vec{K} \) axes is \( \Omega \vec{J} \). (66)

The absolute angular velocity of the satellite is

\[
w = \frac{d\vec{n}}{dt} + \Omega \vec{J} ,
\] (67)

and the components along the body fixed axes \( x, y, z \) are

\[
w_x = [\phi \sin \theta \sin \psi + \theta \cos \psi + \Omega \sin \phi \cos \psi + \Omega \cos \phi \cos \theta \sin \psi] ;
\] (68)

\[
w_y = [\phi \sin \theta \cos \psi - \theta \sin \psi - \Omega \sin \phi \sin \psi + \Omega \cos \phi \cos \theta \cos \psi] ;
\] (69)

\[
w_z = [\phi \cos \theta + \psi - \Omega \cos \phi \sin \theta] .
\] (70)
The Angular Velocity of the Satellite for the Node-System Axes

The angular velocity of the satellite relative to the $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$ axes is given by equation (56). Using equations (49) - (51) to transform the relative angular velocity into component along the node-system axes,

$$\frac{dn}{dt} = \theta \mathbf{i}_\xi + \phi \sin \theta \mathbf{j}_\eta + (\phi \cos \theta + \psi) \mathbf{k}_\zeta.$$  \hspace{1cm} (71)

Again, since the absolute angular velocity is

$$\mathbf{w} = \frac{dn}{dt} + \Omega \mathbf{J},$$

where $\mathbf{J} = \sin \phi \mathbf{i}_\xi + \cos \phi \cos \theta \mathbf{j}_\eta - \cos \phi \sin \theta \mathbf{k}_\zeta$,

the components of $\mathbf{w}$ along the node-system axes are

$$w_{\xi} = [\theta + \Omega \sin \phi];$$ \hspace{1cm} (72)

$$w_{\eta} = [\phi \sin \theta + \Omega \cos \phi \cos \theta];$$ \hspace{1cm} (73)

$$w_{\zeta} = [\phi \cos \theta + \psi - \Omega \cos \phi \sin \theta].$$ \hspace{1cm} (74)
Motion of the Mass Center

Let the position vector of the mass center $G$ of the satellite be given as

$$\vec{R}_G = R_G \vec{K}$$  \hspace{1cm} (75)

as shown in Figure 6.

Further, let the angular orientation of $\vec{R}_G$ be given by the angles $\alpha$ and $\beta$ as shown in Figure 6.

Choose the unit vector $\vec{K}$ to be outward along $\vec{R}_G$; $\vec{T}$ in the direction of the forward motion and in the plane created by the ascending node of the unperturbed orbit and $\vec{R}_G$; and $\vec{J}$ to form a right-handed triad $\vec{I}$, $\vec{J}$, $\vec{K}$ as shown in Figure 6.

The angular velocity of the unit triad is

$$\vec{\omega}_{IJK} = \alpha \vec{J} - \beta \vec{T}.$$  \hspace{1cm} (76)

The velocity of the mass center is

$$\vec{v}_G = \frac{d\vec{R}_G}{dt} = R_G \vec{K} + \vec{\omega}_{IJK} \times \vec{R}_G = R_G \alpha \vec{T} + R_G \beta \vec{J} + R_G \vec{K}.$$  \hspace{1cm} (77)
Figure 6. Motion of the Mass Center.
The acceleration of the mass center is

\[
\ddot{\mathbf{a}}_G = \frac{d\mathbf{v}_G}{dt} = [R_G \ddot{\mathbf{a}} + 2 R_G \dot{\mathbf{a}}] \mathbf{i} + [R_G \ddot{\mathbf{\beta}} + 2 R_G \dot{\mathbf{\beta}}] \mathbf{j} \\
+ [R_G - R_G \ddot{\mathbf{a}} - R_G \ddot{\mathbf{\beta}}] \mathbf{k}
\]

or

\[
\ddot{\mathbf{a}}_G = \frac{1}{R_G} \frac{d}{dt} (R_G^2 \mathbf{a}) \mathbf{i} + \frac{1}{R_G} \frac{d}{dt} (R_G^2 \mathbf{\beta}) \mathbf{j} \\
+ [R_G - R_G \ddot{\mathbf{a}} - R_G \ddot{\mathbf{\beta}}] \mathbf{k}
\]

The general equations of motion of the mass center are as follows.

**The \( \mathbf{k} \) or Radial Equation**

\[
R_G - R_G \ddot{\mathbf{a}} - R_G \ddot{\mathbf{\beta}} = -\frac{g R_e^2}{R_G^2} - \frac{3 g R_e^2}{4 M R_G} [(A + B + C \\
- 3(c_{31}^2 A + c_{32}^2 B + c_{33}^2 C)]
\]

where

\[
M = \sum m_i = \text{Total mass of the satellite} ; \\
c_{31} = \mathbf{i} \cdot \mathbf{k} = \sin \theta \sin \psi ; \\
c_{32} = \mathbf{j} \cdot \mathbf{k} = \sin \theta \cos \psi ; \\
c_{33} = \mathbf{k} \cdot \mathbf{k} = \cos \theta .
\]
The $\mathbf{J}$ Equation

$$\frac{1}{R_G} \frac{d}{dt} (R_G^2 \mathbf{J}) = \frac{3 g R_e^2}{MR_G^4} [c_{31} c_{21} (C - A) + c_{33} c_{22} (C - B)] \quad (82)$$

where in addition to the relations of equation (81)

$$c_{21} = \mathbf{i} \cdot \mathbf{J} = \sin \phi \cos \psi + \cos \phi \cos \theta \sin \psi ;$$

$$c_{22} = \mathbf{j} \cdot \mathbf{J} = -\sin \phi \sin \psi + \cos \phi \cos \theta \cos \psi . \quad (83)$$

The $\mathbf{I}$ Equation

$$\frac{1}{R_G} \frac{d}{dt} (R_G^2 \mathbf{I}) = \frac{3 R_e^2}{MR_G^4} [c_{31} c_{11} (C - A) + c_{32} c_{12} (C - B)] \quad (84)$$

where

$$c_{11} = \mathbf{i} \cdot \mathbf{I} = \cos \phi \cos \psi - \sin \phi \cos \theta \sin \psi ;$$

$$c_{12} = \mathbf{j} \cdot \mathbf{I} = -\cos \phi \sin \psi - \sin \phi \cos \theta \cos \psi . \quad (85)$$

Observe that, since the $c_{ij}$ are functions of the Euler angles $\phi$, $\theta$, $\psi$, the equations of motion of the mass center are coupled to the attitude equations. Due to the complexity and the extreme nonlinearity of the equations of motion, a simplification will be made in order to uncouple the equations of motion of the mass center and the attitude equations.
By examining the radial equation (80), we see that the coupling enters into the equation as a term of the order of magnitude of at least $R^2_G$ smaller than the term $-gR^2_e/R^2_G$.

Since the term $-gR^2_e/R^2_G$ is the force per unit mass on a particle moving in a classical Keplerian orbit, we may interpret the effect of the coupling term as a very small perturbation of the radius vector $\vec{R}_G$. Therefore, in comparison to the radius vector of an unperturbed orbit, the perturbation to the actual radius vector by the coupling term will be considered as negligible. In examining the $\vec{T}$ and $\vec{J}$ equations, we see that the right-hand sides of these equations are due entirely to the coupling terms. If we interpret the coupling terms as producing second order perturbations of the displacement in the directions tangent-to-the-path and normal-to-the-plane of the elementary, Keplerian orbit, then it is reasonable to assume that, within the same degree of neglecting radial perturbations of the radius vector, the perturbations of the displacement in the plane and normal to plane of the elementary orbit may also be neglected.

Therefore, we can consider the orbital parameters to be given by the equations of a particle moving in a vacuum in an inverse square field and that these equations are independent of the attitude equations. Henceforth, the orbital parameters will be considered as known or independently determinable quantities.
The General Attitude Equations

Body-Fixed, Principal Axes

The attitude equations for body-fixed, principal axes are given by Euler's equations:

\[
Aw_x - w_y w_z (B - C) = M_x = 3 \lambda^2 [C - B][\sin \theta \cos \theta \cos \psi]; \quad (86)
\]

\[
Bw_y - w_z w_x (C - A) = M_y = 3 \lambda^2 [A - C][\sin \theta \cos \theta \sin \psi]; \quad (87)
\]

\[
Cw_z - w_x w_y (A - B) = M_z = 3 \lambda^2 [B - A][\sin^2 \theta \sin \psi \cos \psi]; \quad (88)
\]

where

\[
w_x = [\phi \sin \theta \sin \psi + \theta \cos \psi + \Omega \sin \phi \cos \psi + \Omega \cos \phi \cos \theta \sin \psi];
\]

\[
w_y = [\phi \sin \theta \cos \psi - \theta \sin \psi - \Omega \sin \phi \sin \psi + \Omega \cos \phi \cos \theta \cos \psi];
\]

\[
w_z = [\phi \cos \theta + \psi - \Omega \cos \phi \sin \theta].
\]

Differentiating and expanding, equation (86), the first Euler equation,

\[
Aw_x - w_y w_z (B - C) = M_x
\]
\[ 3\chi^2 = [\phi - C - B][\phi - C - B]^T \]

\[ \begin{bmatrix}
\phi 
\end{bmatrix}^T
\begin{bmatrix}
\sin \theta \cos \phi 
\sin \phi 
\sin \phi \cos \theta 
\sin \theta 
\cos \phi 
\cos \phi \sin \theta 
\cos \phi 
\end{bmatrix}
\begin{bmatrix}
\sin \theta \cos \phi 
\sin \phi 
\sin \phi \cos \theta 
\sin \theta 
\cos \phi 
\cos \phi \sin \theta 
\cos \phi 
\end{bmatrix}
\begin{bmatrix}
\phi 
\end{bmatrix}
\]
Equation (87), the second Euler equation,

\[ Bw_y - w w_x (C - A) = M_y \]

\[ = 3 \lambda^2 [A - C] [\sin \theta \cos \theta \sin \psi]. \]
Equation (88), the third Euler equation,
\[
C w_z - w_x w_y (A - B) = M_z \text{ is}
\]
\[
C [\phi \cos \theta - \phi \sin \theta + \psi - \Omega \cos \phi \sin \theta + \Omega \phi \sin \phi \sin \theta \\
- \Omega \theta \cos \phi \cos \theta] \\
- [A - B] [\phi^2 \sin^2 \theta \sin \psi \cos \psi + \phi \theta \sin \theta \cos^2 \psi \\
+ \Omega \phi \sin \phi \sin \theta \cos^2 \psi + \Omega \phi \cos \phi \sin \theta \cos \theta \sin \psi \cos \psi \\
+ \phi \theta \sin \theta \sin^2 \psi - \theta^2 \sin \psi \cos \psi \\
- \Omega \phi \sin \phi \sin \theta \sin^2 \psi - \Omega \theta \sin \phi \sin \psi \cos \psi \\
- \Omega \phi \sin \phi \sin \theta \sin^2 \psi - \Omega \theta \sin \phi \sin \psi \cos \psi \\
- \Omega^2 \sin^2 \phi \sin \psi \cos \psi - \Omega \sin \phi \cos \phi \cos \theta \sin^2 \psi \\
+ \Omega \phi \cos \phi \sin \theta \cos \theta \sin \psi \cos \psi + \Omega \theta \cos \phi \cos \theta \cos^2 \psi \\
+ \Omega^2 \sin \phi \cos \phi \cos \theta \cos^2 \psi + \Omega^2 \cos^2 \phi \cos \theta \sin \psi \cos \psi]
\]
\[
= 3 \lambda^2 \ [B - A] [\sin^2 \theta \sin \psi \cos \psi]. \tag{91}
\]

It is quite apparent that the general attitude equations are highly non-linear and are not amenable to solution. However two integrals of these equations are known. One integral results from the conservation of energy and the second results from the conservation of angular momentum about the $Z$-axis or the local vertical. The second integral results from the fact that the component of the gravitational torque along $Z$ is zero (see equation (28)).
Two Integrals of the Equations of Motion

Conservation of Energy

For a satellite at heights beyond the influence of the atmosphere and assuming that only the force of attraction of the earth is acting on it, the system is conservative and the total energy $E = T + U$ is a constant. Therefore, the integral resulting from the conservation of energy is

$$E = \frac{1}{2} M \left[ R_G^2 + R_G^2 \right] + \frac{1}{2} A w^2 + \frac{1}{2} B w_y^2 + C w_z^2 .$$

Upon expanding, we have

$$E = \frac{1}{2} M \left[ R_G^2 + R_G^2 \right]$$

$$+ \frac{1}{2} A \left[ \phi^2 \sin^2 \theta \sin^2 \psi + \phi \theta \sin \theta \sin \psi \cos \psi \right.$$

$$+ \Omega \phi \sin \phi \sin \theta \sin \psi \cos \psi + \Omega \phi \cos \phi \sin \theta \cos \theta \sin^2 \psi$$

$$+ \phi \theta \sin \theta \sin \psi \cos \psi + \theta^2 \cos^2 \psi + \Omega \theta \sin \phi \cos^2 \psi$$

$$+ \Omega^2 \sin \phi \cos \phi \cos \theta \sin \psi \cos \psi$$

$$+ \Omega \phi \sin \phi \sin \theta \sin \psi \cos \psi + \Omega \phi \cos \phi \sin \theta \cos \theta \sin^2 \psi$$

$$+ \Omega^2 \sin \phi \cos \phi \cos \theta \sin \psi \cos \psi$$

$$+ \Omega \phi \cos \phi \sin \theta \cos \theta \sin^2 \psi + \Omega \theta \cos \phi \cos \theta \sin \psi \cos \psi$$

$$+ \Omega^2 \sin \phi \cos \phi \cos \theta \sin \psi \cos \psi$$

$$+ \Omega \phi \cos \phi \sin \theta \cos \theta \sin^2 \psi + \Omega \theta \cos \phi \cos \theta \sin \psi \cos \psi$$

$$+ \Omega^2 \sin \phi \cos \phi \cos \theta \sin \psi \cos \psi$$
\[ + \frac{1}{2} B \left[ \phi \sin^2 \theta \cos^2 \psi - \phi \theta \sin \theta \sin \psi \cos \psi \right. \\
- \Omega \phi \sin \phi \sin \theta \sin \psi \cos \psi + \Omega \cos \phi \sin \theta \cos \theta \cos^2 \psi \\
\left. - \Omega \phi \sin \phi \sin \theta \sin \psi \cos \psi + \Omega \theta \sin \phi \sin^2 \psi \right] \\
\]

\[ + \frac{1}{2} C \left[ \phi'^2 \cos^2 \theta + \phi \psi \cos \theta - \Omega \phi \cos \phi \sin \theta \cos \theta \\
\phi \psi \cos \theta + \psi^2 - \Omega \psi \cos \phi \sin \theta \\
\left. - \Omega \phi \cos \phi \sin \theta \cos \theta - \Omega \psi \cos \phi \sin \theta \\
\right. \\
+ \Omega^2 \cos^2 \phi \sin^2 \theta \right] \\
\]

\[ - g R^2 \frac{M}{R_G} - g R^2 \left[ (A + B + C) - 3 (A \sin^2 \theta \sin^2 \psi + B \sin^2 \theta \cos^2 \psi + C \cos^2 \theta) \right]. \]
Conservation of Angular Momentum

If the resultant moment of the external forces which act on a body vanishes with respect to its mass center \( G \), then the projections of the angular momentum on coordinate axes through \( G \) must remain constant. It was shown by equation (28) that the component of the gravitational torque along the geocentric vertical through \( G \) vanished. Therefore, the component of the angular momentum of the satellite along this axis must remain constant.

Since the angular momentum with respect to the mass center \( G \) is

\[
\vec{h} = Aw_z \hat{i} + Bw_y \hat{j} + Cw_z \hat{k},
\]

the equation which results from the conservation of angular momentum is

\[
h_z, \text{ a constant } = \vec{h} \cdot \vec{K} = Aw_z \hat{i} \cdot \vec{K} + Bw_y \hat{j} \cdot \vec{K} + Cw_z \hat{k} \cdot \vec{K}.
\]
Therefore,

\[ h_z = A w \sin \theta \sin \varphi + B w \sin \theta \cos \varphi + C w \cos \theta; \]

\[ h_z = A [ \phi \sin^2 \theta \sin^2 \psi + \theta \sin \theta \sin \psi \cos \psi \]
\[ + \Omega \sin \phi \sin \theta \sin \psi \cos \psi + \Omega \cos \phi \sin \theta \cos \theta \sin^2 \psi ] \]
\[ + B [ \phi \sin^2 \theta \cos^2 \psi - \theta \sin \theta \sin \psi \cos \psi \]
\[ - \Omega \sin \phi \sin \theta \sin \psi \cos \psi + \Omega \cos \phi \sin \theta \cos \theta \cos^2 \psi ] \]
\[ + C [ \phi \cos^2 \theta + \psi \cos \theta - \Omega \cos \phi \sin \theta \cos \theta ] . \] \hspace{1cm} (93)

In spite of the knowledge of these two integrals of the attitude equations of motion, the task of solving the system of equations is quite formidable. A search of the literature has failed to produce any published material on the two general integrals (92) and (93) of the general equations of motion. There is no evidence that anyone has bothered to publish them or to attempt to use them in the general study of the attitude motion of a spinning satellite under the influence of the gravity gradient torque. These integrals (92) and (93) are not to be confused with the very well known integrals which are derived from the same dynamical principals, but which are used in the study of the orbit determination or in the study of the motion of a spinning top about a fixed point.
In order to give the reader some idea of what has been done in this area by other investigators, some of the authors and their work are cited here and the more significant aspects of their investigations are briefly discussed. Other authors may be found in the Bibliography.

R. Wolfe and B. Arrow (184) presented the complete coupled equations of motion of the mass center and of the attitude in vector form. They then solved by computer a simplified nonlinear form of these equations. Since the Euler angles corresponding to spin and precession were limited to small motions, this study primarily investigated the nutation or tumbling motion of a satellite under the influence of the gravitational torque. Their more extensive and detailed study confirmed the findings of other authors such as W. B. Klemperer and R. M. L. Baker, Jr. (89) and G. M. Schindler (144) who used less elaborate mathematical models. R. Wolfe and B. Arrow showed the existence of stable oscillations as well as unstable tumbling motions and presented their results in the form of phase plane and phase space plots. Their study also included some results which accounted for the effect of eccentricity of the orbit. This is one of the very few papers which included any coupling of the motion of the mass center to the equations of the attitude motion.
D. M. Schrello (146) derived the general attitude equations of motion for near-earth satellites. These equations included the gravitational torque as well as torques arising from the aerodynamics. However, the equations were only studied for a nonspinning axisymmetric satellite. The results showed that the vehicle motion was characterized by three dimensionless parameters and one inertia parameter.

J. H. Suddath (15) solved analytically a linearized form of the attitude equations for a rectangular-pulse pitching moment and found good agreement with a computer study of the corresponding exact equations over a limited region of interest.

L. H. Grosshoff (66) presented the results of a computer study to determine the secular variation in the spin axis which results from the gravitational torque. His paper, however, only indicated that the spin axis was not as stable as it had first appeared to be to other investigators.

M. E. Keubler (92) studied the gyroscopic motion of an unsymmetrical satellite under no external forces. This investigation studied the transition from spin about the axis of minimum moment of inertia to spin about the axis of maximum moment of inertia by dissipation of internal mechanical energy. The results showed that the angular velocities and nutation angle
are dependent on the energy and symmetry factors. Similar studies on the same topic, as was previously mentioned in the Introduction, were made by L. Meirovitch (102) and W. T. Thomson and G. S. Reiter (166).

D. B. DeBra and R. H. Delp (43) made a thorough study of the attitude stability of a nonspinning, rigid satellite in a circular orbit. The results showed that the stability could be expressed in terms of only two inertia parameters. Their study was made for a satellite of general shape.

All of these studies indicated that the investigators relied heavily on computers for solving their equations. It is quite apparent from the form and complexity of the general equations of motion that the only feasible method of attempting to solve them would be by mechanical means. As yet there has been no evidence that this has been attempted for the complete set of general equations.

Possibly because the equations are so formidable and because there are so many possible sources of perturbing torques, as was pointed out by R. E. Roberson (128), there is a great deal of work being done in industry in mechanically simulating a satellite and the various moments which it might encounter in space. The motions are then recorded and studied. Since most
of this work is classified, and with the exception of one paper by M. D. Olstad, R. Grunberg, W. Blesser and L. Braun, Jr. (116) there has been no published material available on these simulators. Care should be taken to distinguish satellite attitude simulators from the actual satellite attitude control systems about which a great deal has been written.
THE ANALYSIS OF THE SIMPLIFIED ATTITUDE EQUATIONS OF MOTION

The Simplified Attitude Equations

Neglecting products $\Omega \Phi, \Omega \Theta, \Omega \Psi, \Omega^2$ and assuming $\Omega$ is negligible, the attitude equations become

\begin{align*}
A & \left[ \phi \sin \theta \sin \psi + \phi \theta \cos \theta \sin \psi + \phi \psi \sin \theta \cos \psi \right. \\
& - \left. \theta \cos \psi - \theta \psi \sin \psi \right] = 3\lambda^2 \left[ C - B \right] \left[ \sin \theta \cos \theta \cos \psi \right]; \quad (94) \\
B & \left[ \phi \sin \theta \cos \psi + \phi \theta \cos \theta \cos \psi - \phi \psi \sin \theta \sin \psi \right. \\
& - \left. \theta \sin \psi - \theta \psi \cos \psi \right] = 3\lambda^2 \left[ C - A \right] \left[ \sin \theta \cos \theta \sin \psi \right]; \quad (95) \\
C & \left[ \phi \cos \theta - \phi \theta \sin \theta + \psi \right. \\
& - \left. \left[ A - B \right] \left[ \phi^2 \sin^2 \theta \sin \psi \cos \psi + \phi \theta \sin \theta \cos^2 \psi \right. \\
& - \left. \theta^2 \sin \psi \cos \psi \right] = 3\lambda^2 \left[ B - A \right] \left[ \sin^2 \theta \sin \psi \cos \psi \right]. \quad (96)
\end{align*}
Case I. An Unsymmetrical, Spinning Satellite with Steady Precession

Small Oscillations about the Steady Motion

Assume that a steady precession exists for which the time rate of change of the angle $\theta$ is zero. Let the steady motion values of the angle $\theta$ and the steady precession $\phi$ be $\theta_0$, a constant, and $\phi_0$, a constant, and denote the time variable deviations from the steady motion by $\theta_\delta$ and $\phi_\delta$. This prescribed motion is shown in Figure 7, by indicating the motion of the spin axis $\vec{k}$ on a unit sphere. The steady motion is shown by a solid line and the deviation from the steady motion by a dotted line. The instantaneous values of $\theta$ and $\phi$ can then be written as:

\[ \theta = \theta_0 + \theta_\delta; \quad \phi = \phi_0 + \phi_\delta; \]

\[ \theta = \theta_\delta; \quad \phi = \phi_\delta. \quad (97) \quad (98) \]

For small deviations or oscillations about the steady motion, we can make the following approximations:

\[ \theta \approx \theta_0 + \theta_\delta; \]

\[ \phi \approx \phi_0 + \phi_\delta; \quad (99) \]

\[ \phi^2 \approx \phi_0^2 + 2 \phi_0 \phi_\delta; \quad (100) \]

\[ \sin \theta \approx \sin \theta_0 + \theta_\delta \cos \theta_0; \quad (101) \]
Figure 7. The Steady Motion and Small Oscillations about the Steady Motion.
\[
\cos \theta = \cos \theta_0 - \theta_\delta \sin \theta_0 ;
\]
\[
\sin^2 \theta = \sin^2 \theta_0 + 2 \theta_\delta \sin \theta_0 \cos \theta_0 ;
\]
\[
\sin \theta \cos \theta = \sin \theta_0 \cos \theta_0 + \theta_\delta \cos^2 \theta_0 - \theta_\delta \sin^2 \theta_0 .
\]

Also assume that \( \psi \), the spin, = \( \psi_0 \), a constant. This would require an applied torque \( G_z(t) \) about the \( z \)-principal axis that would have to be included in the third attitude equation (96). However, this assumption has the effect of reducing the degrees of freedom to two. Thus, only the first two attitude equations (94) and (95) are required in the study of the small oscillations.
Substituting equations (97) - (104) into the attitude equations (94) and (95), the equations become

\[ A \left[ \phi_\delta \left( \sin \theta_0 + \theta_\delta \cos \theta_0 \right) \sin \psi + \theta_\delta \phi_\delta \left( \cos \theta_0 - \theta_\delta \sin \theta_0 \right) \sin \psi \right. \]

\[ + \left( \phi_\delta + \phi_\delta \right) \left( \psi \right) \left( \sin \theta_0 + \theta_\delta \cos \theta_0 \right) \cos \psi \]

\[ + \theta_\delta \cos \psi - \theta_\delta \psi \sin \psi \]

\[ - \left[ B - C \right] \left[ \phi_\delta^2 + 2 \phi_\delta \phi_\delta \right] \left( \sin \theta_0 \cos \theta_0 + \theta_\delta \cos \theta_0 - \theta_\delta \sin \theta_0 \right) \cos \psi \]

\[ + \left( \phi_\delta + \phi_\delta \right) \left( \psi \right) \left( \sin \theta_0 + \theta_\delta \cos \theta_0 \right) \cos \psi - \theta_\delta \psi \sin \psi \]

\[ = 3 \lambda^2 \left[ C - B \right] \left[ \sin \theta_0 \cos \theta_0 + \theta_\delta \cos \theta_0 - \theta_\delta \sin \theta_0 \right] \cos \psi \]

(105)
and

\[
B[\phi^2 (\sin \theta_0 + \theta_0 \cos \theta_0) \cos \psi + \theta_0 \phi^2 (\cos \theta_0 - \theta_0 \sin \theta_0) \cos \psi
\]

\[
- (\phi + \phi_0) (\psi_0) (\sin \theta_0 + \theta_0 \cos \theta_0) \sin \psi
\]

\[
- \theta_0 \sin \psi - \theta_0 \psi \cos \psi
\]

\[
[G - A] [\phi^2 + 2 \phi \phi_0] (\sin \theta_0 \cos \theta_0 + \theta_0 \cos^2 \theta_0 - \theta_0 \sin^2 \theta_0) \sin \psi
\]

\[
+ \theta_0 \phi^2 (\cos \theta_0 - \theta_0 \sin \theta_0) \cos \psi
\]

\[
+ (\phi + \phi_0) (\psi_0) (\sin \theta_0 + \theta_0 \cos \theta_0) \sin \psi + \theta_0 \psi \cos \psi
\]

\[
= 3 \lambda^2 [A - C] [(\sin \theta_0 \cos \theta_0 + \theta_0 \cos^2 \theta_0 - \theta_0 \sin^2 \theta_0) \sin \psi].
\]

(106)

After expanding, the steady motion is given by the equations

\[
A \phi \psi \sin \theta_0 \cos \psi - [B - C] \phi^2 \sin \theta_0 \cos \theta_0 \cos \psi \]

\[
+ (\psi \sin \theta_0 \cos \psi) = 3 \lambda^2 [C - B] [\sin \theta_0 \cos \theta_0 \cos \psi] \quad (107)
\]

or

\[
A \phi \psi - [B - C] \phi^2 \cos \theta_0 + \psi \frac{3 \lambda^2 [C - B] \cos \theta_0}{2} \quad (108)
\]
and

\[ B \left[ - \phi_0 \psi_0 \sin \theta \sin \psi \right] - [C - A] \left[ \phi_0^2 \sin \theta \cos \theta \sin \psi \right. \\
\left. + \phi_0 \psi_0 \sin \theta \sin \psi \right] = 3 \lambda^2 \left[ A - C \right][\sin \theta \cos \theta \sin \psi] \quad (109) \]

or

\[ B[- \phi_0 \psi_0] - [C - A] \left[ \phi_0^2 \cos \theta \phi_0 \psi_0 \right] = 3 \lambda^2 [A - C][\cos \theta] \quad (110) \]

By eliminating the steady motion terms given by equations (107) and (109), and neglecting products of the small deviational quantities, the equations for the small deviations about the steady motion are

\[ A[ \phi_0 \sin \theta \sin \psi + \theta_\delta \phi_0 \cos \theta \sin \psi + \phi_0 \psi_0 \theta_\delta \cos \theta \cos \psi \]
\[ + \phi_0 \psi_0 \sin \theta \cos \psi + \theta_\delta \psi_0 \sin \psi \]
\[ - [B - C][\theta_\delta \phi_0 \cos \theta \sin \psi + \theta_\delta \phi_0 \sin \theta \cos \theta \cos \psi \]
\[ - \theta_\delta \phi_0 \cos \theta \sin \psi + \phi_0 \psi_0 \theta_\delta \cos \theta \cos \psi \]
\[ + \phi_0 \psi_0 \sin \theta \cos \psi - \theta_\delta \psi_0 \sin \psi \]
\[ = 3 \lambda^2 [C - B] \left[ \theta_\delta \cos \theta - \theta_\delta \sin \theta \cos \psi \right] \quad (111) \]
and
\[ B \left( \phi_0 \sin \theta \cos \psi + \theta_0 \phi_0 \cos \theta \cos \psi - \phi_0 \psi_0 \theta_0 \cos \theta \sin \psi \\
- \phi_0 \psi_0 \sin \theta \sin \psi - \theta_0 \sin \psi - \theta_0 \psi_0 \cos \psi \right) \]

- \[(C - A) \left( \phi_0^2 \theta_0 \cos^2 \theta \sin \psi - \phi_0^2 \theta_0 \sin^2 \theta \sin \psi \\
+ 2 \phi_0 \phi_0 \sin \theta \cos \theta \sin \psi + \theta_0 \phi_0 \cos \theta \cos \psi \\
+ \phi_0 \psi_0 \theta_0 \cos \theta \sin \psi + \phi_0 \psi_0 \sin \theta \sin \psi + \theta_0 \psi_0 \cos \psi \right) \]

\[= 3 \lambda^2 \left( A - C \right) \left[ \theta_0 \cos^2 \theta \sin \psi - \theta_0 \sin^2 \theta \sin \psi \right]. \quad (112) \]

The equations (111) and (112) are still nonlinear due to the terms \( \sin \psi \) and \( \cos \psi \). However, without loss in generality we may assume that the small oscillations are being studied in the neighborhood of the initial position from which \( \psi \) is measured so that \( \psi \) may be taken as small. Since the angle \( \psi \) is a continuous function of time, an interval can always be found for which the assumption of small angles is valid. If \( \psi \) becomes large and invalidates the assumption of small angles, a new time base can be chosen so that \( \psi \) can again be considered small.
The linearized equations for small oscillations about the initial position are

\[
A \left[ \phi_0 \psi_0 \theta_0 \cos \theta_0 + \phi_0 \psi_0 \sin \theta_0 + \theta_0 \right] \\
- [B - C] \left[ \theta_0 \phi_0^2 \left( \cos^2 \theta_0 - \sin^2 \theta_0 \right) + 2 \phi_0 \phi_0 \sin \theta_0 \cos \theta_0 \\
+ \phi_0 \psi_0 \theta_0 \cos \theta_0 + \phi_0 \psi_0 \sin \theta_0 \right] \\
= 3 \lambda^2 [C - B] \theta_0 \left( \cos^2 \theta_0 - \sin^2 \theta_0 \right); \quad (113)
\]

\[
B \left[ \phi_0 \sin \theta_0 + \theta_0 \phi_0 \cos \theta_0 - \theta_0 \psi_0 \right] \\
- [C - A] \left[ \theta_0 \phi_0 \cos \theta_0 + \theta_0 \psi_0 \right] = 0. \quad (114)
\]

Equation (114) may be written

\[
(B \sin \theta_0) \phi_0 = \theta_0 \left[ B \psi_0 - B \phi_0 \cos \theta_0 + (C - A)(\phi_0 \cos \theta_0 + \psi_0) \right]. \quad (115)
\]

Integrating equation (115), we obtain

\[
(B \sin \theta_0) \phi_0 = \theta_0 \left[ (B + C - A) \psi_0 - (B - C + A) \phi_0 \cos \theta_0 \right]. \quad (116)
\]
Substituting (116) into (113), the differential equation for the small nutation $\theta_\delta$ is

$$
\theta_\delta^{\infty} + \frac{1}{A^2B} \left\{ A [ - (A - B)^2 + C^2 ] \psi_0^2 + B [ (B - C)(A + 2(A - C) \cos^2 \theta_0)] \phi_0^2 + A [ A(B - A) + (2A - C)(B - C) - 2(B^2 - C^2)] \phi_0^2 \psi_0^2 \cos \theta_0 + A^2B [(B - C)(2 \cos^2 \theta_0 - 1)(3 \lambda^2)] \right\} \theta_\delta = 0.
$$

Thus the small oscillations are sinusoidal with period

$$
T = \frac{1}{2\pi AB^2} \left\{ A [ - (A - B)^2 + C^2 ] \psi_0^2 + B [(B - C)(A + 2(A - C) \cos^2 \theta_0)] \phi_0^2 + A [ A(B - A) + (2A - C)(B - C) - 2(B^2 - C^2)] \phi_0^2 \psi_0^2 \cos \theta_0 + A^2B [(B - C)(2 \cos^2 \theta_0 - 1)(3 \lambda^2)] \right\}^{1/2}.
$$
and are stable provided the radical in the denominator is real; that is,

\[
\begin{align*}
&\{ A \left[ - (A - B)^2 + C^2 \right] \psi_0^2 \\
&+ B \left[ (B - C)(A + 2(A - C) \cos^2 \theta_0) \right] \phi_0^2 \\
&+ A \left[ A(B - A) + (2A - C)(B - C) - 2(B^2 - C^2) \right] \phi_0 \psi_0 \cos \theta_0 \\
&+ A^2 B \left[ (B - C)(2 \cos^2 \theta_0 - 1) 3 \lambda^2 \right] \} = 0.
\end{align*}
\]

Observe that the stability of the motion depends not only on the principal moments of inertia, \( A, B, C \) but also on the precession rate \( \phi_0 \), the induced spin \( \psi_0 \), and the dip angle \( \theta_0 \). This is in contrast to the results obtained by DeBra and Delp (43) for a nonrotating satellite which depended only on the principal mass moments of inertia \( A, B, C \).

Since,

\[
\phi_\delta = \frac{1}{B \sin \theta_0} \left[ - (A - B - C) \psi_0 - (A + B - C) \phi_0 \cos \theta_0 \right] \theta_\delta,
\]

we can see that \( \phi_\delta \) and \( \phi_\delta \) will also be sinusoidal with the same period of oscillation \( T \).
Steady Precession Rate and Spin Requirement

By solving the first steady motion equation, (108), which may be written

\[ \dot{A} \dot{\phi} - [B - C] [\dot{\psi} \phi + \dot{\phi} \psi] + 3 \lambda^2 [B - C] \cos \theta = 0 \]

or

\[ [B - C] [\cos \theta] \phi^2 + ([B - C] \psi - A \dot{\psi}) \phi - 3 \lambda^2 [B - C] \cos \theta = 0 \]

or

\[ \phi^2 - \frac{[A - B + C] \psi}{[B - C] [\cos \theta]} \phi - 3 \lambda^2 = 0, \]

we can obtain the two precessional speeds.

\[ \phi = \frac{[A - B + C]}{2[B - C] \cos \theta} \pm \left\{ \frac{[A - B + C]^2 \psi^2}{4[B - C]^2 \cos^2 \theta} - 3 \lambda^2 \right\}^{1/2}. \]

This is provided that the induced spin \( \psi \) is great enough to keep the radical of the above equation real. This requirement is satisfied if

\[ \psi^2 \geq \frac{12 \lambda^2 [B - C]^2 \cos^2 \theta}{[A - B + C]^2}. \]

However, observe that the second steady motion equation, (110), can also be solved for the precession speed \( \phi \). By comparing this solution to the one obtained from the equation (108), one finds
that in order that the satellite will have a steady precession, as was initially assumed, a very particular relation must exist between the three inertias $A$, $B$, and $C$. Namely,

$$A + B = C.$$  \hspace{1cm} (126)

This is not unusual since DeBra and Delp (43) showed that for a nonspinning satellite, only two inertial parameters were required to express the stability relations.

The inertia requirement (126) is a particularly interesting relation in that it would be satisfied by any thin-sheet satellite as shown in Figure 8. Such a satellite could be used as a large mirror to reflect solar radiation or possibly as a large reflector for communications purposes. However, even of more significance is the fact that many proposed manned space stations are in the form of relatively thin disks and would probably very nearly satisfy equation (126). This raises the interesting question for future study of how the precession rate and spin requirement are affected if the inertia requirement is "nearly" satisfied.

**Results of Case I**

It was shown that it is possible for an unsymmetrical, spinning satellite to have a steady precession while in orbit. Like a spinning top, it was shown that there are two precession
Figure 8a. Rectangular Thin-Sheet Satellite

Figure 8b. Circular Thin-Sheet Satellite

Figure 8c. Proposed Shape of a Manned Space Station

Figure 8. Thin-Sheet Satellites and Disk-Like Manned Station.
rates as given by equation (124) which are subject to the spin requirement (125) and the inertia requirement (126).

It was also shown that small oscillations about the steady precession were sinusoidal subject to the requirement (119) and that the small deviations \( \theta_\delta \) and \( \phi_\delta \) both oscillated with the same period \( T \) given by equation (118).

From the practical standpoint, the results of Case I would imply that unsymmetrical satellites which are spun for stabilization or for artificial gravity effects will precess under the influence of the gravitational torque if they are not otherwise controlled. If these satellites are to be maintained as stable platforms in space, then it is obvious that continuous attitude control would be a necessity.

As far as it has been possible to determine, the results and analysis of Case I are new to the literature of the attitude motion of a satellite under the influence of the gravity gradient torque.
Case II. Small Disturbances in the Motion of an Unsymmetrical Spinning Satellite

Assume that there is a motion described by \( \phi_0, \theta_0, \psi_0 \) which are functions of time and which satisfy the attitude equations (94) - (96). Denote small departures or variations from this motion by \( \phi_\delta, \theta_\delta, \psi_\delta \) so that we may write at any instant

\[
\phi = \phi_0 + \phi_\delta, \quad \theta = \theta_0 + \theta_\delta, \quad \psi = \psi_0 + \psi_\delta;
\]

\[
\phi_0 = \phi_0, \quad \theta_0 = \theta_0, \quad \psi_0 = \psi_0 \quad (127)
\]

For small departures or variations, we can make the following approximations

\[
sin \theta \approx sin \theta_0 + \theta_\delta \cos \theta_0;
\]

\[
\cos \theta \approx \cos \theta_0 - \theta_\delta \sin \theta_0;
\]

\[
sin \psi \approx \sin \psi_0 + \psi_\delta \cos \psi_0;
\]

\[
\cos \psi \approx \cos \psi_0 - \psi_\delta \sin \psi_0 \quad (128)
\]
Substituting equations (127) and (128) into the attitude equations (94) - (96), expanding and neglecting products of the small variations we get for equation (94),

\[
A \left[ \phi_\theta \sin \theta \sin \psi + \phi_\delta \cos \theta \sin \psi + \phi_\delta \sin \theta \sin \psi \right. \\
+ \left. \phi_\psi \delta \sin \theta \cos \psi \right] \\
+ \phi_\theta \cos \theta \sin \psi - \phi_\theta \delta \sin \sin \psi + \phi_\delta \cos \theta \sin \psi \\
+ \phi_\delta \cos \theta \sin \psi + \phi_\delta \psi \delta \cos \theta \cos \psi \\
+ \phi_\psi \sin \theta \cos \psi + \phi_\psi \delta \cos \theta \cos \psi + \phi_\psi \delta \sin \theta \cos \psi \\
+ \phi_\psi \sin \theta \cos \psi \right. \\
+ \phi_\psi \psi \delta \sin \theta \sin \psi - \phi_\psi \psi \delta \sin \theta \sin \psi \\
+ \phi_\psi \sin \theta \cos \psi + \phi_\psi \delta \cos \psi - \phi_\psi \psi \delta \sin \theta \sin \psi \\
+ \phi_\psi \sin \theta \cos \psi - \phi_\psi \psi \delta \sin \theta \sin \psi - \phi_\psi \psi \delta \cos \theta \cos \psi \right. \\
\left. - [B - C] \left( \phi_\theta^2 \sin \theta \cos \theta \cos \psi + \phi_\delta^2 \cos^2 \theta \cos \psi \right. \\
+ \phi_\theta \delta \sin \theta \cos \theta \cos \psi \right. \\
\left. + 2 \phi_\theta \psi \delta \sin \theta \cos \theta \cos \psi - \phi_\psi \psi \delta \sin \theta \cos \theta \sin \psi \right. \\
\left. - \phi_\theta \cos \theta \sin \psi + \phi_\delta \cos \theta \sin \psi - \phi_\delta \cos \theta \sin \psi \right. \\
\left. - \phi_\psi \cos \theta \sin \psi - \phi_\psi \psi \delta \cos \theta \cos \psi \right. \\
\right]
\]
\[
+ \phi \psi \sin \theta \cos \psi + \phi \theta \cos \theta \cos \psi + \phi \theta \delta \sin \theta \cos \psi \\
+ \phi \delta \psi \sin \theta \cos \psi - \phi \psi \delta \sin \theta \delta \sin \psi \\
- \theta \psi \sin \psi. - \theta \delta \sin \psi - \theta \psi \delta \sin \psi - \theta \delta \psi \cos \psi \\
\]

\[
= 3 \lambda^2 [C - B] [\sin \theta \cos \theta \sin \psi + \theta \delta \cos^2 \theta \sin \psi \\
- \theta \delta \sin^2 \theta \sin \psi + \psi \delta \sin \theta \cos \theta \cos \psi ]. \quad (129)
\]

Similarly, we get for equation (95)

\[
\]

\[
B [\phi \psi \sin \theta \cos \psi + \phi \theta \cos \theta \cos \psi + \phi \theta \delta \sin \theta \cos \psi \\
- \phi \psi \delta \sin \theta \sin \psi \\
+ \phi \theta \cos \theta \cos \psi - \phi \theta \delta \sin \theta \cos \psi + \phi \theta \delta \cos \theta \cos \psi \\
+ \phi \theta \delta \cos \theta \cos \psi - \phi \theta \psi \delta \cos \theta \sin \psi \\
- \phi \psi \sin \theta \sin \psi - \phi \psi \delta \cos \theta \sin \psi - \phi \psi \delta \sin \theta \sin \psi \\
- \phi \psi \delta \sin \theta \sin \psi - \phi \psi \psi \delta \sin \theta \cos \psi \\
- \theta \sin \psi. - \theta \delta \cos \psi - \theta \delta \sin \psi \\
- \theta \psi \cos \psi + \theta \psi \delta \sin \psi - \theta \psi \delta \cos \psi - \theta \psi \cos \psi ]
\]
- [C - A] \left[ \phi_o \theta^2 \sin \theta \cos \theta \sin \psi + \phi_o \delta \cos^2 \theta \sin \psi \right.

\left. - \phi_o \delta \sin^2 \theta \sin \psi \right]

+ 2 \phi_o \delta \sin \theta \cos \theta \sin \psi + \phi_o \delta \sin \theta \cos \theta \cos \psi

+ \phi_o \theta \cos \theta \cos \psi + \phi_o \delta \sin \theta \cos \theta \cos \psi

+ \phi_o \delta \cos \theta \cos \psi - \phi_o \delta \psi \cos \theta \sin \psi

+ \phi_o \psi \sin \theta \sin \psi + \phi_o \psi \delta \cos \theta \sin \psi + \phi_o \psi \sin \theta \sin \psi

+ \phi_o \psi \sin \theta \sin \psi + \phi_o \psi \delta \sin \theta \cos \psi

+ \phi_o \psi \cos \psi - \phi_o \psi \delta \sin \psi + \phi_o \psi \cos \psi + \phi_o \psi \cos \psi \right]

= 3 \lambda^2 \left[ A - C \right] \left[ \sin \theta \cos \theta \sin \psi + \psi \sin \theta \cos \theta \cos \psi \right.

\left. + \psi \cos^2 \theta \sin \psi - \psi \sin^2 \theta \sin \psi \right] , \quad (130)
and for equation (96)

\[
C \left[ \phi \cos \theta - \phi \delta \sin \theta + \phi \delta \cos \theta \\
- \phi \delta \sin \theta - \phi \delta \cos \theta - \phi \delta \sin \theta - \phi \delta \sin \theta \\
+ \psi + \psi \delta \right]

- [A - B] \left[ \phi \delta \sin \theta \sin \psi \cos \psi + 2 \phi \delta \sin \theta \cos \theta \sin \psi \cos \psi \\
+ 2 \phi \delta \sin \theta \sin \psi \cos \psi \\
+ \phi \delta \sin \theta \cos \psi + \phi \delta \cos \theta \cos \psi + \phi \delta \sin \theta \cos \psi \\
+ \phi \delta \sin \theta \cos \psi \sin \theta \sin \psi \cos \psi \\
- \phi \delta \sin \theta \sin \psi - \phi \delta \cos \theta \sin \psi - \phi \delta \sin \theta \sin \psi \\
- \phi \delta \sin \theta \sin \psi - \phi \delta \cos \theta \sin \psi - \phi \delta \sin \theta \sin \psi \cos \psi \\
- \phi \delta \sin \theta \sin \psi \cos \psi + \phi \delta \sin \theta \sin \psi \cos \psi - 2 \phi \delta \sin \theta \sin \psi \cos \psi \right]

= 3 \lambda^2 \left[ B - A \right] \left[ \sin^2 \theta \sin \psi \cos \psi + \psi \delta \sin^2 \theta \cos \psi \\
- \psi \delta \sin^2 \theta \sin \psi + 2 \psi \delta \sin \theta \cos \theta \sin \psi \cos \psi \right]. \quad (131)

The "steady" motion is to be defined by the equations

\[
A \phi_0 \sin \theta \sin \psi + A \phi_0 \theta \cos \theta \sin \psi + A \phi_0 \psi \sin \theta \cos \psi
\]

\[
+ A \theta \psi \cos \psi - A \theta \psi \sin \psi
\]

\[
- [B - C][\phi_0^2 \sin \theta \cos \theta \cos \psi]
\]

\[
+ [B - C][\phi_0 \theta \cos \theta \sin \psi]
\]

\[
- [B - C][\phi_0 \psi \sin \theta \cos \psi]
\]

\[
+ [B - C][\theta \psi \sin \psi] = 3 \lambda^2 [C - B][\sin \theta \cos \theta \sin \theta \sin \psi]
\]

\[
(132)
\]

\[
B[\phi_0 \sin \theta \cos \psi + \phi_0 \theta \cos \theta \cos \psi - \phi_0 \psi \sin \theta \sin \psi
\]

\[
- \theta \psi \sin \psi - \theta \psi \cos \psi
\]

\[
- [C - A][\phi_0^2 \sin \theta \cos \theta \sin \psi + \phi_0 \theta \cos \psi \cos \theta
\]

\[
+ \phi_0 \psi \sin \theta \sin \psi + \theta \psi \cos \psi] = 3 \lambda^2 [A - C][\sin \theta \cos \theta \sin \psi]
\]

\[
(133)
\]
and
\[
C[\phi^0 \cos \theta^0 - \phi^0 \theta^0 \sin \theta^0 + \psi^0]
\]
\[
- [A - B][\phi^2 \sin^2 \theta^0 \sin \psi^0 \cos \psi^0 + \phi^0 \theta^0 \sin \theta^0 \cos^2 \psi^0
\]
\[
- \phi^0 \theta^0 \sin \theta^0 \sin^2 \psi^0 - \theta^2 \sin \psi^0 \cos \psi^0]
\]
\[
= 3 \lambda^2 [B - A][\sin^2 \theta^0 \sin \psi^0 \cos \psi^0].
\] (134)

These equations (133) - (134) for the "steady" motion which are obtained by setting the variations equal to zero are identical to the attitude equations (94) - (96) with the exception of the subscripts.

Observe then that \(\phi^0, \theta^0, \psi^0\) are not to be taken as constants but that they are functions of time which must be found by solving the system of nonlinear, differential equations (132) - (134) for the steady motion. Observe, too, that \(\phi^0, \theta^0, \psi^0\) and \(\phi^o, \theta^o, \psi^o\) are in general also functions of time and must be such as to satisfy the equations (132) - (134).
Collecting like terms in the variations and their derivatives and rearranging, equation (129) may be written

\[
\left\{A \sin \theta \sin \psi\right\} \phi_\delta^\infty
\]

\[
+ \left\{A \left[\cos \theta \sin \psi + \sin \theta \cos \psi\right]\right\} \phi_\delta^\infty
\]

\[- [B - C] \left[2 \phi \sin \theta \cos \theta \cos \psi - \theta \cos \theta \sin \psi \right.
\]

\[+ \psi \sin \theta \cos \psi\} \right\} \phi_\delta^\infty
\]

\[+ \{0\} \phi_\delta^\infty
\]

\[+ \{A \cos \psi\} \theta_\delta^\infty
\]

\[+ \{A \left[\cos \theta \sin \psi - \psi \sin \psi\right] - [B - C] \left[- \phi \cos \theta \sin \psi \right.
\]

\[+ \psi \sin \theta \cos \psi\} \left\} \theta_\delta^\infty
\]

\[+ \{A \left[\cos \theta \sin \psi - \phi \sin \theta \sin \psi + \psi \cos \theta \cos \psi\right]\}
\]

\[- [B - C] \left[\phi^2 \cos^2 \theta \cos \psi - \phi^2 \sin^2 \theta \cos \psi \right.
\]

\[+ \phi \sin \theta \sin \psi + \psi \cos \theta \cos \psi\] \}

\[+ 3 \lambda^2 [B - C] \left[\cos^2 \theta \sin \psi - \sin^2 \theta \sin \psi\right] \right\} \theta_\delta^\infty
\]

\[+ \{0\} \psi_\delta^\infty
\]

\[+ \{A \left[\phi \sin \theta \cos \psi - \theta \sin \psi + \phi \sin \theta \cos \psi - \theta \sin \psi\}\} \psi_\delta^\infty
\]
\[
\begin{align*}
&+ \{ A[\phi_0 \sin \theta \cos \psi_0 + \phi_0 \theta \cos \theta_0 \cos \psi_0 - \phi_0 \psi_0 \sin \theta \sin \psi_0 - \theta \sin \psi_0 - \theta_0 \psi_0 \cos \psi_0 ] \\
&- [B - C][\phi_0 \sin \theta_0 \cos \theta_0 \sin \psi_0 - \phi_0 \theta_0 \cos \theta_0 \cos \psi_0 - \phi_0 \psi_0 \sin \theta_0 \sin \psi_0 - \theta_0 \psi_0 \cos \psi_0 ] \\
&+ 3 \lambda^2 [B - C][\sin \theta_0 \cos \theta_0 \cos \psi_0 ] \} \psi_0. \tag{135}
\end{align*}
\]

Similarly, equation (130) may be written

\[
\begin{align*}
&\{ B[\sin \theta_0 \cos \psi_0 ] } \phi_0 \\
&+ \{ B[\theta_0 \cos \theta_0 \cos \psi_0 - \psi_0 \sin \theta_0 \sin \psi_0 ] \\
&- [C - A][2 \phi_0 \sin \theta_0 \cos \theta_0 \sin \psi_0 + \phi_0 \theta_0 \cos \theta_0 \cos \psi_0 \\
&+ \psi_0 \sin \theta_0 \sin \psi_0 ] } \phi_0 \\
&+ \{ 0 ] } \phi_0 \\
&+ \{ - B[\sin \psi_0 ] } \theta_0 \\
&+ \{ B[\phi_0 \cos \theta_0 \cos \psi_0 - \psi_0 \cos \psi_0 ] \\
&- [C - A][\phi_0 \cos \theta_0 \cos \psi_0 + \psi_0 \cos \psi_0 ] \} \theta_0
\end{align*}
\]
\[ + \{B[\phi_0 \cos \theta \cos \psi - \phi_0 \sin \theta \cos \psi - \phi_0 \psi \cos \theta \sin \psi] \right. \\
\left. - [C-A][\phi_0 \cos^2 \theta \sin \psi - \phi_0 \sin^2 \theta \sin \psi - \phi_0 \theta \sin \theta \cos \psi \\
+ \phi_0 \psi \cos \theta \sin \psi] \right. \\
\left. + 3 \lambda^2 [C-A][\cos^2 \theta \sin \psi - \sin^2 \theta \sin \psi] \} \theta_6 \right. \\
\left. + \{0\} \psi_6 \right. \\
\left. + \{B[-\phi_0 \sin \theta \sin \psi - \theta \cos \psi] \right. \\
\left. - [C-A][\phi_0 \sin \theta \sin \psi + \theta \cos \psi] \} \psi_6 \right. \\
\left. + \{B[-\phi_0 \sin \theta \sin \psi - \phi_0 \theta \cos \theta \sin \psi - \phi_0 \psi \sin \theta \cos \psi \\
- \theta \cos \psi + \theta \psi \sin \psi] \right. \\
\left. - [C-A][\phi_0 \sin \theta \cos \theta \cos \psi - \phi_0 \theta \cos \theta \sin \psi \\
+ \phi_0 \psi \sin \theta \cos \psi - \theta \psi \sin \psi] \right. \\
\left. + 3 \lambda^2 [C-A][\sin \theta \cos \theta \cos \psi] \} \psi_6 \right. \\
\right. \\
\text{Equation (131) may be written}
\]

\[ \{C[\cos \theta_0] \}\phi_6 \]

\[ + \{C[-\theta_0 \sin \theta_0] \}
\]

\[ - [A-B][2 \phi_0 \sin^2 \theta \sin \psi \cos \psi + \theta \sin \theta \cos^2 \psi \\
- \theta \sin \theta \sin^2 \psi] \} \phi_6 \]
The first equation in the small variations may be written

\[ L_{11} \phi_\delta + L_{12} \phi_\delta + 0 \cdot \phi_\delta + M_{11} \theta_\delta + M_{12} \theta_\delta + M_{13} \theta_\delta + 0 \cdot \psi_\delta + N_{12} \psi_\delta + N_{13} \psi_\delta = 0. \]  

(138)
The second equation in the small variations may be written
\[ L_{21} \Phi_6 + L_{22} \Phi_6 + 0 \cdot \Phi_6 + M_{21} \Theta_6 + M_{22} \Theta_6 + M_{23} \Theta_6 + 0 \cdot \psi_6 + N_{22} \psi_6 + N_{23} \psi_6 = 0. \] (139)

The third equation in the small variations may be written
\[ L_{31} \Phi_6 + L_{32} \Phi_6 + 0 \cdot \Theta_6 + M_{32} \Theta_6 + M_{33} \Theta_6 + N_{31} \psi_6 + 0 \cdot \psi_6 + N_{33} \psi_6 = 0. \] (140)

Observe that these are three simultaneous, linear differential equations in the small variations. However \( L_{ij}, M_{ij}, \) and \( N_{ij} \) are functions of time and are the coefficients of the respective variation terms in equations (135) - (137). A summary of these coefficients is given following the discussion of equations (138) - (140).

Assume that \( \Phi, \Theta, \psi \) are analytic, that is all their derivatives with respect to time exist. This assumption would rule out the "looped" or "cusped-type" motions shown in Figure 9. However, all other types of motion, such as the "smooth," elliptic-type motions, monotonic, tumbling motions and steady precession, will satisfy this assumption. Since these latter motions are probably more common in practical satellites, this assumption is not too restrictive from the physical standpoint.
Assume further that the functions of time $L_{ij}$, $M_{ij}$, $N_{ij}$ are expanded in series about any instant of time, $t_o$, which could be selected as $t = 0$ without any loss in generality.

Suppose that a suitably small interval of time is selected about $t_o$ such that a good approximation to the expansions of the functions of time, $L_{ij}$, $M_{ij}$, and $N_{ij}$ is the constant value at $t_o$, then we may consider the coefficients of the equations (138) - (140) as constants at least over this small interval of time.

Let us now study the "stability" of this "derived" system of simultaneous, linear differential equations which now have constant coefficients. Stability studied in this manner is defined as "infinitesimal stability" by J. J. Stoker. Assume

$$
\phi_\delta = L e^{pt}, \quad \theta_\delta = M e^{pt}, \quad \psi_\delta = N e^{pt}; \quad (141)
$$

---

then the equations (138) - (140) yield the system of homogeneous algebraic equations in $p$, $L$, $M$, and $N$

$$(L_{11}p^2 + L_{12}p)L + (M_{11}p^2 + M_{12}p + M_{13})M + (N_{12}p + N_{13})N = 0; \quad (142)$$

$$(L_{21}p^2 + L_{22}p)L + (M_{21}p^2 + M_{22}p + M_{23})M + (N_{22}p + N_{23})N = 0; \quad (143)$$

$$(L_{31}p^2 + L_{32}p)L + (M_{31}p + M_{32})M + (N_{31}p^2 + N_{33})N = 0. \quad (144)$$

If this set of homogeneous algebraic equations is to have a non-trivial solution for $L$, $M$, and $N$, then the determinant of the coefficients must be zero and we obtain the polynomial

$$a_0p^6 + a_1p^5 + a_2p^4 + a_3p^3 + a_4p^2 + a_5p + 0 = 0 \quad (145)$$

where the $a_i$ are functions of $L_{ij}$, $M_{ij}$, and $N_{ij}$ evaluated at $t_0$.

A summary of the $a_i$ is given following the discussion of this study.

Since the characteristic equation has a zero root, this is stable in accordance with the definition of E. J. Routh. It should be noted however that a zero root could easily become a small positive root through a small change in system parameters or through round-off errors in the calculations. For this reason, later authors, such as S. W. McCuskey, prefer to

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6 E. J. Routh, op. cit., p. 219

consider a zero root as unstable. However, for this problem, the zero root arises because the coordinate $\phi$ has become a cyclic coordinate through the process of simplifying the attitude equations. This can be observed from the form of the equations (142) - (144). Therefore this zero root cannot be effected by small changes in system parameters or by round-off errors. For this reason, this zero root will be considered stable. Information about the remaining roots of equation (142) cannot be obtained without numerical values and the aid of a computer. Therefore no conclusion can be made in general about the infinitesimal stability without further study.

**Summary of the $a_i$**

\[
a_0 = \{L_{11}M_{21}N_{31} - L_{21}M_{11}N_{31}\};
\]

\[
a_1 = \{(L_{11}M_{22} + L_{12}M_{21})N_{31} + L_{31}M_{11}N_{22} - (L_{21}M_{12} + L_{22}M_{11})N_{31}\};
\]

\[
a_2 = \{(L_{11}M_{23} + L_{12}M_{22})N_{31} + L_{11}M_{21}N_{33} + L_{21}M_{32}N_{12}
+ (L_{31}M_{12} + L_{32}M_{11})N_{22} + L_{31}M_{11}N_{23}
- (L_{31}M_{22} + L_{32}M_{21})N_{12} - L_{31}M_{21}N_{13} - (L_{21}M_{13} + L_{22}M_{12})N_{31}
- \Gamma (L_{31}M_{23} + L_{32}M_{22})N_{12} - L_{31}M_{21}N_{13} - (L_{21}M_{13} + L_{22}M_{12})N_{31}\};
\]
\[ a_3 = \{ L_{12} M_{23} N_{31} + (L_{11} M_{22} + L_{12} M_{21})N_{33} + L_{21} M_{32} N_{13} \]
\[ + (L_{21} M_{33} + L_{22} M_{32})N_{12} + (L_{31} M_{13} + L_{32} M_{12})N_{22} \]
\[ + (L_{31} M_{12} + L_{32} M_{11})N_{23} - (L_{31} M_{23} + L_{32} M_{22})N_{12} \]
\[ - (L_{31} M_{22} + L_{32} M_{21})N_{13} - L_{22} M_{13} N_{31} \]
\[ - (L_{21} M_{12} + L_{22} M_{11})N_{33} - (L_{11} M_{33} + L_{12} M_{32})N_{22} \]
\[ - L_{11} M_{32} N_{23} \} ; \]

\[ a_4 = \{(L_{11} M_{23} + L_{12} M_{22})N_{33} + L_{22} M_{33} N_{12} \]
\[ + (L_{21} M_{33} + L_{22} M_{32})N_{13} + L_{32} M_{13} N_{22} \]
\[ + (L_{31} M_{13} + L_{32} M_{12})N_{23} - (L_{31} M_{23} + L_{32} M_{22})N_{13} \]
\[ - L_{12} M_{33} N_{22} - (L_{21} M_{13} + L_{22} M_{12})N_{33} \]
\[ - L_{11} M_{32} N_{23} \} ; \]

\[ a_5 = \{(L_{12} M_{23} N_{33} + L_{22} M_{33} N_{13} + L_{32} M_{13} N_{23} - L_{32} M_{23} N_{13} \]
\[ - L_{22} M_{13} N_{33} - L_{12} M_{33} N_{23} \} \).
Summary of the Functions \( L_{ij}, M_{ij}, N_{ij} \)

of Equations (138) - (140)

\[
L_{11} = \{ A[\sin \theta \sin \psi] \} ;
\]

\[
L_{12} = \{ A[\theta \cos \theta \sin \psi + \psi \sin \theta \cos \psi] \}
- [2 \phi \sin \theta \cos \theta \cos \psi - \theta \cos \theta \sin \psi \\
+ \psi \sin \theta \cos \psi][B - C] \};
\]

\[
L_{21} = \{ B[\sin \theta \cos \psi] \} ;
\]

\[
L_{22} = \{ B[\theta \cos \theta \cos \psi - \psi \sin \theta \sin \psi] \\
- [2 \phi \sin \theta \cos \theta \sin \psi + \theta \cos \theta \cos \psi \\
+ \psi \sin \theta \sin \psi][C - A] \};
\]

\[
L_{31} = \{ C[\cos \theta] \} ;
\]

\[
L_{32} = \{ C[-\theta \sin \theta] \\
- [2 \phi \sin^2 \theta \sin \psi \cos \psi + \theta \sin \theta \cos^2 \psi \\
- \theta \sin \theta \sin^2 \psi][A - B] \}.
\]
\[ M_{11} = \{ A[\cos \psi_o] \} ; \]
\[ M_{12} = \{ A[\phi_o \cos \theta_o \sin \psi_o - \psi_o \sin \psi_o] \]
\[ - [B - C][\phi_o \cos \theta_o \sin \psi_o - \psi_o \sin \psi_o] \} ; \]
\[ M_{13} = \{ A[\phi_o \cos \theta_o \sin \psi_o - \phi_o \theta_o \sin \theta_o \sin \psi_o \]
\[ + \phi_o \psi_o \cos \theta_o \cos \psi_o] - [B - C][\phi_o^2 \cos^2 \theta_o \cos \psi_o \]
\[ - \phi_o^2 \sin^2 \theta_o \cos \psi_o + \phi_o \theta_o \sin \theta_o \sin \psi_o \]
\[ + \phi_o \psi_o \cos \theta_o \cos \psi_o \]
\[ + 3 \lambda^2 [B - C][\cos^2 \theta_o \sin \psi_o - \sin^2 \theta_o \sin \psi_o] \} ; \]
\[ M_{21} = \{ -B[\sin \psi_o] \} ; \]
\[ M_{22} = \{ B[\phi_o \cos \theta_o \cos \psi_o - \psi_o \cos \psi_o] \]
\[ - [C - A][\phi_o \cos \theta_o \cos \psi_o + \psi_o \cos \psi_o] \} ; \]
\[ M_{23} = \{ B[\phi_o \cos \theta_o \cos \psi_o - \phi_o \theta_o \sin \theta_o \cos \psi_o - \phi_o \psi_o \cos \theta_o \sin \psi_o] \]
\[ - [C - A][\phi_o^2 \cos^2 \theta_o \sin \psi_o - \phi_o^2 \sin^2 \theta_o \sin \psi_o \]
\[ - \phi_o \theta_o \sin \theta_o \cos \psi_o + \phi_o \psi_o \cos \theta_o \sin \psi_o \]
\[ + 3 \lambda^2 [C - A][\cos^2 \theta_o \sin \psi_o - \sin^2 \theta_o \sin \psi_o] \} ; \]
\[ M_{32} = \{ C[- \phi \sin \theta] \\
- [\phi \sin \theta \cos^2 \psi - \phi \sin \theta \sin^2 \psi \\
- 2 \theta \sin \psi \cos \psi][A - B] \} ; \]

\[ M_{33} = \{ C[- \phi \sin \theta - \phi \theta \cos \theta] \\
- [2 \phi \sin \theta \cos \theta \sin \psi \cos \psi + \phi \theta \cos \theta \cos^2 \psi \\
- \phi \theta \cos \theta \sin^2 \psi][A - B] \\
+ 3 \lambda^2[A - B][2 \sin \theta \cos \theta \sin \psi \cos \psi]\} ; \]

\[ N_{12} = \{ A[\phi \sin \theta \cos \psi - \theta \sin \psi \\
+ \phi \sin \theta \cos \psi - \theta \sin \psi]\} ; \]

\[ N_{13} = \{ A[\phi \sin \theta \cos \psi + \phi \theta \cos \theta \cos \psi \\
- \phi \psi \sin \theta \sin \psi - \theta \sin \psi - \theta \psi \cos \psi] \\
- [B - C][- \phi \sin \theta \cos \theta \sin \psi \\
- \phi \theta \cos \theta \cos \psi - \phi \psi \sin \theta \sin \psi \\
- \theta \psi \cos \psi + 3 \lambda^2[B - C][\sin \theta \cos \theta \cos \psi]\} ; \]

\[ N_{22} = \{ B[- \phi \sin \theta \sin \psi - \theta \cos \psi] \\
- [C - A][\phi \sin \theta \sin \psi + \theta \cos \psi]\} ; \]
\[ N_{23} = \{ B[ - \phi_o \sin \theta \sin \psi_o - \phi_o \theta_o \cos \theta \sin \psi_o \\
- \phi_o \theta_o \psi_o \cos \theta_o \cos \psi_o + \theta_o \psi_o \sin \psi_o ] \\
- [C - A][ \phi_o^2 \sin \theta \cos \theta \cos \psi_o - \phi_o \theta_o \sin \psi_o \cos \theta_o \\
+ \phi_o \theta_o \sin \theta \cos \psi_o - \theta_o \psi_o \sin \psi_o ] \\
+ 3 \lambda^2 [C - A][\sin \theta_o \cos \theta_o \cos \psi_o] \}; \]

\[ N_{31} = \{ C \}; \]

\[ N_{33} = \{ -[A - B][ \phi_o^2 \sin^2 \theta \cos^2 \psi_o - \phi_o^2 \sin^2 \theta \sin^2 \psi_o \\
- 4 \phi_o \theta_o \sin \theta \sin \psi_o \cos \psi_o \\
- \theta_o^2 \cos^2 \psi_o + \theta_o^2 \sin^2 \psi_o ] \\
+ 3 \lambda^2 [A - B][\sin^2 \theta \cos^2 \psi_o - \sin^2 \theta \sin^2 \psi_o] \}. \]
The Study of a Symmetrical Spinning Satellite
-- A "Top" in Space

The Behavioral Study of a Symmetrical Spinning Satellite

The Attitude Equations -- Node-System Axes

Taking advantage of the simplicity afforded by the node-system axes in the study of the motion of a symmetrical body, the angular momentum for these principal axes is

\[ \vec{h} = A w_\xi \hat{\xi} + A w_\eta \hat{\eta} + C w_\zeta \hat{\zeta} \]

where according to equations (72) - (74)

\[ w_\xi = \theta + \Omega \sin \phi \] \hspace{1cm} (72)
\[ w_\eta = \phi \sin \theta + \Omega \cos \phi \cos \theta \] \hspace{1cm} (73)
\[ w_\zeta = \phi \cos \theta - \Omega \cos \phi \sin \theta \] \hspace{1cm} (74)

The angular velocity of the node-system axes is

\[ \vec{\omega}_\xi = w_1 \hat{\xi} + w_2 \hat{\eta} + w_3 \hat{\zeta} \] \hspace{1cm} (146)

where

\[ w_1 = w_\xi = \theta + \Omega \sin \phi \] \hspace{1cm} (147)
\[ w_2 = w_\eta = \phi \sin \theta + \Omega \cos \phi \cos \theta \] \hspace{1cm} (148)
\[ w_3 = w_\zeta - \Omega \cos \phi \sin \theta \] \hspace{1cm} (149)
Euler's equations may be written as

$$\begin{align*}
\overrightarrow{\mathbf{h}} = \frac{\partial \mathbf{h}}{\partial t} \mathbf{\xi} \eta^\mathbf{\xi} + \mathbf{w} \mathbf{\xi} \eta^\mathbf{\xi} \times \overrightarrow{\mathbf{h}} = \mathbf{M}. \quad (148)
\end{align*}$$

Expanding,

$$\begin{align*}
\overrightarrow{\mathbf{h}} = [A w_\xi + C w_\eta w_\xi - A w_3 w_\eta] \mathbf{i} \xi \\
+ [A w_\eta - C w_\xi w_\eta + A w_3 w_\xi] \mathbf{j} \eta \\
+ [C w_\zeta] \mathbf{k} \eta = \mathbf{M}. \quad (149)
\end{align*}$$

Introducing the Euler angles by using (72) - (74), (147) and the moment from (56) - (58), the attitude equations for a spinning, symmetrical satellite are these:

$$\begin{align*}
&\mathbf{A} \mathbf{\theta} + \mathbf{A} \mathbf{\Omega} \sin \phi + \mathbf{A} \mathbf{\Omega} \phi \cos \phi + (\mathbf{C} - \mathbf{A}) \mathbf{s} \phi \sin \theta \\
+ \mathbf{(C} - \mathbf{A}) \mathbf{s} \mathbf{\Omega} \cos \phi \cos \theta + \mathbf{A} \mathbf{\psi} \phi \sin \theta + \mathbf{A} \mathbf{\psi} \mathbf{\Omega} \cos \phi \cos \theta \\
= 3 \lambda^2 (\mathbf{C} - \mathbf{A}) \sin \theta \cos \theta; \quad (150)
\end{align*}$$

$$\begin{align*}
&\mathbf{A} \mathbf{\phi} \sin \theta + \mathbf{A} \mathbf{\phi} \theta \cos \theta + \mathbf{A} \mathbf{\Omega} \cos \phi \cos \theta - \mathbf{A} \mathbf{\Omega} \phi \sin \phi \cos \theta \\
- \mathbf{A} \mathbf{\Omega} \theta \cos \phi \sin \theta - (\mathbf{C} - \mathbf{A}) \mathbf{s} \theta - (\mathbf{C} - \mathbf{A}) \mathbf{s} \mathbf{\Omega} \sin \phi \\
- \mathbf{A} \mathbf{\psi} \theta - \mathbf{A} \mathbf{\psi} \mathbf{\Omega} \sin \phi = 0 \quad (151)
\end{align*}$$

and

$$\begin{align*}
&\mathbf{s} = \text{a constant} = \phi \cos \theta + \psi - \mathbf{\Omega} \cos \phi \sin \theta. \quad (152)
\end{align*}$$
Conservation of Angular Momentum

It was shown in equation (28) that the gravity gradient torque vector is always in a plane perpendicular to the local vertical which lies along the $\vec{K}$-axis. Thus the angular momentum component along this axis, $Z$ must be constant and therefore

$$h_Z = h_\eta \sin \theta + h_\phi \cos \theta = \text{a constant} \tag{153}$$

where

$$h_\eta = A \omega_\eta = A \phi \sin \theta + A \Omega \cos \phi \cos \theta \tag{154}$$

$$h_\phi = C \omega_\phi = Cs = \text{a constant} \tag{155}$$

Therefore,

$$h_Z = A \phi \sin^2 \theta + A \Omega \cos \phi \sin \theta \cos \theta + Cs \cos \theta \tag{156}$$

or

$$\phi + \frac{\Omega \cos \theta}{\sin \theta} \cos \phi = \frac{h_Z - Cs \cos \theta}{A \sin^2 \theta} \tag{157}$$
Conservation of Energy

As previously explained, a second integral of the attitude equations can be obtained from the fact that the sum of the kinetic and potential energies is a constant for a satellite in a vacuum and acted on only by the gravitational field. Thus

\[ T + U = E, \text{ a constant} \quad (158) \]

where

\[
T = \frac{1}{2} M \left( R_G^2 + R_G^2 \Omega^2 \right) + \frac{1}{2} A \omega_\xi^2 + \frac{1}{2} A \omega_\eta^2 + \frac{1}{2} C \omega_\zeta^2 \quad (159)
\]

and \( U \) is given by equation (63).

Therefore,

\[
E = \frac{1}{2} M \left( R_G^2 + R_G^2 \Omega^2 \right) \]

\[
+ \frac{1}{2} A \left[ \theta^2 + 2 \Omega \theta \sin \phi + \Omega^2 \sin^2 \phi + \phi^2 \sin^2 \theta \right] \]

\[
+ 2 \Omega \phi \sin \theta \cos \phi \cos \theta \cos \phi + \Omega^2 \cos^2 \phi \cos^2 \theta \]

\[
+ \frac{1}{2} C s^2 - \frac{g R^2 M}{R_G} - \frac{g R^2 e}{2 R_G^3} \left[ (2A + C) - 3 \left( A \sin^2 \theta + C \cos^2 \theta \right) \right].
\]

(160)
The Behavior of a Spinning, Symmetrical Satellite

Assuming that the terms in $\Omega \theta$, $\Omega \phi$, and $\Omega^2$ are of higher order and may be neglected in equation (160), which results from the conservation of energy, we obtain

$$E = \frac{1}{2} M \left( R_G^2 \Omega^2 + R_G^2 \phi^2 \right) + \frac{1}{2} A \left( \theta^2 + \phi^2 \sin^2 \theta \right) + \frac{1}{2} C \cos^2 \theta
- \frac{g R^2 M}{R_G} - \frac{g R^2}{2 R_G} (2A + C) + \frac{3 g R^2}{2 R_G} (A \sin^2 \theta + C \cos^2 \theta).$$

(161)

Neglecting the term in $\phi$ in equation (157), which results from the conservation of angular momentum, we obtain

$$\phi = \frac{h}{A \sin^2 \theta} - C s \cos \theta$$

(162)

To neglect the term in $\Omega$ corresponds to the physical situation of a satellite which has a long orbital period in comparison to the periods of the attitude motions and which is spinning with a rate $s$ which is more rapid than $\Omega$. This condition has been met by many of our recent unmanned satellites.
Substituting (162) into (161) and rearranging, we have

\[ E - \frac{1}{2} M \left( R_G^2 + R_G^2 \Omega^2 \right) - \frac{1}{2} C s^2 + \frac{g R_e^2 M}{R_G} + \frac{g R_e^2}{2 R_G^3} (2 A + C) \]

\[ = \frac{1}{2} A \theta + \frac{1}{2} A \sin^2 \theta \left( \frac{(h_Z - C s \cos \theta)^2}{A^2 \sin^4 \theta} + \frac{3 g R_e^2}{2 R_G^3} (A \sin^2 \theta + C \cos^2 \theta) \right) \]

\( (163) \)

Observe that this equation is entirely a function of \( \theta \) and independently determinable functions of time \( R_G \) and \( \Omega \). The solution of equation (163) and its subsequent substitution into equation (162) will therefore completely describe the motion of the satellite.

Assuming that the orbit is circular, or near circular, so that \( R_G \) and \( \Omega \) can be taken as constants and then letting

\[ \alpha = \text{a constant} = \frac{2}{A} \left[ E - \frac{1}{2} M (R_G^2 + R_G^2 \Omega^2) - \frac{1}{2} C s^2 \right. \]

\[ + \frac{g R_e^2 M}{R_G} + \frac{g R_e^2}{2 R_G^3} (2 A + C) \left. \right] ; \]

\( (164) \)

\[ \beta = \frac{3 g R_e^2}{R_G^3} , \text{ a constant} ; \]

\( (165) \)

\[ \gamma = \frac{h_Z}{A} , \text{ a constant} ; \]

\( (166) \)

\[ N = \frac{C s}{A} , \text{ a constant} ; \]

\( (167) \)
δ = \frac{C}{A}, \text{ a constant}; \tag{168}

\begin{equation}
\begin{aligned}
u & = \cos \theta, \text{ a variable so that } u = -\theta \sin \theta, \\
\text{equation (163) may be written as}
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
a & = \theta^2 + \left(\frac{\gamma - N \cos \theta}{\sin^2 \theta}\right)^2 + \beta \sin^2 \theta + \beta \frac{C}{A} \cos^2 \theta. \tag{170}
\end{aligned}
\end{equation}

Using equation (169) and rearranging, we have

\begin{equation}
\begin{aligned}
o^2 & = \left[(a - \beta) + \beta (1 - \delta) u^2\right] (1 - u^2) - (\gamma - Nu)^2 \tag{171}
\end{aligned}
\end{equation}

or

\begin{equation}
\begin{aligned}
o^2 & = f(u)
\end{aligned}
\end{equation}

where

\begin{equation}
\begin{aligned}
f(u) & = \left[(a - \beta) + \beta (1 - \delta) u^2\right] (1 - u^2) - (\gamma - Nu)^2. \tag{172}
\end{aligned}
\end{equation}

The solution of equation (171) is given by the integral

\begin{equation}
\begin{aligned}
t & = \int_{u(0)}^{u(t)} \frac{du}{\sqrt{f(u)}} = \int_{u(0)}^{u(t)} \frac{du}{\sqrt{[(a - \beta) + \beta (1 - \delta) u^2](1 - u^2) - (\gamma - Nu)^2}^{1/2}}. \tag{173}
\end{aligned}
\end{equation}

According to Whitaker and Watson the above integral will in general be an elliptic integral and \( u \) will be expressed in terms of elliptic functions. In order for this to be true, \( f(u) \) need satisfy

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\text{Whitaker, E. T. and Watson, G. N., Modern Analysis (London, Cambridge University Press, 1940), Chapter XX, p. 452.}
only a few rather loose requirements. This will almost always be met by the physical problem.

Although the solution for $\theta$ is an elliptic function, the general behavior of the satellite can be obtained by studying the function $f(u)$.

In the physical problem, $u = \cos \theta$ is limited between the values $\pm 1$. However, mathematically $u$ can extend outside this region. For large values of $u$, the dominant term in $f(u)$ is $-\beta (1 - 6)u^4$. Thus for $\delta \gg 1$, $f(u)$ is positive for large positive and negative values of $u$. However, for $\delta \ll 1$, $f(u)$ is negative for large positive or negative values of $u$.

At $u = \pm 1$, $(1 - u^2)$ is zero and

$$f(\pm 1) = -(\sqrt{\gamma N})^2.$$  \hspace{1cm} (174)

It is evident from equation (174) that $f(u)$ at $u = \pm 1$ must always be negative.

Looking at the expression for $f(u)$ in terms of $\theta$,

$$f(u) = u^2 = \theta^2 \sin^2 \theta,$$  \hspace{1cm} (175)

we find that, for real values of $\theta$ and $\theta$, $f(u)$ must be positive.

We therefore conclude that, for the physical problem, $u = \cos \theta$ must lie between $u_1$ and $u_2$ for which $f(u)$ is positive.
Given numerical values for the constants $a$, $\beta$, $\delta$, $\gamma$, and $N$, the actual roots $u_1$ and $u_2$ may be obtained by solving $f(u)$ by numerical methods. Estimates of these roots can be obtained by applying the theorem of Fourier and Budan or the theorem of Sturm. Application of the Rule of Signs of Descartes for various possible combinations of the algebraic signs shows that $f(u)$ has at least two real roots.

The shapes of these possible quartics are shown in Figure 9.

We note that for $\Theta \gg 0$, $\Theta$ must be zero at $u_1$ and $u_2$, which requires that the spin axis moves between the bounding circles $u_1 = \cos \Theta_1$ and $u_2 = \cos \Theta_2$.

The type of curve traced by the spin axis in the region $\Theta_1$ and $\Theta_2$ depends on the relative values of $\gamma$ and $N$. The precession rate can be written as

$$\phi = \frac{\gamma - Nu}{1 - u^2}$$

and since $u$ is less than one for the physical problem, the algebraic sign of $\phi$ depends on $\gamma - Nu$. The different possibilities are shown in Figure 10.

---

Figure 9. Quartic Equation Representing the Motion of a Symmetrical Spinning Satellite under the Influence of Gravitational Torque.
Figure 10a. Steady Precession

Figure 10b. $\gamma > Nu_1$

Figure 10c. $\gamma = Nu_1$

Figure 10d. $\gamma = Nu_1$

$u_2 < u_1 < u_1$

Figure 10. Possible Types of Motion of the Spin Axis.
**Initial Conditions**

If the satellite is placed into orbit so that at \( t = 0 \), \( \theta = \theta \) and \( \phi = 0 \), the values of the constants \( h_Z \) (or \( \gamma \)) and \( E \) (or \( a \)) are

\[
h_Z = C \cos \theta \quad \text{and hence} \quad \gamma = \frac{h_Z}{A} = C \cos \theta = N \cos \theta ;
\]

\[
a = \beta \sin^2 \theta + \beta \frac{C}{A} \cos^2 \theta = \beta \left( 1 + \left( \frac{C}{A} - 1 \right) \cos^2 \theta \right).
\]

Therefore, the precession and nutation rates may be written

\[
\phi = \frac{N (\cos \theta - \cos \theta)}{\sin^2 \theta}
\]

and

\[
\theta^2 = [\cos \theta - \cos \theta] \left[ \beta \left( \frac{C}{A} - 1 \right) (\cos \theta + \cos \theta) - \frac{N^2}{\sin^2 \theta} \right].
\]

The second of the above equations indicates that the right side must be positive. Also, we can observe that \( \theta \) is one on the bounding curves for the nutation.

When \( \theta = \theta \),

\[
\frac{df(u)}{d\theta} \bigg|_{\theta = \theta} = \frac{d \theta^2}{d \theta} = \frac{1}{\sin \theta} [\beta \left( \frac{C}{A} - 1 \right) (2 \cos^2 \theta) (1 - \cos^2 \theta) - N^2].
\]
\[ \beta = \frac{3 g R_e^2}{R_G^3} \leq 1 \text{ and } \cos \theta \leq 1, \text{ then} \]

\[ \frac{df(u)}{d\theta} \bigg|_{\theta = \theta_o} = 0. \quad (190) \]

It is evident from Figure 9 and equation (190) that \( \theta_o = \cos^{-1} u \)

and is the upper bounding curve of the motion of the spin axis

as shown in Figure 10. As a result of this, we can further conclude that the precession is direct.

**The Steady Precession and Spin Requirement**

For steady precession, \( \theta = 0, \phi = \phi_o, \) a constant, and

\( \phi = \phi_o, \) a constant. Assuming a circular orbit, \( \Omega = 0, \) and

using equation (152) to eliminate \( \psi, \) the first attitude equation (150) gives

\[ \phi^2 - \phi \frac{(C_s)}{A \cos \theta_o} + \frac{3 \lambda^2 (C - A)}{A} = 0. \quad (191) \]

Like a spinning top, the two precession rates are

\[ \phi = \frac{+ C_s}{2 A \cos \theta_o} \pm \left[ \frac{C_s^2}{4 A^2 \cos^2 \theta_o} - \frac{3 \lambda^2 (C - A)}{A} \right]^{1/2}. \quad (192) \]
In order for the satellite to have real precession, the radical in equation (192) must be real. Therefore,

\[ C^2 s^2 > 12 \lambda^2 A (C - A) \cos \theta_0 \]  
(193)

or the spin requirement for \( s \) is

\[ s^2 > \frac{12 \lambda^2 A (C - A) \cos \theta_0}{C^2} \]  
(194)

Observe that this spin requirement is always met if \( A > C \) and \( \cos \theta_0 > 0 \).

**Limiting Case of Precession, \( \theta_0 = \frac{\pi}{2} \)**

The limiting case of \( \theta_0 = \frac{\pi}{2} \) must be treated separately since the precession rate as given by equation (193) is undefined. Using equation (152) to eliminate \( \psi \) and substituting \( \theta = \frac{\pi}{2} \), the first two attitude equations (150) and (151) become

\[ A \Omega \sin \phi + 2A \Omega \phi \cos \phi + C s \phi = 0 \]  
(195)

and

\[ A \phi - C s \Omega \sin \phi - A \Omega^2 \sin \phi \cos \phi = 0 \]  
(196)

Neglecting terms in \( \Omega \phi \) and \( \Omega^2 \) in keeping with previous assumptions,

\[ A \Omega \sin \phi + C s \phi = 0 \]  
(197)

and

\[ A \phi - C s \sin \phi = 0 \]  
(198)
Combining equations (198) and (197) we get

\[ \frac{A^2 \phi}{C_s} + C_s \phi = 0 \quad \text{or} \quad \phi + \frac{2 \lambda}{A^2} \phi = 0 . \tag{199} \]

Integrating,

\[ \ln \phi = -\frac{C_s^2}{A^2} t + k \quad \text{or} \quad \phi = K e^{-\frac{C_s^2}{A^2} t} . \tag{200} \]

If at \( t = 0, \phi = 0 \), then \( K = 0 \) and

\[ \phi = 0 . \tag{201} \]

This indicates that for the limiting position of \( \theta = \frac{\pi}{2} \), there is no precession. Also observe that if a small impulse is given to the system at \( t = 0 \) so that \( \phi = \phi_0 \), then

\[ \phi = \phi_0 e^{-\frac{C_s^2}{A^2} t} . \tag{202} \]

This indicates that given a small disturbance at its initial position, the satellite will seek a new position \( \phi_0 \) and then remain there. Thus in the limiting case of \( \theta_0 = \frac{\pi}{2} \), there is no preferred orientation \( \phi \) which the body will take. In this sense, the body may be considered unstable since it neither returns to its initial position or oscillates about it.
Small Oscillations about the Steady Motion

Let the steady-motion values of \( \theta \) and \( \phi \) be \( \theta^o \) and \( \phi^o \), and designating the deviations about the steady values by \( \theta_\delta \) and \( \phi_\delta \), the instantaneous values of \( \theta \) and \( \phi \) are

\[
\theta = \theta^o + \theta_\delta, \quad \phi = \phi^o + \phi_\delta.
\] (203)

For small oscillations, we can make the following approximations:

\[
\theta \phi = (\theta^o + \theta_\delta)(\phi^o + \phi_\delta) \approx \theta_\delta \phi^o;
\]

\[
\sin \theta = \sin (\theta^o + \theta_\delta) = \sin \theta^o \cos \theta_\delta + \cos \theta^o \sin \theta_\delta;
\]

\[
\sin \theta \approx \sin \theta^o + \theta_\delta \cos \theta^o;
\] (204)

\[
\cos \theta = \cos (\theta^o + \theta_\delta) = \cos \theta^o \cos \theta_\delta - \sin \theta^o \sin \theta_\delta;
\]

\[
\cos \theta \approx \cos \theta^o - \theta_\delta \sin \theta^o.
\]

Assuming that the terms in \( \Omega^o, \Omega^o \phi, s \Omega, \) and \( \Omega^2 \) may be neglected in comparison to the other terms in the attitude equations, equation (150) becomes

\[
\begin{align*}
A \theta^o + C s \phi \sin \theta - A \phi^o \sin \theta \cos \theta &= 3 \lambda^2 (C - A) \sin \theta \cos \theta;
\end{align*}
\] (205)

and equation (151) becomes

\[
A \phi \sin \theta + 2 A \theta \phi \cos \theta - C s \theta = 0.
\] (206)
For small oscillations about the steady motion, equation (207) becomes

\[
A \phi_5 \sin \theta + A \phi_5 \theta_5 \cos \theta + 2A \theta_5 \phi_5 \cos \theta - 2A \theta_5 \phi_5 \sin \theta - C_s \theta_5 = 0.
\]

Linearizing,

\[
A \phi_5 \sin \theta + 2A \theta_5 \phi_5 \cos \theta - C_s \theta_5 = 0
\]

or

\[
A \sin \theta \phi_5 = (C_s - 2A \phi_5 \cos \theta) \theta_5.
\]  

Integrating,

\[
A \sin \theta \int_0^{\phi_5} d\phi_5 = (C_s - 2A \phi_5 \cos \theta) \int_0^{\theta_5} d\theta_5,
\]

we get

\[
A \sin \theta \phi_5 = (C_s - 2A \phi_5 \cos \theta) \theta_5
\]

or

\[
\phi_5 = \frac{(C_s - 2A \phi_5 \cos \theta)}{A \sin \theta} \theta_5.
\]  

(208)
For small oscillations about the steady motion, equation (206) is

\[ A \theta_5 + C \sin (\phi_0 + \phi_5) (\sin \theta_0 + \theta_5 \cos \theta_0) - A (\phi_0 + \phi_5) (\sin \theta_0 + \theta_5 \cos \theta_0) (\cos \theta_0 - \theta_5 \sin \theta_0) = 3 \lambda^2 (C - A) (\sin \theta_0 + \theta_5 \cos \theta_0) (\cos \theta_0 - \theta_5 \sin \theta_0). \] (209)

Expanding, the steady motion terms are

\[ C \sin \theta_0 - A \phi_0 \sin \theta_0 = 3 \lambda^2 (C - A) \sin \theta_0 \cos \theta_0. \] (210)

Eliminating the steady motion terms (211) and neglecting products of the small deviational quantities, we get for equation (210)

\[ A \theta_5 + C \sin (\phi_0 + \phi_5) \cos \theta_0 + C \sin (\phi_0 + \phi_5) - A \phi_0 \theta_5 \cos^2 \theta_0 + A \phi_0 \theta_5 \sin \theta_0 \cos \theta_0 = 3 \lambda^2 (C - A) (\sin \theta_0 + \theta_5 \cos^2 \theta_0). \] (211)

Using equation (209) to eliminate \( \phi_5 \) and rearranging, we get

\[ \theta_5 + \left( \frac{[A^2 \phi_0^2 + 3 \lambda^2 A (C - A)](1 - \cos^2 \theta_0) + C^2 \sin^2 \theta_0 - 12 \lambda^2 A (C - A) \cos \theta_0}{A^2} \right) \theta_5 = 0. \] (212)
Thus, the nodding oscillations are sinusoidal with period

\[ T = \frac{2 \pi A}{\left( \left[ A^2 \phi_0^2 + 3 \lambda^2 A(C - A) \right] (1 - \cos^2 \theta_0) + C^2 s^2 - 12 \lambda^2 A(C - A) \cos \theta_0 \right)^{1/2}}. \]  

\( (213) \)

In order for the oscillations to be stable,

\[ \left[ A^2 \phi_0^2 + 3 \lambda^2 A(C - A) \right] (1 - \cos^2 \theta_0) + C^2 s^2 > 12 \lambda^2 A(C - A) \cos \theta_0 \]

or

\[ C^2 s^2 > 12 \lambda^2 A(C - A) \cos \theta_0 - \left[ A^2 \phi_0^2 + 3 \lambda^2 A(C - A) \right] (1 - \cos^2 \theta_0). \]  

\( (214) \)

Observe that when \( \theta_0 = 0 \), which is analogous to a sleeping top, the criterion for stability must be

\[ C^2 s^2 \geq 12 \lambda^2 A(C - A) \quad \text{or} \quad s^2 \geq \frac{12 \lambda^2 A(C - A)}{C^2}. \]  

\( (215) \)

This corresponds to the spin requirement \((195)\) with \( \theta_0 = 0 \) which was obtained in a study of the steady precession of the satellite.

Notice that if \( A > C \), the spin requirement will always be satisfied. This corresponds to a long cylindrical type satellite. Observe, however, that since \( \lambda^2 \) is a very small number, the spin requirement will almost always be met. Although most proposed manned space platforms are in the shape of flat disks \((C \gg A)\), the
spin requirement will be easily met if the platform is given any angular velocity to induce an artificial gravity field within the space platform.
Stability Boundary for $\theta_0 = 0$

The spin requirement for stability about the local vertical is

$$s^2 > 12 \lambda^2 \frac{(C-A)A}{C^2}.$$

The nondimensional "stability boundary" is

$$\frac{s^2}{\lambda^2} = 12 \frac{A}{C} (1 - \frac{A}{C})$$

or

$$w^2 = \frac{12}{\delta} (1 - \frac{1}{\delta}) = \frac{12}{\delta^2} (\delta - 1)$$

where

$$w = \frac{s}{\lambda} \quad \text{and} \quad \delta = \frac{C}{A}.$$

Figure 11. Stability Boundary for a Symmetric, Spinning Satellite at $\theta_0 = 0$. 
Results of the Study of a Symmetrical Spinning Satellite

It was shown that a spinning, symmetrical satellite in orbit under the influence of the gravity gradient torque exhibits all the characteristic motions of the classical top. The possible motions of the spin axes were shown in Figure 10. Like a top, it was found that the satellite could precess steadily and that there were two precession rates given by equation (192)

\[
\phi = \frac{C_s}{2A \cos \theta_o} \pm \left[ \frac{C_s^2}{4A^2 \cos^2 \theta_o} - \frac{3\lambda^2 (C - A)}{A} \right]^{1/2}.
\]

(192)

This equation also yielded the spin requirement

\[
s^2 > \frac{12\lambda^2 A (C - A) \cos \theta_o}{C^2},
\]

(194)

which is definitely a function of the gravitational torque since the quantity \( \lambda^2 \) results from it. Unlike the top, a study of the limiting case of steady precession at \( \theta_o = \frac{\pi}{2} \) revealed that for this orientation the spinning satellite had no precession rate and that it was unstable in the sense that for a small disturbance the corresponding small displacement neither oscillated about the
initial position nor did it return to the initial position. When disturbed, the satellite merely sought another position.

A study of small motions about the steady motion showed that the small motions were oscillatory and stable if the requirement

\[ C^2 s^2 > 12 \lambda^2 A(A - C) \cos \theta - [A^2 \phi^2_o + 3 \lambda^2 A(C - A)] (1 \cos^2 \theta) \]

was met.

As far as it has been possible to determine, the particular analytical approach and these results for the symmetrical, spinning satellite have not been previously published. However, it should be mentioned that the results are not totally unexpected. Many authors, such as R. E. Roberson (136), W. T. Thomson (165) and R. A. Nidey (110), have indicated in numerous papers that the gravity gradient torque would cause the angular momentum vector of a satellite to precess. This has been confirmed by actual observations of the angular motions of the satellites in the Explorer and Tiros series and of Echo I. These observations have been published in papers by R. J. Naumann (107), G. E. K. Lockwood (95), and W. R. Bandeen
and W. P. Manger (12). It is quite possible that the analysis presented here and the assumptions made in it are highly artificial and in the opinion of other investigators did not merit consideration.
CONCLUSIONS

If a satellite or space vehicle is spun in order to induce an artificial gravity field or to stabilize its motion and if it is not otherwise controlled, it has been shown that it is possible, and in general probable, that the satellite will precess and nutate under the influence of the gravitational torque. Under certain conditions, small motions about this steady motion will be stable. These conclusions are drawn from the results of the investigation of three cases of the simplified attitude equations.

The results of Case I show that an unsymmetrical, spinning satellite will precess steadily with the rates

\[
\dot{\phi}_o = \frac{[A - B + C]}{2[B - C] \cos \theta_o} + \frac{[A - B + C]^2 \psi_o^2}{4[B - C]^2 \cos^2 \theta_o} - \frac{1}{3 \lambda^2} \left\{ \frac{[A - B + C]^4}{4[B - C]^4 \cos^2 \theta_o} \right\} \tag{124}
\]

if the spin \( \psi_o^2 \) is such that

\[
\psi_o^2 > \frac{12 \lambda^2 [B - C]^2 \cos \theta_o}{[A - B + C]^2} \tag{125}
\]

and if the unequal inertias satisfy the relation

\[ A + B = C . \tag{126} \]
A study of the small motions about the steady precession shows that the small motions are oscillatory with period

\[
T_o = \frac{1}{2 \pi A B^2} \frac{2 \pi A B^2}{(A - (A - B)^2 + C^2)^2} \psi_0^2 \\
+ B[(B - C) (A + 2(A - C) \cos^2 \theta_0)] \phi_0^2 \\
+ A[A(B - A) + (2A - C)(B - C) - 2(B^2 - C^2)] \phi_0^2 \psi_0 \cos \theta_0 \\
+ A^2 B[(B - C)(2 \cos^2 \theta_0 - 1)3 \lambda^2] \phi_0^2 \psi_0 \cos \theta_0 \\
\]

(119)

and are stable if the radical in the denominator is real. Observe that the precession rate is dependent on the inertias, the dip angle \( \theta_0 \), the gravitational torque \( \lambda^2 \), and the spin rate. Also observe that the spin requirement shows a dependence on the inertias, \( \theta_0 \) and \( \lambda^2 \) and that the stability of small motions depends on all the inertial and dynamical quantities.

The results of Case II, the study of small disturbances in the general motion of an unsymmetrical, spinning satellite, show that the equations for the small variations form a system of second-order, linear differential equations with nonconstant coefficients. The general motion referred to needs only to satisfy the system of nonlinear differential equations (132) - (134). Although one root of the characteristic equation, which arises from the study of the "infinitesimal stability," is zero, it is considered as
stable. This zero results from the fact that in simplifying the attitude equations the coordinate $\phi$ becomes cyclic. Therefore, no changes in the system parameters or round-off errors could make this zero slightly positive. No other specific conclusions can be drawn from the study of the "infinitesimal stability" of the general motion without an actual problem with numerical values.

The results of the study of a symmetrical, spinning satellite under the influence of the gravitational torque show that it exhibits all the characteristic motions of the classical top. These possible motions are shown in Figure 10. Like a top, the satellite can precess steadily and there are two precession rates given by the equation

$$\phi = \frac{+Cs}{2A \cos \theta_o} \pm \left[ \frac{C^2 s^2}{4A^2 \cos^2 \theta_o} - \frac{3\lambda^2 (C - A)}{A} \right]^2.$$  \hspace{1cm} (192)

This equation yields the spin requirement for stable motion as

$$s^2 \geq \frac{12 \lambda^2 A (C - A) \cos \theta_o}{C^2}.$$  \hspace{1cm} (193)

Observe that for long, cylindrical satellite ($A \gg C$), this requirement will always be met.
Unlike the top, a study of the limiting orientation \( \theta_o = \frac{\pi}{2} \) reveals that, for this position, the spinning satellite does not precess. Motion at this orientation is unstable in the sense that a small disturbance does not produce a displacement that oscillates about or returns to the initial position. In fact, given a small disturbance the satellite seeks another position \( \phi \) and remains there.

A study of the small motions about the steady precession shows that the small motions are sinusoidal with the period of oscillation being

\[
T_o = \frac{2 \pi A}{\left( [A^2 \phi^2 + 3 \lambda^2 A(C - A)](1 - \cos^2 \theta_o) + C^2 s^2 - 12 \lambda^2 A(C - A) \cos \theta_o \right)^{1/2}}.
\]  

(213)

In order for the small motions to be stable, it is necessary that

\[
C^2 s^2 > 12 \lambda^2 A(C - A) \cos \theta_o - [A^2 \phi_o^2 + 3 \lambda^2 A(C - A)](1 - \cos^2 \theta_o).
\]

(215)

Observe that for the case \( \theta_o = \frac{\pi}{2} \), the spin requirement (193) is again verified.

A review of the more significant analytical studies is given in a discussion following the presentation of the general equations of motion and two of the integrals of these equations of motion.

This review briefly points out the important aspects of the
investigations by R. Wolfe and B. Arrow (184), D. M. Schrello (146), J. H. Suddath (159), L. H. Grasshoff (66), M. E. Kuebler (92) and D. B. DeBr and R. H. Delp (43). Brief mention is also given to the current trend in industry toward mechanical simulation of a satellite and of the torques that it is likely to encounter in space. The only published paper available on this topic by M. D. Olstad, R. Grunberg, W. Blesser and L. Braun, Jr. (116) is cited.

Although it has been shown that, for the special cases studied, a spinning satellite can have steady motions of precession and nutation and that the small motions about the steady motions can be stabilized, it is very probable that even the steady motions themselves cannot be tolerated for the successful performance of the satellite or space vehicle. Therefore, since the gravitational torque acts continuously and its effect is cumulative, the attitude control engineer will have to design a continuous or at least a periodic control system to correct for this effect. It would be an absolute necessity for satellites which had to be maintained as stable platforms in space for navigation purposes. In the case of manned space vehicles or stations, any moderately rapid precession or nutation would be intolerable. A literature survey shows that there is a considerable wealth of
material on the actual design and analysis of various mechanisms for the control of the attitude of a satellite. Among some of the devices and systems proposed for attitude control are: fly wheels or reaction wheels; jet reaction systems; gyroscopic stabilizers; and "natural" control sources such as the earth's magnetic field and solar radiation pressure. No attempt will be made here to assess or discuss these different systems of control. The Bibliography lists a large number of papers which deal with this phase of attitude motion and control and which the reader may turn to for reference.

There are still many areas of future research in the field of attitude motion and control. Investigation on the effect of the earth's gravitational torque on the motion of satellites is by no means complete. The general, nonlinear attitude equations have yet to be solved. The even more general problem of solving the coupled equations of motion of the mass center and the attitude equations should be more completely investigated (184). Although much has been done in representing the earth as an oblate spheroid and how this affects the orbital parameters, this has yet to be extended to how it affects the attitude motion.
Two additional topics in attitude motion and control for which much research is needed are:

1) the attitude motion of a space vehicle as it is affected by internal moving parts,

2) the attitude motion of a satellite as it is affected by the dissipation of energy due to its own elasticity.

The first problem is of growing significance as the time for manned space vehicles and stations nears. R. E. Roberson in his paper on "The Torques on a Satellite Vehicle from Internal Moving Parts," (129) has laid the foundation for study in this area. The second problem has come to the attention of researchers only very recently and, to date, very little has been written on this problem (102, 166). There is much research that needs to be done in attitude motion and control that can be foreseen or anticipated and probably even much more that is as yet unforeseen.
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I, Paul B. Chin, was born in Canton, Ohio, November 6, 1933. I received my secondary education in Canton, and my undergraduate training at Purdue University, which granted me the degree Bachelor of Science in Engineering Mechanics in 1955. After a year in industry as an engineer in the Dynamics Science Group of North American Aviation, Inc., I became an Instructor in the Department of Engineering Mechanics at The Ohio State University, from whom I was granted the degree Master of Science in 1957. I continued to teach while completing the requirements for the degree Doctor of Philosophy.