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DIFFERENCE EQUATIONS AND ELASTIC PLATES

DISSERTATION
Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
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By
Raymond Leo Foye, B.S.

*****

The Ohio State University
1963

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INTRODUCTION

There is considerable interest in the problem of obtaining the static deflections, vibration characteristics, and bending stresses of variable thickness cantilevered plates with complex boundary curves since there are many instances of the use of this type of structure for missile wings. Several analytical methods have appeared in recent years (1) but most of these were unsuitable as engineering tools for one or more of the following reasons:

1. Logic too complicated,
2. Undue labor involved in their application, or
3. Insufficient generality with regard to plate configuration and thickness variation.

The analysis presented is an approximate one but avoids these three pitfalls.

The method permits the immediate writing of the force-displacement relations in the form

\[ (F) = [A](\omega) \]

where \( (F) \) is a column of static loads acting at a predetermined set of points (called node points) and \( (\omega) \) is the column of corresponding vertical displacements of the plate at the same points. The node points are the points of intersection of the lines of a rectangular finite difference net superposed on the middle plane of the plate.
\([A]\) is not a conventional stiffness matrix, but the latter may be obtained from \([A]\) via a few algebraic operations. The dimension of \([A]\) is generally so great that the electronic digital computer must be relied upon to solve (Eq. 1).

The method, as presented, is subject to the following assumptions and limitations:

1. The plate is thin,
2. The thickness gradient is small,
3. The middle surface is initially plane,
4. The root boundary curve is a straight line,
5. The material is linearly elastic,
6. The deflections are small,
7. Shear deflections are negligible, and
8. There are no resultant forces acting in the plane of the plate.

The first successful attempts at the structural analysis of cantilevered plates followed the Rayleigh-Ritz procedure (3). This method consists of expanding the unknown transverse displacement function in a series of linearly independent functions, each satisfying the root boundary conditions. The coefficients of the series are determined from the condition that the total potential energy be a minimum. The great advantage of the method lies in the fact that it dispenses with the necessity of dealing with the complicated free edge boundary conditions which are automatically satisfied in the minimal process. Unfortunately, considerable labor must be expended to obtain an accurate solution.
Recently, some classical examples, such as the uniformly thick square plate clamped on one edge, have been treated by collocation (21, 22). The method presupposes either the knowledge of a series of solutions to the differential equation for which the first $n$ coefficients are chosen so that the $n$th partial sum satisfies both boundary conditions at $n$ discrete boundary points or the possession of a series of independent functions each satisfying the boundary conditions for which the $n$ coefficients of the $n$th partial sum are chosen so as to satisfy the differential equation at $n$ particular points within the boundaries. The conditions on either option present some difficulties with plates of varying thickness and irregular boundaries.

Another general approach which has met with some measure of success is the finite difference method. Due to the rapid development of computing machines and the good convergence properties of the method a great deal of valuable information on a wide class of boundary value problems has been obtained in recent years. The first known application of the method to elastic plate problems was due to Marcus (5) who obtained approximate solutions to the static deflection of several simply supported rectangular plates. An account of the method appears in reference 6. The same method was later applied to cantilevered plates by Nash (7). Another application of the finite difference method to the determination of the deflections of rectangular cantilevered plates was reported by Livesley and Birchall (8). Their traditional approach of replacing derivatives appearing in the classical plate equation and the appropriate boundary conditions with suitable finite difference approximations of fourth and lower order is difficult to
extend to non-rectangular regions. The difficulty arises when the free boundary of the plate cuts a line of the finite difference net at other than a node point. To enforce the boundary conditions at such an edge it is necessary to use some interpolation procedure to approximate the derivatives of the displacement function between node points. At present, there appears to be no simple, accurate, and unique way of doing this. Nevertheless, solutions to similar problems have been obtained by people with considerable experience in finite difference methods using interpolation. For a more complete account see reference 9.

There have been two recent significant efforts to circumvent the difficulties caused by irregular boundary curves within the framework of the finite difference method. The one considered first is due to Williams (2). This method avoids interpolation by altering the boundary curve so that it does coincide with the lines of the finite difference net thus avoiding the crossing of net lines between node points. The method is general and simple. However, it does not seem to be held in high regard due to the gross alterations of the region that often result and the accompanying loss of accuracy.

The second method, due to Fung (11), is a more subtle approach based on Southwell's earlier formulation of the plate bending problem (12) in terms of two unknown stress functions related through a system of simultaneous second order differential equations. The boundary conditions on the stress functions at the free edges involve only the values of the stress functions themselves and the known edge forces. The two root boundary conditions involve linear combinations of the first partial
derivatives of the stress functions. This type of boundary value problem is readily solved over irregular regions by a simple technique of unequal mesh lengths (9).

The method of Fung is not without its drawbacks. First of all, it is primarily a structural type of analysis which does not yield values of the plate displacements as a direct output. The displacements must be obtained by numerical integrations of the curvatures. Second, there is a computational penalty incurred by the introduction of two stress functions. This considerably increases the number of unknowns and hence the size of the resulting system of difference equations.

Excepting the difficulties encountered at the free edges the finite difference approach would be unrivaled in simplicity. But this weakness of the finite difference method is precisely the forte of the Rayleigh-Ritz method (namely, the handling of the free edge boundary conditions) hence we are led to consider a union of the two techniques with the hope of preserving the desirable characteristics of both. The following section explores this possibility.
THE METHOD

To solve many complex problems in continuum mechanics it is necessary to replace an accurate mathematical model by a simpler one having only a finite number of degrees of freedom. One of the most widely known ways of accomplishing this is the finite difference method which replaces functions and their derivatives by algebraic expressions involving only the values of the functions at a finite number of points in or near the region or interval of interest. This replacement process generally involves some error, hopefully small, hence the expressions which replace the derivatives are called the finite difference approximations. The character of these approximations and their inherent errors will be considered first.

The fact that the difference quotient \( \frac{f(x_i) - f(x_o)}{x_i - x_o} \) approaches the value of \( f'(x_o) \) as \( x_i \) approaches \( x_o \) follows from the definition of the derivative. This quotient may be used to approximate the value of the derivative to any prescribed accuracy by selecting a sufficiently short interval \( |h| = |x_i - x_o| \). Assuming that all the derivatives of \( f \) are continuous an estimate of the sharpness of this approximation, for a given function and interval, may be obtained from Taylor's formula with Lagrange's form of the remainder term, i.e.

\[
(2) \quad f(x_i) = f(x_o) + \frac{f'(x_o)}{1!}(x_i - x_o) + \frac{f''(\overline{x})}{2!}(x_i - x_o)^2
\]

where \( \overline{x} \) lies between \( x_i \) and \( x_o \). From this formula it follows
that

\[ \frac{f(x_i) - f(x_0)}{x_i - x_0} - \frac{f'(x_0)}{2!} \leq \frac{M_2}{2!} |x_i - x_0| \]

where \( M_2 \) is the maximum value of \( |f''| \) in an interval containing \( x_i \) and \( x_0 \). Thus, the error incurred by replacing \( f'(x_0) \) by the finite difference approximation \( \frac{f(x_i) - f(x_0)}{x_i - x_0} \) is seen to be of the same order of magnitude as the distance \( h \). A more explicit statement of the error requires an estimate of the quantity \( M_2 \). Unfortunately, in most of the applications the form of \( f'' \), and hence the magnitude of \( M_2 \), is not known; while the prior evaluation of \( M_2 \) is usually of a degree of difficulty comparable to that posed by the original problem. What is worse, in many applications even the boundedness of \( f'' \) is not assured. The ideal situation would involve the possession of error bounds in terms of the interval length \( h \) and the values of \( f(x_i) \) and \( f(x_0) \). A check on the magnitude of the error would then be a relatively simple matter once a solution for \( f'(x_i) \) and \( f'(x_0) \) had been obtained. At present no such error bounds have been discovered and the prospects for their future revelation seem remote. This difficulty remains as the greatest drawback to the use of the method and a challenge to future investigators.

In summary, an algebraic expression - linear in a finite number of discrete values of a continuous dependent variable - has been obtained which approaches one of the derivatives of the dependent variable as the length of the shortest interval containing all of the corresponding values of the independent variable is made to approach zero. All subsequent difference approximations must be of this form and possess
this property to be so termed. In this particular instance, the explicit form of the difference expression was arrived at by recalling the definition of the derivative. Other more formal means are available for doing this. The widely accepted method uses polynomial interpolation to pass a curve of sufficiently high order through a number of neighboring ordinates. A difference approximation is then obtained by differentiating the interpolational polynomial, usually at some particular point on the shortest interval containing all the discrete independent variables. Lagrange's interpolation formulae is in an especially convenient form for these purposes (23). The order of the approximation, which is the justification for its use, is then established through Taylor's formula as demonstrated in the case of the first derivative.

Before any further discussion, consider an example of the application of these concepts to obtain a valid difference approximation to the second derivative of a function, $f''(x)$, at some point on the x axis in terms of the ordinate at that point and two other equidistant points.

Let $x_w + h = x_o = x_e - h$ where $h > 0$ and $\xi_w = f'(x_w)$, $\xi_o = f'(x_o)$, $\xi_e = f'(x_e)$. We seek an approximation of the form

$$f''(x_o) \approx a \xi_e + b \xi_o + c \xi_w$$

where $a$, $b$, $c$ depend only on $x_o$ and $h$. The unique second order polynomial through the points $(\xi_e, x_e)$, $(\xi_o, x_o)$, $(\xi_w, x_w)$ is

$$P(x) = \frac{(x-x_o)(x-x_w)}{(x_e-x_o)(x_e-x_w)} \xi_e + \frac{(x-x_e)(x-x_w)}{(x_o-x_e)(x_o-x_w)} \xi_o + \frac{(x-x_o)(x-x_e)}{(x_w-x_e)(x_w-x_o)} \xi_w$$
and

\[ P''(x_0) = \frac{2 \xi E}{(x_e - x_0)(x_e - x_W)} + \frac{2 \xi o}{(x_0 - x_e)(x_o - x_W)} + \frac{2 \xi w}{(x_W - x_0)(x_w - x_e)}. \]

Rewriting (Eq. 6),

\[ P''(x_0) = \frac{1}{h^2} \left( \xi E - 2 \xi o + \xi w \right). \]

To justify using the difference approximation \( \frac{1}{h^2} \left( \xi E - 2 \xi o + \xi w \right) \) in place of \( \xi''(x_0) \) consider Taylor's formulae

\[ \xi'' = \xi o + h \xi'(x_0) + \frac{h^2}{2!} \xi''(x_0) + \frac{h^3}{3!} \xi'''(x_0) + \frac{h^4}{4!} \xi''''(x) \]

and

\[ \xi'' = \xi o - h \xi'(x_0) + \frac{h^2}{2!} \xi''(x_0) - \frac{h^3}{3!} \xi'''(x_0) + \frac{h^4}{4!} \xi''''(x_\bar{x}) \]

where \( \bar{x}_1 \) and \( \bar{x}_2 \) both lie between \( x_e \) and \( x_w \). Summing both sides of these equations and adding \(-2\xi o\) to either side gives

\[ \frac{\xi E}{h^2} - 2 \xi o + \xi_W = \frac{h}{2!} \xi''(x_0) + \frac{h^4}{4!} \left\{ \xi''(\bar{x}_1) + \xi''''(\bar{x}_2) \right\} \]

or

\[ \frac{\xi E}{h^2} - 2 \xi o + \xi_W - \xi''(x_0) = \frac{h^2}{4!} \left\{ \xi''(\bar{x}_1) + \xi''''(\bar{x}_2) \right\} = \frac{2}{4!} M_4 \]

where \( M_4 \) is the maximum value of \( \xi'''' \) on \([x_e, x_w]\).

In words this equation tells us that the error in using the difference approximation approaches zero as does \( h^2 \) with diminishing \( h \).

Thus it is a valid, and indeed a rather accurate, representation of
\[ f^{(2)}(x_0) \] provided \( M_4 \) exists.

It is appropriate to mention in passing that difference approximations to a given derivative are not unique. These expressions change not only with the order or degree of the polynomial interpolation but also with the type of functions used as the basis for interpolation.

The time honored approach of polynomial based interpolation has only recently been subject to criticism from some sources (24) on the testimony that it has given poor results on certain classes of problems (for example, some singular differential equations). But, there seems to be no general trend toward alternative derivations, at least in the applications to elliptic equations which are the concern of this study. The gist of the criticism centers around the unending quest for accuracy. The question posed by the critics is this: since, in general, the coefficients of the first few terms of a series of any system of complete functions can be chosen such that the sum of the terms passes through a number of prescribed ordinates why not seek a system of functions which gives a closer difference approximation to a particular derivative than the powers of \( x \)? The answer to this question varies with the particular problem at hand. When considerable prior knowledge as to the character of the solution is available it is sometimes possible to base the interpolation scheme on a set of functions more closely related to the solution than the powers of \( x \). It warrants mention that the few attempts in this direction have not been marked with great success, although the evidence is too scanty to dismiss the question yet. Without some insight into the nature of the true solution, the traditional method seems to have the advantages of simplicity and, in most cases, suitable accuracy.
The polynomial interpolation method necessitates the utilization of a minimum number of $n+1$ ordinates where $n$ is the order of the derivative for which the approximation is sought. More accurate difference approximations may be expected from the introduction of additional ordinates in the interpolation scheme although no conclusive statement to this effect can be made. This qualifying remark is the result of the fact that the introduction of new ordinates usually is accompanied by an increase in the interval over which the derivative in Lagrange's remainder term may range. This could very likely force an increase in the upper bound which may overshadow any reduction in the order of the error (which is only dominant in a limit that is never achieved). For this reason, refined difference approximations are regarded as rather unreliable sources of increased accuracy. For the remainder of this study all difference approximations used will be the most elementary ones obtainable through polynomial interpolation.

The manner in which difference expressions are used to solve ordinary and partial differential equations is well known (9) and will not be considered here. Their usefulness in solving problems in variational calculus is not so well known (4) and will now be discussed.

In the theory of ordinary maxima and minima of a differentiable function $f(x_1, x_2, \ldots, x_n)$ of $n$ independent variables the necessary condition for achieving an extreme value of $f$ in a region of the independent variables is

$$
(\text{II}) \quad \frac{\partial f}{\partial x_i} = 0, \quad (i = 1, 2, \ldots, n).
$$
These equations express the stationary character of the function $f$ at the point in question. Whether the stationary point is a maximum, minimum, or neither can only be decided by further investigation.

The calculus of variations is likewise concerned with the problem of extreme values (or stationary values). Here, however, we have to deal with a new situation, for now the functions no longer depend on a finite number of independent variables but on a whole range of ordinates of some unknown function. Their values cannot be precisely determined by stating the values of a finite number of independent variables; hence they cannot be regarded as functions in the ordinary sense (they are called functionals).

A typical problem in variational calculus might be that of finding the function $y(x)$ taking on the values $y(x_0) = \alpha$, $y(x_n) = \beta$, and minimizing an integral of the form

$$\int_{x_0}^{x_n} F(x, y, y') \, dx.$$  \hspace{1cm} (12)

If such a function uniquely exists the calculus of variations provides a formal procedure for obtaining it. What it does is provide a second order differential equation which $y(x)$ must satisfy along with the conditions $y(x_0) = \alpha$, $y(x_n) = \beta$ if $y(x)$ is to minimize $\int$. This differential equation is called the Euler-Lagrange equation corresponding to the variational problem. Let us now consider a procedure whereby we may obtain an approximation to the solution of this variational problem.
If we divide the interval of integration, $X_0 \leq x \leq X_n$, into $n$ equal parts of length $h = \frac{X_n - X_0}{n}$ by inserting $n-1$ points of division, $X < X_2 < \cdots < X_{n-1}$, then the integrand of (Eq. 12) may be approximated by the step function

$$F\left(\frac{X_i + X_{i-1}}{2}, \frac{Y_i + Y_{i-1}}{2}, \frac{Y_i - Y_{i-1}}{h}\right), \quad (X_{i-1} \leq x \leq X_i)$$

where the subscripts refer to the values of $X$ and $Y$ at the endpoints of the subintervals. The ordinates of the step function represent the values of $F$ obtained by using the average values of $X$ and $Y$ on each $h$ interval and the difference approximation $\frac{Y_i - Y_{i-1}}{h}$ in place of $Y'$. By integrating the step function $F$ over the interval $X_0 \leq x \leq X_n$, the variational problem may be replaced by the ordinary minimal problem

$$\min \sum_{i=1}^{n} h F\left(\frac{X_i + X_{i-1}}{2}, \frac{Y_i + Y_{i-1}}{2}, \frac{Y_i - Y_{i-1}}{h}\right), \quad Y_0 = a, Y_n = b.$$

In all of the cases to be considered here, the above sum will be a polynomial, $P$, in the unknowns, $Y_1, Y_2, \cdots, Y_{n-1}$. A solution may be obtained from the $n-1$ conditions $\frac{\partial P}{\partial Y_i} = 0$. As a concrete example, consider the problem of the minimum of the integral

$$I[Y] = \int_{a}^{b} \left\{ (Y')^2 + c Y \right\} \, dx, \quad Y(0) = Y(1) = 0.$$

First divide the interval $0 \leq x \leq 1$ into four subintervals of equal length as shown in Fig. 1.
Fig. 1 -- Example Problem

From (Eq. 13), referring to Fig. 1

$$\begin{align*}
(Y')^2 + cY &= \left\{ \begin{array}{ll}
16 \left( Y_1 - Y_0 \right)^2 + \frac{c}{2} \left( Y_1 + Y_0 \right), \\
& \quad (X_0 \leq X \leq X_1) \\
16 \left( Y_2 - Y_1 \right)^2 + \frac{c}{2} \left( Y_2 + Y_1 \right), \\
& \quad (X_1 \leq X \leq X_2) \\
16 \left( Y_3 - Y_2 \right)^2 + \frac{c}{2} \left( Y_3 + Y_2 \right), \\
& \quad (X_2 \leq X \leq X_3) \\
16 \left( Y_4 - Y_3 \right)^2 + \frac{c}{2} \left( Y_4 + Y_3 \right), \\
& \quad (X_3 \leq X \leq X_4) 
\end{array} \right. \\
\end{align*}$$

From (Eq. 14),

$$\begin{align*}
P(Y_0, Y_1, \ldots, Y_4) &= 4 \left\{ \left( Y_1 - Y_0 \right)^2 + \left( Y_2 - Y_1 \right)^2 + \left( Y_3 - Y_2 \right)^2 \\
& \quad + \left( Y_4 - Y_3 \right)^2 \right\} + \frac{c}{8} \left\{ \left( Y_1 + Y_0 \right) + \left( Y_2 + Y_1 \right) + \left( Y_3 + Y_2 \right) \\
& \quad + \left( Y_4 + Y_3 \right) \right\}.
\end{align*}$$
Simplifying and enforcing the constraints $y_0 = y_4 = 0$ gives

\[ P(y_1, y_2, y_3) = 8[y_1^2 + y_2^2 + y_3^2 - y_1 y_2 y_3] + \frac{c}{4}(y_1 + y_2 + y_3). \]

Carrying out the minimization gives

\[
\begin{align*}
\frac{\partial P}{\partial y_1} & = 16 y_1 - 8 y_2 + \frac{c}{4} = 0, \\
\frac{\partial P}{\partial y_2} & = 16 y_2 - 8 y_1 - 8 y_3 + \frac{c}{4} = 0, \\
\frac{\partial P}{\partial y_3} & = 16 y_3 - 8 y_2 + \frac{c}{4} = 0.
\end{align*}
\]

The solution of this system is

\[ y_1 = y_3 = -\frac{3}{16} c, \quad y_2 = -\frac{1}{16} c. \]

These are the precise values of the analytic solution $y = \frac{c}{4} (x^2 - x)$, at the points $x_1, x_2, x_3$.

Using this same example, let us consider how the finite difference approach to a variational problem may be used to derive a difference equation corresponding to the Euler equation of the problem without prior reference to the form of the Euler equation. (This was, in fact, the method used by Euler to derive the differential equation.)

Divide the interval $[0, 1]$ into $n$ equal subintervals of length $\frac{1}{n}$ by inserting $n-1$ points of division, $x_1 < x_2 < \cdots < x_{n-1} < x_n$. (Eq. 14) approximates the integral 15 by

\[
P(y_1, y_2, \ldots, y_{n-1}) = \left\{ n(y_i)^2 + \frac{c}{2n} (y_i) \right\} + \cdots
\]

\[ + \left\{ n(y_i - y_{i-1})^2 + \frac{c}{2n} (y_i + y_{i-1}) \right\} + \left\{ n(y_{i+1} - y_i)^2 + \frac{c}{2n} (y_{i+1} + y_i) \right\} + \cdots + \left\{ n(y_{n-1} - y_n)^2 + \frac{c}{2n} (y_{n-1} + y_n) \right\}. \]
The two middle terms on the right-hand side of (Eq. 21) are the only ones involving $\gamma_i$, therefore,

$\frac{\partial P}{\partial \gamma_i} = \frac{c}{n} - 2n(\gamma_{i+1} - 2\gamma_i + \gamma_{i-1}) = 0$

or

$\frac{\gamma_{i+1} - 2\gamma_i + \gamma_{i-1}}{\left(\frac{1}{n}\right)^2} = \frac{c}{2}, \quad (\gamma_i \neq \gamma_1, \gamma_{n-1})$.

(Eq. 23) is the standard difference equation corresponding to the Euler equation of the problem, $2\gamma'' = c$.

The form of the difference equation is altered somewhat when $\gamma_i = \gamma_1$ or $\gamma_{n-1}$, i.e., when $\gamma_i$ is adjacent to a point where $\gamma(x)$ is constrained. Referring to (Eq. 21) we find that

$\frac{\partial P}{\partial \gamma_i} = \frac{c}{n} + 4n\gamma_i - 2n\gamma_2 = 0$

or

$\frac{\gamma_2 - 2\gamma_i}{\left(\frac{1}{n}\right)^2} = \frac{c}{2}$

and

$\frac{\partial P}{\partial \gamma_{n-1}} = \frac{c}{n} + 4n\gamma_{n-1} - 2n\gamma_{n-2} = 0$

or

$\frac{-2\gamma_{n-1} + \gamma_{n-2}}{\left(\frac{1}{n}\right)^2} = \frac{c}{2}$.

These same two equations could have been obtained by writing difference equation 23 at the points $X_1$, $X_{n-1}$ and then enforcing the constraints $\gamma_0 = \gamma_n = 0$. This latter procedure is valid for this particular example but cannot be adopted as a general procedure. As a counter example to its general use consider the simple problem:

$P = \gamma_i^2 + \gamma_i \gamma_2 + \gamma_2^2 - c\gamma_i = \text{min. with } \gamma_2 = -\gamma_i$. 
Enforcing the constraint and then minimizing $P$ gives the correct result:

$$P = \gamma_i^2 - c \gamma_i, \quad \frac{\partial P}{\partial \gamma_i} = 2 \gamma_i - c = 0; \quad \gamma_i = -\gamma_2 = \frac{c}{2}.$$

Minimizing $P$ prior to enforcing the constraint gives the false results:

$$\frac{\partial P}{\partial \gamma_i} = 2 \gamma_i + \gamma_2 - c = 0, \quad \gamma_i - c = 0; \quad \gamma_i = -\gamma_2 = c.$$

In the more complicated problems of the following sections the necessity of adhering to the ordered sequence of

1. Forming the approximation 14 to the integral 12,
2. Enforcing the constraints, and
3. Carrying out the minimal process,

as well as that of selecting suitable independent variables is dispensed with by the introduction of Lagrange multipliers.

Let us now take up the more important consideration of how this method may be used to derive difference equations corresponding to certain boundary conditions. To this end, consider the minimum of integral 15 with $\gamma_0 = 0$ but $\gamma_n$ unspecified. Using equations 13 and 14, the integral 15 may be approximated by

$$P(\gamma_1, \cdots, \gamma_i, \cdots, \gamma_n) = \left\{ n (\gamma_i)^2 + \frac{c}{2n} (\gamma_i) \right\} + \cdots + \left\{ n (\gamma_i - \gamma_{i-1})^2 + \frac{c}{2n} (\gamma_i + \gamma_{i-1}) \right\} + \cdots + \left\{ n (\gamma_n - \gamma_{n-1})^2 + \frac{c}{2n} (\gamma_n + \gamma_{n-1}) \right\}.$$

Note that $P$ now involves the variable $\gamma_n$ which is no longer fixed by the constraints of the problem. Minimizing $P$ with respect to $\gamma_n$ yields a difference equation whose form differs from that of
equation 23. Specifically,

\[
\frac{\gamma_n - \gamma_{n-1}}{\frac{1}{n}} = -\frac{c}{4n} .
\]

As the width of the subinterval \(|X_n - X_{n-1}|\) approaches zero \((n \to \infty)\), this difference equation approaches the boundary condition \(\gamma'(x_n) = 0\). This boundary condition which is satisfied automatically in the process of minimizing \(I\) is called the "natural boundary condition" of the variational problem.

The conventional finite difference approach to the solution of the Euler differential equation of this problem with the end conditions \(\gamma(x_0) = \gamma'(x_n) = 0\) would involve the introduction of a new point on the \(x\) axis: \(X_{n+1} = X_n + \frac{1}{n}\); the enforcement of equation 23 at \(X_n\), i.e.,

\[
\frac{\gamma_{n+1} - 2\gamma_n + \gamma_{n-1}}{\left(\frac{1}{n}\right)^2} = \frac{c}{2} ;
\]

and the satisfaction of the natural boundary condition \(\gamma'(x_n) = 0\) by setting

\(\gamma_{n+1} = \gamma_{n-1}\).

Elimination of variable \(\gamma_{n+1}\) from these last two difference equations gives (Eq. 27) which was arrived at without prior knowledge of the form of the Euler equation or the natural boundary condition. This example illustrates how the minimal technique may be used to derive the difference equations corresponding to both the Euler differential equation and the natural boundary conditions.
The next section will apply the preceding method of deriving difference equations to the solution of beam bending problems. It will also be shown that it is frequently possible to arrive at a single difference equation with variable coefficients which serves as both a finite difference approximation to the Euler equation when applied at all unconstrained points inside the interval of interest and a finite difference approximation to a natural boundary condition when applied at points outside (or coinciding with an end point of) the interval of interest.
APPLICATION TO BEAM PROBLEMS

All linear, elastic structural problems may be formulated as variational problems using the concepts of stress and strain and the theorem of minimum total potential energy (13). For most of the common structural shapes simple integral formulae have been derived which express the total potential energy in terms of the external loads and the unknown displacement functions. For example, the total potential energy of a beam of length $L$ and bending stiffness $EI$, subject to a vertical loading of $q(x)$ pounds per unit length is given by (13)

$$T[\omega] = \int_0^L \left\{ \frac{EI}{2} (\omega'')^2 - q(x) \omega \right\} dx$$

where $\omega(x)$ is the vertical deflection of the neutral axis of the beam. The true deflected form, among all those satisfying the geometric constraints of the supporting structure and compatibility is the one which minimizes the integral $T[\omega]$.

Let us seek an approximate formulation of this structural problem.
Using the approach of the previous section, divide the interval
\( \chi_0 \leq x \leq \chi_0 + L \) into \( n \) equal parts of length \( h = \frac{L}{n} \) by inserting the
points of division \( \chi_i : \chi_i = \chi_0 + i \cdot h \) \( (i = 1, 2, \ldots) \). Letting \( u(T(i)) = u(T(\chi_i)) \),
\( E(T(i)) = E(\chi_i) I(\chi_i) \), and \( q(T(i)) = q(T(\chi_i)) \); and using the finite dif-
ference approximation suggested by equation 9, the integrand of (Eq. 28)
may be approximated by the step function

\[
\frac{E(T(i))}{2} \left\{ \frac{u(T(i+1)) - 2u(T(i)) + u(T(i-1))}{h^2} \right\}^2 - q(T(i)) u(T(i)),
\]

\( (x_i - \frac{h}{2} \leq x \leq x_i + \frac{h}{2}) \).

Strictly speaking, the difference approximations are defined only at
node points but they are continued or extended in this manner to facili-
tate integration. The integral 28 may now be approximated by the integral
of the step function 29; i.e.,

\[
T \approx P = \left[ \frac{E(T(0))}{2} \left\{ \frac{u(1) - 2u(0) + u(-1)}{h^2} \right\}^2 - q(0) u(0) \right] \frac{h}{2}
+ \sum_{i=1}^{n-1} \left[ \frac{E(T(i))}{2} \left\{ \frac{u(i+1) - 2u(i) + u(i-1)}{h^2} \right\}^2 - q(i) u(i) \right] h
+ \left[ \frac{E(T(n))}{2} \left\{ \frac{u(n+1) - 2u(n) + u(n-1)}{h^2} \right\}^2 - q(n) u(n) \right] \frac{h}{2}.
\]

Now, rewrite (Eq. 30) using the symbol \( K(i) \) to represent the length of
the sub-interval \( [x_i - \frac{h}{2}, x_i + \frac{h}{2}] \) which is covered by the neutral
axis of the beam. ( \( K(i) \) will equal \( h \) on all of the sub-intervals
\( [x_i - \frac{h}{2}, x_i + \frac{h}{2}] \) which cover the interval \( [\chi_0, \chi_0 + L] \) except the two
which contain the end points of the beam. There \( K(0) = K(n) = \frac{h}{2} \).

Note that

\[
P = \sum_{i=0}^{n} \left[ \frac{E(T(i))}{2} \left\{ \frac{u(i+1) - 2u(i) + u(i-1)}{h^2} \right\}^2 - q(i) u(i) \right] K(i).
\]
may also be written as

\[ P = \sum_{i=-\infty}^{+\infty} \left[ \frac{E I(i)}{2} \left\{ \sigma(i) - 2\sigma(i+1) + \sigma(i-1) \right\}^2 - q(i) \sigma(i) \right] K(i) \]

because

\[ K(i) = 0 \quad , \quad i \in \{ -n, -n+2, \ldots \} \]

There is much to be gained in this apparent overcomplication of equation 30. \( P \) is now of the same quadratic form in each of the variables \( \sigma(i) \). At length,

\[ P = \cdots + \frac{\sigma(i)}{2} \left\{ \sigma(i) - 2\sigma(i+1) + \sigma(i-1) \right\}^2 - q(i) \sigma(i) K(i) + \frac{\sigma(i+1)}{2} \left\{ \sigma(i+1) - 2\sigma(i) + \sigma(i-1) \right\}^2 - q(i+1) \sigma(i+1) K(i+1) + \cdots \]

where \( \sigma(i) = E I(i) K(i) / h^4 \). The terms shown in equation 33 are the only ones that contain the variable \( \sigma(i) \). Note the advantage of expressing \( P \) in this way. Since \( P \) is of the same form in each variable, \( \sigma(i) \), the equations \( \partial P / \partial \sigma(i) = 0 \) will be of the same form for each \( \sigma(i) \). Explicitly, \( \partial P / \partial \sigma(i) = 0 \) gives

\[ \sigma(i+1) \sigma(i+2) - 2 \{ \sigma(i+1) + \sigma(i) \} \sigma(i+1) + \{ \sigma(i+1) \]

\[ + 4 \sigma(i) + \sigma(i-1) \} \sigma(i) - 2 \{ \sigma(i) + \sigma(i-1) \} \sigma(i-1) \]

\[ + \sigma(i-1) \sigma(i-2) = q(i) K(i) . \]
For a homogeneous beam of uniform cross section, with \( X \) sufficiently far from the ends of the beam, (Eq. 34) reduces to

\[
\begin{align*}
\frac{\partial^4 w(i)}{\partial x^4} + 6 \frac{\partial^4 w(i)}{\partial x^3 \partial y} - 4 \frac{\partial^4 w(i-1)}{\partial x^2 \partial y^2} + 4 \frac{\partial^4 w(i-2)}{\partial x \partial y^3} = \frac{P(i)}{E I} \frac{h^4}{h^4}
\end{align*}
\]

the standard fourth order difference equation corresponding to the Bernoulli-Euler beam equation, \( E I \frac{\partial^4 w}{\partial x^4} = q \). Thus (Eq. 34) may be considered to be a generalization of (Eq. 35) to inhomogeneous beams of non-constant cross section. Since \( P \) is now of the same form in each variable, (Eq. 34) may be applied at points such as \( X^{n+1} \) of Fig. 2 which fall beyond an unconstrained (free) end of a beam where (Eq. 35) does not hold. Hence, (Eq. 34) in some way must also be a difference equation corresponding to the natural boundary conditions, \( \frac{\partial^2 w}{\partial x^2} = 0 \), at the free ends of a beam. Let us interpret equation 34 when \( X \) is near an unconstrained end point, \( X_n \), of a homogeneous beam of uniform cross section (see Fig. 2). When \( i > n + 1 \) equation 34 reduces to the form \( 0 = 0 \). When \( i = n + 1 \), (Eq. 34) becomes

\[
\begin{align*}
\frac{\partial^4 w(n+1)}{\partial x^4} - 2 \frac{\partial^4 w(n)}{\partial x^3 \partial y} + \frac{\partial^4 w(n-1)}{\partial x^2 \partial y^2} = 0
\end{align*}
\]

In the limit (as \( h \rightarrow 0 \)) this equation approaches the force boundary condition \( \frac{\partial^2 w}{\partial x^2}(X_n) = 0 \) (which means that the bending moment at a free end of a beam is zero). When \( i = n \), (Eq. 34) reduces to

\[
\begin{align*}
\frac{\partial^4 w(n+1)}{\partial x^4} - 3 \frac{\partial^4 w(n)}{\partial x^3 \partial y} + 3 \frac{\partial^4 w(n-1)}{\partial x^2 \partial y^2} - \frac{\partial^4 w(n-2)}{\partial x \partial y^3} = -\frac{q(n)}{2 E I} \frac{h^4}{h^4}
\end{align*}
\]
In the limit (as \( h \to 0 \)) equation 37 approaches the force boundary condition \( u''(x_n) = 0 \) (which implies that the shear force at the free end is zero). As it stands, equation 37 states that the shearing force at \( x = x_n - \frac{h}{2} \) is of magnitude \( \frac{q(n) \cdot h}{2} \). Thus (Eq. 37) assumes that \( q(n) \) is the average value of the pressure \( q(x) \) over the interval \( x_n - \frac{h}{2} \leq x \leq x_n \).

In the previous section it was emphasized that the enforcement of the constraints of the problem was a necessary step prior to the minimization of \( P \), hence (Eq. 34), (Eq. 36), (Eq. 37) may only be valid for unconstrained (free-free) beams. Consider first the manner in which the enforcement of a constraint of the form \( u'(i) = 0 \) (a simple knife edge support) effects the form of (Eq. 33) and possibly the form of general difference equation 34.

If \( u'(i) \) is fixed by the constraints then it may be considered as simply a fixed parameter in (Eq. 33). Thus if \( i \neq i' \), equation 34 remains unchanged. When \( u'(i) \) appears in (Eq. 34), for example when \( i = j + 1 \), \( u'(i) \) is not to be regarded as an independent variable any longer but as some constant: zero, to be explicit. Since \( u'(i) \) is no longer an independent variable the equation \( \frac{\partial P}{\partial \omega(i)} = 0 \) should not be enforced. Thus it is apparent that the form of equation 34 is unaffected by the imposition of a number of geometric constraints of the form \( u'(i) = 0 \) and (Eq. 34) may be enforced at each unconstrained node point prior to the imposition of this type of constraint without incurring error.
Now consider the manner in which the enforcement of a clamp type constraint of the form \( \mathcal{U}(x_j) = \mathcal{U}(x_{j+1}) = 0 \) affects the form of (Eq. 33) and possibly (Eq. 34). Approximating the constraint \( \mathcal{U}(x_j) = 0 \) by the difference equation \( \mathcal{U}(j+1) = \mathcal{U}(j) \) and introducing Lagrange multiplier \( \lambda \) the ordinary minimal problem may be restated as

\[
\min \bar{P} = \min \left[ P + \lambda \left( \mathcal{U}(j+1) - \mathcal{U}(j) \right) \right] ; \quad \mathcal{U}(j+1) = \mathcal{U}(j),
\]

Regarding both \( \mathcal{U}(j+1) \) and \( \mathcal{U}(j) \) as independent variables we have

\[
\begin{align*}
\frac{\partial \bar{P}}{\partial \mathcal{U}(j+1)} &= \frac{\partial P}{\partial \mathcal{U}(j+1)} + \lambda = 0, \\
\frac{\partial \bar{P}}{\partial \mathcal{U}(j)} &= \frac{\partial P}{\partial \mathcal{U}(j)} - \lambda = 0.
\end{align*}
\]

Thus, \( \lambda = \frac{\partial P}{\partial \mathcal{U}(j+1)} \) and

\[
\frac{\partial P}{\partial \mathcal{U}(j+1)} + \frac{\partial P}{\partial \mathcal{U}(j)} = 0
\]

becomes the proper difference equation. If it happens that the \( x_j \leq \frac{h}{2} \) interval contains the left end of the beam, equation 40 becomes

\[
\begin{align*}
\mathcal{P}(j+2) \mathcal{U}(j+3) - 2 \{ \mathcal{P}(j+2) + \mathcal{P}(j+1) \} \mathcal{U}(j+2) \\
+ \{ 4 \mathcal{P}(j+2) + 4 \mathcal{P}(j+1) + 2 \mathcal{P}(j) \} \mathcal{U}(j+1) \\
+ 2 \mathcal{P}(j) \mathcal{U}(j) = q(j+1) K(j+1).
\end{align*}
\]

Setting \( \mathcal{U}(j+1) = \mathcal{U}(j+2) \) gives

\[
\begin{align*}
\mathcal{P}(j+2) \mathcal{U}(j+3) - 2 \{ \mathcal{P}(j+2) + \mathcal{P}(j+1) \} \mathcal{U}(j+2) \\
+ \{ 4 \mathcal{P}(j+2) + 4 \mathcal{P}(j+1) + 4 \mathcal{P}(j) \} \mathcal{U}(j+1) \\
= q(j+1) K(j+1).
\end{align*}
\]
The standard general difference equation 34 written at $X_{j+1}$ with $u_j(j+1) = u_j(j-1)$ would have erroneously stated that

$$Q(j+2)u(j+3) - 2\left\{Q(j+2) + Q(j+1)\right\}u(j+2) + \left\{Q(j+2) + 4Q(j+1) + 2Q(j)\right\}u(j+1) = q(j+1)k(j+1).$$

Note however, that (Eq. 43) may be made to coincide with the correct (Eq. 42) simply by doubling the actual value of $Q(j)$ in (Eq. 43). None of the other difference equations involve $Q(j)$ hence no error is introduced by so changing it. As an aid in remembering to double $Q(j)$ in (Eq. 34) when the beam ends at $X_j$ we may imagine the beam to be continuous on both sides of the clamp at $X_j$ when applying (Eq. 34) at $X_{j+1}$. This effectively doubles $Q(j)$. However, in reality, if the beam were continuous at $X_j$ (Eq. 34) would not even be applicable at $X_{j+1}$. (Eq. 40) would be the correct equation.

This leads to one of the curiosities of the generalized finite difference method that has no parallel in the traditional application of difference methods to solving differential equations. From elementary mechanics we know that the solution to a beam problem valid on one side of a clamp constraint is the same whether the beam is continuous across the clamp or not. But the general difference formulation differs in the two cases. Equation 42 is the correct difference equation only if the beam ends in the $X_j \pm \frac{1}{2}$ interval in which case $\frac{\partial}{\partial x}u(j-1)$ is simply $u(j+1) + u(j-1)$. But if the beam is continuous beyond this interval then difference equation 40 is considerably more complicated.
as $\frac{\partial P}{\partial \omega(j-1)}$ now involves other deflections to the left of $X_j$ and the solution for the deflections on both sides of $X_j$ appear to become hopelessly interdependent. The fact that two distinct general difference formulations exist for a problem whose solution we know to be unique need not cause alarm as difference formulations themselves are not unique. The only valid criterion for choosing one difference formulation in preference to another of equal difficulty is that of accuracy. However, in this instance the writer tends to favor the use of general difference equation 42, even on continuous beams, for two reasons. First, it is easier to solve the two separate smaller problems, each valid on one side of the clamp, than the composite problem. Second, the general difference method tacitly assumes that the integrand of the total potential energy can be closely approximated by a step function. But for the case in point, we know that the second derivative of the displacement function is, in general, discontinuous at the point of clamping, hence cannot be closely approximated by a step function unless one of the points of discontinuity of the step function corresponds with the point of discontinuity of the second derivative. Unfortunately, this is not the case here as the discontinuities of the step function approximation to the total potential energy occur half way between the $X_i$ division points while the discontinuity of the integrand occurs directly at one of the $X_i$ division points. The use of (Eq. 42) rather than (Eq. 40) avoids this error because the problem which we are approximating with (Eq. 42) has no discontinuities in its second derivative.
In the special case of the beam of constant bending stiffness, clamped at one end, the general difference equation 34 reduces to the conventional difference equation at all $X_i$ points except $X_{j+1}$, adjacent to the clamp. There, the valid difference equation 42, or 34 with $p(j)$ doubled, also reduces to the conventional difference expression 35 with $\omega(j) = 0$ and $\omega(j+1) = \omega(j-1)$.

The only two other types of constraint which are likely to be encountered are elastic constraints of the form $\mathcal{F}_i = A \omega(x_i)$ or $M_i = B \omega'(x_i)$ where $\mathcal{F}_i$ and $M_i$ are a reacting force and moment respectively, and $A$ and $B$ the constant foundation modulii. These constraints are not workless and generally will alter the form of the difference equations at or near $X_j$ when their contribution of energy is considered.

Note that in this section it was not necessary to consider a free end as coinciding with an $X_i$ division point. There is little significance to this statement in the case of a beam since the length of the interval h may always be conveniently set to some fraction of the total length of the beam, but in the section on plate deflections this fact is of some importance.
SOME THEORETICAL CONSIDERATIONS

Of fundamental concern in the application of the finite difference method is the proof of the nonsingularity of the coefficient matrix and the establishment of the order of magnitude of the discrepancy between the solution to a system of difference equations and the solution to the corresponding differential problem.

The shift operator notation of reference 9 provides a concise means of writing difference equations which does not seem to be generally appreciated. Here the concept will be extended from scalars to matrices with the result that the coefficient matrix of a system of difference equations may be written in terms of sums and products of physically meaningful unitary and diagonal matrices. This notation will then be used to simplify the proof of the existence and uniqueness of solution to a system of general difference equations.

Given a sequence of equidistant points \( X_i (i=1, 2, \ldots, n) \) along an x axis with the discrete values, \( \xi_i \), of a continuous dependent variable corresponding to each \( X_i \), the east shifting operator \( \mathcal{E} \) is defined by \( \mathcal{E} \xi_i = \xi_{i+1} \), \( \mathcal{E} \xi_n = \xi_1 \). This operator, as its name implies, has the effect of shifting our point of interest one station to the right along the x axis. The shifting operator is linear in that

\[
\mathcal{E}(c \xi_i) = c \mathcal{E} \xi_i = c \xi_{i+1},
\]

\[
\mathcal{E}(\xi_i + \xi_j) = \mathcal{E} \xi_i + \mathcal{E} \xi_j = \xi_{i+1} + \xi_{j+1}.
\]
The inverse operator to \( E \) is \( W \), which is defined by
\[
W \xi_i = \xi_{i-1}, \quad W \xi_n = \xi_n.
\]

Some differential expressions and their corresponding difference expressions in this notation are illustrated in the following examples:
\[
\begin{align*}
\frac{d^2 \xi_i}{dx^2} & \quad \cdots \quad \frac{1}{h^2}(E - 2 + W) \xi_i, \\
\frac{d^4 \xi_i}{dx^4} + a \xi_i & \quad \cdots \quad \frac{1}{h^4}(E^2 - 4E + 6 + ah^4 - 4W + W^2) \xi_i.
\end{align*}
\]

Consider now the possibility of generalizing the concept of shift operators from scalars to matrices.

Let \((\xi)\) be a vector whose elements are the aforementioned \(\xi_i\)s, i.e. \((\xi)^T = (\xi_1, \xi_2, \ldots, \xi_n)\). If \((\xi)\) is premultiplied by an \(n\) square matrix formed by moving the first row of the identity matrix to the bottom, i.e.
\[
[\mathcal{E}] = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0
\end{bmatrix}_{n \times n}
\]

the result is a vector which differs from \((\xi)\) in that each \(\xi_i\) element is replaced by the \(\xi_{i+1}\) element and in place of \(\xi_n\) is \(\xi_1\). If \((\xi)^T\) is postmultiplied by the same matrix another row vector results in which \(\xi_i\) is replaced by \(\xi_{i-1}\) and \(\xi_1\) by \(\xi_n\). The
transpose of this new row vector is given by \(( \psi^T[\mathcal{E}] )^T = [\mathcal{E}]^T \psi\)
demonstrating that a west or left shifting matrix is just the transpose of
an east shifting matrix. It can be shown that these two matrices are
also the inverses of each other. The following example shows the ease
with which a complete system of difference equations can be written
using this notation.

Suppose that we wish to replace the differential equation \( \ddot{\psi} + \psi = x \)
on the interval \([0,1]\) subject to the end conditions \( \psi(0) = \dot{\psi}(0) = 0 \)
by a system of difference equations. Partitioning \([0,1]\) into \(n+1\)
equal subintervals by inserting \(n\) points of division at \( \chi_i = i/(n+1) \),
\((i = 1, 2, \ldots, n)\) the difference equation at each of these \(n\) points in
shift operator notation is
\[
(\mathcal{E} - 2 + h^2 + [\mathcal{W}]) \psi_i = h^2 \chi_i, \quad (i = 1, 2, \ldots, n).
\]
Introducing the shifting matrices
\[
[\mathcal{E}] = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & 0
\end{bmatrix}
_{n+2,n+2}, \quad [\mathcal{W}] = [\mathcal{E}]^T
\]
a complete system of difference equations is obtained simply by re-
lacing the shift operators in the previous difference equation by shifting
matrices and the discrete variables by vectors of the same, i.e.
\[
[(\mathcal{E} - (2 - h^2)[1] + [\mathcal{W}]) \psi] = h^2(x)
\]
where
\[
(x)^T = (0, x_1, x_2, \ldots, x_n, 1)
\]
and

$$(\mathbf{f})^T = (\mathbf{f}_0, \mathbf{f}_1, \ldots, \mathbf{f}_n, \mathbf{f}_{n+1}).$$

The equivalence of the two forms may be proved by writing out the general $L$th row of the matrix equation. However, the matrix equation contains two spurious difference equations at $x=0, L$. These can be discarded by omitting the first and last rows of the matrix $[E] - (z-1^2)[I] + [W]$ and the first and last elements of $(\mathbf{x})$. The boundary conditions can be enforced by setting the first and last elements of $(\mathbf{f})$ to zero. In later problems with natural or free end conditions the same difference equation applies at the end points as at the interior points. It will be seen that for this type of problem the system of difference equations obtained by replacing shift operators by shifting matrices and variables by vectors constitutes a complete set of difference equations without the annoying necessity of eliminating extraneous equations at the end points.

Let us now apply these concepts to the proof of the existence and uniqueness of the solution to a system of general difference equations for a beam supported at any two $x_i$ points. This is accomplished by proving that the homogeneous system has only the null solution which is equivalent to showing that the coefficient matrix is non-singular, i.e. has a unique inverse.

To this end, consider first the free-free beam. It will be shown that the homogeneous problem, in this particular case, has only solutions in which all of the displacements lie on a straight line.
In the absence of an applied load the total potential energy of a beam reduces to the strain energy integral:

\[
U = \frac{1}{2} \int E I (\omega''')^2 \, dx.
\]

Dividing the x axis into equal intervals of length h such that the two ends of the beam correspond with division points, the difference approximation to the previous integrand in shift operator notation is

\[
\frac{E I (\omega)}{h^4} \left\{ \left( E - 2 + \mathcal{W} \right) \omega_{\gamma(i)} \right\}^2.
\]

If this quantity is multiplied by \(\frac{K(i)}{2}\) and summed over each \(X_i\) division point for which \(K(i) \neq 0\) the result is the general difference approximation to the integral 44. Consider how this sum might be expressed in matrix notation. A column vector \((\omega_{xx})\), each element consisting of the difference expression \(\frac{1}{h} (E - 2 + \mathcal{W}) \omega_{\gamma(i)}\) at an \(X_i\) point, is given in terms of shifting matrices by

\[
(\omega_{xx}) = \frac{1}{h^2} \left[ [E] - 2 [I] + [\mathcal{W}] \right] (\omega)
\]

where \((\omega)\) is a vector consisting of the displacements of all the \(X_i\) points on the interval covered by the beam and one additional point past each end of the beam. Unfortunately at the left and right \(X_i\) points, beyond the ends of the beam, this vector gives meaningless values for the difference approximation to the second derivative. However, we will soon see that this difficulty is of no consequence. If \((\omega_{xx})\) is premultiplied by a diagonal matrix \([k]\) whose diagonal elements are the \(E I(i)K(i)\) terms associated with each \(X_i\) point involved; the result, \([k] (\omega_{xx})\), is another column vector whose
elements are the product of the difference expression \( \frac{1}{h} ( E - 2 \cdot W ) \omega(i) \) and the value of \( EI(i) K(i) \) at the same \( \chi_i \) point. If \( [k] (\omega_{xx}) \) now be premultiplied by the transposed vector \( (\omega_{xx})^T \), the result is precisely the general difference approximation to the integral

\[
(47) \quad \int EI(\omega'')^2 \, dx
\]

over the length of the beam. Thus (Eq. 44) is approximated in terms of diagonal matrix \( [k] \), unitary matrix \( [E] \), and vector \( (\omega) \) by

\[
(48) \quad U = D = \frac{1}{2} (\omega_{xx})^T [k] (\omega_{xx}) = \frac{1}{2h^4} (\omega)^T [X][k][X](\omega)
\]

where

\[
(49) \quad [X] = \left[ [E] - 2 [1] + [E]^T \right] = [X]^T
\]

Note that the squares of the meaningless values of \( \frac{1}{h} (E - 2 \cdot W) \omega(i) \) at the left and right \( \chi_i \) points are eliminated in forming the product \( (\omega_{xx})^T [k] (\omega_{xx}) \) since the \( K(i) \) associated with these two points is zero. As an example see Fig. 3. The \( \frac{1}{h} (E - 2 \cdot W) \omega(i) \) term associated with the point \( \chi_5 \) is \( \frac{1}{h} \left\{ \omega(4) - 2 \omega(5) + \omega(i) \right\} \) but when this is multiplied by \( EI(5) K(5) = \omega(i) \) in forming \( (\omega_{xx})^T [k] (\omega_{xx}) \) it is eliminated.

Fig. 3 -- Example Beam Problem (II)
\( P \) may be alternately considered as a weighted dot product of the vector \((w_{\mathbf{x}_k})\) with itself. In this regard, it is important to note that since all of the diagonal elements of \([^k]\) are \(\geq 0\), setting \(P = 0\) is equivalent to setting the components of \((w_{\mathbf{x}_k})\) to zero at all node points for which \(K(i) > 0\).

According to the rule for differentiating a quadratic form, the homogeneous system of general difference equations is given by

\[
\frac{1}{2h^4} \frac{\partial}{\partial \mathbf{u}(i)} (\mathbf{w})^T [\mathbf{k}] [\mathbf{x}] (\mathbf{w}) = \frac{1}{h^4} [\mathbf{x}] [\mathbf{k}] [\mathbf{x}] (\mathbf{w}) = 0.
\]

If \((\mathbf{w})\) is a solution to this system, then

\[
\frac{1}{h^4} [\mathbf{x}] [\mathbf{k}] [\mathbf{x}] (\mathbf{w}) = 0.
\]

Premultiplying (Eq. 51) by \(\frac{1}{2h^4} (\mathbf{w})^T\) gives

\[
P = \frac{1}{2h^4} (\mathbf{w})^T [\mathbf{x}] [\mathbf{k}] [\mathbf{x}] (\mathbf{w}) = 0.
\]

But, for this to hold we have already noted that the elements of \((\mathbf{w})_{\mathbf{x}_k} = [\mathbf{x}] (\mathbf{w}) \frac{1}{h^4}\) must vanish at each \(\chi\) point for which \(K(i) > 0\).

Choose any two adjacent \(\chi\) points for example, \(\chi_i\) and \(\chi_{i+1}\), for which neither \(K(i)\) or \(K(i+1)\) equals zero. The two ordinates \(w(i)\) and \(w(i+1)\) may be used to describe a unique straight line containing these two points in the \(w, \chi\) plane. Since \(K(i)\) and \(K(i+1)\) are greater than zero, \(w(i) - 2w(i+1) + w(i+2) = 0\) and \(w(i-1) - 2w(i) + w(i+1) = 0\). These last two statements are precisely the conditions that ordinates \(w(i+2)\) and \(w(i-1)\) lie on the same straight line as \(w(i)\) and \(w(i+1)\). By repetition of this
argument we may conclude that all $\omega_{i}$ deflections involved in the
problem lie on the same straight line through $\omega_{i}$ and $\omega_{i+1}$.
Therefore, we conclude that the unconstrained homogeneous problem has
only the trivial rigid motions for its solutions. In other words, the rank
of the coefficient matrix is $n-2$ where $n$ is the dimension of the coef-
ficient matrix, because the complete solution has the form $\omega_{i} =
a_{i}\omega_{(i)} + b_{i}\omega_{(i+1)}$. The same result can be deduced for a beam
clamped at some node point when the constraint is expressed in the form
$\omega_{i} = \omega_{(i-1)} - \omega_{(i+1)} = 0$.

Setting $\omega_{i}$ and $\omega_{(i+1)}$, or for that matter any other two
displacements, to zero eliminates all but the null solution, i.e. the
system of equations

$$
\begin{cases}
\frac{1}{h^2} \left[ X \right] \left[ k \right] \left[ X \right] (\omega) = 0 \\
\omega_{(i)} = \omega_{(i+1)} = 0 , (i \neq i+1)
\end{cases}
$$

has only $(\omega) = 0$ for its solution. To find a nonsingular $n-2$ square
submatrix in the coefficient matrix of system 52, consider the following
alternate and equivalent statement of the minimal problem obtained by
introducing the Lagrange multipliers $\lambda_{i}$ and $\lambda_{i+1}$, i.e.

$$
\min \left\{ \frac{1}{2h^4} (\omega)^T [X] [k] [X] (\omega) - \lambda_{i} \omega_{i} - \lambda_{i+1} \omega_{i+1} \right\}
$$

with $\omega_{i} = \omega_{(i+1)} = 0$. The solution to this problem is given by

$$
\begin{cases}
\frac{1}{h^2} \left[ X \right] \left[ k \right] \left[ X \right] (\omega) = (\lambda) \\
\omega_{i} = \omega_{(i+1)} = 0
\end{cases}
$$
where \( \lambda \) is the \( n \) vector whose only non-zero elements are \( \lambda(i,.) \) and \( \lambda(i,.) \) appearing in the right-hand side of the two equations obtained by setting to zero the partial derivatives of (Eq. 53) with respect to \( \omega(i,.) \) and \( \omega(i,.) \) respectively. Since these two equations are the only ones involving the two unknown \( \lambda \) they may be set aside and the remaining system of \( n \) equations solved for the \( n \) values of \( \omega(i,.) \). But we know that the only solution to this system is the null one. Another \( n \) square system, equivalent (i.e. having the same solutions) to the latter is obtained by directly substituting the values \( \omega(i,.) = 0 \) and \( \omega(i,.) = 0 \) in the other \( n-2 \) equations thus eliminating these two variables. Deleting the equations \( \omega(i,.) = \omega(i,.) = 0 \), the \( n-2 \) remaining equation involve only \( n-2 \) of the \( \omega(i,.) \) unknowns and again have only the null solution as these \( n-2 \) equations plus the two equations \( \omega(i,.) = \omega(i,.) = 0 \) are completely equivalent to the aforementioned system. Thus, we have found a \( n-2 \) square matrix which must be non-singular. This matrix, is precisely the coefficient matrix obtained by writing the general difference equations at all unconstrained \( X_i \) points and setting \( \omega(i,.) = \omega(i,.) = 0 \). Therefore, the system of non-homogeneous general difference equations always has a unique solution. The same result could have been achieved for any number of pin-type constraints at \( X_i \) points or clamp constraints at end points without substantial innovation.

Now, consider the manner in which a solution to a system of general difference equations approaches the solution to the differential equation as the \( h \) spacing diminishes. For simplicity, we will consider
only beams pinned at both ends. To further facilitate this proof we first consider the finite difference treatment of a simpler differential problem. The relationship between this problem and the beam bending problem will be clarified later. The simpler problem consists of solving the second order differential equation \( y'' = \mathcal{G}(x) \) on the interval \([x_0, x_n]\) with \( y(x_0) = y(x_n) = 0 \). Inserting \( n-1 \) equally spaced points of division of \([x_0, x_n]\) a finite difference analog of the previous differential problem consists of the system of equations

\[
\begin{cases}
y(0) = 0 \\
\frac{1}{h^2} \left\{ y(i+1) - 2y(i) + y(i-1) \right\} = \mathcal{G}_i(i), \quad (i = 1, 2, \ldots, n-1) \\
y(n) = 0 
\end{cases}
\]

where \( h = |x_n - x_0|/n \), \( \mathcal{G}_i(i) = \mathcal{G}(x_0 + ih) \), and \( \mathcal{G}(i) \) hopefully approximates \( y(x_0 + ih) \). Proof of the unique existence of a solution to this system of algebraic equations for finite \( n \) and any prescribed values of \( \mathcal{G}_i(i) \) follows in much the same manner as was demonstrated in the case of the beam bending problem. It will now be shown that for any set of values of \( y(1), y(2), \ldots, y(n-1) \) for which \( \mathcal{G}_1(i), \mathcal{G}_2(i), \ldots, \mathcal{G}_i(n-1) \) are all less than or equal to zero, it will follow that \( y(1), y(2), \ldots, y(n-1) \) are all greater than or equal to zero. To deny this last statement would involve the admission of at least one negative \( y(i) \). But to admit this is to admit to the existence of a negative minimum to the set of \( y(i) \)'s in the sense that for at least one negative \( y(i) \): call it \( y(i^*) \) : \( y(i^*+1) \geq y(i^*) \geq y(i^*-1) \) and \( y(i^*) > \max\{y(i^*+1), y(i^*-1)\} \). Thus \( y(i^*+1) - 2y(i^*) + y(i^*-1) > y(i^*) - 2y(i^*) + y(i^*) = 0 \). But \( y(i^*+1) - 2y(i^*) + y(i^*-1) = h^2 \mathcal{G}_i(i^*) \leq 0 \).
by assumption. Since the quantity $\gamma(i^{\ast}+1) - 2\gamma(i^{\ast}) + \gamma(i^{\ast}-1)$ cannot be both greater than zero and less than or equal to zero, the denial has lead to a contradiction. Therefore, $\gamma(i) \geq 0 \ (i = 1, 2, \ldots, n-1)$.

Once this fact has been established we are able to prove the following result.

Given two solutions, $\gamma_1(i)$ and $\gamma_2(i)$, and their corresponding

$\hat{\gamma}_1(i)$ and $\hat{\gamma}_2(i)$ for which $\hat{\gamma}_1(i) \leq |\hat{\gamma}_2(i)|$ for all $i$; then $\gamma_1(i) \geq |\gamma_2(i)|$

for all $i$. The hypothesis $\hat{\gamma}_1(i) \leq |\hat{\gamma}_2(i)|$ is equivalent to the two statements $\hat{\gamma}_1(i) + \hat{\gamma}_2(i) \leq 0$ and $\hat{\gamma}_1(i) - \hat{\gamma}_2(i) \leq 0$. But, from the results of the previous paragraph we conclude that $\gamma_1(i) + \gamma_2(i) \geq 0$

and $\gamma_1(i) - \gamma_2(i) \geq 0$ or, what is the same, $\gamma_1(i) \geq |\gamma_2(i)|$.

This last result is virtually all that is needed to prove the convergence of the sequence of finite difference solutions to the solution of the differential problem of both the second order equation under consideration and the beam bending problem as $h \to \infty$.

If $\gamma(x_i)$ refers to the discrete values of $\gamma(x)$, the solution to the second order differential problem $\gamma'' = \dot{\dot{\gamma}}(x)$ on $[x_0, x_n]$ with $\gamma(x_0) = \gamma(x_n) = 0$, and $\gamma(i)$ refers to the solution to system 55, then applying Taylor's formula to the function $\gamma(x)$ gives

$$
\begin{align*}
\begin{cases}
\gamma(x_0) = 0 \\
\frac{1}{h^2}\left\{\gamma(x_{i-1}) - 2\gamma(x_i) + \gamma(x_{i+1})\right\} = \gamma''(x_i) - \tau(i) = \gamma(i) - \gamma(i) \\
\gamma(x_n) = 0
\end{cases}
\end{align*}
$$

(56)

where $\tau(i)$ is the remainder term. Substituting the error term $\gamma(i) - \gamma(x_i)$ in place of the solution to system 55 gives
That is, the error term satisfies the same system of equations as does $\gamma(i)$ with the right-hand sides replaced by the remainder terms. Using Lagrange's form of the remainder term we have $\gamma(i) \leq \frac{M_a}{h} \frac{x^2}{2}$ where $M_a = \max \left| f'' \right|$ on $[\gamma_0, \gamma_n]$. Note at this point that the solution to a similar system:

$$
(58) \begin{cases}
\phi(0) = 0 \\
\frac{1}{h^2} \left\{ \phi(i+1) - 2 \phi(i) + \phi(i-1) \right\} = -\frac{h^2 M_a}{12}, (i = 1, 2, \ldots, n-1) \\
\phi(n) = 0
\end{cases}
$$

is given by the discrete values of $\phi(x) = -\frac{2}{h} M_a \frac{(x-x_0)(x-x_n)}{2A}$.

Comparing systems 57 and 58 and recalling the conclusions drawn in the last paragraph, it follows that

$$
(59) \gamma(i) - \gamma(x_i) \leq \phi(x_i) \leq \max |\phi(x)| = \frac{h^2 M_a}{96}
$$

indicating that the error approaches zero as does $\frac{h^2}{96}$.

These results will now be applied to the beam bending problem.

Note that the differential equation governing the transverse displacements of points on the neutral axis of a beam whose stiffness varies along its length may be written as two second order equations rather than a single fourth order one, i.e.
The simplest system of difference equations corresponding to these two differential equations and the pinned end boundary conditions \( \omega(x_o) = \omega(x_n) = M(x_o) = M(x_n) = 0 \) is

\[
\begin{align*}
\omega(0) &= 0 \\
\frac{1}{h^2} \{ M(i+1) - 2M(i) + M(i-1) \} &= q(i), \quad (i = 1, 2, \ldots, n-1) \\
M(n) &= 0 \\
\omega(n) &= 0 \\
\frac{1}{h^2} \{ \omega(i+1) - 2\omega(i) + \omega(i-1) \} &= \frac{M(x_i)}{EI(i)}, \quad (i = 1, 2, \ldots, n-1) \\
\omega(n) &= 0.
\end{align*}
\]

Note that we again make the distinction between \( M(i) \), obtained in solving (Eq. 61), and \( M(x_i) \), the discrete values of \( M(x) \) obtained by solving the differential problem \( M'' = q(x) \) on \([x_o, x_n]\) with \( M(x_o) = M(x_n) = 0 \). The first half of system 61 may be solved for \( M(i) \). Using previously established results the discrepancy between the solution to this algebraic problem and the differential problem, \( M'' = q(x) \) on \([x_o, x_n]\) with \( M(x_o) = M(x_n) = 0 \), is

\[
(62) \quad M(i) - M(x_i) \leq \frac{h^2 M_x}{960}
\]

where \( M_x = \max |q''| \) on \([x_o, x_n]\). Substituting the values of \( M(i) \) in place of \( M(x_i) \) in the second half of system 61 gives
Substituting \( \omega(i) - \omega(x_i) \) in place of \( \omega(i) \) in (Eq. 63) gives

\[
\begin{align*}
\frac{1}{h^2} \left[ \omega(i+1) - 2\omega(i) + \omega(i-1) \right] &= \frac{M(i)}{EI(i)} \quad \text{for } i=1,2,\ldots,n-1 \\
\omega(0) - \omega(x_0) &= 0 \\
\frac{1}{h^2} \left[ \omega(i+1) - 2\omega(i) + \omega(i-1) \right] &= \frac{M(i) - M(x_i) + \xi(i)}{EI(i)} \quad \text{for } i=1,2,\ldots,n-1 \\
\omega(n) - \omega(x_n) &= 0
\end{align*}
\]

(64)

where \( \xi(i) \) is the remainder term of a Taylor series. Using the Lagrange form of the remainder we have

\[
\frac{\xi(i) - \xi(x_i)}{h^2} \leq \frac{h^2}{12} \max \left| \frac{d^3}{dx^3} \right| + \frac{h^2}{12} \max \left| \frac{d^2}{dx^2} \right| \frac{1}{\min \{EI(i)\}}
\]

(65)

The solution of the system of equations

\[
\begin{align*}
\phi(0) &= 0 \\
\frac{1}{h^2} \left[ \phi(i+1) - 2\phi(i) + \phi(i-1) \right] &= -\frac{h}{12} \left( \max \left| \frac{d^2}{dx^2} \right| + \max \left| \frac{d^3}{dx^3} \right| \right) \frac{1}{\min \{EI(i)\}} \\
\phi(n) &= 0
\end{align*}
\]

(66)

is given by the \( x_c \) ordinates of

\[
\phi(x) = \frac{h^2}{12} \left( \max \left| \frac{d^2}{dx^2} \right| + \max \left| \frac{d^3}{dx^3} \right| \right) \frac{1}{\min \{EI(i)\}} (x-x_0)(x-x_n).
\]

(67)

So, from previous conclusions the error term \( \omega(i) - \omega(x_i) \), i.e. the discrepancy between the solution to the algebraic system 63 and the solution to the differential problem of the hinged end beam, is bounded by

\[
\omega(i) - \omega(x_i) \leq \max |\phi(x)| = \frac{h^2}{72} \left( \max \left| \frac{d^2}{dx^2} \right| + \max \left| \frac{d^3}{dx^3} \right| \right) \frac{1}{\min \{EI(i)\}}.
\]

(68)
It will now be shown that the system of equations 61 is equivalent to the system of equations obtained by writing general difference equation 34 at each unconstrained node point and adding the constraints $\omega(0) = \omega(n) = 0$. Therefore, the two systems will be subject to the same error bounds.

Observe that general difference equation 34 may also be written in the form

$$\begin{align*}
\phi(i)\left\{\omega(i+2) - 2\omega(i+1) + \omega(i)\right\} - 2\phi(i-1)\left\{\omega(i+1) - 2\omega(i) + \omega(i-1)\right\} + \omega(i-2) = \phi(i)K(i).
\end{align*}$$

When $i = 2, 3, \ldots, n-2$ the $\phi(i)$ terms in Eq. 69 reduce to $EI(i) = h^3$ and the $K(i)$ terms to $h$. For this case (Eq. 69) may be written

$$\begin{align*}
\frac{1}{h^2}\left\{EI(i+1)\left[\frac{\omega(i+2) - 2\omega(i+1) + \omega(i)}{h^2}\right] - 2EI(i)\left[\frac{\omega(i+1) - 2\omega(i) + \omega(i-1)}{h^2}\right] + EI(i-1)\left[\frac{\omega(i) - 2\omega(i-1) + \omega(i-2)}{h^2}\right]\right\} = \phi(i).
\end{align*}$$

If we let

$$\begin{align*}
M(i) = \frac{EI(i)}{h^2}\left\{\omega(i+1) - 2\omega(i) + \omega(i-1)\right\}, (i = 1, 2, \ldots, n-1)
\end{align*}$$

then equation 70 may be written as

$$\begin{align*}
\frac{1}{h^2}\left\{M(i+1) - 2M(i) + M(i-1)\right\} = \phi(i), (i = 2, 3, \ldots, n-2).
\end{align*}$$
When \( i = 1 \), i.e. \( x = x_0 + h \), equation 69 can be written as

\[
\frac{1}{h^2} \left\{ M(2) - 2M(1) + \frac{EI(0)}{2h^2} [\omega(1) - 2\omega(0) + \omega(-1)] \right\} = q(1)
\]

where \( \omega(-1) = \omega(x_0 + h) \). When \( i = -1 \), i.e. \( x = x_0 - h \) equation 69 becomes

\[
\frac{EI(0)}{2h^2} \left\{ \omega(1) - 2\omega(0) + \omega(-1) \right\} = 0.
\]

Adding (Eq. 73) and (Eq. 74) gives

\[
\frac{1}{h^2} \left\{ M(2) - 2M(1) + M(0) \right\} = q(1)
\]

where

\[
M(0) = \frac{EI(0)}{h^2} \left\{ \omega(1) - 2\omega(0) + \omega(-1) \right\} = 0.
\]

Similarly it can be shown that

\[
M(n) = 0 \quad \text{and}
\]

\[
\frac{1}{h^2} \left\{ M(n) - 2M(n-1) + M(n-2) \right\} = q(n-1)
\]

where \( M(n) = \frac{EI(n)}{h^2} \left\{ \omega(n+1) - 2\omega(n) + \omega(n-1) \right\} \). Equations 72, 75, 76, 77, and 78 give the first half of system 61. Equations 71 and the constraints \( \omega(0) = \omega(n) = 0 \) give the rest of system 61. Thus system 61 is equivalent to the system of general difference equations and both share the error bound given by (Eq. 68).
PLATE BENDING PROBLEMS

The method described in the preceding sections may, with a few innovations, be extended to the problem of solving for the static deflections of plates with small thickness gradients and irregular boundary curves. This particular application of finite difference methods to a variational problem was first given by Houbolt (25) who derived the general difference equation (Eq. 94) presented later in this section. This same equation was derived independently by this writer. The equation alone was presented much earlier by McNeal (20) who apparently deduced it by pure physical reasoning.

Given expressions for the transverse loading $q_i(x,y)$ and bending stiffness $D(x,y)$, the problem of solving for the static deflections of an isotropic, elastic plate which has no initial curvature may be described as that of finding the sufficiently smooth displacement function which does not violate any of the constraints of the supporting structure and which, among all such functions, minimizes the integral

$$
\int_\Omega \left\{ \frac{D}{2} \left[ (\sigma_{xx} + \sigma_{yy})^2 - 2(1-\mu) (\sigma_{xx} \sigma_{yy} - \sigma_{xy}^2) \right] - q \omega \right\} \, dx \, dy.
$$

Integral 79 is called the total potential energy of a plate whose middle plane occupies the region $R$ of the $x, y$ plane (6). The quantity

$$
\iint_R \frac{D}{2} \left\{ (\sigma_{xx} + \sigma_{yy})^2 - 2(1-\mu) \omega_{xx} \omega_{yy} \right\} \, dx \, dy
$$

45
represents the strain energy of bending stored in the deformed plate.

This expression approximates the curvatures of the middle plane by the
second derivatives of the displacement function \( \omega \) with respect to
the coordinates \( x \) and \( y \), and hence is valid only for small displacements.
It also assumes that the displacements in the \( x \) and \( y \) directions are
linear functions of the third coordinate, \( z \). This is valid only for plates
which are thin and have small displacements and small thickness

gradients. The quantity

\[
(81) \quad \sum \int \left[ D(1-\mu)\omega_{xy}^2 \right] dxdy
\]

is an approximation to the strain energy stored in a plate due to twisting.

It assumes, as before, that the displacements in the \( x \) and \( y \) directions
are linear functions of the \( z \) coordinate. The quantity

\[
(82) \quad - \sum \int \rho g \omega dxdy
\]

is the potential of the external pressure or load.

The fact that the strain energy of transverse shear is omitted from
the total potential energy again restricts us to very thin plates whose
transverse shear deformation is negligible in comparison with bending
and twisting deformations. Neglecting the energy of stretching of the
middle plane of the plate assumes there are no external forces acting in
the plane of the plate and that displacements and slopes are small so
that no appreciable resultant forces in the plane of the plate are develop-
ed.

Now let us seek to obtain an approximation to the true displacement
function \( \mathbf{w}(x, y) \) by proceeding along the same lines that were followed in the previous sections. First, form a numerical approximation to the integral 79 by using finite difference approximations to the derivatives contained therein. This approximation is defined only at node points, but it may be extended throughout the region \( \mathbb{R} \) by considering the integrand to be a step function with the ordinates everywhere equal to the ordinate at the nearest node point. Second, minimize the numerical approximation to the total potential energy with respect to each discrete unconstrained displacement. Third, solve the resulting system of linear algebraic equations for the displacements. Let us now carry out these steps in greater detail.

Superpose on the middle plane a square net of mesh width \( h \) selected such that the total number of node points, i.e. points of intersection of the net lines, lying on or adjacent to the region of the middle plane are somewhat less than the dimension of the largest square matrix which we are prepared to invert accurately. Any portion or segment of the plate boundary curve which is constrained should cut the lines of the net only at node points. The latter condition is necessary to permit the accurate mathematical statement of the constraints of the supporting structure. The remaining portions of the boundary curve which are free of constraints need not satisfy this requirement.

Hereafter, we will chiefly be concerned with the discrete values of the displacements at the node points of this net, which we call the primary net. Designate the node points of this net as shown in Fig. 4 and let \( \mathbf{u}(i, j) = \mathbf{w}(x_i, y_j) \).
In order to form a numerical approximation to integral 79 let us first make such an approximation to the quantity 81. This integral may be approximated by considering its integrand to be a step function constant over each square subregion $\bar{R}(i,j)$ of area $h^2$, bounded by the two sets of adjacent net lines $x = x_i$, $x = x_{i+1}$, and $y = y_j$, $y = y_{j+1}$ (see Fig. 4).

![Fig. 4 -- Subregion $\bar{R}(i,j)$](image)

Use of the conventional finite difference approximation

\[(B3) \quad \omega_{xy} \approx \frac{\omega(i+1,j+1) - \omega(i,j+1) + \omega(i,j) - \omega(i+1,j)}{h^2}\]

to the derivative $\omega_{xy}$ on $\bar{R}(i,j)$ serves to define the ordinates of a step function approximation to the integrand of quantity 81 by

\[(B4) \quad (i-\mu) \frac{D(i,j)}{h^2} \left\{ \omega(i+2,j+1) - \omega(i,j+1) + \omega(i,j) - \omega(i+1,j) \right\}^2\]
over \( \bar{R}(i,j) \) where \( \bar{D}(i,j) = D(x_i + \frac{h}{2}, y_j + \frac{h}{2}) \). Integrating step function 84 over area \( \bar{R} \) approximates the integral 81 by

\[
(85) \quad \frac{(1-\mu)}{h^2} \sum_{i,j} \overline{f}(i,j) \left\{ \omega T(i+1,j+1) - \omega T(i,j+1) + \omega T(i+1,j) - \omega T(i,j) \right\}^2
\]

where

\[
(86) \quad \overline{f}(i,j) = \bar{D}(i,j) \left[ \text{Area of middle plane in } \bar{R}(i,j) \right]
\]

and the summation extends over all \((i,j)\) for which \( \overline{f}(i,j) \neq 0 \).

To complete the numerical approximation to (Eq. 79) consider the quantity

\[
(87) \quad \iint_{\bar{R}} \left\{ \frac{D}{2} \left( \alpha \omega_{xx} + \alpha \omega_{yy} \right)^2 - 2(1-\mu) \alpha \omega_{xx} \omega_{yy} \right\} - q \omega d\gamma d\gamma.
\]

To approximate this integral, superpose on the region of the middle plane a secondary net also of mesh width \( h \) with lines parallel to those of the primary net and placed such that node points of the primary net lie at the centers of the square subregions \( \bar{R}(i,j) \) of the secondary net (see Fig. 5).
Using the conventional finite difference approximations

\begin{align*}
\omega_{xx}(x_i, y_j) &\approx \frac{\omega(\mathbf{x}_{i+1, j}) - 2\omega(\mathbf{x}_{i, j}) + \omega(\mathbf{x}_{i-1, j})}{h^2} \\
\omega_{yy}(x_i, y_j) &\approx \frac{\omega(\mathbf{x}_{i, j+1}) - 2\omega(\mathbf{x}_{i, j}) + \omega(\mathbf{x}_{i, j-1})}{h^2}
\end{align*}

the integrand of quantity 87 may be approximated by the step function

\begin{align*}
\frac{\mathcal{D}(i, j)}{2 h^2} &\left[ \left\{ \omega(\mathbf{x}_{i+1, j}) + \omega(\mathbf{x}_{i, j+1}) + \omega(\mathbf{x}_{i-1, j}) + \omega(\mathbf{x}_{i, j-1}) \right\} \\
&- 4 \omega(\mathbf{x}_{i, j}) \right] - 2 (1 - \mu) \left\{ \omega(\mathbf{x}_{i+1, j}) - 2 \omega(\mathbf{x}_{i, j}) + \omega(\mathbf{x}_{i, j-1}) \right\} \\
&+ \omega(\mathbf{x}_{i-1, j}) \left\{ \omega(\mathbf{x}_{i, j+1}) - 2 \omega(\mathbf{x}_{i, j}) + \omega(\mathbf{x}_{i, j-1}) \right\}
\end{align*}

- \varrho(i, j) \omega(i, j)
on $\mathcal{R}(i,j)$ where $D(i,j) = D(x_i, y_j)$, and $\varphi(i,j) = \varphi(x_i,y_j)$.

The integral of step function 90 over the region $\mathcal{R}$ of the middle plane may be considered as an approximation to integral 87. It is given by

\[
\sum_{i,j} \frac{\mathcal{F}(i,j)}{2h^2} \left[ \left\{ \omega(i+1,j) + \omega(i,j+1) + \omega(i-1,j) + \omega(i,j-1) \right\} - 4\omega(i,j) \right]^2 - 2(1-\mu) \left\{ \omega(i+1,j) - 2\omega(i,j) + \omega(i,j-1) \right\} \\
+ \omega(i-1,j) \left\{ \omega(i,j+1) - 2\omega(i,j) + \omega(i,j-1) \right\} \\
- \varphi(i,j) \omega(i,j) \frac{\mathcal{F}(i,j)}{D(i,j)}
\]

where

\[
\mathcal{F}(i,j) = D(i,j) \left[ \text{Area of middle plane in } \mathcal{R}(i,j) \right]
\]

and the summation extends over all $(i,j)$ for which $\mathcal{F}(i,j) \neq 0$.

Adding quantities 85 and 91 we obtain the following approximation to the total potential energy of the plate

\[
T \approx P = \sum_{i,j} \frac{\mathcal{F}(i,j)}{2h^2} \left[ \left\{ \omega(i+1,j) + \omega(i,j+1) + \omega(i-1,j) + \omega(i,j-1) \right\} - 4\omega(i,j) \right]^2 - 2(1-\mu) \left\{ \omega(i+1,j) - 2\omega(i,j) + \omega(i,j-1) \right\} + \omega(i-1,j) \left\{ \omega(i,j+1) - 2\omega(i,j) + \omega(i,j-1) \right\} \\
- \varphi(i,j) \omega(i,j) \frac{\mathcal{F}(i,j)}{D(i,j)} + (1-\mu) \frac{\mathcal{F}(i,j)}{h^2} \left\{ \omega(i+1,j+1) - \omega(i,j+1) + \omega(i,j) \right\} - \omega(i+1,j+1) \right\}^2.
\]
$P$ is a polynomial in the independent $\omega(i,j)$ variables. If the plate is unconstrained, a necessary condition for the total potential energy, as approximated by $P$, to be a minimum is $\partial P/\partial \omega(i,j) = 0$ for all $\omega(i,j)$. Observing that any particular $\omega(i,j)$ is only contained in a total of six terms of the $P$ summation, we find that the condition $\partial P/\partial \omega(i,j) = 0$ yields the following linear equation in $\omega(i,j)$ and the values of the deflections at the twelve closest node points:

\begin{equation}
(93) \quad \left\{ \begin{array}{l}
(\omega(i+2,j) + \omega(i+1,j+1) + \omega(i,j) + \omega(i+1,j-1) \\
-4 \omega(i+1,j) \right\} \frac{E(i+1,j)}{h^4}
+ \left\{ \begin{array}{l}
\omega(i+1,j+2) + \omega(i-1,j+1) + \omega(i,j) \\
-4 \omega(i,j+1) \right\} \frac{E(i,j+1)}{h^4}
+ \left\{ \begin{array}{l}
\omega(i,j) + \omega(i-1,j+1) + \omega(i-2,j) + \omega(i-1,j-1) \\
-4 \omega(i-1,j) \right\} \frac{E(i-1,j)}{h^4}
+ \left\{ \begin{array}{l}
\omega(i+1,j-1) + \omega(i,j) + \omega(i-1,j-1) + \omega(i,j-2) \\
-4 \omega(i,j-1) \right\} \frac{E(i,j-1)}{h^4}
-2(1-\mu) \left\{ \begin{array}{l}
\omega(i+1,j) + \omega(i,j+1) + \omega(i-1,j) + \omega(i,j-1) \\
-4 \omega(i,j) \right\} \frac{E(i,j)}{h^4}
- (1-\mu) \left\{ \begin{array}{l}
\omega(i+1,j+1) - 2 \omega(i+1,j) + \omega(i+1,j-1) \right\} \frac{E(i+1,j)}{h^4}
- (1-\mu) \left\{ \begin{array}{l}
\omega(i+1,j+1) - 2 \omega(i,j+1) + \omega(i-1,j+1) \right\} \frac{E(i,j+1)}{h^4}
- (1-\mu) \left\{ \begin{array}{l}
\omega(i+1,j+1) - 2 \omega(i-1,j) + \omega(i-1,j-1) \right\} \frac{E(i-1,j)}{h^4}
- (1-\mu) \left\{ \begin{array}{l}
\omega(i+1,j-1) - 2 \omega(i,j-1) + \omega(i-1,j-1) \right\} \frac{E(i,j-1)}{h^4}
\end{array} \right. \right.}
\end{equation}
Collecting terms in (Eq. 93) and simplifying gives the following general difference equation:

\begin{align}
& E(i+1, j) \Delta_2(i+2, j) + E(i, j+1) \Delta_2(i, j+2) \\
& + E(i-1, j) \Delta_2(i-2, j) + E(i, j-1) \Delta_2(i, j-2) \\
& + \left\{ \mu E(i+1, j) + \mu E(i, j+1) + 2(1-\mu) \bar{E}(i, j) \right\} \Delta_2(i+1, j+1) \\
& + \left\{ \mu E(i, j+1) + \mu E(i-1, j) + 2(1-\mu) \bar{E}(i-1, j) \right\} \Delta_2(i-1, j+1) \\
& + \left\{ \mu E(i-1, j) + \mu E(i, j-1) + 2(1-\mu) \bar{E}(i, j-1) \right\} \Delta_2(i-1, j-1) \\
& + \left\{ \mu E(i+1, j) + \mu E(i, j-1) + 2(1-\mu) \bar{E}(i, j-1) \right\} \Delta_2(i+1, j-1) \\
& - 2 \left\{ (1+\mu) \left[ E(i+1, j) + E(i, j) \right] + (1-\mu) \left[ \bar{E}(i, j) + \bar{E}(i, j-1) \right] \right\} \Delta_2(i+1, j) \\
& - 2 \left\{ (1-\mu) \left[ E(i, j+1) + E(i, j) \right] + (1-\mu) \left[ \bar{E}(i, j+1) + \bar{E}(i, j) \right] \right\} \Delta_2(i, j+1) \\
& - 2 \left\{ (1-\mu) \left[ E(i-1, j) + E(i, j) \right] + (1-\mu) \left[ \bar{E}(i-1, j) + \bar{E}(i, j) \right] \right\} \Delta_2(i-1, j) \\
& - 2 \left\{ (1-\mu) \left[ E(i, j-1) + E(i, j) \right] + (1-\mu) \left[ \bar{E}(i, j-1) + \bar{E}(i, j) \right] \right\} \Delta_2(i, j-1) \\
& + \left\{ E(i+1, j) + E(i, j+1) + E(i-1, j) + E(i, j-1) + (1+\mu) \bar{E}(i, j) \right\} \Delta_2(i+1, j) \\
& - 2(1-\mu) \left[ \bar{E}(i, j) + \bar{E}(i-1, j) + \bar{E}(i, j-1) + \bar{E}(i, j+1) \right] \Delta_2(i+1, j) \\
& = h^4 q(i, j) \frac{\bar{E}(i, j)}{D(i, j)} .
\end{align}
Let us examine the behavior of equation 94 in some special instances.

First consider the case of a homogeneous, uniformly thick plate. If \((x_i, y_j)\) is not near a free edge than (Eq. 94) reduces to

\[
\begin{align*}
\omega(i+2, j) + \omega(i, j+2) + \omega(i-2, j) + \omega(i, j-2) + 2\left\{\omega(i+1, j+1) + \omega(i-1, j+1) + \omega(i-1, j-1) + \omega(i+1, j-1)\right\} \\
- \delta\left\{\omega(i+1, j) + \omega(i, j+1) + \omega(i-1, j) + \omega(i, j-1)\right\} \\
+ 20 \omega(i, j) = q(i, j) \frac{h^4}{D}
\end{align*}
\]

which is the standard finite difference approximation to the Lagrange equation \(D \nabla^4 \omega = q\), where \(\nabla^4\) is the biharmonic operator.

Now suppose that \((x_i, y_j)\) is adjacent to a free straight edge of the same plate as shown in Fig. 6. In this case (Eq. 94) reduces to

\[
\begin{align*}
\omega(i, j) - 2\omega(i-1, j) + \omega(i-2, j) + \mu\left\{\omega(i-1, j+1) - 2\omega(i-1, j) + \omega(i-1, j-1)\right\} = 0
\end{align*}
\]

the standard finite difference approximation to the free edge boundary condition

\[
\begin{align*}
\omega_x + \mu \omega_{yy} = 0
\end{align*}
\]

which holds at an edge which is parallel to the y axis (6).
Fig. 6 -- Free Edge Plate (I)

Fig. 7 -- Free Edge Plate (II)
Now suppose \((x_i, y_j)\) occurs directly on a free edge as shown in Fig. 7.

In this case (Eq. 94) reduces to

\[
\frac{1}{2} \partial^2 \sigma(u_{i,j+1}) + \frac{1}{2} \partial^2 \sigma(u_{i,j-1}) + \partial^2 \sigma(u_{i+1,j}) + \frac{1}{2} \partial^2 \sigma(u_{i+1,j+1})
\]

\[
+ \frac{1}{2} \partial^2 \sigma(u_{i+1,j-1}) + (1-\mu) \partial^2 \sigma(u_{i,j+1}) + (1-\mu) \partial^2 \sigma(u_{i,j-1})
\]

\[-(1-\mu) \partial^2 \sigma(u_{i+1,j+1}) - \partial^2 \sigma(u_{i,j+1}) - \partial^2 \sigma(u_{i,j-1}) - \partial^2 \sigma(u_{i+1,j-1})
\]

\[
+ 10 \partial^2 \sigma(u_{i,j}) = \sigma(i,j) \frac{h^4}{2D}.
\]

This equation could have been obtained using standard finite difference techniques in the following way.

The two boundary conditions at the free edge in Fig. 7 are the zero moment condition (Eq. 97) and the Kirchoff condition (6)

\[
\sigma_{xx} + (2-\mu) \sigma_{xy} = 0.
\]

The standard difference equations corresponding to (Eq. 97) and (Eq. 99) are (Eq. 96) and the following equation:

\[
\left\{ \partial^2 \sigma(u_{i+1,j}) - 2 \partial^2 \sigma(u_{i+1,j+1}) + \partial^2 \sigma(u_{i+1,j+1}) \right\} - \left\{ \partial^2 \sigma(u_{i,j}) - 2 \partial^2 \sigma(u_{i+1,j}) + \partial^2 \sigma(u_{i,j+1}) \right\}
\]

\[
+ \partial^2 \sigma(u_{i-1,j+1}) + (2-\mu) \left\{ \partial^2 \sigma(u_{i+1,j}) - 2 \partial^2 \sigma(u_{i+1,j+1}) + \partial^2 \sigma(u_{i+1,j+1}) \right\}
\]

\[-(2-\mu) \left\{ \partial^2 \sigma(u_{i+1,j+1}) - 2 \partial^2 \sigma(u_{i+1,j}) + \partial^2 \sigma(u_{i+1,j+1}) \right\} = 0
\]

respectively. Writing the standard difference equation 95 corresponding to \(\sigma_{xx} = \sigma(i,j)\) at point \((x_i, y_j)\) and eliminating the variable \(\sigma(u_{i+2,j})\) with the use of equation 100 gives

\[
\sigma(i,j+2) + \sigma(i,j-2) + 2 \sigma(i,j) + \mu \sigma(i+1,j)
\]

\[
+ \mu \sigma(i+1,j-1) + (4-\mu) \sigma(i-1,j+1) + (4-\mu) \sigma(i-1,j-1)
\]

\[-2(1+\mu) \sigma(i+1,j+1) - 8 \sigma(i,j+1) - 8 \sigma(i,j-1)
\]

\[-2(7-\mu) \sigma(i-1,j+1) + 20 \sigma(i,j) = \sigma(i,j) \frac{h^4}{2D}.
\]
which is the same relation given by equation 98.

Let us now consider the form of general difference equation 94 when applied at a corner of the same type plate (see Fig. 8):

\[
\begin{align*}
&\frac{1}{2} \omega(i-2,j) + \frac{1}{2} \omega(i,j-2) + \frac{K}{2} \omega(i-1,j-1) + \frac{K}{2} \omega(i,j+1) \nonumber \\
&+ (2-\mu) \omega(i-1,j-1) - \frac{(1+\mu)}{2} \omega(i+1,j) - \frac{(1+\mu)}{2} \omega(i,j+1) \\
&- \frac{(7-\mu)}{2} \omega(i,j-1) - \frac{(7-\mu)}{2} \omega(i-1,j) + 5 \omega(i,j) = q(i,j) \frac{h^4}{4D}.
\end{align*}
\]

This equation could have been obtained by writing the conventional finite difference equation 95 at \((x_i, y_j)\); eliminating the variables \(\omega(i+2,j)\) and \(\omega(i,j+2)\) by enforcing the Kirchoff boundary condition, in finite difference form, on both intersecting edges; and eliminating the variable \(\omega(i+1,j+1)\) by enforcing the corner condition of zero twisting moment (8) at \((x_i, y_j)\), i.e.

\[
\omega_{xy} = 0,
\]
in the difference form

(104) \[ \omega(i+1,j+1) - \omega(i-1,j+1) + \omega(i-1,j-1) - \omega(i+1,j-1) = 0. \]

The differential equation of equilibrium of a variable thickness elastic plate is of the form (6)

(105) \[ \nabla^2(D \nabla \omega) - (1-\mu)(D\omega_{yy})_{xx} - 2(D\omega_{xy})_{xy} + (D\omega_{xx})_{yy} = q, \]

where \( \nabla^2 \) is the Laplace operator. After some simple but lengthy manipulations it can be shown that in the interior of \( \mathbb{R} \) difference equation 94 reduces to the conventional difference equation corresponding to (Eq. 105). It remains to be proved that general difference equation 94 is valid when the plate is constrained.

When any number of node points have their displacements fixed by workless constraints of the form \( \omega(i,j) = 0 \), then these displacements may be regarded as fixed parameters in the approximation \( D \) to the total potential energy. The result is that the form of difference equation 94 is unaltered. Since the displacement at the point of constraint is no longer regarded as an independent variable the total potential energy is not to be minimized with respect to that displacement.

If one straight segment of the boundary curve of the plate is constrained by a rigid clamp of the form \( \omega = \omega_x = 0 \) (assuming this edge parallel to the y axis) then we may express the constraint in finite difference form by setting \( \omega(i,j) = 0 \) at those \((x_i, y_j)\) points on the clamped edge, considering the plate and its loading to be symmetric about this root edge, and specifying that the displacements at any node point \((x_i, y_j)\) be equal to the displacement of its corresponding node
point or image point \((X_{i,j}, Y_{i,j})\) across the axis of symmetry. Introducing the Lagrange multiplier \(\lambda(i,j)\) the ordinary minimal problem may be restated as follows:

\[
\min \bar{P} = \min \left[ P + \sum_{i,j} \lambda(i,j) \left( \psi(i,j) - \psi(i',j') \right) \right],
\]

Regarding both \(\psi(i,j)\) and \(\psi(i',j')\) as independent variables we have

\[
\frac{\partial \bar{P}}{\partial \psi(i,j)} = \frac{\partial \bar{P}}{\partial \psi(i',j')} + \lambda(i,j) = 0,
\]

\[
\frac{\partial \bar{P}}{\partial \psi(i',j')} = \frac{\partial \bar{P}}{\partial \psi(i,j')} - \lambda(i',j) = 0.
\]

Eliminating \(\lambda(i',j')\) between (Eq. 107) and (Eq. 108) gives

\[
\frac{\partial \bar{P}}{\partial \psi(i,j)} + \frac{\partial \bar{P}}{\partial \psi(i',j')} = 0
\]

with \(\psi(i,j) = \psi(i',j')\) at all \((X_{i,j}, Y_{i,j})\) off the clamped edge. But taking \(\partial \bar{P}/\partial \psi(i,j)\) and replacing each displacement contained therein by the corresponding displacement of its image point gives \(\partial \bar{P}/\partial \psi(i',j') = \partial \bar{P}/\partial \psi(i,j)\). Thus \(\partial \bar{P}/\partial \psi(i,j) = 0\) and general difference equation 94 remains valid at all unconstrained node points. Consider now two examples of the use of (Eq. 94).

**Example I** - The first example is an application of the difference equation method to the solution of the influence coefficients of a square
plate of uniform thickness, clamped on one side and free on the other three. See Fig. 9.

Fig. 9 -- Cantilevered Square Plate

The root constraints are satisfied by setting $\omega(15) = \omega(9)$. 
Enforcing difference equation 94 at the unconstrained note points

\[ u(1'), \ldots, u(12), u(1'), \ldots, u(12') \]

gives the following coefficient matrix

\[ [K] = \begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix} \]

of system \[ [K](\omega) = \frac{4h^2}{D}(\phi) \]

where

\[ (\omega) = (\omega(1), \omega(2), \ldots, \omega(12), \omega(1'), \omega(1'), \ldots, \omega(12')) \]

\[ (\phi) = (\phi(1), \phi(2), \ldots, \phi(12), \phi(1'), \phi(1'), \ldots, \phi(12')) \]
$$K_\mu = \begin{bmatrix}
1 & 0 & \mu & -2(1+\mu) & \mu & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 2\mu & -4(1+\mu) & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
\mu & 0 & 1 & -2(1+\mu) & 1 & 0 & \mu & 0 & 0 & 0 & 0 & 0 \\
-2(1+\mu) & 2\mu & -2(1+\mu) & 20 & -2(7-\mu) & 2\mu & -2(7-\mu) & 4(2-\mu) & 0 & 2 & 0 & 0 \\
\mu & -4(1+\mu) & 1 & -2(7-\mu) & 39 & 0 & (8-3\mu) & -4(7-\mu) & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 2\mu & 0 & 2 & -4(1+\mu) & 2 & 0 & 2\mu & 0 & 0 \\
1 & 0 & \mu & -2(7-\mu) & (8-3\mu) & -4(1+\mu) & 39 & -4(7-\mu) & 2\mu & -16 & 2(4-\mu) & 0 \\
0 & 2 & 0 & 4(2-\mu) & -4(7-\mu) & 2 & -4(7-\mu) & 76 & 0 & 2(4-\mu) & -32 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\mu & 0 & 2 & -4(1+\mu) & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\mu & 0 & 2(4-\mu) & -32 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\mu & 0 & 2(4-\mu) & 82 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\mu & 0 & 1 
\end{bmatrix}$$
\[ [K_{12}] = \\
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2\mu & 0 & 2 & -16 & 0 & 0 & 2(4-\mu) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2(4-\mu) & 0 & 4 & -32 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \]
Inversion of this matrix (for $\mu = 0.3$) by an electronic digital computer gave the following values for the influence coefficients $\mathcal{C}_{ij}$ as defined by

$$\mathcal{U}(i) = \frac{a^2}{D} \sum_{j} \mathcal{C}_{ij} \mathcal{F}(j)$$

where $\mathcal{F}(j)$ is the magnitude of the force acting at node point $(j)$ and $a$ is the length of one side of the plate.

**TABLE 1.** -- Influence coefficients $\mathcal{C}_{ij}$ for a square cantilevered plate

<table>
<thead>
<tr>
<th>$\mathcal{U}^*$ measured at $i =$</th>
<th>4</th>
<th>5</th>
<th>7</th>
<th>8</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{F}(j)$ applied at $j =$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.518</td>
<td>0.401</td>
<td>0.278</td>
<td>0.217</td>
<td>0.088</td>
<td>0.068</td>
</tr>
<tr>
<td>5</td>
<td>0.401</td>
<td>0.399</td>
<td>0.211</td>
<td>0.208</td>
<td>0.062</td>
<td>0.046</td>
</tr>
<tr>
<td>7</td>
<td>0.278</td>
<td>0.211</td>
<td>0.192</td>
<td>0.132</td>
<td>0.070</td>
<td>0.064</td>
</tr>
<tr>
<td>8</td>
<td>0.217</td>
<td>0.208</td>
<td>0.132</td>
<td>0.128</td>
<td>0.044</td>
<td>0.044</td>
</tr>
<tr>
<td>10</td>
<td>0.088</td>
<td>0.062</td>
<td>0.170</td>
<td>0.044</td>
<td>0.048</td>
<td>0.023</td>
</tr>
<tr>
<td>11</td>
<td>0.068</td>
<td>0.064</td>
<td>0.046</td>
<td>0.044</td>
<td>0.023</td>
<td>0.024</td>
</tr>
<tr>
<td>4'</td>
<td>0.239</td>
<td>0.309</td>
<td>0.112</td>
<td>0.164</td>
<td>0.023</td>
<td>0.047</td>
</tr>
<tr>
<td>5'</td>
<td>0.309</td>
<td>0.352</td>
<td>0.155</td>
<td>0.186</td>
<td>0.040</td>
<td>0.056</td>
</tr>
<tr>
<td>7'</td>
<td>0.112</td>
<td>0.155</td>
<td>0.054</td>
<td>0.088</td>
<td>0.011</td>
<td>0.028</td>
</tr>
<tr>
<td>8'</td>
<td>0.162</td>
<td>0.186</td>
<td>0.088</td>
<td>0.108</td>
<td>0.025</td>
<td>0.038</td>
</tr>
<tr>
<td>10'</td>
<td>0.023</td>
<td>0.044</td>
<td>0.011</td>
<td>0.025</td>
<td>0.002</td>
<td>0.010</td>
</tr>
<tr>
<td>11'</td>
<td>0.048</td>
<td>0.056</td>
<td>0.028</td>
<td>0.036</td>
<td>0.010</td>
<td>0.016</td>
</tr>
</tbody>
</table>

- Experimental values obtained from (17).
Example II.—As a second example, the generalized finite difference method will be applied to the difficult problem of determining the bending moments in a tapered plate. This problem, selected from reference 11, consists of a square plate linearly tapered in one direction and loaded by concentrated forces at the four corners (see Fig. 10). This structure is well suited to the finite difference method because its symmetry and antisymmetry properties conveniently reduce the number of node points by a factor of four. Only one quadrant of the plate need be considered.

Fig. 10 -- Tapered Plate
Using the net of Fig. 11, enforcing difference equation 94 at each unconstrained node point, and recognizing the antisymmetry of the displacements gives the following system of difference equations (for $\mu = 0.3$) written in matrix notation and partitioned as indicated below.

\[
\begin{pmatrix}
\frac{w(1)}{D_0} \\
\vdots \\
\frac{w(16)}{D_0}
\end{pmatrix}
\begin{pmatrix}
A \\
B
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix}
\]
$$[A] = \begin{bmatrix}
3107.8 & -2019 & 256 & 0 & -2653.4 & 1247.4 & 0 & 0 & 500 & 0 & 0 & 0 \\
-2019 & 6687.6 & -2351.8 & 256 & 1359 & -5306.8 & 1397.4 & 0 & 0 & 1000 & 0 & 0 \\
256 & -2351.8 & 6215.6 & -2351.8 & 0 & 1397.4 & -5306.8 & 1397.4 & 0 & 0 & 1000 & 0 \\
0 & 256 & -2351.8 & 5591.6 & 0 & 0 & 1397.4 & -5306.8 & 0 & 0 & 0 & 1000 \\
-2653.4 & 1359 & 0 & 0 & 10076 & -5764 & 1000 & 0 & -5409.8 & 2422.6 & 0 & 0 \\
1247.4 & -5306.8 & 1397.4 & 0 & -6784 & 19562 & -8084 & 1000 & 2531.8 & -10910.6 & 2681.8 & 0 \\
0 & 1397.4 & -5306.8 & 1397.4 & 1000 & -8084 & 20152 & -8084 & 0 & 2681.8 & -10910.6 & 2681.8 \\
0 & 0 & 1397.4 & -5306.8 & 0 & 1000 & -8084 & 19152 & 0 & 0 & 2681.8 & -10910.6 \\
500 & 0 & 0 & 0 & -5409.8 & 2351.8 & 0 & 0 & 17524.8 & -11678.4 & 1728 & 0 \\
0 & 1000 & 0 & 0 & 2422.6 & -10819.6 & 2681.8 & 0 & -11678.4 & 34185.5 & -13924.8 & 1728 \\
0 & 0 & 1000 & 0 & 0 & 3681.8 & -10819.6 & 2681.8 & 1728 & -13924.8 & 35048.6 & -13924.8 \\
0 & 0 & 0 & 0 & 1000 & 0 & 2351.8 & -10819.6 & 0 & 1728 & -13924.8 & 3321.8 \\
0 & 0 & 0 & 0 & 0 & 864 & 0 & 0 & -8889.4 & 4158.2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1728 & 0 & 0 & 4005.8 & -17778.8 & 4417.4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1728 & 0 & 0 & 0 & 4417.4 & -17778.8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1728 & 0 & 0 & 4417.4 & -17778.8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 411.6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 150 & 0 & 0 \\
-332.8 & 128 & 0 & 0 & -1300 & 500 & 0 & 0 & -2246.4 & 864 & 0 & 0 \\
-332.8 & 38.4 & 0 & 0 & 128 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
76.8 & -665.6 & 76.8 & 0 & 0 & 256 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 76.8 & -665.6 & 76.8 & 0 & 0 & 256 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 76.8 & -665.6 & 0 & 0 & 0 & 256 & 0 & 0 & 0 & 0 
\end{bmatrix}$$
The solution to this system of equations as given by the electronic digital computer is

\[
\begin{align*}
d_0 \omega (1) &= 3.4500983 & d_0 \omega (13) &= 0.6449794 \\
d_0 \omega (2) &= 2.5180997 & d_0 \omega (14) &= 0.5105558 \\
d_0 \omega (3) &= 1.6457606 & d_0 \omega (15) &= 0.3502687 \\
d_0 \omega (4) &= 0.8137235 & d_0 \omega (16) &= 0.1791948 \\
d_0 \omega (5) &= 2.2936668 & d_0 \omega (A) &= 0.7521673 \\
d_0 \omega (6) &= 1.7411697 & d_0 \omega (B) &= 1.6330568 \\
d_0 \omega (7) &= 1.1646941 & d_0 \omega (C) &= 2.7731205 \\
d_0 \omega (8) &= 0.5826904 & d_0 \omega (D) &= 4.3820968 \\
d_0 \omega (9) &= 1.3807125 & d_0 \omega (E) &= 4.6065297 \\
d_0 \omega (10) &= 1.0752020 & d_0 \omega (F) &= 3.2771318 \\
d_0 \omega (11) &= 0.7326725 & d_0 \omega (G) &= 2.1147365 \\
d_0 \omega (12) &= 0.3703990 & d_0 \omega (H) &= 1.0392624 \\
\end{align*}
\]

Using these values of the displacements, the bending moments may be computed from the equations

\[
\begin{align*}
M_x &= -D \{ \omega_{xx} + \mu \omega_{yy} \} \approx -16 D \{ \omega(i+1,j) - 2 \omega(i,j) \\
&+ \omega(i-1,j) + 0.3 \omega(i,j-1) - 0.6 \omega(i,j) + 0.3 \omega(i,j+1) \} , \\
M_y &= -D \{ \omega_{yy} + \mu \omega_{xx} \} \approx -16 D \{ \omega(i,j+1) - 2 \omega(i,j) \\
&+ \omega(i,j-1) + 0.3 \omega(i+1,j) - 0.6 \omega(i,j) + 0.3 \omega(i-1,j) \} , \\
M_{xy} &= -D(-\mu) \omega_{xy} \approx -2.8 D \{ \omega(i+1,j+1) - \omega(i-1,j+1) \\
&+ \omega(i-1,j-1) - \omega(i+1,j-1) \} .
\end{align*}
\]
Their values are shown in Fig. 12 along with those which were approximately determined by Fung's method (11). When compared with experimental data the generalized finite difference solution is in better agreement than Fung's solution. This is a remarkable fact when we consider that the generalized finite difference solution involved only a matrix inversion of order 24 while Fung's solution necessitated an inversion of order greater than 100. The plate on which the experimental data was obtained was a 10" square steel plate with a maximum thickness of 1/4" at the center tapering to 1/8" at two opposite sides. Strain gages were placed at node points 5, 7, 9, 10, 11, 13, and 15 of Fig. 11 measuring outer fiber strain in the y direction. The corner loads were 125.7# each. The theoretical and experimental strain values are given in Fig. 13.
<table>
<thead>
<tr>
<th>( M_x ) (GEN. FINITE DIFF.)</th>
<th>( M_{xy} ) (GEN. FINITE DIFF.)</th>
<th>( M_x ) (FENG)</th>
<th>( M_{xy} ) (FENG)</th>
<th>( M_y ) (GEN. FINITE DIFF.)</th>
<th>( M_y ) (FENG)</th>
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<tr>
<td>-0.109</td>
<td>0.177</td>
<td>-0.073</td>
<td>0.387</td>
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<td>0.379</td>
<td>-0.000</td>
<td>0.318</td>
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<td>-0.038</td>
<td>0.791</td>
<td>-0.036</td>
<td>0.683</td>
<td>-0.019</td>
<td>0.624</td>
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<td>0.046</td>
<td>0.607</td>
<td>0.021</td>
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</tr>
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<td>-0.866</td>
<td>-0.185</td>
<td>-0.072</td>
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<tr>
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<tr>
<td>-0.989</td>
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<td>0.071</td>
<td>0.946</td>
<td>0.037</td>
<td>0.924</td>
</tr>
<tr>
<td>-1.086</td>
<td>-0.307</td>
<td>-0.126</td>
<td>-0.128</td>
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<td></td>
</tr>
<tr>
<td>-1.149</td>
<td>-0.633</td>
<td>-0.316</td>
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<tr>
<td>-0.858</td>
<td>-0.502</td>
<td>-0.260</td>
<td>-0.108</td>
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<tr>
<td>-0.952</td>
<td>-0.520</td>
<td>-0.258</td>
<td>-0.108</td>
<td></td>
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<tr>
<td>LINE OF SYMM.</td>
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</tr>
<tr>
<td>0</td>
<td>1.149</td>
<td>1.147</td>
<td>1.144</td>
<td>1.148</td>
<td>1.734</td>
</tr>
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Fig. 12 -- Comparison of Bending-Twisting Moments
Fig. 13 -- Outer Fiber Strain, $\varepsilon_y$
JUSTIFYING THE METHOD

To condense the text of this and later sections, reconsider the shift operator and shifting matrix notations used previously on beam problems. The extensions of these concepts into two dimensions is straightforward. The east shifting operator \( \mathcal{E} \) is defined by
\[
\mathcal{E} \mathcal{Y}(i,j) = \mathcal{Y}(i+1,j) \quad \text{unless } (x_i y_j) \text{ is at the east end of an east-west net line in which case } \mathcal{Y}(i+1,j) \text{ is understood to mean the value of } \mathcal{Y} \text{ at the west end of the net line.}
\]
The corresponding shifting matrix \([\mathcal{E}]\) is again a matrix which transforms the vector \((\mathcal{Y})\) whose elements are the \( \mathcal{Y}(i,j) \) 's into another vector which differs from \((\mathcal{Y})\) in that each \( \mathcal{Y}(i,j) \) is replaced by \( \mathcal{Y}(i+1,j) \). As before, \([\mathcal{E}]\) may be obtained by a rearrangement of the rows of an identity matrix of the appropriate size. \([\mathcal{E}]\) may alternately be considered a partitioned diagonal matrix whose non-zero elements are the one dimensional east shifting matrices for each east-west row of the net. The north shifting operator \( \mathcal{N} \) and matrix \([\mathcal{N}]\) are analogously defined. As before
\[
[u] = [\mathcal{E}]^T \quad \text{and} \quad [\mathfrak{S}] = [\mathcal{N}]^T.
\]

Let us now apply these concepts to the proof of the uniqueness of the solution to a complete system of general difference equations, all of the same form as (Eq. 94). This is achieved by proving that the homogeneous system has only the null solution which is equivalent to showing that the coefficient matrix is non-singular, i.e. has a unique inverse.
In the absence of any applied loads the total potential energy of a deformed plate can be written in the obviously non-negative form:

\[ T[\omega] = \iint_\Omega \frac{D}{2} \left\{ (1-\nu)(\omega_{xx}^2 + \omega_{yy}^2 + 2\omega_{xy}^2) + \nu (\omega_{xx} + \omega_{yy})^2 \right\} \, dx \, dy. \]

This integral may be approximated in terms of the shifting matrices \([N], [G], [E], [W]\) and the vector \((u)\), consisting of the discrete \(\omega^{(i,j)}\) displacements, by

\[ T \approx \mathcal{P} = \frac{1}{2} \left\{ (\omega_{xx})^T [e] (\omega_{xx}) + (\omega_{yy})^T [e] (\omega_{yy}) \right\} + 2 (\omega_{xy})^T [e] (\omega_{xy}) + \frac{\mu}{2} (\omega_{xx} + \omega_{yy})^T [e] (\omega_{xx} + \omega_{yy}) \]

where

\[ (\omega_{xx}) = \frac{1}{\nu} [e] - 2[I] + [W](u) = \frac{1}{\nu} [X](u) \]

\[ (\omega_{yy}) = \frac{1}{\nu} [G] - 2[I] + [S](u) = \frac{1}{\nu} [Y](u) \]

\[ (\omega_{xy}) = \frac{1}{\nu} [N][E] - [N] + [I] - [E](u) = \frac{1}{\nu} [Z](u) \]

and \([e], [e]\) are diagonal matrices whose elements are the \(e(i,j)\) and \(e(i,j)\) quantities respectively. Note that since all diagonal elements of \([e]\) and \([e]\) are greater than or equal to zero, setting \(\mathcal{P} = 0\) is equivalent to setting the components of \((\omega_{xx})\) and \((\omega_{yy})\) to zero at each node point for which \(e(i,j) \neq 0\) and the components of \((\omega_{xy})\) to zero on each net square in which \(e(i,j) \neq 0\). \(\mathcal{P}\) may now be condensed to
\[ P = \frac{1}{2\hbar^4} \begin{bmatrix} K \end{bmatrix} (\omega) \]

where
\[ \begin{bmatrix} K \end{bmatrix} = \begin{bmatrix} [X][\xi][X] + [Y][\eta][Y] + 2[Z][\xi][Z] \\ + \mu [X+Y][\xi][X+Y] \end{bmatrix} \]

According to the rule for differentiating a quadratic form, the homogeneous system of general difference equations is given by

\[ \frac{1}{2\hbar^4} \frac{\partial}{\partial \omega(i,j)} (\omega)^T [K] (\omega) = \frac{1}{\hbar^4} [K] (\omega) = 0. \]

If \((\omega)\) is a solution to this system then \([K] (\omega) = 0\). Premultiplying by \((\omega)^T\) gives \((\omega)^T [K] (\omega) = 0\). But, for this to hold the elements of the difference expressions \((\omega_{xx})\) and \((\omega_{yy})\) must vanish at each net point for which \(\xi(i,j) \neq 0\) and \((\omega_{xy})\) at each net square for which \(\xi(i,j) \neq 0\). Thus, the proof is reduced to that of specifying sufficient conditions on the region of the middle plane to guarantee the vanishing of \((\omega)\) from the vanishing of certain elements of \((\omega_{xx})\), \((\omega_{yy})\), and \((\omega_{xy})\).

To accomplish this we choose three "live" net points, i.e. points for which \(\xi(i,j) \neq 0\), all lying at the corners of a primary net square. The vertical deflections of these three base points define a plane. It will be shown that all net deflections must lie in this base plane.

Assume first that all of the live net points are simply connected, i.e. each live net point can be reached from any other live net point by moving only on line segments of the primary net which
connect adjacent live net points. This assumption is analogous to the notion of simple connectedness of regions that must be used in the uniqueness proof for the solution of the continuum boundary value problem. Another assumption must be made which has no parallel in the corresponding proof for the continuum boundary value problem for reasons which will be clear. Assume that adjacent to every line segment of the primary net which joins two live net points there is a live net area, i.e. a net square for which \( \mathcal{R}(i,j) \neq \emptyset \). The satisfaction of this condition is demonstrated in Fig. 14(a) below. Here, the net points \((x_i, y_i)\) and \((x_{i+1}, y_{i+1})\) are within the middle plane region and both adjacent net areas, \( \mathcal{R}_c(i,j) \) and \( \mathcal{R}_c(i,j-1) \) contain some of the region of the middle plane. A violation of this condition is shown in Fig. 14(b). The line segment \((x_i, y_j : x_{i+1}, y_{j+1})\) connects live net points but neither adjacent area contains any of the region of the middle plane. The difficulty in this situation stems from the fact that a general difference equation unlike a differential equation has a finite region of dependence. The coefficients of a general difference equation written at one end point of such a line segment will be influenced by the presence of the middle plane area associated with the other end point in a completely erroneous manner if there is a gap in the region of the middle plane as shown in Fig. 14(b). This difficulty can always be remedied by refining the net.
We now proceed to prove that the deflections of all live net points must lie in the same plane if the previous assumptions hold.

By virtue of the first of the two previous assumptions any live net point can be linked to the base points through a series of adjacent live net points. It will be shown that the deflections of each of these points lie in the plane defined by the three base point deflections.

Starting at one of the base points, the path from the outlying live net point either meets one of the base points along an extension of one of the two line segments joining the three base points as shown in Fig. 15 or it meets one of the line segments at right angles as shown in Fig. 16.
Fig. 15 -- Simply Connected Region (I)

Fig. 16 -- Simply Connected Regions (II)
In the first case the deflection of point \( (x_{i+1,3}, y_{i+1,3}) \) must lie in the base plane because \( \varepsilon(i+1,3) \neq 0 \) implies that \( \omega(i+1,3) - 2\omega(i+1,3) + \omega(i,1,3) = 0 \) which is the condition that three equally spaced ordinates be collinear.

Furthermore, it can be shown that the deflection of each of the four net points nearest to \( (x_{i+2,3}, y_{i+2,3}) \) also lie in the base plane. Obviously \( \omega(i+1,3) \) does by assumption while \( \omega(i+3,3) \) does also because \( \varepsilon(i+2,3) \neq 0 \) implies \( \omega(i+3,3) - 2\omega(i+2,3) + \omega(i+1,3) = 0 \).

To show that \( \omega(i+2,3) \) and \( \omega(i+2,3) \) lie in the base plane note first that \( \omega(i,3+1) \) does because \( \varepsilon(i,3+1) \) implies that \( 0 = \omega(i,3+1) - 2\omega(i,3) + \omega(i,3-1) \). By the second basic assumption there is some middle plane area associated with one of the two squares \( \overline{R}(i,3) \) and \( \overline{R}(i,3-1) \). If it is the former then \( \omega(i+1,3+1) \) lies in the base plane since \( \omega(i+1,3+1) - \omega(i,3+1) + \omega(i,3) - \omega(i+1,3) \) must vanish and this is precisely the requirements that four ordinates erected at the corners of a rectangle be coplanar. It then follows, since \( \varepsilon(i+1,3) > 0 \), that

\[
\omega(i+1,3+1) - 2\omega(i+1,3) + \omega(i+1,3-1) = 0.
\]

Thus \( \omega(i+1,3-1) \) lies in the base plane.

If there were some middle plane area associated with square \( \overline{R}(i,3-1) \) instead of \( \overline{R}(i,3) \) the same result could have been reached by almost the same chain of logic. The location of \( \omega(i+2,3+1) \) and \( \omega(i+2,3-1) \) in the base plane is deduced by the same logic used on \( \omega(i+1,3+1) \) and \( \omega(i+1,3-1) \). Thus we have shown that \( \omega(i+2,3) \) and the deflection of its four nearest node points lie in the base plane. This argument may be extended along the path of live net points until either the outlying net point is reached or the path turns. Whenever the path turns a new trio of base points is established whose deflections all
lie in the original base plane. The outlying point may possibly be reached through a sequence of steps of this sort. No new arguments are needed to prove the same results when the path from the outlying net point to the base point joins one of the two line segments connecting the adjacent base points at angles as shown in Fig. 16. Thus by a finite sequence of steps we can show not only that the deflection of each live net point lies in a single plane but also that the deflection of each adjacent point does, whether it be a live net point or not. This last statement includes all net points involved in the general difference formulation of any middle plane region satisfying the two basic assumptions with the exception of the one special case shown in Fig. 17 wherein net point \( (x_{i-1}, y_{j+1}) \) is not a live net point itself or adjacent to a live net point but still is involved in the problem since some middle plane area appears in the net square \( \mathcal{R}(i,j) \).

Fig. 17 -- A Free Corner

Since neither \( (x_i, y_{j+1}) \), \( (x_{i+1}, y_j) \), or \( (x_{i-1}, y_{j+1}) \) are live net points \( (x_i, y_j) \) must be if there is some plate in square \( \mathcal{R}(i,j) \). But, by the previous discussion \( \mathcal{U}(i,j), \mathcal{U}(i+1,j), \) and \( \mathcal{U}(i+1,j+1) \) must all be
in the base plane. Therefore, $\mathbf{u}^T \left((i,j) - (i,j+1) + (i,j+1) - (i+1,j)\right) = 0$ and $\mathbf{u}^T (i+1,j+1)$ lies in the base plane too. It has now been shown that the deflection of each node point involved in a general difference formulation lies in the same plane. If the plate is constrained such that the deflections must be zero at a minimum of three non-collinear node points then the null solution is the only coplanar one which does not violate the constraints and hence represents the only solution to the homogeneous problem.

The same minimal problem may be restated as

$$\min \left[ \sum_{i,j} \mathbf{u}^T (i,j) \mathbf{u}^T (i,j) \right] \text{ with } \mathbf{u}^T (i,j) = 0 \text{ for all } (i,j)$$

where $(i,j)$ designates a node point whose deflection must be zero and $\lambda (i,j)$ a Lagrange multiplier. Taking only those equations

$$\frac{\partial}{\partial \mathbf{u}^T (i,j)} \left[ \sum_{i,j} \mathbf{u}^T (i,j) \mathbf{u}^T (i,j) \right] = 0 , \quad (i,j) \neq (i,j)$$

and setting each $\mathbf{u}^T (i,j) = 0$ results in a system of $n$ equations in $n$ unconstrained nodal deflections. Since the two minimal problems are identical, the zero solution is again the only solution. This system is the same as the system of general difference equations obtained by setting $\partial \mathbf{P}/\partial \mathbf{u}^T (i,j) = 0$ for each unconstrained $\mathbf{u}^T (i,j)$ deflection and setting all of the remaining deflections to zero. Thus, a unique solution always exists for any loading.

The problem of proving the convergence of the solutions to the systems of difference equations obtained by diminishing the mesh width to the solution of the continuum boundary value problem for an arbitrary
region $R$ and set of constraints is considerably more difficult than the previous proof and will not be considered here.
ORTHOTROPIC STIFFENED PLATES

This section will be concerned with the extension of the general difference method to problems associated with the bending of antisotropic, sandwich plates reinforced by beam stiffeners. This problem is of interest because this type of construction is commonly used in missile wings. Fig. 18 shows a typical structure of this type with the thickness exaggerated for clarity.

![Diagram of Typical Missile Wing]

Fig. 18 -- Typical Missile Wing

The presence of a core material having difference elastic properties in different directions somewhat complicates the problem since a multitude of stiffness parameters are now associated with each point on the plate.
Consider first the general case of an anisotropic plate whose elastic properties are constant through the thickness.

Assuming all the stress components acting on planes parallel to the middle plane to be negligible, the stress-strain laws become

\[
\begin{align*}
\sigma_x &= a_{11} \varepsilon_x + a_{12} \varepsilon_y + a_{16} \gamma_{xy} \\
\sigma_y &= a_{12} \varepsilon_x + a_{22} \varepsilon_y + a_{26} \gamma_{xy} \\
\tau_{xy} &= a_{16} \varepsilon_x + a_{26} \varepsilon_y + a_{66} \gamma_{xy}
\end{align*}
\]

The symmetry of the coefficient matrix of the previous system follows from the existence of the strain energy density as a function of the final state variables only and the theory of exact differentials. The strain energy

\[
U = \frac{1}{2} \int \int \int_{V_{o1}} (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy} \gamma_{xy}) \, dx \, dy \, dz
\]

becomes

\[
U = \frac{1}{2} \int \int \int_{V_{o1}} (a_{11} \varepsilon_x^2 + a_{22} \varepsilon_y^2 + a_{66} \gamma_{xy}^2 + 2 a_{12} \varepsilon_x \varepsilon_y \\
+ 2 a_{16} \varepsilon_x \gamma_{xy} + 2 a_{26} \varepsilon_y \gamma_{xy}) \, dx \, dy \, dz.
\]

From geometric considerations

\[
\varepsilon_x = -Z \omega_{xx}, \quad \varepsilon_y = -Z \omega_{yy}, \quad \gamma_{xy} = -2 \sqrt{Z} \omega_{xy}
\]

leaving

\[
U = \frac{1}{2} \int \int_{R} \left\{ D_{11} \omega_{xx}^2 + 2 D_{12} \omega_{xx} \omega_{yy} + D_{22} \omega_{yy}^2 + 4 D_{66} \omega_{xy}^2 \\
+ 4 \left( D_{16} \omega_{xx} + D_{26} \omega_{yy} \right) \omega_{xy} \right\} \, dx \, dy
\]
The total potential energy, the sum of equations 114 and 82, is

$$T = \frac{1}{2} \iint_{R} \left\{ D_{11} \omega_{x}^2 + 2D_{12} \omega_{x} \omega_{y} + D_{22} \omega_{y}^2 + 4D_{66} \omega_{xy}^2 + 4 \left( D_{16} \omega_{x} + D_{26} \omega_{y} \right) \omega_{xy} - 2G \omega_{y} \right\} dxdy. $$

Formula 116 is subject to the same limitations as the potential energy expression 79 for isotropic plates.

In the special case of orthotropy with the principal axes parallel to the $x$, $y$ axes

$$
\begin{align*}
\sigma_x &= \frac{E_i}{1-\mu_1\mu_2} \varepsilon_x + \frac{E_2}{1-\mu_1\mu_2} \varepsilon_y \\
\sigma_y &= \frac{E_2}{1-\mu_1\mu_2} \varepsilon_x + \frac{E_i}{1-\mu_1\mu_2} \varepsilon_y \\
\tau_{xy} &= G \gamma_{xy}
\end{align*}
$$

and (Eq. 114) simplifies to

$$U = \frac{1}{2} \iint_{R} \left\{ D_1 \omega_{x}^2 + (\mu_2 D_1 + \mu_1 D_2) \omega_{x} \omega_{y} + D_2 \omega_{y}^2 + 4D_6 \omega_{xy}^2 \right\} dxdy$$

where

$$
\begin{align*}
D_1 &= \frac{E_i h^3}{12 (1-\mu_1\mu_2)} \\
D_2 &= \frac{E_2 h^3}{12 (1-\mu_1\mu_2)} \\
D_6 &= \frac{G h^3}{12}.
\end{align*}
$$

If the directions of the principal axes of orthotropy do not correspond with the coordinate axes then the six stiffness parameters of general anisotropy become linear combinations of the three stiffness parameters.
of orthotropy. Formulae for computing these stiffness parameters are (15)

\[
\begin{align*}
D_{11} &= D_1 \cos^2 \phi + 2D_6 \sin^2 \phi \cos \phi \cos \phi + D_2 \sin^4 \phi \\
D_{22} &= D_1 \sin^4 \phi + 2D_6 \sin^2 \phi \cos \phi + D_2 \cos^4 \phi \\
2D_{66} &= D_6 - \mu_1 D_1 + 2(D_1 + D_2 - 2D_6) \sin^2 \phi \cos^2 \phi \\
D_{12} &= \mu_1 D_2 + (D_1 + D_2 - 2D_6) \sin^2 \phi \cos^2 \phi \\
2D_{16} &= (D_2 \sin^2 \phi - D_1 \sin^2 \phi + D_6 \sin^2 \phi) \sin 2\phi \\
2D_{66} &= (D_2 \sin^2 \phi - D_1 \sin^2 \phi - D_6 \sin^2 \phi) \sin 2\phi
\end{align*}
\]

where $\phi$ is the angle between the x axis and the principal axis referred to by subscript 1.

If the plate consists of an orthotropic core bonded to isotropic cover skins so that there is no relative motion of adjacent points across the interfaces during bending then the composite plate may be treated as a single orthotropic plate. Pertinent formulae, analogous to those already presented, may be derived by regarding the elastic constants as functions of $z$. For three layers geometrically and elastically symmetric about the middle plane formulae 119 become

\[
\begin{align*}
\overline{D}_1 &= \frac{1}{12} \left\{ \frac{E_1 t_z^3}{(1-\mu_1 \mu_2)} + \frac{E_2}{(1-\mu_2^2)} (t_1^3 - t_2^3) \right\} \\
\overline{D}_2 &= \frac{1}{12} \left\{ \frac{E_1 t_z^3}{(1-\mu_1 \mu_2)} + \frac{E_2}{(1-\mu_2^2)} (t_1^3 - t_2^3) \right\} \\
\overline{D}_6 &= \frac{1}{12} \left\{ G_{12} t_z^3 + G(t_1^3 - t_2^3) \right\} + \frac{1}{12} \left\{ \frac{(\mu_1 E_2 + \mu_2 E_1) t_z^3}{2(1-\mu_1 \mu_2)} + \frac{\mu_1 E_2}{(1-\mu_2^2)} (t_1^3 - t_2^3) \right\}
\end{align*}
\]
and in place of the two Poisson ratios appearing in (Eq. 118) and (Eq. 120) substitute the Poisson ratios of the composite material as given by

\[
\begin{align*}
\bar{\mu}_1 &= \frac{1}{12 D_2} \left\{ \frac{(\mu_1 E_2 + \mu_2 E_1) t_2^3}{2 (1 - \mu_1 \mu_2)} + \frac{\mu E}{(1 - \mu_2)} (t_1^3 - t_2^3) \right\} \\
\bar{\mu}_2 &= \frac{1}{12 D_1} \left\{ \frac{(\mu_1 E_2 + \mu_2 E_1) t_2^3}{2 (1 - \mu_1 \mu_2)} + \frac{\mu E}{(1 - \mu_2)} (t_1^3 - t_2^3) \right\}.
\end{align*}
\]

The quantities \( E, G, \mu \) are the usual elastic constants of the isotropic skin material. \( E_1, E_2, \mu_1, \mu_2, G_\perp \) are the usual elastic constants of the orthotropic core with the subscripts referring to the two principal axes of orthotropy in the plane of the plate. \( t_1 \) is the total thickness of the plate. \( t_2 \) is the thickness of the core.

![Fig. 19 -- Sandwich Plate Cross-section](image)

We will now derive a general difference equation which will apply to orthotropic plates. The procedure for doing this is identical to that used in the case of a beam of non-constant cross section and an isotropic plate. First, a functional expression for the total potential
energy of the elastic system is obtained, (Eq. 116). The derivatives appearing in this expression are then replaced by their finite difference approximations and the total potential energy is integrated numerically. To find the true equilibrium position this approximation to the total potential energy is minimized with respect to each discrete unconstrained displacement. This process yields a general finite difference equation which applies at all unconstrained node points of the finite difference net. These steps will now be carried out in the more concise shift-
matrix notation.

Choose a rectangular finite difference net of mesh width $a$ in the $x$ direction and $b$ in the $y$ direction and consider the integrand of (Eq. 116) as a step function, constant over each rectangular subregion bounded by the lines $X = X_i$, $X_i \pm \frac{a}{2}$; $Y = Y_j$, $Y_j \pm \frac{b}{2}$ of the $x$, $y$ plane. Replace the derivatives in the integrand of (Eq. 116) by their finite difference approximations given by (Eq. 83), (Eq. 88), (Eq. 89) and integrate over the region $\mathcal{R}$ to get

$$
\mathbf{T} = \mathbf{P} = (\mathbf{w}^T \left[ \frac{1}{2a^2} \mathbf{X}[X][Y][Z] + \frac{1}{2b^2} \mathbf{Y}[X][Y][Z] \right] + \frac{1}{2a^2} \mathbf{X}[X][Z][Z] + \frac{1}{2b^2} \mathbf{Y}[X][Y][Z] + \frac{2}{b^2} \mathbf{X}[X][Y][W][Z] + \frac{2}{a^2} \mathbf{Y}[X][Y][W][Z] + \frac{2}{b^2} \mathbf{X}[X][Y][W][Z] + \frac{2}{a^2} \mathbf{Y}[X][Y][W][Z] + \frac{2}{a^2} \mathbf{X}[X][Y][W][Z] + \frac{2}{a^2} \mathbf{Y}[X][Y][W][Z] + \frac{2}{a^2} \mathbf{Y}[X][Y][W][Z] + \frac{2}{a^2} \mathbf{Y}[X][Y][W][Z] + \frac{2}{a^2} \mathbf{Y}[X][Y][W][Z] \right] (\mathbf{w}) - (\mathbf{w}^T \left[ \alpha \right] (\mathbf{q})$$
where $[D_{mn}]$ is a diagonal matrix whose main diagonal elements are the values of $D_{mn}$ at each $(x_i, y_i)$ node point; $[\beta], [\sigma], [\delta], [\varepsilon]$ the diagonal matrices whose diagonal elements are the areas of the middle plane contained in the rectangles $\Theta(i,j), \Upsilon(i,j), \Delta(i,j), \varepsilon(i,j)$ (see Fig. 20) defined by

$$
x_i \leq x \leq x'_i + \frac{a}{2}, \quad y_j \leq y \leq y'_j + \frac{b}{2};
$$

$$
x_i \geq x \geq x'_i - \frac{a}{2}, \quad y_j \leq y \leq y'_j + \frac{b}{2};
$$

$$
x_i \geq x \geq x'_i, \quad y_j \geq y \geq y'_j - \frac{b}{2};
$$

$$
x_i \leq x \leq x'_i + \frac{a}{2}, \quad y_j \geq y \geq y'_j - \frac{b}{2};
$$

respectively; and

$$
[\alpha] = [\beta] + [\sigma] + [\delta] + [\varepsilon],
$$

$$
[\bar{\alpha}] = [\beta] + [\varepsilon][\sigma][\delta] + [\eta][\varepsilon][\delta] + [\eta][\varepsilon].
$$

Fig. 20 -- The $\Theta, \Upsilon, \Delta, \varepsilon$ Rectangles
Minimizing $P$ with respect to $w_1^{(i,j)}$ gives the general difference equation

\begin{equation}
\left(12^{(14)}\right)
\begin{aligned}
\frac{1}{\alpha^4} [X][X][P_1][X] &+ \frac{1}{\alpha^4} [Y][X][P_2][Y] &+ \frac{1}{\alpha^4} [X][X][P_3][Y] \\
+ \frac{1}{\alpha^4} [Y][X][P_4][X] &+ \frac{1}{\alpha^4} [Z][X][P_5][Z] &+ \frac{2}{\alpha^4} [X][S][P_6][Z] \\
+ \frac{2}{\alpha^4} [Z][X][P_7][X] &+ \frac{2}{\alpha^4} [X][X][P_8][Z] &+ \frac{2}{\alpha^4} [X][X][P_9][Z] \\
+ \frac{2}{\alpha^4} [X][X][P_{10}][X] &+ \frac{2}{\alpha^4} [X][X][P_{11}][Z] &+ \frac{2}{\alpha^4} [X][X][P_{12}][Z] \\
+ \frac{2}{\alpha^4} [X][X][P_{13}][X] &+ \frac{2}{\alpha^4} [X][X][P_{14}][Z] &+ \frac{2}{\alpha^4} [X][X][P_{15}][Z] \\
+ \frac{2}{\alpha^4} [X][X][P_{16}][X] &+ \frac{2}{\alpha^4} [X][X][P_{17}][Z] &+ \frac{2}{\alpha^4} [X][X][P_{18}][Z] \\
+ \frac{2}{\alpha^4} [X][X][P_{19}][X] &+ \frac{2}{\alpha^4} [X][X][P_{20}][Z] &+ \frac{2}{\alpha^4} [X][X][P_{21}][Z] \\
+ \frac{2}{\alpha^4} [X][X][P_{22}][X] &+ \frac{2}{\alpha^4} [X][X][P_{23}][Z] &+ \frac{2}{\alpha^4} [X][X][P_{24}][Z]
\end{aligned}
\end{equation}

Rearranging terms we get

\begin{equation}
\left(12^{(25)}\right)
\begin{aligned}
\left[X\right] &\left\{ \frac{1}{\alpha^4} [X][P_1][X] + \frac{2}{\alpha^4} [X][P_2][X] + \frac{2}{\alpha^4} [X][P_3][X] \right\} \\
+ \left[Y\right] &\left\{ \frac{2}{\alpha^4} [Y][P_4][Y] + \frac{2}{\alpha^4} [Y][P_5][Y] + \frac{2}{\alpha^4} [Y][P_6][Y] \right\} \\
+ \left[Z\right] &\left\{ \frac{2}{\alpha^4} [Z][P_7][Z] + \frac{2}{\alpha^4} [Z][P_8][Z] + \frac{2}{\alpha^4} [Z][P_9][Z] \right\} \\
+ \left[X\right] &\left\{ \frac{2}{\alpha^4} [X][P_{10}][X] + \frac{2}{\alpha^4} [X][P_{11}][X] + \frac{2}{\alpha^4} [X][P_{12}][X] \right\} \\
+ \left[Y\right] &\left\{ \frac{2}{\alpha^4} [Y][P_{13}][Y] + \frac{2}{\alpha^4} [Y][P_{14}][Y] + \frac{2}{\alpha^4} [Y][P_{15}][Y] \right\} \\
+ \left[Z\right] &\left\{ \frac{2}{\alpha^4} [Z][P_{16}][Z] + \frac{2}{\alpha^4} [Z][P_{17}][Z] + \frac{2}{\alpha^4} [Z][P_{18}][Z] \right\} \\
+ \left[X\right] &\left\{ \frac{2}{\alpha^4} [X][P_{19}][X] + \frac{2}{\alpha^4} [X][P_{20}][X] + \frac{2}{\alpha^4} [X][P_{21}][X] \right\} \\
+ \left[Y\right] &\left\{ \frac{2}{\alpha^4} [Y][P_{22}][Y] + \frac{2}{\alpha^4} [Y][P_{23}][Y] + \frac{2}{\alpha^4} [Y][P_{24}][Y] \right\} \\
+ \left[Z\right] &\left\{ \frac{2}{\alpha^4} [Z][P_{25}][Z] + \frac{2}{\alpha^4} [Z][P_{26}][Z] + \frac{2}{\alpha^4} [Z][P_{27}][Z] \right\}
\end{aligned}
\end{equation}

\begin{equation}
= \left[X\right] (g)
\end{equation}
which reduces in the interior of the plate to the ordinary difference
equation corresponding to the differential equation of bending of an
anisotropic plate of non-constant thickness:

\[
\begin{align*}
(D_{11} u_{xx} + 2 D_{16} u_{xy} + D_{12} u_{yy})_{xx} & \\
+ 2(D_{16} u_{xy} + 2 D_{16} u_{xx} + D_{16} u_{yy})_{xy} & \\
+ (D_{12} u_{xx} + 2 D_{26} u_{xy} + D_{22} u_{yy})_{yy} & = q(x, y).
\end{align*}
\]

Due to its complexity, difference equation 125 is of interest only when
an electronic digital computer is available for setting up the coefficient
matrix.

The effects of a number of straight stiffeners, so arranged that
they intersect the lines of the primary net only at node points, are
easily incorporated into the general difference method if the neutral axes
of the stiffeners lie in the middle plane of the plate before and after
loading and if the twisting resistance of the stiffeners is negligible.
If these conditions are satisfied the stiffeners add a number of terms of
the form

\[
\frac{1}{2} \int_0^L \varepsilon_{xx}^2 \, ds
\]

to the total potential energy 116. Considering the most general case
wherein two beams parallel to the x and y axes, and two diagonal beams
pass through every \((x_i, y_j)\) node point the additional bending energy is
approximated by

\[
\begin{align*}
P' = & \frac{1}{2} (\pi) \left[ \frac{1}{\alpha^2} [X][\mathcal{K}][X] + \frac{1}{\beta^2} [Y][\mathcal{K}][Y] \right. \\
& \left. + \frac{1}{\gamma^2} [U][\mathcal{K}][U] + \frac{1}{\delta^2} [V][\mathcal{K}][V] \right] (\omega)
\end{align*}
\]
where \( c^2 = a^2 + b^2 \); 
\[
[U] = [Y][E] - 2[I] + [G][W] \\
[V] = [Y][W] - 2[I] + [G][E]
\]
and \([A], [K], [K], [K] \) are diagonal matrices corresponding to \([A] \) of a previous section (see page 33) for the east-west, north-south, northeast-southwest, northwest -southeast directed stiffeners respectively. Minimizing \( P' \) with respect to \( \omega^r(i,j) \) gives

\[
\left( \frac{1}{\alpha^4} [X][X] + \frac{1}{\beta^4} [Y][Y] + \frac{1}{\gamma^4} [U][U] \right) \omega = 0. 
\]

Adding this equation to equation 125 gives a complete general difference equation for a stiffened orthotropic plate. This complete equation has been applied to the problem of determining the static influence coefficients of the missile fin shown in Fig. 21. Table 2 gives the values of the influence coefficients determined by a 42 node point analysis as compared with experimental results. Fig. 22 gives the theoretical and experimental values of the five lowest frequencies and their nodal lines. The manner in which this vibration data was obtained from the static influence coefficients is explained in the next section.
Fig. 21 -- Missile Fin
Table 2r-Influence Coefficients (in./lb. x 10^3)

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Fig. 22 -- Nodal Lines and Frequencies

\[ \omega_1 \text{(theor.)} = 30.6 \text{ c.p.s.} \]
\[ \omega_1 \text{(exp.)} = 31.6 \text{ c.p.s.} \]

\[ \omega_2 \text{(theor.)} = 127.4 \text{ c.p.s.} \]
\[ \omega_2 \text{(exp.)} = 130.3 \text{ c.p.s.} \]

\[ \omega_3 \text{(theor.)} = 184.6 \text{ c.p.s.} \]
\[ \omega_3 \text{(exp.)} = 176.1 \text{ c.p.s.} \]

--- nodal lines (theor.)

___ nodal lines (exp.)

\[ \omega_4 \text{(theor.)} = 210.9 \text{ c.p.s.} \]
\[ \omega_4 \text{(exp.)} = 216.4 \text{ c.p.s.} \]
Although satisfactory vibration data was obtained with the theory, correlation between experimental and theoretical influence coefficients was poor. This may, in part, be attributable to misalignment and free play in the three supports as evidenced by the fact that the nodal lines (see Fig. 22) occasionally did not pass through support points.
Consider the problem of obtaining the natural frequencies and the corresponding mode shapes of a flat plate. Assuming that the oscillations are harmonic the pressure term $q(i,j)$ in general difference equations 94 and 125 should be replaced by the dynamic loading term $\sigma(i,j) \omega^4 \omega(i,j)$ where $\sigma(i,j)$ is the mass per unit area of the plate at node point $(x_i, y_j)$ and $\omega^2$ the square of the angular frequency. The enforcement of the general difference equation 94 or 125 at each unconstrained node point results in a system of homogeneous equations of the form

\[
(130) \quad \begin{bmatrix} A_{ii} - \omega^2 M & A_{i2} \\ A_{2i} & A_{22} \end{bmatrix} \begin{bmatrix} \omega' \\ \omega'' \end{bmatrix} = 0
\]

where $[M]$ is a non-singular, diagonal matrix (whose elements are the masses associated with each node point), $(\omega')$ the column of vertical displacements of node points whose associated $\varphi(i,j) \neq 0$, and $(\omega'')$ the column of displacements associated with node points for which $\varphi(i,j) = 0$. After partitioning as indicated in equation 130 it follows that

\[
(131) \quad \begin{bmatrix} A_{ii} \end{bmatrix} - \omega^2 [M] (\omega') + [A_{i2}](\omega'') = 0,
\]

\[
(132) \quad [A_{21}](\omega') + [A_{22}](\omega'') = 0.
\]
Premultiplying (Eq. 132) by \( [A_{xx}]^{-1} \) gives

\[
\omega'' = -[A_{xx}]^{-1} [A_{xl}] (\omega')
\]

Eliminating \( \omega' \) in (Eq. 131), using (Eq. 133), gives

\[
\left[ [A_{ii}] - \omega^2 [M] \right] (\omega') - [A_{ix}] [A_{xx}]^{-1} [A_{xl}] (\omega') = 0, \quad \text{or}
\]

\[
\left[ [A_{ii}] - [A_{ix}] [A_{xx}]^{-1} [A_{xl}] \right] (\omega') - \omega^2 [M] (\omega') = 0.
\]

The matrix

\[
[K] = [A_{ii}] - [A_{ix}] [A_{xx}]^{-1} [A_{xl}]
\]

is commonly called the stiffness matrix or the matrix of stiffness coefficients. Premultiplying (Eq. 135) by the matrix \( [M]^{-1} \) gives

\[
[M]^{-1} [K] - \omega^2 [I] (\omega') = 0.
\]

\( [D_{yn}] = [K]^{-1} [M] \) is called the dynamical matrix. The electronic digital computer may be assigned the task of calculating the elements of the matrix \( [D_{yn}] \) from the preceding formulae and any standard inverse routine, and then obtaining its largest eigenvalues (which are proportional to the reciprocals of the squares of the lowest natural frequencies) and the corresponding eigenvectors (which represent the mode shapes).

**Vibrations of a Triangular Cantilevered Plate.** Consider the problem of determining the lowest natural frequencies of the constant thickness cantilevered plate of Fig. 23.
Fig. 23 −− Triangular Cantilevered Plate

Setting $\mu = 0.3$, $\zeta = \frac{h^4 \sigma E}{D}$, and using the net of Fig. 23 the system of difference equations describing the free harmonic motion of the plate written in matrix form, is

$$
\begin{bmatrix}
[A] - \zeta [B]
\end{bmatrix} \begin{bmatrix}
\mathbf{w}
\end{bmatrix} = \mathbf{0}
$$

where

$$
\begin{bmatrix}
\mathbf{w}
\end{bmatrix}^T = (w_1, w_2, \ldots, w_8, w_{10}, w_{11}, \ldots, w_{16}, w_{17}, w_{18}, w_{20}, w_{21}, \ldots, w_{29}, w_{30}, w_{31}, w_{32})
$$
and \([B]\) is a diagonal matrix whose main diagonal is
\[
(16, 16, 14, 2, 15, 8, 1, 1, 0, 0, 0, 0, 0, 0, 15, 8, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 18, 8, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0).
\]
The lowest frequencies, \(\xi\) (c. p. s.), as given by the electronic digital computer are
\[
\begin{align*}
\xi_1 &= \frac{6.44}{2\pi a^2} \sqrt{D/\sigma} & \quad \left( \frac{6.46}{2\pi a^2} \sqrt{D/\sigma} \right) \\
\xi_2 &= \frac{23.5}{2\pi a^2} \sqrt{D/\sigma} & \quad \left( \frac{27.8}{2\pi a^2} \sqrt{D/\sigma} \right) \\
\xi_3 &= \frac{23.6}{2\pi a^2} \sqrt{D/\sigma} & \quad \left( \frac{28.3}{2\pi a^2} \sqrt{D/\sigma} \right) \\
\xi_4 &= \frac{57.1}{2\pi a^2} \sqrt{D/\sigma} & \quad \left( \frac{73.8}{2\pi a^2} \sqrt{D/\sigma} \right) \\
\xi_5 &= \frac{58.8}{2\pi a^2} \sqrt{D/\sigma} & \quad \left( \frac{67.7}{2\pi a^2} \sqrt{D/\sigma} \right)
\end{align*}
\]
The experimental values from reference 19 are enclosed in parentheses.

As an indication of the rate of convergence of the method as the mesh width decreases, this same plate problem was solved for three difference mesh widths. The results are presented in the following four figures.
Fig. 24 -- First Frequency vs. Mesh Width

Fig. 25 -- Second Frequency vs. Mesh Width
Fig. 26 — Third Frequency vs. Mesh Width

Fig. 27 — Fourth Frequency vs. Mesh Width
LARGER DEFLECTIONS OF PLATES

Progress in the numerical analysis of larger deflections of plates has not kept pace with advances in computer technology. Practical difficulties encountered in reducing the problem to an ordinary differential system have discouraged analog computer handling while the presence of essential non-linearities have had a similar effect on digital computer use. With the latter difficulty in mind, the perturbation method of reference 27 was reassessed. It was found that the method elegantly disposed of the problems arising from the non-linearities and reduced the analysis to a rather simple extension to the small deflection analysis.

The perturbation method has long been a standard tool of non-linear analysis (10). It has been successfully applied to larger deflection problems on several occasions. As in other applications, the method reduces the problem to a sequence of linear problems. These linear problems may be subsequently reduced to matrix form by the general difference method.

The larger deflections of constant thickness isotropic plates are governed by the following field equations (6):

104
where \( \xi, \eta \) are the components of displacement of a point in the middle plane of the plate in the \( x \) and \( y \) directions respectively.

Introducing the ingenious perturbation parameter used by W. Z. Chein (27) on circular plates, we assume the general solution to (Eq. 138) expandable in the form

\[
\begin{align*}
\xi(x,y) &= \bar{\xi}(x,y) W^2 + \bar{\xi}(x,y) W^4 + \ldots \\
\eta(x,y) &= \bar{\eta}(x,y) W^2 + \bar{\eta}(x,y) W^4 + \ldots \\
\omega(x,y) &= \bar{\omega}(x,y) W + \bar{\omega}(x,y) W^3 + \ldots \\
q(x,y) &= (a_1 W + a_2 W^3 + \ldots) \zeta(x,y)
\end{align*}
\]

where \( W \) is the vertical displacement of a particular point \( (x_0,y_0) \) in the middle plane of the plate and \( \zeta(x,y) = q(x,y)/\max |q(x,y)| \). Odd powers of \( W \) were omitted from the \( \xi, \eta, \omega \) expansions because they are symmetric functions of \( W \). \( q, \omega \) are antisymmetric.
Inserting (Eq. 139) in (Eq. 138) and equating the coefficient of each power of $\mathbf{W}$ to zero in each equation gives

$$\begin{align*}
\nabla^4 \bar{\mathbf{V}} &= \frac{a_P}{D}, \\
2 \bar{u}_x + (1-\mu) \bar{v}_y + (1+\mu) \bar{u}_x + 2 \bar{w}_x \bar{w}_x + (1-\mu) \bar{w}_x \bar{w}_y \\
&\quad + \bar{w}_x \bar{w}_y = 0, \\
\text{and} \\
2 \bar{v}_y + (1-\mu) \bar{w}_x + (1+\mu) \bar{v}_x + 2 \bar{w}_x \bar{w}_y + (1-\mu) \bar{w}_x \bar{w}_y \\
&\quad + \bar{w}_x \bar{w}_y = 0, \\
\nabla^4 \bar{\mathbf{V}} &= \frac{a_P}{D} + \frac{12}{h^4} \left\{ \bar{u}_x \left( \bar{u}_x + \mu \bar{v}_x + \frac{1}{2} \bar{w}_x^2 + \frac{\mu}{2} \bar{w}_y^2 \right) \\
&\quad + (1-\mu) \bar{w}_y \left( \bar{u}_y + \bar{v}_x + \bar{w}_x \bar{w}_y \right) \\
&\quad + \bar{w}_y \left( \bar{u}_y + \mu \bar{v}_x + \frac{1}{2} \bar{w}_x^2 + \frac{\mu}{2} \bar{w}_y^2 \right) \right\}, \\
\end{align*}$$

Inserting (Eq. 139) in the appropriate homogeneous boundary conditions on $u, \sigma, \mathbf{w}$ and equating to zero the coefficients of the powers of $\mathbf{W}$ gives the boundary conditions on $\bar{u}, \bar{v}, \bar{w}, \bar{\mathbf{w}}, \cdots$. For small deflections we might reasonably neglect all powers of $\mathbf{W}$ greater than one. In this case system 139 and 140 reduces to

$$\begin{align*}
\nabla^4 \bar{\mathbf{V}} &= a_P; \\
q &= a_P W \mathbf{W}; \\
\mathbf{w} &= \bar{\mathbf{w}} W
\end{align*}$$

or by substitution

$$\nabla^4 \mathbf{w} = \nabla^4 (W \bar{\mathbf{w}}) = W \nabla^4 \bar{\mathbf{w}} = \frac{a_P}{D} \mathbf{W} = \frac{q}{D}.$$  

The last line contains the conventional small deflection equation.

If only powers of $\mathbf{W}$ greater than two are neglected the solution for $\mathbf{w}$ is still the small deflection solution but a first approximation to
is now available by integrating the second and third equations of system 140.

If only powers of \( W \) greater than three are neglected, the first true non-linear effects are encountered in the relation

\[
\text{max. } |q(x, \gamma)| = a_1 W + a_2 W^3.
\]

It will be demonstrated in the remainder of this section that for many purposes this degree of approximation may be adequate.

At this stage, the steps necessary to obtain a solution become sufficiently numerous to obscure their physical meaning. It is perhaps for this reason that subsequent analysts have seemed hesitant in applying this method to other plate problems. Let us now consider each step in some detail.

For the case considered system 139 reduces to

\[
\begin{aligned}
\mathcal{u}(x, \gamma) &= \overline{u}(x, \gamma) \frac{W^2}{2}, \\
\mathcal{u}(x, \gamma) &= \overline{u}(x, \gamma) \frac{W^2}{2}, \\
\mathcal{u}(x, \gamma) &= \overline{u}(x, \gamma) \frac{W}{2} + \overline{u}(x, \gamma) \frac{W^3}{3}, \\
\text{max. } |q(x, \gamma)| &= a_1 W + a_2 W^3
\end{aligned}
\]

and system 140 reduces to only those equations given at length. The first step involves the solution of the differential equation \( D^2 \overline{u} \overline{u} = a_1 \overline{\phi} \) with the appropriate homogeneous boundary conditions for arbitrary \( a_1 \).

From the third equation of (Eq. 142) and the meaning of \( W \), it is apparent that \( a_1 \) must be chosen such that \( \overline{u}(x, \gamma, a_1) = 1 \). The constant \( a_1 \) now fixed has a precise interpretation. It is the amplitude of
the pressure mode necessary to produce a unit vertical displacement at point \( (x_0, y_0) \) using small deflection theory. Obviously the point should not be made to correspond with a zero point of \( \overline{w}(x, y, a_i) \).

To avoid this difficulty choose \( (x_0, y_0) \) to correspond with a maximum point of \( \overline{w}(x, y, a_i) \). \( \overline{w} \) hereafter relates to this particular choice of \( (x_0, y_0) \) and \( a_i \). The second and third equations of (Eq. 140) may now be solved for the functions \( \overline{u}, \overline{v} \). Comparing the second and third equations of (Eq. 140) with those of (Eq. 138) we see that \( \overline{u}, \overline{v} \) are just the lateral displacements corresponding to the unit transverse displacement mode, \( \overline{w} \). Note from the form of the second and third equations of (Eq. 140) that if the amplitude of \( \overline{w} \) were doubled the amplitudes of \( \overline{u}, \overline{v} \) would quadruple, i.e. the amplitudes of \( \overline{u}, \overline{v} \) are proportional to the square of the \( \overline{w} \) amplitude. Considering the first two equations of (Eq. 142), we see that the final in-plane displacements (to the degree of approximation considered) are computed on the assumption that the linear small deflection solution is a good enough approximation to the form, but not necessarily the amplitude, of the true vertical displacements to serve as a basis for this computation.

The next step involves the function \( \overline{w} \). From the form of the final equation of (Eq. 140) we see that this function may be regarded as a small deflection solution resulting from the sum of two static pressures. The first of these is \( a_2 \overline{p}(x, y) \) which is of the same form as the original pressure distribution \( \overline{p} \) but of yet undetermined amplitude. The second pressure distribution corresponds to the vertical
components of the middle plane forces resulting from the unit deflection mode \( \bar{u} \) and its corresponding in-plane displacements \( \bar{u}, \bar{v} \).

The solution for \( \bar{w} \) may similarly be regarded as the sum of two displacement functions corresponding to the two pressures. The first of these displacements, \( \bar{w}_1(x, y; \alpha) \), is the same as \( \bar{w}(x, y; \alpha) \) with \( \alpha \) in place of \( \alpha \). The second, \( \bar{w}_2(x, y) \), is the small vertical displacement due solely to the presence of middle plane forces resulting from \( \bar{u}, \bar{v}, \bar{w} \). Observe from the form of the final equation of (Eq. 140) that the amplitude of \( \bar{w}_2 \) is proportional to the cube of the amplitude of \( \bar{w} \) since the amplitude of the \( \bar{u}, \bar{v} \) modes are proportional to the square of the \( \bar{w} \) amplitude. Now consider how all three aforementioned vertical displacement functions \( \bar{w}_1, \bar{w}_2, \bar{w}_3 \) may be combined to approximate the larger non-linear displacements. For convenience assume in what follows that the final deflection, \( \bar{w} \), of point \( (x_0, y_0) \) is prescribed and it is desired to find the amplitude of the unit pressure \( f(x, y) \) necessary to maintain that given deflection at \( (x_0, y_0) \).

According to small deflection theory the vertical plate deflections are \( \bar{w}_1 \bar{w} \). But additional deflections, \( \bar{w}_2 \bar{w}^3 \), arise with the development of middle plane forces resulting from the \( \bar{w}_1 \bar{w} \) deflections (and their corresponding in-plane deflections, \( \bar{u} \bar{w}^2 \) and \( \bar{v} \bar{w}^2 \)) which are neglected in small deflection theory and in general tend to reduce the vertical deflections. However, if we simply add the deflections \( \bar{w}_1 \bar{w} \) and \( \bar{w}_2 \bar{w}^3 \) we find that the deflection of point \( (x_0, y_0) \) no longer has the prescribed value of \( \bar{w} \) on which we have based our
computations of the middle plane forces that gave rise to the \( \overline{w}_x W^3 \) deflections. Consider now the third displacement component \( \overline{w}_i(x, y, \alpha_2) \) and how its arbitrary factor \( \alpha_2 \) may be selected such that when it is added to the deflections \( \overline{w}_W + \overline{w}_a W^3 \) the previous objection is removed. If \( \alpha_2 \) , upon which depends linearly, be chosen such that \( \overline{w}_i(x_0, y_0, \alpha_2) = -\overline{w}_2(x_0, y_0) \) then \( \alpha_2 \) may be interpreted physically as that additional increment of the unit pressure mode, beyond that dictated by linear small deflection theory, necessary to nullify the vertical deflection \( \overline{w}_2(x_0, y_0) \), due solely to the middle plane forces resulting from \( \overline{u}, \overline{w}, \overline{w}_r \). \( \overline{w}_i \) hereafter refers to this particular choice of \( \alpha_2 \). Adding the displacement \( \overline{w}_i W^3 \) to \( \overline{w}_W + \overline{w}_a W^3 \) we find that \( (\overline{w}_i + \overline{w}_a) W^3 = 0 \) at point \( (x_0, y_0) \). Thus the vertical displacement at \( (x_0, y_0) \) becomes \( W \) as required and the computation of the middle plane forces and their resulting vertical displacements are based on the correct displacement amplitude. The fact that the computation of the middle plane forces is also based on the wrong form of the vertical displacement function has not been corrected. This error could perhaps best be reduced by an iterative process which seems to be the obvious next step in improving the method. This iteration would take the vertical deflections \( \overline{w}_W + \overline{w} W^3 \) and recompute the middle plane forces and their resulting changes on the linear, small deflection solution, etc. This process is similar to an iteration scheme used by von Karman (28) on the same problem but differs in one very important point. von Karman used the small deflection solution in total to compute the middle plane forces for the
second iteration. Chein's method relies only on the form of the linear solution and not its amplitude for this purpose. As might be expected, von Karman's iterations diverged when the non-linearities became significant because the small deflection solution in this case grossly overestimates the deflection amplitudes and subsequent calculations for the resulting in-plane forces, which are sensitive to the displacement amplitudes, were thrown into even greater error. Iterations on Chein's method should avoid this difficulty.

The amplitude of the unit pressure mode $\mathcal{P}(x,y)$ necessary to maintain a deflection of $\mathcal{W}$ at $(x_0, y_0)$ is the sum of that necessary to overcome the traditional bending stiffness, as given by small deflection theory, plus that necessary to counteract the stiffening effect of the middle plane forces. Neglecting the middle plane forces, we recall that the pressure necessary to maintain a unit vertical deflection at $(x_0, y_0)$ was $\alpha_1 \mathcal{P}(x, y)$. Since the pressure and deflection amplitudes are linearly related, specifying a vertical deflection of $\mathcal{W}$ at $(x_0, y_0)$ results in a pressure of $\alpha_1 \mathcal{P}(x, y) \mathcal{W}$. Considering middle plane forces alone, we recall that a vertical deflection of $\mathcal{W}$ at $(x_0, y_0)$ gave rise to a secondary pressure system which reduced the deflection at $(x_0, y_0)$ by the amount $\overline{\mathcal{W}}_x(x_0, y_0) \mathcal{W}^3$, $\overline{\mathcal{W}}_x(x_0, y_0)$ being the reduction in vertical displacement at $(x_0, y_0)$ for $\mathcal{W} = 1$. The amplitude of the unit pressure $\mathcal{P}(x, y)$ necessary to restore the vertical deflection at $(x_0, y_0)$ to $\mathcal{W}$ after considering middle plane forces is proportional to the amplitude of the secondary displacements resulting from these middle plane forces. Since a secondary displacement
amplitude of \( \bar{w}_2(x_0, y_0) \) (resulting from a unit \( W \) displacement) results in a pressure distribution \( \alpha_2 \varphi(kx) \); a secondary displacement amplitude of magnitude \( \bar{w}_2(x_0, y_0) W^3 \) results in an additional pressure \( \alpha_2 \varphi(kx) W^3 \). The final pressure, \( \varphi(kx) W^3 \), is the sum of pressures \( \alpha \varphi(kx) W \) and \( \alpha_2 \varphi(kx) W^3 \). This fact is expressed in the last equation of (Eq. 142) which in most applications will be regarded as an equation for \( W \), given \( \max. |\varphi(kx)| \), rather than the other way around. Let us next consider an example problem before attempting a difference formulation of the general problem.

**Sample Problem.** We apply the method to the problem of determining the maximum deflection of a simply supported square plate subject to a uniform vertical pressure. The edges of the plate may rotate freely but not translate in the middle plane, i.e. \( \varphi = \psi = \zeta = 0 \) at the edges are the kinematic constraints. Also \( \varphi_{xx} = \varphi_{yy} = 0 \) at the edges.

![Fig. 28 -- Simply Supported Plate](image-url)
The boundary conditions on $\bar{\psi}, \bar{\psi}, \ldots$ are $\bar{\psi} = \bar{\psi} = \ldots = 0$ at $x, y = \pm \ell$. $\bar{\psi}_{xx} = \bar{\psi}_{xx} = \ldots = 0$ at $x = \pm \ell$, and $\bar{\psi}_{yy} = \bar{\psi}_{yy} = \ldots = 0$ at $y = \pm \ell$.

The boundary conditions on $\bar{\mu}, \bar{\mu}, \ldots$ are $\bar{\mu} = \bar{\mu} = \ldots = 0$ at $x, y = \pm \ell$. Using the net of Fig. 28, and the various symmetry relations, the simplest difference approximation to (Eq. 140) is (for $\mu = 0.3$)

$$
\begin{bmatrix}
20 & -32 & 8 \\
-8 & 24 & -16 \\
2 & -16 & 20
\end{bmatrix}
\begin{bmatrix}
\bar{\psi}(1) \\
\bar{\psi}(2) \\
\bar{\psi}(3)
\end{bmatrix}
= \frac{a_1 c^4}{16 D}
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
$$

$$
\begin{bmatrix}
12.1 & -2.8 \\
-2.8 & 21.6
\end{bmatrix}
\begin{bmatrix}
\bar{\mu}(x) \\
\bar{\mu}(3)
\end{bmatrix}
= -\frac{2}{c^4}
\begin{bmatrix}
2.0 \bar{\psi}(1) - 5.4 \bar{\psi}(1) \bar{\psi}(2) + 1.4 \bar{\psi}(2) \bar{\psi}(3) \\
5.4 \bar{\psi}(x) + 0.7 \bar{\psi}(1) \bar{\psi}(2) - 10.8 \bar{\psi}(2) \bar{\psi}(3)
\end{bmatrix}
$$

Solving the first three equations for $\bar{\psi}(1), \bar{\psi}(2), \bar{\psi}(3)$ gives

$$
\bar{\psi}(1) = \frac{33}{32} \left( \frac{a_1 c^4}{16 D} \right), \quad \bar{\psi}(2) = \frac{21}{28} \left( \frac{a_1 c^4}{16 D} \right), \quad \bar{\psi}(3) = \frac{35}{64} \left( \frac{a_1 c^4}{16 D} \right).
$$

Setting $\bar{\psi}(1) = 1$ gives

$$
a_1 = \frac{32}{33} \left( \frac{16 D}{c^4} \right), \quad \bar{\psi}(2) = \frac{4\ell}{6\ell}, \quad \bar{\psi}(3) = \frac{35}{6\ell}.
$$
Solving the next two equations for $\bar{u}(2)$, $\bar{u}(3)$ gives

$$ c\bar{u}(2) = 0.2206, \quad c\bar{u}(3) = 0.1062. $$

Solving for $\bar{w}(1)$, $\bar{w}(2)$, $\bar{w}(3)$ we get

$$ \bar{w}(1) = \frac{33}{32}(\frac{a_2c^4}{16D}) - \frac{1.225}{h^2}, \quad \bar{w}(2) = \frac{21}{28}(\frac{a_2c^4}{16D}) - \frac{0.853}{h^2}, \quad \bar{w}(3) = \frac{35}{64}(\frac{a_2c^4}{16D}) - \frac{0.59}{h^2}. $$

Setting $\bar{w}(1) = 0$ gives $a_2 = \frac{1.19}{h^2}(\frac{16D}{c^4})$.

Finally, from (Eq. 141)

$$ 19\left(\frac{W}{h}\right)^3 + 15.5\left(\frac{W}{h}\right) = \frac{9c^4}{hD} $$

This solution is plotted in Fig. 29 for comparison with the linear and non-linear solutions taken from reference 6.
Consider now the general difference form of the larger deflection problem. The plate is assumed to be isotropic with a slight thickness gradient. The external loads are normal to the middle plane.

Using the expansions 139, boundary-value problems for $u, \bar{u}, \ldots$, $\bar{u}, \bar{\bar{u}}, \ldots, \bar{w}, \bar{\bar{w}}, \ldots$ may be obtained by substituting (Eq. 139) in the field equations and homogeneous boundary conditions for $u, \bar{u}, \bar{w}$ and equating the coefficients of successive powers of $W$ to zero, or
alternately, to expand the total potential energy in powers of \( \mathbf{W} \) and

equate to zero the first variation of successive coefficients of the expansion. The latter approach permits the derivation of the general
difference equations of the problem.

In the larger deflection range, the total potential energy of the
plate is given by (6)

\[
(143) \quad \mathcal{T}[\mu, \nu, \sigma] = \frac{1}{2} \left\{ \int \int \mathcal{D} \left\{ \left( \omega_{xx} + \omega_{yy} \right)^2 - 2(1-\mu) \left( \omega_{xx} \omega_{yy} - \omega_{xy}^2 \right) \right\} \right. \\
\left. + \frac{12}{\pi^2} \left\{ \omega_x^2 + \omega_y^2 + \omega_y^2 + \omega_y^2 + \frac{1}{4} \left( \omega_x^2 + \omega_y^2 \right)^2 \right\} \right. \\
+ 2\mu \omega_x \omega_y + \mu \omega_y^2 + \mu \omega_x^2 + \frac{(1-\mu)}{2} \left( \omega_y^2 + \omega_x^2 \right) \\
\left. + \omega_y \omega_x + (1-\mu)(\omega_x \omega_y + \omega_x \omega_y \omega_y) \right\} - 2q \omega \mathbf{W} \right. dx dy.
\]

Expanding \( \mathcal{T} \) in powers of \( \mathbf{W} \) gives

\[
(144) \quad \mathcal{T} = \left\{ \frac{1}{2} \left\{ \int \int \mathcal{D} \left\{ \left( \omega_{xx} + \omega_{yy} \right)^2 - 2(1-\mu) \left( \omega_{xx} \omega_{yy} - \omega_{xy}^2 \right) \right\} \right. \\
\left. + \frac{12}{\pi^2} \left\{ \omega_x^2 + \omega_y^2 + \omega_y^2 + \omega_y^2 + \frac{1}{4} \left( \omega_x^2 + \omega_y^2 \right)^2 \right\} \right. \\
+ 2\mu \omega_x \omega_y + \mu \omega_y^2 + \mu \omega_x^2 + \frac{(1-\mu)}{2} \left( \omega_y^2 + \omega_x^2 \right) \\
\left. + \omega_y \omega_x + (1-\mu)(\omega_x \omega_y + \omega_x \omega_y \omega_y) \right\} - 2q \omega \mathbf{W} \right. dx dy \right\} \mathbf{W}^2 \\
+ \left\{ \frac{1}{2} \left\{ \int \int \mathcal{D} \left\{ \left( \omega_{xx} + \omega_{yy} \right)^2 - 2(1-\mu) \left( \omega_{xx} \omega_{yy} - \omega_{xy}^2 \right) \right\} \right. \\
\left. + \frac{12}{\pi^2} \left\{ \omega_x^2 + \omega_y^2 + \omega_y^2 + \omega_y^2 + \frac{1}{4} \left( \omega_x^2 + \omega_y^2 \right)^2 \right\} \right. \\
+ 2\mu \omega_x \omega_y + \mu \omega_y^2 + \mu \omega_x^2 + \frac{(1-\mu)}{2} \left( \omega_y^2 + \omega_x^2 \right) \\
\left. + \omega_y \omega_x + (1-\mu)(\omega_x \omega_y + \omega_x \omega_y \omega_y) \right\} - 2q \omega \mathbf{W} \right. dx dy \right\} \mathbf{W}^4 \\
+ \ldots
\]
Equating to zero the first variation of the $W^2$ coefficient gives the field equation and natural boundary conditions for determining $\bar{\omega}$. It is interesting to note that the boundary value problem for $\bar{\omega}$ may be derived in a different way by considering the coefficient of the $W^4$ term which we designate hereafter as $I_4$. Setting $S I_4 = 0$ for all comparison functions of the form $\bar{\omega} + \epsilon S \bar{\omega}$ leads to the correct boundary value problem for $\bar{\omega}$ rather than $\bar{\omega}$ which was varied. Conversely setting $S I_4 = 0$ for comparison functions $\bar{\omega} + \epsilon S \bar{\omega}$ yields the correct boundary value problem for $\bar{\omega}$. Setting $S I_4 = 0$ for variations of the form $\bar{\omega} + \epsilon S \bar{\omega}$ and $\bar{\omega} + \epsilon_z S \bar{\omega}$ gives the necessary equations for determining $\bar{\omega}, \bar{\omega}$. Therefore, the statement $S I_4 = 0$ completely defines the perturbation problem if we are content to consider only powers of $W$ less than four in expansions 139. This is convenient because it permits the derivation of the four general difference equations corresponding to the functions $\bar{\omega}, \bar{\omega}, \bar{\omega}, \bar{\omega}$ from a single minimal problem. Using shift matrices the general difference equations are:

$$
\begin{align*}
[A](\bar{\omega}) &= \alpha_1 [x](\bar{\rho}), \\
[B](\bar{\omega}) + [C](\bar{\rho}) &= (\bar{\gamma}_1), \\
[B'](\bar{\omega}) + [C'](\bar{\omega}) &= (\bar{\gamma}_2), \\
[A](\bar{\omega}) &= \alpha_2 [x](\bar{\rho}) + (\bar{\gamma}_2)
\end{align*}
$$

where

$$
[A] = \frac{1}{\alpha_1} [x] + \frac{K}{\alpha_2} [y] [D][x][x] + \frac{2(\mu - \lambda)}{\alpha_1 \beta_2} [z]'[D][x][z] \\
+ \frac{K}{\beta_2} [y] + \frac{K}{\alpha_2} [x] [D][x][y],
$$
\[
\begin{align*}
[B] &= \frac{1}{2} \left( k + \mu \right) \left( x^2 y + \frac{T}{2} \right) \nu \left( x^2 y + \frac{T}{2} \right), \\
[C] &= \frac{1}{2} \left( k + \mu \right) \left( x^2 y + \frac{T}{2} \right) \nu \left( x^2 y + \frac{T}{2} \right), \\
[C'] &= [C]^T
\end{align*}
\]

and
\[
\begin{align*}
[x] &= [E] - [1], \\
[y] &= [n] - [1], \\
[\alpha] &= [\beta] + [E][x] + [E][y] + [E], \\
[\alpha'] &= [\beta] + [\alpha] + [n][\beta] + [n][E].
\end{align*}
\]

\([E]\) is a diagonal matrix with each diagonal element consisting of the quantity \(\frac{12D}{t^2}\) evaluated at one of the \((x_i, y_i)\) points.

\([n]\) is a diagonal matrix with each diagonal element consisting of the quantity \(\frac{12D}{t^2}\) evaluated at one of the \((x_i + \frac{a}{2}, y_i)\) points.

\([\beta]\) is a diagonal matrix with each diagonal element consisting of the quantity \(\frac{12D}{t^2}\) evaluated at one of the \((x_i, y_i + \frac{b}{2})\) points.
The elements \( \mathbf{f}_1(i, j), \mathbf{f}_2(i, j), \mathbf{f}_3(i, j) \) of the force vectors \( (\mathbf{f}_1), (\mathbf{f}_2), (\mathbf{f}_3) \) are given by

\[
\mathbf{f}_1(i, j) = -\frac{1}{2a} \Theta(i+\frac{1}{2}, j) \alpha(i+\frac{1}{2}, j) \overrightarrow{\mathbf{u}}_x(i+\frac{1}{2}, j) + \frac{1}{2a} \Theta(i-\frac{1}{2}, j) \alpha(i-\frac{1}{2}, j) \overrightarrow{\mathbf{u}}_x(i-\frac{1}{2}, j) - \frac{\mu}{2a} \Theta(i+\frac{1}{2}, j) \alpha(i+\frac{1}{2}, j) \overrightarrow{\mathbf{u}}_y(i+\frac{1}{2}, j) + \frac{\mu}{2a} \Theta(i-\frac{1}{2}, j) \alpha(i-\frac{1}{2}, j) \overrightarrow{\mathbf{u}}_y(i-\frac{1}{2}, j) - \frac{(1-\mu)}{2a} \Theta(i+\frac{1}{2}, j) \alpha(i+\frac{1}{2}, j) \overrightarrow{\mathbf{u}}_x(i+\frac{1}{2}, j) \overrightarrow{\mathbf{u}}_y(i+\frac{1}{2}, j) + \frac{(1-\mu)}{2a} \Theta(i-\frac{1}{2}, j) \alpha(i-\frac{1}{2}, j) \overrightarrow{\mathbf{u}}_x(i-\frac{1}{2}, j) \overrightarrow{\mathbf{u}}_y(i-\frac{1}{2}, j),
\]

\[
\mathbf{f}_2(i, j) = -\frac{1}{2b} \Theta(i, j+\frac{1}{2}) \alpha(i, j+\frac{1}{2}) \overrightarrow{\mathbf{u}}_y(i, j+\frac{1}{2}) + \frac{1}{2b} \Theta(i, j-\frac{1}{2}) \alpha(i, j-\frac{1}{2}) \overrightarrow{\mathbf{u}}_y(i, j-\frac{1}{2}) - \frac{\mu}{2b} \Theta(i, j+\frac{1}{2}) \alpha(i, j+\frac{1}{2}) \overrightarrow{\mathbf{u}}_x(i, j+\frac{1}{2}) + \frac{\mu}{2b} \Theta(i, j-\frac{1}{2}) \alpha(i, j-\frac{1}{2}) \overrightarrow{\mathbf{u}}_x(i, j-\frac{1}{2}) - \frac{(1-\mu)}{2b} \Theta(i, j+\frac{1}{2}) \alpha(i, j+\frac{1}{2}) \overrightarrow{\mathbf{u}}_x(i, j+\frac{1}{2}) \overrightarrow{\mathbf{u}}_y(i, j+\frac{1}{2}) + \frac{(1-\mu)}{2b} \Theta(i, j-\frac{1}{2}) \alpha(i, j-\frac{1}{2}) \overrightarrow{\mathbf{u}}_x(i, j-\frac{1}{2}) \overrightarrow{\mathbf{u}}_y(i, j-\frac{1}{2}),
\]

\[
\mathbf{f}_3(i, j) = -\frac{1}{a} \Theta(i+\frac{1}{2}, j) \alpha(i+\frac{1}{2}, j) \overrightarrow{\mathbf{u}}_x(i+\frac{1}{2}, j) + \frac{1}{a} \Theta(i-\frac{1}{2}, j) \alpha(i-\frac{1}{2}, j) \overrightarrow{\mathbf{u}}_x(i-\frac{1}{2}, j) - \frac{1}{b} \Theta(i, j+\frac{1}{2}) \alpha(i, j+\frac{1}{2}) \overrightarrow{\mathbf{u}}_y(i, j+\frac{1}{2}) + \frac{1}{b} \Theta(i, j-\frac{1}{2}) \alpha(i, j-\frac{1}{2}) \overrightarrow{\mathbf{u}}_y(i, j-\frac{1}{2}) - \frac{1}{a} \Theta(i+\frac{1}{2}, j) \alpha(i+\frac{1}{2}, j) \overrightarrow{\mathbf{u}}_x(i+\frac{1}{2}, j) \overrightarrow{\mathbf{u}}_y(i+\frac{1}{2}, j) + \frac{1}{a} \Theta(i-\frac{1}{2}, j) \alpha(i-\frac{1}{2}, j) \overrightarrow{\mathbf{u}}_x(i-\frac{1}{2}, j) \overrightarrow{\mathbf{u}}_y(i-\frac{1}{2}, j) - \frac{1}{b} \Theta(i, j+\frac{1}{2}) \alpha(i, j+\frac{1}{2}) \overrightarrow{\mathbf{u}}_x(i, j+\frac{1}{2}) \overrightarrow{\mathbf{u}}_y(i, j+\frac{1}{2}) + \frac{1}{b} \Theta(i, j-\frac{1}{2}) \alpha(i, j-\frac{1}{2}) \overrightarrow{\mathbf{u}}_x(i, j-\frac{1}{2}) \overrightarrow{\mathbf{u}}_y(i, j-\frac{1}{2}).
\]
where $D = \frac{12 \Phi}{b}$. 

The quantities in $\xi_1, \xi_2, \xi_3$ such as $\alpha(i, j + \frac{1}{2}) \bar{\omega}(i, j + \frac{1}{2}) \bar{\omega}(i, j + \frac{1}{2})$ which are not defined by their finite difference approximations at point $(x_i, y_i + \frac{1}{2})$ can be replaced by their average value at the four nearest points of definition, i.e.

$$
\begin{align*}
\varphi(i, j) &= \frac{\bar{\varphi}(i, j + 1) - \bar{\varphi}(i, j)}{a} \left\{ \frac{\bar{\varphi}(i, j + 1) - \bar{\varphi}(i, j)}{a} \right\} \\
\xi_1(i, j) &= \frac{\bar{\xi}_1(i, j + 1) - \bar{\xi}_1(i, j)}{a} \left\{ \frac{\bar{\xi}_1(i, j + 1) - \bar{\xi}_1(i, j)}{a} \right\} \\
\xi_2(i, j) &= \frac{\bar{\xi}_2(i, j + 1) - \bar{\xi}_2(i, j)}{a} \left\{ \frac{\bar{\xi}_2(i, j + 1) - \bar{\xi}_2(i, j)}{a} \right\} \\
\xi_3(i, j) &= \frac{\bar{\xi}_3(i, j + 1) - \bar{\xi}_3(i, j)}{a} \left\{ \frac{\bar{\xi}_3(i, j + 1) - \bar{\xi}_3(i, j)}{a} \right\} \\
\end{align*}
$$
A disturbing fact regarding the perturbation method is the dependence of the solution upon the choice of the point \((x_0, y_0)\). For convenience \((x_0, y_0)\) was selected to coincide with a maximum point of the small deflection solution, but the only actual requirement on the choice of \((x_0, y_0)\) was that it not correspond with a zero point of the small deflection solution. With this arbitrariness in the location of \((x_0, y_0)\) we would like to think that its choice was of no consequence to the solution, but this is not the case if the perturbation expansions are truncated and the linear small deflection solution is of a different form than the non-linear solution. In fact, the greater the discrepancy between the forms of the linear and non-linear solutions the more sensitive the perturbation solution is to changes in \((x_0, y_0)\). Is there then any more logical basis for the choice of \((x_0, y_0)\)? No answer to this question has appeared to this writer's knowledge, but a measure of intuition and consideration of the results of applications seem to indicate that if primarily interested in the maximum plate deflection or the vertical deflection at some particular point, good results are obtained by choosing the same point to be the \((x_0, y_0)\) point. However, the question is admittedly open to further study. In the event that the forms of the linear and non-linear solutions differ radically the validity of a perturbation solution obtained by neglecting powers of \(W\) greater than three in the expansions 139 is in doubt, and in such cases the retention of higher power terms seems advisable.

In this section and the preceding sections the general difference method has been explored with many sample problems treated
demonstrating that the method can yield good approximate information on the elastic characteristics of plates. The chief advantage of the method is its generality. Using this method a high speed digital computer can be programmed to solve a wide class of plate problems. This is desirable in industries with many varied applications of plate type structures. Whether or not the method consistently yields superior accuracy over comparable methods for the same amount of labor or machine time has yet to be established. Further theoretical study and much more data is needed to satisfactorily answer this question.
REFERENCES


AUTOBIOGRAPHY

I, Raymond Leo Foye, was born in Lowell, Massachusetts, May 21, 1933. I received my secondary-school education at Keith Academy, Lowell, Massachusetts, and my undergraduate training at Lowell Technological Institute where I received the degree of Bachelor of Science in Textile Engineering in 1955. I held the position of Instructor in Engineering Mechanics from 1955-1956 at Cornell University and from 1961-1962 at The Ohio State University while completing the requirements for the Doctor of Philosophy degree.