CARLSON, Robert Lee, 1924—
A STUDY OF COLUMN CREEP.

The Ohio State University, Ph.D., 1962
Engineering Mechanics

University Microfilms, Inc., Ann Arbor, Michigan
A STUDY OF COLUMN CREEP

DISSERTATION

Presented in Partial Fulfillment of the Requirements
for the Degree Doctor of Philosophy in the
Graduate School of The Ohio State University

by

Robert Lee Carlson, B. S., M. S.

The Ohio State University
1962

Approved by

[Signature]

Adviser
Department of Engineering Mechanics
PREFACE AND ACKNOWLEDGMENTS

The author would like to acknowledge the guidance and encouragement of Professor S. B. Folk, his adviser. Part of the analytical and experimental results presented here were obtained under a program sponsored by the Aeronautical Research Laboratory of the United States Air Force, Contract No. AF 33(616)-6301, at Battelle Memorial Institute.
## CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>PREFACE AND ACKNOWLEDGMENTS</td>
<td>ii</td>
</tr>
<tr>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>FAILURE CRITERIA ANALYSIS</td>
<td>15</td>
</tr>
<tr>
<td>GENERAL DISCUSSION</td>
<td>16</td>
</tr>
<tr>
<td>Inspection Procedures</td>
<td>16</td>
</tr>
<tr>
<td>A New Inspection Procedure</td>
<td>20</td>
</tr>
<tr>
<td>APPLICATION OF THEORY TO A COLUMN MODEL</td>
<td>25</td>
</tr>
<tr>
<td>Analysis</td>
<td>25</td>
</tr>
<tr>
<td>Interpretation of Analysis</td>
<td>35</td>
</tr>
<tr>
<td>Relation of Results to General Column</td>
<td>40</td>
</tr>
<tr>
<td>Experimental Results</td>
<td>54</td>
</tr>
<tr>
<td>Calculated and Experimental Inspection Results</td>
<td>79</td>
</tr>
<tr>
<td>COLUMN-CREEP ANALYSIS</td>
<td>84</td>
</tr>
<tr>
<td>DEVELOPMENT OF A STRAIN-RATE RELATION</td>
<td>85</td>
</tr>
<tr>
<td>Creep-Recovery Effects</td>
<td>85</td>
</tr>
<tr>
<td>A Strain-Rate Relation</td>
<td>96</td>
</tr>
<tr>
<td>APPLICATION OF THE STRAIN-RATE RELATION TO THE COLUMN MODEL</td>
<td>107</td>
</tr>
<tr>
<td>Analysis</td>
<td>107</td>
</tr>
<tr>
<td>Relation of Results to General Column</td>
<td>110</td>
</tr>
<tr>
<td>Column-Creep Experiments</td>
<td>122</td>
</tr>
<tr>
<td>Calculated and Experimental Results</td>
<td>127</td>
</tr>
<tr>
<td>CONTENTS (contd)</td>
<td>Page</td>
</tr>
<tr>
<td>----------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>SUMMARY AND CONCLUSIONS</td>
<td>137</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>144</td>
</tr>
<tr>
<td>AUTOBIOGRAPHY</td>
<td>148</td>
</tr>
<tr>
<td></td>
<td>TABLES</td>
</tr>
<tr>
<td>---</td>
<td>----------------------------------------------------------------------</td>
</tr>
<tr>
<td>1</td>
<td>Inspection-Experiment Data</td>
</tr>
<tr>
<td>2</td>
<td>Summary of Inspection-Experiment Results</td>
</tr>
<tr>
<td>3</td>
<td>Summary of Column-Creep Data</td>
</tr>
<tr>
<td>4</td>
<td>Comparison of Theoretical Capacities</td>
</tr>
</tbody>
</table>
## FIGURES

<table>
<thead>
<tr>
<th>Figure Number</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Column Average Stress versus Deflection</td>
<td>22</td>
</tr>
<tr>
<td>2</td>
<td>Idealized H-Section Column</td>
<td>27</td>
</tr>
<tr>
<td>3</td>
<td>Load and Configuration Details during Loading Inspection</td>
<td>46</td>
</tr>
<tr>
<td>4</td>
<td>Stress Distributions Before and After Inspections</td>
<td>46</td>
</tr>
<tr>
<td>5</td>
<td>Compression Stress-Strain Curves for Room Temperature and 325° F</td>
<td>57</td>
</tr>
<tr>
<td>6</td>
<td>Stress versus Strain after Different Amounts of Creep at 10,000 psi</td>
<td>61</td>
</tr>
<tr>
<td>7</td>
<td>Tangent Modulus versus Stress for Different Amounts of Pre-Creep</td>
<td>63</td>
</tr>
<tr>
<td>8</td>
<td>Column-Creep-Buckling Test Stand with Furnace</td>
<td>65</td>
</tr>
<tr>
<td>9</td>
<td>Close-Up View of End Cap and Yoke</td>
<td>66</td>
</tr>
<tr>
<td>10</td>
<td>History of Interrupted Column-Creep-Buckling Experiment</td>
<td>69</td>
</tr>
<tr>
<td>11</td>
<td>Compressive Creep and Recovery Behavior of Aluminum Alloy 7075-0 at 325° F</td>
<td>91</td>
</tr>
<tr>
<td>12</td>
<td>Graphical Representation of Creep Behavior</td>
<td>98</td>
</tr>
<tr>
<td>13</td>
<td>Compressive Creep Strain versus Time for Aluminum Alloy 7075-0 at 325° F</td>
<td>105</td>
</tr>
<tr>
<td>14</td>
<td>Relationship between Creep Strain Rate and Stress</td>
<td>106</td>
</tr>
<tr>
<td>15</td>
<td>Relationship between Stress and Pseudo-Inelastic Strain</td>
<td>106</td>
</tr>
<tr>
<td>FIGURE</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>16</td>
<td>Change in Stress During Time Interval $\Delta t$</td>
<td>114</td>
</tr>
<tr>
<td>17</td>
<td>Average Stress versus Failure Time for Slenderness Ratio of 75</td>
<td>125</td>
</tr>
<tr>
<td>18</td>
<td>Average Stress versus Failure Time for Slenderness Ratio of 60</td>
<td>126</td>
</tr>
</tbody>
</table>
INTRODUCTION

The creep of engineering materials has been recognized for many years as a factor which must be considered in the design of structures exposed to elevated temperatures. Numerous design rules and procedures have been advanced to account for creep effects, and many machines and structures have been designed to operate at elevated temperatures. Designing to allow for creep is not, therefore, a new problem.

In recent years, however, there has been a concerted effort to change the character of the design processes. Past design practices primarily emphasized the elimination of creep as a problem; i.e., materials and cross sections were selected which eliminated the dangers of the creep problem. The stress intensities were obtained from an elastic stress analysis since the stress values so determined represented the initial state. Making use of a combined stress theory such as the distortion-energy criterion, the point of maximum distortion energy could be located, and a corresponding or equivalent value of uniform, uniaxial stress could be deduced. A review of available tensile creep data would then establish what materials and what cross-
sections would be necessary to eliminate creep as a problem. It will, of course, be recognized that this type of design approach essentially parallels that which is frequently used to avoid the possibility of plastic flow in structures and machines.

The recent desire for a change in the above procedure can be traced to a change in the type of problems that are arising. Formerly, problems involved primarily stationary units, for which conservative design was both allowable and desirable. With the advent of high-speed aircraft and missiles, it is apparent that a different design philosophy is necessary. The added weight accompanying conservative design results in decreases in efficiencies which are difficult to accept. In addition to differences in weight requirements, there are also drastic differences in the lifetimes involved. In stationary power-plant units, for example, lifetimes are projected in terms of years and it is frequently allowable to base an analysis on secondary creep or steady-state creep behavior. In missile applications, lifetimes may be of the order of minutes and hence, the primary state of creep becomes a factor of importance. In secondary-stage creep, the creep rate is constant, whereas for the primary stage, the creep rate is continuously decreasing. The primary stage creep is, therefore, the more difficult to incorporate in an analysis.
In contemplating a mathematical theory for stress analyses including the effect of creep or time-dependent flow, it is helpful to review briefly the structure of the mathematical theory of elasticity. In elasticity it is possible to formulate a system of equations which provide a basis for analysis. These are --

1. Equations of equilibrium.
2. Compatibility relations.
3. Stress-strain relations.
4. Boundary conditions.

If creep effects are introduced, the system used for elasticity theory must be modified. Upon examination of the above list of equations it can be readily recognized that the differences between the theory of elasticity and a theory incorporating creep effects arise in the second and in the third items; i.e., the compatibility, and the stress-strain relations.

In elasticity the strains considered are small and a normal strain can be defined as the partial derivative of the displacement in the given direction with respect to the coordinate for the direction. In creep, the strains can no longer be considered small; thus incremental strains must be considered. A normal strain rate, then, is defined as the partial derivative of the displacement rate with respect to the given coordinate.
The difference between elasticity and creep in terms of the stress-strain relations is more involved. Hooke's Law no longer provides a satisfactory description of the material response. In addition to an elastic component of strain, it is necessary to include a creep-strain component. Actually, to be general, it would be desirable to include also a plastic strain component -- a time-dependent inelastic strain -- to describe loading above the elastic limit. Using summation notation, the total strain $\varepsilon_{ij}$, where $i, j = 1, 2, 3$, may be written as

$$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p + \varepsilon_{ij}^c,$$

where superscripts $e$, $p$, and $c$ designate respectively, the elastic, plastic, and creep-strain components. Since in formulating relationships for the various components, it is found convenient to work in terms of strain rates, the above representation appears as

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^e + \dot{\varepsilon}_{ij}^p + \dot{\varepsilon}_{ij}^c,$$

where the dot refers to differentiation with respect to time.

Expressions for the term $\dot{\varepsilon}_{ij}^e$ can be readily obtained by differentiation of Hooke's Law. The possible forms of a relation for the term $\dot{\varepsilon}_{ij}^p$ form a sizable part of the mathematical theory of plasticity. To formulate a relation, it is necessary to adopt a flow...
condition which establishes in functional form the conditions for the onset of flow in terms of a general stress state. This in turn is introduced in a flow rule which relates the rates of flow to the stresses. One might, for example, use the von Mises flow condition\(^1\) with the von Mises flow rule (1).\(^2\) This procedure is usually described as the application of the von Mises flow condition and its associated flow rule. Alternately, it is possible to use the Tresca flow condition (maximum shear-stress theory) and its associated flow rule (2). The use of the different flow rules usually results in somewhat different solutions and frequently a choice between these two possibilities is based on an examination of the relative difficulties associated with the alternate solutions. It is of interest to note also that a number of solutions using unnatural combinations -- the Tresca flow condition with the von Mises flow rule -- have appeared in the literature. This type of mixing does not satisfy certain energy requirements so it appears to be less satisfactory from a consistency viewpoint.

---

\(^1\) The form of this condition is equivalent to that obtained for the distortion-energy theory.

\(^2\) Numbers with parentheses refer to references listed at the end of the text.
A number of relations have been proposed for the term $\dot{\varepsilon}_{ij}^c$, which represents the creep-rate term. Two more or less independent approaches have been utilized in these proposals. In one, the theory of linear viscoelasticity has been developed, and in the second the procedures utilized for developing flow rules for plasticity theory have been adopted and used to develop creep-rate relations. In the former procedure, the material response is characterized by the use of spring and dash-pot elements in various combinations (parallel and series). These laws provide linear relationships in which a summation of derivatives of the deviations of stresses with respect to time are equal to the sum of terms of the derivatives of the deviator of strain with respect to time (3). In the solution of problems, the number of elements in the model (and the number of viscosity constants) and hence the number of terms in the relation used is frequently kept small for the sake of simplicity.

A very thorough review of the creep-rate based on plasticity-theory procedures is presented in a review by Johnson (4). One of the characteristics of the relationships in this category is that they are usually nonlinear. This, of course, results in considerable complications in applications to stress analyses involving creep effects.

The complexities of the forms of the relations for the plastic strain rate and the creep strain rate have to date precluded the
possibility of using a strain-rate function which includes all three strain-rate components (elastic, plastic, and creep) in bodies involving variable triaxial or even biaxial stress states. In plasticity alone, the number of solutions including elastic and plastic components is far outnumbered by the solutions including only the plastic component. Similarly, in creep-stress analysis the number of solutions involving combined stresses is greater for those including the creep-rate component than those including the elastic component as well.

Because of the simplifications or idealizations that are necessary in order to obtain solutions, it can be expected that applications to structural problems involving materials with responses which are not completely represented by the strain-rate law will produce results which may vary considerably in accuracy. It is necessary, therefore, to adopt a philosophy of approach which is very clearly illustrated in a paper by Patel, Venkatraman, and Hodge (5). They summarize their viewpoint in the following statement:

As pointed out in the Introduction, the study of the creep phenomenon is still in its infancy. The expressions adopted for creep strains, even from carefully controlled experiments, are only approximate. An exact and pains-taking analytical solution of the creep problem, based on these expressions, may differ considerably from experimental results. Consequently, in the solution of
creep problems it would not be reasonable to expect an accuracy comparable to that in linear elasticity. However, creep has to be taken into account in many modern engineering problems, even if only to within an order of magnitude.

The preceding discussion and the comments of Patel, et. al., tend to suggest, perhaps, that in many problems involving creep, the analyst should be content with solutions in which only a creep-rate term is used. Although this may often be necessary from a practical standpoint, it can be unsatisfactory in some instances. It is possible, for example, to cite problems in which this procedure will, in essence, eliminate important features of a problem. A few of these problems will be described to illustrate this point.

As one example, consider a body loaded to produce internal stresses which vary from point to point, and let the problem be the determination of the residual-stress distribution remaining in the body if the external loading is removed after a given period. A beam under a bending moment serves as a simple example. In a problem of this type, the required solution cannot be made unless both creep strain and elastic-strain components are included in the material response law. Without elasticity, the body merely changes

\[3\text{ The solution of Patel et al (Reference 5) adopted this method of solution.}\]
shape and no elastic adjustments occur upon unloading. Since most material bodies possess some elasticity, solutions which neglect elastic strains eliminate this characteristic behavior.

As a second example, consider the problem of relaxation. Although this can occur in complex forms, it is convenient to cite the case of the bolt or rivet which is tightened, and because of creep which occurs subsequently, relaxes or loses tension. If the initial separation remains fixed, the relaxation process consists of an exchange of elastic strain (a reduction) for creep strain (an increase) such that the sum remains constant. The presence of the elastic component is, therefore, an essential part of the problem. In the bolt problem, the formulation of the response law is not difficult. In more complex problems involving relaxation, the formulation is more difficult, however. Neglect of the elastic strains in these problems will result in solutions which lose a characteristic of the total behavior.

As a final example, results obtained in the present investigations on the problem of creep buckling in columns will be cited. By using a strain-rate law which includes elastic, plastic, and creep strain rate components in the solution of the problem, it is shown
that three modes of column behavior can be observed for different combinations of components as follows:

1. Elastic, plastic, and creep components present
2. Elastic and creep components present (plastic component dropped)
3. Creep component present (elastic and plastic components dropped)

The details of the behavior observed are presented in the body of the dissertation. This example and the previous ones, however, serve to indicate that although it may be necessary to accept approximate solutions to many creep problems, care must be exercised in evaluating the results to determine whether important features of the problem have been lost.

The preceding discussion has served as a background for the discussion of the problem investigated in this dissertation -- the behavior of columns subject to creep. The phenomenon of creep buckling can manifest itself in many structural elements. Any element, in fact, which is subject to elastic instability, can be expected to become unstable at elevated temperatures due to creep under loads less than those required for short-time or elastic theory buckling. Although some work has been conducted on elementary forms of creep buckling in cylindrical shells and plates, the
greater part of past research effort has been on creep buckling as it is manifested in columns. Since this is the topic of interest here, the review of past work will be confined to results obtained for columns.

Both analytical and experimental studies have been conducted on column-creep buckling. The aim of most of the analytical studies has been to obtain a solution that can provide deflection histories, and also critical or collapse times. From such solutions, it should be possible to study the basic features of column action during creep buckling. Although such studies are possible, there are questions about the general applicability of the results. This stems from the fact that it has been necessary to use either deformation laws that apply for idealized material (viscoelastic material, for example (6), or empirical deformation laws (7, 8, 9)). The extent to which conclusions regarding the details of column action apply to real columns is, therefore, not known.

Most of the experimental creep-buckling studies (10, 11) have been conducted on columns made from commercially available structural alloys. The results, therefore, are representative of the behavior that can be expected from real materials. In most previous studies, the primary objective has been either to obtain qualitative support for a particular method of solution or to obtain extensive data for a range of values of such variables as slenderness
ratio, temperature, and imperfection. Little attention has been devoted to a study of the details of column action.

Results from previous experimental studies have been useful in a number of ways. They have provided an introduction to the phenomenon of creep buckling. They have indicated the relative effects of some of the more important variables. They have made it possible to evaluate and compare the relative resistance to creep buckling of a number of structurally useful alloys.

Many of the analyses used to date are of such a nature as to make the subsequent solutions approximate in spite of the fact that the mathematics may be exact. The various models proposed have incorporated at least one and usually two of the following simplifications and approximations:

1. The column cross section is assumed to be an I-beam with a web of zero thickness.
2. The deformation law is of restricted applicability.
3. The column is assumed to possess characteristics that make it possible to analyze the creep-buckling action by the use of equations similar to those that have been evolved for inelastic, time-independent buckling.

The simplification under Point 1 is in itself acceptable, and should be capable of representing the mechanics of column action.
Shanley (12), in fact, used a form of this model to clarify the details of plastic buckling action. Point 1, however, has been used in conjunction with either Point 2 or Point 3. A final evaluation, then, depends on the validity of Points 2 and 3.

A number of different deformation laws have been used in solutions to the column-creep-buckling problem. The use of the various laws, in fact, is the primary reason for essentially different solutions. In terms of being capable of describing, in a relatively simple manner, the essential behavior of metals, it appears that, at least for the present, the deformation law proposed by Odqvist (13) and used by Hoff (14) possesses the most potential for providing a practical and realistic solution to the column-creep-buckling problem.

The assumptions used under Point 3 appear to make it possible to bypass the difficulties associated with selecting and using a particular form of creep law. Methods utilizing variations of this type of approach have been presented by Shanley (15) and Gerard (16). Although these methods will be discussed in greater detail later, it can be noted here that they each possess the advantage of simplicity. They are, therefore, attractive and have received attention in a number of applications. Since they involve
basic assumptions regarding the details of column action during creep buckling, their limitations should be clearly understood before they are utilized.

The preceding discussion briefly summarizes the nature of the research that has been conducted, and it implicitly indicates several areas in which additional research could contribute to an improved understanding of the phenomenon of creep buckling.

One of the areas of study indicated involves the mechanics of column action during creep buckling. A more detailed knowledge of column action should provide a means of evaluating the accuracy of the various simplifying assumptions that have been suggested in previous work. It should also provide a basis for formulating and checking new simplifications and approximations and may therefore be of particular value in still more complex structural elements involving plates and shells.

The objective of the program described by this dissertation has been to provide a description of the details of column action during creep buckling. In the presentation to follow, the results have been divided into two main sections. In the first section, FAILURE CRITERIA ANALYSIS, the conditions necessary for buckling and methods for inspecting for stability are covered. In the second section, COLUMN-CREEP ANALYSIS, the deflection history of a column creeping under a constant load is treated.
FAILURE CRITERIA ANALYSIS

In terms of the perspective to be developed in this section, buckling -- due to creep -- is an event that occurs at the critical time or what might be described as the end of the column lifetime. The properties of the column prior to buckling are also of interest in this development, however, so more specifically, we are concerned not just with creep buckling, but with the mechanics of a column whose deflections are increasing with time due to creep.

One method of deducing the properties of a creeping column is to perform an inspection at the time of interest. The present work begins, therefore, with a consideration of procedures for inspecting a creeping column for stability. The studies of Rabotnov and Shesterikov (17) and those of Fraeijs de Veubeke (18) and Hoff (14) are interpreted as examples of two essentially different possible procedures. An inspection procedure similar to, but more general than, that of Fraeijs de Veubeke is proposed, and an analysis of the mechanics involved is performed. The equations developed are then used to illustrate the essential differences in column action between purely viscous materials, viscoelastic materials, and materials which possess both time-dependent and
nonlinear time-independent components of deformation. Finally, the results of column-creep experiments, in which inspections were performed, are described, and a discussion of the correlation with the analysis is presented.

**General Discussion**

**Inspection Procedures**

Analyses of the column-creep problem have been based either on initially imperfect columns or on initially perfect (straight, axially loaded) columns. In many respects, an analysis based on imperfect columns is more attractive, since real columns are imperfect. Also, in the limit, it may be possible to approach perfection analytically as closely as desired in the imperfect-column analysis. The perfect-column analysis, in contrast, cannot be analytically extended to describe the imperfect column. Although time-dependent behavior is of interest here, it may be noted that the above distinction also applies to time-independent behavior as well.

In a sense, the above reasoning may be misleading. It tends to suggest, perhaps, that perfect-column analysis is of little value. A review of static (no time effects) column theory indicates that this is not the case; rather, there is something to be gained from considering the perfect column in static buckling.
The value of the perfect column in creep-buckling analysis is, however, less obvious. Several creep-buckling theories (15, 16, 17) use a perfect column as a basis for analysis. Hoff (19), however, has demonstrated that these theories do not provide equivalent results. It is possible that the differences in results could arise from differences in the procedure used to inspect for stability. One must then, however, examine the acceptability of the different inspection procedures.

A comparison of the details of inspection procedures is difficult. Only the theory of Rabotnov and Shesterikov (17) appears to be explicit in this respect. Their inspection procedure is stated in detail. The theories of Shanley (15) and Gerard (16) involve implicit assumptions regarding column action. To a large extent, the use of these assumptions has been supported on the basis of providing satisfactory correlations with experimental data. Although this may be encouraging, it is not sufficient to dispel the reservations that exist regarding development of the analyses.

The theories developed by Shanley (15) and Gerard (16) postulate that a perfect column that is initially stable in the straight form may become susceptible to buckling after a certain amount of creep has occurred. This would be possible if the effective bending stiffness decreased with increasing time. In these developments, the concept of a time-dependent modulus is then introduced to reflect the
decrease in bending stiffness. The formulae for static or short-time buckling conditions then become available for creep-buckling predictions. Following the outlined procedure, Shanley proposed a time-dependent modulus based on the use of tangents to the isochronous stress-strain curves, and Gerard proposed a modulus based on the use of secants to the isochronous stress-strain curves.

Although the selection of the appropriate time-dependent modulus apparently can be arbitrary, the basic approach involved is not unsound. The inspection procedure associated with this approach is essentially "static" in nature; i.e., inertia effects are neglected. For it, one is concerned with the column response (at the time of inspection) as a disturbance is administered. By way of contrast, the dynamic inspection procedure of Rabotnov and Shesterikov is concerned with the column response (at the time of inspection) after a disturbance is administered.

For the static procedure, one is concerned with the response developed during an instantaneous inspection. Permitting the inspection to occur noninstantaneously may occur as a possibility. The inspection then involves deflection rates, however, and this does not lead to a definitive procedure.

---

4 Isochronous stress-strain curves are plots of stress versus strain with time as a parameter. Each curve is a plot of stress versus the total strain (elastic, plastic, creep) that occurs for the given time.
The use of a static inspection procedure was discussed in 1953 by Carlson (20), who considered the stability of a perfect column subject to creep. It was concluded that "a perfect column that is loaded to a value of load which is less than its tangent-modulus load will become unstable in time only if the static stress-strain properties are affected by either the creep that has occurred or by metallurgical change." This statement focused attention on the importance of the short-time stress-strain properties, whereas in the application of the theories of Gerard and Shanley, the amount of creep that has occurred governs the results.

Recently, Fraeijs de Veubeke (18) applied a static inspection procedure to imperfect creeping columns and produced, as the imperfection became vanishingly small, the same result predicted by Carlson (20). Fraeijs de Veubeke's analysis also includes a description of column behavior for loads greater than the tangent-modulus load.

The analysis of Fraeijs de Veubeke (18) is an extension of earlier work by Hoff (14). Hoff observed that, for a material possessing properties found in metals and alloys (time-independent elastic and inelastic strain, time-dependent inelastic strain or creep), the critical deflection depended only on the time-independent properties. Knowing the critical deflection, the critical time could be computed by the use of the basic equation of motion.
Fraeijs de Veubeke showed that the critical deflection could be obtained by "inspecting" the column for stability at a deflection, \( w \), by the consideration of a neighboring configuration, \( w + dw \). Although the final result that he obtained was available from Hoff’s work, the new treatment helped to focus attention on the details of the creep-buckling process.

As noted above, the inspection procedure of Fraeijs de Veubeke involves an examination during an instantaneous incremental increase in deflection. In the present study, a more general inspection procedure was utilized. This involved an increase in the column load, which, of course, also produces a corresponding deflection increase. This procedure is simpler to achieve experimentally than deflection increases under constant load, and as will be shown, includes the Fraeijs de Veubeke result as a special case.

A New Inspection Procedure

As a basis for this discussion, consider an imperfect column made of a material possessing the mechanical properties common to structural metals and alloys. In terms of strain components, this means that the effects of elastic plastic, and creep terms must be considered.

Under a constant load or average stress, and after a certain period of time during which the column deflection is increasing
steadily because of creep, collapse will occur, i.e., the column will no longer be able to support its load. To illustrate how creep buckling is in essence similar to static or short-time column buckling, reference will be made to Figure 1, which is a plot of column average stress versus column deflection. The solid curve, starting at the origin, describes the short-time (no creep) variation of average stress with deflection. For a column load producing an average stress such as \( \sigma_1 \), stability can be examined by an infinitesimal increase in stress, \( d\sigma \). If \( \frac{d\sigma}{dy} \) is greater than zero, the column is considered stable under \( \sigma_1 \). When the column average stress is increased, the level designated as \( \sigma_m \) will ultimately be reached. Here,

\[
\frac{d\sigma}{dy} = 0 ,
\]

and the column is considered unstable, i.e., collapse will occur and deflections greater than that corresponding to the maximum average stress can be realized only for average stresses less than \( \sigma_m \) (see the dashed curve). This interpretation of time-independent stability has been presented by Bleich (24) and by Hoff (22).

The stability of a column whose deflection is steadily increasing due to creep (under a constant load) can be examined in essentially the same manner. Initially, the column is loaded to an average stress of \( \sigma_1 \). At time zero, the column is stable in the same sense
Figure 1. Column Average Stress versus Deflection
as previously. If no creep occurs, the stress-deflection point remains on the solid curve. If creep can occur, however, and the load remains constant, the stress-deflection point will move horizontally and to the right with increasing time. A sequence of times, $t_1$, $t_2$, ... can be depicted, then, as in Figure 1. To examine the stability of the loaded column after a time $t_1$ has elapsed, the inspection described earlier can be utilized, i.e., an infinitesimal increase in stress, $d\sigma$, can be applied instantaneously (since creep can occur, the stress increase should not take any time).

If, as is indicated at $t_1$, the value $\frac{d\sigma}{dy}$ is greater than zero, the column is stable at the time $t_1$. Subsequent examinations at $t_2$, $t_3$, ... would indicate that, as the column deflection increases with time, the value of $\frac{d\sigma}{dy}$ would decrease, and ultimately become zero at the critical or failure time $t_1$.

The physical significance of the collapse phenomenon is that the internal column fibers are no longer able to provide an internal moment that can resist the external moment, i.e., their capacity to resist the external moment is exceeded. In the static or short-time case, this limiting moment is reached by an increasing column load and accompanying deflection increases. For the creep-buckling case, the moment is increased to a limiting value by deflection increases.
alone, as the external load remains constant. In both instances, however, the internal-moment capacity is ultimately exhausted.

It should be emphasized that in the preceding discussion, the conclusions regarding the state of the column at any given time are based on an externally observable response (the type of response predicted at $t_1$ has been experimentally observed, i.e., the sudden loss of stability, characteristic of attaining $\sigma_m$, is observed at $t_1$). The prevailing internal stress distribution is not explicitly specified. In an actual column, the internal history may be too complex (due to relaxation, creep recovery, strain hardening due to creep, and so on) to permit an exact specification of the stress distribution. This, however, is a limitation of available analytical methods, not of the concepts involved in the inspection procedure. The inspection procedure, therefore, is valid for materials of the type specified regardless of whether or not we can solve for the prevailing stress distribution.

The problem from this point becomes one of developing a model that can simulate the actual column behavior in an acceptable manner. As will be seen in the subsequent discussion, an analysis incorporating the necessary material properties can be performed and it will be seen that the results are in agreement with those depicted in Figure 1.
Application of Theory to a Column Model Analysis

The difficulties attendant with the solution of stress analysis problems involving creep were reviewed in the Introduction. In problems involving triaxial or even biaxial states of stress, it was seen that some difficulties are bypassed by the use of a response relationship that includes only a creep-strain component, i.e., the elastic and plastic components are ignored. The discussion of the preceding section suggests that these components may be necessary if the analysis is to provide a description of behavior that agrees with that which is observed. Fortunately, the problem for analysis involves uniaxial stresses, and this provides a considerable reduction in mathematical difficulties. Experience has indicated (8, 9) that for columns with solid rectangular cross sections, even analyses including two of the three strain components (elastic and creep) become extremely complex, and can be solved only by numerical procedures which incorporate the use of computer facilities. Thus, although the problem is solvable, there are important restrictions:

1. The nature of the numerical solutions requires that one specific problem be solved at a time. This means that general interrelationships between the various parameters are difficult to detect. Important characteristic features may thus be missed.
2. As noted above, the previous solutions ignored the plastic strain component, and it will be shown that the emission of this component robs the analysis of an important result.

In view of the above considerations, it appears that if a solution of the type desired is to be obtained, a simplification of the column model must be considered. This trend of thought leads naturally to the use of a column model similar to that used successfully by Shanley (12) in his well-known paper on inelastic column theory. This model was, of course, capable of illustrating the essential features of the inelastic column. A variation of that model has been used in a number of creep-buckling studies (6, 7, 14, 18), and it offers the simplification desired here. The model referred to is shown in Figure 2. In the analysis, the following definitions are used:

\[ \begin{align*}
A & \quad \text{Total cross-sectional area of column} \\
L & \quad \text{Length of column} \\
h & \quad \text{Distance between flanges of column} \\
x = \frac{x'}{L} & \quad \text{Nondimensional axial coordinate} \\
w_o = \frac{2}{h} y_o & \quad \text{Nondimensional initial, unloaded deviation from straightness (imperfection)} \\
w = \frac{2}{h} y & \quad \text{Nondimensional deflection due to load} \\
P & \quad \text{Axial force (negative when compressive)}
\end{align*} \]
Figure 2. Idealized H-Section Column.
The creep law used as a basis for analysis is that proposed by Odqvist (13) and used by Hoff (14) in his analysis.

\[
\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + k_1 \left( \frac{\sigma}{\mu} \right)^m \frac{\dot{\sigma}}{\mu} + \dot{F}.
\]  

(1)

The constant \( k_1 \) assumes values necessary to describe the manner of loading.

When \( \sigma < 0, \dot{\sigma} < 0 \), \( k_1 = (-1)^m \)

When \( \sigma < 0, \dot{\sigma} > 0 \), \( k_1 = 0 \).

It is of interest to note that Odqvist used the second term on the right of Equation (1) to represent primary creep. It is not, however, as is primary creep, a time-dependent term as multiplication of Equation (1) by an infinitesimal change in time, \( dt \), readily reveals.
It can more realistically be represented as a term due to time-independent plastic flow. Differentiation of the Ramberg-Osgood equation with respect to time, in fact, produces the first two terms of Equation (1).

For the conditions considered, \( F \), the creep strain-rate function, will be negative, and the constant \( m \) will be considered an odd integer. The analysis can, of course, be performed if \( m \) is even simply by changing the sign of \( k_1 \).

As the column creeps, the strain rates in the flanges will be

\[
\varepsilon_c = \frac{\dot{\varepsilon}_c}{E} = \left(-\frac{c}{\mu}\right)^m \frac{\dot{\sigma}_c}{\mu} + F_c , \tag{2}
\]

\[
\varepsilon_t = \frac{\dot{\varepsilon}_t}{E} = \left(-\frac{t}{\mu}\right)^m \frac{\dot{\sigma}_t}{\mu} + F_t \quad \text{if } \dot{\varepsilon}_t < 0 , \tag{3}
\]

and

\[
\dot{\varepsilon}_t = \frac{\dot{\sigma}_t}{E} + F_t \quad \text{if } \dot{\varepsilon}_t > 0 . \tag{3'}
\]

The moment, \( M \), developed at any section of the column by the load \( P \) is

\[
M = -P(y + y_o) = -\frac{Ph}{2} (w + w_o) \tag{4}
\]

The flange stresses are

\[
\sigma_c = \frac{P}{A} - \frac{2M}{hA} = \sigma (1 + w + w_o) \tag{5}
\]

and

\[
\sigma_t = \frac{P}{A} + \frac{2M}{hA} = \sigma (1 - w - w_o) . \tag{6}
\]
The curvature relation for the problem is

$$\varepsilon_t - \varepsilon_c = -\frac{1}{2} \left( \frac{h}{L} \right)^2 \frac{\partial^2 w}{\partial x^2} = -\frac{1}{2} \left( \frac{h}{L} \right)^2 c,$$

(7)

where $c$ is the curvature of the column.

Equations (2), (3), (5), (6), and (7) form the basis for analysis. These equations will now be operated on to convert them into a form convenient for an analysis of the inspection procedure.

Equations (5), (6), and (7) can also be written

$$\dot{\sigma}_c = \sigma \dot{w} + \left( 1 + w + w_o \right) \dot{\sigma},$$

(5')

$$\dot{\sigma}_t = \dot{\sigma} \left( 1 - w - w_o \right) - \sigma \dot{w},$$

(6')

and

$$\dot{\varepsilon}_t - \dot{\varepsilon}_c = -\frac{1}{2} \left( \frac{h}{L} \right)^2 \dot{c}$$

(7')

When the column is inspected by an increase, $d\sigma$, in the column average stress, the states of loading in the column flanges will be functions of the column deflection, $w$. The stress change in the concave flange will occur as a loading for all increases in $d\sigma$. The
change of stress in the convex face, $d\sigma_t$, is given by Equation (6').

An inspection of this equation reveals that, if

$$|\sigma dw| > |(1 - w - w_o) d\sigma|,$$

$d\sigma_t$ will be a decreasing compression or "unloading". This will be designated as Case II. If

$$|\sigma dw| < |(1 - w - w_o) d\sigma|,$$

d$\sigma_t$ will be an increasing compression or "loading". This will be designated as Case I.

Case I (Loading on Convex Flange)

For Case I, the conditions require that the differential strain rate law for $\dot{c}_t$ describe loading. Hence, the relation

$$\dot{\epsilon}_t = \frac{\dot{\sigma}_t}{E} - \left(\frac{\sigma_t}{\mu}\right)^m \frac{\dot{\epsilon}_t}{\mu} + F_t$$

must be used. Through the use of Equations (2), (3), (5'), (6'), and (7'), the equation governing the "inspection" is obtained:

$$\frac{1}{2} E \left(\frac{h}{L}\right)^2 \dot{c} = \left(2\sigma - E(\frac{\sigma}{\mu})^m + 1 \left[ (1 + w + w_o)^m + (1 - w - w_o)^m \right] \right) \dot{\epsilon}_t$$

$$+ \left[ 2(w + w_o) - \frac{E}{\mu} \left(\frac{\sigma}{\mu}\right)^m \left[ (1 + w + w_o)^{m+1} - (1 - w - w_o)^{m+1} \right] \right] \dot{\epsilon}_t - F_t + F_c$$

(8)
Let the initial imperfection be

\[ w_0 = a_0 \sin \pi x . \]

To obtain results from Equation (8) it proves convenient to assume that \( w \) can be satisfactorily expressed as a sine function. This assumption has been checked experimentally (24) and has been found to provide a good approximation for the given conditions. This aspect will be discussed in greater detail in the section on experimental results. Assume, then, that

\[ w = a \sin \pi x , \quad (9) \]

so that

\[ \dot{w} = \dot{a} \sin \pi x . \]

Let

\[ \varepsilon_E = -\frac{2}{4} \left( \frac{h}{L} \right)^2. \]

Dividing Equation (8) by \( \dot{\varepsilon} \), making use of Equation (9), and noting that for an instantaneous increase in the average stress, \( d\sigma \), the time-dependent quantities \( \frac{dF_c}{d\sigma} \) and \( \frac{dF_t}{d\sigma} \) are zero yields

\[
\frac{d\sigma}{dw} = \frac{-2 E \varepsilon_E + 2\sigma - E (\gamma')^m + 1 \left\{ \left[ 1 + (a + a_0) \sin \pi x \right]^m + \left[ 1 - (a + a_0) \sin \pi x \right]^m \right\}}{E (\gamma')^m \left\{ \left[ 1 + (a + a_0) \sin \pi x \right]^{m+1} - \left[ 1 - (a + a_0) \sin \pi x \right]^{m+1} \right\} - 2(a + a_0) \sin \pi x} \quad (10)
\]
For Case I, Equation (10) provides a method for computing the slopes of loading inspection curves of the type shown in Figure 1. This completes the analysis for inspections in which the compressive loading in both the convex and concave flanges increases.

Case II (Unloading on Convex Flange)

This case deals with the problem when the stress in the convex flange is decreasing, or unloading. In this development, it is assumed that the given inequality holds along the entire convex flange. Actually, the region of unloading begins at the column midpoint and grows toward the column ends. It will be seen, however, that for the condition of primary interest, unloading occurs all along the convex flange. The condition to be utilized is then

$$\sigma dw > \left(1 - w - w_o\right) d\sigma$$

and the following relation must be used:

$$\dot{\varepsilon}_t = \frac{\sigma_t}{E} + \dot{F}_t$$

(3')

Through the use of Equations (2), (3'), (5'), (6'), and (7'), the equation governing the inspection is obtained:

$$-\frac{1}{2} \left(\frac{h}{L}\right)^2 \ddot{\varepsilon} = \left[-2 \frac{\sigma}{E} + \left(\frac{\sigma}{\mu}\right)^{m+1} (1+w+w_o) \right] \dot{w}$$

$$+ E \left(\frac{\sigma}{\mu}\right)^m \left[(1+w+w_o)^{m+1} - 2\mu (w+w_o)\right] \frac{\sigma}{E\mu} + \dot{F}_t - \dot{F}_c$$

(11)
Applying the procedure used in the development of Equation (10) yields

\[ \frac{d\sigma}{dw} = \frac{2\sigma - 2E\epsilon \sigma E - E^{(\frac{\sigma}{\mu})} \mu \left[1 + \left(a + a_0\right) \sin \pi x\right] \mu}{\mu^{(\frac{\sigma}{\mu})} \left[1 + \left(a + a_0\right) \sin \pi x\right] \mu + 1 - 2\left(a + a_0\right) \sin \pi x} \]  \quad (12)

For case II, Equation (12) provides a method for computing the slopes of the loading inspection curves of the type shown in Figure 1.

As noted in the preceding section, the critical time occurs when

\[ \frac{d\sigma}{dw} = 0. \]  \quad (13)

Setting Equation (12) equal to zero yields

\[ + 2\sigma - 2E\epsilon \sigma E - E^{(\frac{\sigma}{\mu})} \mu \left[1 + \left(a + a_0\right) \sin \pi x\right] \mu = 0 \]

As a sample calculation, let \( m = 1 \) and consider a one-point collocation that satisfies the equation at \( x = \frac{1}{2} \) (the column midpoint). The following result is then obtained for the total critical deflection:

\[ (a + a_0) \text{ crit.} = 2 \left(\frac{E}{\sigma}\right)^2 \left[\frac{\sigma}{E} - \frac{1}{2} \left(\frac{\sigma}{\mu}\right)^2\right]. \]  \quad (14)

\[ ^5 \text{The nondimensional deflection, } w, \text{ corresponds to the deflection, } y, \text{ of Figure 1; i.e., the conditions } \frac{d\sigma}{dw} = 0 \text{ and } \frac{d\sigma}{dy} = 0 \text{ are equivalent.} \]
For the conditions specified, this corresponds to the value of critical deflection that would be obtained from Hoff's solution (14) with the following qualifications:

1. Hoff uses \( w \) as the total deflection, whereas in this analysis \( (w + w_o) \) is the total deflection. Setting \( a_o = 0 \), therefore, makes the terms equivalent.

2. Hoff defines \( \varepsilon_E = \frac{\pi^2}{4} \left( \frac{h}{L} \right)^2 \), whereas here we have used \( \varepsilon_E = -\frac{\pi^2}{4} \left( \frac{h}{L} \right)^2 \). Substituting the defined quantities back into the corresponding equations for critical deflections would produce equivalent results.

Interpretation of Analysis

The equivalence of results noted above for two different procedures might have been expected, since at the critical time (see Figure 1), the inspection procedure discussed here coincides with that of Fr ae ijs de Veubeke, i.e., \( w \) increases, but \( \sigma \) remains constant.

From an experimental point of view, however, times less than the critical are easier to examine, and the present analysis provides a method for accomplishing the examination. The examination, in turn provides a relatively direct check on the validity of the concept that the initial stress-strain curve governs instantaneous, incremental inspections. By increasing the column load quickly at a time less than the critical time, experimental values can be
obtained for $\frac{d\sigma}{dw}$. Using the basic material properties, an appropriate computed value can be obtained from Equations (10) or (12). These values can then be compared.

Previous checks (25) on the validity of the concepts involved have been less direct. They involve a comparison of experimental and computed curves of column load versus critical time. This type of check is less direct in that it introduces the computation of the critical time. In effect, this results in an accumulation of additional assumptions, which, in turn, should be checked.

From a practical point of view, it may also be noted that disturbances can be expected to be transmitted through the ends of the column as load increases. One would be interested, therefore, in knowing the "stiffness" remaining after creep deflections had occurred and, in fact, may wish to specify a limiting deflection based on an allowable reserve of stiffness.

The analysis conducted to this point has produced only a part of the solution of the problem. One must proceed further for a complete analysis in which the critical time is required. It is worth noting, however, at this point that the stiffness and the critical deflection were computed without specifying the creep-strain rate function $F$. This could have been a power term in $\sigma$, a strain-hardening
relation involving the stress and the creep strain, or a more general relationship. This part of the solution will be considered in a subsequent section of the dissertation.

If the nonlinear time-independent term had not appeared in Equation (1) and subsequently in (2') and (3'), the inspection criterion would not have evolved as it did. A column of a viscous material [only a function F would appear on the right side of Equation (1)], for example, would be infinitely rigid to an instantaneous inspection.

A column of viscoelastic material would merely adjust, with the necessary elastic stress changes, to an instantaneous inspection. There would, moreover, be no maximum condition to the average stress. This can be deduced from Equation (12) by letting \( \mu \to \infty \), and noting that only as \( \alpha \to \infty \) would \( \frac{d\sigma}{dw} \) at \( x = \frac{1}{2} \) approach zero. 6

For the type of behavior characterized by the preceding analysis, the nonlinear time-independent term is, therefore, essential. It is of interest to note, then, that the details of failure or collapse for the different materials cited are fundamentally different from one another.

---

6 Since the curvature relations (7) and (7') are approximate, permitting "\( a \)" to become very large in Equation (12) is not actually an acceptable procedure. It does, nevertheless, provide insight into the relative differences for the various types of materials.
In the inspection procedure used in the analysis presented here, and that by Fræijs de Veubeke, it is tacitly assumed that

\[ \frac{d\epsilon}{d\sigma} = f(\sigma) d\sigma , \]

or

\[ \frac{d\sigma}{d\epsilon} = \frac{1}{f(\sigma)} , \]

regardless of the time or the creep strain that has accrued. Excluding metallurgical changes, which would obviously require a generalization of this expression, it is possible that this assumption is, at best, an approximation even for so-called metallurgically stable materials. It is possible, for example, that a relation of the type

\[ \frac{d\sigma}{d\epsilon} = g(\sigma, \epsilon) \]

where \( \epsilon \) is the creep strain, may be effective. This would not invalidate the inspection procedure. It would, however, require that the modified value of \( \frac{d\sigma}{d\epsilon} \) at the moment of inspection be known. Even though we are concerned only with metallurgically stable materials, it is still proper to examine this assumption more closely.

Two references were found in the literature that, although they do not duplicate the conditions of interest completely, possess value in examining the validity of the assumption involved. The studies described by the references involve experiments in which tensile specimens were crept various amounts at elevated temperatures. They were subsequently tested at room temperature to
determine the effect on short-time stress-strain behavior of the accrued creep strain. Although these studies were conducted under either a constant stress or a constant load, and the effect on the room-temperature rather than "at temperature" stress-strain properties was found, the trends observed are translatable in terms of the present problem.

In the first study to be cited, Lloyd, Hazlett, and Parker (26) obtained results on commercial "A" nickel that had been precrept at 700 F. The effect of precreep was to raise the stress-strain curve, and the greater the precreep, the higher the stress-strain curve. Although the shift in the stress-strain curves is significant, it may be noted that for a given level of stress, the difference between the slope of curves for precrept materials and the slope of the zero-precreep curve appears to be relatively small. The latter difference, rather than the difference in the level of the stress-strain curves, is, of course, the difference of interest here.

In the second study of interest, Sherby, Goldberg, and Dorn (27) obtained extensive results on high-purity aluminum precrept at several elevated temperatures. Here again, although prior creep raises the stress-strain curve, the differences in slope for a given stress appear to be small.
For the results reviewed, it would appear that \( \frac{d\sigma}{d\epsilon} \) probably is a function, not only of the stress, but also of the creep strain. This indicates that a solution for these two materials is based on the assumption that \( \frac{d\sigma}{d\epsilon} \) is a function only of stress would be approximate. It should be noted, however, that both the precreep strains and the smallest strains shown for the stress-strain data of the references cited are greater than those of interest to this study. The details of the manner in which the curves would begin to separate in the region of interest cannot, therefore, be deduced. On the basis of these data, however, it is reasonable to expect that some difference in slopes would exist. Further, the necessity of additional research in this area is indicated. Results from such experiments are presented in a subsequent section.

Relation of Results to General Column

The results of the preceding analysis are applicable for the column model utilized, i.e., the two-flange cross-section model. It is appropriate, however, to inquire about the relation of the results to columns with solid cross sections.

In the formulation of the new inspection procedure illustrated in Figure 1, no restriction regarding the cross section was made. Qualitatively then, the features of column behavior deduced from the analysis should be present in columns of solid cross section. This
reasoning parallels that of Shanley (12) in the application of the results obtained for a column model to the general problem of inelastic column buckling.

It would be desirable, of course, to be able to obtain quantitative results from the analysis that would be applicable to other column cross sections. At attempt at this type of application has been made by Chapman, Erickson, and Hoff (25). The procedure which they used was to replace the solid cross section by an equivalent two-flange cross section. The equivalence utilized was based on the requirement that the bending stiffness of the actual column and the model column be the same. For linear elastic behavior this, in turn, results in the requirement that the total cross-sectional areas be the same and that flange separation be twice the radius of gyration of the solid cross section. If, then, the depth of a rectangular cross section is denoted by $d$ and the flange separation by $h$, it follows that

$$h = \frac{\sqrt{3}}{3} d.$$  

So long as the stress distribution acting on the actual cross section is linear, the above equivalence is valid. In column creep, however, the stress distribution can be expected to become nonlinear with time, so that the use of the stated equivalence means that the analysis is approximate.
It would be helpful if an analysis of the approximation involved could be made. An effort, therefore, has been directed toward developing a method for providing an estimate of the error in terms of the governing parameters.

Before introducing that analysis, it will be helpful to consider a similar, but more simple problem first. Consider a beam of solid cross section with two orthogonal axes of symmetry whose origin is the section centroid. For bending about one of the axes of symmetry, let the problem be the determination of an equivalent two-flange beam. Let the material response law be given by the nonlinear relation

$$\varepsilon = \left( \frac{\sigma}{\lambda} \right)^n,$$

where \(n\) and \(\lambda\) are material constants. If the tensile and compressive properties of the material are governed by the same law, it is only necessary to consider half of the cross section.

For the solid cross section

$$\varepsilon = cz,$$

where

- \(\varepsilon\) is the strain,
- \(c\) is the curvature,

and

- \(z\) is the distance from the neutral axis.
Using the strain relationship,

\[ \sigma = \lambda (cz)^n . \]

For a beam of depth, d, and width, b, the bending moment \( M_s \), can be written

\[ M_s = 2 \int_0^{d/2} \frac{1}{n + 1} dz = 2 \lambda c \int_0^{d/2} b z^n dz , \]

or

\[ M_s = \lambda c^n I_s , \]

where

\[ I_s = 2 \int_0^{d/2} b z^n dz . \]

For the two-flange cross section with a total area, A, and a flange separation, h, the bending moment, \( M_I \), is

\[ M_I = 2 \left( \frac{A}{2} \sigma \right) \left( \frac{h}{2} \right) = \frac{1}{2} Ah \sigma , \]

or

\[ M_I = \frac{1}{2} Ah \lambda \left( \frac{h}{2} c \right)^n = \lambda c^n I_I , \]
where
\[ I_I = A \left( \frac{h}{2} \right)^n. \]

For the two beams to be equivalent in bending, it is necessary that the rate of change of bending moments with respect to curvature change be the same, or that
\[
\frac{dM_s}{dc} = \frac{dM_I}{dc}.
\]

It follows, then, that
\[
\frac{1}{n} \lambda c \left( \frac{1-n}{n} \right) I_I = \frac{1}{n} \lambda c \left( \frac{1-n}{n} \right) I_s,
\]
or
\[ I_I = I_s. \]

Although the expressions for \( I_s \) and \( I_I \) are similar to the common area moments of inertia, the specific form is different due to the nonlinear stress-strain relationship. If \( n = 1 \), i.e., a linear law is used, the definitions become the commonly used ones.

If the solid cross section is rectangular,
\[
I_s = \frac{2n + 1}{2nb \left( \frac{d}{2} \right)^n} \frac{n}{(2n+1)}.
\]
By requiring that the total areas of the two cross sections be equal, \( A = bd \), and

\[
I = bd \left( \frac{h}{2} \right)^n.
\]

By the requirement that \( I_s = I \), it follows that the relationship

\[
h = d \left( \frac{2n+1}{n} \right)^{n+1}
\]

provides the necessary equivalence for the given problem. It is of interest to note, that as \( n \) goes from unity to infinity, the ratio \( \frac{h}{d} \) decreases from \( \frac{\sqrt{3}}{3} \) to \( \frac{1}{2} \).

The preceding example provides a simple example of an interpretation of equivalence between a solid cross section and a simple two-flange cross section. Considering the problem as it occurs in columns quickly reveals that the presence of a net force, in addition to the bending moment, complicates the problem. The same basic ideas still apply, however.

In the development to follow, it will be helpful to refer to Figures 3 and 4. In Figure 3a the instantaneous inspection has caused the column deflection to change from \( y \) to \( y + dy \). The corresponding change on a column load versus midpoint deflection diagram is shown in Figure 3b.
Figure 3. Load and Configuration Details during Loading Inspection.

Figure 4. Stress Distributions Before and After Inspections.
The stress distribution acting on a rectangular cross section before (dashed curve) and after (solid curve) inspection are represented in Figure 4a. It should be noted that due to creep, the stress distribution before inspection cannot be mapped on a stress-strain diagram, i.e., the corresponding stress and strain values would not, in general, be on the stress-strain curve. The stress changes occurring during an instantaneous inspection are indicated in Figure 4b.

As the column creeps, the degree of nonlinearity of the stress distribution increases. It should, therefore, be most pronounced at the time of failure or collapse. This, however, corresponds to an inspection for which \( \frac{dP}{dy} = 0 \), so the instantaneous bending during the "inspection" occurs under a constant load. If, now, the inelastic loading indicated in Figure 4b is assumed to be governed by an average tangent modulus, the nonlinear loading curve can be approximated by the dashed line. An application of von Karman's (28) double-modulus type analysis is then possible.

The equivalence of interest requires that the derivative

\[
\frac{dM}{dc},
\]

where \( M \) is the bending moment and \( c \) is the curvature, for the actual and the model column be the same for the instantaneous inspection.
Since $M = P_y$, and we are concerned now with an inspection under
constant load

$$dM = P\,dy,$$

and

$$\frac{dM}{dc} = \frac{P\,dy}{dc}.$$

From a von Karman double-modulus analysis the following
equation can be written

$$\frac{dM}{dc} = E_R I,$$

where $E_R$ is the reduced modulus and $I$ is the area moment of in-
ertia of the cross section with respect to the centroidal axis.

For a rectangular cross section (28)

$$E_R I = \frac{4EE_T}{(\sqrt{E} + \sqrt{E_T})^2} \left( \frac{bd^3}{12} \right)$$

(15)

where

- $E$ is the modulus of elasticity in compression,
- $E_T$ is the tangent modulus
- $b$ is the width of the cross section,

and

- $d$ is the depth of the cross section.
The tangent modulus varies, of course, with loading stress. The effective loading stress on solid cross section will be taken as the value which is acting on the concave flange of the equivalent two-flange cross section.

For a two-flange cross section (28)

\[
E_R I = \frac{2EE_T}{(E+E_T)} \left( \frac{Ah^2}{4} \right)
\]

(16)

For equivalence the values of \(E_R I\) of Equations (15) and (16) must be equal. It follows then, that

\[
h = \frac{\sqrt{6}}{3} \left( \sqrt{1 + \frac{E_T}{E}} - 1 + \sqrt{\frac{E_T}{E}} \right)
\]

(17)

As would be expected, Equation (17) reduces to

\[
h = \frac{\sqrt{3}}{3} \ d
\]

for \(E = E_T\).

In order to use Equation (17) for estimating the correct relationship between \(h\) and \(d\) at the time of the inspection, the appropriate value \(E_T\) must be selected. \(E_T\), however, depends on the level of stress acting on the region of the cross section which undergoes "loading". Since the condition of interest occurs at the critical time, the "critical deflection" is involved. The manner in which \(h\) varies
with $E_T$ in Equation (17) can be deduced, however, simply by letting the ratio of $\frac{E_T}{E}$ decrease from unity to zero. For this variation of the ratio, $h$ increases from $\frac{\sqrt{3}}{3} \, d$ to $\frac{\sqrt{6}}{3} \, d$. The tabulation below includes results from intermediate values of the ratio.

<table>
<thead>
<tr>
<th>$\frac{E_T}{E}$</th>
<th>$h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.577 , d</td>
</tr>
<tr>
<td>0.5</td>
<td>0.585 , d</td>
</tr>
<tr>
<td>0.2</td>
<td>0.618 , d</td>
</tr>
<tr>
<td>0</td>
<td>0.816 , d</td>
</tr>
</tbody>
</table>

The above tabulation reveals two particularly interesting facts. The effective flange separation increases with decreasing $E_T$. The increase in $h$ is slow, i.e., a fifty per cent reduction in $E_T$ results in about a one per cent increase in $h$.

The manner in which $h$ can be expected to vary for a given column will be discussed after the introduction of the experimental results of this investigation. There are, however, several other points concerning Equation (17) which should be discussed first. From Equation (17), which we will call Case 2, it has been seen that $h$ increases as the degree of nonlinearity (as measured by the decrease in $E_T$) increases. In the example of the pure bending of a beam with a nonlinear response law, which we will call Case 1, it was found that $h$ decreased with increasing nonlinearity. It is
apparent from these results that generalizations about h with respect to the nonlinearity of the stress distribution cannot be deduced unless a more detailed examination of the results is made.

It will be convenient in the discussion which follows to note that Equation (17) could also be obtained for the pure bending of a beam in which the loading is elastic, but in which the compression modulus is different from the tension modulus for the beam material. Realizing this, we can compare results for the two cases under the same type of loading.

In comparing the stress distributions for the two cases, two differences are prominent. For Case 1, the stress distribution is an odd nonlinear function with respect to the bending axis. For Case 2, there is no symmetry and the tensile and compressive segments are each linear.

The physical reason for the decrease in h for Case 1 can be understood when it is recognized that, as the nonlinearity of the stress distribution increases, the interior fibers of the beam become increasingly efficient in terms of the proportion of the total bending carried. As a consequence, the flanges of the equivalent beam move closer together. 7

7The distance of a flange from the bending axis is given as the position (in the solid cross section relative to the bending axis) in which half the area can be concentrated. The stress on the flange equals that for the corresponding point in the solid cross section.
For Case 2, it is apparent that the mechanism of shifting "inward" of the proportion of the bending moment does not occur because stress distributions are linear with respect to the bending axis. As the ratio \( \frac{E_T}{E} \) decreases, the distance of the axis of bending from the centroidal axis increases (for \( \frac{E_T}{E} = 1 \), they coincide). From Equation (17) it is apparent that as the modulus \( E_T \) is relaxed from the value \( E \), the flange separation must be increased to maintain equivalence.

From the preceding discussion, it can be concluded that --

1. For stress distributions of the Case 1 type, the effective flange separation decreases.

2. For stress distributions of the Case 2 type, the effective flange separation increases.

The type of stress distribution present in the column during an inspection is actually a combination of Cases 1 and 2. The response on each side of the axis of bending differs because of the manner of loading and hence different moduli, \( E \) and \( E_T \), are operative. This aspect corresponds to Case 2. In addition, however, the stress distribution will be nonlinear in the sense of Case 1 on the "loading" side of the cross-section. It can be seen, then, that there are two opposing effects, one tending to increase the effective flange separation and the other tending to decrease it. The net effect should then be less than the greater of the separate effects.
It is important to note, however, that rather drastic departures from linearity for either Case 1 or Case 2 separately are necessary to produce a five per cent change in the effective flange separation.

The tendency of one effect to compensate for the other is beneficial in terms of the quantitative value of the analysis in which a rectangular cross section is replaced by the two-flange model. Although it is difficult to compare the relative influence of the two effects, it can be anticipated that both effects will become more important in changing the effective flange separation as the bending component of the loading increases. For relatively small deflections, and hence for comparatively small bending components of load, the two-flange cross section should, therefore, be a good replacement for the rectangular cross section. The relation of the two-flange cross section to the rectangular cross section will be discussed further in connection with the results obtained from column experiments.

It should be emphasized that the above conclusions are based on a comparison of the two-flange cross section and a rectangular cross section. A review of the analyses leading to these conclusions indicates, however, that the effects governing the replacement of other cross sections by their two-flange equivalents could be examined in a similar manner.
Experimental Results

The analysis and discussion of the preceding sections have indicated the types of experiments that should be conducted for an evaluation of the procedures developed. Three types of experiments were conducted. These included: the uniform, uniaxial compression test for obtaining the static or short-time stress-strain curve; experiments designed to determine the effect of creep on strain-hardening characteristics; column-creep experiments in which loading inspections are performed.

The selection of a material for the experimental studies was based on several requirements. To simplify experimental procedures, it was desirable to conduct experiments at a relatively low temperature -- if possible, no greater than about 350°F. Moreover, the material should creep sufficiently at the test temperature to produce significant losses in column capacity in relatively short times. At the same time, the material should have a good resistance to creep at room temperature, to permit alignment adjustments with trial loading without the danger of introducing initial curvatures. Finally, the material should be metallurgically stable at the temperature used in the experiments.

On the basis of a review of readily available materials that appeared to be capable of satisfying these requirements, two
aluminum alloys, 7075-0 and 5052-0, were selected for preliminary evaluation. The annealed condition, as designated by the -0, was chosen to assure metallurgical stability. The preliminary evaluation consisted of selected tensile creep tests. The preliminary tests indicated that, for a temperature of 325° F, the creep resistance of the 5052-0 alloy was significantly greater than that of the 7075-0 alloy. The latter alloy possessed acceptable room-temperature properties, could be used in a stable condition, and had a relatively low resistance to creep at a convenient test temperature. It was selected, therefore, as the material for the experimental studies. All specimens were machined from the same heat of bar stock.

The details regarding the equipment, experimental procedures and results obtained for each of the three types of experiments conducted will be discussed separately in the following sections.

**Compression Stress-Strain Results**

The constants for the time-independent inelastic part of the deformation law were obtained from the short-time compression stress-strain curve. To obtain this information, specimens were prepared from the bar stock of aluminum alloy 7075-0. The compression test specimen was designed with a 0.505-inch-diameter reduced section and 1-inch gage length. The ends which were 0.900-inch diameter, were ground flat and parallel to within +0.0001 inch.
A cylindrical furnace with continuous circular windings was used in heating these specimens for testing. The temperature distribution over the specimen length was $\pm 2^\circ F$ from the center-controlling thermocouple. During a test, the temperature variation was less than $\pm 4^\circ F$. A universal hydraulic testing machine was used to apply the compressive loads.

Each compression specimen was equipped with two Bakelite AB-3 strain gages mounted on opposite sides in the reduced section. Prior to elevated-temperature tests, the specimen was placed on the testing machine and aligned at room temperature. Each gage was read independently for this purpose. After being at temperature for at least 1/2 hour, the specimen was loaded several times to about 25 per cent of the proportional limit in order to obtain Young's modulus. The two gages were then connected in series and fed electrically into an autographic recorder. The specimen was then tested to beyond the 0.2 per cent yield at a strain rate of 0.002 inch per inch per minute.

The results of two tests conducted at $325^\circ F$ are shown in Figure 5 as solid curves. Also included in Figure 5 is a room-temperature short-time test result. The plotted points represent values obtained from a Ramberg-Osgood-type stress-strain relationship in which

$$
\epsilon = \frac{\sigma}{E} + \left(\frac{\sigma}{\sigma_0}\right)^n,
$$
Figure 5. Compression Stress-Strain Curves for Room Temperature and 325° F.
where

\( \varepsilon \) is the total strain

\( E \) is Young's modulus

\( \sigma_0 \) and \( n \) are parameters describing the inelastic behavior.

For the plotted points of Figure 5, the following values were used for the constants:

\[
E = 9.1 \times 10^6 \text{ psi}
\]

\[
\sigma_0 = 58 \times 10^3 \text{ psi}
\]

\[
n = 6.
\]

The form of the deformation law in the body of the report differs from the above relationship because of the differentiation with respect to time. By manipulating the inelastic parts of these equations, it can be shown that \( m \) and \( \mu \) of Equation (1) are given as

\[
m = n - 1
\]

and

\[
\mu = \frac{\sigma_0}{n^{1/n}}.
\]

It follows then that

\[
m = 5
\]

and

\[
\mu = 43.1 \times 10^3 \text{ psi}.
\]
Effect of Creep of Strain Hardening

In a previous section, the results of investigations by Lloyd et al. (26) on commercial nickel and Sherby et al. (27) on high-purity aluminum were cited to illustrate how the rate of strain hardening can be increased by precreep. The materials involved in these studies were different from that used in the present program. Also, the ranges of strain both in precreep and in the recorded stress-strain diagram were greater than those of interest here. For these reasons, it was felt advisable to conduct studies more directly applicable to the present problem.

In the experiments designed to study the effect of creep on strain hardening, compression specimens with 0.400 inch by 0.500 inch rectangular cross sections and 1.25 inches long were used. All experiments were conducted on a 5000-pound-capacity Baldwin-Tate-Emery screw-testing machine.

The test temperature of 325°F was obtained by the use of a circulating-air oven. Temperature calibrations indicated that the total temperature variation with time over the gage length was less than ±2°F.

Strain was recorded autographically on a Baldwin stress-strain recorder. The signal to the recorder was obtained from two Bake-lite AB-3-type electrical-resistance strain gages bonded to the
0.500-inch faces of the specimens. The procedure used for applying these gages and an evaluation of their use in this type of application is described in Reference (24).

The experimental procedure consisted of loading the specimens to 10,000 psi with a speed-control setting giving a strain rate of 0.001 inch per inch per minute in the elastic range. After holding the stress constant at 10,000 pse (manual adjustments of the control setting were made to maintain the load) for a predetermined amount of creep strain, loading was resumed at the same head-rate setting used in loading to 10,000 psi.

Stress-strain curves obtained from the load-deformation records are shown in Figure 6 for the four experiments conducted. For the curve identified as P-0, loading was stopped at 10,000 psi, but immediately resumed. This procedure was followed with what was intended to be the base curve to eliminate differences that might originate in stopping and restarting, i.e., with the exception of the amount of time spent and the creep accrued at 10,000 psi, the procedures for each experiment were the same. It should be noted that unloading after the period of creep under constant load was avoided. The complicating effects of creep recovery, therefore, were not introduced.

It is apparent from a comparison of the curves of Figure 6 that the rate of strain hardening subsequent to creep at 10,000 psi is
Figure 6. Stress versus Strain after Different Amounts of Creep at 10,000 psi.
increased significantly over the curve for no creep, P-0. To determine the extent of this effect more directly, a plot of the tangent modulus versus stress curves for P-0 and P-1 is presented in Figure 7. It is of interest to note that the corresponding curves of P-2 and P-3 lie within the accuracy of graphical slope determinations for the curve P-1. This suggests that the increase in the rate of strain hardening reaches a "steady-state" level after relatively small amounts of precreep. The over-all effect could, therefore, be interpreted as the sum of a hardening due to creep and a softening due to exposure to temperature. For all amounts of creep, hardening predominates. Beyond a certain value of creep strain, however, the softening effect nullifies any additional hardening that may be introduced.

These results indicate that the increased hardening accompanying precreep as observed for nickel (26) and aluminum (27) also exists for Aluminum Alloy 7075-0. The effect of creep on strain hardening is pertinent to the analysis of the column-creep problem. The existence of the effect and its influence on the analytical results will be discussed subsequently in greater detail in connection with the column-creep problem.

**Column Experiments**

The analysis and the discussion in preceding sections have indicated the type of experiment that can be utilized to examine the
Figure 7. Tangent Modulus versus Stress for Different Amounts of Precrеп.
properties of a column subject to creep. Since the change in the resistance to bending -- as measured by the instantaneous derivative of the average stress with respect to the deflection -- serves as the basis for the examination procedure developed in this study, it follows that column-creep buckling experiments in which the average stress is suddenly increased are required. The results of experiments designed to provide the necessary data are presented in the section that follows.

All experiments were conducted at a temperature of 325° F on specimens machined from the aluminum alloy 7075-0. The length of all column specimens was 10,000 inches. Three slenderness-ratio values were used: 60, 75, and 90. These values were achieved by the use of rectangular cross sections of three sizes: 0.600 inch by 0.577 inch, 0.480 inch by 0.463 inch, and 0.400 inch by 0.385 inch.

The column experiment unit is a lever-arm, dead-weight type, as shown in Figure 8. The load is transmitted from a loading pan through a lever arm to the top-loading plunger. From the top-loading plunger, the load is transferred to the specimen through the knife-edge arrangement in the end caps. The loading circuit is completed through the lower support and a load cell mounted on the frame beneath the furnace.

The end caps are based on a design originating in the work of Mathauser and Brooks (10). Figure 9 shows the details of the end caps and the knife edges through which the load is transmitted.
Figure 8. Column-creep-buckling Test Stand with Furnace
Figure 9. Close-up View of End Cap and Yoke
design of the end caps places the knife edges in the plane of the ends of the columns. The end of the column specimens bears against a hardened plate insert. Adjustments for column-end eccentricity can be made by the use of the screw mechanism. The end caps and yokes were made of the 7075-T6 aluminum alloy, except for hardened-steel inserts and knife edges.

The lateral deflection of the column midpoint was measured by the use of a linear differential transformer positioned outside the furnace. The coil of the transformer was mounted on a fixture that was attached to the column end caps. The core was connected by a yoke to the column midpoint. By this means, only the deflection relative to the column ends was measured.

The signal from the transformer was fed into a 6-point recorder, which, when only one channel was used, printed every 5 seconds. Measurements could be obtained between printings by visually observing a pointer attached to the printing mechanism. Two ranges were available. For one range, the chart width of 11 inches represented a core movement of 0.01 inch with the smallest division representing 0.0005 inch. For the second range, the chart width represented a movement of 0.1 inch with the smallest division representing 0.005 inch. Calibration of the unit indicated an accuracy of \( \pm 2 \) per cent.
The furnace for the column test unit is of the circulating-air type and is shown in Figure 8. Heating elements are fixed to two of the furnace walls behind baffles. A fan is used to force the air between the baffles and the heating elements, and then across the specimen. Good temperature distribution was achieved for the column specimens with this type of system. Temperature calibrations of the furnace indicated that the total temperature variation with time over the gage lengths was less than $+2^\circ F$.

The details of the experiment being considered can be illustrated by reference to Figure 10. The graphs of Figure 10 describe the history of an interrupted column-creep buckling experiment in terms of column load, lateral deflection, and time. First, the column was loaded quickly (in the experiments conducted, the total load was transferred from a hydraulic jack to the column test stand in about 10 seconds) to point 0, which represents the zero-time condition. With the load remaining constant, the column was free to deflect with time as indicated in the lower graph. At a time corresponding to the point designated as A, the column load was increased to the magnitude indicated by B. The transition from A to B, which ideally occurs instantaneously, results in the deflection increase shown. The slopes of the curve at points along A-B are the measures of bending stiffness referred to above. It can be noted that in the range B-C the load has been indicated as being
Figure 10. History of Interrupted Column-Creep-Buckling Experiment.
constant. Since there is nothing in the basic analysis of the problem that indicates that additional interruptions beyond B could not be made, a second interruption is indicated between C and D. It may also be noted that, since the equation for the derivative of the average stress with respect to the deflection is a function of the variables, average stress and deflection, at the midpoint, it also should be possible to compute the slope at any point along A to B or C to D. Of course, the proper values of stress and deflection must be used.

For the experiments conducted, imperfections were controlled by adjustments at room temperature prior to testing. Two types of measurements were taken. In one type, strains at quarter points of the column were measured, in order to make certain that the adjustments made produced deflection configurations that were symmetric with respect to the center of the columns. The second measurement obtained was the column deflection at midlength. The load-deflection data were analyzed by the use of the Southwell plot (28) to determine the effective imperfections. After a satisfactory alignment was achieved, the column specimens were heated to the test temperature of 325° F. After the temperature had stabilized (since the material was in a metallurgically stable condition, no special precautions regarding the temperature-stabilization period were necessary), the experiments were conducted according to the procedure indicated in Figure 10.
A summary of the experiments conducted in this part of the program is presented in Table 1. The inspection loading was applied incrementally in a period of 15 to 20 seconds. In all instances, the values of midpoint deflection and average stress correspond to the beginning of the second increment of load applied during the inspection. This procedure eliminated in several instances erratic results observed for the first increment, so, for the sake of uniformity, it was adopted for all results. The number of inspections performed ranged from one to three per specimen.

The experimental values of the slope \( \frac{d\sigma}{dy} \) at \( x' = \frac{L}{2} \) given in Table 1 were obtained from plots of load versus midpoint deflection obtained during the inspections. Plots for these data are presented in Reference (29). The values in the last column of Table 1 are the computed slopes of the tangents to the loading curve at the appropriate stress level.

In a previous section, an analysis was presented for computing the quantity \( \frac{d\sigma}{dy} \). Before comparing computed values with the experimental values, three factors which have an influence on such a comparison will be reviewed and the results will be presented. The three factors are

1. The effect of the change in the effective flange separation
2. The effect of accrued creep on the strain-hardening characteristics
## Table 1. Inspection-Experiment Data

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Imperfection $10^{-3}$ inch</th>
<th>Interruption Time minutes</th>
<th>Stress, $\sigma$ psi</th>
<th>Midpoint Deflection $y(\frac{L}{2})$</th>
<th>$\frac{d\sigma}{dy}$ at $x' = \frac{L}{2}$</th>
<th>$\frac{d\sigma}{dy}$ at $x' = \frac{L}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>60-1-1</td>
<td>7.0</td>
<td>87.0</td>
<td>-8360</td>
<td>28.7</td>
<td>-255</td>
<td>-415</td>
</tr>
<tr>
<td>60-2-1</td>
<td>4.0</td>
<td>22.0</td>
<td>-8070</td>
<td>11.0</td>
<td>-720</td>
<td>-895</td>
</tr>
<tr>
<td>60-2-2</td>
<td>4.0</td>
<td>33.0</td>
<td>-8450</td>
<td>17.4</td>
<td>-370</td>
<td>-550</td>
</tr>
<tr>
<td>60-2-3</td>
<td>4.0</td>
<td>41.5</td>
<td>-8710</td>
<td>28.7</td>
<td>-275</td>
<td>-405</td>
</tr>
<tr>
<td>60-3-1</td>
<td>3.0</td>
<td>23.5</td>
<td>-7870</td>
<td>12.1</td>
<td>-620</td>
<td>-820</td>
</tr>
<tr>
<td>60-3-2</td>
<td>3.0</td>
<td>32.0</td>
<td>-8300</td>
<td>21.5</td>
<td>-325</td>
<td>-520</td>
</tr>
<tr>
<td>75-1-1</td>
<td>5.0</td>
<td>41.0</td>
<td>-6580</td>
<td>10.1</td>
<td>-630</td>
<td>-660</td>
</tr>
<tr>
<td>75-1-2</td>
<td>5.0</td>
<td>53.0</td>
<td>-7300</td>
<td>18.2</td>
<td>-370</td>
<td>-405</td>
</tr>
<tr>
<td>75-3-1</td>
<td>2.0</td>
<td>44.0</td>
<td>-7800</td>
<td>5.3</td>
<td>-1125</td>
<td>-1140</td>
</tr>
<tr>
<td>75-3-2</td>
<td>2.0</td>
<td>58.0</td>
<td>-8470</td>
<td>20.7</td>
<td>-270</td>
<td>-240</td>
</tr>
<tr>
<td>90-3-1</td>
<td>1.0</td>
<td>104.0</td>
<td>-6880</td>
<td>15.2</td>
<td>-160</td>
<td>-235</td>
</tr>
<tr>
<td>90-4-1</td>
<td>2.0</td>
<td>48.0</td>
<td>-6290</td>
<td>9.5</td>
<td>-455</td>
<td>-540</td>
</tr>
<tr>
<td>90-4-2</td>
<td>2.0</td>
<td>60.0</td>
<td>-6480</td>
<td>13.3</td>
<td>-275</td>
<td>-325</td>
</tr>
<tr>
<td>90-5-1</td>
<td>4.0</td>
<td>27.5</td>
<td>-5830</td>
<td>12.9</td>
<td>-275</td>
<td>-310</td>
</tr>
<tr>
<td>90-5-2</td>
<td>4.0</td>
<td>37.0</td>
<td>-6030</td>
<td>18.6</td>
<td>-180</td>
<td>-210</td>
</tr>
<tr>
<td>90-5-3</td>
<td>4.0</td>
<td>51.0</td>
<td>-6220</td>
<td>41.3</td>
<td>-62</td>
<td>-90</td>
</tr>
</tbody>
</table>

(a) In accordance with page 178 of Reference (4), the effective imperfection is assumed to be equivalent to an initial curvature given by a half sine wave.
3. The effect of time on events that ideally should be instantaneous.

Each of the above factors will be discussed separately in the sections that follow.

To assess the influence of the first factor above, the procedures developed in this dissertation for determining effective flange separations will be applied to a column of the type for which results are included in Table 1.

Consider first the type of nonlinearity designated as Case 2 (see Figures 3 and 4). As a typical example, consider a column with the following specifications:

- slenderness ratio, \( \lambda = 60 \)
- cross-section depth, \( d = 0.577 \) inch
- cross-section width, \( b = 0.600 \) inch
- average stress, \( \sigma = -8500 \) psi
- initial imperfection, \( y_o = \left( \frac{L}{2} \right) = 0.005 \) inch

For the example, it is desirable to consider a deflection which is considerably larger than those for which the interruptions listed in Table 1 were made. Thus, let the deflection be 0.200 inch. To compute a value of effective flange separation, for the equivalent two-flange column model, an average value of \( E_T \) for the loading region of the stress distribution (see Figure 4b) must be obtained.
The value of $E_T$ depends on the level of stress acting (see Figure 4a), so it is necessary to obtain an estimate of this value. The following formula will be used to obtain this estimate:

$$\sigma_L = \sigma + \frac{\sqrt{3}}{3} \left( \frac{6M}{bd^2} \right) = \sigma + \frac{2\sqrt{3}}{d} (\sigma)(y)$$

where $\sigma_L$ is the effective loading stress,

$y$ is the total deflection,

and the remaining terms have been defined previously. The factor of $\frac{\sqrt{3}}{3}$ was selected in accordance with the discussion preceding the development of Equation (17). This method of computing $\sigma_L$ is, thus, based on the use of a sum of the direct average stress, and a stress component derived from a linear bending stress distribution. The value of the bending component is taken as that which occurs at a distance $\frac{\sqrt{3}}{6} d$ from the axis of bending, or from the centroidal axis, since they coincide for a linear bending distribution. This method for obtaining an estimate of $\sigma_L$ can be justified for two reasons.

1. If the actual type of stress distribution is considered (see Figure 4b), it is found that, for equal bending moments, this method tends to yield a bending stress component that is on the high side. When added to $\sigma$, therefore, it provides a $\sigma_L$ which is in error on the high side. This in turn yields an average $E_T$ which is lower than it should be. A more accurate estimate of $h$ would, then, be less than
the computed one, or closer to the value of \( \frac{\sqrt{3}}{3} d \). For the purpose of this discussion, it follows that a more accurate estimate is not warranted.

2. The calculation of \( h \) from Equation (17) is relatively insensitive to changes in \( E_T \). If the value of the bending component yielding what should be an average \( E_T \) is somewhat in error, the error in \( h \) will not be very significant.

From the data given for the example

\[
\sigma = -8500 \\
d = 0.577 \text{ inch} \\
b = 0.600 \text{ inch} \\
y = 0.205 \text{ inch}
\]

For these values \( \sigma_L = 19,000 \text{ psi} \). The corresponding value of \( E_T = 2.0 \times 10^6 \text{ psi} \). From Equation (17), the value of effective flange separation corresponding to this value of \( E_T \) is \( h = 1.06 \frac{\sqrt{3}}{3} d \).

From the above results, it can be concluded that the type of nonlinearity associated with Case 2, which is characterized by a shifting of the axis of bending from the centroidal axis, does not introduce large deviations from the approximate equivalence relation of \( h = \frac{\sqrt{3}}{3} d \) for relatively large deflections. For deflections of the order of those listed in Table 1, hence, the influence of the Case 2 type nonlinearity should be small.
The influence of the Case 1 type of nonlinearity on the effective flange separation is more difficult to assess quantitatively. From the equation
\[ h = \left[ \left( \frac{2n+1}{n} \right)^{-\frac{n}{n+1}} \right] d \]
for bending, it can be determined that the effective flange separation decreases from \( h = \frac{\sqrt{3}}{3} \) \( d \) for \( n = 1 \) to \( h = 0.94 \left( \frac{\sqrt{3}}{3} \right) \) \( d \) for \( n = 2 \). This is a decrease of 6 per cent.

Unfortunately, it is not possible to select what might be considered an appropriate value of \( n \) for the Case 1 type of nonlinearity. The above example merely serves to illustrate that a reduction of \( h \) is possible for some conditions. For the conditions of loading for the column, the features present are much more closely represented by Case 2 type nonlinearity. The possible contribution of an effect which would tend to reduce the effective flange separation cannot, therefore, be evaluated in terms of the elementary examples used. In view of the uncertainty regarding possible decreases in \( h \), the following method of reasoning will be adopted. It will tentatively be assumed that effects are present which cause the effective flange separation to decrease. If this assumption is correct, calculations using a value of \( h = \frac{\sqrt{3}}{3} \) \( d \) should predict greater column stiffnesses than are actually observed. By this indirect method, of course, the influence of the Case 1 type of nonlinearity must be deduced from the experimental column data.
The results of experiments designed to study the effect of creep on the strain-hardening behavior of the material have been presented earlier in the dissertation. These results, which are summarized in Figures 6 and 7, suggest that significant effects probably should be observed. Since a quantitative analysis is not possible with the limited data available, it will be necessary to deduce from the column-inspection data the degree of influence of this factor.

The effect of time on events that ideally should be instantaneous involves two of the experiments conducted; the uniform compression test, and the column-inspection experiments.

A possible source of error arises from the tacit assumption that the strain of the short-time compression stress-strain curve has no time-dependent component. In order to minimize this effect, the stress-strain curve was recorded autographically to permit the use of a relatively high strain rate (0.002 inch per inch per minute). This effect could contribute to a lowering of the stress-strain curve from which the inelastic constants were obtained.

 Practically, the loading inspection cannot occur instantaneously; time must elapse, and consequently, time-dependent deformation can occur.

It will be noted that the latter effect, although due to rate of loading, would produce a result opposite to that associated with the
stress-strain test. To compare the relative significance of these two rate effects, the strain rates will be compared. For the compressive stress-strain test, the strain rate was 0.002 inch per inch per minute. The strain rate during the interruptions in the column experiments must be estimated. Considering a half-sine-wave incremental increase in deflection, the strain rate on the concave face at the column midpoint (due to the increase in direct loading and bending) is estimated to be of the order of 0.0004 inch per inch per minute. A comparison of these values suggests that the rate of loading of the columns during the interruptions would tend to be more critical.

As noted previously, the column-loading inspections required a period of from 15 to 20 seconds. Since the column deflection was increasing with time prior to the inspection, it must be expected that some of the deflection increase during the inspection was time dependent. For an increase in stress, $\Delta \sigma$, the observed increase in deflection, $\Delta y$, can be written as

$$\Delta y_1 = \Delta y + \Delta y_2,$$

where $\Delta y$ is the increase due to time-independent deformation and $\Delta y_2$ is the increase due to time-dependent deformation. The corrected slope involving only time-independent deformation then can be written as

$$\frac{\Delta \sigma}{\Delta y} = \frac{\Delta \sigma}{\Delta y_1 - \Delta y_2}.$$
But the time-dependent component can be written as

\[ \Delta y_2 = \frac{\Delta y_2}{\Delta t} \frac{1}{\frac{\Delta \sigma}{\Delta t}} \Delta \sigma . \]  

(19)

The terms for \( \Delta y_2 \) in Equation (19) can be obtained from experimental curves of the type shown in Figure 10. The value \( \frac{\Delta y_2}{\Delta t} \) will be taken as the slope of the deflection-time curve just prior to the loading inspection. The corresponding rate just after the inspection was always somewhat greater. The inspection slope was taken after the first increment of load, however, so the rate preceding the inspection should more nearly apply. The value of \( \frac{\Delta \sigma}{\Delta t} \) will be taken as the average loading rate during the inspection. Values of \( \frac{d\sigma}{dy} \) corrected by the use of Equations (18) and (19) are given in the last column of Table 1. In most instances, the correction is quite significant and must, therefore be considered.

Calculated and Experimental Inspection Results

It has been shown that "column stiffness", \( \frac{d\sigma}{dw} \), during a loading inspection, can be written as:

\[
\frac{d\sigma}{dw} = \frac{2\sigma - 2E\varepsilon_x - E\left(\frac{g}{\mu}\right)^m + 1 \left\{ [1 + (a + a_o) \sin \pi x]^{m+1} - [1 - (a + a_o) \sin \pi x]^m \right\}}{2(a + a_o) \sin \pi x - 2(a + a_o) \sin \pi x} .
\]
This relation applies for loading in both the convex and concave flanges:

\[ \left| \sigma \, dw \right| < \left| (1 - w - w_0) \, d\sigma \right| , \]

and this condition applied for all experiments conducted.

By considering \( x = \frac{1}{2} \) (this corresponds to \( x' = \frac{L}{2} \)), and making use of the relation

\[ \frac{d\sigma}{dy} = \frac{2 \, d\sigma}{h \, dw} , \]

values of \( \frac{d\sigma}{dy} \) can be computed for the conditions given in Table 1.

These computations have been made and the results are entered in column 2 of Table 2. Third column of Table 2 was obtained by letting \( \mu \to \infty \) in the above equation and performing the calculation with the terms that remain. This is equivalent to what would be obtained for a completely elastic response during the loading inspection.

In view of data presented in a preceding section (see Figure 7), it should be expected that the rate of strain hardening should be increased over that for the virgin material by accrued creep. The two calculations (columns 2 and 3) should, therefore, provide bounds on the value of \( \frac{d\sigma}{dy} \) observed in the experiments.
TABLE 2. SUMMARY OF INSPECTION-EXPERIMENT RESULTS

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Calculated $\frac{d\sigma}{dy}$ at $x' = \frac{L}{2}$ $10^3$ psi inch$^{-1}$</th>
<th>Calculated $\frac{d\sigma}{dy}$ at $x' = \frac{L}{2}$ for $\mu \to \infty$ $10^3$ psi inch$^{-1}$</th>
<th>Experimental $\frac{d\sigma}{dy}$ at $x' = \frac{L}{2}$ $10^3$ psi inch$^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>60-1-1</td>
<td>-305</td>
<td>-470</td>
<td>-415</td>
</tr>
<tr>
<td>60-2-1</td>
<td>-855</td>
<td>-1130</td>
<td>-895</td>
</tr>
<tr>
<td>60-2-2</td>
<td>-520</td>
<td>-775</td>
<td>-550</td>
</tr>
<tr>
<td>60-2-3</td>
<td>-345</td>
<td>-500</td>
<td>-405</td>
</tr>
<tr>
<td>60-3-1</td>
<td>-880</td>
<td>-1130</td>
<td>-820</td>
</tr>
<tr>
<td>60-3-2</td>
<td>-490</td>
<td>-685</td>
<td>-520</td>
</tr>
<tr>
<td>75-1-1</td>
<td>-565</td>
<td>-630</td>
<td>-660</td>
</tr>
<tr>
<td>75-1-2</td>
<td>-310</td>
<td>-380</td>
<td>-405</td>
</tr>
<tr>
<td>75-3-1</td>
<td>-870</td>
<td>-1130</td>
<td>-1140</td>
</tr>
<tr>
<td>75-3-2</td>
<td>-220</td>
<td>-335</td>
<td>-240</td>
</tr>
<tr>
<td>90-3-1</td>
<td>-220</td>
<td>-260</td>
<td>-235</td>
</tr>
<tr>
<td>90-4-1</td>
<td>-375</td>
<td>-415</td>
<td>-540</td>
</tr>
<tr>
<td>90-4-2</td>
<td>-265</td>
<td>-300</td>
<td>-325</td>
</tr>
<tr>
<td>90-5-1</td>
<td>-290</td>
<td>-310</td>
<td>-310</td>
</tr>
<tr>
<td>90-5-2</td>
<td>-200</td>
<td>-220</td>
<td>-210</td>
</tr>
<tr>
<td>90-5-3</td>
<td>-90</td>
<td>-105</td>
<td>-90</td>
</tr>
</tbody>
</table>
The corrected experimental values of $\frac{d\sigma}{dy}$ are presented in the fourth column. In comparing results in the second, third, and fourth columns, it will be seen that 10 out of 16 inspections satisfy the stiffness inequalities.

$$\text{calc. stiff. (elastic-plastic)} \leq \text{exp. stiff.} \leq \text{calc. stiff. (elastic)}.$$  

Of the remaining six, five of the corrected values are within 8 percent of either the lower or upper bound. Only one experimental value does not satisfy the inequality on the left.

From these results, it appears that the strain hardening that accompanies creep has a pronounced effect on the results. In essence, the proportional limit is raised and the column response tends towards that which would be expected for an elastic response ($\mu \rightarrow \infty$).

It is of interest to note that the results for the larger slenderness ratio values, 75 and 90, tend to cluster close to the upper limit, whereas those for the slenderness ratio of 60 tend to cluster close to the lower limit. One of the primary differences between these column sizes is the stress level to which the columns are loaded. The level of the stresses tends to increase with decreasing slenderness ratio. It is believed that this trend toward the "elastic-plastic" stiffness with increasing stress level is associated with the nature of the increase in strain hardening due to creep. It would be expected that the approach to the "elastic" stiffness would be more nearly
complete for stresses just above the virgin proportional limit than for stresses well above the virgin proportional limit. Thus, if stresses are low enough, what would have been an elastic-plastic loading response in the virgin material could become an elastic response for precrept material. For higher stress levels, some increase in stiffness (above the calculated elastic-plastic stiffness) is, thus, generally observed, but the completely elastic response is not achieved.

The above results indicate that the analytical methods developed can be used successfully to estimate bounds for the instantaneous stiffness of a creeping column. It appears that the approximation involved in the replacement of the solid cross section by the two-flange cross section has considerably less influence on the results than the effect of creep on the strain-hardening characteristics of the material. This conclusion applied, of course, for the range of deflections experienced in the inspection experiments. Although this conclusion could be altered for situations in which deflections were of the order of the critical deflection, it will be shown in subsequent sections of this report that it is probably preferable to base design of deflections that are several orders of magnitude less than the critical deflection. This procedure would allow for a reserve of column stiffness for possible sudden overloading.
COLUMN-CREEP ANALYSIS

The phase of the present investigation described in the preceding sections dealt with the time-independent properties of a creeping column, i.e., the behavior of the column in terms of its response to an instantaneous increase in load. The studies to be described in the following sections are devoted to an analysis of the behavior of the column while it is creeping. The analysis of this problem involves the total response of the column material, i.e., both the time-independent and the time-dependent properties. This means that the strain rate must include elastic, plastic, and creep components.

In a column creeping under a constant load, part of the column cross section undergoes unloading. Since this can be expected to introduce creep-recovery effects (which have been ignored in previous analyses), a review of the literature pertaining to this behavior is presented. Results of a creep-recovery experiment on the material used in this investigation are presented, and the effects that such behavior can introduce into the analysis of the creeping column are deduced.
A strain-rate relation making use of the creep-recovery studies is introduced, and a procedure for obtaining bounds on the actual strain is developed. The strain-rate relation is then used in an analysis of column creep of the two-flange model. The relation of the analysis based on the column model to the behavior of columns with solid cross sections is then analyzed.

Results from column-creep experiments are presented, and the analysis developed is applied. A comparison of calculated and experimental results is made, and the results are discussed. The analyses of Shanley and Gerard are also applied and compared with the analysis developed here.

**Development of a Strain-Rate Relation**

**Creep-Recovery Effects**

**Review of Literature**

Creep, as characterized by a strain-versus-time curve, usually is described by three stages within which the behavior has specific properties. During the primary stage, the creep rate is continuously decreasing. In the second or steady-state stage, the creep rate is a constant. If the loading is tensile, a final or tertiary stage exists that is characterized by a creep rate that increases continuously to fracture.
In discussions of creep behavior, it is convenient to separate the creep of the first and second stages into two components. These components are designated as the transient and the steady-state components. The steady-state component is a linear function of time, and its rate is the rate in effect during the second stage. The transient component is obtained by subtracting the steady-state component from the total creep. The strain rate associated with the transient component reduces to zero at the beginning of the second stage. The terminology associated with this method of analysis will be useful in the subsequent review of experimental studies.

Barrett (30) and Smith (31) indicate that the creep that occurs during the primary stage can either be partly or wholly recovered in polycrystalline metals, i.e., upon load removal, the specimen will tend to return to its original length in a time about equal to the duration of loading. Creep recovery, however, is not generally observed for single crystals. Also, to observe this behavior, the temperature after unloading should be maintained. Recoverable strain is apparently negligible if the original creep rate is high, or if prior to unloading, a low rate is followed by a high rate.

An explanation of creep recovery is given by Barrett (30) as follows:

Recoverable primary creep is understandable in terms of grain-boundary flow; it could be the amount of flow that occurs before it is brought to a stop by the keying points
at grain junctions. It is recoverable because residual stress, built up during this flow, cause reversed flow in the boundaries when the external load is removed. Some recoverable flow may be expected on slip bands as well as grain boundaries and this may occur even when grain-boundary flow is negligible. The residual stresses left around the slip bands that have operated cause 'after-effects' if they relax, and if they remain until a reversed stress is applied, they cause the 'Bauschinger effect', i.e., the lowered yield stress when plastic strain in tension is followed by compressive stressing or vice versa.

A review of available data on creep recovery suggests that after unloading, the portion of the creep recovered is that associated with the transient creep component. This was observed by Chalmers (32) for fine-grained tin, for which essentially all of the transient creep was recovered in a period equal to the period under load. The recovery versus time curve, in fact, very nearly duplicated the transient creep versus time curve.

Additional work by Chalmers (32) indicated that creep recovery in single crystals of tin did not occur. Experiments conducted on large-grained specimens indicated that although some recovery occurred, it was less than that which could be associated with a transient component of creep.

Studies conducted by Ke(33) on high-purity aluminum at 175°C indicated that in an unloaded period equal to the loaded period, about 93 per cent of the deformation acquired was recovered. If an attempt is made to subtract a steady-state component, the recovered deformation can be increased to about 97 per cent of the transient creep.
The studies of Chalmers and Ke involved low stresses and small creep deformations. Studies conducted on a 0.17 per cent carbon steel by Johnson (34) yielded results indicating that although the effect of creep recovery was important at low stresses, it was relatively less important at high stresses. Johnson found that for the steel he studied, the proportion of the total creep that was recovered upon unloading increased as the stress level was decreased. He found no simple relation between creep strain and recovery strain, however.

In addition to experiments in which unloading was complete, a number of studies have been conducted in which unloading was partial. This type of experiment is of particular interest here, since unloading in the convex flange of the column model, though continuous, is only partial over any given time interval.

The specific behavior observed upon partial unloading depends upon the percentage reduction of the initial stress. Certain general characteristics are, however, noted. Following partial unloading, there is an induction period during which the creep rate is small, or even negative if the stress reduction is relatively large. Eventually, a new creep rate, apparently characteristic of the new stress, is achieved and creep behavior common to constant stress experiments is reestablished.

Three examples of creep recovery after partial unloading will be cited. The first refers to experiments conducted by Tapsell and
Prosser (35) on a nickel-chromium-molybdenum steel tested at 840° F. After creeping under a stress of 22,400 psi for about 15 hours, the specimen was unloaded to a stress of 896 psi. After about 8 hours, approximately 43 per cent of the time-dependent strain present at 15 hours had been "recovered".

In an experiment by Lubahn (36) on a chromium-molybdenum-vanadium steel at 1000° F, a 50 per cent decrease in the initial stress resulted in an induction period of about 4 hours, during which the strain decreased. After the induction period, the strain rate tended to zero. Although there was a slight increase in rate subsequently, it was scarcely detectable.

As a final example, data obtained by Carreker, et al. (37) in an experiment on lead at 124° C will be cited. Initially, the stress acting was 456 psi, and creep had reached the steady-state rate. At what was designated as zero time, the stress was increased to 468 psi. At 90 minutes, the stress was reduced to the original value of 456 psi. For a period of about 10 minutes, no change in strain was observed. The creep rate then rose slowly to a value that was within 4 per cent of the value found to be characteristic of the stress at 456 psi prior to zero time.

The preceding examples generally indicate that creep-recovery effects are significant for a number of materials. There is indication in the work of Johnson that at high stress levels creep recovery
is not, comparatively, as important as at low stress levels. Neverthe­
less, the recovery effects that he observed even at higher stress
levels were significant. It may also be noted that the stresses acting
on columns are frequently in what would be considered a low stress-
level range.

A Creep-Recovery Experiment

The results of the preceding section suggest that the degree of
creep recovery can vary with the metal involved, the stress, and the
temperature. To obtain some measure of the probable importance
of recovery for the Aluminum Alloy 7075-0 in the present program,
a creep-recovery experiment was conducted.

The results of the experiment are presented in Figure 11. The
compression specimen used had a 0.400-inch by 0.500-inch rectangu­
lar cross section and was 1.25 inches long. Loading was in uniform
compression -- the values being indicated in Figure 11 -- and the
temperature was 325°F. The plotted data points are the averages
of strain readings obtained by the use of electrical-resistance strain
gages of the Bakelite AB-3 type mounted on opposite faces of the
specimen. The performance of this gage for the conditions of the
type involved here has been evaluated and is reported in Appendix I
of Reference (24). The differences of the individual strain values
from the average values plotted was less than 2 per cent.
Figure 11. Compressive Creep and Recovery Behavior of Aluminum Alloy 7075-0 at 325° F.
The creep strain accrued during the initial 30-minute period under a stress of 8000 psi was 185 microinches per inch. The stress was then successively reduced to 2170 psi, 125 psi, and finally to zero. During the periods under each of the above reduced stresses, recovery occurred. The amount of recovery is the difference between the 185 microinches per inch of creep and the permanent strain remaining after complete unloading, 83 microinches per inch. The recovery was, therefore, 102 microinches per inch or 55 per cent of the creep initially accrued.

For the given alloy and temperature, the 8000-psi stress is of the order than can be expected to occur in the convex flange of columns with slenderness ratios of about 60. For larger slenderness ratios the corresponding stresses would be lower and recovery effects could be expected to be relatively more pronounced. In terms of stress level, then, the results are applicable to column-creep studies.

In the convex flange of a creeping column, the stress history differs from that of Figure 11 in that the stress value decreases continuously from zero time. Although different histories of unloading over the given period could be expected to result in different final values of permanent strain, the basic features of the unloading
process are similar. The behavior observed can, therefore, be considered representative, and it can be concluded that recovery effects can be expected to be significant in column creep.

Recovery Effects During Column Creep

The question of creep recovery only enters considerations involving the convex flange. In the two-flange column model the total strain rate in the convex flange, \( \dot{\varepsilon}_t \), can be expressed as the algebraic sum of the elastic strain rate, \( \dot{\varepsilon}_t^e \), the creep rate, \( \dot{\varepsilon}_t^c \), and the recovery rate, \( \dot{\varepsilon}_t^r \),

\[
\dot{\varepsilon}_t = \dot{\varepsilon}_t^e + \dot{\varepsilon}_t^c + \dot{\varepsilon}_t^r.
\]

The terms on the right are all functions of the decreasing stress \( \sigma_t \) or the stress rate \( \dot{\sigma}_t \) (a compressive stress is defined here as a negative quantity. The notation \( \dot{\sigma}_t > 0 \), therefore, indicates a decrease in the compressive stress.).

The total strain rate in the concave flange, \( \sigma_c \), can be expressed as

\[
\dot{\varepsilon}_c = \dot{\varepsilon}_c^e + \dot{\varepsilon}_c^p + \dot{\varepsilon}_c^c,
\]

where

\( \dot{\varepsilon}_c^e \) is the elastic strain-rate function

\( \dot{\varepsilon}_c^p \) is the plastic strain-rate function,

and

\( \dot{\varepsilon}_c^c \) is the creep-rate function.
In the concave flange $\dot{\varepsilon}_c < 0$, i.e., the compressive stress is increasing so that no recovery term is present and the possibility of plastic deformation must be included.

The rate of change with respect to time of the curvature, $\dot{c}$, is proportional to the difference of the strain rates in the two flanges, so that

$$\dot{\varepsilon}_t - \dot{\varepsilon}_c = \dot{c},$$

or

$$\dot{c} = (\dot{\varepsilon}_t^c - \dot{\varepsilon}_c^c) - \dot{\varepsilon}_c^p + [(\varepsilon_t^c + \varepsilon_r^c) - \varepsilon_c^c]. \quad (20)$$

Assumptions regarding creep recovery do not involve the first two quantities on the right. The third or bracket term, therefore, can be used to study the effect of such assumptions on the curvature, $c$.

If it is assumed that no creep recovery occurs, then the bracket term becomes

$$[\dot{\varepsilon}_t^c - \dot{\varepsilon}_c^c].$$

If it is assumed that the net effect of creep and creep recovery in the convex flange is zero, then the bracket term reduces to

$$[- \dot{\varepsilon}_c^c].$$
Taking signs (compressive strains are defined as negative quantities) and relative magnitudes into account, it can be seen that

$$[-\dot{\varepsilon}_c^c] \geq [(\dot{\varepsilon}_t^c + \dot{\varepsilon}_t^r) - \dot{\varepsilon}_c^c] \geq [\dot{\varepsilon}_t^c - \dot{\varepsilon}_c^c].$$  \hspace{1cm} (21)

This inequality is valid as long as the convex flange is loaded in compression. The three terms on the right side of Equation (20) are positive quantities, as are each of the quantities in the inequality (21).

A conservative expression for curvature rate then would be

$$\dot{C} \sim (\dot{\varepsilon}_t^e - \dot{\varepsilon}_c^e) - \dot{\varepsilon}_c^p + (-\dot{\varepsilon}_c^c). \hspace{1cm} (22)$$

A nonconservative expression for curvature would be

$$\dot{C} \sim (\dot{\varepsilon}_t^e - \dot{\varepsilon}_c^e) - \dot{\varepsilon}_c^p + (\dot{\varepsilon}_t^c - \dot{\varepsilon}_c^c). \hspace{1cm} (22')$$

The use of an expression of the type indicated in (22), in addition to being conservative (where the estimated rate is greater than the actual rate), results in a simplification of computations involving the convex flange. The strain-rate relation of the convex flange becomes simply

$$\dot{\varepsilon}_t = \frac{\dot{\varepsilon}_t^t}{E}.$$

In other words, as the column deflects in time, and the stress in the convex flange decreases, only the elastic strain changes corresponding to the stress changes are considered. For "slender" and
"intermediate" columns, for which the stresses in the convex flange are relatively low, this is probably a good approximation. For "short" columns, the stresses in the convex flange are relatively higher and the approximation will tend to be comparatively more conservative. The degree of conservatism refers, of course, to the difference between the predicted curvatures and the actual curvatures.

It should be noted that the error associated with the approximate expression for \( \dot{\varepsilon}_t \) (total strain rate in convex flanges) will not result in an error of corresponding magnitude in the rate of curvature change, \( \dot{\gamma} \). The error in \( \dot{\gamma} \) can be considerably less due to the fact that \( \dot{\varepsilon}_c \) will be greater, and often much greater than \( \dot{\varepsilon}_t \). The value of \( \dot{\gamma} \) is, therefore, more "strongly" dependent on the value of \( \dot{\varepsilon}_c \), and typical calculations indicate, for example, that a 10 per cent error in \( \dot{\varepsilon}_t \) can result in an error in \( \dot{\gamma} \) of the order of 3 per cent or less.

A Strain-Rate Relation

In formulating solutions to stress-analysis problems involving creep, it is necessary to establish a relationship among stress, strain, and time. Relatively simple relationships involving strain-rate and stress can be used for this purpose. In the first phase of this dissertation the strain-rate relation

\[
\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \left( \frac{\sigma}{\mu} \right) m \frac{\dot{\sigma}}{\mu} + F
\]

was introduced for cases in which \( \dot{\sigma} > 0 \).
If the creep-rate function, $F$, is selected to be a power function in stress such that $F = \left( \frac{\sigma}{\lambda} \right)^n$, the strain rate can be written as

$$\dot{\varepsilon} = \frac{\sigma}{E} + \left( \frac{\sigma}{\mu} \right)^m \frac{\dot{\sigma}}{\mu} + \left( \frac{\sigma}{\lambda} \right)^n,$$

or

$$\varepsilon = \frac{\sigma}{E} + \left( \frac{\sigma}{\mu} \right)^{m+1} \frac{1}{(m+1)} + \int_0^t \left( \frac{\sigma}{\lambda} \right)^n \, dt. \quad (23)$$

To visualize the various components of Equation (23), consider Figure 12a which is a strain-time plot for a constant stress experiment. Upon loading, the strain increases from zero to the value indicated by the Point "A". For a short period after the application of the load, the strain rate will decrease (the primary stage). At a point such as "B" the strain rate often achieves a steady-state condition and remains constant as indicated between "B" and "C."

To represent a curve of the type shown in Figure 12a by Equation (23) one can utilize a simplification which replaces O-A-B-C by O-D-C. The first two terms then, can be used to represent the strain O-D and the increases with time from D to C can be represented by the final term.

The accuracy of the above type of representation depends, of course, on the error introduced by replacing A-B by A-D-B. If times less than the time at B are of interest, the approximation can be poor. Also, the construction utilized in Figure 12a requires that a "steady-state" exist.
Figure 12. Graphical Representations of Creep Behavior.
In many problems of interest to the stress analyst, the period of interest falls within the so-called primary stage. This is the case in the present investigation, so it must be expected that the use of the approximation indicated in Figure 12a could yield poor results.

In view of the attractiveness of the simplicity of Equation (23), alternative interpretations have been investigated. The solid curve of Figure 12b is typical of the strain-time curves obtained in the present investigation. In the given example, the time $t = t^1$ represents the upper limit for the duration of the experiments, i.e., in the case of the creep buckling of interest here, the experiments would be designed to be completed in times of $t < t^1$.

Although there is no steady-state behavior indicated in Figure 12b, it is still possible, to approximate the curve A-B by straight lines and thus utilize a relation of the type indicated in Equation (22). The dashed line A-B and the dashed line C-D, which is parallel to A-B and tangent to the solid curve serve, respectively, as lower and upper limits to the solid curve A-B. It should be noted that by the proposed procedure, the two limiting estimates would depend on the time $t^1$, since the position of B is given by the intersection of the creep curve and a vertical line at $t = t^1$.

In order to relate the limiting lines of Figure 12b to the components of Equation (23), it is convenient to refer to Figures 12b and 12c. The solid curve O-A-F of Figure 12c is the so-called short-time
stress-strain curve for the given material. Consider first the representation of the lower limiting path O-A-B of Figure 12b. The strain O-A is given by

\[ \frac{\sigma}{E} + \frac{(\frac{\sigma}{\mu})^{m+1}}{(m+1)} \]

The time-dependent strains may then be represented (whether or not this particular relation is appropriate must, of course, be determined from the given data) by a term of the type

\[ \int_{0}^{t} (\frac{\sigma}{\lambda})^{n} dt \]

Consider now the representation of the upper limiting path O-A-C-D. The strain A-C must now be grouped with the strain O-A. Since each stress level will have an additional increment of strain beyond that given by the short-time stress-strain curve, a pseudo stress-strain curve, shown dashed in Figure 12c, will be developed. The strain O-C can be represented by

\[ \frac{\sigma}{E} + \frac{(\frac{\sigma'}{\mu'})^{m'+1}}{(m'+1)} \]

where \(m'\) and \(\mu'\) are the values corresponding to the dashed curve of Figure 12c. The time-dependent strains again may be represented by
since the slopes of the two dashed lines of Figure 12b are the same.

From the above, we can summarize and write for the lower limiting relation

\[ \dot{\varepsilon}_L = \frac{\dot{\sigma}}{E} + \left( \frac{\sigma}{\mu} \right)^m \frac{\dot{\sigma}}{\mu} + \left( \frac{\sigma}{\lambda} \right)^n, \tag{24} \]

and for the upper limiting relation

\[ \dot{\varepsilon}_u = \frac{\dot{\sigma}}{E} + \left( \frac{\sigma}{\mu'} \right)^{m'} \frac{\dot{\sigma}}{\mu'} + \left( \frac{\sigma}{\lambda} \right)^n. \tag{25} \]

Equations (24) and (25) provide relatively simple analytical expressions for upper and lower bound estimates of strain rate.

There may be some reluctance to use a steady-state expression for describing a behavior which is not steady-state (Figure 12b). It may be noted, however, that whereas a steady-state does exist in Figure 12a, Equation (23) is still not exact due to the "misfit" during the primary stage. In essence, then, one is actually dealing with empirical curve fitting, and any advantage gained by being able to attribute a basic significance to the creep-strain term is offset by the poor agreement with actual values. On the other hand, it is readily seen that constructions of the type shown in Figure 12b applied to Figure 12a for times equal to or less than that for Point "B" will yield considerably better approximations than that obtained by the use of A-D-B.
Creep Data

In order to develop strain-time relationships of the type illustrated in Figure 12b, it is necessary to have short-time stress-strain data and creep data. Short-time stress-strain data for the 7075-0 aluminum alloy at 325°F were presented earlier in the form of the stress-strain curves of Figure 5. Creep data from tests in which specimens were loaded in uniform compression were obtained as part of the present investigation. These results will be presented below.

The compression specimens used were the same as that used for the creep-recovery experiment, i.e., they had a 0.400-inch by 0.500-inch rectangular cross section and were 1.25 inches long. The creep specimens had two shallow V-grooves in opposite faces to receive knife-edges from an extensometer used to measure strain. The V-grooves were 1 inch apart, and, thus, provided a 1-inch gage length.

The creep-stand unit, used for the compression-creep tests under a uniform uniaxial stress, was of the lever-arm dead-weight type. The load was applied at the end of the lever arm through a knife-edge arrangement. Two of the three contacts on the lever were knife edges, and the third point was a 1/2-inch steel-ball contact. The load was transmitted from the lever arm through the 1/2-inch steel ball to a hardened plunger and thus to the specimen.

The loading plunger was a hardened cylinder with a ball seat on the top face and a flat finish on the bottom face. A ball bushing (ball
bushings are arranged in six rows that are parallel to the axis of the plunger) was used to guide the hardened plunger and, at the same time, allow free axial movement. A rectangular column with a hardened end plate was used as a base support for the specimen.

The strain extensometer was attached to the 1-inch gage section of the specimen. The extensometer arms were used to transfer the motion of deformation of the specimen to a linear differential transformer outside the furnace. These arms were spring loaded to keep the knife edges in contact with the specimen. The transformer was electrically connected to a time-cycle recorder. The total movement corresponding to the chart width of 11 inches was 0.01 inch. The smallest division was 0.00005 inch.

The test temperature of 325° F was obtained by the use of a circulating-air oven similar to that used with the column test unit (see Figure 8). Temperature calibrations indicated that the total temperature variation with time over the gage lengths was less than ±2° F. A Foxboro potentiometer temperature controller was used to control the temperature. The reference thermocouple was maintained in an ice bath.

The design of the loading stand permitted the plane of the lower pedestal to be altered to permit good alignment of the test specimens. The alignment was adjusted at room temperature prior to testing by
the application of trial loads. Strains during alignment were obtained by the use of electric resistance strain gages, Type A-7, mounted to the two 0.400-inch faces. When good alignment was achieved, the extensometer was attached, the specimen was heated to the test temperature, and the preselected amount of compressive load was applied by means of a hydraulic jack.

The results obtained are plotted in Figure 13 as curves of creep strain versus time. The curves were obtained from printed test records which were, in essence, continuous since the recorder printed every 5 seconds.

The data of Figures 5 and 13 were analyzed in accordance with the procedure indicated in Figures 12b and 12c, and the plots developed for obtaining the empirical constants of Equations (24) and (25) are shown in Figures 14 and 15. A value of $t_1 = 5$ hours (see Figure 12b) was used in developing these plots. The values of the constants corresponding to these plots are

\[
\begin{align*}
E &= 9.1 \times 10^6 \text{ psi} \\
\mu &= 43.1 \times 10^3 \text{ psi} \\
m &= 5 \\
\lambda &= 5.3 \times 10^4 \text{ psi} \\
n &= 5 \\
\mu' &= 52.3 \times 10^3 \text{ psi} \\
m' &= 3
\end{align*}
\]
Figure 13. Compressive Creep Strain versus Time for Aluminum Alloy 7075-0 at 325°F.
Figure 14. Relationship between Creep Strain Rate and Stress.

Figure 15. Relationship between Stress and Pseudoinelastic Strain.
The lines drawn through the data points in Figures 14 and 15 provide good fits with the data, except for low values of stress in Figure 14. At low stresses, the values of strain rate indicated by the line are lower than the experimental value.

Application of the Strain-Rate Relation to the Column Model

Analysis

In the analysis to follow, the strain-rate law discussed in the preceding section will be used to analyze column creep of the column model shown in Figure 2. The definitions used previously will be the same.

Since the exponents \( m \) and \( n \) are odd for the material used in the experiments for this investigation, the analysis will be developed for odd values of \( m \) and \( n \). For even values, only appropriate changes in sign would be necessary to develop the analysis.

The strain-rate function for the concave flange can be written as

\[
\dot{\varepsilon}_{c} = \frac{\dot{\sigma}}{E} - \frac{(c)_{m}}{\mu} \frac{\dot{\sigma}}{\mu} + \frac{(c)_{n}}{\lambda} .
\]  

(26)

For the strain-rate function for the convex flange, it will be assumed that only the first term, the elastic component, is active. This choice is based on factors discussed in the section on creep-recovery effects. There, it was noted that due to creep recovery
accompanying unloading, the inclusion of the third term (for creep strain rate) yields a total strain rate that is greater than that which actually occurs. It was also shown that the inclusion of the third term in a column-creep analysis would result in a nonconservative estimate of deflections, i.e., at a given time, the actual deflection would tend to be greater, due to creep recovery, than that predicted by such an analysis. Assuming no creep in the convex flange is the opposite limiting extreme and it correspondingly should provide a conservative estimate of deflection in the analysis, i.e., the computed deflection would tend to be greater than the actual deflection. The strain-rate function can then be written as

$$\dot{\epsilon}_t = \frac{\dot{\sigma}_t}{E}.$$  

(27)

The moment, $M$, developed at any section of the column (see Figure 2) by the load $P$ is

$$M = -P(y + y_o) = -\frac{Ph}{2}(w + w_o).$$

The flange stresses are

$$\sigma_c = \frac{P}{A} - \frac{2M}{hA} = \sigma(1 + w + w_o),$$  

(28)

and

$$\sigma_t = \frac{P}{A} + \frac{2M}{hA} = \sigma(1 - w - w_o).$$  

(29)
The curvature relation is

$$\dot{\epsilon}_t - \dot{\epsilon}_c = - \frac{1}{2} \left( \frac{h}{L} \right)^2 \frac{\partial^3 w}{\partial t \partial x^2}.$$  \hspace{1cm} (30)

Substituting (28) in (26) and (29) in (27) and using the result in (30) gives the following equation of motion:

$$- \frac{1}{2} \left( \frac{h}{L} \right)^2 \frac{\partial^3 w}{\partial t \partial x^2} = - 2 \frac{\sigma}{E} \frac{\partial w}{\partial t} + \left( \frac{\sigma}{\mu} \right)^{m+1} \left( 1 + w + w_0 \right)^m \frac{\partial w}{\partial t} - \left( \frac{\sigma}{\lambda} \right)^n \left( 1 + w + w_0 \right)^n.$$  \hspace{1cm} (31)

It was shown experimentally in Reference (24) that after a short time, the column configuration very nearly assumes a half-sine wave. It will, therefore, be assumed that

$$w_0 = a \sin \omega x,$$

$$w = a \sin \pi x .$$

If \( x = \frac{1}{2} \) is considered, (31) becomes

$$\frac{\pi^2}{2} \left( \frac{h}{L} \right)^2 \frac{da}{dt} = - 2 \frac{\sigma}{E} \frac{da}{dt} + \left( \frac{\sigma}{\mu} \right)^{m+1} \left( 1 + a + a_0 \right)^m \frac{da}{dt} - \left( \frac{\sigma}{\lambda} \right)^n \left( 1 + a + a_0 \right)^n .$$

Letting \( \epsilon_E = - \frac{\pi^2}{4} \left( \frac{h}{L} \right)^2 \), this can be written

$$t = \int_{a(0)}^{a} \frac{1}{\left( \frac{\sigma}{\lambda} \right)^n \left( 1 + a + a_0 \right)^n} \left[ 2 \left( \epsilon_E - \frac{\sigma}{E} \right) + \left( \frac{\sigma}{\mu} \right)^{m+1} \left( 1 + a + a_0 \right)^m \right] \frac{da}{\left( \frac{\sigma}{\mu} \right)^{m+1} \left( 1 + a + a_0 \right)^m} ,$$  \hspace{1cm} (32)

where \( a(0) \) is the instantaneous deflection caused by the application of the load at \( t = 0 \).
Equation (32) can be integrated to yield a relation between time and the midpoint deflection \( a \). Since the lower limit strain-rate relation was used, the computed deflection should be less than the actual deflection.

Letting

\[
A = 2\left(\varepsilon_E - \frac{\sigma}{E}\right),
\]
\[
B = \left(\frac{\sigma}{\mu}\right)^m + 1,
\]
\[
C = \left(\frac{\sigma}{\lambda}\right)^n, \quad (1 + a) = b, \quad (m - n) = p,
\]

and integrating yields

\[
t = \frac{A}{C(1 - n)} \left[ (b + a)^1 - n - [b + a(0)]^{1 - n} \right] + \frac{B}{C(p + 1)} \left[ (b + a)^{p + 1} - [b + a(0)]^{p + 1} \right]
\]

(33)

Substitution of \( m' \) for \( m \) and \( \mu' \) for \( \mu \) in the above yields an expression for an upper limit, i.e., the computed deflection should be greater than the actual deflection. It should be noted that in terms of column capacity (lifetime for a given stress) the upper limit strain-rate relation yields a lower limit to capacity because the strain is greater than the actual strain (see Figure 12b). The lower limit strain rate relation yields an upper limit to capacity.

Relation of Results to General Column

In an earlier section of this dissertation, the use of the two-flange column model during inspections was discussed, and a procedure for
analyzing the mechanics of the time-independent behavior involved was
presented. For equivalence during inspections between the model
cross section and a solid cross section, it was noted that the quantity
\( \frac{dM}{dc} \) must be the same for the two cross sections.

Equivalence conditions for time-dependent behavior will differ
from those required for time-independent behavior, but the basic ideas
are similar. It will be helpful in illustrating this to treat, as before,
the simple problem of the pure bending of a beam. Whereas the re-
response law used previously was

\[ \varepsilon = \left( \frac{\sigma}{\lambda} \right)^n, \]

consider now the law

\[ \dot{\varepsilon} = \left( \frac{\sigma}{\alpha} \right)^m \]

where \( \alpha \) and \( m \) are material constants.

Following the development previously used for the solid cross
section and the two-flange cross section, it can readily be shown that

\[ M_s = 2ac \int_0^{h/2} \frac{1}{m} b(z)^m + 1 \, dz \]

or

\[ M_s = \frac{1}{m} \cdot I_s. \]
Similarly,

\[ M_I = a \dot{c} m I_I. \]

\( I_s \) and \( I_I \) differ from the previous definitions only in the replacement of \( \lambda \) by \( a \) and \( n \) by \( m \).

A comparison of these results with those obtained for the time-independent behavior discussed previously indicates that the relations obtained differ only for the curvature term. Replacing \( c \) in the previous results by \( \dot{c} \) yields the results obtained above. Correspondingly, for the solid and two-flange cross sections to be equivalent, it should now be required that

\[ \frac{dM_s}{d\dot{c}} = \frac{dM_I}{d\dot{c}}. \]

As previously, the condition for equivalence under pure bending requires that

\[ I_I = I_s. \]

It follows from this treatment that for equivalence between a beam of rectangular cross section and the two-flange cross section the relationship

\[ h = (\frac{2m+1}{m}) \frac{m}{m+1} \]

\[ d \]
must hold as previously. For bending of a beam with the same non-linear response in tension and compression, the effective flange separation, \( h \), decreases with increasing \( m \).

The equivalence condition for column creep also requires the equality of \( \frac{dM}{dc} \) values. The analysis is, however, complex due to the presence of a direct load, and the necessity for using a more complete strain-rate relationship than that used in the examples.

To determine what may be accomplished in the way of analysis, a study of the basic characteristics of the problem has been made. The nonlinearity of the stress distribution acting on the solid cross section at the time, \( t \), and a change in stress during a time interval, \( \Delta t \), are features of the problem which must be included in an analysis. The analysis that follows is, therefore, concerned with the stress distribution at a time \( t \) and that at time \( t + \Delta t \). These are illustrated schematically in Figure 16a. For the given conditions, the deflection is increasing under a constant column load. During the time interval \( \Delta t \), therefore, the compressive stress increases near the concave face of the bent column and decreases near the convex face.

Let the strain-rate law be represented in functional form as

\[
\dot{\varepsilon} = f(\sigma, \dot{\sigma}) .
\]
Figure 16. Change in Stress During Time Interval $\Delta t$. 

(a) Stress distribution at time $t$. 
(b) Stress distribution at time $t + \Delta t$. 

$\Delta \sigma$ is the change in stress, $\sigma$ is the stress, $h_1$ and $h_2$ are the lengths along the time interval. 

$\Delta \sigma_c$ is the change in stress component.
Consider a power-series expansion of $\dot{\varepsilon}$ about particular values of 

$\sigma = \sigma_1$, and $\sigma = \dot{\sigma}_1$.

$$
\dot{\varepsilon} = f(\sigma_1, \dot{\sigma}_1) + \left[ (\sigma - \sigma_1) f_\sigma(\sigma_1, \dot{\sigma}_1) + (\dot{\sigma} - \dot{\sigma}_1) f_\dot{\sigma}(\sigma_1, \dot{\sigma}_1) \right] + \cdots
$$

where the subscripts on $f$ refer to partial differentiation with respect to $\sigma$ and $\dot{\sigma}$.

Letting

$f(\sigma_1, \dot{\sigma}_1) = A_1$

$f_\sigma(\sigma_1, \dot{\sigma}_1) = B_1$

and $f_\dot{\sigma}(\sigma_1, \dot{\sigma}_1) = C_1$

the above expression can be written as

$$
\dot{\varepsilon} = A_1 + B_1 (\sigma - \sigma_1) + C_1 (\dot{\sigma} - \dot{\sigma}_1) + \cdots.
$$

A change in the strain rate $\Delta \dot{\varepsilon}$ can then be written as

$$
\Delta \dot{\varepsilon} = B_1 \Delta \sigma + C_1 \Delta \dot{\sigma} + \cdots. \quad (34)
$$

Before proceeding further, it is desirable to consider some of the properties of a creeping column. Initially, the stress across the cross section will be compressive, varying from a maximum on the concave face to a minimum on the convex face. As time proceeds, the stresses near the concave face will increase continuously and the stresses near the convex face will decrease continuously. It may be
anticipated then, that the response of the column fibers near the opposite faces will be governed by different laws, i.e., one for increasing stresses and one for decreasing stresses. In this respect the behavior is analogous to that designated as Case 2 type behavior for time-independent behavior. In Equation (34), $\dot{\varepsilon}$ is analogous to $\varepsilon$ for the time-independent case. Unfortunately, however, only the term involving $\sigma$ on the right has its counterpart in the time-independent case. An analysis which incorporates more than the first term on the right does not appear possible.

Since the higher order terms could be dropped by limiting consideration to small deviations of $(\sigma - \sigma_1)$ and $(\dot{\sigma} - \dot{\sigma}_1)$, the term restricting further analysis is $C_1 \Delta \dot{\sigma}$. This term could be dropped if there were periods in which the fiber stresses could be represented as

$$\sigma = kt,$$

where $k$ is a constant which depends on the position in the cross section. The stress rate is then

$$\dot{\sigma} = k, \text{ and } \Delta \dot{\sigma} = 0.$$

A linearization of Equation (34) then yields

$$\Delta \dot{\varepsilon} = B_1 \Delta \sigma.$$  (35)
The accuracy of Equation (35) depends not only upon the validity of the assumption that $\Delta \dot{\phi} = 0$, but also upon $f(\sigma, \dot{\sigma})$ and the stress distribution acting. In spite of these rather important restrictions, the analysis will be carried through to illustrate the influence of a nonlinearity of the type designated as Case 2 in the previous analysis.

Equation (35) can be used to represent the change in strain rate accompanying a change in stress near the convex region of Figure 16a. The value of $B_1$ refers to some value of stress such as $P_1$.

In a similar manner, a relationship for a point $P_2$ can be written as

$$\Delta \dot{\epsilon} = B_2 \Delta \sigma.$$  \hspace{1cm} (35')

By requiring that plane sections remain plane during bending, the total strains on the concave face, $\epsilon_c$, and on the convex face, $\epsilon_t$, can be written as

$$\epsilon_c = \epsilon_o + \epsilon_c$$

and

$$\epsilon_t = \epsilon_o + \epsilon_t,$$

where $\epsilon_o$ is the strain at 0 in Figure 16a, and $\epsilon_c$ and $\epsilon_t$ are bending strains ($\epsilon_c < 0$, $\epsilon_t > 0$).

Introducing the curvature, $c$, it also follows that

$$\dot{\epsilon}_c = h_2 \dot{c},$$

$$\dot{\epsilon}_t = - h_1 \dot{c},$$

$$\Delta \dot{\epsilon}_c = h_2 \Delta \dot{c},$$

$$\Delta \dot{\epsilon}_t = - h_1 \Delta \dot{c}.$$

But $\Delta \epsilon = \Delta \dot{\epsilon}_o + \Delta \dot{\epsilon}_c$, and $\Delta \epsilon = \Delta \dot{\epsilon}_o + \Delta \dot{\epsilon}_t$. 

\hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{
At point 0, however, \( \Delta \tau = 0 \) during \( \Delta t \), so

\[
\Delta \hat{c}_c = 0 ,
\]

and

\[
\Delta \hat{c}_t = h_2 \Delta \hat{c} , \quad \Delta \hat{c}_t = - h_1 \Delta \hat{c} .
\]

By use of these relations and Equations (35) and (35'), it follows that

\[
\Delta \tau_c = \frac{h_2 \Delta \hat{c}}{B_2}
\]

and

\[
\Delta \tau_t = \frac{- h_1 \Delta \hat{c}}{B_1} .
\]

At this point, it can be recognized that the development above parallels the von Karman double-modulus analysis. That is, the following correspondence can be noted:

<table>
<thead>
<tr>
<th>Term in Above Analysis</th>
<th>Corresponding Term in Double-Modulus Analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{B_1} )</td>
<td>Elastic Modulus</td>
</tr>
<tr>
<td>( \frac{1}{B_2} )</td>
<td>Tangent Modulus</td>
</tr>
<tr>
<td>( \hat{c} )</td>
<td>Curvature</td>
</tr>
</tbody>
</table>
The change in bending moment, \( \Delta M \) (under a constant load), corresponding to the change in rate of change of curvature, \( \Delta \dot{c} \), during the interval of time, \( \Delta t \), can then be written as

\[
\Delta M = E' R I \Delta \dot{c}
\]

where \( E' \) is defined in terms of the cross section and the creep constants, and \( I \) is the area moment of inertia of the cross section with respect to the centroidal axis.

For a rectangular cross section

\[
E'_R I = \frac{4}{(\sqrt{B_1} + \sqrt{B_2})^2} \left( \frac{bd^3}{12} \right), \tag{36}
\]

and for a two-flange cross section

\[
E'_R I = \frac{2}{(B_1 + B_2)} \left( \frac{Ah^2}{4} \right). \tag{37}
\]

For the rectangular and two-flange cross sections to be equivalent within the time period discussed, the expressions for \( E'_R I \) of
Equations (36) and (37) must be equal. It follows then that \( h \) is related to \( d \) through the equation

\[
h = \frac{\sqrt{6}}{3} \frac{\sqrt{1 + \frac{B_1}{B_2}}}{1 + \sqrt{\frac{B_1}{B_2}}} \ d
\]  

(38)

To compute \( h \), it is necessary to evaluate \( B_1 \) and \( B_2 \). To illustrate this computation, let \( f(\sigma, \dot{\sigma}) \) for the "loading" region of the cross section be

\[
\dot{\varepsilon} = \frac{\dot{\sigma}}{E} - m \frac{\sigma}{\mu} \dot{\sigma} + n \frac{\sigma}{\lambda} \dot{\sigma}
\]

when \( m \) and \( n \) are odd integers.

Then

\[
B_2 = f(\sigma_2, \dot{\sigma}_2) = - m \frac{\sigma_2}{\mu} \dot{\sigma}_2 + n \frac{\sigma_2}{\lambda} \dot{\sigma}_2
\]  

(39)

For the "unloading" region of the cross section, let

\[
\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\lambda} \dot{\sigma}
\]

and

\[
B_1 = f(\sigma_1, \dot{\sigma}_1) = \frac{n}{\lambda} \left( \frac{\sigma_1}{\lambda} \right)^{n-1}
\]  

(40)
Since for the conditions of loading $B_1 < B_2$, it would follow that as bending proceeded, $h$ would increase. This result is analogous to that obtained in the development of Equation (17).

The assumptions and thus limitations of the preceding analysis preclude its application for obtaining quantitative results. Its primary value lies in illustrating the fact that the time-dependent behavior has certain features that are common to the time-independent behavior. It may be expected, therefore, that two effects may be operative in a comparison of a rectangular cross-section and a two-flange cross section. For one type of nonlinearity, the rate of change of stress with respect to the distance from the axis of bending decreases with increasing distance. For this case, the equivalent flange area moves inward, tending to reduce the flange separation. A second type of nonlinearity results from a difference in material response on either side of the axis of bending. This causes a shift of the axis of bending away from the centroidal axis, and results in an increase in the effective flange separation.

The above analysis cannot be applied directly to the column creep-buckling problem to obtain quantitative results. The results obtained should, however, prove to be of value in interpreting the experimental results to be presented in the next section.
Column-Creep Experiments

The equipment and procedures used in conducting the column-creep experiments were the same as those used for the column-inspection experiments (see Figure 8) with one exception. There were no interruptions during these experiments, i.e., the column load remained constant throughout.

Two series of experiments were conducted. In the first series, the column specimens with a slenderness ratio of 75 were used, and all of the adjusted values of column imperfection were of the same order. Values of column load were selected to provide the variation of capacity with time to failure or collapse for the given value of imperfection.

In the second series of experiments, the column specimens with a slenderness ratio of 60 were used. For this series, however, three distinct values of imperfection were obtained. The three groups of imperfections within the series were of different orders of magnitude, and they were designed to provide a description of the variation of capacity behavior with imperfection.

A summary of the data for the two series of experiments are presented in Table 3.
<table>
<thead>
<tr>
<th>Specimen</th>
<th>Average Stress (psi)</th>
<th>$y_o \left(\frac{L}{2}\right) \times 10^{-3}$ inch</th>
<th>Critical Time (minutes)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>First Series</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>75-5</td>
<td>-6800</td>
<td>4.5</td>
<td>300</td>
</tr>
<tr>
<td>75-7</td>
<td>-6800</td>
<td>7.5</td>
<td>140</td>
</tr>
<tr>
<td>75-8</td>
<td>-7610</td>
<td>7.5</td>
<td>86</td>
</tr>
<tr>
<td>75-9</td>
<td>-7610</td>
<td>4.5</td>
<td>72</td>
</tr>
<tr>
<td>75-10</td>
<td>-8650</td>
<td>7.5</td>
<td>6</td>
</tr>
<tr>
<td>75-11</td>
<td>-7610</td>
<td>7.5</td>
<td>32</td>
</tr>
<tr>
<td><strong>Second Series</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>60-7</td>
<td>-8730</td>
<td>1</td>
<td>84</td>
</tr>
<tr>
<td>60-8</td>
<td>-9520</td>
<td>1</td>
<td>21</td>
</tr>
<tr>
<td>60-5</td>
<td>-7950</td>
<td>7</td>
<td>42</td>
</tr>
<tr>
<td>60-11</td>
<td>-6940</td>
<td>7</td>
<td>240</td>
</tr>
<tr>
<td>60-6</td>
<td>-5260</td>
<td>50</td>
<td>224</td>
</tr>
<tr>
<td>60-10</td>
<td>-5780</td>
<td>50</td>
<td>106</td>
</tr>
</tbody>
</table>
To illustrate more readily the trend of the results, the data points for the two series of experiments have been plotted in Figures 17 and 18. In terms of reproducibility, it is more revealing to examine Figure 17 than Figure 18, because there are more data points per imperfection. Although the data in Figure 17 appear to exhibit a reasonable trend in terms of increasing failure time with decreasing average stress, a more detailed examination reveals that some scatter or lack of reproducibility is present. Specimens 75-8, 75-9, and 75-11 in particular illustrate this fact. Thus, though a general trend can be discerned, the data taken together, much like fatigue data, are capable only of indicating the probable average stress versus failure time relationship. Individual experiments, considered alone, do not possess a high degree of reliability. This is not surprising, since the column-creep experiments are, compared for example to standard tensile tests, a very complex, and involve a number of variables which are difficult to control precisely.

The curves of Figures 17 and 18 are the results of an application of the analysis presented in an earlier section. These results will be discussed in the next section.
Figure 17. Average Stress versus Failure Time for Slenderness Ratio of 75.
Figure 18. Average Stress versus Failure Time for Slenderness Ratio of 60.
Calculated and Experimental Results

An Application of the analysis yielding Equation (33) of a previous section will be made in this section. The constants necessary for these computations are presented on page 104. Computations making use of the two limiting strain-rate relations illustrated in Figure 12 b and 12c have been performed.

As noted in the section on the use of the two-flange model during inspections, computations of the critical deflection should be corrected to allow for a possible change in the effective flange separation. To avoid this problem, times in Equation (33) have been computed for a deflection of \( y(\frac{L}{2}) = 0.10 \) inch. A review of the column-creep experiment records reveals that the times corresponding to this value of deflection are very close to the actual collapse time. A value of flange separation, \( h = \frac{\sqrt{3}}{3} d \), where \( d \) is the depth of the rectangular cross section, was used.

The computed limit curves for the slenderness ratio of 75 are plotted in Figure 17. For the computations a value of \( y_o(\frac{L}{2}) = 0.0075 \) inch was used. A comparison of the experimental results and the computed curves reveals that in all but one instance the experimental points are below the lower curve. The one exception is for Specimen 75-10 (see Table 3) which lies between the computed curves.

A review of the various aspects of the problem indicates two factors which could contribute to the trend observed. One lies in a
comparison of the creep data and the creep expression used, and the
other originates with the effective flange-separation value used in the
computation.

In discussing the creep expression, it is helpful to refer to
Figure 14 which served as a basis for the derivation of the creep-reла­tion constants. From Figure 14, it can be seen that for data below
about -8000 psi, the line underestimates the creep strain rate. From
Table 3, it can be seen that with one exception, the level of average
stresses is significantly below -8000 psi.

It should be noted that part of the behavior observed for Speci­men 75-10 no doubt stems from the fact that upper-limit calculation
is most conservative for times that are short relative to the time, $t_1$,
of Figure 12b. In the computation, $t_1 = 300$ minutes and the failure
time involved was 6 minutes. The effect to be expected from this is
apparent from Figure 12b.

The effect that the strain-rate relation might have on the com­
putational results of Figure 17 can be estimated rather simply by
inspection of Equation (33). It is seen that the creep-rate constant $\lambda$
is raised to the power $n$ in both terms which involves constant $C$. It
does not appear elsewhere in the relation. A change in $\lambda$ will result
in a movement of the solid line of Figure 14 to the right or left, but
parallel to the line shown. (It is assumed that $n$ remains constant in
these considerations). If the value of $\lambda$ is decreased from $5.3 \times 10^4$ psi
to a value of $\lambda' = 4.86 \times 10^4$ psi, the line will pass through the point having a stress of -7000 psi. Although this second line does not represent the data values for the points for higher stresses, it could be expected to provide an improved representation for stresses in the neighborhood of -7000 psi.

If only the value of $\lambda$ is changed to $\lambda'$ in Equation (33), it is possible to obtain an estimate of the error in time that could be associated with the use of the value of $\lambda$. This computation has been performed and it has been found that the times obtained using $\lambda'$ will be 60 per cent of the times obtained using $\lambda$. Reducing the lower-bound-curve time values by this amount yields the dashed curve of Figure 17. It is seen that now only two of the six column data points lie below the dashed curve; however, it appears that the modified estimate has not provided all of the correction that is required. This suggests that there may be some error associated with the assumed flange separation; i.e., the effective flange separation is apparently less than the assumed value of $h = \frac{\sqrt{3}}{3} d$.

The experiments conducted on the column specimens with a slenderness ratio of 60 were designed to indicate the effect of imperfection on column capacity. Although the two experiments conducted for each imperfection do not permit an accurate deduction of the capacity versus failure-time curve for any of the three values of
im perfection, they do indicate the change in the level of the corresponding capacity curves. This was the objective of this series of experiments.

The computations performed to obtain the curves of Figure 17 suggested that the lower bound estimate tends to be above the experimental points. In performing the corresponding calculations for the data of Figure 18, it was found that the same trend persisted. To limit the number of curves in Figure 18, therefore, it was felt sufficient to include only the lower bound estimate results.

An examination of Figure 18 reveals that each pair of data points is below its corresponding computed curve. The computed lower bound is not, therefore, an actual lower bound.

It will be noted in Figure 18 that the difference between the computed curves and the corresponding experimental points increases as the imperfection increases. Since the stress level decreases as the imperfection increases, the previous discussion regarding the error in the strain-rate relation can be expected to apply for the lowest curve. The level of stresses for specimens 60-10 and 60-6 is, in fact, even lower than those referred to previously in Figure 17. An even more drastic correction than that applied previously may, therefore, be expected.

The creep-rate relation used should be more accurate at the higher stress levels. The experimental points are still below the computed
curve, however, so it appears that for the imperfections of 0.007 inch and 0.001 inch, the effective flange separation is less than the assumed value of $h = \frac{\sqrt{3}}{3} d$.

Although the magnitudes of the computed values are in error, the decrease in capacity with increasing imperfection is reflected in the computed curves. To appreciate the value of this property, and to help assess the results in terms of the level of development of this area of analysis, the alternative methods of analysis proposed by Shanley (15) and Gerard (16) have been performed. It should be noted that each of these theories yields a formula which includes a geometric description of the actual cross section. For a hinged-end column of length, $L$, the critical stress at time, $t$, can be written

$$\sigma_c(t) = \frac{\pi^2 E(t)}{\left(\frac{L}{r}\right)^2}$$

where $E(t)$ is the appropriate time-dependent modulus, and $r$ is the radius of gyration of the column cross section.

Inspection of this formula reveals that there is no means of introducing the effect of column imperfection. For a given material, temperature, and slenderness ratio, a single curve will be obtained.

The curves for these two theories, Shanley's time-dependent tangent modulus and Gerard's time-dependent secant modulus, are presented in Figure 18. For times greater than about 60 minutes, the
curve for Shanley's theory is close to the data and the lower bound
curve for the smallest imperfection. For the larger imperfections,
however, the time-dependent tangent modulus curve is quite noncon­servative.

The curve for Gerard's theory, being considerably above all of
the other results, is grossly nonconservative. For imperfections less
than those considered here, it could, of course, be expected to yield
an improved correlation.

A comparison of the results for each of the theories is presented
is Table 4. The percentage error is based on the experimental values.

An examination of the results of Table 4 indicates that in spite
of the fact that the present theory utilizes an idealized cross section,
the results obtained reflect more accurately the experimental behav­ior observed than those obtainable with the theories of Shanley and
Gerard. The use of Equation (33), thus, provides an improved method
for estimating the capacity of columns subject to creep.

At this point, it would be of interest to discuss what are the most
effective means of applying this type of analysis. A review of the re­sults obtained indicates that the applicability of the present analysis is
best when imperfections are small. Hence, one approach would be to
make certain that workmanship, both in fabrication and installation,
be closely controlled to achieve small imperfections.
### Table 4. Comparison of Theoretical Capacities

<table>
<thead>
<tr>
<th>Specimen</th>
<th>Experimental psi</th>
<th>Present Theory psi</th>
<th>Error per cent</th>
<th>Shanley's Theory psi</th>
<th>Error per cent</th>
<th>Gerard's Theory psi</th>
<th>Error per cent</th>
</tr>
</thead>
<tbody>
<tr>
<td>60-7</td>
<td>-8370</td>
<td>-9450</td>
<td>8</td>
<td>-8850</td>
<td>1</td>
<td>-11800</td>
<td>35</td>
</tr>
<tr>
<td>60-8</td>
<td>-9520</td>
<td>-9910</td>
<td>4</td>
<td>-10500</td>
<td>10</td>
<td>-14900</td>
<td>57</td>
</tr>
<tr>
<td>60-5</td>
<td>-7950</td>
<td>-9160</td>
<td>15</td>
<td>-9700</td>
<td>22</td>
<td>-13200</td>
<td>66</td>
</tr>
<tr>
<td>60-11</td>
<td>-6940</td>
<td>-8100</td>
<td>17</td>
<td>-8530</td>
<td>23</td>
<td>-10800</td>
<td>56</td>
</tr>
<tr>
<td>60-6</td>
<td>-5260</td>
<td>-6420</td>
<td>22</td>
<td>-8950</td>
<td>70</td>
<td>-10800</td>
<td>110</td>
</tr>
<tr>
<td>60-10</td>
<td>-5780</td>
<td>-6880</td>
<td>19</td>
<td>-8600</td>
<td>49</td>
<td>-11500</td>
<td>99</td>
</tr>
</tbody>
</table>
Also, in the construction of capacity versus lifetime curves, the
deflection corresponding to collapse should not be used as a basis for
computation. Limiting deflections considerably smaller should be
used. As shown in the first section of the dissertation, the uncer-
tainty of the applicability of the model analysis increases with increas-
ing deflection. This uncertainty can, however, be decreased by con-
fining limiting deflections to smaller values.

In an actual structure, it is quite probable that the built-in im-
perfection can only be estimated. Although it should be possible to
estimate the order of magnitude of the imperfection, there is likely
to be some degree of uncertainty associated with the estimate. A
possible procedure for computing the capacity in such cases would be
to use an imperfection value, which is one order of magnitude greater
than the expected values. From Figure 18, it can be seen that this
procedure would be satisfactory, i.e., curves based on imperfection
values 10 times greater than the actual imperfection would provide a
margin of safety in terms of capacity.

The final topic to be discussed will be the selection of a factor
of safety. In nearly all design work, some form of a factor of safety
is used. This may consist of exaggerating the expected loads, or it
may involve the selection of cross sections which will assure that the
stresses are a fraction (for example, two-thirds) of the limiting values.
In an area such as creep buckling, the raw data that are fed into an analysis (creep data) are not likely to possess the accuracy of the elastic modulus in more standard types of design. Also, the methods of analysis available are not as satisfactory as those which are available through elasticity. It seems essential, therefore, that some form of a factor of safety be utilized in this type of design.

Three types of analysis have been discussed here: Shanley's theory, Gerard's theory, the analysis developed in this investigation. The type of safety factor which would be required for each of these methods of analysis can be anticipated by examining the results in Table 4. The safety factor that would be necessary to make Gerard's theory acceptable for any but the very smallest imperfections would have to be very large. Shanley's theory could, on the other hand, be used in conjunction with acceptable factors of safety for the two smallest imperfections. The factor of safety necessary for the largest imperfection would have to be large.

In contrast to the foregoing theories, the analysis developed here could be used in conjunction with acceptable factors of safety for all three values of imperfection. This difference reflects, of course, the fact that the effect of the value of imperfection has been included in the analysis.

In summary, the results presented in Figure 18 indicate, when interpreted in terms of the factors introduced in the computations, that
the analysis yielding Equation (33) provides a simple and satisfactory procedure for computing the capacity of columns subject to creep. It may also be added that in contrast to several somewhat simpler proposed methods (15, 16), the present analysis includes the effect of column imperfection, and as a consequence, provides a more satisfactory description of column-creep behavior over a range of imperfections.
SUMMARY AND CONCLUSIONS

The analysis of structures exposed to elevated temperatures has been a design problem for many years. Exposure to such conditions can create a number of problems. Corrosion processes, for example, are accelerated. Metallurgical changes which can cause embrittlement may occur. Time-dependent deformation or creep under load may be introduced. The investigation described in this dissertation considers a problem that arises when creep effects become important.

The problem studied is that of a column subject to creep under a constant load. The work described is divided into two sections. In the first section, conditions necessary for buckling and methods for inspecting the stability are covered. In the second section, an analysis of the deflection history of a column creeping under a constant load is made.

The first section begins with a discussion of procedures for examining the stability of a creeping column. The studies of Rabotnov and Shesterikov and those of Fraeijs de Veubeke are interpreted as examples of two essentially different possible procedures. The procedure of Rabotnov and Shesterikov involves a dynamic inspection, and it is concerned with the nature of the column response after a
disturbance is administered. The procedure of Fraeijs de Veubeke is static, and it is concerned with the nature of the column response developed as a disturbance is administered.

An alternative, static inspection procedure involving sudden load increases is proposed, and it is noted that the type of column response predicted is actually observed for columns whose deformation response is typical of that for metals, i.e., elastic, plastic, and creep components of strain are present. It is then observed that since the agreement between the postulated response and the experimental response is qualitative, it would be desirable to perform an analysis which demonstrates the response analytically. After reviewing the problem, it is concluded that an analysis which incorporates a realistic creep-rate law (elastic, plastic, and creep components) can be most effectively accomplished by the use of the two-flange cross-section column model.

An analysis, starting with an initially imperfect column, is performed, and it is shown that the response postulated earlier can be given an analytical representation. The response of the column to an inspection is expressed as the rate of change of the average stress with respect to the deflection. Failure or collapse occurs when this expression becomes zero. Since the expression developed
reflects the column stiffness at the time of the inspection, a value of zero simply indicates that the column has lost the capacity to resist further bending.

The stiffness expression developed is useful because it provides a means of evaluating the properties of a column prior to collapse. It should be of practical interest also because it provides a means of specifying a limiting deflection based on an allowable reserve of stiffness.

Since the strain-rate law used in the analysis contained elastic, plastic, and viscous components, the results can be used to determine the properties of columns made of a variety of materials. In particular, results for a purely viscous material and a viscoelastic material are shown to differ fundamentally from those for the general material (elastic, plastic, and viscous). For the general material, the inspection stiffness decreases with increasing deflection and becomes zero at a finite or critical value of deflection. The inspection stiffness for the viscoelastic stiffness decreases with increasing deflection, but approaches zero only as the deflection becomes unbounded. For the purely viscous material the column stiffness is infinitely rigid for an instantaneous inspection.

In the theory developed, the strain-rate law tacitly implies that creep which occurs during the course of a column lifetime does not modify the plastic or strain-hardening characteristics of the column
material. Since data in the literature indicate that this assumption may not be correct, experiments were conducted to study such effects on the aluminum alloy 7075-0, the material used in the experimental program. The results obtained are in agreement with those available in the literature and they indicate that the rate of strain hardening increases with precreep. This result indicates that the stiffness predicted by the column-inspection theory should be less than that which would be observed in an actual column because the rate of strain hardening of the material after creeping should be greater than that of the uncrept or virgin material. The stiffness estimate should, therefore, be conservative.

The analysis developed makes use of the idealized two-flange column model. The relation of the results obtained to more general cross sections is, however, of interest. As a consequence, the particular case of the relation of results for a rectangular cross section to those for the column model are investigated. As a first approximation, it is shown that for a rectangular cross section whose depth is d, the effective flange separation for the equivalent model may be taken as \( h = \frac{\sqrt{3}}{3} d \). From a consideration of the nature of the stress distribution acting on the rectangular cross section, it is deduced that this should be a good approximation for deflections less than the critical deflection.
To evaluate the theory upon which the inspection procedure proposed is based, interrupted column-creep experiments were conducted. Experiments were conducted at 325° F on aluminum alloy 7075-0 columns with slenderness ratios of 60, 75, and 90. The results of the experiments are analyzed by the use of the theory developed, and it is shown that the instantaneous stiffness can be estimated successfully. In addition, a systematic appraisal of the factors which can be expected to influence the results is made. It is concluded that the replacement of the solid rectangular cross section by the two-flange cross section has considerably less influence than the effect of creep on the strain-hardening characteristics of the material.

The second section of the dissertation is concerned with an analysis of a column creeping under a constant load. In formulating a strain-rate law for the analysis, an attempt is made to include elastic, plastic, and creep components of strain rate. Since the creep rate is, in general, a function of time, the use of a "steady-state" term only can be expected to result in significant error. To retain the simplicity of this approach, and to provide an estimate of the error in this type of representation, a method which provides upper and lower bounds to the actual behavior is proposed. It is shown that the method can be adjusted to the time period of interest, and can, therefore, result in considerably less error than less flexible, available methods.
In considering the column-creep problem, it is noted that unloading occurs on part of the column cross section. Since this can be expected to result in creep recovery, a review of available literature on this subject was made. The results indicate that the amounts of creep recovery depend on the experimental condition and the material. To evaluate creep-recovery effects which could be expected in this program, an experiment on the aluminum alloy 7075-0 at 325°F, in which a uniform compressive stress is decreased in increments, was performed. The amount of recovery observed was appreciable so it was concluded that the influence of creep recovery should be considered in the analysis.

A consideration of the influence of creep-recovery effects on column creep reveals that if creep recovery is ignored, the analysis will be nonconservative. That is, the predicted deflection will be less than the actual deflection. Since creep recovery would be difficult to include in the analysis, it is proposed that only elastic behavior be considered in the unloading region. This, in effect, implies that all creep is recovered. It is shown that this should result in a conservative analysis.

An analysis based on the above considerations is applied to the two-flange column model, and an explicit relation for time in terms of deflection with average stress, column imperfection, slenderness
ratio, and material properties as parameters is developed. Use of
the derived relation with upper or lower bound estimates for creep
strain yields bounds for column-creep behavior.

The relation of the results of the column-creep analysis on the
two-flange cross section to more general columns is discussed. It is
shown, by consideration of a rectangular cross section, that the con­
siderations which apply are similar to those encountered in the same
comparison for instantaneous column properties. It is concluded,
however, that since the analysis cannot be used to obtain quantitative
results, comparisons must be based on experimental results.

The results of column-creep experiments at 325° F on aluminum
alloy 7075-0 columns with slenderness ratios of 60 and 75 are pre­
sented. An application of the analysis developed in this dissertation
and those of Shanley and Gerard are made and compared with the ex­
perimental results. It is shown that only the present theory provides
satisfactory estimates for all imperfections. This results from the
fact that the present analysis includes the effect of imperfection,
whereas, the analyses of Shanley and Gerard do not. It is concluded
that in terms of the factors introduced in the computations, the analy­
sis proposed provides a relatively simple and satisfactory procedure
for computing the capacity of columns subject to creep.
REFERENCES


I, Robert Lee Carlson, was born in Gary, Indiana, May 22, 1924. I received my secondary-school education at Emerson School in Gary, and my undergraduate schooling at Purdue University from which I received a Bachelor of Science degree in Mechanical Engineering in 1948. In 1950 I received a Master of Science degree in Engineering Mechanics from Purdue University. I enrolled as a graduate student at The Ohio State University for the Spring Quarter of 1958. Since that time, I have been completing the requirements for the Doctor of Philosophy degree while working at the Battelle Memorial Institute.