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THE DEVELOPMENT OF THE
PHILOSOPHY OF LOGIC
FROM 1880 TO 1906

DISSERTATION
Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Gwendolyn Duell Bowne, B. A., M. A.

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INTRODUCTION

The development of mathematical logic in the late nineteenth and early twentieth centuries involved both the construction of new symbolic "languages" of logic, and the presentation of various interpretations of the philosophical importance of these constructs. In this study I have avoided the technical development of mathematical logic, which can reasonably be regarded as a chapter in the history of mathematics rather than of philosophy. The metaphysical and epistemological views which were associated with the mathematical techniques are a complex and fascinating topic in themselves; and this study is devoted primarily to tracing out the arguments which were carried on among the "mathematical philosophers" between 1885 and 1906. In this interval the original "logistic" philosophy was proposed, criticized, and finally abandoned by its leading proponents, Louis Couturat in France and Bertrand Russell in England.

Most of the interest in mathematical philosophy in this period appears to have been in France. The discussions on which this study is based are primarily taken from the Revue de Métaphysique et des Morales and from other French journals of the period. Translations given are my own, but where the passages involved are lengthy or complex, the original is included.
Although some discussion is included where it seems necessary to clarify the issues, I have concentrated upon following the course of the argument as it took place and presenting the views which were actually advanced. There are, of course, value judgments involved in the selection of important arguments and of central figures. I have decided to use the contemporary estimates as a guide rather than those of later periods. "Unconscious influences" are not philosophically interesting to any one who distinguishes philosophy from psychology; and a philosopher's actual statement of his reasons for holding a view and of the arguments which have influenced his thought deserves to be taken seriously.

From this point of view, the central figures in the early development of the philosophy of logic were Louis Couturat, Pierre Boutroux, and Henri Poincaré in France; and Bertrand Russell and G. E. Moore in England. The other major party to the argument was Immanuel Kant, who had died in 1804 but still exercised a powerful influence upon philosophy.
CHAPTER I

BRADLEY'S LOGIC

In 1883 the Absolute Idealist philosophy dominated philosophical thought, as reflected in the professional journals of the time. As a leading representative of that viewpoint, F. H. Bradley had a characteristic approach not only to problems of ethics and metaphysics, but also to logic. His *The Principles of Logic*, published in 1883, was extremely influential among philosophers, although there were even at this time several other schools of logicians who refused to accept the Hegelian ideal of logic. The old tradition of formal logic still claimed the allegiance of some students; and the inductive logic of J. S. Mill was a new contender in the field. There was also some very new work in the field of formal logic being carried on, in Bradley's opinion, as a variety of mathematics. As such, he believed, it was incapable of acquiring any philosophical significance.

Most of the questions which were at issue between the idealist logic and a suddenly revitalised formal logic during the subsequent twenty-five years are, I believe, still open.

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Indeed, they seem to have once again aroused the interest of philosophical logicians. In the earlier debates on these questions, Bertrand Russell took a particularly active part. His writings are also of special interest, since he began as a partisan of the idealistic logic and became for a time the leading defender of the new formal logic. Russell's reasons for such a reactionary procedure are to be found in the debates of this transition period between the dominance of the idealistic logic and the dominance of mathematical logic. Bradley's reasons for his own views appear in his Logic.

Although Bradley expressed his views in a vocabulary which is alien to most English-speaking readers of 1963, the views he held are apparently as modern now as then.¹ His rejection of the old tradition of formal logic was on the basis of its philosophical inadequacy,² a defect which he thought also appeared in such newer formal logics as the "Equational Logic" of Professor Jevons³ which he regarded as the best example of that mathematical approach.

Bradley believed that the business of logic was to give an account of "reasoning in general," which no formal logic,

¹Consider P. F. Strawson's writings on Logic.

²Bradley also dismissed J. S. Mill's inductive logic for similar reasons.

³W. R. Jevons, a contemporary of George Boole. Bradley refers to his Principles of Science in a number of discussions in the Logic.
whether syllogistic or equational could ever do. Formal logic, Bradley argued, had the fatal defect of being unable to present an accurate picture of actual reasoning processes. Equational logic was, he believed, limited by its mathematical nature to dealing with "those problems which accommodate themselves to numerical reasoning,"¹ and could not include other obviously valid inferences which did not fit the subject-attribute pattern.² Bradley's examples of recalcitrant inferences were such relational arguments as: A is to the right of B, and B is to the right of C, so A is to the right of C.

The inadequacy of equational logic could, Bradley said, only be even more striking in mathematical logic. Therefore, the mathematicians' variants on the work of Jevons could safely be dismissed by philosophical logicians without study. At least, this was the course Bradley proposed for himself, since he lacked, he said, the mathematical training which would have made it possible for him to follow the work of the mathematicians. He stated that, given the proper training, he would have found it "a pleasure to have seen how the defects of the Equational Theory appeared in mathematical form." In a passage that hints at philosophical claims made by the mathematical logicians, he continued: "If I knew perhaps what mathematics were, I would see how there is nothing special

¹Bradley, Bk. II, Pt. II, chap. iv, par. 1, p. 343.

or limited about them, and how they are the soul of logic in general and (for all I know) of metaphysics too. But such a logic, Bradley stated, would not provide "any account (adequate or inadequate) of reasoning in general" and therefore could not properly claim to be logic at all.

Bradley was convinced that whatever account of reasoning in general a logic might manage to give could not, at any rate, be a set of formal rules of inference. He argued that formal logicians, whether they were "friends of the syllogism" or preferred the equational logic, emphasized the importance of perceived identities in reasoning. They reasoned on the basis of universal connections between attributes such that "given one in a subject, you must have the other also." Bradley agreed that this was one extremely important type of reasoning. But, he argued, not all inferences were of this subject-attribute type. Formal logic had once claimed, he said, using the past tense as is appropriate in speaking of the dead, to provide us with "not merely principles of reasoning, but actual canons and tests of inference. Within the pale we were secure of salvation, and on the outside it was heresy to doubt that you were lost." Therefore Bradley argued, the existence of valid relational inference is a fatal blow.

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1Bradley, Bk. II, Pt. II, chap. iv, par. 23, p. 360. 
to the formal logicians. Rather than canons, Bradley suggested logic can hope to provide only principles of inference which "give us under each head of inference the general and abstract form of the operation. . . It is not merely that the terms are left blank, for the special relations of the premises and conclusion are also left blank."¹

As a further argument against a search for canons and tests of valid inference, Bradley argued that it would first be necessary "to make a list of valid inferences."² I can find no explicit reason for this statement but it follows from Bradley's belief that every actual inference involves a unique situation and that the validity of the inference may depend upon some of its specific characteristics rather than on characteristics which could be shared with others. However, his demand for a list of all valid inferences might be based upon an argument from continuity: every argument of Form A differs from every argument of Form B. But every argument of Form A, Type 1, differs from every argument of Form A, Type 2. If we need separate canons for A and B, why not separate canons for A₁ and A₂? But arguments of

¹Bradley, Bk. II, Pt. I, chap. iv, par. 5, 245.
²Bradley, ibid., par. 6, 246.
Type \(A_1\) differ from those of \(Type \ A_2\), and so on to the limit of classification, which is the unique individual. Such a line of reasoning seems plausible as a reconstruction of Bradley's thought by analogy with other arguments Bradley used, but he did not present it.

The completed list of all valid inferences would be, Bradley pointed out, the completion of an infinite process, and was therefore intrinsically unobtainable. Thus, no canon of inference could ever be constructed, and reasoning could never be formal, in the sense that all valid inferences could be exhibited as instances of certain types of reasoning and all logical conclusions could be "anticipated in theory and reduced to formulas."

Inference could not, in Bradley's opinion, be formal even in the sense that some formal principles of valid reasoning could be given. All logical principles must, he said, contain a material element. "For without a difference, such as that between the letters A and B, or again between the A in two several positions, you cannot state or think these principles, and the nature of these differences is clearly material." 

Nevertheless, he believed, in the sense that there may be in every argument an "active" and a "passive" element, a relevant and an irrelevant portion, reasoning can be regarded

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1Bradley, Bk. III, Pt. II, chap. 1, par. 2, p. 469.
Each argument contains an "active principle of inference" which can be extracted and stated as a general axiom. But the actual principle of inference is "a function, not a datum."\(^1\) It is a way of acting, not a content of the mind, and must not be confused with a major premise.

Logic has its first principles; but they were, in Bradley's opinion, "postulates, assumptions whose truth we cannot here scrutinize, but on which our intellects are forced to embark, if they mean to serve us in the voyage of life."\(^2\) Examples of such postulates are: that the content of a logical conclusion is not dependent upon my drawing the conclusion; and that some processes of arranging elements do not change the character of the elements arranged. But these necessary postulates of reasoning ultimately are not true, and all discursive thinking which "must work with ideas or universal contents of thought,"\(^3\) falsifies Reality.

It appears that in Bradley's opinion, formal logic and mathematical logic, as well as mathematics itself, dealt with problems of a lower order of generality than those which were the proper study of Logic as a philosophical discipline.

\(^1\)Bradley, Bk. III, Pt. II, chap. i, par. 8, p. 474.

\(^2\)Bradley, Bk. III, Pt. II, chap. iii, par. 2, p. 499. Bradley thought that examination of these postulates was the business of metaphysics.

\(^3\)Bradley, Bk. III, Pt. II, chap. iv, par. 8, p. 526.
A belief that study of mathematics might give an insight into problems of metaphysics and epistemology seems to have been held only by mathematicians and by a very small group of French philosophers who were also interested in mathematics.

However, there were active formal logicians. In the contemporary philosophical journals, such logicians as Hugh MacColl and Mrs. Ladd Franklin were debating fine points of logical procedure. For example, Lewis Carroll's Paradox of Carr, Brown, and Allen produced a particularly lively discussion. There were also other kinds of "logic" suggested as a supplement to or substitute for formal logic. Some German writers were proposing that an accurate picture of human reasoning processes, which they apparently thought of as logic, required a "logic of the emotions." Pragmatist views of logic were also discussed in the English and American philosophical journals. Logic was usually distinguished from the Algebra of Logic; and the only sign of a connection between mathematics and philosophy in the thought of professional philosophers at the time is a discussion in the Revue de Métaphysique et des Morales, where Delboeuf, Renouvier, Calinon, and Poincaré were carrying on a debate about the philosophical implications of non-Euclidean geometry in a journal intended for an audience which included professional philosophers.

In The Monist, C. S. Peirce and Ernst Schröder were at this time working on a logic of relations. However, Peirce
and Schröder apparently were working as mathematicians and aroused little response either among the formal logicians or the philosophers. The Monist published several articles by Peirce and Schröder over the period from 1896-1898. These articles not only suggested modifications of the algebra of logic but also hinted at philosophical implications of such a logic. Both Peirce and Schröder thought that their new logical researches had metaphysical implications; but both were reticent about spelling them out.

In his article "The Regenerated Logic," Peirce suggested some of his own metaphysical views. They seem not at all revolutionary for the period, except for his suggestion that mathematics must be regarded as the foundation of all philosophy. Mathematics, he said, is certain because it does not deal with facts at all, but only with objects which are "the mind's own creation." Logic, said Peirce, rests upon mathematics, and deals with facts "implied in the supposition of an unlimited applicability of language." Metaphysics in turn should rest upon logic. Logic attempts to find in assertions those "essential elements" which are independent of linguistic structure, and to find out what conditions assertions must meet in order that they may correspond to reality. It is also, said Peirce, the business of logic to consider how one question leads to another and how questions arise. He

did not make it quite clear whether or not he regarded this part of the business of logic as a psychological inquiry.

Peirce in this article made the distinction between "icons" and "indices" which corresponds in his "logic of relatives" to the distinction between predicates and subjects: "icons" as images of characteristics, and "indices" as non-descriptive denoting words indicating an object simply as an unique "hecedity" which is "in relation to the assertion" a subject. Assertion Peirce thought of as a connection of subject and predicate; the sign of the connection was the copula.

Actual discussion of relations occurs in the article, "The Logic of Relatives."¹ Peirce there suggested that relationship statements could be thought of either as involving many subjects or one collective subject.² He also suggested that by the study of relationship statements it should be possible to generalize formal logic and to develop all formal logic from the logic of relations. Peirce objected to Cantor's set theory, and offered a proof that there are more possible combinations of objects than there are objects themselves.

The greatest response to the logical writings of Peirce came from Schröder, who credited his logic of relatives to Peirce. Schröder published in The Monist a logician's appraisal of modern developments in logic, with particular

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²An ordered couple would be such a collective subject.
emphasis on the most widely-known contemporary movement—Peano's pasigraphy.\(^1\) "Pasigraphy," "universal language" or "writing," was Peano's own term for his work. Schröder quoted Peano as saying that he was trying to realize the ideal language suggested by Leibniz, but I do not know just how seriously this statement of Peano's purposes should be taken. In Leibniz's view, the Universal Characteristic was to be adequate to express all human reasoning. Peano's efforts were concentrated upon the speech of mathematics, which he attempted to translate into his own symbolic language. In Peano's new language, precisely-defined concepts were to be manipulated according to set rules of calculation. Schröder's statement that he thought of his efforts as directed toward the Universal Characteristic suggests that Peano had actually set himself a larger problem than that of translating all notions and reasoning procedures of mathematics by means of primitive logical notions and purely logical operations.

However, it had already proved unexpectedly difficult to accomplish the translation even of mathematics. Peano had at first, according to Schröder's article, thought his work successfully

\(^1\)Ernst Schröder, "On Pasigraphy: Its Present State and the Pasigraphic Movement in Italy," \textit{The Monist}, IX (October, 1898).
completed in his *Formulaire des Mathématiques*. He had later, said Schröder, found that he had not succeeded. However, Schröder commented, the failure should be blamed only on an inadequate symbolism. He stated: "As an individual opinion of mine, perhaps not as yet shared by many . . . I consider pure Mathematics to be only one branch of general logic, the branch originating from the creation on Number. . . . This view is confirmed by the fact, that under the pasigraphic aspect Arithmetic can do without any peculiar categories or primitive notions (such as multitude, number, finiteness, lines, function, Abbildung or one-to-one correspondence, addition, etc.)"\(^1\)

As is usual in these early discussions, Peirce in his articles asserted that logic is a branch of mathematics and Schröder asserted that mathematics is a branch of logic without any apparent feeling that their views were not identical. The thesis that mathematics and logic are segments of the same deductive system seems less controversial throughout these earlier years than any statement of their relative priority.\(^2\)

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1Schröder, *The Monist*, IX, 47.

2It has been suggested by some historians of logic that recognition of this fact is an unfriendly concentration upon an unimportant detail. Considering that one of the main points of disagreement between followers of Russell, Hilbert, and Brouwer a few years later is precisely this point of priority. I feel that it cannot properly be classified among the nit-pickers' problems, at least from the historical point-of-view.
In his discussion of the work of the modern logicians, Schröder mentioned Frege, but not favorably. He regarded Frege as one "who heedless of anything accomplished in the same direction by others, took immense pains to perform what had already been much better done and was therefore superseded from the outset, thus delivering a still-born child."

The child was, of course, later resuscitated by Russell, who objected on metaphysical grounds to the logic of relations advanced by Peirce and Schröder. However, it is clear that the influence of Frege on the development of modern logic was not felt during the early years. Probably his influence on logic was first felt by way of Russell's interest in his work.

In contrast to Frege, Schröder apparently had some contemporary audience. By way of his effect on Schröder, it appears that Peirce also received some favorable attention among those who were willing to take formal logic seriously. However, this group apparently did not include many professional philosophers, or even many mathematicians. The orthodox logic of the late nineteenth century was Bradley's logic. Historically, two particularly interesting exponents of that orthodoxy are Bertrand Russell and Alfred North Whitehead.
CHAPTER II

BERTRAND RUSSELL: FOUNDATIONS OF GEOMETRY

As a student, Bertrand Russell\(^1\) studied both philosophy and mathematics. His first book\(^2\) was a revised version of the fellowship dissertation accepted by Trinity College, Cambridge, in 1895. In this book, he defended the Kantian philosophy of mathematics, which he described as the view "that certain knowledge, independent of experience, was possible about the real world" and is exemplified in geometry, against the attacks of the "metageometers." These were geometers such as Riemann and Helmholtz, who argued that the consistency of non-Euclidean geometry was a refutation of Kant's philosophy of mathematics.

Russell thought the issue was between Kantian "idealism" and "empiricism." The empiricists' reply to the idealist

\(^1\)Russell was born in 1872. His "Hegelian period," which is described in this section, was brief and soon repudiated.

was "the apparently paradoxical assertion that Geometry at bottom, had no certainty of a different kind from that of Mechanics — only the perpetual presence of spatial impressions, they said, made our experience of the truth of the axioms so wide as to seem absolute certainty."¹

The Kantian thesis involves, said Russell, two independent assertions about space. To clarify the discussion, Russell proposed to separate them as the assertions: first, that space is a\textit{ priori}; and, second, that it is\textit{ subjective}. He proposed the definitions: "a priori applies to any piece of knowledge which, though perhaps elicited by experience, is logically presupposed in experience; subjective applies to any mental state whose immediate cause lies, not in the external world, but within the limits of the subject. The latter definition, of course, is framed exclusively for Psychology; from the point of view of physical Science all mental states are subjective."² Subjectivity and "apriority" must, then, be separated; and "the test of a priority will be purely logical."³ Would experience be impossible, if a certain axiom or postulate were denied?

Russell believed that he could establish within geometry

¹Russell, Sect. 1.
²Russell, Sect. 2.
³Russell, Sect. 5. The \textit{logic} appealed to here is apparently Transcendental Logic.
a distinction between "the necessary and the purely assertorical" among its axioms and postulates. This distinction would be established by finding either the fundamental logical postulate of the science or the essential nature of the subject-matter of the science. These would, Russell thought, in the end be found to coincide and to be the one ground of the necessity of those axioms found to have that character.

The logical consistency of non-Euclidean geometries was, Russell argued, irrelevant to Kant's thesis that Euclidean geometry is necessarily true: "for Kant is careful to argue that geometrical reasoning, by virtue of our intuition of space, is synthetic, and cannot, though a priori, be upheld by the principle of contradiction alone." Non-Euclidean geometers must prove, before they could deny the certainty that Kant had attributed to Euclidean geometry, "that we can frame an intuition of non-Euclidean spaces," or they must use philosophical arguments to "disprove the first two arguments of the Aesthetik or attack the synthetic a priori and its connection with subjectivity."

Russell believed that philosophical attacks both on the arguments of the Aesthetik and on the subjectivity of the synthetic a priori could be pushed with some success. The distinction between synthetic and analytic judgments could

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1Russell, *Geometry*, Sections 7, 8.

2Ibid., Section 54.
not stand; the principle of contradiction could prove nothing except on the assumption of the possibility of experience; and Kant's arguments established the necessity, not of Euclidean space but merely of "some form of externality — which may be sensational or intuitional, but not merely conceptual."

The distinction between synthetic and analytic judgments, Russell pointed out, was "completely rejected by most modern logicians." His examples were Bradley and Bosanquet. For the arguments proving the untenability of the distinction, Russell referred the reader to the *Logic* of Bradley, or to Bosanquet. However, Russell also supplied a brief summary of the argument and of the logical principles which he accepted as incompatible with that distinction.

"Logic, at the present day, arrogates to itself at once a wider and a narrower sphere than Kant allowed to it. Wider, because it believes itself capable of condemning any false principle as postulate; narrower, because it believes that its law of contradiction, without a given whole or a given hypothesis, is powerless, and that two terms, *per se*, though they may be different, cannot be contradictories, but acquire this relation only by combination in a whole about which something is known, or by connection with a postulate which,

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1Bernard Bosanquet. I do not have the date of the first edition of his logic, but there is a review of *The Essentials of Logic* in *Mind*, 1895.
for some reason, must be preserved. Thus no judgment, per se, is either analytic or synthetic, for the severance of a judgment from its context robs it of its vitality, and makes it not truly a judgment at all. But in its proper context it is neither purely synthetic nor purely analytic; for while it is the further determination of a given whole, and thus in so far analytic, it also involves the emergence of new relations within this whole, and is so far synthetic.¹

Russell held that the concept of a mathematical point was self-contradictory, although it is difficult to understand what "self-contradiction" could have meant to him within the logical framework just presented. Therefore, he believed, the treatment of geometry as a study of properties of a manifold, or set of points, while technically useful, was philosophically misleading. At this time, Russell suggested that Space should be thought of as a hypostatized abstraction from "a peculiar and abstract kind of matter, which is not regarded as possessing any causal qualities, as exerting or as subject to the action of forces." He took this position because, he said, the idea of space involved both "the contradictions inherent in the notion of the continuum, and the contradictions which spring from the fact that space, while it must, to be knowable, be pure relativity, must also, it would seem, since it is immediately experienced, be something

¹Russell, Sect. 56, p. 59.
more than mere relations. Actual simple elements of matter, Russell thought, would not be contradictory since they, unlike points, could be unextended without contradiction.

In his chapter on the philosophical consequences of the logical possibility of constructing consistent non-Euclidean geometries, Russell returned to a more extended interpretation of Kant's Aesthetik. From this discussion it is possible to obtain a portrait of "Russell's Kant," who, although not accepted as an accurate historical portrait by all the antagonists, provides a background against which the points of specific disagreement become more clear.

It is, said Russell, the Kantian contention that experience of objects as different but related requires, as a given element in sense-perception, space as a "form of externality." This form must be regarded as "given" in the sense that it is neither a "conception" nor an "inference" but is "discoverable, through analysis, by attention to the object of sense-perception." It is Kant's position, also, that space must not be part of the actual "data of sense" but an element added to the data of sense, by "a subjective intuition." More accurately, said Russell, Space must be regarded as an intuited element to which data of sense are added, and as being both psychologically and logically prior.

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1Russell, Sect. 199.

2Russell, 180.
to objects in space. Kant's view that space is non-sensational was, Russell believed, irrelevant to his logic, being a part of his rather dubious psychological theories. The logical or epistemological issue was, Russell said, whether any "form of externality, which renders experience of diversity in relation possible, can be merely conceptual."

In considering Kant's arguments, Russell thought it necessary to distinguish between the externality of sensations to the self and the mutual externality of sensed objects. He thought that the first kind of externality was to be dismissed as derivative, although "essential in the subsequent distinction of Self and not-Self."¹

As for the second type of externality, Russell stated that the logic of Bradley and Bosanquet had proved "that all knowledge involves a recognition of diversity in relation, or, if we prefer it, of identity in difference." From this premise, he argued, it follows at once that the object of any knowledge must be complex and "not a mere particular." That is, what is known must be recognized as an "identity in difference," a unification of many qualities into one individual. It is, therefore, an object which is, in Bradley's terminology, a universal -- a concrete universal, of course, not an abstract one. But if the "object of a cognition" must be complex, that is, must contain recognizably different

¹Russell, Sect. 189.
characteristics, at least some of the "objects of perception" must also be complex.

Russell argued for this conclusion as follows: "Knowledge must start from perception."¹ But, then, either we know only our present perceptions or we can relate our present perceptions to other perceptions by contrast and comparison. The present perception, if a "mere particular," could not, as proved above, be known. Knowledge gained by comparison and contrast of perceptions requires a perception other than the present perception with which we can contrast or compare it. But, argued Russell, the other perception, to be other, must have occurred at a different time. Knowledge gained by comparison and contrast of perceptions therefore involves a form of externality relating and separating perceptions themselves. It also, said Russell, requires the abandonment of the hypothesis that the present perception can be a "mere particular." "For the power of comparing it with another perception involves a point of identity between the two, and thus renders both complex." Even without this argument, Russell stated, it follows from the nature of time as continuous that our present perception involves "the internal complexity involved in duration throughout the specious present: its mere particularity and its simplicity

¹The premise is unquantified, but is clearly intended to apply to "all knowledge."
are lost." Also, any perception refers to its own past and future, hence, "its self-subsistence is also lost."

This argument, continued Russell, proves only the necessity of time for the possibility of knowledge. But internal complexity of the momentary experience, which actually exists, requires "the possibility of a diversity of simultaneously existing things," at least qualities. An additional form of externality must be given in experience before we could think of causation and change.

Briefly, the argument is: "All knowledge is obtained by inference from the This of sense-perception." But inference depends upon identity in difference. The "This of sense-perception" must therefore be a complex, having adjectives which refer "beyond itself." But this, as was shown above, can happen only by means of a form of externality. Therefore, argued Russell, the a priori axioms of geometry are established "as necessarily having existential import and validity in any intelligible world." The axioms of geometry which Russell considered a priori were not, however, the axioms of Euclid, but those "common to Euclidean and non-Euclidean" metrical geometry, and all the axioms of projective geometry.

The only flaw Russell saw in the Kantian arguments was

1Russell, Sect. 190.
2Russell, Sect. 139.
a failure to recognise that the form of externality which had been proved to be necessary for experience was not necessarily space, insofar as space is a particular kind of form of externality. The peculiar characteristics of space are, said Russell, "a mere experienced fact." However, "to suppose two things simultaneously in the same position" in our actual space and time is a logical contradiction, since "we infer real diversity, i.e. the existence of different things, only from position in space and time." We have not, Russell continued, "constructed the data of sense out of logic." Instead, "logic is dependent, as regards its application, on the nature of those data." Why the nature of the data happens to be suited to logical analysis is, said Russell, a metaphysical question which he will avoid in this argument.

Russell had admitted that space was a contradictory notion, and concluded that it must be regarded as an hypos­tatized abstraction. He argued that geometry must therefore be thought of as dealing with "spatial figures and matter," in the abstract sense discussed above. This abstract matter must be actually differentiated into units or "atoms" which are absolutely simple, "containing" no relations, but standing in spatial relations with other atoms. Such simple elements appear "to Geometry as a point," a self-contra­dictory unextended bit of extension. Actually, Russell said, they are unextended but not bits of space, and contradiction is avoided.
Thus, Kant's infinite given whole must, Russell thought, be dismissed as "a psychological illusion." The mathematical antinomies make it impossible to regard it as either a subjective intuition or an independent reality. Space must, then, be resolved into "felt relations" of spatial order which unavoidably seem to be "more than mere relations." Empty space, Russell suggested, must be thought of as "a mere name for the logical possibility of spatial relations." This is true because space "contains only one aspect of a relation, namely the aspect of diversity; but spatial order, by its reference to matter becomes more concrete, and contains also the element of unity, arising out of the connection of different material atoms. Spatial order, then, consists of relations in the ordinary sense; its merely spatial element, however . . . is only one aspect of a relation, but an aspect which, in the concrete, must be inseparably bound up with the other aspect."

As appears in the summary above, Russell gave, in this first book, a clear statement of the philosophical views he was soon to reject. He also gave a statement of some of his own views which shows how well his own metaphysical position would fit in with the Platonism of G. E. Moore, and with the mathematical treatment of the continuum which he still rejected as philosophically unsound. During this early period in his thought, Russell had been primarily interested in geometry. He accepted the prevailing philosophical opinions,
as is reasonable in a young man in his early twenties. But, being mathematically sophisticated, he had devoted himself to proving the essential irrelevance to metaphysics and epistemology of modern mathematical developments, while Bradley and other mathematically innocent philosophers simply took that irrelevance for granted. He was, however, already aware of recent developments in set theory. In 1896, he wrote: "Cantor's second class of numbers, by which he hopes to exhaust continua, begins with the first number larger than any of the first class; but as the first class (the ordinary natural numbers) has no upper limit, it is hard to see how the second class is ever to begin. Cantor's attempts, indeed, seem to have proved, more conclusively than ever, that no legitimate extension of number can suffice for the adequate treatment of continua."¹

Apparently, Russell regarded the absence of an adequate mathematical treatment of continua as a refutation of one Kantian position — that space is an infinite given whole. Louis Couturat in France at this time regarded the existence of an adequate mathematical theory of infinite numbers as a refutation of the entire Critical Philosophy. There was a disagreement on a point of fact — the adequacy of Cantor's

theory -- as a basis for differing evaluations of the Kantian philosophy of mathematics. Since Russell and Couturat soon became allies in promoting "the logistic movement," it is interesting to observe the details of their original disagreement. Throughout, however, they seem to have been agreed upon one point: if Cantor is right, Kant is wrong.
CHAPTER III

COUTURAT'S DE L'INFINI MATHÉMATIQUE: METAPHYSICS

Louis Couturat,¹ an outspoken advocate of "scientific philosophy," published his first book in 1896. Like Russell's Foundations of Geometry, the book was Couturat's dissertation; and, like Russell's, Couturat's book reflected a concern with mathematics as well as with philosophy. But Couturat's interests were in Analysis and Set Theory rather than in Geometry, and his studies led him to adopt a very unusual position. In De l'Infini Mathématique,² he gave a favorable discussion of Cantor's work up to that time, and insisted that the idea of an infinite number was clear, distinct, and free from absurdity.

Throughout his brief philosophical career, Louis Couturat was preoccupied with the refutation of Kant. However, his enemy was "irrationalism" as such -- any view

¹1868-1914. Couturat taught for two years at the University of Caen, giving a course in modern mathematical logic in 1898. He was an active contributor to the Revue de Métaphysique et des Morales until his death, and was also active in organizing the first two International Congresses of Philosophy.

which accepted the possibility that reality might be self-contradictory or that reason might not be able to think about it without running into self-contradiction. Couturat apparently accepted the entire Kantian psychology; but he hoped to dispose of Kantian irrationalism by mathematical analysis and reinstate metaphysics as a legitimate philosophical pursuit. Among his Kantian enemies he explicitly included Russell and, by implication, Bradley; for Russell in his *Foundations of Geometry* accepted and appealed to Bradley’s *Logic*.

*Foundations of Geometry* appeared to Couturat to be: "l'Esthétique transcendentale de Kant, revuë, corrigée, et complétée à la lumière de la Métageométrie," and to be thoroughly admirable, even though mistaken. Russell had proved, Couturat thought, that non-Euclidean geometry only strengthened the Kantian view that in addition to the rational a priori there is an intuitive and sensible a priori which is the foundation of the synthetic judgments of pure mathematics. However, he thought Russell’s introduction of an abstract matter must be an effect of "a residue of realism and empiricism, incompatible with the otherwise logical and coherent views of H. Russell." Also, he was dissatisfied with Russell’s treatment of the antinomies of space. As appears in *De l'Infini Mathématique*, it was in the new theory of infinite numbers that

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Couturat thought he had found a refutation of Kant and the Critical Philosophy; but Russell still rejected that theory.

In his Preface and Introduction to *Infini*, Couturat stated that the object of philosophy is to justify scientific method. However, not only scientific laws but also the laws of logic are subject to review by metaphysics. Both can be considered as true only "so far as they conform to rational principles," the "a priori forms of reason." From several equally logical chains of concepts, philosophy must choose the most rational — that is, the one which introduces the most unity, insight, and harmony into our ideas while connecting them with the "primordial and simple ideas."

In any such system of logically connected prepositions, Couturat stated, the "most natural and the most philosophical order" is one which bases the entire system of concepts on "real principles," that is, on principles which are really "self-evident and irreducible." As for formal logic, Couturat did not yet assign it a high philosophical status, in spite of his interest in mathematics. For him, logic and "the Art of Reasoning" were still distinct; in fact, they were the business of separate Faculties.

Couturat explained his use of the word "Reason" as referring to the thinking self "which organizes the infinitely variegated chaos of phenomenal consciousness." From this conscious datum, Reason extracts the elements which it then systematizes according to its own laws. The resulting system
is "objective reality." Operations of Reason are known, Couturat held, only by their products. The principles guiding the operations can be felt but not seen. Therefore, examination of psychological facts can never reveal to us those rules "which unite our ideas, guide our judgments and our reasonings, and make us distinguish truth from falsity." Empiricists, who expect "to discover the hidden laws of reason by observing the undulating and moving surface of consciousness resemble those who attempt to explain the movement of a ship by the eddies and the choppy water it raises, without recognition of the rudder which directs it and the propeller which pushes it ahead." The true method of understanding the operations of the human mind is neither introspective nor physiological psychology, but careful analysis of its constructs, particularly science.

Criticism of thought, Couturat believed, must rest upon a Theory of Knowledge, which would distinguish between the degrees of knowledge and between several "faculties of knowledge" having differing degrees of authority. Reason in particular must be carefully distinguished from "the understanding, which abstracts and generalizes, judges and reasons about concepts, and which is the properly logical and analytic faculty, while the reason is the faculty of pure ideas and of a priori synthetic principles."

But Theory of Knowledge, Couturat said, naturally involves a metaphysics. If reason is the supreme judge of truth, it is
also the supreme judge of reality. Reality cannot be unintelligible: "because the real for us is what we think is true, and the true is that which reason understands and affirms as existing. Unknowable noumena arise from realist prejudices (sustained also by moral requirements the value of which we need not estimate) which consider things as existing outside and independent of the spirit." From a purely speculative point of view, Couturat held, such things are unjustifiable. We do not know that our sensations require transcendent causes, as Kant thought they did, for we do not have any reason to identify sensation and intuition with "receptivity." For a logical idealism, sensations are caused by the whole organized system of sensations. Reality is immanent in the mind, and is a construct of Reason.

Couturat held that the only possible proof that a lawful system of thought constructed by Reason is not real would be a proof that that system is self-contradictory.¹ "That is why the Antinomy of Pure Reason is really the citadel of transcendental idealism."

Therefore, Couturat thought, Metaphysics could be rehabilitated if we could prove that reason never contradicts itself and that a priori synthetic principles depend upon a pure intellectual intuition. Of course, Couturat pointed out, with

¹Couturat maintained throughout the subsequent discussions that the burden of proof is on the side of the accuser, and that a system is to be regarded as consistent until proved inconsistent.
the abandonment of the thing-in-itself, metaphysics becomes essentially philosophy of science. "Who is to tell us that the a priori synthetic judgements of metaphysics are different from those constituting pure mathematics or pure physics?"

Accordingly, Critique of Science should lead us through Theory of Knowledge to a philosophy of nature and a philosophy of mind.¹

Couturat saw mathematics as "la science rationelle et a priori par excellence." For even though mathematical science included some experimentally-obtained postulates, analysis rested solely on the idea of number and could be constructed entirely a priori. Only the application of numbers to magnitudes involved a passage beyond the a priori.

Couturat argued that both number and magnitude were a priori concepts. In its efforts to express all magnitudes in terms of number, mathematics had been forced to generalize the concept of number continually until the most recent generalization included the concept of infinite number. Infinite numbers are as necessary to mathematics as irrational numbers; and are philosophically unobjectionable.

I shall postpone discussion of Couturat's justification of the concept of infinity to Chapter IV of this dissertation. The philosophical consequences which he believed to follow

¹The summary above is based on Couturat's "Introduction" and "Preface" to De l'Infini mathématique.
from Cantor's work will be considered as separate from the technical problem of establishing the theory itself.

Couturat argued that there are a priori synthetic axioms at the base of the sciences both of number and of magnitude. But the ideas themselves, as well as the axioms concerning them, "are independent of all sensuous form." This view, if true, he felt to require that "one of the fundamental theses of the Kantian critique be renounced; that is, that there is no kind of intuition other than sensuous intuition, and that the judgments of mathematics could not be based on anything other than the a priori forms of sensibility."

The word "idea" as he used it in his book was, said Couturat, to be taken "in a Platonist sense, and signifies a universal, abstract type which sensible things imitate or in which they participate more or less imperfectly; only it must be held that if we find these ideas in concrete objects and in their relations, it is because reason introduces these relations into things, imposes a priori forms and fashions them, so to speak, in the image of its pure ideas. (Couturat's footnote: This means that it is necessary to correct the objectivist realism of Plato by the transcendental idealism of Kant.)"

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1Couturat, De l'Infini, Pt. II, Bk. II, chap. 1, sect. 18.
2Couturat, De l'Infini, Pt. II, Bk. III, chap.11, sect. 15.
The Kantian theory that number rests upon intuitions of time or space, Couturat held, rests on the view that we can think only by forming imaginative images representing the concept. If that were true, it would be necessary to form an image representing number. Since everyone agrees that we cannot form an infinite image, the theory of infinite numbers is of great importance in deciding the question whether the idea of number rests on intuition.

These preceding passages are a key to Couturat's interpretation of the Kantian philosophy which explain his subsequent criticism. He attributes to Kant the view that we cannot think without imagery (apparently equating Denken and Erkennen). Couturat's arguments rest heavily upon the Schematism; and it appears that when Couturat attacked Kant he was attacking the view described above—that all thought is imagery. This view he characterized as an appeal to "intuition." However, he regarded intellectual intuition as necessary, and possible. Also, he subscribed to a thoroughgoing idealism which did not differentiate between the constructs of Reason and an independent reality.

Although Couturat thought that the idea of number was a priori, he also regarded it as a creature rather of the Understanding than of Reason. Whole numbers represent, he stated, "the extension of a genus or species of which a
concept represents the comprehension.\textsuperscript{1} Together these two aspects, number and the concept, give "the two faces or moments of the complex operation called "abstraction and generalization." For instance, we can only count what are regarded as examples of some one concept.\textsuperscript{2} Both numbers and concepts are formed by the understanding by ignoring actual individual differences between things and reading into them rational characteristics which they do not possess. Therefore, both numbers and concepts are "inadequate to reality." That is, the process of counting a collection requires that it be regarded as at once One and Many. Its elements must be at once similar and different. But concrete objects could not have such characteristics without contradiction. Therefore, number can never be "objectively given, but always subjectively constructed by the understanding in virtue of its own laws." In another passage Couturat asserted: "It would be absurd to attribute any objective value to this artificial procedure of the imagination operating on the sensuously given; and it is as manifest an error to realize number as to realize concepts." He pointed out that while everyone in modern days was convinced of the absurdity of Plato's system, it was necessary to remind ourselves that

\textsuperscript{1}Couturat, \textit{De l'Infini}, Pt.II, Bk.IV, chap.i, sect.4.

\textsuperscript{2}For example, two apples, five men, or ten things.
the system of Pythagoras was equally absurd.\(^1\)

Couturat pointed out, also, that although his dis-
cussion seemed to distinguish between nature and the mind,
he wished to assert not that there is a world external to
the mind which must somehow be grasped by the mind, but
rather that there is a distinction to be made between the
ways of knowing. Mind must be equated with understanding,
the faculty of forming abstract and general concepts. But
we know these concepts to be inadequate to reality because
we have another way of apprehending reality —by the faculty
of reason. From Reason we obtain a priori ideas which could
never arise from experience, such as infinity and continuity.
Furthermore, such ideas could never arise from a Kantian pure
(sensuous) intuition.\(^2\)

The reasons to be advanced against the Kantians were
not, Couturat believed, absolutely conclusive. But they were
highly probable. For example, continuity belongs not only to
space and time but to all magnitudes both intensive and exten-
sive, even those which cannot be directly intuited, like
magnetic forces. Really, the question is not whether space
and time are continuous, but why we are forced to think of
movements in space and time as continuous. The axiom of con-
tinuity, which is the basis of this way of thinking, must,

\(^1\)Couturat, De l'Infini, Pt.II, Bk. IV, chap. i, sect. 10.
\(^2\)Couturat, De l'Infini, Pt.II, Bk. IV, chap. iii, sect.2.
Couturat stated, simply be one of the rules of reason. This axiom has no proof, and is independent of the Law of Contradiction. It must simply be seen. Knowledge of it, and of any such axiom of reason is "neither discursive nor deductive, it is appropriate to call it, in opposition to sensuous intuition (pure or empirical) a rational intuition." 

All such ideas of reason were, Couturat believed, "precise and rigorous" in their application to concrete reality. The real world really appears to us as infinite and continuous. Only the efforts of the understanding to reduce these ideas to concepts and numbers lead to the belief that infinity and continuity are irrational notions. The world of sensation is a continuous mass; it is divided and intellectualized by the understanding and then resolved again into a continuous mass by reason. But the end product is not, Couturat asserted, a simple repetition of the world of sensation; it is a world "made transparent to reason." The natural world must conform to reason's laws because reason constructs it. But the supposed necessity that the world be finite is based upon sensation and imagination rather than upon reason.

Kant's Antinomies can be resolved, Couturat argued, by recognizing the distinction between reason and imagination.

1 Apparently Couturat means by "the axiom of continuity" not the salitus non datur, but Dedekind's Postulate. At any rate, he equates them in several statements.

2 Couturat, De l'Infini, Pt. II, Bk. IV, Chap. iii, sect. 15.
"The indefinite is not the image of the infinite, because it is always finite; it is only the parody of it. Infinity is a rational idea of which no adequate image can be found and which cannot be "constructed" in sensuous intuition. The fact that this idea exists and is neither empty nor contradictory, refutes simultaneously the aristotelian axiom "No idea without an image," and the kantian axiom, "No concepts without intuition." 1

The Antinomies fall, according to Couturat, because the actual infinite is not self-contradictory. The theses involving the finitude of the world in space and time may be rejected, the antitheses accepted, and no conflict remains. The theses were proved by Kant only by reduction of the antitheses to absurdity. Our new mathematical knowledge makes the reduction no longer valid. Kant's argument involved, Couturat held, an insistence that the possibility of a realized infinite series of concrete objects cannot even be thought without contradiction. The weakness Couturat saw in his argument lay in the movement from thought of a completed infinite series to the completion of an infinite series of thoughts. If the thought of the series as a whole had to be constructed by a successive synthesis, said Couturat, an infinite series would indeed be inconceivable; but Reason does not operate in that way. A magnitude is

1Couturat, De l'Infini, Pt. II, Bk. IV, chap. iv, sects. 9, 13.
given as a whole first; then it may be inadequately expressed in the understanding by numbering its parts.

Couturat believed that the construction of Cantor's theory of infinite numbers had finally freed reason of Kant's charge of self-contradiction. Thus, reason was free to indulge in metaphysics again with a good conscience. But the details of Couturat's defense of Cantor are also, for that reason, important to his argument; and it is necessary to consider *De l'Infini Mathématique*’s justification of infinite number.
CHAPTER IV

DE L'INFINI MATHEMATIQUE: INFINITE NUMBERS

Couturat argued in his Preface to De l'Infini that all scientific developments were legitimate, and were merely to be justified by philosophy. The theory of infinite numbers was an accomplished fact in mathematics. He proceeded to justify it, in two different ways: first, by showing that it was necessary for the purposes of mathematics; and, second, by showing that it did not involve self-contradiction.

Infinite numbers are, Couturat argued, required by mathematicians in their efforts to express in numbers all magnitudes. Magnitude, however, must be accepted as a primitive and indefinable concept, an Idea of Reason. Magnitudes can be subjected to number at all only by means of measurement, that is, the selection of an arbitrary unit and the statement of how many units of that type exhaust the given magnitude. This process cannot, Couturat argued, be essential to the meaning of magnitude.

Any science of magnitude involves, said Couturat, such a priori synthetic premises as the fundamental axiom of equality: two magnitudes of the same kind are either equal or unequal. This axiom involves another rational idea,
equality, upon which all the seemingly arbitrary mathematical definitions of equality rest. Also, there are other axioms of equality and inequality which cannot be "reduced to the type of the analytic judgment, which is: A is A," and which are both synthetic and a priori.

Besides the axioms of equality, there are, Couturat held, many other axioms which are like them "indemonstrable, and nevertheless possess an evidence almost equal to that of analytic judgments. Among these axioms is the "Axiom of Continuity," which is expressed in Dedekind's Postulate.1 Because of this axiom, mathematicians have been required by reason to admit irrational numbers to express numerically incommensurable magnitudes. But, Couturat continued, this does not mean that number is to be defined in terms of measurement, or that the ratio between two magnitudes can be defined in terms of numbers. It is absurd to attempt to define everything, and nonsense to insist that every magnitude "is composed of indivisible elements, and that every magnitude contains a finite and determinate number of these elements."2

In the discussion summarized above, Couturat appealed both to a priori concepts and to a priori synthetic axioms. But, to satisfy the Kantians, he stated, some answer must be given to the question how a synthetic judgment can be

1Couturat, De l'Infini, Pt. II, Bk. II, chap. iii, sect. 8.
both a priori and not based on intuition. If infinity is to
be defended, the answer must be, said Couturat, something other
than the Kantian appeal to the imagination. The correct an-
swer is, Couturat held, recognition of intellectual intuition.

But demonstration that the concepts of, for example, num-
ber and magnitude are a priori required further argument. Af-
ter all, if "whole number" is indefinable, Couturat must ex-
plain and refute the mathematicians' attempts to define it.
He proceeded by examining two different types of attempted
definition of "whole number," which he characterized as the
"rationalist" and the "empiricist" theories. The issue be-
tween them he regarded as "the old quarrel between realism
and nominalism."

The empiricists, in Couturat's terminology, were those
who, like Dedekind, Helmholtz, Kronecker, and Gauss, regard
the concept of number as "entirely independant of representa-
tions or intuitions of space and time" -- "free creations
of the human spirit." Couturat's quotation is credited to
R. Dedekind's Was Sind und Was Sollen die Zahlen? Their
emphasis was, Couturat said, entirely on the detachment of
numbers from any sensible experience, and he recognized his
own paradoxical use of terms in labeling as empiricist a
view which seems "radically a priorist."

However, said Couturat, justification for the label is
found on a closer examination of the view. The a priori
element appealed to is the sequence of ordinal numbers,
regarded as meaningless symbols arranged in a conventional order, and as having only two characteristics — a shape and a position in an order of succession. Cardinal numbers are obtained by correlating real objects with ordinal numbers, and, thus, arise from experience. The a priori element is thought of as completely empty and devoid of content, so that "application of arithmetical symbols to the physical world can be made only by experimental synthesis of the concrete and the abstract, which reintroduces into the analytic systems a matter given in perception and expressed by postulates; thus all the axioms first posited as arbitrary conventions or rational principles reappear as experimental, and therefore contingent, truths."\(^1\)

But there is, said Couturat, a vicious circle in this attempt to derive the idea of cardinal number from an experimental synthesis rather than from a synthesis a priori. The operation of counting on which the attempt rests may be defined as setting up a one-to-one correspondence between the objects of the collection and the series of ordinals; but each number and each object must first be recognized to be one thing. Therefore, the idea of unity must be a priori. Furthermore, in order that the result of application to a set of objects of a series of ordinals should be a cardinal

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\(^1\)Couturat, *De l'Infini*, Pt. II, Bk. II, chap. 1, sect. 5.
number, we must regard the series of ordinals up to a certain number as containing the idea of the cardinal number as well. For if the idea is not already present, argued Couturat, the act of counting could never create it: "for how could the idea of that number originate from the juxtaposition of two series which, separately, have no cardinal number?" But, if the series of numbers itself has a cardinal, so does the set of objects, and one should be able to extract the idea from the concrete set without using the ordinals at all. Either way, "one extracts from the empirical act of counting only what one has put into it in advance; and if one thinks to have made a cardinal number appear from it, it is because it has previously been slipped into it." In short, said Couturat, this theory recognizes that people count things; "but it cannot explain that fact, because it ignores its logical significance and rational import."1

Opposed to this empiricist theory was the "rationalist" view, which defined number in an apparently empiricist manner as "a collection of unities," an idea extracted from experience. Stolz was an example of this approach, which Couturat also rejected. An attempt to define a number as "a plurality of unities" committed its advocates, he argued, to the view that "plurality" and "number" had the same meaning, and that every collection is a number. Thus, two equivalent classes actually were the same number, which is absurd.

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1Couturat, De l'Infini, 331.
However, said Couturat, Cantor defined the cardinal number as the common concept of equivalent classes; and this he believed to be the correct view. The idea of cardinal number does not arise from the contemplation of the collections, since the idea of the numerical equivalence of two collections is logically posterior to the idea of a particular cardinal number. Unity is not a sensible quality among sensible qualities, and can never be extracted from sensation; thus, the idea of unity and with it of any number as a unity must be "a pure form of reason which the mind imposes a priori on its objects solely in virtue of thinking them." The problem of defining number must, then, be thought of as an attempt to give an explicit formulation to an implicit idea — "to find a logical formula for this rational idea." The problem is, Couturat said, a question "not of formal logic but of reason," and must be resolved by a special method of analysis which is not the deductive method of science.

Fundamentally, Couturat concluded, mathematics cannot define number; it can define only the conditions of equality among numbers. "All the definitions of mathematics are purely nominal and consequently always presuppose the concept which they appear to construct. That is why the logical concepts elaborated by the scholar necessarily involve rational ideas which by their very nature elude all scientific

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1Couturat, De l'Infini, 355.
definition. This explains that opposition between the logical order and the rational order which is found everywhere in the principles of Mathematics.¹

The Kantian theory that number rests on an intuition of time or space, Couturat argued, depends upon the theory that we can think only by forming imaginative images representing the concept. If that were true, it must be possible to form an image representing number. Obviously, the theory of infinite numbers is relevant to this question of whether or not the idea of number rests upon an appeal to the imagination, and thus, on Couturat's interpretation, to the truth of the Kantian philosophy of mathematics. That is, Kantians could not admit that there is an idea of an infinite number. No empiricist, Couturat argued, could admit that idea, for all the ordinals which can actually be reached by counting through the natural numbers are finite. Since every cardinal number is, on that theory, derived from an ordinal, there can be no infinite cardinal numbers, either. Since a denial of infinite numbers would, Couturat felt, cripple mathematics, he regarded this as a strong objection to empiricism. As a way out for the empiricist, he suggested that the admission of a second series of transfinite ordinals following all the natural numbers would be consistent with their views, for since all numbers are for them only conventions, such a

¹Couturat, De l'Infini, Pt. II, Bk. I, chap. iii, sect. 10.
convention could be adopted as well as any other. For the rationalists, who need not think of each number as obtained from its predecessor, there should be nothing shocking in the idea that there are numbers which could never be reached by successive additions of one.

But there were many other objections which had been advanced against the concept of infinite number, and since Couturat's object was to "justify the same idea a priori and by purely philosophical reasons," showing "that nothing prohibits conceiving of infinite number," he presented a debate between an advocate and an opponent of the theory, in the form of a lengthy dialogue. In the course of the dialogue, many objections were brought up and countered. The objection that "abstract infinite numbers" do not have all the characteristics of other whole numbers Couturat dismissed as a rather silly suggestion that the infinite must be finite. The Finitist advanced Galileo's argument: there must be a square of each natural number; therefore, if the series of natural numbers is endless, the series of square numbers will have as many members as the entire series. But the series of square numbers contains only some of the natural numbers, so this consequence is absurd, and the series of natural numbers cannot be infinite. But Couturat's Infinitist countered by pointing out that the argument really proves that the series of natural numbers must be infinite. For each square number is itself a natural number, and hence has its own square, ad infinitum.
To avoid other threatened contradictions, Couturat pointed out a distinction between the plurality of elements in an infinite collection, its power, and that plurality arranged in an infinite linear series, that is, well-ordered -- its number. The number must be supposed, Couturat stated, to vary with the order of the elements. Therefore, in order that an infinite set have a number at all, it must be specified in its definition what order its members are to be arranged in. The Axiom of Whole and Part simply does not apply to powers, said Couturat.

Finally, Couturat argued, the fact that Cantor's theory of trans-finite numbers existed and contained no inconsistency was itself a proof that abstract infinite number was a possible concept. As for concrete infinite numbers, or, rather, infinite collections of objects, to suppose that such exist is simply, said Couturat, to suppose a series of objects corresponding to the natural numbers. But the set of natural numbers itself is infinite, and composes an infinite collection.

The assertion that "it is evident" that all numbers must be obtainable by successive addition of one has no force, Couturat stated, against anyone who is not an empiricist -- that is, against anyone who believes that a collection has a number even before it has been counted. We might not know that it was infinite until the counting was done; but, said
Couturat, "it is simply a question of time. Give me an infinite time, and I will undertake to enumerate an infinite collection."\(^1\)

The unity of an infinite series is, Couturat argued, to be found in its proceeding from one law of formation. Thus, the series can be regarded as "engendered by one single act of the mind" and its terms are connected by an "intelligible connection." Consequently, he said, the law confers upon the series a rational unity, by collecting the terms under one idea and including them in one definition. "It doesn't matter whether the terms of the series are finite or infinite in number, as long as we have a way to construct them all; and the way to do it is furnished in a formula for a single operation (indefinitely repeated addition of one) from which appear one by one all the whole numbers, and into which, inversely, they can all be made to return."

Finitist: I think you are exaggerating; the formula will allow you to construct all the whole numbers you want, but always a finite number. You will never get all the whole numbers. The series of natural numbers, just because it is endless and therefore always incomplete, makes up an indefinite plurality but not a closed collection, a totality.

Infinitist: You seem to think that a closed collection is necessarily finite; that is a mistake. One can conceive of infinite sets as closed and as exactly bounded as finite sets . . . . In general, any collection of objects is closed and bounded

\(^1\)Couturat, *De l'Infini*, Pt. II, Bk. III, chap. 11, sect. 4.
when we know precisely which objects belong to it and which objects do not belong to it; and when we can distinguish the objects belonging to it from one another and recognize their unity and their identity. It is under those conditions that a set is well-defined and has a determinate number. But the series of natural numbers is a well-defined set. It is a collection of numbers, all different, in which each one of them is a distinct unity. It doesn't even lack a cardinal number, the infinite number.

Sect. 9. Finitist: But, once again, you can't count all the whole numbers, nor assemble them all together in that synthetic act of apprehension which, according to you, is necessary for the formation of a number; you can never include the set of these numbers in one intuition so that you can perceive the totality.

Infinitist: You are mistaken. To be able to consider the series of natural numbers as a whole, and to apprehend it in its totality, it is enough to know for one thing, what a whole number is, and on the other hand, to know how to distinguish any whole number from one that is not whole. But these are things that you would be ashamed not to know, and any uncertainty or hesitation about them is impossible. That being the case, you have as I do the idea of the totality of whole numbers, that is, of a well-defined set, of a closed collection having a definite number. Besides, didn't we both talk, all along, of all the whole numbers and didn't we know exactly what we meant by it? And if I were to undertake to enumerate them in order, wouldn't you be able to tell if I repeat or forget one? You see perfectly well that you know them all.¹

In the remaining two sections of this important chapter, the distinction between the potential and the actual infinite entered into discussion. The law of formation gives, the Finitist argued, a possibility of always finding new numbers, but it does not give the numbers themselves, "nor do they

¹Couturat, De l'Infini, Pt.II, Bk.III, chap.11, sect. 8-9.
exist ready-made in your thoughts." To this, the Infinitist replied that the objection is valid only against an empiricist view of number, "where the whole numbers are given one by one and in succession, like cards or counters. In the rationalist theory, on the contrary, all the numbers are given at once in the law of formation which is a general and uniform rule."

Numbers cannot, he continued, be defined by addition, since addition requires numbers in its own definition. The successive addition of one "is not a way of creating numbers, but only a way of finding them, following a uniform rule, and arranging them in an order which is natural and useful, but not at all necessary. And because each one of them is essentially independent of the others, their existence as a whole does not depend on their order of succession; it is, in principle, simultaneous. . . . To assimilate the infinite set of these ideas to the indefinite set of the numbers which can be represented empirically is to confuse the sign with the thing signified; it is to identify the idea with its material image."

Couturat's discussion in De l'Infini presents a philosophical defense of the rationality of the procedures used to introduce infinite numbers, entirely uninhibited by the fear of paradoxes — unlike Russell's discussions in Principles of Mathematics. It may show why Russell argued in his later discussions with Poincaré that the paradoxes were due not to

\[\text{Couturat, De l'Infini, Pt.II, Bk.III, chap.ii, sect. 10-14.}\]
logic but to common-sense,¹ or "intuition." For the time being, in 1897, Russell remained unconvinced by Couturat's defense of infinity. In his review of Couturat's book for *Mind* of that year, Russell repeated his earlier assertion that infinite numbers could never have "philosophical validity", no matter how useful they might be to mathematical analysis.² However, he applauded Couturat's efforts to defend an unpopular cause.


²Closing sections of Russell's review of *De l'Infini*, *Mind*, VI, p.118.
CHAPTER V

A. N. WHITEHEAD:

A TREATISE ON UNIVERSAL ALGEBRA
AND LOGICISM

In their first books, Russell and Couturat discussed only two alternative philosophies of mathematics. However, as early as 1895 mathematicians were discussing three alternatives. The issue did not appear to them as between rationalism and empiricism, but between Formalism, Logicism, and Intuitionism. The Logicist\(^1\) point of view was represented by Peano and his associates in Italy and, as shown above, by Peirce and Schröder. It was a philosophy which denied the distinction between the methods of logic and the methods of mathematics which still seemed natural to both Russell and Couturat.

Couturat had prepared a course of lectures on mathematical logic for the year 1896, but his earliest published comment on mathematical logic appears to be his review of

\(^1\)The distinction between Logicism and Logistic is discussed in Chapter VIII of this dissertation.
Peano's *La Logique Mathématique*. Couturat spoke of Peano's new symbolic language as an attempt to produce the "Universal Characteristic" of Leibniz. He suggested that a successful universal characteristic would have two valuable aspects, one "stenographic" and the other "algorithmic." That is, he said, to the extent that its symbols actually succeeded in symbolizing ideas directly "without the fallacious intermediary of words," they would permit the replacement of reasoning by calculation. Logical derivation of consequences, instead of requiring careful thought, would then become a variety of mechanical transformation of formulae according to rules of manipulation. The stenographic advantages are obvious.

Couturat then presented a summary of the logical notations in use among mathematical logicians since Boole, and suggested that the logical calculus would be simplified by defining addition, multiplication and negation separately instead of trying to derive one from another. He also suggested that the attempt to unify class calculus and propositional calculus was a mistake, and expressed doubt that Peano's distinction between class membership and class inclusion indicated any important logical difference in relationships. At bottom, he said, both expressed the relation of "inclusion or subsumption" and we need only adopt a special sign to indicate when

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we are talking about individuals. However, he felt that the new sign had a technical justification since it made the notation simpler.

Couturat expressed surprise at the number of axioms Peano found necessary in his system, and suggested that a different choice of axioms would make many of the supposed axioms deducible. However, he concluded that with a bit more attention to the algorithmic functions of the proposed symbolic language, Peano's work might become philosophically interesting and valuable. But in order to achieve general interest, mathematical logic, or the symbolic language\(^1\) (which Peano called "pasigraphy") would need to stop restricting itself to an attempt to express the "logically very simple" reasonings of mathematics.

Where Peano's work had failed to do so, Whitehead's A Treatise on Universal Algebra with Applications\(^2\) reviewed by Couturat in 1900\(^3\) apparently convinced Couturat that mathematical logic was already philosophically important. In

\(^1\) Mathematical logic, as a language, is discussed in Chapter VIII of this dissertation, as "logistic." The new name was proposed for the discipline in 1904.

\(^2\) Whitehead, A., A Treatise on Universal Algebra with Applications (Cambridge, 1898).

\(^3\) L. Couturat, Review "Le Algèbre universelle de M. Whitehead," Revue de Métaphysique et des Moralees, XIII (1900), 323-352.
his review, Couturat stated as "the philosophical conclusion which appears from Mr. Whitehead's own exposition" the view that all mathematics is based upon the Algebra of Logic. He did not say that this was the conclusion Whitehead drew from his own exposition, but the implication is that Couturat believed it to be so.

Couturat asserted: "The Algebra of Logic is, at bottom, the theory of sets, considered from the point of view of their mutual inclusion and exclusion and that is why the theorems of logic permit three interpretations: as concepts, as propositions, and as sets of points or spatial regions. From a point of view even more abstract and general, the Algebra of Logic appears as the foundation of mathematics, or as the most elementary branch of mathematics, prior to the sciences of number, of order, and of magnitude, which are all its dependents and tributaries: it is the mathematics of the whole and the part."

I believe from the discussions mentioned above that serious consideration of the thesis that logic was the foundation of mathematics was new even to Couturat, the great advocate of philosophical study of mathematics. However, the view had been advanced by Schröder in 1898; and it had been criticized by Felix Klein at the Chicago Exposition Conference on Mathematics in 1893.1 The situation shows the

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1Klein regarded himself as an "intuitionist," opposed to the "excessive rigor and purism" of Formalists and Logicians.
justice of Couturat's complaint that mathematicians and philosophers had little contact with one another's thought. Without Couturat's interest, Whitehead, who at this time wrote not as a "philosophe" but as one of the "savants," a mathematician, would probably have been disregarded except by other mathematicians. Even Russell, who had read the proofs of the geometrical parts of Whitehead's book before publication\(^1\) seems to have been unimpressed by its philosophical implications.

As a work on the Algebra of Logic, Whitehead's book was reviewed by other students of logic, among them Hugh MacColl, whose review gives a striking contrast to that of Couturat.\(^2\) MacColl found Whitehead's book interesting but

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\(^1\)According to Whitehead's Preface to *Universal Algebra*. Whitehead (1861-1947) was at this time teaching at Cambridge.

\(^2\)Hugh MacColl, Review of *A Treatise on Universal Algebra* by A. N. Whitehead, Mind, VIII (1899). MacColl was an active formal logician, but his views have been either ignored or absorbed into the main line of subsequent developments. His symbolism and his ideas of logical reasoning were more complicated than those of most of his contemporary logicians, and he was not a mathematician. He advocated a calculus of statements, not of terms or things, since he held that any introduction of concrete things or of classes reduced the generality of "pure logic" and changed it into an "applied logic" which required special restrictions on the rules of inference. In view of the fact that restrictions have had to be introduced in order to avoid the paradoxes, it may well be that his insistence upon the differences between the logic of statements, the logic of things, and the logic of classes would reward further study. He also introduced ideas of modality. However, what he himself says of the relationship between his logic and Whitehead's symbolic treatment of "Existential Import of Propositions" seems to be true in general of his thought and the new development in formal logic: "... I find that they move along different paths and have but few points in common." Whether his ideas were better or worse than the ones which were accepted, they were at any rate uninfluential.
alien, revealing not only the value of the extensional algebra but also its limits. He saw no great philosophical import in the book.

Indeed, Whitehead appears to have been deliberately avoiding philosophical discussions in Universal Algebra, except for his Preface and introductory first chapter. There, he presented a definition of mathematics as: "the development of all types of formal, necessary, deductive reasoning."

Formal, because "the meaning of the proposition forms no part of the investigation"; necessary, because "it has followed the rule" of inference; and deductive, in the sense that "it is based on definitions which, as far as the validity of the reasoning is concerned (apart from any existential import) need only the test of self-consistency."

Whitehead regarded mathematical definitions as either conventions or, if existential, "the result of an act of pure abstraction," in which case "they require for verification more than the mere test of self-consistency." The difficulties of determining whether a given definition was existential or conventional could be escaped, he said, by regarding them all "either as referring to a world of ideas created by convention, or as referring exactly or approximately to the world of existing things." His logic in this book Whitehead attributed to "Mill, Jevons, Lotze and Bradley"; his symbolic logic

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1Whitehead, Universal Algebra, vi.
to "Boole, Schröder, Venn"; and the whole concept of a universal algebra to Hamilton and De Morgan. His views on symbolism he attributed to Stout.

Whitehead stated that all algebras, logical algebra included, were to be treated in his book in their character of calculi, a calculus being defined as "the art of the manipulation of substitutive signs according to fixed rules and of the deduction therefrom of true propositions." Substitutive signs he defined as "counters" having no intrinsic meaning but interpretable in terms of those entities for which they were substituted, their use in reasoning being "to economize thought." The non-frivolous kinds of calculi, said Whitehead, are those which resemble some calculus used to deal with "things and relations of things."

Propositions occur in a calculus, said Whitehead, in a form which is convenient for purposes of deduction, although that form may be artificial and not suitable to the expression of all propositions without exception. The symbols may be regarded as equivalent for the limited purposes of a calculus without being "barren identities." Indeed, Whitehead stated, "equivalence . . . implies non-identity as its general case. Identity may be conceived as a special limiting case of equivalence." Also, he said, "from another point of view all things which for any purpose can be conceived as equivalent

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form the extension (in the logical sense) of some universal conception. And conversely the collection of objects which together form the extension of some universal conception can for some purpose be treated as equivalent.¹

The operations of a calculus were, said Whitehead, to be thought of as methods of derivation: "a thing a will be said to result from an operation on other things a d e etc. when a is presented to the mind as the result of the presentations of a d and e etc. under certain conditions; and these conditions are phenomenal events or mental activities which it is convenient to separate in idea into a group by themselves and consider as defining the nature of the operation which is performed on a d e etc..."²

The advantages of reasoning by way of a calculus rather than by way of inference as defined by Bradley were, said Whitehead, that while "inference is an ideal combination or construction within the mind of the reasoner which results in the intuitive evidence of a new fact or relation between the data," when the calculus was used "this process of combination is externally performed by the combination of the concrete symbols, with the result of a new fact respecting the symbols which arises for sensuous perception." When this new fact is treated as a symbol carrying a meaning, said

¹Whitehead, Universal Algebra, 6.
²Whitehead, Universal Algebra, 8.
Whitehead, it is found to mean "the fact which would have been intuitively evident in the process of inference." Whitehead here referred to C. S. Peirce.¹

As for the value of the algebra of logic to "Formal Logic, conceived as the Art of Deductive Reasoning," Whitehead stated: "it seems obvious that a calculus — beyond its suggestiveness — can add nothing to the theory of Reasoning. For the use of a calculus is after all nothing but a way of avoiding reasoning by the help of the manipulation of symbols."²

I believe it is clear that Whitehead was at this time trying to leave philosophical arguments to the philosophers, although he did say that the algebra of logic was the simplest and most fundamental kind of algebra. He in this book did not seem to find any conflict between the disavowal that the algebra of logic can be helpful to formal logic, and his earlier statement that all formal, necessary, and deductive reasonings are the subject matter of mathematics. Apparently he did not believe that "the Art of Deductive Reasoning" fell into this category. Philosophy, inductive reasoning, and imaginative literature were the three explicit exceptions he made to his suggestion that almost all kinds of "serious thought" should be regarded as "mathematics developed by means of a calculus." Nevertheless, Couturat attributed to Whitehead


²Whitehead, Universal Algebra, 99.
the view that pure mathematics is to be regarded as "the general method applicable to all concrete sciences, as the science of pure and abstract reasoning, in a word, as the universal Logic." \(^1\)

Couturat's main support for this interpretation was Whitehead's argument that mathematics cannot be restricted to the study of quantity and number. Couturat was of the opinion that this development of an abstract, or uninterpreted algebra was the final culmination of a long line of thought, a development hinted at in Boole's discovery of the "formal analogy of the laws of logic and the laws of algebra." Leibniz' vision of a Universal Characteristic, Descartes' Universal Mathematics, "these prophetic dreams are embodied to some extent in the work of M. Whitehead; he provides a scientific content and positive applications to these prophetic intuitions, which could have been regarded for a long time as chimaeras of the metaphysicians; he shows that these great rationalists were right, by confirming and illustrating the cartesian idea of mathematics as the universal science."

Once again, Couturat apparently saw no distinction between the statements that logic was the foundation of mathematics and that mathematics was the foundation of logic. In this article he enthusiastically supported both.

One of the first favorable reactions to Couturat's assertion of the unity of mathematics and logic came from his philosophical opponents, the Kantians. In Germany, Paul Natorp's attention was drawn to the works of Whitehead, and of Hermann Grassmann, by Couturat's review. Natorp published an article calling the attention of German-speaking philosophers to these new developments in mathematics and to their importance for philosophy.

Mathematicians and philosophers of mathematics now appeared, Natorp commented, to be engaged in a search for "Grundbegriffe, letzte Begriffe des Erkennens." They were, therefore, he said, engaged on the traditional business of "what used to be called Dialectic and is now called Logic."

This was particularly true, Natorp thought, of Grassmann's work; for Whitehead's book was more "formalist" and less philosophical. Natorp stated that Grassmann, unlike Whitehead, was not content to accept arbitrary conventions as his starting points. He searched for real beginnings from which all would follow in logical order, "in stetigem Fortschritt von einem unmittelbare Anfang her."

The logicist thesis that there is a continuous passage from logic to foundations of mathematics seemed to Natorp reasonable and justified, even though it lacked a proof.

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Indeed, said Natorp, purely mathematical methods are in order only within a given body of theory. The search for "necessary and sufficient conditions of a science," the Platonic removal of the hypotheses, is a logical rather than a mathematical inquiry. When these first principles of a science are reached, Natorp continued, "die Logik fordert für sie, als synthetische Sätze, wie Kant sagt, 'wo nicht einen Beweis' (der hier in der That ausgeschlossen ist) 'doch wenigstens eine Deduktion der Rechtmäßigkeit ihrer Behauptung,' sie fordert, nach Plato, den Rückgang auf 'voraussetzunglose,' d. h. auf solche letzte Voraussetzungen von denen es möglich ist, sich zu überzeugen, dass sie nicht wiederum andere, fundamentalere voraussetzen, nämlich auf die schlechthin fundamentalen Verfahrensweisen des 'Denkens,' d. i. gesetz- mässigen Vorstellens der Gegenstände überhaupt die sie in einer begrenzten Zahl reiner Grundfunctionen des Denkens (Kategorien) festzulegen sucht.\(^1\)

\(^1\)Logic demands for them, since they are synthetic propositions, as Kant says, "where not a proof" (which is in fact excluded here) "still at least a justification of the legitimacy of accepting them." Logic demands, following Plato, going back to the "unconditioned," that is, to the final conditions which it is possible to convince oneself do not in turn presuppose others still more fundamental. That is, logic requires our going back to the absolutely fundamental ways in which "thought," meaning lawful contemplation of objects in general, operates; and these logic tries to lay down in a definite number of pure fundamental operations of thought (Categories).
This hospitable reaction to the unity thesis of the logicians obviously depended upon Natorp's interpretation of what the logical and the mathematical methods were. He apparently regarded logic, as Couturat had in De l'Infini, as an inquiry to be strictly distinguished from Formal Logic, "that superannuated and archeological discipline." That a philosophical theory asserting the dependence of mathematics upon logic would be regarded as a refutation of the philosophy of Kant apparently never occurred to Natorp. For him, as a neo-Kantian, it seemed rather a newly-discovered consequence of Kant's own principles.

1The epithets were Couturat's.
CHAPTER VI

THE LOGIC OF G. E. MOORE

The Logicist philosophy of mathematics had now become a problem for philosophers. Its claims were urged at the First International Congress of Philosophy in 1900, where the passigraphers\(^1\) from Italy read a number of papers explaining points of their logic, or, rather, their new language. Bertrand Russell also read a paper at the Congress in which he also advocated a new logic. Russell's logic, however, was unrelated to the symbolic developments. It was a new metaphysical logic, an alternative to the logic of Bradley, and was based upon philosophical views advanced by G. E. Moore. Those views were, in Russell's opinion, logical, because they assigned a new philosophical status to relations.

The new theory of relations which appeared in Russell's paper, "L'Idée d'Ordre et la Position Absolue dans l'Espace

\(^1\) This was Peano's term for the work he was doing, and is usually translated by commentators of the period as "universal writing."
et le Temps,\textsuperscript{1} also formed part of the framework Russell used in his criticism of the philosophy of Leibniz in his book, \textit{A Critical Exposition of the Philosophy of Leibniz}.\textsuperscript{2}

R. Latta's\textsuperscript{3} review of Russell's book in \textit{Mind}, 1901,\textsuperscript{4} commented rather sadly upon the fact that Russell's own view on logic was not expounded there. It was, Latta said, obviously so different from the accepted logic that the reader was unable, in evaluating Russell's criticism of Leibniz, to make the usual "rough allowance for 'windage'" by allowing for the opinions of the critic.\textsuperscript{5} For example, among the unexplained terms Russell used, said Latta, was the word "proposition," where most logicians spoke of "judgments." Also, Russell's attack upon the "subject-predicate logic" left, Latta said,

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\textsuperscript{2}B. Russell, \textit{A Critical Exposition of the Philosophy of Leibniz} (2d ed.; London: George Allen and Unwin Ltd., 1937). The first edition was published in 1900 by the University Press at Cambridge.

\textsuperscript{3}Latta was a professor of philosophy who had also written on Leibniz.


\textsuperscript{5}ibid., 526.
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the impression that he thought that all propositions were relational and had neither subject nor predicate. Russell also denied that there are any analytic propositions. Latta regarded the logical issues raised by such a theory as very important: "If Mr. Russell's account of the relation between knowledge and truth is correct, the Kantian revolution is reduced to absurdity, and the highest category (whatever it may be) cannot be mind or self-consciousness or anything that suggests knowledge. For truth is represented as essentially independent of all knowledge, though in some unexplained way it comes, by being believed, to be an element in the constitution of knowledge."

It appears that Latta was disturbed because Russell was reopening the closed issue between realism and idealism. It may seem strange that this issue should be spoken of as a point of logic, but in philosophical discussions at the turn of the century any distinction between the "logical" and the "metaphysical" or "epistemological" aspects of a problem is extremely vague. The distinction was recognized to exist. For instance, Bradley acknowledged the difficulty of making sure just what is the business of logic, but was sure of some clearly logical and some clearly metaphysical points. Both Russell and Couturat wanted to demonstrate that the metaphysics of Leibniz rested upon his logical views rather than the logic on the metaphysics. Then, as now, there was no suggestion of a precise definition for either discipline.
However, apparently both Latte and Russell considered views such as those which G. E. Moore expressed in his articles, "On the Nature of Judgment," and "Necessity," to be concerned with logic, and I shall from this point refrain from inserting quotation marks and use the word without apology where it was used in the contemporary arguments, since fair warning of an unusual usage has been given. After all, the word is used in some fairly odd ways in modern discussions, too.

The two articles by G. E. Moore were published in 1899\(^1\) and 1900\(^2\) respectively, and may be supposed to present some of the considerations which led Russell to reject Bradleian logical views which he had previously accepted. The most striking direct assault on one of Bradley's positions is in the article on "The Nature of Judgment." There, Moore argues against Bradley's view that "the universal meaning" or "concept" is an abstraction from ideas, or "mental facts," in favor of a view that concepts are transcendent Platonic entities. This kind of realism was as unpopular as the other; but Moore attacked Bradley's conceptualism with the following argument, which he spoke of as a variant on the Third Man.


\(^2\)G. E. Moore, "Necessity," Mind, IX (1900). Moore was born in 1875, and was "several years junior" to Russell at Cambridge. His strong influence on Russell's thought is spoken of by Russell in his autobiographical sketch in the Schilpp volume, The Philosophy of Bertrand Russell. Moore died in 1958.
When I have an idea of something, said Moore, that idea has a specific content, "a character which is different or distinguishable from that of other mental facts." Bradley's view was, said Moore, that in judgment part of this character must be cut off and "fixed by the mind." But, argued Moore, before we can perform such an abscession, we must know the character of the whole idea, at least in part. (If this character is to be known only by making a judgment) I must have already cut off part of an idea of this whole idea. But the character of the whole idea of the present idea must have then been known, at least in part, before I could cut off a part of it. (If this character is to be known only by making a judgment) I must have already cut off part of an idea of the idea of the idea. But before I could cut off part of this content -- and so on, into the infinite regress. Moore's conclusion was that "the idea used in judgment" cannot be "a part of the content of any psychological idea whatever."

The hypothesis I have inserted in parentheses in the summary above is not used in Moore's article, but it is an essential part of the argument. Without it, and given the hypothesis that the regress in question is a vicious one, the conclusion of Moore's argument would be: either some

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1Moore, "Judgment," Mind, VIII (1899). The portion of this sentence in single quotes was quoted by Moore from Bradley, with approval.
knowledge is not a result of judgment, or the concept used in judgment "is not a mental fact nor any part of a mental fact." Certainly for Russell¹ and probably for Moore, any "actual infinite" was regarded at this period as involving a contradiction. Bradley also, although he thought Reality to be infinitely complex, regarded the idea of an infinite number or an infinite series as a contradiction.² Bradley would, therefore, probably not have tried to present the regress as harmless. But, since his theories seem not only to permit but to require that some knowledge be not a result of judgment, the argument is not conclusive against him.

Moore also gave a briefer and stronger version of his argument: "Identity of content is presupposed in any reasoning; and to explain the identity of content between two facts by supposing that content to be part of (or all of) the content of a third fact, must involve a vicious circle. For in order that the content of the third fact may perform this office, it must already be supposed like the contents of the other two, i. e., having something in common with them, and this community of content is exactly what it was proposed to explain."

Moore resolved the difficulty by concluding that concepts are not mental states, but are "possible objects of thought,"

¹Russell, Philosophy of Leibniz, 115.

independent entities capable of entering into external relations with a knowing subject, or with several knowing subjects. Judging is a way of knowing propositions, which are complex concepts. Truth and falsity are indefinable properties of these complexes, and must simply be immediately recognized. Existence is one concept among others: "All that exists is thus composed of concepts necessarily related to one another in specific manners, and likewise to the concept of existence."

Moore's proof of this part of his theory was: The concept is necessary to truth or falsehood, as Bradley has proved to everyone's satisfaction. But concepts are neither existents nor parts of existents.¹ This has been proved by the Third Man. Therefore, argued Moore, truth is not a relation between existents and propositions (complex concepts), because if the proposition that truth is a relation between a proposition and an existent were true, it could not be because of a relationship between itself and an existent without a vicious circle.² But truth must involve two terms and a relation between them. Therefore, Moore concluded, it must be that truth must be a property of complex concepts.

¹Yet existents are only complex concepts. I have no idea how to give this article of Moore's a self-consistent interpretation.

²Since this argument involves substituting an entire definition for one of the variables contained in the definition, it is of particular interest in the light of later developments.
"Facts" were useless in this discussion, said Moore, because they must be expressed in propositions before they could be used in argument; and these propositions must be true, if they agree with the facts. The rules of logic themselves must be expressed in true propositions. Moore did not say why, but I suppose it is because if the rules of logic are accurate statements about the nature of the world, the propositions expressing them must be true. But, said Moore, only concepts can enter into propositions. Therefore, since truth must be an element in propositions, the only conclusion is that truth is itself a simple concept, logically prior to any proposition.

Given these conclusions, Moore stated, we can see what ways of reasoning will be logically permissible. Logic may validly start with complex propositions and determine what simpler propositions are "involved" in them; or it may start from a "simple" proposition¹ and add new concepts to it to obtain a more complex proposition—"which is the properly deductive procedure exhibited in the propositions of Euclid." In this method, true conclusions follow only from true premises; in the first method, analysis of a false proposition may disclose true constituents. Both processes are "synthetic, in the sense that the results arrived at are

¹Simple propositions are of course only relatively simple, being complex concepts.
different from the premises and merely related to them.\textsuperscript{1}

Moore at this time was obviously not quite sure whether or not concepts could be "created" or "constructed"; for he speaks not of new knowledge following a logical inquiry, but of new propositions.

Both things and ideas must be, Moore continued, thought of as "composed of nothing but concepts. A thing becomes intelligible first when it is analyzed into its constituent concepts." An existent is "a true existential proposition," and "perception" is "cognition" of such propositions. Of course, propositions are not, he pointed out, "assertions" or "affirmations" but "the combination of concepts which is affirmed." Inference is the perception of other logical connections.

This view is, said Moore, a Transcendentalism. Kant's Transcendentalism, he stated, "rests upon the distinction between empirical and a priori propositions," which corresponds to the distinction between categorical and hypothetical propositions. Kant's distinction does not, Moore believed, rest on the necessity of the propositions; for only propositions not involving the notions of substance and attribute involve no necessity at all. That is, argued Moore, the assertions that the subject must be a substance and the

\textsuperscript{1}Moore, "Judgment," Mind, 1899, p.182.
predicate must be an attribute are "involved in" all categorical judgments, thus introducing an element of necessity into any of them. The real basis of the distinction between the empirical and the a priori is, Moore believed, that empirical judgments are those into which empirical concepts "enter", empirical concepts being concepts which can exist in parts of time. For even pure existential propositions, asserting the existence of a simple concept, are, he believed, absolutely necessary. "Red exists now" either states, Moore asserted, a necessary connection of concepts or an impossibility. But, said Moore, Kant has proved that all particular propositions involve space or time or the categories, whether these particular propositions are true or false. Therefore, these general concepts are as immediately known as the empirical ones.

Finally, Moore concluded, both our mind and the world must be regarded as "less ultimate" than judgment, and judgment itself as less ultimate than logical ideas, or concepts, which are "the only substantive or subject."

It is confusing that Moore chose to refer to his Platonic objects as concepts. Although for Bradley the concept was the content of a thought, rather than itself a thought, it was still dependent upon thought for its existence -as was, of course, everything else. Moore apparently wished to reject this Idealism; but his use of the word concept hints at,
to adopt Couturat's phrase, "a residue of conceptualism and idealism in the new Platonist logic." His discussion of existence is especially puzzling, and suggests that Moore, like the young Russell,\(^1\) had "seen" the validity of the Ontological Argument.

The status of necessity appeared to Moore himself to require further clarification. He had insisted in his article on Judgment upon the necessity of all truth; the second article\(^2\) is to explain the meaning of "necessity." The sense in which this meaning was to be discovered was not, Moore warned, that of enumerating "all those predicates which the word is commonly used to signify; for the only test that a word is correctly defined is common usage."\(^3\)

It was to be the kind of definition in which the "something in common" possessed by all those predicates must be "seen." In the effort to reach this intellectual vision of "necessity," then, Moore continued, we must examine its instances. Moore classed these into three groups: necessary connections,
necessary things, and necessary propositions. What have they in common? He concluded that they are all "presupposed by" or "logically prior to" some true proposition. No proposition is, Moore believed, necessary in itself, but only in connection with other propositions. The connection between the propositions is itself necessary only because it is presupposed by the true proposition that such a connection holds: "... the general principle that what follows from a truth is itself true is necessary, because it is implied in every argument. That any one thing does follow from any other is, indeed, not always a necessary proposition: but that, if it does follow, then, if the first be true, the second is also true, is a necessary proposition. It is logically prior to any statement such as: Since this, then that. And such statements are not among the least common of truths."

The sense in which a thing can be necessary, said Moore, can only be "that from the proposition 'The One exists' there is a valid inference to the proposition 'The other existed' or 'will exist!' It it really does follow that, since one thing exists, another has existed or will exist, what more

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1 Moore, "Necessity," Mind, p. 301. "Logical priority" itself, Moore believed, must simply "be seen in an instance, in order to be recognised."

necessary relation can be desired?"^1 Hence, Moore concluded, all necessity is logical.

In the course of the argument, Moore also found time to consider the problem of analytic and synthetic truths. He argued that either "analytic" and "necessary" have the same meaning, or "analytic truths are necessary" is a synthetic proposition, and only its truth guarantees the necessity of an analytic proposition. Further, "any proposition, it would seem, must contain at least two different terms and their relation; and, this being so, the relation may always be denied of the two terms without contradiction." Self-contradiction of a single proposition Moore thought impossible, so an analytic proposition could not be one whose contradictory is self-contradictory. "If, on the other hand, we take the definition that it is a proposition of which the predicate is contained in the subject, then either its meaning is that the predicate is united in some way with the other predicates, which along with it define the subject, in which case the analytic proposition is as synthetic as you please; or else the predicate is simply identical with the subject."^2 But identities are not propositions at all.\(^3\) Therefore, no analytic propositions exist.

^1 Moore, "Necessity," *Mind*, IX (1900), 303.

^2 Ibid., 295.

^3 For Moore, this was true by definition.
After some consideration, it seems to me that the only thing to do with Moore's third argument on this topic, since I am committed to dragging it from its decent obscurity, is to quote it without comment, while pointing out that the traditional Three Laws of Thought were commonly regarded at this time as equivalent.

Moreover, the law of contradiction itself, than which nothing is commonly supposed to be more plainly analytic, is certainly synthetic. For suppose some one to hold that Not every proposition is either true or false. You cannot deny that this is a proposition, unless you are also willing to allow that the law which it contradicts is not a proposition; and he may perfectly well maintain that this is one of those propositions which is true, and the contradictory of which, your law, is false, although this is not the case with every proposition. Whereas, if you urge that it is included in the notion of a proposition that it should be either true or false, either your law becomes a pure tautology and not a proposition, or else there is something else in the notion of a proposition beside the property that it is either true or false, and then you are asserting a synthetic connection between this property and those others.¹

I believe it is clear from the above discussion of views advanced by Bradley, Moore, Couturat, and Russell that the

¹Moore, "Necessity," Mind IX (1900), 295.
distinction between analytic and synthetic propositions was not favorably regarded at the end of the nineteenth century. Russell's earliest attack upon the distinction was based on Bradley's arguments against it; but Bradley had held that all propositions were both analytic and synthetic. The new logic of Moore thus parted company with the new logic of Bradley not only on their solution to the problem of universals but on the issue of the analytic-synthetic distinction of propositions.

There seems to have been a great unanimity among the logicians of this period on their rejection of the distinction between analytic and synthetic truths. There is less agreement on the reasons why this distinction must be rejected. Bradley based his rejection upon his belief that such a distinction, like the distinction between the One and the Many, was ultimately indefensible. Every ostensible individual was simultaneously a multiplicity, and every ostensible multiplicity simultaneously an unity. The Absolute, of course, would be neither. In the same way, every "judgment" was for Bradley simultaneously analytic and synthetic. Furthermore, no judgment considered in itself was either.

Russell interpreted Bradley's view as meaning that every judgment is at once "a further determination of a given whole" and to that extent analytic, and a recognition of "the emergence of new relations within this whole," which is the meaning of "synthetic." That is, analytic judgments take apart a unit
already constructed, and synthetic judgments create a new
unit. Bradley actually spoke of mental experiments anal-
ogous to a chemical experiment when he spoke of inference,
so it seems fair to take these expressions literally.

Couturat apparently shared this understanding of the
terms, although he insisted that the "ideal experiments"
must not be thought of as involving imagery. Whitehead ex-
pressly accepted the "ideal experiment" view of inference,
which was to consider it "synthetic" in the sense assigned
to that word above. It seems that this sense of synthetic
could properly be called "creative." There is no apparent
distinction between a synthetic truth and a synthetic
judgment. In this logical context, however, a request for
such a distinction would be out of order. The idealist
logic involves the denial of any reality independent of the
mind, and with it of any truth independent of judgment.

The meanings assigned to the terms analytic and synthetic
by G. E. Moore are, of course, different from those of the
idealists. The concept of creation of new truths was, or at
least should have been, nonsense for Moore; his early articles
seem to leave no place even for new judgments. Yet, surpris-
ingly, he denied the existence not of synthetic but of
analytic truths, or, rather, propositions.

Moore defined a "synthetic" process of inference \(^1\) as one

\(^1\) There seems no room for processes in Moore's theory, either.
which leads to a conclusion not "new" but "other than" the premises. All propositions which differ in any respect are "different" and "merely related" to one another. Presumably Moore also thought of the relation of logical equivalence as a "mere relation." Therefore, no reasoning from one proposition to another could be other than synthetic in this sense, by definition. Identities were, again by definition, not propositions, and apparently their assertion was not a process of inference. Considering that propositions were for Moore to be actually built out of concepts, as a building is made of bricks, his point becomes obvious. We could not construct a building by using the same brick 55,000 times.

Russell, however, introduced yet another meaning of these words in his Philosophy of Leibniz. Analytic truths must follow from the law of contradiction alone, without the addition even of a premise. Whatever is non-analytic is synthetic. Thus, all inference, as well as every proposition, is synthetic. The logic is impeccable; but there is ground to complain about the definitions.

The attack upon "subject-predicate logic" which Latta found new and startling in Russell's Leibniz should not have seemed strange in itself; Bradley had attacked formal logic in the same terms. But even the idealist Bradley had protested against viewing the world as "a bloodless ballet of categories"; the really striking innovation in the new logic

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1See Chapter I above.
of Russell and Moore was their doing just that with the denial of any metaphysical distinction between concepts and existents.

Latta suggested that Russell's criticisms of Leibniz were based upon "a profound study of modern mathematical logic." However, it was the logic of Moore and Bradley, not of Peano, which Russell used in his criticisms of Leibniz' "subject-predicate logic." The criticism had been introduced by Bradley, and Bradley had emphasized the importance of relational inferences. Russell's main criticisms of Leibniz were: (1) there are some irreducibly relational propositions; and (2) given that fact, there is no use for the concept of substance in philosophy. He also urged that Leibniz' principles forced him into the absurdity of admitting infinite numbers, which was a reductio of his whole system.

In his argument designed to show the absurdity of Leibniz' philosophy, Russell pointed out that Leibniz believed in "the actual infinite" although he rejected infinite number.\(^1\) His attempted escape from the contradictory concept of infinite number was, Russell pointed out, by the argument that infinite aggregates were not "wholes" and hence had neither magnitude nor number.\(^2\) For Leibniz, said Russell, nothing actual is continuous, in the sense of having indeterminate parts; actual aggregates must have simple,

\(^1\)Russell, Leibniz, 109.

\(^2\)Ibid., 110.
definite parts. Both aggregates and relations are regarded by Leibniz, Russell said, as not "really real." Yet, any application of number, other than one, requires that a plurality of substances be perceived as a unit. Leibniz argued, said Russell, that such a unity of monads existed only "for perception." Russell objected that if it is only as perceived that there are a number of monads, there is only one real substance; but if there are many monads, the assertion that there are a number of them is true. But that assertion cannot be reduced to subject-predicate form. Therefore, concluded Russell, "the basis for the use of substance has fallen through, and the assertion of infinite aggregates, with all its contradictions, becomes quite inevitable for Leibniz."  

Russell's proof that Leibniz' concept of substance "is derivative from the logical notion of subject and predicate" was as follows:

In the thought of Spinoza and Descartes, "the notion of substance, though not by them clearly analyzed into its elements, was not an ultimate simple notion, but a notion dependent, in some undefined manner, upon the purely logical notion of subject and predicate. The attributes of a substance are the predicates of a subject; and it is supposed

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1 Russell, Leibniz, 115.  
2 Ibid., 117.  
3 Ibid., 12.
that predicates cannot exist without their subject, though the subject can exist without them. Hence the subject becomes that whose existence does not depend upon any other existent.\footnote{Russell, \textit{Leibniz}, 41.}

This argument, if such it is, that substance and subject are identified in the thought of Spinoza and Descartes rests, so far as I can see, upon Russell's assertion: "The attributes of a substance are the predicates of a subject," which itself has no supporting reasons.

Russell pointed out that Leibniz, arguing against Locke, held that, "we conceive several predicates in one and the same subject, and this is all that is meant by the words support or \textit{substratum}, which Locke is using as synonymous with substance."\footnote{\textit{Ibid.}, 42.} This quotation seems strong support of Russell's interpretation of Leibniz' thought. But Russell then argues that substance has been used to indicate not only the logical subject but that which persists through change. But change "implies," Russell said, "a subject which has preserved its identity while altering its qualities. This notion of the subject of change is, therefore, not independent of subject and predicate, but subsequent to it; it is the notion of subject and predicate applied to what is in time. It is this special form of the logical subject, combined with the doctrine that there are terms which can only be subjects and
not predicates, which constitutes the notion of substance as Leibniz employs it."¹

It seems to me that Russell's argument rests upon the use of the word "subject" to refer to a constituent of a proposition, and also to an ontological constituent of a real world "in time." Since this identification is what was supposed to be proved, the argument here begs the question, except in its historical context. For it seems clear that it takes for granted Moore's identification of existents and propositions, as discussed above.

Russell went on to point out that "any term may be made a subject" although some, for instance, "the term I" appear incapable of attribution to any other term."² He did not recognize, at least in print, that this was an admission of at least part of the thesis he was combating. He suggested that with Leibniz' recognition that "the unity of the law" of change is both our sign that one substance is changing and that "which constitutes it the same substance," he could not have any further need for substances. The "unity of the law" could now serve without any need for "the appeal to subject and predicate."³

However, commented Russell, for some reason Leibniz wished to retain a distinction between the monad and the sum

¹Russell, Leibniz, 42. ²Ibid., 43. ³Ibid., 48.
of its predicates. Russell suggested that he did so because the Ego seemed to him self-evidently "one subject." He then concluded that, on the basis of the preceding considerations it can be seen that: "the ground for assuming substances -- and this is a very important point -- is purely and solely logical," depending on the "purely logical tenet" that predicates require subjects.

To properly evaluate this statement of Russell's, I believe that it is necessary to remember the wide scope he assigned in this period to the field of logic. For instance, he considered the view that an entity is "a true existential proposition," that is, a complex Platonic idea, to be a point of logic. One of Leibniz' logical grounds for assuming substance was asserted by Russell to be his belief that the existence of at least one substance was self-evident. To one not antecedently convinced that substances and qualities are identical with their conceptual counterparts, the argument seems to beg the question.

Another argument Russell offered against the subject-predicate logic was: in the statement "This is a man," the word "this" must mean something, and only its meaning is capable of distinguishing which substance we are speaking of. Yet, the meaning of such a word is, he said, usually some reference to time or place, so that 'this is human'

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1Russell, Leibniz, 49. 2ibid., 50.
would reduce itself to 'humanity exists here' . . . but he regarded time and place as themselves ultimately reducible to predicates. Thus the substance remains, apart from its predicates, wholly destitute of meaning. ¹ As to the way in which a term wholly destitute of meaning can be logically employed, or can be valuable in Metaphysics, I confess that I share Locke's wonder.⁶

Russell introduced the familiar arguments against the possibility of reducing all propositions to subject-predicate form, and concluded that it is contradictory to suppose substances, or subjects of propositions, real but time and space unreal. This argument also must be understood in the light of Moore's logic, where propositions are to be considered as non-verbal entities, and as the actual constituents of the world. Apparently, Russell had not completely excluded from his thinking all traces of the older logic of judgments, since he seems to make a distinction between a proposition and its verbal expression or symbolization. Of course, it is difficult to find a place in Moore's Platonic universe for knowledge or knowers, as well as for such anomalies as language.

I believe that the extent to which Russell's arguments in Leibniz are to be disregarded in a study of the development of the philosophies based on modern, that is, mathematical logic can only be determined by a detailed study of how

¹"Meaning" is apparently here to be equated with "conceptual content," as it was for Bradley.
far they were disregarded or discredited in that development. It is also difficult to reconcile the arguments summarized above with the discussion of matter in Leibniz, which seems to show Russell already at odds with Moore's metaphysics.

The definition of substance, and "how substances must differ if there be many" are, said Russell, logical problems like the question of "the nature of propositions."¹ But "questions as to the actual world" lead to the consideration of matter, and for Leibniz to the invention of "monads as real units by which the continuum was rendered discrete."² Leibniz here started, said Russell, from common-sense ideas of matter and space and transformed them "into something quite different, namely, unextended substances and their perceptions."³ Russell did not suggest that this common-sense idea might also be a part of Leibniz' idea of substance, although he had identified "substance" and "logical subject" on similar grounds. However, he did believe that common-sense and logic cooperated in leading Leibniz to his metaphysics. It must be remembered that Russell had little respect for "common-sense" at this stage in his thought.

At this point Leibniz was led into error, said Russell, by his refusal to admit "the antimony of infinite division."⁴

¹Russell, Leibniz, 70. ²Ibid., 72.
³Ibid., 74. ⁴Ibid., 84.
Infinite number is self-contradictory, asserted Russell. To avoid asserting that there are an infinite number of monads, Leibniz, who was committed to "infinite aggregates" was required, said Russell, to assert that "aggregates" are not real beings. But, said Russell, such a procedure is self-contradictory, since "calling it a multitude" involves "the assertion of a whole." Leibniz' denial of the unity of an aggregate was, Russell said, a deduction from the subject-predicate logic which collapsed without the doctrine of substances.

Another extremely interesting passage from Leibniz emphasizes the great difference between the views Russell held at that time and those he expressed in Principles of Mathematics. It has to do with Leibniz' vision of a Universal Characteristic.

"What he desired was evidently akin to the modern Science of Symbolic Logic, which is definitely a branch of Mathematics, and was developed by Boole under the impression that he was dealing with the 'Laws of Thought.' As a mathematical idea -- as a universal Algebra, embracing Formal Logic, ordinary Algebra, and Geometry as special cases -- Leibniz's conception has shown itself in the highest degree useful. But as a method of pursuing philosophy, it had the

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1 Russell, Leibniz, 115. 2 Ibid., 117.

3 This argument is certainly repudiated in later works. Consider the "No-Class Theory" Russell proposed later.
formalist defect which results from a belief in analytic propositions and which led Spinoza to employ a geometrical method." This error, said Russell, was a misconception of the philosophical method. Leibniz had supposed that what was required was a convenient method of deduction. But, in fact, said Russell, definable ideas and proveable propositions are of only secondary philosophical interest. In dealing with the indefinable and indemonstrable, "no method is available save intuition." Therefore, Russell concluded, the Universal Characteristic was a dream which showed only that Leibniz completely misunderstood the nature of philosophical thought. It was Russell's belief that the basis of Leibniz' error was his acceptance of analytic propositions, and he felt that that belief in its turn rested upon subject-predicate logic.

In order that there should be analytic propositions at all, Russell continued, we must have a definable subject, which must be a species rather than an individual; and the subject must always be complex, "a collection of attributes." However, "and this is the weak point of the doctrine of analytic judgments," any such collection must be a collection of mutually compatible predicates. Thus, analytic judgments presuppose

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1 Russell, Leibniz, 170-171.
2 Ibid., 18.
a non-analytic judgment of compatibility. This requirement may also be stated, Russell said, as a demand that definitions be of "possible objects."

Leibniz held, said Russell, that definition is "analysis into indefinable simple ideas" which are the constituents of the complex ideas to be defined; and this Russell believed was correct. But Leibniz also believed, said Russell, that the 'primary principles' which 'cannot be proved, and indeed have no need of proof,'" are "identical or analytic," which is, Russell argued, false.

Leibniz's distinction between real and nominal definitions Russell interpreted as the distinction between those definitions which have been shown to be consistent and those which have no such proof that their object is possible. But, Russell stated, for Leibniz all simple ideas are compatible; therefore, all collections of simple ideas are consistent; and any complex idea must be possible. Simple ideas are, by virtue of the very definition of analyticity, incapable of analytic relationships with one another. Their relations must, then, be synthetic. Thus, every real definition involves the synthetic proposition

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1 Russell, Leibniz, 19.
2 Russell, Leibniz, 168-169.
3 Russell, Leibniz, 20.
that its constituents are compatible. Russell expanded on this point by discussing "the round square." This is a complex idea which involves the proposition "round and square are compatible." The contradiction arises only because of a "synthetic relation of incompatibility" between those ideas, a fact which is "almost admitted by Leibniz." It follows, Russell argued, that "even if 2+1 be indeed the meaning of 3, still the proposition that 2+1 is possible is necessarily synthetic. A possible idea cannot, in the last analysis, be merely an idea which is not contradictory; for the contradiction itself must always be deduced from synthetic propositions. And hence the propositions of Arithmetic, as Kant discovered, are one and all synthetic."¹

In general, Russell concluded, no proposition at all can follow from the Law of Contradiction alone, "except the proposition that there is truth, or that some proposition is true;" for anything further, would involve some premise, such as 'this is a proposition,' which "does not follow from the law of contradiction. Thus the doctrine of analytic propositions seems wholly mistaken." Even such a proposition as "an equilateral rectangle is a rectangle," supposed to be analytic, is actually a 'judgment of whole and part,' in which the constituents of the whole have a kind of unity

¹Russell, Leibniz, 21.
destroyed by analysis: "Thus even here, in so far as the subject is one, the judgment does not follow from the Law of Contradiction alone."\(^1\)

The question whether there are necessary propositions can, then, only be a search for necessary synthetic propositions. But, said Russell following Moore: all true propositions are necessary. The difference is in the way we come to know them, whether by perception or reasoning;\(^2\) and at that the difference is exaggerated. For: "It is supposed that in a priori knowledge we know a proposition, while in perception we know an existent. This is false. We know a proposition equally in both cases. In perception we know the proposition that something exists. It is evident that we do not merely know the something, whatever it be, for this is equally present in mere imagination. What distinguishes perception is the knowledge that the something exists. And indeed whatever can be known must be true, and therefore must be a proposition."\(^3\) From this Russell concluded that knowledge is never caused by what is known, which is a fact;

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\(^1\)Russell, *Leibniz*, 22.
\(^3\)Russell, *Leibniz*, 164.
but by what *exists* and is not known. He held that this reduces the distinction between perception and intellectual knowledge so greatly that we must now hold either that both kinds of knowledge are innate or that both are acquired from unknown causes. The *objects* of knowledge, or ideas, must, however, be recognized as not mental and not existent at all; otherwise, no two people could know the same thing, without an endless regress.\(^2\)

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1. The admission of existents distinct from propositions is a departure from Moore's position.

CHAPTER VII
THE FIRST CONGRESS OF PHILOSOPHY
SUBJECTS, PREDICATES, AND RELATIONS

Moore's belief in Platonic ideas, but not his belief that existence was one concept among others; Bradley's rejection of subject-predicate logic; and Moore's rejection of analytic propositions appear as the most striking elements of the "new logic" Russell took with him to the First International Congress of Philosophy. At the conference he presented a paper on relations, with a brief reference to the relational logic as symbolized by Schröder. He also became interested in the logical views of Peano and his associates which were presented in a number of papers read at the Congress.

These papers by the mathematical logicians appear to have been designed to be interesting to philosophers as well as to mathematicians, with little emphasis on technical details. For example, Peano\(^1\) presented a very brief paper on "Les Définitions Mathématiques." Definition must, he

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\(^1\)G. Peano, 1858–1932. Peano's work was well-known among mathematicians by this time.
stated, be an equation between a name and some previously known terms. What terms are to be supposed known gives rise to problems for any science; yet only mathematical logic begins with a list of the terms which are to be regarded as known. Such a list could easily be given for any science: for example, in Geometry, we need only list the undefined specifically Geometrical ideas and the ideas of logic "indicated by the words, without, of, to... and by the grammatical forms contained in the propositions." Mathematics uses only a few words and rules of grammar, said Peano, and even so there are more words used than ideas represented. For example, many words, such as "product", "factor", etc. show the one idea represented in algebra by the symbol "x". For an enumeration of the primitive ideas for at least a large portion of mathematical science, Peano referred his hearers to his Formulaire de Mathématique.

Setting aside the known ideas, he continued, a definition is to be thought of as an equation between two different signs; and any such equation is "a possible definition." The definition may also be of a whole expression rather than of a single sign. Then, given all such "possible definitions" in a science, we can arrange them in various orders, which will make different symbols "primitive," that is, indefinable in

terms of earlier ideas. It is convenient to select the order which uses the fewest primitive ideas. Finally it must be possible to replace any defined sign by its definition; otherwise the definition is incomplete. Thus definable terms are theoretically unnecessary in the formulation of the science, and are only a kind of abbreviation to be avoided when convenient.

The summary I have given above reflects the general uncertainty as to whether what is being defined is a symbol or an idea which appeared in Peano's paper. But it is quite clear that for Peano a symbol or idea was "primitive" only in relation to some particular deductive system. For him, "logical priority" was a relative and not an absolute term, which seems to be a point of conflict with Moore's metaphysics and with Russell's Philosophy of Leibniz.

Burali-Forti also discussed definitions, specifying three different ways of giving "logical definitions." For Burali-Forti, what is to be defined is an object. He also gave an explanation, but not a definition, of "definition": "To define any object x means: give one or several logical

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1C. Burali-Forti. Born 1861. I do not have a date of death. He is spoken of as a member of Peano's "school," and had published his paradox of the greatest ordinal number in 1897.

2It seems likely that "object" meant for him "object of knowledge."
relations containing $x$ such that, for a given element $y$, it is possible to assert or deny the relation $x = y$. In other words, $x$ is defined when all the properties of $x$ may be deduced from the logical relations in question.

The equality sign is used to mean identity in "all properties." However, he also spoke of a "nominal definition" as an equation between the object and "an expression formed with previously known elements," and as a definition of words. "Definition by postulates," the second type of definition, he described as definition of a group of objects or words by specifying the logical relations holding between them.

"Definition by abstraction" Burali-Forti thought of as a way of defining an operation. An operation is "any sign $f$ which placed, (for example) before some element $x$ of a, produces a well-determined element $fx \ldots". For example, \texttt{sin, cos, tang, log,} are operations for real numbers; \texttt{length of} is an operation for lines; \texttt{mass of, temperature of} are operations for bodies." The definition consists in a statement of the class to which it may be applied and which elements of the class are made identical to any element $x$ by the operation. That is, it is made clear for what $y$ or $y's x$ equals $fy$.

In the light of these explanations of various ways of defining, Burali-Forti proposed a precise usage for "concept"
and "intuition." A concept is any object which can have a
nominal definition, and whatever must be defined in one of the
other ways is an intuition.

1 Padoa's paper contained as part "une Introduction Logique
da une théorie deductive quelconque." He began with the idea of
class as primitive, and commented: "For us, a class is com-
pletely known when it is known which individuals belong to it."
The meaning of "x is an a" is, always, "x is an individual
belonging to the class a." Classes are equal when they con-
tain the same individuals, and individuals are equal, in
Peano's sense of logical identity, when they belong to the
same classes. The class not-a is the class containing every
individual which does not belong to a. Thus, "x is not a mem-
ber of a" means "x is a member of not-a." Therefore, the class
"a or non-a" includes every object.

For Padoa it was symbols which were to be defined by ar-
bitrarily selected non-defined symbols, in terms of which
the entire deductive system must be theoretically expressible.
There must also, said Padoa, be some undemonstrated proposi-
tions. The meaning of the symbols was unimportant. For,
said Padoa, although it is obvious that a deductive system
must gain any practical importance it may have only from the

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1 Padoa was an Italian mathematician of the group called
by Russell "the disciples of Peano." The paper mentioned here
is in the Bibliothèque du Congrès International.
"ideas" and "facts" its symbols may represent, where "facts" are "relations between the ideas," yet such interpretations are irrelevant, said Padoa, to logic. Logic is concerned "not with the empirical knowledge of the properties of things, but the formal knowledge of relations between the symbols." The formal deductive system is "an abstraction from all its interpretations" in the sense that a general theory is an abstraction from special theories. Any proposition proved for the general case is proved for the special cases under it; and the independence of the primitive symbols and propositions may be shown by finding proper interpretations which verify the system minus the symbol or proposition under test.

Pieri discussed "La Géométrie comme Système Purement Logique," a study which should be the consideration of abstract logical relationships, completely free, "like arithmetic" from any appeal to intuition. Its objects should be, said Pieri, "pure creations of our mind and its postulates simple acts of our will." That means that no appeal is to be made to "spatial intuition" or to the "evidence of the premises." Where sensuous imagery is impossible, as in symbolic logic, an appeal to evidence is senseless. We

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1Padoa, Bibliothèque du Congrès, 319.
2Mario Pieri, born 1860, another associate of Peano. His article is in the Bibliothèque du Congrès International.
3Bibliothèque du Congrès, 374.
4Ibid., 376.
5Such an appeal was for Pieri to sensuous imagery.
must simply trust, in such cases, to "the most strict observance of the rules of the pure deductive method, renovated and reinforced in our days by the algebra of Logic."

According to the committee for the Second International Congress of Philosophy, these papers by the logicians produced a great deal of interest in the subject. Traces of this interest are difficult to find in the journals, but the interest aroused in Russell soon showed its effects in print.

Russell had come to the Congress with new metaphysical convictions, which he expressed in his paper, "L'Idée d'Ordre et la Position Absolue dans l'Espace et le Temps." After the Congress, he studied Peano's works, which, by way of their impact on Russell now became influential among the professional philosophers. From the viewpoint of contemporary philosophers, the philosophers of mathematical logic before 1904 were Russell and Couturat. The other philosophies of mathematics were apparently not taken seriously by philosophers.

From the journals, the leading philosophical issue of the time appears to have been the debate between pragmatists and absolute idealists, who had a strikingly similar logical theory. Formal logic was looked down upon by both, and symbolic logic was even less acceptable. By adhering to one of the three opposing philosophies of mathematics, Russell inherited

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1Couturat regarded himself as a simple follower of Russell and Peano.
an argument with other mathematicians; by accepting "the mathematical method" at all, he also inherited a foreign war.

Some of the views of the logicians fitted very neatly into Russell's new metaphysics. Comparison of his original paper and the revised version he published in *Mind* the next year demonstrates both their compatibility and this historical independence.

The original paper combined the discussions later given in two separate articles. The second of the two articles is very nearly a simple translation of the later part of the original paper; the first article is devoted to a new exposition of the notion of series and of relations in general. It has little in common with the earlier discussion except a conviction that relations are ultimate and irreducible, at least in mathematical thought.

This article, "On the Notion of Order," began with the comment that modern mathematicians, especially Cantor, have so developed the theory of order that philosophers can no longer avoid recognizing its philosophical importance. The importance of these theories was, according to Russell, due to the fact that philosophers "have in general professed a theory of relations which, if it were correct, would render

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2Both articles appeared in *Mind* X (1901).
series logically impossible. Any possibility of introducing order into a set of terms, said Russell, "depends entirely upon the fact that there are sets of terms having intrinsic order, with which any other set of terms can be correlated."

Examples of intrinsically ordered sets which Russell gives are "the integers," "series of times," and "series of places."

"These have an order independent of our caprice — they form what I shall call independent or self-sufficient series. The casual terms correlated with them form, on the contrary, only a series by correlation." This distinction between two types of series also appeared in the earlier paper, but the new examples accepted as proven the point for which he then argued.

In the original paper, Russell also discussed formal properties of relations with some reference to the work of Schröder. This discussion of relations was revised and greatly expanded in "On the Notion of Order." Russell still stressed asymmetry as a necessary characteristic of order-generating relations but he suggested that transitivity may not be necessary, giving as a counter-example "the doubtful and complex case of genealogy."

Since both identity and diversity are symmetrical, Russell pointed out, it is obviously "formally impossible to reduce asymmetrical relations to identity or diversity."

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which would be necessary in order to reduce them to terms of substance and accident. Russell said that De Morgan had proved in a paper published in *Cambridge Philosophical Transactions*, Volume X, page 345, that all symmetrical transitive relations "are reducible, as may be proved in each particular case, to possession of a common property, or identity of content. This again, on examination, is found to consist of sameness of relation to the so-called common property."\(^1\) However, this does not hold for asymmetrical relations. Also, some relations have a peculiar property called "sense," which cannot be reduced to the formal property of asymmetry, since some relations having sense are not asymmetrical, notably in the case of "right and left."\(^2\)

It is this fundamental difference of sense, said Russell, which requires rejection of the philosophical dogma that relations are really internal states of things.

Russell presented a version of Moore's argument for the ultimacy of relations: Let A and B be related by an asymmetrical relation R, such as "greater than." We attempt to express this by the adjectives b and a. However, A has b only with respect to B, and B has a only with respect to A. Neither alone has either adjective. But, by hypothesis, they are otherwise identical. Therefore, A differs from B but

\(^1\) Russell, "Order," *Mind* X (1901), 32.

\(^2\) ibid., 57.
has no "specific point of difference," which is a contradiction, unless relations are not reducible to adjectives.

Neither can we adopt, argued Russell, "the monistic theory of relations, and say that they give really an adjective of the whole composed of the related terms" for "an analogous difficulty" arises. We must be able to distinguish the "whole" in which A is greater than B from that in which B is greater than A. But, said Russell, we can only do this by retaining an asymmetrical relation between parts of the whole. But in this case, we have failed to reduce relations to adjectives.

The article continued with a discussion of particular series which differs from the earlier paper but contains some points already discussed in the Revue de Métaphysique et des Morales, to which Russell made several references. He summed up the central point of his article as "to be an independent series is to have a distinguished place among entities."

Russell's article on position in time and space and his Congrès paper, like Leibniz, emphasized arguments against "subject-predicate logic," which he described as the view that every true proposition ascribes a predicate to a subject. Russell, modifying a point from Leibniz, argued in the article that such a view is self-contradictory, since it must both admit and deny that "there are predicates" is a true proposition. He concluded that belief in internality of relations "arises from neglect to observe the eternal self-identity of all logical concepts or Platonic ideas, which alone form the
constituents of propositions." Predication itself, he said,
is an irreducible relation; and no relation can ever modify
either of its terms.

Russell drew in his article on "position" a distinction
between "being," which is "a general attribute of everything,"
and "existence," which is a relationship some things have to
"existence." Both true and false propositions have being,
as have numbers and all "objects of thought." For, he held,
"all knowledge must be recognition, on pain of being mere
delusion; Arithmetic must be discovered in just the same sense
in which Columbus discovered the West Indies, and we no more
create numbers than he created the Indians . . . . Whatever
can be thought of has being, and its being is a pre-
condition, not a result, of its being thought of." 2

The Principle of Identity of Indiscernibles must, Russell
argued, be rejected, as depending on subject-predicate logic
and on the equally-mistaken views that there is a difference
of kind between subjects and predicates, and that all dif-
ference between subjects must be a difference of predicates.
For: "before two subjects can differ as to predicates, they
must already be two; and thus the immediate diversity is
prior to that obtained from diversity of predicates." He

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1. Russell's arguments against internal relations will
be considered at length later in this chapter.

2. Russell, "Is Position in Time and Space Absolute or
Relative?" Mind X (1900), 312.
argued that the distinction between the thing and its qualities, as well as the idea of "interaction" depends on "a false and scholastic" subject-predicate logic. All things eternally have all the relations which they have.\(^1\) Also, he said, Moore had proved that there is no absolute meaning to "necessity." However, some propositions are self-evident, such as the law of contradiction, and some propositions are "obviously untrue." They are to be accepted or rejected on these grounds alone.\(^2\)

The arguments advanced by Russell against the subject-predicate logic deserve to be considered at length, being as they are, a corner-stone of the new logic.

Subject-predicate logic, the position which Russell attributed to all earlier philosophers with the possible exception of Plato, was characterized by Russell as the assertion that there are two and only two ultimately different sorts of entities -- subjects, or substances; and predicates, or qualities -- the term used in each pair depending upon whether the point of view being taken in logical or metaphysical. One of Russell's arguments against this view was that relations

\(^1\)Russell, "Position," *Mind* X (1900), 315-317.

\(^2\)A comparison of Russell's views with the paper of Pieri, summarized on pages 103 and 104 of this dissertation will show that Russell's views were not shared by all logicians, whether or not they logically should have been. It is, however, very difficult to determine precisely how "Formalist" Peano and the other "Logicians" actually were.
are a third entity as ultimate as either of the other two. He used in addition to the arguments already presented, an argument that the relation of diversity must hold between substances, whether or not it is known to hold, so that relations cannot be the work of the mind. His argument that two substances could not differ in predicates unless they were already numerically diverse, and that the monist alternative is self-contradictory have been summarized above.

Russell argued as follows to prove the necessary externality of relations: if $R$ holds between $A$ and $B$, it must hold between precisely these two terms and not between a different pair of terms, say, $C$ and $D$. But if the relation modified its terms, this condition would fail.

"To say that two terms which are related would be different if they were not related, is to say something perfectly barren; for if they were different, they would be other, and it would not be the terms in question, but a different pair, that would be unrelated."

It appears from this argument, I believe, that while Russell explicitly rejected the Identity of Indiscernibles he was strongly convinced of the Diversity of the Discernible. That is, qualitative identity does not insure numerical identity; but qualitative diversity insures numerical diversity. From this principle, a metaphysics of instantaneous atoms or a Parmenidean One must follow, since there is no allowance for time or change. At the moment,
Russell and Moore seem to have been committed to the one eternal plenum of Platonic Ideas as the ultimate reality. Of the immediate logical consequences of his principles, Russell recognized and accepted the denial of change,¹ but was apparently in the process of changing his mind.

The arguments against any fundamental distinction between subjects and predicates must be extracted from earlier papers by both Moore and Russell. They had both argued that it must be possible for predicates to become the logical subjects of a proposition, which is sufficient to disprove any contention that there are entities which must always be predicates and never be subjects. This, however, as appears above, is not the conclusion they drew. What they asserted was that "there is no essential difference between subjects and predicates."² To support this proposition they require, it seems to me, an argument to prove that there is nothing which must always be a subject and can never be a predicate. I have so far found no argument adequate to this proposition in their writings. Their arguments would prove, although it is not explicitly pointed out by them, that relations can also be logical subjects, but this still does not give the desired conclusion. The only line of thought I have been able to postulate at this point leads back to Moore's view

¹Russell, "Position," Mind X (1900), 309.
²Ibid., 313.
that whatever is knowable is a concept. It is possible that from this they concluded that both subjects and predicates, as well as "things" and "ideas," were concepts and hence must be "essentially" identical. However, Russell had already abandoned Moore on this point. The argument against "subject-predicate logic" as it appears up to this time can, I believe, only be regarded as a non sequitur, unless the shifting meanings assigned to "subject-predicate logic" qualify it rather as a fallacy of equivocation.

After the great impression made on him by the mathematical logicians at the Congress in 1900, Russell began a study of Peano's work. He denied in a later article that the logic of Peano was a variant of the Algebra of Logic or of the Universal Characteristic which he had criticized in Leibniz, rather than abandoning his scorn for those pursuits. Peano's logic was, he said, "distinct from and logically prior to, the subject which Leibniz calls 'Universal Mathematics' and which Whitehead calls 'Universal Algebra.'" Universal Algebra studies "deductions from the assumption of a synthesis obeying such and such laws, but otherwise undefined;" it "employs deduction and the logical kinds of synthesis, which are explicitly dealt with in the Logical Calculus." The formal laws of the Universal Algebra are, said Russell, to be stated by logic in terms of logical constants. He had, he stated,

1Russell, "Recent Work on the Philosophy of Leibniz," Mind XII (1903), 187.
succeeded in recent work in reducing these constants to eight. In terms of these "every notion occurring throughout the whole science can be defined. Thus all mathematics is merely the study of these eight notions; and the Logical Calculus is a name for the more elementary parts of this study." However, he added, "due to the fact that propositions are synthetic" the propositions which must be accepted as indemonstrable in mathematics "instead of being one, appear to number about twenty." He stated that in logic "indemonstrable" was to be understood as meaning "not demonstrated." But we could not test, he said, any one of these principles by supposing it false and deducing absurd consequences from the supposition. For "all our axioms are concerned with the principles of deduction, so that, if any one of them be true, the consequences which might seem to follow from denying it do not follow as a matter of fact. Thus from the hypothesis that a true principle of deduction is false, valid inference is impossible."^2

Russell also mentioned the Kantian philosophy of mathematics in passing. He was sure that it had been "conclusively disproved," no longer by Bradley but by modern "logicized" mathematics. Presumably his reasons for this belief were

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^1 I believe he meant "organic unities with emergent properties."

those later given by Couturat in his once-and-for-all refutation.\(^1\) The labors of Weierstrass, Cantor, and Peano, said Russell, had conclusively proved that: "algebra and geometry do not contain an independent contribution to method, and mathematics is concerned solely with logical implications."\(^2\)

The absolute self-identity of concepts is, said Russell, necessary to explain change, for "change in an identical content means difference in its relations to different moments of time; but the content must remain strictly self-identical, and this self-identity is logically prior to change, not subsequent to it."\(^3\) "The constancy of law" is "absolute timeless self-identity" of a relation. Thus, said Russell, the new mathematics of Weierstrass and Cantor shows how plurality and change can be admitted even though "the Eleatic postulates as to the rational conditions of being" are accepted.\(^4\)

As yet, no contributor to Mind had been inspired to take issue with Moore and Russell. However, there was dissatisfaction expressed in other journals, notably in France.

Pierre Boutroux,\(^5\) writing a review of Russell's A

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\(^1\) L. Couturat, "La Philosophie des Mathématiques de Kant," Revue de Métaphysique et des Morales XII (1904).

\(^2\) Russell, "Recent Work," Mind XIII (1903), 192

\(^3\) Ibid., 194

\(^4\) Ibid., 197

\(^5\) 1850-1922. I believe he was the "mathematician" mentioned by E. T. Bell in Men of Mathematics (New York: Simon & Schuster, 1937) p. 532, the nephew of Henri Poincaré.
Critical Exposition of the Philosophy of Leibniz. accepted
Russell's criticism of Leibniz in that book as based upon
"a profound study of general logic and especially the logic
of mathematics."¹ Actually, the criticism of "subject-predi­
cate logic" contained in Russell's book is not based upon the
imposing logical system of Principia Mathematica. Leibniz
antedates Russell's studies of the logic of mathematics. It
is doubtful that Russell could properly have been regarded in
1901 even as a representative of the "logic of the sciences"
which Boutroux wished to criticise, although he subsequently
joined Couturat in that position.

It was the identification of logic with metaphysics to
which Boutroux especially objected in Russell's book. For
instance, he noted that Russell insisted that the only philo­
sophical reason for introducing a metaphysical substance was
the need for a logical subject of propositions. Boutroux ob­
jected: "It is not allowable, coming after Descartes and Leibniz
to be so dogmatic; for these philosophers have shown that the
world of mathematics need not be identical with the actual world.
Thus purely logical considerations have no weight in meta­
physics, and we are not able to admit without more explanation
the criticisms which Russell addresses to Leibniz."²

¹P. Boutroux, "Exposé Critique de la philosophie de Leibniz
par B. Russell," Revue de Métaphysique et des Morales, IX (1901),
331
²Ibid., 333
Boutroux' central objection to mathematical logic and the logicians appears later in the review: he believed them to be too "nominalistic." They failed, he thought, to recognize the independence of mathematical fact. Mathematics is not, Boutroux said, a "free creation of the human spirit," but is a recreation of an "ideal" world. Clearly, Boutroux here attributed to Russell the opinions expressed by Pieri at the Congress and earlier by Dedekind rather than the extreme Platonism which Russell and Moore were advocating at this time. Perhaps Boutroux failed to recognize the Platonism of his opponents because of his conviction that "logicians" were logically committed to a "Lockean" view of mathematical thought. He contrasts his own view that mathematics studies relations of necessary connection between ideas where the connections found are dependent upon the ideas studied with the view of anyone who believes that formal logic is possible. To believe in formal logic, said Boutroux, it is necessary to believe that mathematics deals only with simple ideas extracted from experience and with arbitrary combinations of those ideas. The combinations must be thought of as "free" contributions of the mind, ways of manipulating ideas which may be isolated and analyzed, and which can proceed in the same way no matter which simple ideas are fed into the machinery. Boutroux thought logicians "even worse than Kantians," since they believed that even the patterns of
manipulation were arbitrary rather than set by the nature of the mind. For anyone sharing Boutroux' opinions of the nature of necessary connections between ideas, any study of necessary connections in general would be impossible, and with it any formal logic.

Other philosophers of the period also opposed the "mathematical method" in print. A series of articles by Yves Moisant in the Revue de Philosophie gave a particularly detailed statement of the objections, and although its inclusion at this point involves a slight breach of chronology, it provides a useful supplement to the argument between the logicians and Boutroux.

Moisant saw the logical philosophy of Russell's Principles of Mathematics as just another example of the "mathematical rationalism" which had characterized all modern philosophy since Descartes. An ideal of "one science, one method," of a unified deductive system, united all "mathematicism" from Descartes to Couturat, according to Moisant. These mathematicist philosophers, whether rationalists or empiricists, searched for "the ultimate element, the atom." They differed only with regard to whether the elements were to be facts or ideas. All of them expected to construct everything from the simple elements, which could have to one another only the simple relations of "juxtaposition or identity." The "scholastic view" that there could be terms
at once "really distinct and intrinsically united" was sup­
planted by the modern "reductionism" and "mathematical method." But, argued Moisant, the mathematical method could never give
that vision of the unity of the many which was the true goal
of philosophy.¹

Unlike most philosophers who shared his opinions of the
mathematical method, Moisant thought that logicism should be
taken seriously as a philosophical position. Also, he was
apparently quite sure what the method of mathematics was.
However, the most lively debates in which logicians were en­
gaged during this period were not with philosophers but with
other mathematicians, and were about the nature of the method
of mathematics.

The questions: "Does the mathematical method give us
knowledge of reality?" and "What is the mathematical method?"
are quite distinct from one another, Moisant discussed the
first, Boutroux the second. But it was still another question
which preoccupied Russell's discussions: "Given that the
mathematical method gives us knowledge of Reality, what, then,
must be the nature of Reality?" Of course, a satisfactory
answer to Russell's question required prior agreement as to

¹Y. Moisant, "Un caractère de la philosophie moderne,
le mathématisme," Revue de Philosophie (Mai, 1904). Also
his "La Pensée philosophique et la Pensée mathématique,"
Rdp (Janvier, 1905), and his "La Pensée philosophique et la
Pensée mathématique," Rdp (Fevrier, 1905).
what the mathematical method is and whether it does give knowledge about Reality. The logicist interpretation of the mathematical method as deduction from the principles and primitive concepts of formal logic contained two controversial points: (1) the deductive nature of mathematics; and (2) the source of its fundamental premises and concepts.

In the subsequent arguments about whether intuition had been banished from mathematics, both sides seem to have ignored the distinction between these questions. Russell argued that the non-intuitive nature of mathematical reasoning is proved by a proof that mathematics introduces no new methods of reasoning. Some of his opponents argued that the intuitive character of mathematical reasoning is proved by the fact that it introduces new concepts and rests upon undefined primitive postulates, whether those postulates are to be regarded as mathematical or logical. The two positions are perfectly compatible, except for the decision as to whether or not such reasonings are "intuitive".

There was also, however, perfectly genuine disagreement about the methods, as distinct from the content, of mathematical and logical reasoning. This appears in the later arguments with Boutroux and with Poincaré, who denied Russell's thesis that all valid reasoning is deduction — that is, is analytic in Couturat's sense of following from given premises by the laws of logic without requiring further information.
The question as to whether or not mathematics reveals anything about the nature of reality was one on which there was little unanimity among the logicists themselves. Pieri thought of mathematics and logic as "free creations of the human spirit"; so, apparently, did Couturat—at least initially. But for Couturat, all Reality was a creation of the human spirit. Russell's argument was that given the new theory of infinite numbers it was no longer necessary to suppose that mathematics did not give knowledge of Reality. The arguments that it did give such knowledge involve a return to the old question of the status of the Platonic Forms. Here the Platonist wing of the logicians and the Kantians stood opposed as Realists and Conceptualists; but there were also Conceptualist logicians, such as Pieri, and Platonist Kantians, such as Boutroux. This was possible because the issue between deduction and intuition as theories of mathematical method was also regarded as a conflict between logicism and Kantianism.

Russell's view at this time seems to have been that logicism as a theory of mathematical method required Platonism as a theory of the nature of Reality, taking for granted that mathematics gives insight into the nature of Reality. Although many logicists were Platonists, the connection between the two does not seem logically necessary. Probably Platonism and
logicism were actually united in the eyes of many philosophers by the fact that after 1900 Russell accepted and argued for both.
CHAPTER VIII

LOGICISM AND LOGISTIC

The publication of Russell's *Principles of Mathematics* in 1903 marked a highpoint of "logistic" enthusiasm in philosophy, although it probably showed a reduction in Russell's own hopes. Couturat entertained great philosophical expectations; and the results Russell expected to obtain from his extension of the mathematical and logical developments of the last half of the nineteenth century were striking enough. Because of Peano's work in symbolic logic, said Russell: "the Kantian view, which asserted that mathematical reasoning is not strictly formal, but always used intuitions, i.e., the a priori knowledge of space and time", was now "capable of a final and irrevocable refutation." Contrary to Kant's view, "all mathematics can be strictly and formally deduced" from "ten principles of deduction and ten other premises of a general logical nature", and "all the entities that occur in mathematics can be defined in terms of those that occur in the above twenty premises."

The mathematics to which he is referring, Russell continued, is not to be confused with, for example, Euclidean geometry, the actual propositions of which "do not follow from the
principles of logic alone; and the perception of this fact led Kant to his innovations in the theory of knowledge." For "the question whether the axioms and demonstrations of Euclid hold of actual space or not" was no concern, said Russell, of pure mathematics. The proposition, "given the Euclidean axioms, the Euclidean propositions follow" is, however, a proposition of pure mathematics. This type of mathematics always deals with relations of formal implication between propositional functions constructed only of variables and logical constants. "The fact is that, when once the apparatus of logic has been accepted, all mathematics necessarily follows." The task to be undertaken in Principles was, then, as Russell saw it, "to show that all mathematics follows from symbolic logic, and secondly to discover, as far as possible, what are the principles of symbolic logic itself." Russell's thesis in the book was, then, "that all pure mathematics deals exclusively with concepts definable in terms of a very small number of fundamental logical concepts, and that all its propositions are deducible from a very small number of fundamental logical principles." It was, he noted, controversial.

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among mathematicians, but among philosophers "it is almost universally denied."

The purely mathematical part of this proposed program had been carried on for some time, as is shown above in this dissertation, with associated philosophical discussions. However, apparently Russell and Couturat were the first or the most influential philosophers who saw it as a weapon with which to overthrow an entrenched philosophy, "Kantian" for Couturat and "Hegelian" for Russell, which was the enemy of all clear-thinking.

In Principles Russell stated his view that all mathematics is a form of logic as a truth by definition, so long as the discussion was limited to pure mathematics. He recognized that the definition stood in need of a defense, and proposed to justify it by showing "that whatever has, in the past, been regarded as pure mathematics, is included in our definition, and that whatever else is included possesses those marks by which mathematics is commonly though vaguely distinguished from other studies. The definition professes to be, not an arbitrary decision to use a common word in an uncommon signification, but rather a precise analysis of the ideas which, more or less unconsciously, are implied in the ordinary employment of the term." The questions of philosophy of mathematics would, said Russell, thus be reduced to questions in
philosophy of logic where "the discussion must be resumed by philosophy."¹

Peano's logic, said Russell, had already shown the formal deducibility of all pure mathematics. In this field, no statement depending for its truth upon "the particular nature" of its terms would appear; the only constants of pure mathematics were, said Russell, relations.² These relations were to be classified according to the types of logical deductions they made possible. Logical constants must be, said Russell, relations which hold even when every term in the proposition asserting them has been changed into a variable having "an absolutely unrestricted field: any conceivable entity may be substituted for any one of our variables without impairing the truth of our proposition."³ Russell still thought of propositions as "complex concepts,"⁴ which might be asserted both psychologically and logically.

It was the logical notion of denoting which Russell now saw as the real root of "all theories of substance, of the subject-predicate logic, and of the opposition between things and ideas, discursive thought and immediate perception."⁵

¹Russell, Principles, 4.

²Part of Russell's task was, accordingly, to prove that "if x is the square root of 1, then x is 1" is not pure, however mathematical it may be.

³Russell, Principles, 7. ⁴ibid., 34-35.

⁵By inference, he rejected all these distinctions.
Bradley's view was, said Russell, "that all words stand for ideas having what he calls meaning, and that in every judgment there is something, the true subject of the judgment, which is not an idea and does not have meaning." Russell rejected this as a confused compound of psychology and logic, and as based on the out-moded subject-predicate logic. His proposed alternative view was: "words all have meaning, in the simple sense that they are symbols which stand for something other than themselves. But a proposition, unless it happens to be linguistic, does not itself contain words: it contains the entities indicated by words." The kind of meaning having philosophical importance, and which is indicated by denoting, is, said Russell, a characteristic not of words but of a peculiar kind of concepts which are "symbolic in their own logical nature."

For Russell, "concept" was no longer used to refer to any object of thought, as it had been for Moore. Neither, of course, did he think of them as in any sense mental. They were, he specified, one of two different types of "term."

"Term" was introduced by Russell as synonymous with "the words unit, individual, and entity." It was to apply to "anything . . . that can be mentioned." But, among terms, we were to distinguish between concepts and things.

1This statement was later modified by the introduction of the still-wider "object." See Principles, page 55.
All terms, said Russell, are "in fact, possessed of all the properties commonly assigned to substances or substantives. Every term, to begin with, is a logical subject: it is, for example, the subject of the proposition that itself is one. Again every term is immutable and indestructible. What a term is, it is, and no change can be conceived in it which would not destroy its identity and make it another term." But, among the units or individuals which "can never occur otherwise than as a term in a proposition," Russell noted one group with the peculiar characteristic of being able to occur in propositions either as subjects or in another way, in which the proposition was no longer about that concept. The examples Russell gave were, "Socrates is human," and "Humanity belongs to Socrates." These are, he said, equivalent but distinct propositions; only the second of the two is "about" humanity.

This appears to me to be a statement that some terms can only be subjects of propositions, but others can be predicates, or, to give an equivalent statement, that there are some terms which must be always a subject and never a predicate. If Russell's assertion that metaphysical substances are merely logical subjects were taken seriously, he would seem to have little quarrel left with the subject-predicate logic. Certainly, all concepts are no longer "essentially identical." However, the terminological embarrassment of such a statement was avoided by giving a restricted meaning to the word "predicate:" "Predicates, then, are concepts, other than verbs, which occur in
propositions having only one term or subject." The reasons which had led Russell to abandon the original views of Moore were not given; and the amount of change that had taken place in Russell's views was obscured by the continued attack on Bradley, who had rejected subject-predicate logic himself. I suspect on the basis of Russell's discussion of denoting concepts that he wished, in contrast to Moore, to read back into the logical notion of a subject all that was contained in the metaphysical notion of a substance. Moore had originally intended, it seems to me, to denude the substance of all non-logical properties. But Russell's objections to such concepts as "a man" as real subjects, and hence as ingredients in propositions, seems to be that they are not substantial enough.

At the time of writing Principles, Russell believed in the existence of complex wholes, called unities, which were composed of but not simple juxtapositions of simple ideas. Only propositions, he said, had this kind of unity, and were thus not completely determined as soon as their constituents were known. Also, they could not be reconstituted by mere enumeration of their parts. Such unities always involved a relation actually relating terms, said Russell. Therefore, analysis of propositions was always "falsification," although Russell noted that he meant by this only that "though analysis

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1 Russell, Principles, 139. His opponents' charge that he did not recognize such unities was untrue. It is possible that he was, however, inconsistent in doing so.
gives us the truth, and nothing but the truth, yet it can never give us the whole truth."

But classes were a problem. Russell suggested that they must be only aggregates, if they could be thought of as wholes at all: "classes must be taken extensionally to the degree involved in their being determinate when their members are given." Yet, "a class seems to be not many terms, but to be itself a single term, even when many terms are members of the class. This difficulty would seem to indicate that the class cannot be identified with all its members but is rather to be regarded as the whole which they compose." Classes can also, said Russell, be logical subjects of propositions. But whatever is a logical subject is a term, by his earlier definition. In what is either a delicate distinction or a contradiction, Russell suggested in this difficult situation that a class "of many terms can be a logical subject without being arithmetically one,"\(^1\) although it must be, according to the earlier discussion, a unit in order to be a term.\(^2\)

Introduction of this complication into the treatment of classes was due to Russell's discovery of the paradox of the


\(^2\)Ibid., 43.
class of all classes which are not members of themselves. Principles of Mathematics, accordingly, was a revision of Russell's original beliefs in an attempt to deal with an unexpected difficulty. His opinions about the status of logical principles as well as the details of his logical system were apparently changing rapidly as he attempted to resolve the problem. The articles published during this period by Russell and his philosophical opponents give a better picture of this phase of the development of logical philosophy than Principles itself, for this reason. But the logicism of Principles, its identification of logic and mathematics, Russell believed had been firmly established in spite of all paradoxes.

Logicism was a philosophy of mathematics asserting the unity of mathematics and logic and the priority of logic. Russell's version involved holding that all mathematics could be strictly and formally deduced from principles and premises of logic and all mathematical entities defined in terms of the entities occurring in the logical principles. He was not, however, simply a logicist. Russell had come to accept Peano's symbolic approach and to use the "new language" of pasigraphy. He was, therefore, also an advocate of "logistic."
Logistic was a theory about the nature of all valid reasoning. The term was suggested at the Second International Congress of Philosophy by Couturat and others as a replacement for "mathematical logic." It was intended to apply to the work of Peano and his associates, and also to Schröder and Peirce, that is, to those logicians who were devoting themselves to the construction of a new and precise language which would replace for the purposes of reasoning the vague and ambiguous expressions of a natural language. The linguistic aspect of logistic was essential to it, at least in Couturat's estimation. For example, he denied that Hilbert was a logistician on the grounds that Hilbert never used any symbols in his works. There is reason to believe that it was really the linguistic aspect of logistic which attracted Couturat.

If logistic were to be merely the creation of a language, it would seem that the question of what was to be expressed in the language should be answered independently. However, it happened to be the case that the logicians were attempting to translate mathematics into the new language rather than, for example, political theory. Indeed, Couturat at first regarded this narrowness of interest on the part of logistic as deplorable; after all, a Universal Characteristic should be universal. But the peculiar characteristics of the proposed language naturally tied it
closely to mathematics, since Peano's pasigraphy was an adaptation of the Algebra of Logic. The new symbolic language was mathematical. But if the techniques of a calculus should be found suitable to the expression of all valid reasoning, it would be because all valid reasoning was calculation.

Logicism claimed that all mathematical reasoning could be reduced to formal logic. But it would be possible to be a Logicist without holding that all reasoning is reducible to formal logic. However, if mathematics could not be so reduced, the more general claim would also fail. Therefore, it was reasonable for the argument to center upon the unexpectedly difficult attempt to translate mathematics into logic. The logistic enterprise, oddly enough, was actually attempting to reduce logic to mathematics, as a way-station in reducing mathematics to logic and all valid reasoning to mathematics. It might be likened, unkindly, to an effort to prove that "eggs is eggs;" and similarly the logistic assertion, while equivalent to "mathematics is mathematics" cannot be taken as a simple tautology.

It was clearly essential to the logistic program that all "entities" of mathematics be definable in terms of the "entities" of logic. Therefore, the problem of definition was a major issue. However, it is extremely difficult to determine just what either the logisticians or their opponents
thought was being done when a definition was presented. Peano's use of the term has been discussed above. Russell in *Principles* specifically stated his intention to use it in one way, but in the course of the book also used it in another.

Definition of a "propositional function"—a complex concept of a relation holding between variables which may be transformed into a proposition involving the same relation—was to be taken as fundamental, Russell stated.\(^1\) The definition of denoting concepts, for example, is by showing how they are derived from class-concepts. Class-concepts, in turn, were to be derived from propositional functions: "and a is a class-concept when "x is an a" is a propositional function."\(^2\) Propositional functions, or some of them, could also be defined. That is, they could be stated as implying and implied by "a propositional function which has either been accepted as indefinable or has been defined in terms of indefinables."

Assuming that Russell accepted the same view toward equivalence that Whitehead expressed in *Universal Algebra*, he was not here asserting the identity of the defined concept and the defining concept but merely their interchangeability.

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\(^1\)Russell, *Principles*, Sect. 16.

\(^2\)Ibid., Sect. 73.
in certain contexts for certain purposes. Presumably, Russell's propositional functions would be thought of as interchangeable in the context of a logical argument for purposes of logical deduction. Russell's defense of his definition of disjunction and negation supports this interpretation. There he stated that his definitions do not "give the true philosophical analysis of the matter." Again, he pointed out on page 27 that the definability of mathematical notions in terms of logical ones was not by any means, as it was in philosophy, "an analysis of the idea to be defined into constituent ideas," but rather "pointing out a fixed relation to a fixed term, of which only one term is capable: this term is then defined by means of the fixed relation and the fixed term. The point in which this differs from philosophical definition may be elucidated by the remark that the mathematical definition does not point out the term in question, and that only what may be called philosophical insight reveals which it is among all the terms there are."

The sense in which all mathematics was to be definable in terms of mathematical logic appears, then, to be the sense in which all the regions of the earth are definable in terms of their latitude and longitude. The sense in which mathematical propositions were deducible from the propositions of

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logic could then be that they were separate segments of the same deductive system. The deductive system in question would be a pre-existing, independent structure, both its elements and their relations being independent of our "perceiving" them. **Inference** would be perception of the relationships in this system; in fact: "the mind is as purely receptive in inference as common sense supposes it to be in perception of sensible objects." But, with introduction of **inference** as a way of knowing the system, there is need for a more complex theory. Whatever the structure of the system might be in itself, at least some of the relationships in it sometimes come to be known. The argument between logicians and their opponents also concerned the way such knowledge was to be acquired. Russell insisted at this time that it was to be obtained only by **deduction**.

It is apparent from the above discussion of definitions that in Russell's usage each definition required a new act of "philosophical insight" as its supplement in order that the term defined be "perceived." Yet Russell maintained that his theory banished "all appeal to intuition" from mathematics. One way of interpreting this claim might be as maintaining

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only that no appeal to intuition need be made during the actual process of reasoning. Perhaps all valid inference was to be the following out of set patterns of manipulation which required no attention to the character of the idea manipulated, in which each term counted as one, like each Utilitarian man. I believe this was actually part of the view he wished to maintain, and it was the view ascribed to him by contemporary opponents. However, it must be recognized that Russell and Couturat also insisted upon a distinction between "intuition" and "philosophical insight" or "perception" of concepts. The "Kantian or sensuous intuition" was opposed by them to "intellectual intuition" which they accepted.

Russell’s position may also have involved a belief that the perception of a term was irrelevant to its function in the process of inference. Indeed, it must have done so to be consistent. For the interchangeability in a process of deduction of non-identical equivalents is based upon an assumption that some differences in the perceived nature of the terms do not affect their use in deduction. Russell was neither required nor entitled to claim that logically equivalent concepts have only logical properties. He could maintain as a reasonable and modest thesis that only logical properties are relevant to logical reasoning. Other characteristics of the perceived entities would then be irrelevant to logic without it being necessary to deny that what can be defined
may nevertheless also be an entity in its own right. However, in other passages Russell asserted that defined entities are not "fundamental"; and his opponents were not unreasonable in assuming that he meant to deny them real individuality. That is, he appeared to believe that defined entities were actually illusory, having no unity or emergent properties. Even a heap of bricks has some properties other than those belonging to its component bricks, such as its shape. There is an obvious change from the locating concept of definition to a peculiar sort of creation theory in these passages. I think a fairer interpretation would be that he thought of the defined entities as having a unity, but not an ultimate, or dependable one. Why their peculiar individual characters should be dismissed as irrelevant to all reasoning about them I do not understand. But if we were trying to find out only what place they occupied in a network of logical relationships these characteristics could seem irrelevant. I can, for example, locate the latitude and longitude of Budapest and Hong Kong and calculate the distance between them with no attention to the peculiar qualities of either city. The issue between logicism and intuition, on this interpretation, would be whether all the relationships studied by mathematics were of the sort that required no perception of the actual terms.
There was no agreement that even logic itself studied relations of this kind, which I believe could properly be called "formal" relationships. But the immediate controversy stemmed from the logistic contention not that logic was a formal science but that mathematics was so, too. They were committed, too, it seems, to arguing that biology was a formal science; but only the mathematicians rose to object. They were most immediately concerned, of course, because of the close relationship between logistic and logicism.

In *Universal Algebra* Whitehead had maintained that some logic was a branch of pure mathematics—specifically, that the algebra of logic was a branch of universal algebra. But he there carefully distinguished the algebra of logic from "The Art of Reasoning." Whitehead's definition of pure mathematics as given in that book required that all formal deductive logic be considered a branch of pure mathematics, since pure mathematics was to include "all types of formal, necessary, deductive reasoning." Russell's definition of mathematics in *Principles* required, on the contrary, that pure mathematics be derivative from logic, since it was to contain all and only "propositions of the form 'p implies q,' where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants." But the definition also required that logic itself be contained in
pure mathematics, at least to the extent that logic itself can be expressed in a deductive system. Couturat had been convinced by his study of Whitehead that mathematics was the general logic of all reasoning, a conclusion which was not, as has been pointed out above, the view asserted by Whitehead. Couturat's opinion was that all reasoning was mathematical, which is not the same as a view that all mathematical reasoning is an application of the principles of a deductive system. It differs even more greatly from the view that all mathematical reasonings can be translated into the deductive form and that all mathematical ideas obtained in the reasoning process will be "reducible" to the primitive ideas of the logical system.

Any of the three views on the relation of logic and mathematics could be made a truth-by-definition by specifying that we are discussing only valid reasoning and giving an appropriate definition of validity. But these views were offered not as definitions of valid reasoning in mathematics but as truths about valid reasoning. Conformity to the principles of a logical calculus was not to be regarded as the only criterion of good reasoning; rather, reasoning independently recognized to be valid was expected to also be valid by the criteria of the deductive system of mathematical logic. Thus, the deductive system and its rules would appear to be supererogatory, or at best to have a
purely aesthetic value. The non-deductive recognition of good reasoning would be actually taken as fundamental, and in case of conflicts between that good judgment and the deductive system it should be the system which was to be discarded or amended. Yet, Russell at this time asserted the contrary; in case of a conflict between logic and common sense, it was the latter which should "commit suicide." I believe his paper in 1913 gives a clearer statement of precisely what Russell hoped for from the deductive system, although I am not sure he was quite clear about his aims himself in 1901. The system was to be the guide in reasoning beyond the area where the self-evidence of the logical connections could be seen. In order to learn the correct deductive pattern to apply to the world, it was necessary to appeal to intuitive evidence as the data which might lead us to discover the correct laws of inference. But these laws, once found, could then be used to correct observation, like a scientific law. Objections to such a program could, then, take the form of denying that a proposed deductive system was the correct one; or of denying that any such general laws of inference exist. The subsequent discussions produced both types of objection.

Clearly, if "logicism" were to be defined in the weak sense later suggested by Russell, that is, merely as asserting the unity of logic and mathematics, even a
philosopher who held that deductive systems have a purely aesthetic value could be a "logicist". He would need only to assert the theoretically-possible unification of mathematics and logic, without considering why such a translation should be attempted. But a less ascetic view which expresses a belief that reduction of mathematics to logic is for some reason desirable is a position having greater philosophical interest. Also, it seems historically true that Russell and Couturat originally wished to advance the deductive system as an ideal form of reasoning, in terms of some ideal. Couturat obviously intended to claim that deductive systems were a standard to which all good reasoning must conform-- that only thinking which fit the pattern of the universal algebra was logical.\(^1\) It must be remembered that for Couturat, as for Whitehead, logical thinking was not the "Art of Reasoning"; it was for him only an inferior segment of that art, a process attributable to the lower faculty of Understanding. Couturat's position appears less extreme under that qualification. However, the explanation of his metaphysics and epistemology is embedded in \textit{De L'Infini Mathématique}, a book of about the same size.

\(^1\)Of course, if the union of logic and mathematics is not thought of as symmetrical, Couturat's position is not a logicist thesis at all, but the converse of such a theory.
as Volume I of Principia Mathematica and bristling to only a slightly lesser degree with mathematical formulae. Probably not many of his philosophical opponents had struggled through it.

In his much less frightening book, Principles of Mathematics, Russell stated the view that "all mathematics is logic," and Couturat endorsed his view whole-heartedly. Russell also maintained that all good reasoning was deduction,¹ without making special reservations in favor of reason. Apparently these qualifications soon faded from the view of even Couturat himself, and the opponents were clear sighted about the basic issues in ignoring it.

The position of this branch of logicism after Russell, and the view which most philosophers referred to in discussing the movement, appears to have been: that mathematics is a formal deductive system; that it is a deductive system based solely upon the concepts and principles of symbolic logic; and that only deductive reasoning is logical. The further conclusion that only logical reasoning is philosophically respectable was probably taken for granted by all concerned at least part of the time, although both Russell and Couturat were actually committed to its denial (principles of reason and self-evident truths are not arrived at by deduction.)

¹B. Russell, Principles, 11.
Philosophical reactions to *Principles of Mathematics*, as indicated by contemporary journals, ranged from enthusiastic approval to complete rejection. Couturat thought it a landmark in all philosophy. Milhaud took a middle-of-the-road position, commenting that it was not surprising that Russell should succeed in deducing number concepts from a logic which already contained number concepts in its primitive propositions. But Hausdorff dismissed the book as a piece of hair-splitting scholasticism: "Ein scholastischer Scharfsinn, der eingebildete Probleme sieht und die wirkliche übersieht, feiert in diesem Werke seine Orgien der Subtilität." Reading it he regarded as simply a test of patience, since its discussions were "spitz und doch nicht klar." He complained that rather than presenting a formal deductive system such as Peano's, where the primitive ideas and principles were selected in a purely arbitrary manner, Russell displayed "eine im mittelalterlichen Sinne realistische, eine apriorististische Tendenz, die uns swingen will, die characteres des Fundamentalen und des Abgeleiteten in bestimmter Weise zu verteilen." Among the foolish questions he found Russell discussing was: "ob ein Ganzes nur die Summe seiner Teile oder noch etwas außerdem

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sei." All such questions are nonsense, said Hausdorff, at least to a mathematician, and require no answer "soweit man mit ihnen Mathematik und nicht Metaphysik treiben will." He also found some technical innovations in Russell's logic which he thought were a poor idea. He blamed the introduction of unlimited substitution of variables and of formal implication for the contradictions, and objected because Russell's definitions always involved negative terms and "incompletable totalities." However, Hausdorff granted Russell one bit of mathematical understanding; he understood the modern concept of infinity.

At the other extreme, Couturat, who had no objection to a philosopher's doing metaphysics rather than mathematics, hailed Russell's book with delight. From this time on, he characterized himself as a mere follower of the great logicians, Russell and Peano. He published a long series of articles, "Les Principes des Mathématiques," which was explicitly presented as simply a summary and recapitulation in French of Russell's book. However, in his summary Couturat also added to the precision of the views expressed and made the metaphysical statements more uncompromising.

Couturat stated that modern logic and mathematics, as a simple matter of fact, do coalesce in set theory, group theory,

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and theory of functions, so long as mathematics is recognized to be a pure "hypothetico-deductive" system. It still remained, he said, in contrast to Russell's statement that it had been done, to prove formally that all mathematics, as such a set of "hypothetically necessary" propositions, was deductively obtainable from the principles and concepts of logic. However, he said, the proof would be given in a forth-coming book which Russell and Whitehead were preparing.

Couturat, again in contrast with Russell, regarded the axioms of a deductive system as essentially arbitrary, sufficient but not necessary as foundations for the consequences to be drawn from them and neither claiming nor requiring any superior degree of self-evidence. Definitions he spoke of as "conventions of notation, abbreviations." Also, he asserted that the principles of logic discovered by modern logistic were reflections of the actual processes of all valid reasoning. "Undoubtedly these principles have been unconsciously applied for centuries by all men who reason properly, and especially by the mathematicians; but they have not been noted and explicitly formulated until our day, and it is in this that Logic really consists (as a science, not as a natural disposition of the mind)."

The ideal of logic, as Couturat saw it, was to formulate all the valid ways of reasoning used in science or in everyday life. He expected that logistic would be able to achieve this goal, and that it could be achieved only in a symbolic language. The symbols of modern logic were the only practical way to insure that reasoning was kept purely formal: "that is, to separate the form of the arguments from their content, to make it independent of the meaning of the terms and propositions." The efficacy of the symbolic method had, Couturat pointed out, already been demonstrated by its success in recognizing the invalidity of four modes of syllogistic reasoning (universal to particular) which had passed as valid for centuries because of their verbal form. Other fundamental errors in traditional logic, such as its equating the three laws of thought and its founding all syllogistic reasoning on assertions of identity have also been uncovered. Furthermore, stated Couturat, it had been possible for Russell and Whitehead, starting from purely logical principles, to demonstrate formally all of Cantor's set theory: "thus confirming the logical validity of that theory, and purging it of all postulates and all appeals to intuition."

Couturat, who seems to have been unconcerned with the more technical and mathematical side of logistic, saw the

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demonstration of set theory simply as a triumph. For Russell, however, there were other implications. By this route, the paradoxes of set theory became problems of logic; for a logical system having contradictory consequences is an untrustworthy system. If the proofs were valid, and if set theory contained internal contradictions, the logical system from which it had been deduced also stood condemned until its flaw was found and eliminated.
CHAPTER IX

LOGISTIC AND INTUITION

THE SECOND CONGRESS OF PHILOSOPHY

Russell’s *Principles of Mathematics*, was considered by him "addressed in equal measure to the philosopher and the mathematician."¹ In it he proposed to demonstrate with "all the certainty and precision of which mathematical demonstrations are capable," the thesis that "all pure mathematics deals exclusively with concepts definable in terms of a very small number of fundamental logical concepts, and that all its propositions are deducible from a very small number of fundamental logical principles."² This thesis was, he stated, "very recent among mathematicians and is almost universally denied by philosophers." His views were enthusiastically supported by Couturat, who, as a fervent disciple of Russell and "the new logic," played an important part in the presentation of these mathematical

¹Russell, *Principles*, xvi.

²Ibid., xv.
studies as a properly philosophical subject matter. The sessions on "Logic and Philosophy of Science" at the Second Annual Congress of Philosophy in 1904 were dominated by the advocates of "logistic", who presented it as not only a legitimate part of philosophy but the only legitimate kind of logic.

The Second International Congress of Philosophy opened in 1904, during an interval when Russell was apparently maintaining a thoughtful silence. The unity of mathematics and logic and the non-intuitive character of logical reasoning were argued for there in a number of papers, but little new was added to the views already expressed by Russell and Couturat. The character of this Second Congress differs strikingly in its logic and history of the sciences section from that of the First; there were no technical papers by the mathematicians, although Peano was again present. The philosophical logicians had taken up the battle for the time being—and it may not be entirely irrelevant that the mathematicians now had a serious difficulty on their hands.

It seems likely, after Russell's discussion in Principles, that most students of logic were aware of the existence of the paradoxes. Russell was still struggling with the technical

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problem, but was sure that some solution would be found. The conviction that mathematics and logic were one system, at first based on the belief that a formal proof of their union within one deductive system had been found, had now asserted its independence and was serving as a foundation for the hope that a formal proof of their unity could be found.

One prevailing attitude toward the modern logic, a simple refusal to recognize its existence, was represented at the Congress by Professor Windelband. In his paper, he asserted that every science gives rise to its own particular logic, and that since history had now become a science the moment had come to produce a "logic of history." He dismissed the researches of modern mathematically-oriented logic as a "game."¹

Couturat, who regarded the new logic as the final refutation of the Kantian mathematical philosophy, which he thought was in its turn the foundation of the entire Critical Philosophy, was the spokesman at the Congress for "logistic." This new name for the study previously called mathematical logic was proposed at the Second Congress, by Couturat and several others.

Among the opponents of logistic at the Congress, Pierre Boutroux again argued for the need of intuition in mathematics.

¹W. Windelband, "Le problème actuel de la logique et de la théorie de la connaissance par rapport aux sciences de la nature et de la culture," Actes du IIe Congrès International de Philosophie (Genève: Kundig, 1904).
He attempted to establish this need by showing that mathematics is an "open system," while logic is and must be "closed."

Boutroux defined the view to which he objected as the assertion that "all the notions of pure mathematics can be reduced to ten or so primitive notions, of which they are analytic combinations."\(^1\) I think that it is obvious that this resembles very closely Russell's statement of the logistic thesis in *Principles*, although Boutroux did not cite Russell explicitly as his opponent. Also, where Russell said, "definable in terms of," Boutroux said, "are analytic combinations of," a considerably less ambiguous statement which may not have accurately reflected Russell's views at this time.

Boutroux stated: "The logician sees in the selection of postulates nothing but a preliminary and immediate operation, proceeding all logical work, in the proper sense; for the mathematician, on the contrary, it is precisely in the choice of definitions and postulates that the true discovery resides. It is through the introduction of new notions, much more than through the transformations of symbols that mathematics continues to advance. In other words, the real task of the mathematician is in the search for postulates; the whole question is to know

where he is to find them: to say that those postulates are only particular cases of logical postulates is, it seems to me, to dodge the problem. ¹

Apart from the fact that Boutroux was interested in an aspect of mathematical thinking which had not yet occupied the logicians to any extent, that of finding new basic concepts, it seems to me that the main point of his objections was that mathematical discovery requires that attention be paid to the content of the ideas considered. Awareness of the individual character of an idea, or concept, was, for him, not logical or deductive but intuitive.

He stated, in another passage: "Behind that scaffolding of symbols that we pile up indefinitely one upon the other -- as a skilled juggler amuses himself by increasing the difficulty of his exercises -- there are laws, single and not composite, for which the adequate formula escapes us, while they are nevertheless present to us, and we employ our best efforts trying to translate them into our analytic language. He who only looks at the scaffolding may perhaps suppose that mathematics is nothing but a skillful construct in which the parts fit together nicely. But that would be forgetting that to

¹All papers at the IIème Congrès were published in French in the official record, Actes du IIème Congrès International de Philosophie, as were all the articles in the Revue de Métaphysique et des Morales. References in this dissertation are to the original papers, but the quotations in English are my own translations.
guide so many efforts there must be a goal toward which they converge, a model which they tend to realize: there must be, in other words, an intuition which determines the choice of analytic combinations which we retain in preference to an infinity of alternatives equally possible and ingenious but worthless to the mathematician. The intuition required is, Boutroux specified, "the cartesian or the kantian; it is not, naturally, sensible intuition."²

The logical point of view in mathematics is, said Boutroux, adopted by those "who attach more value to a demonstration than to a result. They establish no order of rank among their problems. To them any question is a good one, provided that they can employ their ingenuity upon it." The other type of mathematicians are less "struck by elegance and rigor" because they are "convinced that mathematical facts have another existence besides that conferred on them by a demonstration," and believe that "instead of an infinity of artificial problems there are some that are fruitful and necessary; they do not believe themselves to have the right to substitute fanciful constructions for the objective realities of their science."³

It seems to me that Boutroux' charge was that the logistic

¹Boutroux, "Correspondance," Revue de Métaphysique et des Morales, XII (1904), 918-919.
²Ibid., 915. ³Ibid., 920.
preoccupation with correct procedures of reasoning was incompatible with the belief that there are "mathematical facts having another kind of existence besides that conferred on them by our demonstrations." Couturat responded to the criticism by insisting that the logisticians did believe themselves to be studying pre-existing mathematical facts. He argued that Peano and his associates did not select their postulates simply at random; they were attempting to discover the primitive notions and primitive postulates of a given system.\(^1\)

This suggestion that there are absolutely primitive elements in a logistic system must be, I believe, regarded as Couturat's private opinion. Certainly Peano regarded the primitives as more arbitrarily selected that Couturat's statement indicates. For instance, Peano held that "logical relation" could be deduced from "mathematical function" or "mathematical function" from "logical relation" with equal validity.\(^2\) In contrast, Couturat apparently believed that mathematics is an application of the logic of relations but the logic of relations is not to be thought of as an application of set theory. He followed in this interpretation Russell rather than Peano. For Peano, as for Whitehead in Universal Algebra, logic was a branch of mathematics. But for Couturat and to some extent for Russell,

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\(^1\)L. Couturat, "La Section de Logique et Philosophie des Sciences au Congrès de Genève," Revue de Métaphysique et des Morales, XII (1904), 1047.

\(^2\)Ibid., 1046. Couturat recorded Peano's comment.
it was also a study of "the laws of thought" underlying all criticism of reasoning. From this standpoint, Couturat and Itelson could make the comment -- which must have been infuriating to the opposition -- that after all it is impossible to attack logistic, since to argue against it one must already admit its validity. Logistic is, said Couturat, the study of the "logical procedure of the mind, its own resources, and so-to-speak, internal activities which it applies to all types of knowledge," although the best place to observe this activity is in mathematics, "the most beautiful application of logical laws and the striking proof of their fecundity."¹

Couturat also objected to Boutroux' characterization of logic as a mechanical manipulation of symbols, and as a closed system. The "mathematical nominalists," said Couturat, see "symbols and conventions" where the logicians see "primitive concepts and principles." Furthermore, Couturat argued, the logicians do not object to infinite systems, and they are perfectly willing to admit the existence of intellectual intuition. It is the Kantians who are forced to deny such a thing; and, said Couturat, identifying the cartesian and the kantian intuitions as Boutroux did was a confusion of the sensible and the intelligible.²

In a subsequent article on mathematical correspondence,

¹ibid., 1065. ²ibid., 1053.
Boutroux clarified his objections to the proposed unification of mathematics and logic. Couturat as spokesman for the logicians at the Congress had insisted that mathematics was "the formal science of relations of order." Boutroux' countercontention was that mathematics is not and cannot be a "formal" science at all.

The argument between Couturat and Boutroux as to whether correspondence is a definable or a primitive notion illustrates their opposing points of view. Couturat there advanced as exhaustive and exclusive the alternatives: "either it is definable or it needs no definition." Boutroux rejected both. He argued that between the two views "that want all the notions of mathematics to be explicitly (quantitatively) defined and the doctrine of the logicians who claim to reason about undefined notions, there is place for a third way of looking at it: mathematical notions are seen as intuitive facts of which no definition can give a complete idea, which are not the immediate object of our science but the model that our synthetic constructs try to copy." Mathematics is, he argued, a constantly changing science; but logic is static. The ideas of mathematics are always incompletely defined.

It is, Boutroux continued, difficult to present a clear definition of either logic or mathematics. Some logicians

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2 Ibid., 625.
3 Ibid., 629.
think logic an independent science with its own subject matter; some think it part of all other sciences, or even of all thinking. But it is also to be identified with some calculus. The point of agreement between these different views of logic is the admission that logic cannot provide its own initial elements. Mathematics in contrast finds its postulates by analysis of a given body of fact. Logic is devoid of a given subject matter, and we discover what is a part of logic only by developing the consequences of its postulates.

Logical analysis is actually, Boutroux concluded, the reverse of the mathematical method. The ideas of mathematics are intuitively given and have a material content. It is the business of mathematics to increase the amount of such material knowledge. Logic, on the other hand, attempts to "reduce" given ideas to simpler ideas, producing merely different ways of expressing the same content. Therefore, Boutroux concluded: "In order that there be progress in a science, it is necessary for both the logicians and the intuitive researchers to have gone that way, one going upstream, the other downstream." The distinction between mathematics and logic is the distinction between "the progressive enrichment of the field of pure science" and "the impeccable logical concatenation of the propositions constituting the science."

1ibid., 637.
Boutroux also argued that in order to translate mathematics into a "logical system" the translation must be effected in a finite number of words. This conception of a logical system he attributed to the works of M. J. Drach, Kronecker, and Borel who defined a logical system in this way. To Couturat's response that logic finds no contradictions in the idea of infinite numbers Boutroux commented: "one wonders what sort of thing, within a real science (one which is practically realized), a definition in the statement of which an infinite number of words occurred could possibly be."

However, concluded Boutroux, belief that mathematics is a logical system in the above sense would be "in perfect agreement with the requirements of the science, on the condition that the system be considered as capable of expansion and that effort is made to enrich it continually."¹

As for the process of intuition by which we know mathematical truths, it is, Boutroux stated, a kind of perception, an act "neither free nor arbitrary," and both Descartes and Kant understood it in this sense. It can only be negatively defined as what is neither a logical operation nor a sensation, but a "suprasensible experience." In this way, the truths of mathematics are intuitive; and they are neither defined nor "undefined" in the sense of being "without content."

¹ibid., 623.
At this point, Russell entered the debate, with the express intention of clearing up misunderstandings about his views.¹

Boutroux's idea of mathematics as a changing science rested, Russell stated, on a confusion between what is known about mathematics and mathematics itself. New complex ideas of logical constants can be formed without affecting the existence of earlier ideas, "just as new Carolingians can be born without affecting the existence of Charlemagne."² What changes in such cases is, said Russell, our knowledge. If we apply the same name to different ideas at different times it does not follow that the ideas designated by the names have also changed. Indeed, said Russell, if the ideas changed we could not recognize that the word had changed its meaning.

Russell suggested that Boutroux's criticism of the idea of a mathematical "function" was an example of this confusion. The idea of a function is not, he said, "obscure and ambiguous." It is, rather, a generic term of which the specific intensional and extensional functions used by Peano are not impoverishments as Boutroux suggested, but species. That the word "function" may have many meanings is "a fact which might interest a lexicographer." It is of no particular interest to a student of logic.

²Ibid., 910.
Definition of some particular function may be difficult, said Russell, but that does not affect the meaning of the word "function" itself.¹ "It seems to me," Russell continued, "that M. P. Boutroux attaches too much importance to names. He seems to think that if the word "function" is used first in one sense, then in another, the object designated at first by that name has become the object designated later by that name. But the objects, clearly, are not at all affected by the names we are pleased to give them and remain unchanged whatever the variations in our terminology and our knowledge."²

I think that it is clear from a study of Boutroux' argument that what he wished to assert was that two senses of one word were sometimes related as two views of Mount Rainier, and that one such view might be better than another. Russell spoke as though no such relationship could exist between mathematical objects." Indeed, it seems that Boutroux actually asserted that we have in mathematics what Russell denied was possible, a present intuition of what the future concepts of mathematics will be. For Boutroux only the degree of precision with which the concept was seen changed. Therefore, the position

¹ibid., 909.
²ibid., 910.
Boutroux expressed was not countered but simply by-passed by Russell's answer. In these discussions, Boutroux seems to have been primarily interested in the character of our knowledge of mathematics and Russell in the nature of the objects of that knowledge, as Russell suggested. But the disagreement between them was more fundamental, and involved differing concepts of the relationship between thought and the objects of thought. There was already present in Russell's thought a hint of the "neutral monism" which he explicitly adopted later. It was, for him, the same Platonic object "in" or "out" of the mind -- at least, at times, for he still seems to mean something mental by the word "idea." 1 Boutroux consistently distinguished between a Platonic object and an idea of that object. It is not surprising that they failed to understand one another's arguments.

For example, Russell complained that he found Boutroux's views on the subject of intuition very hard to understand. He decided that Boutroux must mean only to assert that there are some indefinable and indemonstrable elements in mathematics. 2 But, he said, Boutroux seemed to believe that we have an unchanging, infinitely complex idea of Correspondence against which we test all proposed definitions of that idea. "But isn't it evident that none of our ideas is infinitely complex?"

1 The ambiguity of this word in philosophical contexts is particularly striking in these arguments.

2 Russell, "la relation," Revue de Métaphysique etc., 913.
The suggestion that we might have such ideas was, said Russell, "novel," but there was no reason to accept it.¹

However, said Russell, the really basic difficulty in Boutroux's views was that he believed both that to think about correspondence we must have defined it and that every such definition was "false."² If a definition is false, it may be false in a way which will make all our conclusions based upon it false. Therefore, we can know absolutely nothing about correspondence. So, concluded Russell, Boutroux is forced to assert that we know absolutely nothing about correspondence and yet know enough about it to judge the adequacy of our definition, which is self-contradictory. Russell suggested that the theory could be made tenable by asserting that correspondence is indefinable and that the supposed definitions are actually primitive propositions about it. "But that is not Boutroux's theory, and, actually, there is every reason to believe that correspondence is not indefinable."

Russell suggested that the problem arose from Boutroux's idea of what a definition is. Boutroux, said Russell, wanted a complete list of all the attributes common to the members of any given class as a definition of the class, or rather, of its

¹ibid., 912.

²Russell's objection here seems disingenuous, since Boutroux was asserting only the "inadequacy" Russell himself thought was present in any analysis of a proposition. Boutroux thought all concepts had the same kind of unity.
concept. Then, said Russell, because we cannot provide any necessarily-connected set of attributes common and peculiar to the class, Boutroux concluded that we could never define the concept. However, stated Russell, the answer is that the concept is to be defined as, "all the concepts which apply to these things." ¹ The number of concepts applying to all the members of a class will, he continued, always be equal to the number of "all the entities." We do not need to enumerate either the members of the class or the subordinate concepts in order to define the class; we only need to know that a class concept having as its extension the given class can be constructed by means of concepts already known.

Setting aside for the moment Russell's use here of a creation theory of definition, his statement appears very odd when applied to a concrete instance. He was asserting that we can define, for instance, "color," as: "all the class concepts which apply to this collection of objects," without any need to specify "this collection of objects." The collection could not be specified by giving the class-concept, since that is what is to be defined; and he also said that the individuals making up the collection need not be explicitly enumerated. Such a definition seems to me to leave the concept in question completely

¹This is not a direct quotation from Russell.
unlocatable as well as unconstructed, and I hesitate to trust
either my translation of my interpretation of the passage.
Therefore, I will quote the original from the *Revue de Métaphysique et des Morales*.

Russell stated, in reply to the view attributed to Boutroux in the summary above:

"... nous répondrions que la compréhension, en ce sense, est simplement une extension de concepts-classes, à savoir l'extension de 'tous les concepts-classes qui ont l'extension donnée.' Ce n'est pas là une entité logique fondamentale, et il n'est jamais nécessaire ni possible de connaître tous les concepts-classes ayant une extension donnée, puisque leur nombre est toujours égal au nombre de toutes les entités. Et ce n'est pas la possibilité de connaître cette extension qu'on entend, c'est qu'un concept-classe, dont l'extension est la classe donnée, peut être construit au moyen de concepts déjà connus. En ce sens, rien de ce que M. P. Boutroux a dit ne tend à montrer que la classe des correspondances est indéfinissable."

How one is to know that the extensions of two classes coincide without knowing either all the members of both classes or having a criterion for selecting the members has always seemed to me one of the major mysteries of modern logic, in spite of the importance to the subject of assuming that it is possible. Russell did not attempt to justify his principle in this article, but his use of it did not lead to objections during this particular argument.

On the other hand, the arguments of Boutroux seem directed

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1Russell, "la relation," *Revue de Métaphysique etc.*., 915.
against a position which had not been explicitly defended by the logicians up to this time. He shared with the Platonist wing of the group, apparently without recognizing the agreement, the view that the subject matter of mathematics is a system of Platonic ideas. However, he rejected the suggestion that increased knowledge of the system can be obtained by following mechanically along the patterns of deductive inference.

If the logicians had been willing to specify that they were concerned only with logical structure and were prepared to admit that the ideas of the system might also have non-logical properties and non-structural necessary connections, some of their opponents might have been mollified. But the logicians were also committed to the views that only logical structure was the subject of reasoning in mathematics, and that only mathematical reasoning was good reasoning. They could not consistently have made the concessions required by the defenders of intuition in mathematics. Such men as Boutroux were convinced, I believe, that in the act of "perception" of the particular Platonic idea filling a particular place in the logical system, some non-structural aspect of that idea was perceived, and necessary connections between that content and other ideas in the system were also recognised and made the basis of future reasonings about the system.

It was this conflict, I believe, which also divided the proponents and the opponents of the use of symbols in logic.
The opponents of logistic wanted to know not only where to find a given concept in a logical structure, but what the content of the concept was. Further, they were convinced that no analysis of the concept would be a complete picture of it. The ideas stated in the analysis, or definition, would be really present in the organic whole that was the original concept. But, as in Russell's propositions, the concept would include some unique character derived from the actual union of those ideas, from the presence of a "relation really relating" them. When Russell was consistently applying his mapping theory of definition, he seemed to feel that the content of the defined concept could always be seen just by looking in the right logical direction. Also, he held that the content was irrelevant to the purely structural aspects of the system, as "to the right of" is independent of the color of the objects so related. Russell thought of the Platonic system as a structure like that of a crystal, where the elements are separated as well as united. But his opponents thought of it as analogous to a jigsaw puzzle where the actual character of the element determines its place in the system.

Also, Russell from time to time apparently slipped into thinking of definition as a kind of creation, such that the defining concepts were the actual ingredients of the concept defined. This particular view reflects, I believe, an intrusion of the ontological approach into a logical discussion. Once again, the subject was being thought of as a
substance. The definition was considered not only as a presentation of the characteristics of a concept but the constituent parts of a thing. Russell's concepts were things built up out of other things; and yet he wished to distinguish between a concept and a mental event of knowing the concept, although he also wanted the same thing to be present both in the Platonic system and "in" the mind.

Thus, another issue arose between logistic and its Kantian\(^1\) and Hegelian\(^2\) opponents. The logicians frequently suggested that there was one and only one way of obtaining new knowledge about the system of concepts — deduction. Russell and Couturat certainly said this in many passages in their writings. The explicit formulation of the rules of deduction given in a logistic system was expected to recapitulate the rules which always had been followed by good reasoners so that good reasoning could be carried out consciously and at will. Poincaré attacked logistic primarily at this point.

There was, finally, the issue of Platonic realism. It did not arise, actually, between Russell and Boutroux, although Boutroux thought that it did. Search for absolutely primitive elements of a system might reasonably be undertaken by a conceptualist, and a Platonist could regard the primitive elements of his deductive system as arbitrarily selected. It is a question

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\(^1\)Boutroux disliked Kantian philosophy, but his views would be classified here by Russell and Couturat.

\(^2\)Bradley strongly denied that he was a Hegelian.
only of whether the world of mathematical objects and the process of mathematical reasoning are regarded as logically symmetrical or as logically asymmetrical. That is, we might suppose that given certain logical elements it will be possible to perceive only their logical consequences, not the elements from which they follow logically. This view is actually implied in the assertion by Russell and Couturat that we reason only by deduction. In such a case discovery of some logically prior elements would be absolutely essential to knowledge of that part of the System depending upon them; and knowledge of the whole system would demand knowledge of the absolutely first principles.¹

There could also be psychological views which would lead a Conceptualist mathematician to a search for absolutely primitive concepts and propositions as against Peano's conventionalist view. But Peano was not required as a conventionalist to reject Platonism. He need only hold that the system is symmetrical.

Couturat held in his answers to Boutroux that the primitive elements are not arbitrary; and in his Principles he

¹The relation between cause and effect on any view which holds that the same effect may have different causes, but that the same cause always produces the same effect is of this asymmetrical character. Given the effect alone, it would be impossible to infer the cause; but from the cause the effect could be inferred.
held that they were. Russell suggested that there were absolutely primitive and intrinsically indefinable elements; and he also seems to have believed that there was an element of convention in their selection. Simultaneous assertion of these two views exposes a great deal of unguarded area to the onslaughts of the enemy. There is some justification for Couturat's claim that Boutroux in his distinction between the mathematical and the logical attitudes toward definitions and postulates\(^1\) showed a fundamental misunderstanding of logistic. But the misunderstanding was aggravated by the fact that the logisticians were still working both sides of the street.

\(^1\)"Between the point of view of the logician and that of the mathematician there is an essential difference: the logician sees in the determination of postulates nothing but a preliminary, immediate operation preceding the properly logical work; for the mathematician, on the contrary, it is precisely in the choice of definitions and postulates that true discovery resides."
CHAPTER IX

THE LOGISTIC REFUTATION OF KANT

Russell had up to this time hailed three different "modern logics" as providing a final refutation of Kant, first Bradley, then G. E. Moore, and finally Peano. In 1904, Couturat undertook the task of providing a final and ceremonial statement of that refutation. The Rezue de Metaphysique et des Morales published a special issue in commemoration of the Kant Centennial, and Couturat contributed a long article\(^1\) devoted to the refutation of that philosophy. Because the logistic school subsequently referred to Couturat's article as their authority for denying, for example, the distinction between analytic and synthetic judgments, the argument of that article must be presented as clearly as possible.

Couturat's target in his attack on Kant's philosophy of mathematics was the entire "critical philosophy," which to him meant an irrationalistic "moralism" exalting "reasons

\(^1\) L. Couturat, "Philosophie des Mathématiques de Kant", Rezue de Métaphysique et des Morales, XII (1904).
of the heart above the "lifeless" reasoning of the intellect. Talk about "the logic of the emotions" infuriated him, and his attack upon Kant was made in the name of rationalism, but of a rationalism which was to include "the method of the sciences" as a part of formal logic.

The entire Kantian system rests, said Couturat, upon the Transcendental Aesthetic. For: "Wenn die mathematische Urtheile nicht synthetisch sind, so fehlt Kant's ganzer Vernunftkritik der Boden." This quotation from Zimmermann was taken by Couturat as a heading for his article. He then posed the Kantian position as follows:

The method of mathematics is rational understanding through construction of concepts. Concepts are constructed by showing the a priori intuition corresponding to them. The only intuitions a priori are space and time. Only concepts of magnitude can be constructed a priori: figure, duration, and number. Thus mathematical and only mathematical judgments can be simultaneously "synthétiques (comme les jugements empiriques) et a priori (comme les jugements analytiques.)" It also follows from this analysis that only mathematics can have "axioms" i.e., immediately known a priori synthetic propositions. Further, only mathematics can have definitions which are incontestable, for in other sciences the intuited object of the concept can only be described.
Therefore, logic and mathematics are completely separate. Logic rests only on the principle of contradiction and produces only analytic propositions; mathematics rests on intuition and its propositions are synthetic.

Couturat began with an examination of Kant's distinction between analytic and synthetic judgments. Kant's definition of the analytic judgment as one in which "the predicate is not contained in the subject" must be, said Couturat, supplemented immediately by the case where the predicate is partially included in the subject.\(^1\) Also, we now know that not all judgments have the subject-predicate form, and that not all judgments can be reduced to relations of inclusion between two concepts. Further, hypothetical and disjunctive judgments applying to not two but many concepts do not fit the classification. "La définition de Kant est donc absolument insuffisant en principe."

The distinction between analytic and synthetic judgments had thus been shown, said Couturat, to be at best incomplete. It could hold only for such judgments as can be regarded as asserting a relationship between concepts thought of as assemblages of "partial concepts"; and it can be shown to be

\(^1\)Note that this idea of 'partial inclusion' makes sense only in terms of classes in extension.
inapplicable even to subject-predicate judgments, the most likely candidates.

Kant obviously thinks of a concept, said Couturat, as an "assemblage" of partial concepts which are its "essential characters." But that view is false. Therefore, the distinction between analytic and synthetic judgments which rests on it must be rejected. Kant's "popular" statement of the distinction in the Prolegomena as the distinction between explicative and amplicative judgments must simply be regarded as a confusion of the issue. It leads to the view that logic is sterile and can produce only "useless tautologies."

There is, said Couturat, a logical and a psychological interpretation of Kant's distinction. It sometimes seems that Kant intended the distinction to be psychological, referring to what we actually think. But Kant distinguished between a logical connection of concepts and their necessary connection in our thinking, so it cannot be interpreted psychologically. After considering the alternatives, Couturat decided to accept Vaihinger's interpretation, and concluded that Kant intended to contrast "what we think more or less implicitly in a concept and the way we think of it" with that which "is logically contained in it,

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\(^1\)Couturat asserted, but did not argue for this point.
whether or not we actually think it." This logical concept is, said Couturat, simply the definition. Thus, it is the fact of being or not being contained in the definition of a concept which is the basis of Kant's distinction between the analytic and synthetic judgment.

It seems to me that when Couturat spoke here of definition of a concept he meant a statement of the essential properties of the thing defined, those without which it could not be that kind of thing. But since he is speaking of a concept, he might mean not a statement but the properties themselves, and be thinking of definition as construction. I believe that one of those "scholastic" pedantries so scorned by modern philosophers would have been of some value in this discussion—the distinction between definitions and essences. If the issue had been clear, Couturat's denial that there was any such distinction, which follows the line proposed by G. E. Moore, would have been more clearly recognizable as asserting a new metaphysical position.

The principle of all analytic judgments, continued Couturat, was stated by Kant to be the Law of Contradiction. However, said Couturat, Kant did not state the Law of Contradiction, but actually used a much more complicated principle. For, said Couturat, to say that we must not deny a of ab does not require that we affirm a of ab. For that we
require a second principle, that of identity. Kant, said Couturat, probably didn't know the difference between the two laws, since he did not distinguish between identical and analytic statements. Analytic but not identical statements really rest on the law of simplification.¹ The traditional formal logic resting only on identity is, thus, "a false conception of logic," and is "absolutely sterile, because it only permits passage from the same to the same, and justifies only idle tautologies."²

Couturat did not press this argument that there is one and only one real "Law of Contradiction," and that any other principle called by that name must be an imposter. The fairer interpretation of Kant's meaning is, Couturat continued, that he thought all deductive reasonings took place according to the laws of logic, whatever those laws may be. Today we know, said Couturat, that there are about twenty of them, all independent. Furthermore, we know, Couturat stated, that these laws are insufficient by themselves to provide any reasoning whatever, being completely devoid of subject matter.³

The matter must be introduced into a formal deductive system

¹Couturat, "Kant," Revue de Métaphysique et des Morales, XII (1904), 329.
²Ibid., 330.
³Apparently he believed that Kant was unaware of this.
by definitions; so analytic judgments are those following from
definitions by the laws of logic without the introduction of
new "given" elements. This statement gave, Couturat believed,
"the spirit if not the letter of Kant's doctrine."¹

The question of analytic and synthetic definitions, Couturat
continued, is a separate issue. Of these, analytic definitions
were supposed to be definitions of pre-existing concepts while
synthetic definitions constructed new concepts. In this sense,
said Couturat, Kant held that all mathematical definitions were
synthetic. But, argued Couturat, even admitting this it would
not follow that mathematical judgments are synthetic, in the
sense appropriate to judgments. How the concepts used in judg­
ment are obtained is irrelevant to what is done with them in
the reasoning process. Besides, Couturat concluded, this point
is really unimportant, since Kant was mistaken about mathemat­
cal definitions. They do not construct new concepts at all,
but only introduce new symbols to be used in place of old sym­
bols. Definitions are not propositions, but abbreviations;
and nothing can be deduced from them alone but only from the
group of propositions for which they stand.²

¹Couturat, "Kant," Revue de Métaphysique etc., 330.
²Ibid., 334. Since Couturat had stated that the only way
to provide the laws of logic with any material to operate on
was by way of definitions and was now asserting that definitions
introduce no new matter either, the conclusion appears to be
that no material can ever be introduced into a deductive system.
The formalist conclusion that mathematical and logical systems
are only reasoning about symbols seems to follow; but Couturat
would probably not have been willing to accept it.
After some discussion of Kant's probable understanding of "pure mathematics," Couturat decided that he thought it included only arithmetic and geometry. Kant believed the propositions of arithmetic to be synthetic, Couturat argued, only because he took an empiricist view of arithmetic. All the propositions of arithmetic actually follow, said Couturat, from the definitions and the laws of logic, and accordingly are analytic; to prove this, Couturat gave a formal derivation of $7 + 5 = 12$, by substitution in formulae of defined identities.

Kant thought each such formula of arithmetic to be based on an immediate intuition, which committed him, Couturat pointed out, to the "shocking consequence" that there are an infinite number of self-evident truths. This "is hardly conformable with the idea of a rational science." If these truths were really self-evident, anyway, said Couturat, we shouldn't have to do long calculations to find them out. But, Couturat suggests, Kant's error was natural given his faulty conception of logic.

Couturat then added a new discussion of $7 + 5$. If I think of 7, the operation of addition, and of 5, he said, I already necessarily think the number 12, unless I am thinking

\[\text{\textsuperscript{1}}\text{ibid.}, 340.\]
Only of symbols. Number concepts are simply concepts of collections; concepts can be thought of without being imaged; their relationships are eternal truths, whether we are aware of them or not; and the identity of 7 + 5 and 12 is one of those eternal truths.\(^1\)

Kant's argument, Couturat continued, is really an assertion that seven, plus, five, are "partial concepts" which are to be united to compose a "whole concept." Kant distinguished between "addition" and the "uniting" which he thought could only be done in intuition. He was confusing addition and logical multiplication, said Couturat. For, if one thinks properly of the combining of two classes into a third, which is what happens when we add seven and five, this process of thought does not involve "going outside the concept" but "realizing in the mind" that combining of classes. Whatever our imagery, which is logically irrelevant, 7 + 5 "contains by definition the concept 12, or better, is identical with it."\(^2\)

\(^1\)This is a curious departure from his previous attempt to reduce mathematics to the manipulation of symbols, and is, I suppose, an attempt to cope with any lingering doubts that his argument might not allow for the possibility that the symbols are associated with concepts.

\(^2\)Couturat, "Kant," Revue de Métaphysique etc., 343.

\(^3\)Couturat must now cope with the Kantian who might suppose that his discussion of "realizing in the mind" an actual adding of one class to another is an admission of precisely what Kant said earlier.
This view was opposed to Kantian philosophy in Couturat's opinion since Kant confused ideas appearing to Reason with images given to intuition. Surely, Couturat said, Kant cannot really have meant to go so far as to rest the certainty of syllogistic reasoning on "intuition," equating evidence and intuitive certainty, like any "simple empiricist." There must be, Couturat argued, purely logical and intellectual demonstrations. Even in Geometry we need not appeal to any pictorial thinking; to do so is actually conducive to error. Intuition, empirical or a priori is only "the singular representation of a unique and perfectly determinate figure." Then, any reasoning based on intuition is really about only one specific figure, or it is about a general concept merely represented by the image—and in that case, the reasoning is not based on intuition.

Kant's antinomies of pure reason Couturat regarded as unimportant, resting on a false idea of the infinite which he had already disposed of in De L'Infini Mathématique.

1Couturat, "Kant," Revue de Métaphysique etc., 357.
2Ibid., 366.
3Ibid., 367.
4Ibid., 377.
Kant had "arbitrarily introduced the notion of time into number and magnitude"; therefore, his idea of the infinite "is an indirect refutation of his philosophy of mathematics."

From this discussion, Couturat concluded that Kant was wrong and Leibniz right about logic and mathematics. Kant had accepted Aristotelean logic without any criticism, not realizing that mathematics is a purely formal science, not "the science of number and quantity" but a "hypothetico-deductive system." Now, after Boole's discovery of pure mathematics, we know, said Couturat, that "there is only one logic, the logic of deduction, of which the methods called inductive are only an application, for there is only one single way to link truths together in a formal and necessary manner."¹ Couturat suggested that Kant's preoccupation with subordinating reason to faith led him into his fatal mistake of over-empiricism.

Russell, who was still insisting that all truths are synthetic, must have found Couturat an embarrassing ally at this point. However, Russell meant by synthetic "containing more than is contained in the premises," and accepted a difference in symbols as an indication of the required newness of the conclusion, while for Couturat difference in symbols was unimportant as compared with the identity of the symbolized concept.

¹ibid., 379-382.
It was apparently Couturat’s article on Kant that provoked Poincaré into undertaking a public refutation of the logistic thesis. He commented on that article: "On voit bien que c’est le centenaire de la mort de Kant," and proceeded to a defense of Kant and an assault upon logistic.¹ On the basis of Couturat’s articles interpreting the work of Russell and Peano, Poincaré stated the view to which he was opposed as: "that there are no synthetic a priori judgments, that mathematics is entirely reducible to logic and that intuition plays no part at all." Couturat, said Poincaré, had announced that all these points had been proved by the new logistic; but his belief that logistic had proved them was wrong.

Poincaré’s attacks centered upon the view that the mathematical method is deductive, "analytic" in the sense defined by Couturat above. Poincaré was not actually opposed

to the view that logic and mathematics are a unified whole, for he argued that logic as well as mathematics requires synthetic rather than analytic judgments. However, he probably held that while their principles were of the same synthetic a priori type, their methods were different at least in degree, and could not have been classified as a logicist. Nevertheless, he concentrated upon proving the inescapably "intuitive" character of logical reasoning itself. He mentioned the need for intuitions of contradiction and consistency, but his argument centered upon the status of the principle of mathematical induction.

The new logic of mathematics is intended to be purely formal in character, stated Poincaré, pointing out the work of Hilbert, where to demonstrate a theorem, "it is not necessary or even useful to know what it means." For Hilbert's purposes, that of reducing an already established system, geometry, to the minimum number of axioms, this is perfectly true, said Poincaré. It is true for the analoguous efforts in arithmetic and analysis. However, Poincaré stated, the most that any of these reductions can show is "that all the theorems can be deduced by purely analytic methods, by simple logical combinations of a finite number of axioms, and that the axioms are only conventions." It remains the business of philosophy to consider the origin of and the reasons for the adoption of such conventions.
Choice between equally logical alternative postulates in geometry is not, said Poincaré, made by logical rules; but, "guided by a sure instinct, or by some vague awareness of some more profound and hidden geometry, which is the only thing giving value to the constructed edifice," the right alternatives are chosen.¹ This instinct which is necessary for mathematical discovery may perhaps be dispensed with in the study of what is already discovered and made part of the science. Could logistic then, if it abandoned the claim to be a way of discovering new truths, maintain that it can demonstrate all the mathematical truths that have been discovered? No, answered Poincaré. The principle of mathematical induction is required to prove many propositions of mathematics; and this principle is not reducible to logic. There are also other principles of the same kind, which are a problem to the logician because they cannot be admitted either as axioms or as a priori synthetic judgments, and must therefore be made out to be definitions.

Definition by postulates, as opposed to direct definition is a legitimate mathematical procedure, Poincaré thought, but only under certain conditions. Given a set of postulates concerning a system of notions, all but one of them previously defined, we may consider that the set of postulates defines an

¹H. Poincaré, "Logique," Revue de Métaphysique etc., XIII (1905), 617.
unknown as "that object which satisfies the postulated conditions." That is, we may do so if we can show that there can be such an object—-that is, that the postulate set does not imply a contradiction. If we cannot prove the consistency of the set of postulates proposed as a definition, continued Poincaré, we must assume as an axiom that the set is consistent if we insist that it is a definition. Accordingly, each such definition contains an axiom in disguise, as Mill had argued on the wrong grounds.

Poincaré did not indicate why he believed it impossible for an inconsistent set of postulates to be a definition. I believe that he held a creation theory of definition, and believed that self-contradictory objects cannot be created. He also held that any non-contradictory object "existed" for mathematical purposes, whether it has actually been defined or not.¹ Poincaré's discussion of mathematical induction as a problem for Logistic rested upon his demand for consistency proofs for any proposed definition.

¹Meinong, with whose work Russell at this time agreed, believed that self-contradictory objects also had being; and Moore had denied that self-contradiction of a single proposition or entity was possible. The issue was a serious one, although not so much for Logicism as for Platonism.
To prove the consistency of a set of postulates, Poincaré said, it is usual to try to find an object satisfying the stated conditions, that is, to find an actual case in which all the postulates of the system are true at once. When no such interpretation can be found, he continued, it is necessary to show that no contradictory consequences can be deduced from the postulate system. If the number of consequences of the system is finite, actually checking each pair of propositions it contains is logically possible. But if the number of logical consequences of the postulate system is infinite, no such actual check is possible. In such cases, said Poincaré, appeal must be made to the principle of mathematical induction: "If a property is true of the number 1, and it can be shown to be true of n + 1 whenever it is true of n, it is true for all the whole numbers."

Now, said Poincaré, in the particular case where the system of postulates the consistency of which we are trying to establish is a system including the principle of mathematical induction itself, no proof using that principle can legitimately be presented. This was, Poincaré said, the position in which logistic now found itself, and "whether or not the puzzle is insoluble, it is one they have not solved."

Also, he continued, a mathematical definition is a definition of a mathematical object. Any statement that a concrete

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1 Poincaré, "Logique," Revue de Métaphysique etc., 519.
object satisfies or approximately satisfies the definition of a mathematical object is, said Poincaré, a new truth, not a definition. As such, it can only be verified by experience.¹ There is, he continued, an analogous difficulty within pure mathematics in determining when the mathematical objects defined by two different definitions are identical. The logicians had, he said, made an unjustified identification of two such defined objects in their definitions of number.² These definitions, he added, all had the further defect of being circular, in that all of them employed the notion of some particular kind of number to define number in general.³

Couturat's view of the philosophical importance of Peano's pasigraphy would, said Poincaré, "astonish Peano himself," and was quite unjustified. However, suggested Poincaré, there was a most interesting use of this language by M. Burali-Forti to be found in Volume XI of the Rendiconti del circolo mathematico di Palermo, "Una Questione sui numeri transfiniti." Poincaré recommended this to his readers' attention as the most important of all the articles that had been written in the "new language." It was, he said, "the first example of those antinomies which are encountered in the study of transfinite numbers and which have been for some years the despair of mathematicians." The

¹Poincaré, "Logique," Revue de Métaphysique etc., 821.
²Ibid., 821.
³The circle was, I assume, that things could not be classified as kinds of numbers unless first identified as numbers.
results of Burali-Forti contradict those of Cantor. But, he
pointed out, both were arrived at by formally valid logical de-
ductions. Nevertheless, one must be false. The conclusion, said
Poincaré, must be that application of the new rules of logic was
not enough to preserve us from errors in reasoning. But, surely
the rules of logic cannot be deceitful; so some misleading ap-
peal to intuition must still be concealed in the symbolic
language.¹ The implication that Peano’s logistic has therefore
failed in its attempt to introduce rigor and avoid appeal to
intuition in mathematical reasoning was left by Poincaré for the
reader to draw.

Poincaré said that he was prepared to admit that the new
logic of Russell was certainly an extension of the traditional
logic, even to the extent of giving “a new meaning to that word.”
Truths irreducible to logic in the old sense might, he thought,
very well be reducible to logic in the new sense. But, he in-
sisted, Russell’s logic rested on indefinable notions and on
indemonstrable principles which were synthetic a priori judgments.
Thus, the appeal to intuition had simply been transplanted from
mathematics to logic. Before the indefinables could be regarded
as introduced by definition, we would need to have a consistency
proof of a system involving an infinite number of consequences;
and to give such a proof, the principle of mathematical induction
would have to be accepted as an a priori synthetic judgment.

If we all agree with Russell and Couturat that logic is based upon synthetic a priori judgments, continued Poincaré, could they then derive all the rest of mathematics from their original supply of such judgments without further appeal to intuition? He believed that this, too, must be denied. For, he argued, while they could reach on the basis of the original logical principles some parts of algebra, they could not reach theory of numbers, analysis, or geometry. Poincaré rejected Couturat's contention that Peano's postulates for arithmetic were only definitions, because one of those postulates was the principle of mathematical induction. Thus, the set could not be provided with a consistency proof, and could not be considered a definition.

Also, Poincaré said, in their attempts to establish mathematical induction as a definition of whole numbers and then to use the principle of mathematical induction in their proofs, the logicians were guilty of equivocation. They were ostensibly committed to the view that "whole numbers" are "the numbers satisfying mathematical induction." That is all they were supposed to know about whole numbers. How, then, asked Poincaré, do they know that the number of steps in any series of deductions is always a whole number? If they cannot establish this proposition independently, he argued, they cannot know that their chains of reasoning satisfy mathematical induction. But they know that the number of premises is a whole number only
by knowing that the argument from induction is valid. Poincaré suggested that they had failed to notice this lack of rigor because they were actually using in their reasonings the old "vulgar" and "intuitive" meaning of whole number, which was to have been replaced by the new definition.

Poincaré stated that Hilbert had made some attempt to deal with these difficulties, but that Russell and Couturat had not even seen the problem. Couturat, he said, relied on Russell; but Russell confused the proposition, "a number can be defined by recurrence," with the proposition, "upon numbers so defined one can reason by recurrence." 1

Poincaré pointed out that for Hilbert logic was not prior to arithmetic but must be developed simultaneously. He believed this difference was partly due to the fact that Hilbert thought of "class" as an arithmetical concept, while Russell regarded it as logical. Also, said Poincaré, Hilbert's definitions were explicitly intended to be definitions of symbols only. Thus, only the "objects" and combinations of objects which were explicitly introduced in the course of developing his system were regarded as "existing" to serve as values for the variables of a formula obtained at any given stage of the development. Accordingly, he said, each introduction of a new "object" changed the extension of the axioms, and required that they be retested, and either asserted again or modified. 1

Russell, on the contrary, was, said Poincaré, willing to accept any object whatsoever, whether previously defined or not, as a value for his variables. Poincaré thought this attitude was due to Russell's thinking in terms of "comprehension" rather than extension.\(^1\) That is, he said, Russell started with the general idea of being and specified it by degrees; but Hilbert did not. Why? Because, said Poincaré, Hilbert wanted to avoid contradictions and to avoid the paradoxes of set theory which he believed would follow from the intensional viewpoint of Russell. The issue between them, said Poincaré, was one for logic to decide. But, whatever that decision might be, Hilbert was also required to use the principle of mathematical induction to justify his definition of whole number, and he also had no proof of the consistency of his definition.\(^2\) He simply, said Poincaré, introduced mathematical induction as "granted" to be consistent with his other postulates. He, also, had no justification for applying to actual successions of arguments a principle which he had introduced as applying only to symbols.

For, continued Poincaré, whatever else in a deductive system might be regarded as conventional, the demand for consistency could not. If we know before defining it what we mean by a "contradiction," the meaning of that word is not

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\(^1\)I do not believe Poincaré recognized Russell's Platonism.

conventional; if we do not know, there would be no sense in asking whether the proposed definition of the word was itself consistent.

The importance of the principle of mathematical induction in the argument was, said Poincaré, that the whole theory of infinite numbers rested upon it. It was, he said, the only way of proving the Theorem of Bernstein, "if the elements of a class A can be put in one-to-one correspondence with the elements of a proper part of class B and the elements of a proper part of A can be put into one-to-one correspondence with the class B, the class A is equivalent to the class B," which was the foundation of the theory of infinite cardinals. The theorem, said Poincaré, is proved by setting up an endless series of correspondences and using the principle of mathematical induction. Therefore: "If M. Couturat knows a different proof of Bernstein's Theorem, let him publish it quickly, it is an important mathematical discovery. But if he doesn't know one, let him stop saying that the theory of infinite numbers is constructed without the principle of induction. Let him not write that MM. Russell and Whitehead have been able to formally demonstrate, starting from purely logical principles, all the propositions of that theory and purge it of every postulate and every appeal to intuition. If they had been able at the same time to purge it of every contradiction, they would have done us signal
service; alas! mathematicians still argue about that theory without nearing an agreement."\(^1\)

Poincaré dismissed Couturat's claim that Russell and Hilbert had definitely decided the debate between Kant and Leibniz and ruined the Kantian theory of mathematics as clearly inaccurate. "I don't know if they really believed they had, but if they did believe so, they were wrong."

Poincaré's article took it for granted, as his use of the past tense shows, that the opinions of the proponents of mathematical logic were now being reformulated after their disastrous encounter with the paradoxes. At any rate, there was now an alternative variety of logic, Hilbert's, to be reckoned with; and when Russell eventually stepped into the discussion, he admitted that he now held a different view. For the time being, however, the defense of logistic was left in the hands of Couturat, who could not believe that the paradoxes were such a serious problem as Poincaré made out.

Couturat characterized himself as a "popularizer and vulgarizer" of the doctrines of "the masters of logic," who was substituting for them in the argument at their request.\(^2\) It was reasonable that he do so, he pointed out, since Poincaré had drawn his opinions of Russell and Peano from Couturat's

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own expositions rather than from the primary sources. As for Hilbert, said Couturat, he was not a Logician at all: "he has never employed any logical calculus in his works," and he could not be granted to be even a pioneer in deductive systems.

Besides, said Couturat, Poincaré was unreasonable in his demand that logicians make no mistakes in their demonstrations. Burali-Forti's paradox was really a problem of set theory rather than logic, anyway. Couturat added that Burali-Forti himself believed that his paradox arose from the definition of "class," and Russell thought that it could be solved by eliminating the concept of "class." Further, said Couturat, the paradox was not a criticism of logistic but of intuition, for the principle that every concept determines a class as its extension had seemed intuitively evident, and yet lead to a contradiction. Discovery of this fact should, Couturat said, be counted as a triumph for Logistic.

Couturat wavered between two positions on the question of discovery. He first asserted that Logic was not concerned with the psychological fact of discovery of new truths but only with the subsequent verification of what was discovered. He then said that Poincaré's "sure instinct" was only unconscious logical reasoning. "The reason that discovers is like, and at bottom identical with, the reason that demonstrates, without which it

could never verify the workings of the former; and those workings are not discoveries, that is, are not true, except on that condition.\(^1\) The new, "extended" logic was, Couturat said, simply a deeper insight into "the principles that have always guided those reasonings recognized as good by that rational instinct to which M. Poincaré attaches such great value." The new rules of logic, he said, were no more "synthetic a priori judgments"\(^2\) than the law of identity, which had been considered the "type of analytic judgments."\(^3\) Why, he asked, were such laws considered "intuitive truths" when discovered by modern logicians and "logical truths" when stated by Aristotle? All logicians, he said, recognized a need for intellectual intuition of the relations of ideas; but a Kantian spatial intuition was not used either in logic or in mathematics.\(^4\) At any rate, the complete translatibility of mathematics into the symbolic language of logic had already been proved by Peano, said Couturat, so Poincaré's objections to its possibility were futile.

Couturat flatly denied that mathematical definitions needed to prove or to presuppose the existence of the objects defined. That is, he argued both that contradictory definitions are definitions, and that mathematical existence was not absence of

\(^1\)Ibid.

\(^2\)The rules for determining analytic truth were generally spoken of at the time as being themselves analytic truths.

\(^3\)Couturat, "Logistique," *Revue de Métaphysique et de Morale*, (1906), 218.

\(^4\)Ibid., 219.
contradiction. It was rather, he stated, "the non-emptiness of a class." Individuals, the basis of such non-emptiness, were always thought of as existing and could never be defined. More precisely, it was senseless to say that an individual exists.

If the definition of a class is contradictory, said Couturat, no individual will fit the definition, and the class will not exist -- but still has being. But it is not true that every non-contradictory definition is a definition of a class having members. As for consistency proofs, said Couturat, they are impossible in principle and cannot be demanded. A system of logic must be assumed consistent until a contradiction has been proved in it. Inconsistency, Couturat added, could not even be defined without a vicious circle: "postulates are inconsistent when they have inconsistent consequences."

He suggested that Poincaré's belief in mathematical induction as "the type of mathematical reasoning" was due to a mistaken view that mathematical reasoning is linear, that is, a series of syllogisms of which each conclusion reappears as premise of a new argument. Besides the fact that not all deduction is by syllogism, said Couturat, the deductions characteristic of

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1 Couturat's reason for this, based on statements by Russell, was that existence is contained in the concept of an individual, a new ontological argument.
mathematics proceed instead by sorites. The order in which propositions are proved is irrelevant to their logical connections. If no contradictions appear after we have followed our deductive rules for n deductions, it is not from the properties of n that we conclude that no contradiction will occur next time we use the rules; the principle of mathematical induction is completely irrelevant. The procedure actually used is simple induction. Mathematical induction gives an absolutely certain conclusion; but all judgments from past to future are only probable, and use "common induction."  

Couturat also objected that Poincaré had not proved his charge of equivocation in definitions, and suggested that Poincaré had been misled yet again by his continual supposition that logisticians were reasoning about "words" and "phrases" which had a meaning in current use. But, said Couturat, logistic actually uses symbols instead of words--to avoid "illogical associations" with ordinary words.  

Couturat concluded by saying that this whole question was really only a matter of logic, the question whether all mathematics could be deduced from logic, and Poincaré was quite wrong in supposing it relevant to the "debate between Kant and Leibniz."

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1 Couturat, "Logistique," Revue de Métaphysique et de Morale, (1906), 239.
2 Ibid., 241.
3 Ibid., 247.
It is difficult to see any justification for this claim that a question which was absolutely decisive against Kant when answered in one way should be regarded as irrelevant when the answer seems to be going in the other way. It would, naturally, not be logically conclusive, since from if $p$ then $q$ and not-$p$ one cannot conclude not-$q$. But it is certainly relevant to a claim that $q$ is true to argue that $p$ is false, if $p$ is the only reason we advance for asserting $q$.

Couturat's attempt to deal with the paradoxes was also unsatisfactory. He attempted to answer Poincaré's conclusion that the rules of Logistic were not infallible by evading the issue with a suggestion that the paradoxes arose from some undiscovered errors in applying the rules of Logistic. They were, of course, paradoxes because they were logically valid.

Couturat did not specify in this argument whether in speaking of logic he was referring to the principles stated by Russell and Peano, or to those implicit rules of right reason with which he had, at one time, publicly identified those postulates. Poincaré was reasonable in assuming as he did that Couturat still made this identification, although Couturat's reference to induction as a kind of valid reasoning is startling, and actually suggested a new attitude among the logicians.
In his reply to Couturat, Poincaré concentrated upon two issues: does the demonstration of the principle of mathematical induction involve an appeal to intuition? and are the rules of Logistic infallible means of discovering new truths? Logistic demands, Poincaré stated, that its rules be followed rigidly, "blindly", in the constructing of any logical demonstration. This is necessary because there could be no rule for deciding when the rules apply, and there was to be no appeal to intuitive, or non-demonstrative reasoning. In order to justify a demand for such supreme authority, the rules must claim infallibility. Nevertheless, in following these rules the logisticians had encountered the paradoxes. If we were to "change the rules" to avoid the paradoxes, said Poincaré, as Russell now suggested, it must be because these rules are not infallible. Therefore, the rules of symbolic logic discovered by Russell and Peano were not infallible, not the eternal rules of right reason, and the logistic of Peano had been destroyed.

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2 Poincaré's argument is effective against the claim that the postulates and methods of proof proposed by Peano were a final and accurate statement of all good mathematical reasoning. The claim that mathematical reasoning was the only good reasoning was not, so far as I have discovered, made by Peano himself, but only by Russell and Couturat.
If Russell succeeded in solving the paradoxes, added Poincaré, it would be not by the use of logistic, but by his own "mathematical intuition.\textsuperscript{1}

After a brief comment on Couturat's suggestion that self-contradiction of a postulate set was not a very important problem, Poincaré took up the technical details of such consistency proofs as had been offered by Hilbert; the attempts to prove the principle of mathematical induction which had been made by Whitehead and Russell, by Burali-Forti, and by Zermelo; and Russell's early attempts to revise the principles of logistic in order to avoid the paradoxes. Poincaré's conclusion was that they were all inadequate, although he believed that Russell's refusal to permit any further use of "non-predicative" definitions, which required in their statement the notion of the class supposedly to be defined, was correct. He pointed out, however, that without use of such definitions Cantor's set theory collapsed, and suggested that it was the admission of the "actual infinite" which was the real root of all the trouble. At any rate, Poincaré said it was clear that there must be some authority which over-rideres the rules of logistic as a judge of good reasoning.

\textsuperscript{1}Poincaré, "Les Mathématiques," \textit{Revue de Métaphysique etc.}, (1906) 296.
Obviously, it was time for one of "the masters of logistic" to speak for himself on the issue. Russell had published several attempts to revise logistic sufficiently to avoid the paradoxes. He turned now to the debate with Poincaré with the insistence that Poincaré had simply misunderstood the whole logistic position.¹

Poincaré supposed, said Russell, that the logicians presented their rules as procedures to be trusted implicitly, "blindly" followed, and productive of demonstrations which were infallible. This mistake about the nature of logistic was, Russell admits, a natural one: "In fact, I shared it until I came upon the contradictions."²

What Russell now believed was that: "the method of Logistic is essentially the same as that of every other science. It admits of the same fallibility, the same uncertainty, the same mixture of induction and deduction, and the same necessity to appeal for confirmation of the principles, to the general agreement of calculated results with observation. Its object is not to eliminate 'intuition', but to control and systematize its employment, to eliminate the errors to which its uncontrolled employment gives room, and to discover general laws from which

²Ibid., 630.
one can, by deduction, obtain results never contradicted by intuition and, in the crucial cases, confirmed by it. In all that Logistic is on exactly the same footing as, for example, astronomy, except that in astronomy the verification is effectuated not by intuition but by sensation. The "primitive propositions" from which the deductions of Logistic begin should, if possible, be evident to intuition; but this is not indispensible and in any event is not the only reason for their adoption. That reason is inductive, that is, that among their known consequences (including themselves) many appear to intuition as true, none appear false, and those which appear true cannot be deduced (as far as one can see) from some system of indemonstrable propositions inconsistent with the system in question . . . If intuition were infallible this complicated procedure of verification would not be necessary. But intuition is not infallible, as the contradictions prove. Therefore, there always remains an element of uncertainty just as in astronomy. In time it can be immensely diminished; but infallibility is assured to no mortal, even if, as M. Poincaré counsels him, he abstains carefully from making his arguments conclusive."

Russell insisted that this did not really represent so great a departure from his own position in Principles, since he had discussed the paradoxes there, and that he did not regard it as a destruction of the work of Peano. However, to
select only a few of the similarly-oriented statements from *Principles of Mathematics*, Russell had then held that "what can be mathematically demonstrated is true," and believed that common sense "ought to commit suicide" when faced with a contrary logical proof. Any sense in which his new views did not represent a radical departure from these views escapes the eye. It must be assumed that he now wished to reduce his logistic thesis to the statement that it was technically possible to deduce mathematics from logic, which he still maintained and, indeed, continued to maintain.

The rules of logic were, Russell continued, to be followed blindly, certainly, but only until they produced a contradiction. Then we would know that the rules were wrong and required modification. On the other hand, we must be careful to apply the rules not only where we were intuitively convinced that they were adequate, but also to judiciously selected crucial cases where they could be tested, like any other hypothesis. In the process of correction, said Russell, perhaps we would find a new set of principles "in any event nearer the truth than the old."

What, then, was the relationship between the Logistic established in this way and the ordinary mathematics? It was, said Russell, the business of Logistic to discover the principles used in mathematical reasoning, to show how they were used there, and to draw further interesting consequences.

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The generalizations of mathematical principles which were used by Cantor in constructing his set theory still seemed intuitively evident, said Russell. But, as "some people prefer to have true logical rules," the logisticians applied those principles beyond the area where they were sure that they would hold, in order to find their limitations. Until such limits were known, said Russell, every application of the rules was hazardous.

A large part of Russell's article was also devoted to an attempt to explain and to eliminate the paradoxes. However, the technical modifications he suggested were superceded by later ones; and whatever technical modifications were introduced into the formulation of the rules of Logistic could not be as revolutionary as the change in Russell's philosophy of logic. Considering the state of affairs at the time, it seems appropriate that Poincaré have the last word in this summary, as he had in the debate in the Revue de Métaphysique et des Morales.

Characterizing Russell's new position, Poincaré pointed out that for him now "logistic is no more than an auxiliary of intuition. It even needs intuition, not only in its beginnings but at each step; not only to ask it for new principles but as an incessant control, just as a theory in mathematical physics has value only when it has been confirmed by experience. If logisticians apply their rules blindly, it is to demonstrate better that they are false, and in this they are not doing a useless task."
"We find ourselves far from the eight notions and twenty propositions which enclose all human thought or at least all mathematical thought. We find ourselves far from the *Exegi monumentum aere perennius* of M. Couturat (no need to remind readers of this Revue that it is of M. Russell that M. Couturat spoke); far from the *Centennial of the death of Kant*. I rather think that Russell even goes a bit too far.**

At any rate, Poincaré concluded, the old logistic was dead. Before the new could be criticized, he said, it must first be born. He had received interesting letters from Hilbert and Zermelo which he would need time to study, and Russell's own view was in process of change.

There was no further debate with Poincaré. In a rather surprising passage, Jorgenson awards the decision in the argument to Russell on the grounds that no one later took up Poincaré's position. Since Poincaré had opposed the metaphysical claims of logistic and his leading opponent had completely abandoned those claims, further argument was not necessary for Poincaré. On the question of Cantor's theories, it cannot be said that no one later took up Poincaré's view: "Il n'y a pas de infini actual." However, it is true that the

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opposition to logistic and infinite numbers which now entered
the field did not regard itself as the successor of Poincaré.

Brouwer's _Over de Grondlagen der Wiskunde_, published in
1907, referred at length to the debate between Russell and
Poincaré. Brouwer disagreed with both parties to the debate.
He objected to Poincaré's type of intuitionism, on the grounds
that Poincaré had missed the real source of all the difficulties
of logistic and Cantorism. This basic flaw was, said Brouwer,
"the confusion of the act of constructing mathematics with the
language of mathematics."¹

Brouwer continued: "The blunder of logistic is this:
that it creates only a construct of words that can never be
carried over into the real mathematics."² Poincaré himself
showed, according to Brouwer, a mistaken belief that mathematical
existence meant only being "exempt de contradiction." Really,
said Brouwer, mathematical existence means intuitively constructed
"and if an accompanying language is free from contradiction is
not only unimportant in itself, but also no criterion of math­
ematical existence."³

Brouwer explained his own view of the relation between math­
ematics and logistic by presenting a discussion of the various

¹L. E. J. Brouwer, _Over de Grondlagen der Wiskunde_
(Amsterdam-Leipzig: Maas & Van Suchtelen, 1907), 176. I have
translated the quoted passages from the Dutch.
²Ibid., 176. ³Ibid., 177.
stages in the development of mathematics and logic. The first stage was, he said, the pure constructing of intuitive mathematical systems "which if they are applied are expressed outwardly in life and seen in the world as mathematical." The second stage was, then, "the linguistic parallel of mathematics: mathematical speech or writing." At the third level, the language itself became the object of study. Brouwer characterized this third level as "the mathematical view of language: logical constructions in language are noticed, brought up according to principles from the common logic or the expansion of it through logic of relations, the logistic; but the elements of those linguistic constructs are linguistic accompaniments of mathematical construction of relations."

Logistic as a system appeared, said Brouwer, at an even later fourth stage, where the meaning of the logical elements of the third stage was no longer considered. Then, from these elements a "second-order mathematical system," the system of logistic, was constructed. As a fifth stage, a language of logistic in which to state the elements and principles of the system, would be constructed. Sixth, said Brouwer, mathematics itself could be thought of as speech, as Hilbert did. Then, it would be possible to regard the speech of the sixth level as itself composed of meaningless symbols, and construct a third-order mathematics. This would lead to an eighth level linguistic accompaniment of the third-order mathematics. Brouwer suggested
that the process could go on indefinitely, but the entire structure rested upon the real mathematics as its only foundation.

Mathematics itself rests, said Brouwer, upon the intuited continuum, a "flowing" which is the foundation of "the measurable continuum." He described the fundamental intuition as an "over-intuition" of unity in difference, of persistence in change, and held that the idea of the continuum must be prior to any attempts to define that idea or to express it in numerical terms.

Brouwer considered the construction of concepts in intuition as a process of creation very similar to Bradley's "ideal experiments." Juxtaposition in thought of concepts, he said, sometimes but not always caused them to coalesce into new concepts. Whether or not the union would occur could only be discovered by trying it out. Intuitive constructions must be attempted; and only where the concepts did actually coalesce into a new unit could mathematics safely accept and work with the newly-constructed concept, or use it as an element in further constructions.

Brouwer also suggested that men are able to see their lives in mathematical terms as part of a causal intuition. This ability

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1Brouwer, Grondlagen, 173-175.

2Ibid., 5.
also rested, he believed, upon the intuition of time.\footnote{1} In the intellect, said Brouwer, successions were perceived as mathematical series. Man introduced more regularity into nature by deliberate isolation of causal chains, and could even construct pure mathematics by considering this habit of his intellect. Then, when the occasion arose, these pure systems could be projected into actuality as it was experienced. But no system of mathematics was given in experience; not even, he emphasized, the unmeasured time continuum.\footnote{1}

In opposition to Kant's view that the characteristic postulates of Euclidean geometry were synthetic a priori principles, Brouwer argued that the human mind could apply any geometry it chose to its experience. He denied that the experienced world forced any particular system of measurement on us.\footnote{2} Euclidean geometry he regarded as simply a universally accepted system which could easily be replaced. The only a priori synthetic judgments there, the only necessities to which experience must conform, rested, said Brouwer, on the "over-intuition" of the unity-in-plurality of time.\footnote{3} Thus the flow of time was the basis of all mathematics and of the synthetic a priori judgments stating what was required in order that

\footnote{1}{Brouwer, \textit{Grondlagen}, 118 and 98.}
\footnote{2}{ibid., 116-117.}
\footnote{3}{ibid., 119.}
mathematics be possible. Brouwer listed three of these a priori
synthetic judgments: (1) the possibility of mathematical syn­
thesis, the thought of a unity-in-multiplicity and of successive
construction of new unities-in-multiplicity; (2) the possibility
of "interpolation," as in seeing the connection between two
elements of the same whole; (3) the principle of mathematical
induction.

Brouwer concluded, accordingly, that the laws of logic were
dependent upon mathematics, not conversely. Logical postulates
were, he thought, only able to help reveal the structure of a
system already constructed. Logic must begin by taking some
system of thought for granted and then try to find starting
points from which such a system could be constructed. But it
studies only words; and those words are "only the accompaniment
of a wordless mathematical constructing." In this process of
construction, which was the real mathematics, contradiction was
impossible. A verbal contradiction was simply a reflection of
a breakdown in the process, a situation where "no place can be
found in the already-given fundamental structure for the pro­
posed structure."

The Law of Excluded Middle was, said Brouwer, a principle
of logic and as such applied only to words. It need not be
applied to the process of mathematical construction. This, he

\[^1\text{Brouwer, Grondlagen, 119-120.} \quad ^2\text{ibid., 125.}\]
said, would solve the Paradoxes. But, at any rate, neither Russell's logic nor Cantor's set theory had any real mathematical system as a foundation; and logistic itself, being only a translation of the language of mathematics, was only a derivative of mathematical procedures.

\[1\] Brouwer, *Grundlagen*, 163.

\[2\] Ibid., 164. Since construction of an infinite aggregate could never be completed, Brouwer's principles would throw at least a suspicion on any theory of "infinite cardinals."
CHAPTER XII

CONCLUSIONS

This study ends with the end of the first phase in the development of the philosophy of modern logic. The major problems that have divided the logicians in the later years of the twentieth century appear more clearly at the beginning of the period. Further, it seems to me that these early developments in philosophy of logic are significant for modern philosophy in general, for many characteristic attitudes toward philosophy at the present day rest upon philosophy of logic.

However, there is controversy as to whether there is any importance in the history of philosophy at all. Probably a "decent respect for the opinions of mankind" requires some justification for this long inquiry.

While the discovery of truth may very well be completely independent of any knowledge of the course of previous investigations into a subject, the giving of reasons to support the new discovery is usually facilitated by historical knowledge. Knowledge of the truth gained by "inspiration" I regard as not philosophically interesting; for philosophical inquiry is, or ought to be, a search for reasons. In the attempt to understand
the reasoning in many modern philosophical arguments, one encounters from time to time an appeal to "the discoveries of modern logic." These like any other appeal to authority, pose something of a problem.

The obvious alternatives in such a situation are to give a respectful assent to the pronouncement; to reject the authority; or to question whether the authority actually did make that pronouncement and if so what its reasons were. Granted that there cannot be a reason for everything, it seems to me sensible to distinguish between the self-evident truths and the reasoned conclusions used in any argument. Either may reasonably be asked to show its credentials, although different credentials are expected. I have been in this study attempting to find out what metaphysical and epistemological pronouncements have actually been made by modern logic, and which of them are to be taken as self-evident and which as proven by argument. One of the most difficult parts of the task has been, as might be expected, to find out what "modern logic" is, and who speaks for it with authority.

Modern logic, as represented by contemporary authors writing about logic, shows a singular lack of unanimity on almost all philosophical topics. However, no one has a good word for F. H. Bradley and the Idealist Logicians. It seemed reasonable, therefore, to assume that the unifying thread of
modern logic might be found and defined by contrast with the logic of Bradley. By studying the period when logicians decided to abandon that logic, it might be discovered how the new metaphysics was to follow from the new logic, and why both the metaphysics and the logic were superior to the old.

From the study I have made and discussed above, it seems clear that the metaphysical and epistemological principles asserted by the various types of modern logic have been throughout the development premises rather than conclusions. That is, the logic has been developed to suit the metaphysics rather than the metaphysics to suit the logic. This seems to me an interesting result. It has been denied at least by implication in any argument which has appealed to modern logic to support epistemological and metaphysical (or anti-metaphysical) positions, and in any assertion that logic is somehow a more respectable subject than metaphysics. Nevertheless, the metaphysical preoccupation of modern logic seems quite obvious from the evidence of this study; and in order to get a fresh view of a philosophical situation which has been obscured by argument, it is sometimes useful to point out the obvious.

There can, for instance, be no doubt that Brouwer's objections to the mathematical logic of Russell and Whitehead were metaphysical and epistemological objections. Russell rejected the treatment of relations as characteristics of complex
subjects proposed by Peirce and Schröder, by his own statement, because it was metaphysically unsound. Mathematicians such as Kronecker and Poincaré, rejected Cantor's work on metaphysical grounds and regarded the paradoxes which were eventually discovered as only natural in an intrinsically absurd theory.

If logic were still thought of in the broad sense of Bradley and the early Russell, these controversies might quite innocently be classified as logical. But when logic is thought of as a study of techniques of manipulating symbols, it becomes absurd to classify any of these arguments as concerned with points of logic. Furthermore, to do so in the present context inhibits free discussion and study of the issues involved. It transforms these ancient philosophical problems into parts of a mathematical mystique to be understood only by initiates who have learned the intricacies of both set theory and mathematical logic. Actually, the central problems of philosophy of logic, to eliminate that nerve-wracking word, "metaphysics," are not questions about the derivation of one formula from another but about the nature of the world and of our knowledge of the world. They belong to the field of general philosophy rather than to mathematics. The technical developments based upon the proposed answers to these questions are interesting, but their primary function in the philosophical argument seems merely to have been as potential reductio's of the original hypotheses.
It may be that any conclusions about the course of philosophy of logic drawn at this point are premature. By 1913 Russell had recovered from the despondency of 1906; and certainly, mathematical logic continued to be studied by philosophers and was increasingly cited as an authority for various metaphysical and epistemological conclusions. Yet, the results of this study seem to me largely negative, so far as it has been an attempt to find the source of these philosophical successes. For example, I think it is a common assertion that "modern logic has refuted Kant." But the intuitionist philosophy of mathematics was flourishing in 1906 and had, at least temporarily, silenced the more wide-reaching claims of mathematical logic. Of course, the arguments against the claims of logistic were presented largely in languages other than English, and Russell's withdrawals from his earlier positions seem usually to have been unnoticed in the discussions, except by Poincaré. The situation, however, gives rise to an uncomfortable suspicion that an appeal to the discoveries of modern logic in a philosophical discussion is a poor reason, unless logic has something better to offer in the years after 1906 than they had in the years before.

Also, I think the complexity of the modern logical development needs emphasizing. Apparently there never was, from the philosophical point of view, a modern logic to appeal to, only
modern logics. There was no general agreement on philosophy of logic, or even on proper techniques, even among the mathe­matically-inclined logicians. Where there is no consensus of opinion to appeal to, individual arguments must be examined more carefully. In this connection, it is possible at long range to isolate some issues which apparently were not clear to the participants in the arguments of these early years. For example, the claim that all good reasoning is deduction or that deductive logic was the whole "Art of Reasoning" was definitely abandoned by Russell and Couturat by 1906. It is not clear just what sort of union they now asserted between logic and mathematics. I have suggested that the deductive system was to serve with intuition in a kind of checks-and-balances system like, as Russell said, any other empirical science. But this may be an inadequate statement.

The meaning in the Philosophy of logic of "defining" one entity in terms of others also remains obscure. Poincaré’s articles must have made the logisticians conscious that there was a problem. They gave at least 3 different answers, as discussed above, but the relationships between these views in their thought are not clear. Brouwer’s book, summarized in Chapter XI of this dissertation, gives an unusually complete working-out of the implications of the "creation" theory of definition; but he would not have agreed that the new concept was a less "fundamental logical entity" than its
parts, as Russell held when he was thinking in terms of creative
definition. The view that definition was simple substitution
of symbols appears to have been taken most seriously by Hilbert,
and Poincaré had pointed out difficulties in his view. Russell's
mapping theory never seems to have been taken seriously in the
arguments.

I think that a clear statement of just what a definition
is would clarify a great many of the issues in philosophy of
logic; but I suspect that no such statement could have been
agreed upon by all the parties to these early disputes. The
problem of the status of defined entities and the question of
the validity of analysis as a philosophical method depended on
the answer given to the old problem of the One and the Many.
I think the participants to these disputes recognized that they
were disputing basic issues of metaphysics.

The metaphysical and epistemological issues arising from
his Platonism were at this time beginning to claim a great
deal of Russell's attention, as is shown in his article on
"denoting."¹ The metaphysics of Russell's logistic as it stood
was clearly inadequate and needed further development; the
epistemology, too, was still very casual. Apparently, Russell
regarded his statements on these questions as only provisional,
to be replaced by better insights later. The one issue on which
he felt himself firmly committed, and which he later advanced

as the source of all the philosophical achievements of logistic, was Cantor's theory of infinite numbers.

Russell argued in 1913, after publication of the first edition of *Principia Mathematica* that his logistic had now established Cantor's theory firmly. By doing so, it had, he said, established both Platonic Realism and Realism as opposed to Idealism. In this renewed metaphysical claim, there are two separate contentions: (1) that the problem of the paradoxes and other obstacles to the acceptance of Cantor's theories had been eliminated; and (2) that Cantor's theory disproved the philosophy of Kant.

There had still been no real debate on the second question. If such debate arose, it was later in the development of the philosophy of logic. Couturat's interpretation of Kantian views was apparently taken as authoritative by the logicians. The theory of infinite numbers was accepted as refuting any claim that all thought was calling up of images which were copies of previous sensations, at least by the advocates of logistic. But on the historical question whether this view is Kantian the argument simply took the form of assertion and counter-assertion. It seems to me a bit odd that Russell should have at first criticized Kant for an irrational belief that there might be "infinite given wholes," only to insist later that Kant could not possibly believe that there were infinite given wholes. Whatever change of philosophy Russell himself may have undergone
in the interval between the two statements should not have af­
fected the logical implications of Kant's views. Any decision
as to whether the views of Kant and the views of Cantor actually
are incompatible must, I believe, be based on a study of the
views of Kant and Cantor, and not on the pronouncements of modern
logic or modern logicians.

If these theories are incompatible, the question as to which
is correct is far from closed. Indeed, at present the intuitionists seem to be ahead on points. Whether they have better ar­
guments than the defenders of infinite numbers, or simply more
stubborn metaphysical convictions can be determined only by a
study of the next stage in the development of the philosophy of
modern logic.
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My full name is Gwendolyn Ethel Duell Bowne. I was born in Spokane, Washington, on September 15, 1925. I attended Harrison High School in Harrison, Idaho, and after graduation from high school completed a course at Kinman Business University in Spokane, Washington. I entered the University of Washington, Seattle, Washington, in February of 1946, after working for several years for the Spokane Air Technical Service Command. I received the B.A. degree from the University of Washington in 1948, and the M.A. degree from the University of Washington in 1951. While completing work toward these two degrees, I worked as undergraduate and as graduate assistant. I attended Cornell University from 1951 to 1953 as a candidate for the Ph.D. degree, but did not complete the requirements. While at Cornell, I worked as graduate assistant and as Instructor. In 1961 I entered the Ohio State University, and was in residence through spring quarter of 1962 while completing the requirements for the Ph. D. degree. Since autumn semester, 1962-63, I have been employed as Instructor in Philosophy at Edinboro State College, Edinboro, Pennsylvania.