FEIRCE DECOMPOSITION IN SIMPLE LIE-ADMISSIBLE
POWER-ASSOCIATIVE ALGEBRAS

DISSERTATION

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INTRODUCTION

The class of Lie-admissible power-associative rings includes all associative rings, Lie rings, Jordan rings, rings of type $(\gamma, \delta)$, anti-flexible rings, and others. The structure theory of such rings is therefore likely to be most complicated.

Rings of type $(\gamma, \delta)$ have been studied by Albert [5], Kleinfeld [10, 11], Kokoris [15, 16], and Maneri [20]. It has been shown that if $e$ is an idempotent of a simple $(\gamma, \delta)$ ring $A$ then $A$ has a Peirce decomposition,

$$A = A_{11} + A_{10} + A_{01} + A_{00},$$

where $A_{ij} = \{ x \in A \mid ex = ix, xe = jx \}$. Simple $(\gamma, \delta)$ rings with an idempotent $e$ not the identity element have been shown to be associative.

Kleinfeld [12] introduced the term associator dependent for a ring which is third-power-associative and satisfies an identity of the form

$$\alpha_1(x,y,z) + \alpha_2(y,z,x) + \alpha_3(z,x,y) + \alpha_4(x,z,y)$$
$$+ \alpha_5(z,y,x) + \alpha_6(y,x,z) = 0$$

where the $\alpha_i$ are constants, not all equal, in some field of scalars and

$$(x,y,z) = xy \cdot z - x \cdot yz.$$
In [14] it was shown that every associator dependent ring satisfies (I) \((x,y,z) + (y,z,x) + (z,x,y) = 0\); (II) \((y,x,x) = (x,y,x)\); or (III) \(\alpha(y,x,x) - (\alpha + 1)(x,y,x) + (x,x,y) = 0\), \(\alpha\) a fixed scalar.

Third-power-associative rings \(A\) which satisfy (III) with \(\alpha \neq 1, -1/2, -2\) were proved to have a Peirce decomposition

\[ A = A_1 + A_0 + A_{01} + A_{00} \]

with respect to an idempotent \(e\) of \(A\). Simple rings with characteristic prime to 6 and having no idempotent \(e\) such that \(A_{10} + A_{01} \neq 0\) were shown to be alternative. Kleinfeld [9] showed that a simple alternative ring is either associative or a Cayley-Dickson algebra over its center. Thus there are no new simple rings in this class.

Actually the authors note in the above that, except for the case \(\alpha = -1\), the hypothesis \(A_{10} + A_{01} \neq 0\) can be replaced by the assumption that \(e\) is not an identity element. The case \(\alpha = -1\) yields the so-called anti-flexible law.

Kosier [18] showed that anti-flexible power-associative rings are also Lie-admissible. He displayed examples of new simple algebras in this class. As indicated by the above these algebras have the property that

\[ A = A_{11} + A_{00} \]

in every Peirce decomposition.
If $\alpha = 1$ in (III) one obtains the flexible law,
\[(x,y,x) = 0.\]

In [21] Oehmke showed that a simple flexible power-associative finite-dimensional algebra over a field of characteristic zero has a unity element and therefore has a degree. He shows that for algebras $A$ of degree greater than two $A^+$ is a simple Jordan algebra. Kokoris [17] extended this result to algebras of degree two.

Kleinfeld and Kokoris [13] proved that the only algebras of degree one are one-dimensional fields.

Laufer and Tomber [19] proved that if $A$ is a flexible power-associative algebra over an algebraically closed field of characteristic zero and if $A^(-)$ is a simple Lie algebra, then $A = A^(-)$. The hypotheses of power-associativity and $A^(-)$ being a Lie algebra imply also that $A$ satisfies (I).

If $\alpha = - (1/2)$ in (III) one gets the left alternative law; $\alpha = -2$ leads to the right alternative law. Rings of these types are anti-isomorphic. Albert [6, 7] has shown that any semi-simple finite-dimensional right alternative algebra of characteristic not two is alternative. Maneri [20] proved that a right alternative ring of characteristic prime to 6 which satisfies (I) and has an idempotent $e$, not a unity element, is alternative.

The identity (II) gives a class of rings anti-isomorphic to those satisfying (III) with $\alpha = 0$. Thus we are left to consider
(I). This gives the class of Lie-admissible rings which, even if power-associativity is also assumed, is a very big class. In view of some of the known results on finite-dimensional algebras satisfying these assumptions, a reasonable additional hypothesis would seem to be to assume the existence of a Peirce decomposition.

It is the purpose of this dissertation to investigate simple power-associative finite-dimensional algebras $A$ satisfying (I) and having an idempotent $e$, not a unity element, which gives a Peirce decomposition of $A$.

Chapter I of the dissertation is devoted to basic notions and deriving multiplicative relations satisfied by the modules $A_{ij}$ in the Peirce decomposition of $A$.

In Chapter 2 it is proved that if $A_{00} = 0$ in the Peirce decomposition of the finite-dimensional algebra $A$ over an algebraically closed field of characteristic prime to 6 then $A$ is isomorphic to a certain three-dimensional algebra first introduced by Weiner [22].

Another theorem is proved (without the assumption that the scalar field $\Phi$ is algebraically closed) under the strong assumptions that $A_{11} = \Phi e$, $A_{10} = 0$, and $A_{01} \neq 0$ in the Peirce decomposition. This is done in Chapter 3. The algebra in this case is of dimension $s + 2$ where $s$ is a positive integer.

The fourth chapter is devoted to examples of simple algebras.
1. DEFINITIONS AND IDENTITIES

By an algebra we mean a finite dimensional vector space equipped with a multiplication which is bilinear over the field of scalars of the vector space. This definition of an algebra is the customary one except that we delete the assumption that multiplication is associative and we always assume finite dimensionality. By the characteristic of an algebra we refer to the characteristic of the field of scalars of the algebra.

If \( x, y, z \) are elements of an arbitrary algebra \( A = (A, +, \cdot) \) we define the associator
\[
(x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z)
\]
and the commutator
\[
(x, y) = x \cdot y - y \cdot x.
\]
One easily verifies that these functions are linear in each variable and are related by the identity
\[
(x \cdot y, z) + (y \cdot z, x) + (z \cdot x, y) = (x, y, z) + (y, z, x) + (z, x, y).
\]

A Lie algebra is by definition an algebra \( (L, +, \cdot) \) which satisfies the identities
\[
(x \cdot x) = 0
\]
\[
(x \cdot y) \cdot z + (y \cdot z) \cdot x + (z \cdot x) \cdot y = 0.
\]

Every algebra \( A \) has an attached algebra \( A^{(-)} \) which is obtained by introducing commutation as a multiplicative operation in the
vector space $A$. $A$ is termed Lie-admissible in case $A(-)$ is a Lie algebra. $A(-)$ always satisfies $(x,x) = 0$ so $A(-)$ is a Lie algebra if and only if

$$(2) \quad (xy - yx,z) + (yz - xy,x) + (zx - xz,y) = 0$$

is an identical relation in $A$.

An algebra $A$ over a field $\Phi$ is said to be power-associative provided each $x$ in $A$ generates an associative subalgebra $A_x$. Since the powers of an element in a power-associative algebra are unambiguously defined it makes sense to speak of nilpotence of an element. A non-nil power-associative algebra contains a non-nilpotent element $x$ which is also a non-nilpotent element of the associative algebra $A_x$. Therefore by a well-known [8, p. 15] theorem on associative algebras $A_x$, hence $A$, contains an idempotent.

Albert [4, p. 560] has shown that if $e$ is an idempotent element of a power-associative algebra $A$ of characteristic different from two, three, or five then $A$ may be written as the supplementary sum

$$(3) \quad A = A_e(1) + A_e(1/2) + A_e(0)$$

of submodules $A_e(\lambda)$ where

$$A_e(\lambda) = \{ x \in A \mid ex + xe = 2\lambda x \}.$$

Moreover the submodules $A_e(0)$ and $A_e(1)$ are orthogonal and such that

$$e \cdot x_1 = x_1 \cdot e = x_1$$
$$e \cdot x_0 = x_0 \cdot e = 0$$

for all elements $x_1$ in $A_e(1)$ and $x_0$ in $A_e(0)$. 

Let $A$ be a Lie-admissible power-associative algebra and suppose that $A$ contains an idempotent element $e$ with the property that for every $x$ in $A$

$$(e,e,x) = (e,x,e) = (x,e,e) = 0.$$ 

The reason for imposing this last condition is that it enables us to prove exactly as for associative algebras that $A$ decomposes as a vector space direct sum

$$A = A_{11} + A_{10} + A_{01} + A_{00}$$

where $A_{i,j} = \{ x \in A \mid ex = ix, xe = jx \}$ for $i,j = 0,1$. This so-called Peirce decomposition can be an aid in determining the multiplicative structure of $A$. Of course if $e$ is the identity element of $A$ then $A_{10} = A_{01} = A_{00} = 0$ and $A = A_{11}$ so the Peirce decomposition tells us nothing.

Comparing the Peirce decomposition with the decomposition

$$(3)$$

it is easy to see that $A_e(1) = A_{11}$, $A_e(0) = A_{00}$ and $A_e(1/2) = A_{10} + A_{01}$. However, we emphasize that the Peirce decomposition is assumed whereas $(3)$ is provable from the assumption of power-associativity. For an example of a simple Lie-admissible power-associative algebra possessing an idempotent for which $(4)$ fails to hold see Chapter 4.

Albert [3, Lemma 1] noted that when the field of scalars contains at least three distinct elements an algebra satisfying

$$(5)$$

also satisfies the multilinear identity...
(6) \((xy + yx, z) + (yz + zy, x) + (zx + xz, y) = 0\)
obtained by linearizing (5). Conversely putting \(z = y = x\) in (6)
yields (5) so long as the characteristic of the algebra is not two
or three.

An algebra of characteristic not two which is Lie-admissible
and third-power-associative satisfies the identity
(7) \((x, y, z) + (y, z, x) + (z, x, y) = 0\)
obtained by adding (2) and (6) and deleting a factor of two in the
result. Conversely any algebra which satisfies (7) is Lie-admissible
and if the characteristic is not three is third-power-associative.

We assume throughout that the characteristic of \(A\) is not two
or three. Then

\[H(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y) = (x, y, z) + (y, z, x)\]
\[+ (z, x, y) = 0\]

for all \(x, y, z\) in \(A\). Also the fourth-power-associativity identities
\((x^2, x, x) = 0\) and \((x, x, x^2) = 0\) may be linearized to yield

\[P(a, b, x, y) = \sum (a, b, x, y) = 0\]

and

\[Q(a, b, x, y) = \sum (a, b, x, y) = 0\]

respectively. The \(\sum\) appearing here in both cases indicates a
sum to be taken over all twenty-four permutations of the elements
\(a, b, x, y\).

We write \(x_{ij}\) for the component of the element \(x\) in the module
\(A_{ij}\) of the Peirce decomposition of \(A\) with respect to \(e\). Our study
begins with the derivation of some basic multiplicative properties of the modules $A_{ij}$.

LEMMA 1. If $i \neq j$ then

\begin{align*}
(8) & \quad x_{ij}y_{jj} = 0, \\
(9) & \quad x_{ij}y_{ij} = (x_{ij}y_{ij})_{ij} + (x_{ij}y_{ij})_{jj} \in A_{ij} + A_{jj}, \\
(10) & \quad y_{ij}x_{ij} = (x_{ij}y_{ij})_{jj} \in A_{jj}, \\
(11) & \quad y_{ji}x_{ii} = (y_{ji}x_{ii})_{ij} + (y_{ji}x_{ii})_{jj} \in A_{ji} + A_{jj}, \\
(12) & \quad x_{ii}y_{jj} = (y_{jj}x_{ii})_{jj} \in A_{jj}, \\
(13) & \quad x_{ij}y_{ij} = y_{ij}x_{ij} \in A_{ii} + A_{jj}, \\
(14) & \quad x_{ij}y_{ji} \in A_{ii} + A_{jj}, \\
(15) & \quad x_{ii}y_{ii} \in A_{ii} + A_{jj}, \\
(16) & \quad x_{ii}y_{ii} + y_{ii}x_{ii} \in A_{ii}, \\
(17) & \quad x_{ii}^2 \in A_{ii}, \\
(18) & \quad (A_{ij}, A_{ii} + A_{ji} + A_{jj}) = 0, \\
(19) & \quad (A_{ij}, A_{jj}, A_{ij}) = 0, \\
(20) & \quad (A_{ij}, A_{jj}, A_{jj}) = 0, \\
(21) & \quad (A_{jj}, A_{jj}, A_{jj}) = 0, \\
(22) & \quad a_{ij}x_{ij}y_{ij} = x_{ij}y_{ij}^2a_{ii} = (1/2)((a_{ii}x_{ij}y_{ij})_{ii} + (a_{ii}y_{ij}x_{ij})_{11}), \\
(23) & \quad a_{ii}x_{ij}y_{ij} = x_{ij}y_{ij}a_{ii} = (1/2)((x_{ij}a_{ii}y_{ij})_{ij} + (y_{ij}a_{ii}x_{ij})_{ii}), \\
(24) & \quad (a_{ii}(x_{ij}y_{ij} + y_{ij}x_{ij}))_{ii} = (a_{ii}x_{ij}y_{ij})_{ii} + (y_{ij}a_{ii}x_{ij})_{ii}, \\
\text{and} \\
(25) & \quad ((x_{ij}y_{ij} + y_{ij}x_{ij})a_{ii})_{ij} = (y_{ij}a_{ii}x_{ij})_{11} + (x_{ij}y_{ij}a_{ii})_{11}.
\end{align*}
PROOF. Since the function $P$ is identically zero
\[
0 = (1/2)P(x,y,e,e) = (xy + yx,e,e) + (xe + ex,e,y)
+ (xe + ex,y,e) + (ey + ye,e,x) + (ey + ye,x,e)
+ (e,x,y) + (e,y,x).
\]
The associator $(xy + yx,e,e) = 0$ by (4). Hence the above becomes
\[
(26) \quad (xe + ex,e,y) + (xe + ex,y,e) + (ey + ye,e,x)
+ (ey + ye,x,e) + (e,x,y) + (e,y,x) = 0.
\]
Similarly
\[
0 = (1/2)Q(x,y,e,e) = (e,e,xy + yx) + (e,yxe + ex)
+ (y,e,xe + ex) + (e,x,ey + ye) + (x,e,ey + ye)
+ (e,x,y) + (e,y,x)
\]
and $(e,e,xy + yx) = 0$, so
\[
(27) \quad (e,y,xe + ex) + (y,e,xe + ex) + (e,x,ey + ye)
+ (x,e,ey + ye) + (x,y,e) + (y,x,e) = 0.
\]
To prove (8) put $x = x_{11}, y = y_{1j}$ in (26) and (27) obtaining
\[
(28) \quad 21(x_{11},e,y) + 21(y_{1j},e,x_{11}) + 2j(y_{1j},e,x_{11})
+ 2j(x_{11},y_{1j},e) + (e,x_{11},y_{1j}) + (e,y_{1j},x_{11}) = 0
\]
and
\[
(29) \quad 2j(e,y_{1j},x_{11}) + 2j(y_{1j},e,x_{11}) + 2j(e,x_{11},y_{1j})
+ 2j(x_{11},e,y_{1j}) + (x_{11},y_{1j},e) + (y_{1j},x_{11},e) = 0
\]
respectively. Subtracting (29) from (28) gives
\[
(30) \quad (2i - 1)(x_{11},y_{1j},e) + 2j(y_{1j},e,x_{11})
+ (1 - 2j)(e,x_{11},y_{1j}) + (2j - 1)(y_{1j},x_{11},e)
+ 2i(x_{11},e,y_{1j}) + (1 - 2i)(e,y_{1j},x_{11}) = 0.
\]
\((j - 1)H(x_{11}, y_{jj}, e) = 0\) yields an identity which we write as

\[(31) \quad (j - 1)(x_{11}, y_{jj}, e) + (j - 1)(y_{jj}, e, x_{11}) + (j - 1)(e, x_{11}, y_{jj}) = 0.\]

Also \((1 - j)H(y_{jj}, x_{11}, e) = 0\) yields

\[(32) \quad (1 - j)(y_{jj}, x_{11}, e) + (1 - j)(x_{11}, e, y_{jj}) + (1 - j)(e, y_{jj}, x_{11}) = 0.\]

Now adding (30), (31), (32) and using the fact that \(i + j = 1\) we obtain

\[3(j - 1)(y_{jj}, e, x_{11}) + 3(1 - j)(x_{11}, e, y_{jj}) = 0\]

which, since \(i - j = \pm 1\) and the characteristic of \(A\) is not three, implies

\[(x_{11}, e, y_{jj}) = (y_{jj}, e, x_{11}).\]

Therefore

\[0 = H(e, x_{11}, y_{jj}) = (e, x_{11}, y_{jj}) + (x_{11}, y_{jj}, e) + (x_{11}, e, y_{jj}) = 2(1 - j)x_{11}y_{jj} + x_{11}y_{jj}e - e\cdot x_{11}y_{jj} = (21 - 2j)(x_{11}y_{jj})_{11} + (21 - 2j)(x_{11}y_{jj})_{10} + (1 + 21 - 2j)(x_{11}y_{jj})_{01} + (21 - 2j)(x_{11}y_{jj})_{00}.\]

By equating each component to zero we obtain (8).

Next put \(x = x_{11}, y = y_{ij}\) in (26). Since \(i + j = 1\) this gives

\[(33) \quad 2i(x_{11}, e, y_{ij}) + 2i(x_{11}, y_{ij}, e) + (y_{ij}, e, x_{11}) + (y_{ij}, x_{11}, e) + (e, x_{11}, y_{ij}) + (e, y_{ij}, x_{11}) = 0.\]

Subtract from (33) the identity
\[ H(x_{ij}, y_{ij}, e) + H(y_{ij}, x_{ij}, e) = (x_{ij}, y_{ij}, e) + (x_{ij}, e, y_{ij}) + (e, x_{ij}, y_{ij}) + (y_{ij}, x_{ij}, e) + (x_{ij}, e, y_{ij}) + (e, y_{ij}, x_{ij}) = 0. \]

There results
\[ (21 - 1)((x_{ij}, e, y_{ij}) + (x_{ij}, y_{ij}, e)) = 0. \]

Now \( (x_{ij}, e, y_{ij}) = 0 \) and since \( 21 - 1 = \pm 1 \), (34) reduces to
\[ (x_{ij}, y_{ij}, e) = 0 \]
or
\[ x_{ij}y_{ij}e = j(x_{ij}y_{ij}). \]

This is equivalent to (9).

For every element \( a \) we have
\[ (35) \quad (a, e) = (a_{ii} + a_{ij} + a_{ji} + a_{jj}, e) = (j - 1)a_{ij} + (i - j)a_{ji}. \]

By (9), \( (x_{ij}y_{ij})_{ji} = 0 \) and therefore
\[ (x_{ij}y_{ij}, e) = (j - 1)(x_{ij}y_{ij})_{ij}. \]

Hence
\[ 0 = H(y_{ij}, e, x_{ij}) = (y_{ij}e, x_{ij}) + (ex_{ij}, y_{ij}) + (x_{ij}y_{ij}, e) = (j - 1)(y_{ij}x_{ij} - x_{ij}y_{ij} + (x_{ij}y_{ij})_{ij}). \]

Since \( j - 1 = \pm 1 \) it follows, again using (9), that
\[ y_{ij}x_{ij} = (x_{ij}y_{ij})_{ij}. \]

This is (10).

There is associated with the algebra \( A = (A, +, \cdot) \) an anti-isomorphic (under the identity mapping) algebra \( A^\# = (A, +, \cdot) \) in which multiplication is defined by
\[ x^\#y = y \cdot x. \]
$A^\#$ has all the properties of $A$ including a Peirce decomposition.

Indeed $(A^\#)_{00} = A_{00}$ and $(A^\#)_{11} = A_{11}$, but $(A^\#)_{10} = A_{01}$ and $(A^\#)_{01} = A_{10}$. This fact obviates the proof of (11) and (12) since these are the analogues of (9) and (10) respectively in $A^\#$.

From (35) we see that $(a,e) = 0$ implies $a_{ij} = a_{ji} = 0$. Now

$$0 = H(x_{ij}, y_{ij}, e) + H(y_{ij}, x_{ij}, e) = (x_{ij} y_{ij} + y_{ij} x_{ij}, e)$$

$$+ (i + j)(x_{ij}, y_{ij}) + (i + j)(y_{ij}, x_{ij})$$

$$= (x_{ij} y_{ij} + y_{ij} x_{ij}, e)$$

so $x_{ij} y_{ij} + y_{ij} x_{ij}$ is in $A_{ii} + A_{jj}$. This means in particular that

$$(x_{ij} y_{ij})_{uv} = -(y_{ij} x_{ij})_{uv}$$

when $u \neq v$. Thus, again using (35),

$$0 = H(x_{ij}, y_{ij}, e) = (x_{ij} y_{ij}, e) + (j - i)(y_{ij}, x_{ij})$$

$$= (j - i)(x_{ij} y_{ij})_{ij} + (i - j)(x_{ij} y_{ij})_{ji}$$

$$+ (j - i)(y_{ij} x_{ij})_{ij} - (j - i)(x_{ij} y_{ij})_{ji}$$

$$- 2(j - i)(x_{ij} y_{ij})_{ij} - 2(j - i)(x_{ij} y_{ij})_{ji}$$

$$+ (j - i)(y_{ij} x_{ij})_{jj} - (j - i)(x_{ij} y_{ij})_{jj}.$$  

We find by equating components in the same module to zero that

$$(x_{ij} y_{ij})_{uv} = (y_{ij} x_{ij})_{uv} = 0$$

when $u \neq v$ and

$$(x_{ij} y_{ij})_{uu} = (y_{ij} x_{ij})_{uu}$$

for $u = 0, 1$. This proves (13).

Now for arbitrary $u, v = 0, 1$,

$$0 = H(x_{uv}, y_{vu}, e) = (x_{uv} y_{vu}, e) + u(y_{vu}, x_{uv})$$

$$+ u(x_{uv}, y_{vu}) = (x_{uv} y_{vu}, e).$$

It follows from (35) that
\[ x_{uv}y_{vu} \in A_{11} + A_{jj}. \]

Taking \( u = i, v = j \) gives (14) and taking \( u = v = i \) gives (15).

To derive (16) put \( x = x_{11}, y = y_{11} \) in (26) to obtain

\[
21((x_{11},e,y_{11}) + (x_{11},y_{11},e) + (y_{11},e,x_{11}) + (y_{11},x_{11},e))
+ (e,x_{11},y_{11}) + (e,y_{11},x_{11}) = 0.
\]

Subtracting the identity

\[
21(H(x_{11},y_{11},e) + H(y_{11},x_{11},e)) = 0
\]

from (36) gives

\[
(1 - 21)((e,x_{11},y_{11}) + (e,y_{11},x_{11})) = 0
\]

which implies

\[
e(x_{11}y_{11} + y_{11}x_{11}) = e(x_{11}y_{11} + y_{11}x_{11}).
\]

In view of (15) this suffices to establish (16). Putting \( y_{11} = (1/2)x_{11} \) in (16) gives (17).

An observation which will be useful in the sequel is that

(9) and (10) together imply

\[
(x_{11},y_{1j}) \in A_{1j}
\]

and (11) and (12) together imply

\[
(x_{11},y_{1j}) \in A_{j1}.
\]

To prove (18) we first write \( H(x_{1j},y_{1j},a) = 0 \) in the form

\[
(x_{1j}y_{1j},a) + (y_{1j},a,x_{1j}) + (a,x_{1j},y_{1j}) = 0.
\]

By (10) and (38),

\[
(y_{1j},a,x_{1j}) = ((a_{11},y_{1j},jj),x_{1j}) \in A_{1j};
\]

by (9), (13), and (38),
\( (a_{ij}x_{ij}, y_{ij}) = ((a_{ij}x_{ij})_{ij} + (a_{ij}x_{ij})_{jj}, y_{ij}) \)
\[ = ((a_{ij}x_{ij})_{jj}, y_{ij}) \subseteq A_{ij}; \]
and by (13), (8), and (15),
\( (x_{ij}y_{ij}, a_{ij}) = ((x_{ij}y_{ij})_{ii} + (x_{ij}y_{ij})_{jj}, a_{ij}) \)
\[ = ((x_{ij}y_{ij})_{ij}, a_{ij}) \subseteq A_{ij} + A_{jj}. \]
Thus putting \( a = a_{ij} \) in (39) gives a sum of three commutators, the first of which is in \( A_{ij} + A_{jj} \) and the others in \( A_{ij} \), equal to zero. We may equate the \( A_{ij} + A_{jj} \) component to zero to obtain
\( (x_{ij}y_{ij}, a_{ij}) = 0, \)
whence
\( (A_{ij}^2, A_{ij}) = 0. \)
Putting \( a = a_{ij} \) in (39) we obtain three commutators which we now evaluate separately. We find that by (13), (37), and (38),
\( (x_{ij}y_{ij}, a_{ij}) = ((x_{ij}y_{ij})_{ii} + (x_{ij}y_{ij})_{jj}, a_{ij}) \subseteq A_{ij}; \)
whereas by (14), (37), and (38),
\( (y_{ij}a_{ij}, x_{ij}) = ((y_{ij}a_{ij})_{ii} + (y_{ij}a_{ij})_{jj}, x_{ij}) \subseteq A_{ij}, \)
and
\( (a_{ij}x_{ij}, y_{ij}) = ((a_{ij}x_{ij})_{ii} + (a_{ij}x_{ij})_{jj}, y_{ij}) \subseteq A_{ij}. \)
Again equating components to zero in (39) we find \( (x_{ij}y_{ij}, a_{ij}) = 0 \) or
\( (A_{ij}^2, A_{ij}) = 0. \)
To complete the proof of (18) we take \( a = a_{jj} \) in (39). By (13), (8), and (15),
\[(x_{ij}, y_{ij}, a_{jj}) = ((x_{ij}, y_{ij})_{11} + (x_{ij}, y_{ij})_{jj}, a_{jj})
\]
\[= ((x_{ij}, y_{ij})_{jj}, a_{jj}) \in A_{11} + A_{jj};\]
by (11), (13), and (37),
\[((y_{ij}, a_{jj})_{11}, x_{ij}) = ((y_{ij}, a_{jj})_{11} + (y_{ij}, a_{jj})_{11}, x_{ij})
\]
\[= ((y_{ij}, a_{jj})_{11}, x_{ij}) \in A_{1j};\]
and by (12) and (37),
\[(a_{jj}, x_{ij}, y_{ij}) = ((x_{ij}, a_{jj})_{11}, y_{ij}) \in A_{ij}.
\]
As before we may conclude that \[(x_{ij}, y_{ij}, a_{jj}) = 0\]
or
\[(A_{1j}^2, A_{jj}) = 0.\]
This completes the proof of (18).

The proofs of (19) through (21) depend upon the identity
\[H(x_{ij}, y_{ij}, a_{jj}) = 0, \text{ which we write as}\]
\[(40) \quad (x_{ij}, y_{ij}, a_{jj}) + (y_{ij}, a_{jj}, x_{ij}) + (a_{jj}, x_{ij}, y_{ij}) = 0.\]
By (14), (8), and (15),
\[(x_{ij}, y_{ij}, a_{jj}) = ((x_{ij}, y_{ij})_{11} + (x_{ij}, y_{ij})_{jj}, a_{jj})
\]
\[= ((x_{ij}, y_{ij})_{jj}, a_{jj}) \in A_{11} + A_{jj};\]
by (10) and (37),
\[(y_{ij}, a_{jj}, x_{ij}) = ((a_{jj}, y_{ij})_{11}, x_{ij}) \in A_{1j};\]
and by (12) and (38),
\[(a_{jj}, x_{ij}, y_{ij}) = ((x_{ij}, a_{jj})_{11}, y_{ij}) \in A_{jj}.
\]
By equating the components in \(A_{11} + A_{jj}, A_{1j},\) and \(A_{jj}\) of the left-hand side of (40) to zero we obtain (19), (20), and (21) respectively.
To derive (22) we first write

\[ P(e, x_{ij}, y_{ij}) - H(x_{ij}, y_{ij}) \]

\[ - H(x_{ij}, y_{ij}) = 0 \]

which, since \(2i - 1 = i - j\), reduces to

\[(i - j)(a_{ii}x_{ij} + y_{ij}) + (i - j)(a_{ii}x_{ij} + y_{ij})\]

\[ + (a_{ii}x_{ij} + x_{ij}a_{ii}y_{ij}) + (a_{ii}x_{ij} + x_{ij}a_{ii}y_{ij})\]

\[ + (a_{ii}y_{ij} + y_{ij}a_{ii}x_{ij}) + (a_{ii}y_{ij} + y_{ij}a_{ii}x_{ij})\]

\[ + 2(x_{ij}y_{ij}, e, a_{ii}x_{ij}) + 2(x_{ij}y_{ij}, e, a_{ii}x_{ij}) = 0.\]

Here we have used (13) to write

\[ x_{ij}y_{ij} + y_{ij}x_{ij} = 2x_{ij}y_{ij}. \]

By (18), (13), (18), and (16),

\[ x_{ij}y_{ij} + e_{i} = (1/2)(x_{ij}y_{ij} + a_{ii}x_{ij}y_{ij}) \in A_{ii}. \]

Thus by (13) and (18),

\[ (x_{ij}y_{ij}, e, a_{ii}x_{ij}) = 0. \]

Moreover, since \(x_{ij}a_{ii} \in A_{ij}\) and \(x_{ij}a_{ii}y_{ij} \in A_{ii}\), we find that

\[ (x_{ij}a_{ii}x_{ij}, e, y_{ij}) = 0. \]

Since the same relation holds with \(x_{ij}\) and \(y_{ij}\) interchanged, (41) becomes, again using \(x_{ij}y_{ij} = y_{ij}x_{ij}\),

\[ (a_{ii}x_{ij}, e, y_{ij}) + (a_{ii}x_{ij}, e, y_{ij}) + (i - j)a_{ii}x_{ij}y_{ij}\]

\[ + (a_{ii}y_{ij}, e, x_{ij}) + (a_{ii}y_{ij}, e, x_{ij}) + (i - j)a_{ii}y_{ij}x_{ij}\]

\[ + 2(j - 1)a_{ii}x_{ij}y_{ij} = 0. \]

But since \(a_{ii}x_{ij} \in A_{ij} + A_{ij}\) and \(a_{ii}x_{ij}y_{ij} \in A_{ii} + A_{ij}\),

\[ (a_{ii}x_{ij}, e, y_{ij}) + (a_{ii}x_{ij}, e, y_{ij}) + (i - j)a_{ii}x_{ij}y_{ij}\]

\[ = (i - j)(a_{ii}x_{ij}, y_{ij})_{ii}. \]

Thus, and similarly, (42) becomes
and (22) follows. (23) is the analogue of (22) in an anti-isomorphic algebra, \( A^\# \).

To obtain (24) and (25) we begin with the identity
\[
P(a_{ij}, e, x_{ij}, y_{ji}) - H(a_{ij}, x_{ij}, y_{ji}) - H(a_{ij}, y_{ji}, x_{ij}) = 0
\]
which we record as
\[
(i - j)(a_{ij} x_{ij} y_{ji}) + (i - j)(a_{ij} y_{ji} x_{ij})
\]
\[
+ (a_{ij} x_{ij} + x_{ij} a_{ii}, e, y_{ji}) + (a_{ij} x_{ij} + x_{ij} a_{ii}, y_{ji}, e)
\]
\[
+ (a_{ij} y_{ji} + y_{ji} a_{11}, e, x_{ij}) + (a_{ij} y_{ji} + y_{ji} a_{11}, x_{ij}, e)
\]
\[
+ (x_{ij} y_{ji} + y_{ji} x_{ij}, e, a_{11}) + (x_{ij} y_{ji} + y_{ji} x_{ij}, a_{11}, e) = 0.
\]

Since \( a_{ij} x_{ij} + x_{ij} a_{11} \in A_{ij} + A_{j1} \),
\[
(a_{ij} x_{ij} + x_{ij} a_{11}, e, y_{ji}) = 0.
\]

Since \( x_{ij} y_{ji} + y_{ji} x_{ij} \in A_{11} + A_{j1} \),
\[
(x_{ij} y_{ji} + y_{ji} x_{ij}, e, a_{11}) = 0.
\]

Also, \( x_{ij} a_{11} \in A_{j1}, a_{11} x_{ij}, y_{ji} \in A_{j1} + A_{11} \), so
\[
(x_{ij} a_{11}, y_{ji}, e) = 0,
\]

and
\[
a_{11} y_{ji} \in A_{j1}, a_{11} y_{ji}, x_{ij} \in A_{11},
\]

so
\[
(a_{11} y_{ji}, e, x_{ij}) + (a_{11} y_{ji}, x_{ij}, e) = 0.
\]

Thus (43) reduces to
(44) \((a_{11}x_{1j}y_{j1}, e) + (1 - J)a_{11}x_{1j}y_{j1} + (y_{j1}a_{11}, e, x_{1j}) + (y_{j1}a_{11}, x_{1j}, e) + (i - J)a_{11}y_{j1}x_{1j} + (x_{1j}y_{j1} + y_{j1}x_{1j}, a_{11}, e) - (1 - J)a_{11}(x_{1j}y_{j1} + y_{j1}x_{1j}) = 0.\)

Now since \(a_{11}x_{1j}y_{j1} \leq (A_{1j} + A_{jj})y_{j1} \leq A_{11} + A_{j1} + A_{jj},\)
we have

(45) \((a_{11}x_{1j}y_{j1}, e) + (1 - J)a_{11}x_{1j}y_{j1} = (1 - J)(a_{11}x_{1j}y_{j1})_{1j} + (1 - J)(a_{11}x_{1j}y_{j1})_{j1}.\)

By (9) and (14),

\[(a_{11}x_{1j}y_{j1})_{j1} = ((a_{11}x_{1j})_{1j}y_{j1})_{j1} + ((a_{11}x_{1j})_{jj}y_{j1})_{j1} = ((a_{11}x_{1j})_{jj}y_{j1})_{j1};\]

by (9) and (10),

\[((a_{11}x_{1j})_{jj}y_{j1})_{j1} = ((a_{11}x_{1j})_{jj}, y_{j1});\]

by (10),

\[((a_{11}x_{1j})_{jj}, y_{j1}) = (x_{1j}a_{11}, y_{j1});\]

and by (20),

\((x_{1j}a_{11}, y_{j1}) = 0.\)

Thus

\[(a_{11}x_{1j}y_{j1})_{j1} = 0,\]

which, along with (45), enables us to rewrite (44) as

(46) \((1 - J)(a_{11}x_{1j}y_{j1})_{1j} + (y_{j1}a_{11}, e, x_{1j}) + (y_{j1}a_{11}, x_{1j}, e) + (i - J)a_{11}y_{j1}x_{1j} + (x_{1j}y_{j1} + y_{j1}x_{1j}, a_{11}, e) - (1 - J)a_{11}(x_{1j}y_{j1} + y_{j1}x_{1j}) = 0.\)

By (11),
(\(y_{ji}a_{i1}e, x_{ij}\)) = 1(y_{ji}a_{i1})_{ji}x_{ij} + J(y_{ji}a_{i1})_{jj}x_{ij}
- 1(y_{ji}a_{i1})_{ji}x_{ij} - J(y_{ji}a_{i1})_{jj}x_{ij} = (J - 1)(y_{ji}a_{i1})_{jj}x_{ij};

and by (12),
(\(1 - J\))\(a_{i1}y_{ji}x_{ij}\) = (\(1 - J\))(\(y_{ji}a_{i1}\))_{jj}x_{ij}.

Thus
(\(y_{ji}a_{i1}e, x_{ij}\)) + (\(1 - J\))\(a_{i1}y_{ji}x_{ij}\) = 0

and (46) becomes
(47) (\(1 - J\))(\(a_{i1}x_{ij}y_{ji}\))_{ii} + (\(y_{ji}a_{i1}x_{ij}\))_{ii} + \(x_{ij}y_{ji} + y_{ji}x_{ij}\) - (\(1 - J\))\(a_{i1}(x_{ij}y_{ji} + y_{ji}x_{ij})\) = 0.

Since \(y_{ji}a_{i1}x_{ij}\) \(\in (A_{ji} + A_{jj})A_{ij} \subseteq A_{ii} + A_{jj},\)

(\(y_{ji}a_{i1}x_{ij}\))_{ii} - (\(y_{ji}a_{i1}x_{ij}\))_{jj} = (\(1 - J\))(\(y_{ji}a_{i1}x_{ij}\))_{ii}

and (47) reduces to
(48) (\(1 - J\))(\(a_{i1}x_{ij}y_{ji}\))_{ii} + (\(1 - J\))(\(y_{ji}a_{i1}x_{ij}\))_{ii} + \(x_{ij}y_{ji} + y_{ji}x_{ij}\) - (\(1 - J\))\(a_{i1}(x_{ij}y_{ji} + y_{ji}x_{ij})\) = 0.

If \((x_{ij}y_{ji} + y_{ji}x_{ij})\)\(a_{i1} = z_{i1} + z_{jj}\) then \(a_{i1}(x_{ij}y_{ji} + y_{ji}x_{ij})\)

= \(w_{ii} - z_{jj}\) because of (8) and (16). Thus
(\(x_{ij}y_{ji} + y_{ji}x_{ij}\))\(a_{i1} - (1 - J)a_{i1}(x_{ij}y_{ji} + y_{ji}x_{ij})\)

= \(iz_{i1} + Jz_{jj} - iz_{i1} - iz_{jj} - (1 - J)w_{ii} + (1 - J)z_{jj}\)

= \(-(1 - J)((a_{i1}(x_{ij}y_{ji} + y_{ji}x_{ij}))_{ii}).\)

Together with (48) this implies (24). Then
0 = H(x_{ij}, y_{j1}, a_{i1}) + H(y_{j1}, x_{ij}, a_{i1}) = (x_{ij}y_{j1} + y_{j1}x_{ij}, a_{i1})
  + (a_{i1}x_{ij} + x_{ij}a_{i1}, y_{j1}) + (a_{i1}y_{j1} + y_{j1}a_{i1}, x_{ij})

and by (20) and (21)

(x_{ij}a_{i1}, y_{j1}) = (a_{i1}y_{j1}, x_{ij}) = 0.

Hence

(x_{ij}y_{j1} + y_{j1}x_{ij})a_{i1} - a_{i1}(x_{ij}y_{j1} + y_{j1}x_{ij}) = a_{i1}x_{ij}, y_{j1}
  + y_{j1}, a_{i1}, x_{ij} - y_{j1}, a_{i1}, x_{ij} + x_{ij}, y_{j1}, a_{i1}.

Equating the components in $A_{i1}$ and using (24) gives (25). This completes the proof of Lemma 1.
2. A STRUCTURE THEOREM

In this chapter we assume that $A$ is a simple algebra and that $A_{00} = 0$. An example of such an algebra is the algebra $B$ with basis $\{e, x, y\}$ such that $e^2 = e$, $ex = x$, $ye = y$, $xy = -yx = e$, and all other products of basis elements are zero. Every element $a$ of $B$ has the form

$$a = \alpha e + \beta x + \gamma y$$

where $\alpha$, $\beta$, $\gamma$ are scalars. If $a$ is a nonzero element of an ideal $I$ of $B$, then $I$ also contains $ea + ae - a = \alpha e$, $x(a - ea) = \gamma e$, and $(a - ae)y = \beta e$. Since the scalars $\alpha$, $\beta$, and $\gamma$ are not all zero, $e \notin I$. Hence also $ex = x$ and $ye = y$ are in $I$ so that $I = B$. Thus the only ideals of $B$ are $B$ and the zero ideal. Therefore $B$ is simple.

If

$$a_r = \alpha_r e + \beta_r x + \gamma_r y$$

for $r = i, j, k$, then

$$a_i a_j = (\alpha_i \alpha_j + \beta_i \gamma_j - \gamma_i \beta_j)e + (\alpha_i \beta_j)x + (\gamma_i \gamma_j)y.$$  

In particular $a_1^2 = \alpha_1 a_1$, from which we see that $B$ is power-associative and in fact

$$a_1^n = \alpha_1^{n-1} a_1$$

for every positive integer $n$. Moreover
\[(a_i, a_j) = 2(\alpha_1 \beta_j - \gamma_1 \beta_j)e + (\alpha_1 \beta_j - \gamma_1 \beta_j)x + (\gamma_1 \alpha_j - \gamma_j \alpha_1)y\]

and

\[
((a_i, a_j), a_k) = 2[(\alpha_1 \beta_j - \gamma_1 \beta_j) \gamma_k - (\gamma_1 \alpha_j - \gamma_j \alpha_1) \beta_k]e + [2(\beta_1 \gamma_j - \gamma_1 \beta_j) \beta_k - \alpha_k(\alpha_1 \beta_j - \gamma_1 \beta_j)]x + [(\gamma_1 \alpha_j - \gamma_j \alpha_1) \alpha_k - 2 \gamma_k(\beta_1 \gamma_j - \gamma_1 \beta_j)]y.
\]

A short computation shows that

\[
((a_i, a_j), a_k) + ((a_j, a_k), a_i) + ((a_k, a_i), a_j) = 0
\]

or \(B\) is Lie-admissible. In fact a closer look at \(B(\cdot)\), in which 
\((e, x) = x, (e, y) = -y,\) and \((x, y) = 2e,\) reveals that so long as the field of scalars has characteristic different from two, \(B(\cdot)\) is a simple Lie algebra.

It is evident that \(B\) has a Peirce decomposition with respect to the idempotent \(e\).

**THEOREM 1.** If \(A\) is a simple Lie-admissible power-associative algebra over an algebraically closed field \(\mathbb{F}\) with characteristic prime to 6, and if \(A\) has an idempotent \(e\) satisfying (4) and such that \(A_{00} = 0\) in the resulting Peirce decomposition, then either \(e\) is a unity element of \(A\) or \(A \cong B\), where \(B\) is the three-dimensional algebra described above.

**PROOF.** The multiplication properties in Lemma 1 will play an important role in our proof. We shall make repeated use of the module multiplication table.
which compactly exhibits the information in (9) through (14) for the case that $A_{00} = 0$. We note also that (13) says

$$(50) \quad (A_{10} A_{10}) = (A_{01} A_{01}) = 0$$

and (18) says

$$(51) \quad (A_{10}^2 A_{11} + A_{01}) = (A_{01}^2 A_{11} + A_{10}) = 0.$$  

From (19) we have

$$(52) \quad (A_{01} A_{10} A_{11}) = 0$$

and (22) through (25) specialize to

$$(53) \quad a_{11} x_{10} y_{10} = x_{10} y_{10} a_{11} = (1/2)(a_{11} x_{10} y_{10} + a_{11} y_{10} x_{10}),$$

$$(54) \quad a_{11} x_{01} y_{01} = x_{01} y_{01} a_{11} = (1/2)(x_{01} a_{11} y_{01} + y_{01} a_{11} x_{01}),$$

$$(55) \quad a_{11} (x_{10} y_{01} + y_{01} x_{10}) = a_{11} x_{10} y_{01} + y_{01} a_{11} x_{10},$$

and

$$(56) \quad (x_{10} y_{01} + y_{01} x_{10}) a_{11} = y_{01} a_{11} x_{10} + x_{10} y_{01} a_{11}$$

respectively.

We assume throughout that $e$ is not a unity element for $A$ and begin by proving
LEMMA 2. \( A_{11} \) is an associative subalgebra of \( A \) and moreover,

\[(A, A_{11}, A_{11}) = (A_{11}, A, A_{11}) = (A_{11}, A_{11}, A) = 0.\]

PROOF.

\[H(x_{11}, y_{11}, a_{1j}) = (x_{11}, y_{11}, a_{1j}) + (y_{11}, a_{1j}, x_{11}) \]

\[+ (a_{1j}, x_{11}, y_{11}) = 0;\]

and if \( i \neq j \) then, by (49), two of the associators are equal to zero, whence all three are equal to zero. We also see from (49) that \( A_{11} \) is a subalgebra of \( A \). Thus it suffices to show that \( A_{11} \) is associative.

We assert that the subspace

\[I = (A_{11}, A_{11}, A_{11}) + (A_{11}, A_{11}, A_{11})A_{11}\]

is an ideal of \( A \). To prove this assertion we need the fact, readily verified by expansion, that the function

\[T(a, x, y, b) = (ax, y, b) - (a, xy, b) \]

\[+ (a, x, yb) - a(xy, b) - (a, x, y)b\]

is identically zero in any non-associative algebra. Thus

\[0 = T(a_{mn}, x_{11}, y_{11}, b_{ij}) = (a_{mn}x_{11}, y_{11}, b_{ij}) - (a_{mn}x_{11}, y_{11}, b_{ij}) \]

\[+ (a_{mn}x_{11}, y_{11}, b_{ij}) - a_{mn}(x_{11}, y_{11}, b_{ij}) - (a_{mn}x_{11}, y_{11}, b_{ij}).\]

If \( m + n = 1 + j = 2 \), then it follows that

\[A_{11}(A_{11}, A_{11}, A_{11}) \subseteq I.\]

If \( m + n = 2 \) and \( 1 + j = 1 \), then from (49) and the observation in the first paragraph above we get

\[(A_{11}, A_{11}, A_{11})A_{ij} = 0.\]

If \( m + n = 1 \) and \( 1 + j = 2 \), we get

\[A_{mn}(A_{11}, A_{11}, A_{11}) = 0.\]

Thus
(A_{11}, A_{11}, A_{11}) A \subseteq I,

and

\[ A(A_{11}, A_{11}, A_{11}) \subseteq I. \]

Now \((A_{11}, A_{11}, A), (A_{11}, A, A_{11}),\) and \((A, A_{11}, A_{11})\) are each contained in \(I\) and consequently

\[ (A_{11}, A_{11}, A_{11}) A_{11} \subseteq ((A_{11}, A_{11}, A_{11}), A_{11}, A) \]
\[ + (A_{11}, A_{11}, A_{11}) A_{11} \subseteq I. \]

We have shown that \(IA \subseteq I.\) Finally

\[ A \cdot (A_{11}, A_{11}, A_{11}) A_{11} \subseteq (A_{11}, A_{11}, A_{11}), A_{11} \]
\[ + A(A_{11}, A_{11}, A_{11}), A_{11} \subseteq I + IA_{11} \subseteq I, \]

and it follows that \(AI \subseteq I.\) Hence \(I\) is an ideal of \(A.\) If \(A = I \leq A_{11}\) then \(e\) is a unity element for \(A,\) which contradicts our assumption. Therefore \(I = 0\) and in particular \((A_{11}, A_{11}, A_{11}) = 0\) which proves Lemma 2.

Next we show

**Lemma 3.** \(A_{ij}^2 = 0\) for \(i \neq j.\)

**Proof.** We prove that \(A_{10}^2 = 0.\) Then \(A_{01}^2 = 0\) by the anti-isomorphism principle.

First we prove that \(J = A_{10}^2 + A_{10}^2 A_{10}\) is an ideal of \(A.\)

By (53),

\[ A_{11} A_{10}^2 = A_{10} A_{11}^2 \subseteq A_{10}^2; \]

by (49),

\[ A_{10} \cdot A_{10}^2 \subseteq A_{10} A_{11} = 0; \]

and by (51) and (49),
\[ A_{01}^2 A_{10} = A_{10}^2 A_{01} \subseteq A_{11}^2 A_{01} = 0. \]

Since \( A_{10}^2 A_{10} \subseteq J \) by definition, we have \( A_{10}^2 A \subseteq J \) and \( A_{10}^2 \subseteq J \).

By (49),
\[ (A_{10}^2 A_{10})_{11} \subseteq A_{11} A_{10} \cdot A_{11} \subseteq A_{10}^2 A_{11} = 0. \]

By (49), (57), and the paragraph above,
\[ A_{11} (A_{10}^2 A_{10}) = (A_{11} A_{10}^2) A_{10} \subseteq A_{10}^2 A_{10} \subseteq J. \]

Again by (49),
\[ (A_{10}^2 A_{10}) A_{10} = A_{10} (A_{10}^2 A_{10}) \subseteq A_{10} \subseteq J. \]

Letting \( a_{11} = u_{10} y_{10} \) in (55), we have
\[ u_{10} y_{10} (x_{10} y_{01} + y_{01} x_{10}) = (u_{10} y_{10} y_{01}) y_{01} + (u_{10} y_{10} x_{10}) x_{10}. \]

But by (51) and (49),
\[ y_{01} u_{10} y_{10} = u_{10} y_{10} y_{01} = 0. \]

Hence
\[ (u_{10} y_{10} x_{10}) y_{01} = u_{10} y_{10} (x_{10} y_{01} + y_{01} x_{10}) \subseteq A_{10}^2 A_{11} \subseteq J \]
by the above, and so
\[ A_{10}^2 A_{10} A_{01} \subseteq J. \]

Finally,
\[ 0 = H(u_{10} y_{10} x_{10} y_{01}) = (u_{10} y_{10} y_{01}) \]
\[ + (x_{10} y_{01}, u_{10} y_{10}) + (y_{01}, u_{10} y_{10}, x_{10}), \]
and
\[ y_{01} u_{10} y_{10} = 0 \]
as above. Also
\[ (x_{10} y_{01}, u_{10} y_{10}) \subseteq (A_{11}, A_{10}^2) = 0 \]
by (49) and (51), so
\[ y_{01}(u_{10}v_{10}x_{10}) = (u_{10}v_{10}x_{10})y_{01} \in A_{10}^2 A_{10} \cdot A_{01} \leq J. \]

Thus \( J \) is an ideal of \( A \).

Since \( A \) is simple either \( J = 0 \) or \( J = A \). If \( J = A_{10}^2 A_{10} + A_{10}^2 = A \) then \( A_{10}^2 = A_{11} \), \( A_{11} A_{10} = A_{10} \) and \( A_{01} = 0 \). Since \( e \in A_{10}^2 \), we may write

\[ e = \sum_{i=1}^{t} x_{10}(i) y_{10} \]

Now for any \( x_{10}, y_{10} \) we have

(58)
\[
0 = H(x_{10}, x_{10}, y_{10}) = (x_{10}^2 y_{10}) + (x_{10} y_{10}^2) + (y_{10} x_{10}^2) = x_{10}^2 y_{10} + 2x_{10} y_{10}^2 + 2x_{10} y_{10}^2
\]
by (49). Also

\[
0 = (1/4)P(x_{10}, x_{10}, y_{10}) = (x_{10}^2 y_{10}) + 2(x_{10} y_{10}^2) + (y_{10} x_{10}^2)
\]

is in the center of the associative subalgebra \( A_{11} \), we obtain

\[
x_{10}^2 y_{10}^2 = -2(x_{10} y_{10})^2.
\]

It follows that

\[
(x_{10} y_{10})^4 = (1/4)(x_{10}^2 y_{10}^2)^2 = (1/4)x_{10}^4 y_{10}^4 = 0
\]

since \( x_{10}^3 = x_{10} \cdot x_{10}^2 = 0 \).

But then we have

\[
e = e^{3t+1} = \left[ \sum_{i=1}^{t} x_{10}(i) y_{10} \right]^{3t+1} = 0
\]
since every term in the multinomial expansion must contain, for
some \( j \), a factor \( \binom{j}{10} \binom{j}{10} \) = 0. From this contradiction we conclude
that \( J = 0 \). Thus \( A_{10}^2 = 0 \) and the lemma is proved.

In view of Lemma 3 we may now replace (49) with the table

<table>
<thead>
<tr>
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<th>( A_{11} )</th>
<th>( A_{10} )</th>
<th>( A_{01} )</th>
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<td>( A_{11} )</td>
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</tr>
<tr>
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<td>( A_{11} )</td>
</tr>
<tr>
<td>( A_{01} )</td>
<td>( A_{01} )</td>
<td>( A_{11} )</td>
<td>0</td>
</tr>
</tbody>
</table>

We observe in passing that if either \( A_{10} \) or \( A_{01} \) is zero then
the other is an ideal of \( A \). If both are zero then \( e \) is a unity
element for \( A = A_{11} \).

**Lemma 4.** If \( B_{11} \) is an ideal of \( A_{11} \) then

\[
L = B_{11} + B_{11}A_{10} + A_{01}B_{11} + A_{01}B_{11}A_{10} + B_{11}A_{10}A_{01} + (A_{01}B_{11}A_{10})A_{11}
\]

is an ideal of \( A \).

**Proof.** Let \( B_{11} \) be an ideal of \( A_{11} \). Then

\[
A_{11}B_{11} + B_{11}A_{11} \subseteq B_{11} \subseteq L.
\]

By (59),

\[
A_{10}B_{11} + B_{11}A_{10} = B_{11}A_{10} \subseteq L
\]

and
\[ A_{01}B_{11} + B_{11}A_{01} = A_{01}B_{11} \leq L. \]

Thus
\[ AB_{11} + B_{11}A \leq L. \]

By (59),
\[ B_{11}A_{10} - A_{10}A_{11} = 0; \]
by (57) and the fact that \( B_{11} \) is an ideal of \( A_{11} \),
\[ A_{11}B_{11}A_{10} = A_{11}B_{11} \leq B_{11}A_{10} \leq L; \]
and by (50) and (59),
\[ A_{10}B_{11}A_{10} = B_{11}A_{10} \leq A_{10} \leq 0. \]

Noting that
\[ A_{01}B_{11}A_{10} + B_{11}A_{10} \leq L \]
by the definition of \( L \) we conclude that
\[ A_{01}B_{11}A_{10} \leq L \]
and
\[ B_{11}A_{10} \leq L. \]

Let \( b_{11} \in B_{11} \). By (55),
\[ y_{01}b_{11}x_{10} = b_{11}(x_{10}y_{01} + y_{01}x_{10}) - b_{11}x_{10}y_{01}. \]
Since \( x_{10}y_{01} + y_{01}x_{10} \in A_{11} \) and \( B_{11} \) is an ideal of \( A_{11} \) the above implies
\[ A_{01}B_{11}A_{10} \leq B_{11} + B_{11}A_{10} \leq L. \]
Similarly, by (56),
\[ x_{10}y_{01}b_{11} = (x_{10}y_{01} + y_{01}x_{10})b_{11} - y_{01}b_{11}x_{10}, \]
which implies
\[ A_{10}A_{10}B_{11} \leq B_{11} + A_{01}B_{11}A_{10}. \]
Moreover by (57) and the fact that $B_{11}$ is an ideal of $A_{11}$,

$$A_{01}B_{11}A_{11} = A_{01}B_{11}A_{11} \subseteq A_{01}B_{11} \subseteq L;$$

by (59),

$$A_{11}A_{01}B_{11} \subseteq A_{11}A_{01} = 0;$$

and by (59) and (50),

$$A_{01}B_{11}A_{01} \subseteq A_{01}B_{11}A_{01} \subseteq A_{01}^2 = 0.$$ 

Thus $A_{01}B_{11}A_{11}$ and $A_{01}B_{11}A_{11}A_{01}$ are contained in $L$.

Now

$$(A_{01}B_{11}A_{11})A_{11} \subseteq L$$

by definition of $L$, and since, by (59) and (52),

$$(A_{01}B_{11}A_{11})A_{11} \subseteq (A_{01}B_{11}A_{11}) = 0,$$

it follows that

$$A_{11}(A_{01}B_{11}A_{11}) \subseteq L.$$

We see by (59) that

$$A_{01}B_{11}A_{11} \subseteq A_{11}$$

and hence

$$A_{11}(A_{01}B_{11}A_{11}) \subseteq A_{10}A_{11} = 0.$$ 

Therefore

$$(A_{01}B_{11}A_{11})A_{11} = (A_{01}B_{11}A_{11}, A_{10})$$

and by using the function $H$,

$$(A_{01}B_{11}A_{11}, A_{10}) \subseteq (B_{11}A_{10}A_{10}A_{01}) + (A_{10}A_{01}B_{11}A_{10}).$$

Now $B_{11}A_{10} \subseteq A_{10}$ so

$$B_{11}A_{10}A_{10} \subseteq A_{10}^2 = 0.$$
by (59). Moreover \( A_{10}A_{01} \leq A_{11} \) and it follows from the second paragraph of this proof that

\[(A_{01}B_{11}A_{10}) \leq L.\]

Consequently we have

\[(A_{01}B_{11}A_{10})A_{10} \leq L.\]

Also

\[(A_{01}B_{11}A_{10})A_{01} \leq A_{11}A_{01} = 0\]

and since (61) implies

\[A_{01}B_{11}A_{10} \leq B_{11} + A_{10}A_{01}B_{11}\]

we have

\[(A_{01}B_{11}A_{10}) \leq A_{01}B_{11} + A_{01}(A_{10}A_{01}B_{11}).\]

By (59) and use of the function \( H \),

\[A_{01}(A_{10}A_{01}B_{11}) = -(A_{10}A_{01}B_{11}A_{10}) = (A_{10}A_{01}B_{11}, A_{01})\]

\[\leq (A_{01}B_{11}A_{01}, A_{10}) + (A_{01}A_{10}A_{01}B_{11});\]

by (59),

\[(A_{01}A_{10}A_{01}B_{11}) \leq (A_{01}A_{10}) = 0\]

and

\[(A_{01}A_{10}A_{01}B_{11}) \leq (A_{11}A_{01}B_{11});\]

and by the third paragraph of this proof

\[(A_{11}A_{01}B_{11}) \leq L.\]

Since \( A_{01}B_{11} \leq L \), (61) now implies

\[A_{01}(A_{01}B_{11}A_{10}) \leq L.\]

We thus have
\[ A(A_{01} \cdot B_{11}A_{10}) \subseteq L \]

and

\[ (A_{01} \cdot B_{11}A_{10})A \subseteq L. \]

Let \( b_{11} \in B_{11} \). Then \( b_{11}x_{10} \in A_{10} \) and by (56),

\[ (b_{11}x_{10} \cdot y_{01} + y_{01} \cdot b_{11}x_{10})a_{11} = y_{01}(a_{11} \cdot b_{11}x_{10}) \]

\[ + b_{11}x_{10} \cdot y_{01}a_{11}. \]

By (57) and the fact that \( B_{11} \) is an ideal of \( A_{11} \),

\[ y_{01}(a_{11} \cdot b_{11}x_{10}) = y_{01}(a_{11}b_{11} \cdot x_{10}) \in A_{01} \cdot B_{11}A_{10} \subseteq L. \]

Also \( y_{01}a_{11} \in A_{01} \) so that

\[ b_{11}x_{10} \cdot y_{01}a_{11} \in B_{11}A_{10} \cdot A_{01} \subseteq L. \]

Since

\[ (y_{01} \cdot b_{11}x_{10})a_{11} \in (A_{01} \cdot B_{11}A_{10})A_{11} \subseteq L, \]

it now follows from (65) that

\[ (b_{11}x_{10} \cdot y_{01})a_{11} \subseteq L. \]

Next we note that (55) implies

\[ a_{11}(b_{11}x_{10} \cdot y_{01} + y_{01} \cdot b_{11}x_{10}) = (a_{11} \cdot b_{11}x_{10})y_{01} \]

\[ + y_{01}a_{11} \cdot b_{11}x_{10}, \]

and

\[ a_{11}(y_{01} \cdot b_{11}x_{10}) = (y_{01} \cdot b_{11}x_{10})a_{11} \in (A_{01} \cdot B_{11}A_{10})A_{11} \subseteq L \]

by (59), (52), and the definition of \( L \). Since \( y_{01}a_{11} \in A_{01}, \)

\[ y_{01}a_{11} \cdot b_{11}x_{10} \in A_{01} \cdot B_{11}A_{10} \subseteq L, \]

and by (57) and the fact that \( B_{11} \) is an ideal of \( A_{11} \),

\[ (a_{11} \cdot b_{11}x_{10})y_{01} = (a_{11}b_{11} \cdot x_{10})y_{01} \in B_{11}A_{10} \cdot A_{01} \subseteq L. \]

It now follows from (66) that \( a_{11}(b_{11}x_{10} \cdot y_{01}) \in L \) or
Now \( B_{11}A_{10} \leq A_{10} \) so \( B_{11}A_{10} \cdot A_{01} \leq A_{10}A_{01} \leq A_{11} \) and consequently
\[
A_{10}(B_{11}A_{10} \cdot A_{01}) \leq A_{10}A_{11} = 0.
\]

It then follows that
\[
(B_{11}A_{10} \cdot A_{01})A_{10} = (B_{11}A_{10} \cdot A_{01}, A_{10}).
\]

By use of the function \( H \) we find
\[
(B_{11}A_{10} \cdot A_{01}, A_{10}) \leq (A_{01}A_{10}, B_{11}A_{10}) + (A_{10}B_{11}A_{10}, A_{01}).
\]

Now by (59),
\[
A_{10} \cdot B_{11}A_{10} \leq A_{10}^2 = 0,
\]
and by (62)
\[
(A_{01}A_{10} \cdot B_{11}A_{10}) \leq L.
\]

Thus
\[
(B_{11}A_{10} \cdot A_{01})A_{10} \leq L.
\]

Also \( B_{11}A_{10} \leq A_{10}, B_{11}A_{10} \cdot A_{01} \leq A_{10}A_{01} \leq A_{11} \), and therefore
\[
(B_{11}A_{10} \cdot A_{01})A_{01} \leq A_{11}A_{01} = 0.
\]

Now (60) implies that
\[
B_{11}A_{10} \cdot A_{01} \leq B_{11} + A_{01}B_{11} \cdot A_{10}.
\]

Therefore
\[
A_{01}(B_{11}A_{10} \cdot A_{01}) \leq A_{01}(B_{11} + A_{01}B_{11} \cdot A_{10}).
\]

Since \( A_{01}B_{11} \leq L \), it follows that \( A_{01}(B_{11}A_{10} \cdot A_{01}) \leq L \) if and only if \( A_{01}(A_{01}B_{11} \cdot A_{10}) \leq L \). Now \( A_{01}B_{11} \leq A_{01} \) implies \( A_{01}B_{11} \cdot A_{10} \leq A_{01}A_{10} \leq A_{11} \) implies \( (A_{01}B_{11} \cdot A_{10})A_{01} = 0 \). Therefore
\[
A_{01}(A_{01}B_{11} \cdot A_{10}) = -(A_{01}B_{11} \cdot A_{10}, A_{01}),
\]

and by using the function \( H \),
\[ A_{01}(A_{01}B_{11}A_{10}) \leq (A_{10}A_{01}, A_{10}B_{11}) + (A_{01}A_{01}B_{11}, A_{10}). \]

But \( A_{01}B_{11} \leq A_{01} \) so \( A_{01}A_{01}B_{11} \leq A_{01}^2 = 0 \) by Lemma 3. Also, by (64),

\[ (A_{10}A_{01}, A_{01}B_{11}) \leq L. \]

Hence

\[ A_{01}(B_{11}A_{10}, A_{01}) \leq L. \]

This completes the argument that

\[ A(B_{11}A_{10}, A_{01}) \leq L \]

and that

\[ (B_{11}A_{10}, A_{01}) \leq L. \]

By (59) and (52),

\[ (A_{01}B_{11}A_{10}, A_{11}) \leq (A_{01}A_{10}, A_{11}) = 0. \]

Thus we see that \( A_{11}(A_{01}B_{11}A_{10}) = (A_{01}B_{11}A_{10})A_{11} \). Therefore it suffices to show that \( (A_{01}B_{11}A_{10})A_{11} \cdot A \) and \( A \cdot A_{11}(A_{01}B_{11}A_{10}) \) are contained in \( L \). By (59), (57), and the fourth paragraph of this proof,

\[ (A_{01}B_{11}A_{10})A_{11} \cdot A = (A_{01}B_{11}A_{10})A_{11}A \]

\[ \leq (A_{01}B_{11}A_{10})A \leq L, \]

and

\[ A \cdot A_{11}(A_{01}B_{11}A_{10}) = AA_{11}(A_{01}B_{11}A_{10}) \]

\[ \leq A(A_{01}B_{11}A_{10}) \leq L. \]

This completes the proof that \( L \) is an ideal of \( A \).

**Lemma 5.** \( A_{11} \) is a simple algebra.

**Proof.** Lemma 4 shows that if \( B_{11} \) is a non-zero ideal of \( A_{11} \) then
and

\[ B_{11}A_{10} = A_{10} \]

From (57) it follows that for every positive integer \( n \)

\[ R^n_{11}A_{10} = A_{10} \]

and

\[ A_{01}R_{11} = A_{01} \]

Thus \( A_{11} \) can have no non-zero nilpotent ideals. In particular then, the radical (which is defined to be the maximal nilpotent ideal) of \( A_{11} \) is zero and consequently \( A_{11} \) is a semi-simple (associative) algebra.

By a well-known [8, p. 27] theorem of Wedderburn, \( A_{11} \) is the direct sum of simple algebras. If \( B_{11} \) and \( C_{11} \) are distinct (hence orthogonal) simple components, we have, by (57) and the facts that

\[ B_{11}C_{11} = 0 \text{ and } B_{11}A_{10} = C_{11}A_{10} = A_{10}, \]

which as we have seen cannot be the case. Thus the number of simple components is one, which means that \( A_{11} \) is simple, proving Lemma 5.

By Wedderburn's theorem there is a \( \phi \)-isomorphism between the simple associative algebra \( A_{11} \) and the algebra \( M_n(D) \) of \( n \) by \( n \) matrices over \( D \), where \( D \) is a division algebra over \( \Phi \). When \( \Phi \) is algebraically closed, \( D = \Phi \). We shall not invoke the hypothesis that \( \Phi \) is algebraically closed until a later stage of the proof. Doing this will enable us to prove a stronger theorem in Chapter 3.
Let the matrix $E_{ij}$, having a one in the $i$-th row and $j$-th column and zeros elsewhere, correspond in the algebra isomorphism to the element $e_{ij}$ of $A_{11}$. Then the identity matrix corresponds to the element $\sum_{i=1}^{n} e_{ii}$ and since $e$ is the unique identity element of $A_{11}$ we must have

$$e = \sum_{i=1}^{n} e_{ii}.$$ 

We may identify $D$ as a matrix subalgebra of $M_n(D)$ over $\Phi$ consisting of the matrices

$$\text{diag}(d, \cdots, d)$$

where $d \in D$.

Let $d_1, \cdots, d_t$ be elements in $A_{11}$ corresponding to basis elements of the division algebra $D$ over $\Phi$. By the above we may think of $D$ as being a subalgebra of $A_{11}$. Consequently we shall refer to the elements $d_m$ as being "in" $D$.

The arbitrary element of $A_{11}$ may be written in the form

$$\sum_{m=1}^{t} \sum_{p,q=1}^{n} \gamma^{mpq} d_m e_{pq}$$

where $\gamma^{mpq} \in \Phi$. To multiply such elements we use the distributive laws and the rule

$$(d_m e_{pq})(d_r e_{uv}) = \delta_{qu} d_m d_r e_{pv}.$$ 

The conditions $A_{11}A_{10} \subseteq A_{10}$ and $(A_{11}, A_{11}, A_{10}) = 0$ imply that $A_{11}$ may be thought of as a set of (left) linear transformations on
the vector space $A_{10}$ over $\Phi$. From this viewpoint the elements $e_{ii}$, $i = 1, \ldots, n$, form a supplementary set of orthogonal projections. Consequently $A_{10}$ may be decomposed into the vector space direct sum
\[ \sum_{i=1}^{n} R_{10}(e_{ii}), \]
where $R_{10}(e_{ii})$ denotes the range space of the operator $e_{ii}$. Therefore we have a basis
\[ \{ x_{1}^{(1)}, \ldots, x_{1}^{(n)}, \ldots, x_{n}^{(1)}, \ldots, x_{n}^{(n)} \} \]
for $A_{10}$ over $\Phi$ with the special property
\[ e_{ii} x_{j}^{(k)} = \delta_{ij} x_{j}^{(k)} \]
for all $k$.

Note that if $i \neq j$, then for all $m, k$,
\[ e_{mi} x_{j}^{(k)} = e_{mi} e_{ii} x_{j}^{(k)} = 0. \]
Here and in the following we make free use of (57).

For all $m, j, k$,
\[ e_{mj} x_{j}^{(k)} = e_{mm} e_{mj} x_{j}^{(k)}. \]
Thus
\[ e_{mj} x_{j}^{(k)} \in R_{10}(e_{mm}). \]
If $0 \neq d \in D$, then
\[ x_{j}^{(k)} = e_{jj} x_{j}^{(k)} = (d^{-1} e_{jm})(de_{mj}) x_{j}^{(k)}, \]
and consequently,
\[ de_{mj} x_{j}^{(k)} \neq 0. \]

$A_{11}$ may also be viewed as a set of (right) linear transformations on $A_{01}$ and $A_{01}$ has a basis.
\{ y_1^{(1)}, \ldots, y_1^{(d_1)}, \ldots, y_n^{(1)}, \ldots, y_n^{(d_n)} \}

over \( \Phi \) such that

\begin{equation}
(y_j^{(k)})_{e_{11}} = \delta_{ij} y_j^{(k)}. \tag{71}
\end{equation}

By (52), \( A_{11} A_{10} \) is contained in the center of \( A_{11} \). The center of \( A_{11} \) consists of those elements of \( A_{11} \) which correspond to scalar matrices with an element from the center of \( D \) down the diagonal.

Thus we may write

\begin{equation}
y_j^{(k)} x_u^{(v)} = d_{jkuv}. \tag{72}
\end{equation}

where the \( d_{jkuv} \) are in the center of \( D \). By (59) we may also suppose

\begin{equation}
x_u^{(v)} y_j^{(k)} = \sum_{m=1}^{t} \sum_{p,q=1}^{n} \gamma_{mpq}^{kuv} (d_{mnpq}) \tag{73}
\end{equation}

where the \( \gamma_{mpq}^{kuv} \) are in \( \Phi \).

Now

\[
0 = R(x_u^{(v)}, y_j^{(k)} x_r^{(s)}) = (x_u^{(v)} y_j^{(k)} x_r^{(s)}) + (y_j^{(k)} x_r^{(s)} x_u^{(v)}) + (x_r^{(s)} x_u^{(v)} y_j^{(k)})
\]

and the last commutator is zero by Lemma 3. Using (59), the above reduces to

\begin{equation}
x_u^{(v)} y_j^{(k)} x_r^{(s)} + y_j^{(k)} x_r^{(s)} x_u^{(v)} = 0; \tag{74}
\end{equation}

and substituting from (72) and (73) into this we obtain

\[
\sum_{m=1}^{t} \sum_{p,q=1}^{n} \gamma_{mpq}^{kuv} (d_{mnpq} x_r^{(s)}) + d_{jkr u} x_u^{(v)} = 0.
\]

Using (68) we have
We may now prove

**LEMMA 6.** $A_{11} = D$.

**PROOF.** The lemma asserts that $n = 1$. Assume $n \neq 1$. Then we may select a $v$ such that $v \neq u$, and consequently

$$e_{wv}(d_{jkrsv}x_v^k) = e_{wv}(d_{jkrsv}u^x_u^k) = 0.$$  

Thus multiplication of (75) on the left by $e_{wv}$ yields

$$
\sum_{m=1}^t v_{mwr} d_{jkrsv} x_r^s = 0
$$

or

$$
(\sum_{m=1}^t v_{mwr} d_{jkrsv}) e_{wv} x_r^s = 0.
$$

By (70) we conclude that

$$
\sum_{m=1}^t v_{mwr} d_{jkrsv} = d_{jkrsv} = 0
$$

and then, since $\{d_1, \ldots, d_t\}$ is linearly independent over $\Phi$,

$$
(\sum_{m=1}^t v_{mwr} d_{jkrsv}) = 0.
$$

This holds for all $j, k, u, v, m, r$ and all $v \neq u$. Thus (75) may now be written

$$
(\sum_{m=1}^t v_{mwr} d_{jkrsv}) e_{wv} x_r^s + d_{jkrsv} x_v^k = 0.
$$

By (55),

$$
e_{wv}(x_u^k y_j^k + y_j^k x_u^k) = e_{wv} x_u^k y_j^k + y_j^k e_{wv} x_u^k.
$$
Since \( v \neq u \) the right-hand side of (78) is, by (67), (71), and (72),

\[ \delta_{v} d_{ju v} \]

By (76), (73), and (72), we have

\[ x_{u}^{(v)} y_{j}^{(k)} + y_{j}^{(k)} x_{u}^{(v)} = \sum_{m=1}^{t} \sum_{q=1}^{n} \gamma_{ju v} d_{m e u q} + d_{ju v} \]

and therefore the left-hand side of (78) is

\[ d_{ju v} \]

Thus (78) implies

(79) \[ d_{ju v} = \delta_{v} d_{ju v} \]

If \( v \neq j \) this says

\[ d_{ju v} = 0, \]

which implies

(80) \[ d_{ju v} = 0. \]

If \( v = j \) then (79) yields

\[ d_{ju v} (\sum_{i \neq j} e_{i i}) = 0, \]

which again implies (80). Thus (80) holds in any case. Recall that there was no restriction on the subscripts \( j, k, u, v \) appearing in (80). Our only assumption is that there be a \( v \) different from \( u \); that is, that \( n > 1 \). In view of (80), (77) becomes

\[ (\sum_{m=1}^{t} \gamma_{ju v} d_{m e u r}) x_{r}^{(s)} = 0, \]

which, by (70), implies

\[ \sum_{m=1}^{t} \gamma_{ju v} d_{m} = 0. \]
Since \( \{d_1, \ldots, d_t\} \) is linearly independent over \( \Phi \), 
\[
\text{(61)} \quad \gamma_{jkuv} = 0.
\]

In view of \( (60) \), \( (72) \) becomes 
\[
y_j^*(v) = 0;
\]
and by \( (76) \) and \( (81) \), \( (73) \) becomes 
\[
x_u^{(v)}y_j^{(k)} = 0.
\]

Thus \( A_{01}A_{10} = 0 = A_{10}A_{01} \). But then by \( (59) \), 
\[
AA_{10} = (A_{11} + A_{10} + A_{01})A_{10} = A_{11}A_{10} \subseteq A_{10}
\]
and 
\[
A_{10} = A_{10}(A_{11} + A_{10} + A_{01}) = 0.
\]

Thus \( A_{10} \) is an ideal of \( A \). We noted earlier (following the proof of Lemma 3) that \( A_{10} \neq 0 \). Since \( A_{10} \) clearly does not contain \( e \), \( A_{10} \neq A \). Thus the simplicity of \( A \) is contradicted. This contradiction arose from the assumption that \( n > 1 \). Therefore \( n = 1 \), and the proof of Lemma 6 is complete.

We now use our hypothesis that \( \Phi \) is algebraically closed to conclude that \( A_{11} \) is isomorphic to \( \Phi \). Since \( e \) is the identity element of \( A_{11} \) we have \( A_{11} = \Phi e \).

**Lemma 7.** \( A_{10} \) and \( A_{01} \) are each one-dimensional over \( \Phi \).

**Proof.** Let \( x^{(v)} \) be any non-zero element of \( A_{10} \). If \( x^{(s)} \) is another element of \( A_{10} \) such that \( \{x^{(v)}, x^{(s)}\} \) is linearly independent over \( \Phi \), then for any \( y^{(k)} \) in \( A_{01} \) we have
(82) $x^{(v)} y^{(k)} x^{(s)} + y^{(k)} x^{(s)} x^{(v)} = 0$,  
which is just (74) with the (now) superfluous subscripts omitted.

Since both $x^{(v)} y^{(k)}$ and $y^{(k)} x^{(s)}$ are elements of $\Phi e$, (82) asserts a dependence relation over $\Phi$ between $x^{(s)}$ and $x^{(v)}$, which were taken to be linearly independent. Therefore $x^{(v)} y^{(k)} = y^{(k)} x^{(s)} = 0$.

Interchanging $v$ and $s$ in (82) and applying the above argument again gives $x^{(s)} y^{(k)} = y^{(k)} x^{(v)} = 0$.

Now $x^{(v)}$ was an arbitrary element of $A_{10}$ and $y^{(k)}$ arbitrary in $A_{01}$ and we have shown that if $A_{10}$ has dimension greater than one over $\Phi$ then

$$x^{(v)} y^{(k)} y^{(k)} x^{(v)} = 0.$$ 

But this implies $A_{10} A_{01} = A_{01} A_{10} = 0$ which, as was shown in the proof of Lemma 6, cannot be the case. Therefore $A_{10}$ is one-dimensional over $\Phi$. Also $A_{01} = A_{10}^\#$ is one-dimensional over $\Phi$.

We now have $A_{11} = \Phi e$, $A_{10} = \Phi x$, and $A_{01} = \Phi y$. A change of notation in (82) yields

$$(xy + yx)x = 0.$$ 

Since $xy + yx$ is in $\Phi e$, we conclude that

$$xy + yx = 0.$$ 

Without loss of generality we may take

$$xy = -yx = e.$$ 

This completes the proof of Theorem 1.
3. ANOTHER STRUCTURE THEOREM

Suppose now that $A$ is a simple algebra such that $A_{00}$ is one-
dimensional over $\Phi$. If $A_{00} = \Phi z$ then $z^2 \in A_{00}$ by (17) so

$$z^2 = \alpha z,$$

and without loss of generality, we may take $\alpha = 0$ or 1.

We shall assume that $z^2 = z$ and make free use of this as well

as the fact, expressed in (8), that

$$zA_{11} = A_{11}z = 0.$$

Since $(1/2)Q(e,z,z,x_{10}) = 0$ we have

$$(83) \quad (e,z,x_{10}) + (z,e,x_{10}) + (x_{10},e,z)$$

$$+ (e,x_{10},z) + (z,z,x_{10}) = 0.$$

Since $zx_{10} \in A_{11},$

$$(e,z,x_{10}) = (z,e,x_{10}) = 0.$$

Also $(x_{10},e,z) = 0$ and since $x_{10}z \in A_{10} + A_{11},$

$$(e,x_{10},z) = 0.$$

Thus (83) reduces to

$$(e,z,x_{10}z) + (z,e,x_{10}z) + (z,z,x_{10}) = 0.$$

Now $x_{10}z \in A_{10} + A_{11}, z \cdot x_{10}z \in z(A_{10} + A_{11}) \subseteq A_{11},$ and

$z \cdot x_{10} \in zA_{11} = 0$. Therefore

$$- z \cdot x_{10}z - z \cdot x_{10}z + zx_{10} = 0$$

or

$$(84) \quad zx_{10} = 2z \cdot x_{10}z.$$

But
0 = (1/6)Q(z,z,z,x_{10}) - H(z,z,x_{10}) = (z,z,zx_{10} + x_{10}z) - (z,z,x_{10}),
and since zA_{10} \subseteq A_{11},
z\cdot zx_{10} = z(z\cdot x_{10}z) = 0.
Thus we have
z\cdot x_{10}^2 - zx_{10} = 0
or
zx_{10} = z\cdot x_{10}^2.
Together with (84) this implies
(85) \quad zx_{10} = 0.
Then also
(86) \quad x_{10}^2 \in A_{10}
by (11) and (12). Consideration of the anti-isomorphic algebra
gives
(87) \quad y_{01}z = 0
and
(88) \quad xy_{01} \in A_{01}.
Now f = e + z is an idempotent element of A and for every
basis element x_{10} of A_{10} we have
(f,f,x_{10}) = fx_{10} - f\cdot fx_{10} = fx_{10} - fx_{10} = 0;
(f,x_{10}^2) = fx_{10}^2 - f\cdot x_{10}^2 = x_{10}^2 - x_{10}^2 = 0
and, since H(f,f,x_{10}) = 0,
(x_{10},f,f) = 0.
Similarly, for every element y_{01} of A_{01}, we have
Since \( f \) acts as a unit element on \( A_{11} + A_{00} \) we see that
\[
(f, f, A) = (f, A, f) = (A, f, f) = 0.
\]
That is, \( A \) has a Peirce decomposition,
\[
A = A_{11}(f) + A_{10}(f) + A_{01}(f) + A_{00}(f),
\]
with respect to \( f \).

If \( x \in A \), write \( x = x \downarrow + x_{10} + x_{01} + x_{00} \) where \( x_{ij} \in A_{ij}(e) \).

Then \( fx = x \downarrow + x_{10} + x_{01} + x_{00} \) and \( xf = x \downarrow + x_{10} + x_{01} + x_{00} \).

Thus \( x \in A_{00}(f) \) if and only if \( x = 0 \). That is, \( A_{00}(f) = 0 \). Now \( e \) and \( z \) are non-zero elements of \( A_{11}(f) \) and \( ez = 0 \); so we see that \( A_{11}(f) \) cannot be a division algebra. It follows from the proof of Theorem 1 that \( f = e + z \) must be an identity element for \( A \).

This implies in particular that \( x_{10}^2 = x_{10} \) and \( z y_{01} = y_{01} \) for every \( x_{10}, y_{01} \) in \( A_{10}, A_{01} \) respectively.

Note also that \( A \) has a Peirce decomposition with respect to the idempotent \( z = f - e \). Observe moreover that in this decomposition
\[
A_{11}(z) = \Phi z
\]
is one-dimensional. This means that our investigation may be completed by determining all algebras of this type. This is attempted next. We consider simple algebras \( A \) for which \( A_{11} = \Phi e \) is one-dimensional. For such algebras we have

**Lemma 8.** The modules \( A_{ij} \) multiply as indicated in the table which follows.
PROOF. The only thing really new here is that $A_{00} A_{10} = A_{01} A_{00} = 0$, and $A_{10} A_{00} \leq A_{10}$. Use (11), (12), and the assumption that $A_{11} = \Phi e$ to write

$$a_{00} x_{10} = \alpha e$$

and

$$x_{10} a_{00} = \alpha e + u_{10}.$$

Then, by (18) and (13),

$$0 = H(x_{10}, x_{10}, a_{00}) = (x_{10}^2, a_{00}) + (a_{00} x_{10} + x_{10} a_{00}, x_{10})$$

$$= (2 \alpha e + u_{10}, x_{10}) = 2 \alpha (e, x_{10}) = 2 \alpha x_{10}.$$

It follows that $\alpha = 0$. Hence $A_{00} A_{10} = 0$ and $A_{10} A_{00} \leq A_{10}$. By consideration of $A^\#$ it follows that $A_{01} A_{00} = 0$, and $A_{00} A_{01} \leq A_{01}$.

Observe that if $A_{10} + A_{01} = 0$ then $\Phi e$ is an ideal of $A = \Phi e + A_{00}$, whence $A = \Phi e$ is a field with unity element $e$.

Next we consider an algebra $A$ for which $A_{10} = 0$ and $A_{01} \neq 0$ (algebras with $A_{10} \neq 0$ and $A_{01} = 0$ are anti-isomorphic to these).

An example of such an algebra is the algebra $G$ which follows.
Let $G$ be the algebra with basis $e, z, y_1, \ldots, y_s$ and with multiplication defined by $e^2 = e$, $z^2 = z$, $ez = ze = y_1z = ey_1 = 0$, $y_1e = y_1$ and $y_iy_j = y_jy_i = \alpha_{ij}(e + z)$ where the $\alpha_{ij}$ are scalars such that the symmetric matrix $(\alpha_{ij})$ is non-singular.

First we note that $e + z$ is an identity element for $G$. Write $e + z = 1$. We next show that $G$ is simple.

Let $I \neq G$ be an ideal of $G$ and suppose

$$a = \varepsilon e + \gamma z + \sum_{i=1}^{s} \eta_i y_i$$

is an element of $I$. Then $eae = \varepsilon e \in I$ and unless $\varepsilon = 0$, every $y_1 = y_1e$ is in $I$. But then every $y_iy_j = \alpha_{ij} \cdot 1$ is in $I$ and since not all the $\alpha_{ij}$ can be zero, $1 \in I$ whence $I = G$. Thus $\varepsilon = 0$.

Similarly $zaz = \gamma z \in I$ and unless $\gamma = 0$, every $y_i = zy_i$ is in $I$. This again implies that $I = G$, so we conclude that $\gamma = 0$. Thus if $I$ contains a non-zero element $a$, $a$ must have the form

$$a = \sum_{i=1}^{s} \eta_i y_i.$$

But then for each $j$, $ay_j = \sum_{i=1}^{s} \eta_i \alpha_{ij} \cdot 1 = 0$. But this says that

$$(\eta_1, \ldots, \eta_s)(\alpha_{ij}) = (0, \ldots, 0)$$

which, since $\det(\alpha_{ij}) \neq 0$ is possible only if each $\eta_i = 0$. Thus $a = 0$ and consequently $I = 0$. Therefore $G$ is simple.

Let $a' = \varepsilon'e + \gamma'z + \sum_{i=1}^{s} \eta_i' y_i$ with a similar notation for the elements $a'', a'''$ of $G$. Then we compute
Let $G$ be a quadratic algebra. Thus $G$ is power-associative.

Next observe that

$$(a', a'') = \sum_{i=1}^{s} (\eta_i' \eta_i'' - \eta_i' \eta_i e' - \eta_i'' \eta_i) y_i$$

and

$$(y_i, a''') = (\eta_i'' - \eta_i''' y_i.$$

Thus

$$((a', a''), a''') = \sum_{i=1}^{s} (\eta_i' \eta_i'' - \eta_i' \eta_i e' - \eta_i'' \eta_i e') (\eta_i''' - \eta_i''') y_i$$

from which one finds that

$$((a', a''), a''') + ((a'', a'''), a') + ((a''', a'), a'') = 0.$$

Thus $G$ is Lie-admissible.

**THEOREM 2.** Let $\mathbb{F}$ be a field of characteristic not 2 or 3 and let $A$ be a simple Lie-admissible power-associative algebra over $\mathbb{F}$ possessing an idempotent $e$ such that $(e, e, A) = (e, A, e) = (A, e, e) = 0$. If $A_{11} = \mathbb{F} e$, $A_{10} = 0$, and $A_{01} \neq 0$ in the resulting Peirce decomposition then $A$ is an algebra of the type described above.
PROOF. Note that $A_{00} \neq 0$ under our hypotheses for otherwise, as noted following Lemma 3, $A_{10} = 0$ implies that $e$ is a unity element for $A$. We shall show that $A_{00}$ is one-dimensional over $\Phi$.

Consider the subspace

$$I = \Phi e + A_{10} + (A_{01}^2)_{00}.$$  

By (8) and (22),

$$a_{00}(x_{01}y_{01})_{00} = a_{00}x_{01}y_{01}$$

$$= (1/2)((a_{00}x_{01}y_{01})_{00} + (a_{00}y_{01}x_{01})_{00})$$

$$= x_{01}y_{01}a_{00} = (x_{01}y_{01})_{00}a_{00},$$

which implies

$$A_{00}(A_{01}^2)_{00} = (A_{01}^2)_{00}A_{00} \subseteq (A_{01}^2)_{00}.$$  

Using the multiplicative properties stated in Lemma 8,

$$(\Phi e + A_{01})(A_{01}^2)_{00} = 0,$$

$$(A_{01}^2)_{00}(\Phi e + A_{01}) \subseteq A_{01} \subseteq I,$$

$$\Phi e \cdot A = \Phi e(\Phi e + A_{01} + A_{00}) = \Phi e \subseteq I,$$

$$A \cdot \Phi e = (\Phi e + A_{01} + A_{00})\Phi e = \Phi e + A_{01} \subseteq I,$$

$$A_{01}^2 = A_{01}(\Phi e + A_{01} + A_{00}) = A_{01} + A_{01}^2 \subseteq I,$$

and

$$AA_{01} = (\Phi e + A_{01} + A_{00})A_{01} \subseteq A_{01}^2 + A_{01} \subseteq I.$$

Thus $I$ is an ideal of $A$. Since $e \in I$, $I \neq 0$. Since $A$ is simple, $I = A$. Therefore

$$A_{00} = (A_{10}^2)_{00};$$

and by (18),

$$(a_{00}, a_{00}) = 0.$$
The module multiplication table for \( A \) has the following form:

<table>
<thead>
<tr>
<th></th>
<th>( \Phi e )</th>
<th>( A_{01} )</th>
<th>( A_{00} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Phi e )</td>
<td>( \Phi e )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( A_{01} )</td>
<td>( A_{01} )</td>
<td>( \Phi e + A_{00} )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( A_{00} )</td>
<td>( 0 )</td>
<td>( A_{01} )</td>
<td>( A_{00} )</td>
</tr>
</tbody>
</table>

from which we see that if \( A_{01}^2 \subseteq A_{00} \), then \( A_{01} + A_{00} \) is an ideal of \( A \). Since \( e \notin A_{01} + A_{00} \), we can have only \( A_{01} + A_{00} = 0 \). But then \( A = \Phi e \) is a field with unity element \( e \), contradicting our assumption that \( e \) is not a unity element. Therefore \( A_{01}^2 \notin A_{00} \).

Since \( a_{01}b_{01} = b_{01}a_{01} \),

\[
a_{01}b_{01} = (1/2)(a_{01} + b_{01})^2 - (1/2)a_{01}^2 - (1/2)b_{01}^2,
\]

and consequently every element of \( A_{01}^2 \) may be written in the form \( \sum \alpha_1 y_1^2 \) where \( y_1 \in A_{01} \). Since \( A_{01}^2 \notin A_{00} \) we may assume that \( A_{01} \) contains an element \( y_{01} \) such that

\[
y_{01}^2 = \alpha e + z_{00} \quad \alpha \neq 0.
\]

By associativity of cubes and (90),

\[
\alpha y_{01} = y_{01}y_{01} = y_{01}^3 = y_{01}y_{01} = z_{00}y_{01}.
\]

Therefore \( z_{00} \neq 0 \). Moreover, by (90), fourth-power-associativity, and the above,

\[
\alpha^2 e + (z_{00})^2 = y_{01}y_{01} = y_{01}y_{01} = (\alpha y_{01})y_{01} = \alpha^2 e + \alpha z_{00}.
\]
Hence \((z'_{00})^2 = \alpha z'_{00}\). Since \(\alpha \neq 0\), we are now able to write
\[(92) \quad y_{01}^2 = \alpha (e + z_{00})\]
where \(z_{00} = \alpha^{-1}z'_{00}\) and consequently
\[(93) \quad z_{00}^2 = (\alpha^{-1}z'_{00})^2 = \alpha^{-2}(z'_{00})^2 = \alpha^{-2}(\alpha z'_{00}) = \alpha^{-1}z_{00} = z_{00}^2\]
Also it follows from (91) that
\[(94) \quad z_{00}y_{01} = y_{01}\]
Our next step is to show that
\[A_{01}^2 = \Phi(e + z_{00})\]
where \(z_{00}\) is the idempotent element above.
Let \(a_{01}\) be an arbitrary element of \(A_{01}\). In view of our earlier remarks it will suffice to show that \(a_{01}^2 \in \Phi(e + z_{00})\). We have
\[y_{01}^2 = \alpha (e + z_{00})\]
Let
\[(95) \quad a_{01}^2 = \beta e + b_{00}\]
and
\[(96) \quad a_{01}y_{01} = y_{01}a_{01} = \xi e + c_{00}\]
Since
\[(1/6)Q(y_{01}, y_{01}, y_{01}, a_{01}) = 0,\]
we have
\[(97) \quad (a_{01}y_{01}^2 + (y_{01}a_{01}y_{01}) + (y_{01}a_{01}y_{01} + y_{01}a_{01}) = 0.\]
By (92), (96), and (90), we have
\[(a_{01}, y_{01}, y_{01}^2) = \alpha'(a_{01}, y_{01}, e + z_{00})
= \alpha'(a_{01}, y_{01}(e + z_{00}) - a_{01} y_{01})
= \alpha'((y e + c_{00}z_{00} - y e - c_{00})
= \alpha'(c_{00}z_{00} - c_{00});
\]

\[(y_{01}, a_{01}, y_{01}^2) = \alpha(y_{01}, a_{01}, e + z_{00})
= \alpha((y e + c_{00})(e + z_{00}) - y_{01} a_{01})
= \alpha((y e + c_{00}z_{00} - y e - c_{00})
= \alpha'(c_{00}z_{00} - c_{00});
\]

and

\[(y_{01}, y_{01}, a_{01} y_{01} + y_{01} a_{01})
= 2(y_{01}, y_{01}, y e + c_{00})
= 2(\alpha(e + z_{00})(y e + c_{00}) - y_{01} y y_{01})
= 2(\alpha y e + \alpha z_{00} c_{00} - \gamma d e - \gamma d z_{00})
= 2 \alpha(z_{00} c_{00} - \gamma z_{00}).
\]

Thus (97) becomes

\[2 \alpha(c_{00}z_{00} - c_{00}) + 2 \alpha(z_{00} c_{00} - \gamma z_{00}) = 0\]

which, by (89) and the fact that \(\alpha \neq 0\), implies

(98) \[c_{00}z_{00} = (1/2)(c_{00} + \gamma z_{00}).\]

Now for any \(s_{00}, t_{00}\) in the subalgebra \(A_{00}\) and any \(x_{01}\) in \(A_{01}\) we have, by (90),

\[(x_{01}, s_{00}, t_{00}) = (t_{00}, x_{01}, s_{00}) = 0.\]

Therefore \(0 = H(t_{00}, x_{01}, s_{00}) = (s_{00}, t_{00}, x_{01})\), whence

(99) \[(A_{00}, A_{00}, A_{01}) = 0.\]

By (99) and (94),
\[ c_{00}^2 y_{01} = c_{00}^2 y_{01} = c_{00}^2 y_{01}; \]

by (98) and (94),

\[ c_{00}^2 y_{01} = (1/2)(c_{00} + \delta z_{00}) y_{01} = (1/2)c_{00} y_{01} + (1/2)\gamma y_{01}. \]

Thus

(100) \[ c_{00} y_{01} = \gamma y_{01}. \]

Now by (92), (96), and (100),

\[
0 = H(y_{01}, y_{01}, a_{01}) = (y_{01}^2, a_{01}) + 2(y_{01} a_{01}, y_{01}) \\
= \alpha (r, a_{01}) + \alpha (z_{00} a_{01}, y_{01}) + 2 \gamma (e, y_{01}) + 2(c_{00}, y_{01}) \\
= - \alpha a_{01} + \alpha z_{00} a_{01} - 2 \gamma y_{01} + 2 \gamma y_{01} \\
= \alpha (z_{00} a_{01} - a_{01})
\]

and since \( \alpha \neq 0 \),

(101) \[ z_{00} a_{01} = a_{01}. \]

Recalling that \( a_{01} \) was an arbitrary element of \( A_{01} \) we have, by (90), (22), (101), and (96),

(102) \[ z_{00}^2 c_{00} = z_{00} a_{01} y_{01} \]

\[
= (1/2)((z_{00} a_{01} y_{01})_{00} + (z_{00} y_{01} a_{01})_{00}) \\
= (1/2)((a_{01} y_{01})_{00} + (y_{01} a_{01})_{00}) \\
= c_{00}^2.
\]

On the other hand, by (92), (90), (22), and (100),

(103) \[ z_{00}^2 c_{00} = \alpha^{-1}(y_{01}^2)_{00} c_{00} = \alpha^{-1}(y_{01}^2 c_{00}) \]

\[
= \alpha^{-1}(c_{00} y_{01} y_{01})_{00} \\
= \alpha^{-1} \gamma (y_{01}^2)_{00} \\
= \alpha^{-1} \gamma \alpha z_{00} = \gamma z_{00}.
\]

Comparing (102) and (103) we get
Next, by (95), (96), (104), and (101),
\[ 0 = \Pi(a_{01}, a_{01}, y_{01}) = (a_{01}^2 y_{01}) + 2(a_{01} y_{01} a_{01}) \
= \beta(e, y_{01}) + (b_{00} y_{01}) + 2 \gamma(e, a_{01}) + 2 \gamma(z_{00} a_{01}) \
= -\beta y_{01} + b_{00} y_{01} - 2 \gamma a_{01} + 2 \gamma a_{01} \
= b_{00} y_{01} - \beta y_{01}. \]

Thus
\[ b_{00} y_{01} = \beta y_{01}. \]

Finally, by (90), (95), (22), and (101),
\[ z_{00} b_{00} = z_{00}^2 a_{01} = (z_{00} a_{01} a_{01})_0 = (a_{01}^2)_0 = b_{00}; \]
but by (90), (92), (22), and (105),
\[ z_{00} b_{00} = \alpha^{-1}(y_{01})_{00} b_{00} = \alpha^{-1} y_{01} b_{00} \
= \alpha^{-1}(b_{00} y_{01} y_{01})_{00} = \alpha^{-1} \beta (y_{01}^2)_{00} \
= \alpha^{-1} \beta \alpha z_{00} = \beta z_{00}. \]

By comparing (106) and (107) we see that
\[ b_{00} = \beta z_{00}. \]

Thus we have shown that for every \( a_{01} \) in \( A_{01} \),
\[ a_{01}^2 = \beta(e + z_{00}). \]

Hence by our earlier comment,
\[ A_{01}^2 \subseteq \Phi(e + z_{00}). \]

Since we have already shown that \((A_{01}^2)_{00} = A_{00}\) we conclude that \( A_{00} = \Phi z_{00} \) is one-dimensional. Moreover our work shows that \( z_{00} \)
is an idempotent and for every \( y_{01} \) in \( A_{01} \),
\[ y_{01} z_{00} = 0 \]
and
\[ z_{00} y_{01} = y_{01}. \]

To complete the proof of Theorem 2 let \( \{ y_1, \ldots, y_s \} \) be a basis of \( A_{01} \) over \( \phi \). Suppose

\[ y_i y_j = \alpha_{ij}(e + z) \]

where \( z = z_{00} \). Since \( y_j y_i = y_i y_j \), the matrix \( (\alpha_{ij}) \) is symmetric.

Suppose \( (\alpha_{ij}) \) is singular. Then there is a non-zero vector

\[(\eta_1, \ldots, \eta_s)\]

in its null-space. But then the element

\[ a = \sum_{i=1}^{s} \eta_i y_i \]

has the property that \( ea = 0 \), \( ae = a \), \( za = a \), \( az = 0 \), and

\[ ay_j = \sum_{i=1}^{s} \eta_i \alpha_{ij} = 0 = y_j a. \]

Hence \( \phi a \) is an ideal of \( A \). This is impossible since \( A \) is simple and \( \phi a \not\subseteq A_{01} \neq A \). Therefore the matrix \( (\alpha_{ij}) \) is non-singular and \( A \) has precisely the properties of the example G.
4. EXAMPLES

In this chapter we present more examples of simple Lie-admissible power-associative algebras.

We remarked in Chapter 1 that not every simple Lie-admissible power-associative has an idempotent, not the identity, giving a Peirce decomposition. It will be convenient to look at one such class of algebras here.

Albert [1, 2] characterizes a reduced central simple Jordan algebra of degree two over a field $\Phi$ of characteristic not two as an algebra $J$ with basis $\{e, z, u_1, \ldots, u_n\}$, $n \geq 1$, over $\Phi$ where the basis elements multiply according to the table

<table>
<thead>
<tr>
<th></th>
<th>$e$</th>
<th>$\ldots u_j \ldots$</th>
<th>$z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e$</td>
<td>$e$</td>
<td>$u_j$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\ldots$</td>
<td>$\frac{u_1}{2}$</td>
<td>$\gamma_{ij}(e + z)$</td>
<td>$\frac{u_1}{2}$</td>
</tr>
<tr>
<td>$u_i$</td>
<td>$\ldots$</td>
<td>$\frac{u_i}{2}$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$z$</td>
<td>$0$</td>
<td>$\frac{u_j}{2}$</td>
<td>$z$</td>
</tr>
</tbody>
</table>

with the matrix $(\gamma_{ij})$ symmetric and non-singular.
The (for us) salient facts about such an algebra \( J \) are that \( J \) is commutative (hence Lie-admissible), has an identity element \( e + z \), is simple and quadratic. The first two properties are evident from the multiplication table. To see that \( J \) is quadratic, let

\[
x = \xi e + \sum_{j=1}^{n} \mu_j u_j + \gamma z
\]

be an arbitrary element of \( J \). Then

\[
x^2 = \xi ex + \sum_{j=1}^{n} \mu_j u_j x + \gamma zx
\]

\[
= \xi \left( \xi e + \sum_{i=1}^{n} (1/2) \mu_i u_i \right)
\]

\[
+ \sum_{i=1}^{n} \mu_i \left( \xi u_i + \sum_{j=1}^{n} \mu_j \xi_{ij} (e+z) + (1/2) \gamma u_i \right)
\]

\[
+ \gamma \left( \sum_{j=1}^{n} (1/2) \mu_j u_j + \gamma z \right)
\]

\[
= \xi^2 e + \sum_{j=1}^{n} \xi \mu_j u_j + \gamma z + \gamma \sum_{j=1}^{n} \mu_j u_j
\]

\[
+ \sum_{j=1}^{n} \mu_i \mu_j \xi_{ij} (e+z)
\]

\[
= (\xi + \gamma) x + \left[ \sum_{j=1}^{n} \mu_i \mu_j \xi_{ij} - \xi \gamma \right] (e+z).
\]

Since \( J \) is quadratic it is easy to see that \( J \) is power-associative. Indeed, for every \( x \), the subspace \( \Phi \cdot x + \Phi \cdot (e + z) \) is a subalgebra, is associative, and contains \( J_x \).

To prove that \( J \) is simple we first note that if an ideal of \( J \) contains either \( e \) or \( z \) then it contains every \( u_i = 2e u_i = 2z u_i \) and
therefore contains every $\gamma_{ij}(e + z)$. Since the matrix $(\gamma_{ij})$ is non-singular not every $\gamma_{ij}$ can be zero, and consequently any ideal containing $e$ or $z$ contains $e + z$, the identity element of $J$. Thus the only such ideal is $J$ itself.

Let

$$x = \xi e + \sum_{j=1}^{n} \mu_j u_j + \zeta z$$

be an element of an ideal $I$ of $J$. Then $I$ also contains

$$ex = \xi e + (1/2)\sum_{j=1}^{n} \mu_j u_j,$$

and

$$zx = (1/2)\sum_{j=1}^{n} \mu_j u_j + \zeta z.$$ 

Therefore $I$ contains

$$ex - zx = \xi e - \zeta z,$$

$$e(ex - zx) = \xi e,$$

and

$$z(ex - zx) = \zeta z.$$ 

Hence unless $\xi = 0$, $I$ contains $e$ and by the above, $I = J$.

Similarly unless $\zeta = 0$, $I = J$. Assuming that $I \neq J$ we find that $I$ can contain only elements of the form

$$x = \sum_{i=1}^{n} \mu_i u_i.$$ 

But then $I$ contains, for every $j$,

$$xu_j = \sum_{i=1}^{n} \mu_i \gamma_{ij}(e + z).$$
Since I \neq J, \ e + z \notin I \ and, for every j, we must have

$$\sum_{i=1}^{n} \mu_{i} \xi_{ij} = 0.$$ 

This says that the vector $$(\mu_{1}, \ldots, \mu_{n})$$ is in the null space of the matrix $$(\xi_{ij})$$. Since $$(\xi_{ij})$$ is non-singular $$(\mu_{1}, \ldots, \mu_{n}) = (0, \ldots, 0)$$. This says that \(x = 0\). Hence if I \neq J then I = 0.

Therefore \(J\) is simple.

\(J\) does not have a Peirce decomposition with respect to the idempotent \(e\) since

$$(e,e,u_{1}) = (1/4)u_{1} \neq 0.$$ 

Indeed if a commutative algebra \(A\) has a Peirce decomposition with respect to an idempotent \(e\) then

$$A_{10} = eA_{10} = A_{10}e = 0$$ 

and

$$A_{01} = A_{01}e = eA_{01} = 0.$$ 

Moreover, it is implicit in (16) that \(A_{11}\) and \(A_{00}\) are subalgebras when \(A\) is commutative. But then

$$A = A_{11} \oplus A_{00}$$ 

and, if \(A\) is simple, we can only have \(A = A_{11}\), in which case \(e\) is an identity element for \(A\).

Kosier [16] recently studied algebras which satisfy the so-called anti-flexible identity

$$(x,y,z) = (z,y,x).$$
Assuming power-associativity he proves that such an algebra is Lie-admissible and that every idempotent yields a Peirce decomposition. He proves that simple algebras of this type are associative if $A_{10} + A_{01} \neq 0$ for some idempotent $e$, and introduces examples of new simple algebras having $A_{10} + A_{01} = 0$. These are defined as the supplementary sum of $n$ orthogonal subspaces $A_i$, $i = 1, \ldots, n$ where $A_i$ has a basis $\{e_i, u_i, v_i\}$ such that $e_i$ is the unity of $A_i$, $u_i^2 = v_i^2 = 0$, and $u_i v_i = -v_i u_i = e_1 + \cdots + e_n$ the unity element of the algebra. Kosier also uses such an algebra to construct another having nil radical (maximal nil ideal) zero but which is not a direct sum of simple algebras. This example is Lie-admissible, power-associative, and has a Peirce decomposition. It has no unity element.

Any set $\{e_1, \ldots, e_n\}$ of mutually orthogonal idempotents in an algebra $A$ is linearly independent since if $\alpha_1 e_1 + \cdots + \alpha_n e_n = 0$ then $e_i(\alpha_1 e_1 + \cdots + \alpha_n e_n) = \alpha_i e_i = 0$ implies $\alpha_i = 0$ for $i = 1, \ldots, n$. A finite-dimensional algebra therefore has a maximum number of mutually orthogonal idempotents.

If an algebra $A$ has no identity element then it is said to be of degree zero. The three-dimensional algebra $B$ in Theorem 1 is of degree zero. If an algebra $A$ has an identity element $u$ then the degree of $A$ is the maximum number of mutually orthogonal idempotents whose sum is $u$. By the preceding paragraph we see that every
(finite-dimensional) algebra $A$ has a (finite) degree. The algebra $J$ is of degree two. Kosier's examples may have arbitrary positive degree.

We proceed now to present more examples of simple Lie-admissible power-associative algebras having Peirce decomposition. These appear to be new.

Let $\mathbb{F}$ be a field with characteristic different from two.

Let $A_1$ be the vector space over $\mathbb{F}$ with basis

$$\{ e_1, z_1, x_1, \cdots, x_n, y_1, \cdots, y_t \}$$

$s, t \geq 1$, and for $i = 2, \cdots, m$, let $A_i$ be the vector space over $\mathbb{F}$ with basis

$$\{ e_i, z_i, w_{1i}, \cdots, w_{ni} \},$$

$n_i \geq 1$. Define the algebra $A$ to be the orthogonal sum

$$A = A_1 + \cdots + A_m$$

of the (vector) subspaces $A_i$, where the multiplication in $A_1$ is defined by the table

<table>
<thead>
<tr>
<th></th>
<th>$e_1$</th>
<th>$\cdots$</th>
<th>$x_k$</th>
<th>$\cdots$</th>
<th>$y_k$</th>
<th>$\cdots$</th>
<th>$z_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_1$</td>
<td>$e_1$</td>
<td>$x_k$</td>
<td>$\cdots$</td>
<td>$y_k$</td>
<td>$\cdots$</td>
<td>$z_1$</td>
<td></td>
</tr>
<tr>
<td>$x_j$</td>
<td>$0$</td>
<td>$\lambda_{jk}(e_1+z_1)$</td>
<td>$\cdots$</td>
<td>$0$</td>
<td>$\cdots$</td>
<td>$x_j$</td>
<td></td>
</tr>
<tr>
<td>$y_j$</td>
<td>$y_j$</td>
<td>$\zeta_{kj}e_1+\delta_{kj}z_1-(1/2)\sum_{n=2}^{m}(e_{n}+z_{n})$</td>
<td>$\cdots$</td>
<td>$\mu_{jk}(e_1+z_1)$</td>
<td>$y_k$</td>
<td>$\cdots$</td>
<td>$z_1$</td>
</tr>
<tr>
<td>$z_1$</td>
<td>$0$</td>
<td>$0$</td>
<td>$\cdots$</td>
<td>$y_k$</td>
<td>$\cdots$</td>
<td>$z_1$</td>
<td></td>
</tr>
</tbody>
</table>
and multiplication in $A_i$ for $i > 1$ is defined by the table

<table>
<thead>
<tr>
<th></th>
<th>$e_i$</th>
<th>$\cdots$</th>
<th>$v_{ik}$</th>
<th>$\cdots$</th>
<th>$z_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_i$</td>
<td>$e_i$</td>
<td>$(1/2)(v_{ik} - \sum_{n \neq i} (e_n + z_n))$</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$v_{ij}$</td>
<td>$(1/2)(v_{ij} + \sum_{n \neq i} (e_n + z_n))$</td>
<td>$\zeta_{jk} (e_i + z_i)$</td>
<td>$(1/2)(v_{ij} - \sum_{n \neq i} (e_n + z_n))$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$z_i$</td>
<td>0</td>
<td>$(1/2)(v_{ik} + \sum_{n \neq i} (e_n + z_n))$</td>
<td>$z_i$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The Greek letters in the tables are scalars which satisfy the following conditions:

(I) For all $i$ and $j$,

\[
\lambda_{ij} = \lambda_{ji},
\mu_{ij} = \mu_{ji},
\alpha_{ij} + \tilde{\alpha}_{ij} = \beta_{ij} + \tilde{\beta}_{ij}.
\]

(II) If $(\lambda_{ij})$ and $(\mu_{ij})$ are zero matrices then $(\alpha_{ij})$ or $(\tilde{\alpha}_{ij})$ and $(\beta_{ij})$ or $(\tilde{\beta}_{ij})$ is not a zero matrix.

(III) The $s$ by $(s + 4t)$ composite matrix

\[
((\lambda_{ij}) (\alpha_{ij}) (\tilde{\alpha}_{ij}) (\beta_{ij}) (\tilde{\beta}_{ij}))
\]

has linearly independent (with respect to $\Phi$) rows.

(IV) The $(t + 4s)$ by $t$ composite matrix

\[
((\mu_{ij}) (\alpha_{ij})^T (\tilde{\alpha}_{ij})^T (\beta_{ij})^T (\tilde{\beta}_{ij})^T)^T,
\]

where $T$ denotes transpose, has linearly independent rows.
(v) For each \( k = 2, \cdots, m \), the matrix \( (\xi^{(k)}_{ij} + \eta^{(k)}_{ij}) \) is non-singular.

For our demonstration of the properties of \( A \) we shall take a simplified case; namely where \( \alpha_{ij} = \beta_{ij} = \gamma_{ij} = \delta_{ij} = 0 \) and 
\( \lambda_{ij} = \mu_{ij} = \xi^{(k)}_{ij} = \delta_{ij} \) the Kronecker delta. These assumptions are stronger than conditions (I) through (V) but are much easier to deal with in our calculations.

The algebra \( A \) has an attached algebra \( A^{(\cdot)} \) which is the same vector space as \( A \) but with a multiplication \((\cdot)\) defined by
\[
x \cdot y = \frac{1}{2} (xy + yx)
\]
where juxtaposition denotes the product in \( A \). Examining the multiplication tables for \( A \) we see that \( A^{(\cdot)} \) is the direct sum of subalgebras \( A_{i}^{(\cdot)} \), \( i = 1, \cdots, m \), each of which is a reduced central simple Jordan algebra of degree two. The identity element of \( A_{i}^{(\cdot)} \) is
\[
u_{i} = e_{i} + z_{i}
\]
and this is also an identity element for \( A_{i} \). The element
\[
u = u_{1} + \cdots + u_{m}
\]
is the identity element for \( A \) and for \( A^{(\cdot)} \).

Every element of \( A \) may be written uniquely in the form
\[a = a_{1} + \cdots + a_{m}\]
with \( a_{i} \) in \( A_{i} \). Noting that squares in \( A \) agree with squares in \( A^{(\cdot)} \) we have by orthogonality of the \( A_{i} \),
\[
a^{2} = a_{1}^{2} + \cdots + a_{m}^{2}
\]
\[= (\sigma(a_{1})a_{1} + \tau(a_{1})u_{1}) + \cdots + (\sigma(a_{m})a_{m} + \tau(a_{m})u_{m}).\]
Thus we see that the subspace
is a subalgebra, is associative, and contains \( A_a \). By definition, 
\( A \) is power-associative.

Observe that 
\[
u = x_1^2 + 2x_1y_1 \in A_1^2
\]
and, for \( i > 1 \)
\[
u = v_{11}^2 + z_{11}v_{11} - e_{11}v_{11} \in A_1^2.
\]
Thus \( u \in A_1^2 \) for \( i = 1, \ldots, m \).

If \( I \) is an ideal of \( A \) then \( I \) is also an ideal of the semi-simple Jordan algebra \( A^+ \) and must therefore be a sum of certain of the \( A_i \). But if \( A_i \leq I \) then \( A_i^2 \leq I \) and since \( u \in A_i^2 \) for \( i = 1, \ldots, m \), it follows that the only ideals of \( A \) are 0 and \( A \).

If \( i \neq j \) then by orthogonality
\[
(A_i, A_j) = 0.
\]
Moreover from the tables we see
\[
(A_i, A_i) \leq A_i + \sum_{n \neq i} \Phi^i u_n
\]
so that \( (A_i, A_i), A_j) = 0 \) and therefore
\[
((a', a''), a''') + ((a'', a'''), a') + ((a''', a'), a'') = 0
\]
holds in \( A \) if and only if it holds when the elements \( a', a'', a''' \)
are all in the same \( A_i \). For \( i > 1 \),
\[
(A_i, A_i) \leq \sum_{n \neq i} \Phi u_n
\]
so
and (108) holds. Thus we fix our attention on the identity (108) for elements of \( A_1 \).

Let

\[ a' = \epsilon' e_1 + y' z_1 + \sum_{i=1}^{s} \frac{1}{i!} x_i + \sum_{j=1}^{t} \eta_j y_j \]

with a similar notation for the elements \( a'' \) and \( a''' \) of \( A_1 \). Then

\[
(a', a'') = \frac{1}{2} \sum_{i=1}^{s} \sum_{j=1}^{t} (\frac{1}{i!} \eta_j'' - \eta_j' \frac{1}{i!}) (u_2 + \cdots + u_n)
\]

\[ + \sum_{i=1}^{s} [(\epsilon' - y') \frac{1}{i!} - \frac{1}{i!} (\epsilon'' - y'')] x_i \]

\[ + \sum_{j=1}^{t} [(y' - \epsilon') \eta_j'' - \eta_j (y'' - \epsilon'')] y_j, \]

and therefore

\[
((a', a''), a''') = \mu(a', a'', a''') \sum_{n=2}^{m} u_n + \sum_{i=1}^{s} \frac{1}{i!} (a', a'', a''') x_i
\]

\[ + \sum_{j=1}^{t} \eta_j (a', a'', a''') y_j \]

where

\[
\mu(a', a'', a''') = \frac{1}{2} \sum_{i=1}^{s} \sum_{j=1}^{t} \left\{ [(\epsilon' - y') \frac{1}{i!} - \frac{1}{i!} (\epsilon'' - y'')] \eta_j''
\]

\[ - [(y' - \epsilon') \eta_j'' - \eta_j (y'' - \epsilon'')] \frac{1}{i!} \right\} ;
\]

\[ \frac{1}{i!} (a', a'', a''') = - \left[ (\epsilon' - y') \frac{1}{i!} - \frac{1}{i!} (\epsilon'' - y'') \right] (\epsilon''' - y'''); \]

and

\[
\eta_j (a', a'', a''') = - \left[ (y' - \epsilon') \eta_j'' - \eta_j (y'' - \epsilon'') \right] (y''' - \epsilon''').
\]

A short calculation reveals that

\[ \Pi (a', a'', a''') + \Pi (a'', a''', a') + \Pi (a''', a', a'') = 0 \]
when \( \tau \) is any one of \( \mu \), \( f_1 \), or \( \eta_j \). This says that (108) holds in \( A_1 \). By our earlier remarks this suffices to show that \( A \) is Lie-admissible.

We have demonstrated that \( A \) is simple power-associative and Lie-admissible. It is evident that \( A \) has Peirce decomposition with respect to the idempotent \( e_1 \) and in fact with respect to every idempotent of the form

\[
e = e_1 + u_1 + \cdots + u_n
\]

where \( \{i_1, \ldots, i_n\} \) is a subset of \( \{2, \ldots, m\} \). Indeed in the decomposition with respect to \( e \) we have

\[
\begin{align*}
A_{10}(e) &= \sum_{j=1}^{s} \Phi \cdot x_j, \\
A_{01}(e) &= \sum_{j=1}^{t} \Phi \cdot y_j, \\
A_{11}(e) &= \Phi e_1 + \sum_{j=1}^{n} A_{1j},
\end{align*}
\]

and \( A_{00}(e) \) is the sum of \( \Phi z_1 \) and those subspaces \( A_i \) whose indices run over the complement of the set \( \{i_1, \ldots, i_n\} \) in \( \{2, \ldots, m\} \).

There is an interesting special case imbedded in the preceding. The algebra

\[
C = A_2 + \cdots + A_m, \quad m \geq 3,
\]

as vector space, but with the multiplication table redefined for each \( A_i \) so as to delete \( e_1 + z_1 \) as an entry, is easily seen from
the proofs given for $A$ to have the properties of simplicity, power-associativity, and Lie-admissibility. In fact $C$ has the interesting property

\[ ((c,c),c) = 0. \]

Moreover relative to every idempotent $e = \sum_{j} u_{i_{j}}$ $C$ has a Peirce decomposition

\[ C = C_{11} + C_{00} \]

where $C_{11} = \sum_{j} A_{i_{j}}$.

The algebra $G$ arising in Theorem 2 is another special case of the algebras presented here; namely the case $m = 1$. When $m = 1$ one may take either $s = 0$ or $t = 0$ (but not both) and still have a simple algebra.
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