This dissertation has been microfilmed exactly as received

HULBERT, Lewis Eugene, 1924—
THE NUMERICAL SOLUTION OF TWO-
DIMENSIONAL PROBLEMS OF THE THEORY
OF ELASTICITY.

The Ohio State University, Ph.D., 1963
Engineering Mechanics

University Microfilms, Inc., Ann Arbor, Michigan
THE NUMERICAL SOLUTION OF TWO-DIMENSIONAL
PROBLEMS OF THE THEORY OF ELASTICITY

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Lewis Eugene Hulbert, B.Sc., M.Sc.

*****

The Ohio State University
1963

Approved by

[Signature]
Adviser
Department of Engineering
Mechanics
ACKNOWLEDGMENTS

I would like to express my appreciation to my major adviser, Francis W. Niedenfuhr for his guidance and encouragement during the course of research on this dissertation.

I would like to express my gratitude to the Numerical Computation Laboratory of The Ohio State University for providing the computing services necessary for performing the research. I would also like to express my gratitude to Battelle Memorial Institute for the financial assistance that they provided me under their education program.

Finally, I would like to thank my wife, Beatrice, for her understanding, patience, and encouragement, without which this dissertation could not have been written.
## CONTENTS

ACKNOWLEDGMENTS ............................................. II  

LIST OF TABLES ......................................... v  

LIST OF ILLUSTRATIONS ........................................ vili  

INTRODUCTION ............................................... 1  

Chapter

1. DEVELOPMENT OF THE MATHEMATICAL EQUATIONS OF THE FIRST FUNDAMENTAL PROBLEMS OF THE PLANE THEORY OF ELASTICITY .............................. 6  
   The equations of the three-dimensional theory of elasticity .............................. 6  
   Plane strain ........................................... 9  
   The generalized plane stress state ........................................... 13  
   Equations of the plane problems in terms of stresses ................................... 19  
   Displacements in multiply connected regions.  
      Michell's equations ................................. 24  

REFERENCES ........................................... 55  

2. THE STRESS FUNCTION FOR PLANE PROBLEMS .......... 57  
   Reduction of the plane problem to a problem without body forces .............................. 57  
   The series solution of the biharmonic equation in polar coordinates .............................. 68  
   The stress function in multiply connected regions .............................. 85  
   The symmetric stress functions for multiply connected regions .............................. 91  

REFERENCES ........................................... 115
### CONTENTS

(Continued)

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3. SATISFACTION OF THE BOUNDARY CONDITIONS. THE POINT MATCHING APPROACH</td>
<td>116</td>
</tr>
<tr>
<td>The point matching approach</td>
<td>119</td>
</tr>
<tr>
<td>The classical approach for satisfying boundary conditions</td>
<td>129</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>132</td>
</tr>
<tr>
<td>4. NUMERICAL RESULTS</td>
<td>134</td>
</tr>
<tr>
<td>Circular holes in an infinite plate arranged with cylindrical symmetry</td>
<td>137</td>
</tr>
<tr>
<td>Circular holes arranged with cylindrical symmetry in a finite circular plate</td>
<td>172</td>
</tr>
<tr>
<td>Problems involving noncircular holes</td>
<td>202</td>
</tr>
<tr>
<td>Problems involving infinite rows of holes</td>
<td>211</td>
</tr>
<tr>
<td>Torsion problems</td>
<td>219</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>228</td>
</tr>
<tr>
<td>5. THE COMPUTING PROGRAM</td>
<td>230</td>
</tr>
<tr>
<td>Program I</td>
<td>231</td>
</tr>
<tr>
<td>Program II</td>
<td>244</td>
</tr>
<tr>
<td>Program III</td>
<td>246</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>276</td>
</tr>
<tr>
<td>AUTOBIOGRAPHY</td>
<td>280</td>
</tr>
<tr>
<td>Table</td>
<td>Description</td>
</tr>
<tr>
<td>-------</td>
<td>-------------</td>
</tr>
<tr>
<td>1.</td>
<td>Boundary values of $\sigma_0/p$ on the outside hole of three holes in a line in an infinite plate loaded in uniaxial tension normal to the line of centers of the holes</td>
</tr>
<tr>
<td>2.</td>
<td>Boundary values of $\sigma_0/p$ on the center hole of three holes in a line in the infinite plate loaded in uniaxial tension normal to the line of centers of the holes</td>
</tr>
<tr>
<td>3.</td>
<td>Values of $\sigma_x/p$ and $\sigma_0/p$ on the x-axis between the center and outside holes of three holes in a line in the infinite plate loaded in uniaxial tension normal to the line of centers of the holes</td>
</tr>
<tr>
<td>4.</td>
<td>Boundary values of $\sigma_0/p$ on the semicircle of the basic symmetry element for two, four, and six holes symmetrically arranged along the arc of a circle in the infinite plate</td>
</tr>
<tr>
<td>5.</td>
<td>Boundary values of $\sigma_0/p$ on $r_0$ for seven and nineteen pressurized holes in the infinite plate</td>
</tr>
<tr>
<td>6.</td>
<td>Boundary values of $\sigma_0/p$ on $r_1,0$ for seven and nineteen pressurized holes in the infinite plate</td>
</tr>
<tr>
<td>7.</td>
<td>Boundary values of $\sigma_0/p$ on $r_2,0$ and $r_3,0$ of nineteen pressurized holes in the infinite plate</td>
</tr>
<tr>
<td>8.</td>
<td>Values of the stresses calculated at selected interior points of the seven-hole problem</td>
</tr>
<tr>
<td>9.</td>
<td>Values of the stresses calculated at selected interior points of the nineteen-hole problem</td>
</tr>
<tr>
<td>10.</td>
<td>Boundary values of $\sigma_0/p$ on the intersection of the x-axis with the inner and outer boundaries of the eccentric hole problems</td>
</tr>
</tbody>
</table>
### LIST OF TABLES

(Continued)

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>11. Boundary values of $\sigma_\theta/\rho$ on the hole boundaries for three and four holes in a circular plate with ratio of hole radius to exterior radius of 0.20</td>
<td>180</td>
</tr>
<tr>
<td>12. Boundary values of $\sigma_\theta/\rho$ for four holes in a circular plate with ratio of hole radius to exterior boundary radius of 0.25</td>
<td>181</td>
</tr>
<tr>
<td>13. Boundary values of the $\sigma_\theta/\rho$ on the exterior boundary of problems involving three and four holes in a circular plate</td>
<td>182</td>
</tr>
<tr>
<td>14. Stresses on the central hole of problems involving seven circular holes in a circular plate</td>
<td>185</td>
</tr>
<tr>
<td>15. Stresses on the exterior boundary of problems involving seven circular holes in a circular plate</td>
<td>185</td>
</tr>
<tr>
<td>16. Stresses on the eccentric hole of problems involving seven circular holes in a circular plate</td>
<td>186</td>
</tr>
<tr>
<td>17. Boundary values of $R\sigma_C/\rho$ for problems involving six circular holes in a circular plate loaded by six concentrated tensile forces</td>
<td>197</td>
</tr>
<tr>
<td>18. Boundary values of $\sigma_\theta/\rho$ at points of the boundary of a pressurized star-shaped hole in the infinite plate as a function of $x$</td>
<td>207</td>
</tr>
<tr>
<td>19. Boundary values of $\sigma_C/\rho$ for an infinite row of pressurized holes as calculated by the computing program and given by Howland</td>
<td>214</td>
</tr>
<tr>
<td>20. Boundary values of $\sigma_C/\rho$ for two rows of pressurized holes in the infinite plate</td>
<td>217</td>
</tr>
<tr>
<td>21. Boundary shear stresses on the eccentric holes of circular tubes with longitudinal circular holes as a function of $\alpha = \pi - \Theta$</td>
<td>225</td>
</tr>
<tr>
<td>Table</td>
<td>Page</td>
</tr>
<tr>
<td>-------</td>
<td>------</td>
</tr>
<tr>
<td>22. Boundary shear stresses on the center holes of circular tubes with longitudinal circular holes as a function of $\alpha = \pi - \theta$</td>
<td>226</td>
</tr>
<tr>
<td>23. Boundary shear stresses on the exterior boundary of circular tubes with longitudinal circular holes as a function of $\alpha = \pi - \theta$</td>
<td>227</td>
</tr>
<tr>
<td>24. Card input to Program I</td>
<td>232</td>
</tr>
<tr>
<td>25. Card input to Program III</td>
<td>247</td>
</tr>
<tr>
<td>26. Function table for subroutine DPHI(\phi)</td>
<td>253</td>
</tr>
<tr>
<td>27. Function table for subroutine DPHIDX</td>
<td>254</td>
</tr>
<tr>
<td>28. Function table for subroutine DPHIDY</td>
<td>255</td>
</tr>
<tr>
<td>29. Function table for subroutine DPHIDX2</td>
<td>256</td>
</tr>
<tr>
<td>30. Function table for subroutine DPHDY2</td>
<td>257</td>
</tr>
<tr>
<td>31. Function table for subroutine DPHDXY</td>
<td>258</td>
</tr>
</tbody>
</table>
# LIST OF ILLUSTRATIONS

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Illustration of plane strain</td>
<td>9</td>
</tr>
<tr>
<td>2</td>
<td>Illustration of generalized plane stress</td>
<td>14</td>
</tr>
<tr>
<td>3</td>
<td>Illustration of the method of introducing a cut in a ring</td>
<td>25</td>
</tr>
<tr>
<td>4</td>
<td>The physical interpretation of dislocations</td>
<td>29</td>
</tr>
<tr>
<td>5</td>
<td>The boundary coordinate system</td>
<td>37</td>
</tr>
<tr>
<td>6</td>
<td>A line integral circuit around a simply connected region</td>
<td>44</td>
</tr>
<tr>
<td>7</td>
<td>Line integral circuits enclosing a hole</td>
<td>48</td>
</tr>
<tr>
<td>8</td>
<td>Calculation of dislocations in the general multiply connected region</td>
<td>51</td>
</tr>
<tr>
<td>9</td>
<td>The circuit for evaluating $d_0$</td>
<td>73</td>
</tr>
<tr>
<td>10</td>
<td>Two cases for which the origin is external to the region</td>
<td>73</td>
</tr>
<tr>
<td>11</td>
<td>The circular path for evaluating the dislocations</td>
<td>79</td>
</tr>
<tr>
<td>12</td>
<td>A region containing an arbitrary number of arbitrary circular holes</td>
<td>86</td>
</tr>
<tr>
<td>13</td>
<td>A group of holes with cylindrical symmetry</td>
<td>92</td>
</tr>
<tr>
<td>14</td>
<td>A row of holes with translational symmetry</td>
<td>92</td>
</tr>
<tr>
<td>15</td>
<td>A point set with cylindrical periodicity</td>
<td>94</td>
</tr>
<tr>
<td>16</td>
<td>A set of points with cylindrical symmetry</td>
<td>98</td>
</tr>
<tr>
<td>17</td>
<td>The angular functions $\psi_j$ for cylindrical symmetry</td>
<td>101</td>
</tr>
<tr>
<td>18</td>
<td>The functions $\psi^*$ for cylindrical symmetry</td>
<td>105</td>
</tr>
</tbody>
</table>
### LIST OF ILLUSTRATIONS

(Continued)

<table>
<thead>
<tr>
<th>Figure</th>
<th>Illustration Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>19.</td>
<td>$\psi$ for translational symmetry</td>
<td>110</td>
</tr>
<tr>
<td>20.</td>
<td>The doubly periodic array of cooling holes</td>
<td>121</td>
</tr>
<tr>
<td>21.</td>
<td>Three holes in uniaxial tension</td>
<td>138</td>
</tr>
<tr>
<td>22.</td>
<td>Boundary values of $\sigma_c/p$ for three equal colinear holes in an infinite plate with uniaxial tension, $b/a = .15$</td>
<td>145</td>
</tr>
<tr>
<td>23.</td>
<td>Boundary values of $\sigma_c/p$ for three equal colinear holes in an infinite plate with uniaxial tension, $b/a = .33$</td>
<td>146</td>
</tr>
<tr>
<td>24.</td>
<td>Values of $\sigma_x/p$ and $\sigma_y/p$ on the x-axis between the holes for three equal colinear holes in an infinite plate with uniaxial tension, $b/a = .15$</td>
<td>147</td>
</tr>
<tr>
<td>25.</td>
<td>Values of $\sigma_x/p$ and $\sigma_y/p$ on the x-axis between the holes for three equal colinear holes in an infinite plate with uniaxial tension, $b/a = .83$</td>
<td>147</td>
</tr>
<tr>
<td>26.</td>
<td>The two-hole problem</td>
<td>149</td>
</tr>
<tr>
<td>27.</td>
<td>The four-hole problem</td>
<td>149</td>
</tr>
<tr>
<td>28.</td>
<td>The six-hole problem</td>
<td>150</td>
</tr>
<tr>
<td>29.</td>
<td>Boundary values of $\sigma_\theta/p$ for two holes in an infinite plate loaded by internal pressure</td>
<td>152</td>
</tr>
<tr>
<td>30.</td>
<td>Boundary values of $\sigma_\theta/p$ for four holes in an infinite plate loaded by internal pressure</td>
<td>153</td>
</tr>
<tr>
<td>31.</td>
<td>Boundary values of $\sigma_\theta/p$ for six holes in an infinite plate loaded by internal pressure</td>
<td>154</td>
</tr>
<tr>
<td>32.</td>
<td>The seven-hole problem</td>
<td>157</td>
</tr>
<tr>
<td>33.</td>
<td>The nineteen-hole problem</td>
<td>157</td>
</tr>
<tr>
<td>34.</td>
<td>Boundary values of $\sigma_\theta/p$ for seven holes in an infinite plate loaded by internal pressure</td>
<td>164</td>
</tr>
</tbody>
</table>

ix
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>35.</td>
<td>Boundary values of $\sigma_\phi/p$ on boundaries $n_0$ and $n_1,0$ of nineteen pressurized holes in the infinite plate</td>
<td>165</td>
</tr>
<tr>
<td>36.</td>
<td>Boundary values of $\sigma_\phi/p$ on boundary $n_2,0$ of nineteen pressurized holes in the infinite plate</td>
<td>166</td>
</tr>
<tr>
<td>37.</td>
<td>Boundary values of $\sigma_\phi/p$ on boundary $r_3,0$ of nineteen pressurized holes in the infinite plate</td>
<td>167</td>
</tr>
<tr>
<td>38.</td>
<td>The tube sheet with 50 per cent ligament efficiency</td>
<td>168</td>
</tr>
<tr>
<td>39.</td>
<td>The eccentric hole problem</td>
<td>174</td>
</tr>
<tr>
<td>40.</td>
<td>Three circular holes in a circular plate</td>
<td>178</td>
</tr>
<tr>
<td>41.</td>
<td>Four circular holes in a circular plate</td>
<td>178</td>
</tr>
<tr>
<td>42.</td>
<td>Seven circular holes in a circular plate</td>
<td>183</td>
</tr>
<tr>
<td>43.</td>
<td>Boundary values of $\sigma_\phi/p$ on the center hole and exterior boundary of a circular plate with seven pressurized holes</td>
<td>187</td>
</tr>
<tr>
<td>44.</td>
<td>Boundary values of $\sigma_\phi/p$ on the center hole and exterior boundary of a circular plate with seven holes with pressure in the central hole only</td>
<td>188</td>
</tr>
<tr>
<td>45.</td>
<td>Boundary values of $\sigma_\phi/p$ on eccentric holes of a circular plate with seven pressurized holes</td>
<td>189</td>
</tr>
<tr>
<td>46.</td>
<td>Boundary values of $\sigma_\phi/p$ on eccentric hole in a circular plate with seven holes: pressure in the central hole only</td>
<td>190</td>
</tr>
<tr>
<td>47.</td>
<td>The six-hole problem of a finite plate showing two positions of application of concentrated loads</td>
<td>192</td>
</tr>
<tr>
<td>48.</td>
<td>Theoretical and experimental values of $R\sigma_\phi/p$ on the hole boundaries of a circular plate having six holes and loaded by six concentrated forces; Problem 1</td>
<td>198</td>
</tr>
</tbody>
</table>
LIST OF ILLUSTRATIONS
(Continued)

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>49.</td>
<td>Theoretical and experimental values of $R\sigma/p$ on the hole boundaries of</td>
<td></td>
</tr>
<tr>
<td></td>
<td>a circular plate having six holes and loaded by six concentrated forces;</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Problem 2</td>
<td>199</td>
</tr>
<tr>
<td>50.</td>
<td>Theoretical and experimental values of $R\sigma/p$ on the hole boundaries of</td>
<td></td>
</tr>
<tr>
<td></td>
<td>a circular plate having six holes and loaded by six concentrated forces;</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Problem 3</td>
<td>200</td>
</tr>
<tr>
<td>51.</td>
<td>Theoretical and experimental values of $R\sigma/p$ on the hole boundaries of</td>
<td></td>
</tr>
<tr>
<td></td>
<td>a circular plate having six holes and loaded by six concentrated forces;</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Problem 4</td>
<td>201</td>
</tr>
<tr>
<td>52.</td>
<td>The star-shaped hole</td>
<td>206</td>
</tr>
<tr>
<td>53.</td>
<td>The hole formed by two intersecting circles</td>
<td>208</td>
</tr>
<tr>
<td>54.</td>
<td>Two infinite rows of holes</td>
<td>215</td>
</tr>
<tr>
<td>55.</td>
<td>Boundary values of $R\sigma/p$ for two infinite rows of pressurized holes</td>
<td>218</td>
</tr>
<tr>
<td>56.</td>
<td>The cross section of a circular cylinder with three longitudinal circular holes</td>
<td>223</td>
</tr>
<tr>
<td>57.</td>
<td>The cross section of a circular cylinder with five longitudinal circular holes</td>
<td>223</td>
</tr>
<tr>
<td>58.</td>
<td>Over-all flow chart for Program I</td>
<td>259</td>
</tr>
<tr>
<td>59.</td>
<td>Flow chart for Panel I, Program I: initialization</td>
<td>260</td>
</tr>
<tr>
<td>60.</td>
<td>Flow chart for Panel II, Program II: symmetry poles for rotational symmetry</td>
<td>261</td>
</tr>
<tr>
<td>61.</td>
<td>Flow chart for Panel III, Program I: symmetry poles for translational symmetry</td>
<td>262</td>
</tr>
<tr>
<td>62.</td>
<td>Flow chart for Panel IV, Program I: set up boundary conditions and write each</td>
<td>263</td>
</tr>
<tr>
<td></td>
<td>condition on tape</td>
<td></td>
</tr>
</tbody>
</table>
LIST OF ILLUSTRATIONS
(Continued)

Figure                  Page
63. Flow chart for Subroutine BØNDRY (conic sections)  .  264
64. Flow chart for Subroutine BC1  . . . . . . . . . . . . . . . 265
65. Main flow chart for derivative subroutines  . . . . . . . 266
66. Flow chart for Panel I, derivative subroutines  . . . . 267
67. Flow chart for subroutine ANGLE  . . . . . . . . . . . . . 268
68. Over-all flow chart for Program II  . . . . . . . . . . . . 269
69. Flow chart for Panel I, Program II: read matrix
tape and check for errors  . . . . . . . . . . . . . . . . . . . . 270
70. Flow chart for Panel II, Program II: calculate A'A,
A'B and overwrite them on A and B  . . . . . . . . . . . . . 271
71. Over-all flow chart for Program III  . . . . . . . . . . . . 272
72. Flow chart for Panel I, Program III: initialization  . 273
73. Flow chart for Subroutine BDYSIG  . . . . . . . . . . . . . 274
74. Flow chart for Subroutine SIGINT  . . . . . . . . . . . . . 275
The study of the stress concentration effects of holes and cutouts is of fundamental importance in modern engineering design. The analysis of problems involving three-dimensional stress distributions is extremely complex and only a very few special solutions are known. Very often, however, an object under study will have a configuration such that the stress distributions in the object will be essentially two dimensional.

The present work is concerned with the numerical solution of plane two-dimensional stress problems. These arise in the study of thin plates loaded by in-plane forces, and long prisms loaded on the lateral surfaces by loads applied on the generators of the prism. The first case is called generalized plane stress and the second is called plane strain. We will show in Chapter 1 that there is a mathematical analogy between the statements of problems for these two cases. We will consider in this dissertation only problems for which the stresses or the external forces are specified on the boundary. Problems of this type have been called by Muskhelishvili "the first fundamental problem of plane elasticity theory", and we shall use this notation also.
The mathematical statement of the first fundamental problem of plane elasticity theory will be derived in a general form in Chapter 1 of this dissertation. At the beginning of Chapter 2, we will transform the statement of the problem in such a way that the basic differential equations are considerably simplified. At this point, we will introduce the Airy stress function. We will show that, in terms of the stress function, the statement of the first fundamental problem is reduced to a single fourth-order differential equation and certain boundary conditions. The differential equation is the well-known biharmonic equation

\[ \nabla^4 \phi \equiv \left( \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^2} \right) \phi = 0 \]

Complete series forms of the stress function satisfying the biharmonic equation have been known in a number of coordinate systems for over a half century. The application of these stress-function series to the solution of boundary value problems in plane elasticity theory has been studied very intensively by a great number of investigators. However, in trying to use the stress-function series to solve boundary problems, it is usually found that an enormous amount of algebra and numerical calculations must be performed in order to determine the coefficients of the terms of the stress function so that the boundary conditions are satisfied. Thus, the number of plane elasticity problems that have been solved is actually quite small. If the region of the problem
is simply or doubly connected, the Kolosov-Muskheilishvili complex variable technique can be used to find the solution to the boundary problem. However, if the region has higher connectivity\(^1\), this technique is no longer applicable.

In this dissertation, we shall be mainly concerned with solving boundary problems for such multiply connected regions. In Chapter 2, we will consider certain forms of the stress-function series that are applicable to stress problems for many different types of multiply connected regions. These stress functions are somewhat more general than are usually found in the literature. We will write the derivations of these stress functions almost entirely in real notation, although using complex notation would have been simpler. The reason for this is that we will use these stress functions in a computing program.

In Chapter 3, we will discuss the point matching technique. This is a powerful numerical technique for satisfying the boundary conditions. Although the solutions obtained by this technique are approximate, they are very accurate if the series chosen for the given problem has a sufficiently rapid convergence.

The numerical calculations, necessary in the application of the point matching technique, are easily carried out on a digital computer.

---

\(^1\)In this dissertation, the connectivity of a region will be taken equal to the number of simple closed curves bounding the region. For example, a finite region with two holes or an infinite region with three holes will be called triply connected.
electronic computer. In Chapter 5, we will describe the computer program written as part of this dissertation. This program was designed to solve even complicated stress problems with a minimum amount of input data.

We used the computer program to solve a variety of stress problems. The numerical results are described in Chapter 4. In a number of cases, the problems have been solved by other approaches. In these cases, we will compare the results obtained with our computer program with the results obtained by others. For certain problems, the results appear to be new. These problems are discussed in detail.

It is felt that the results demonstrate the power of the point matching technique that was used. The only problems for which the obtained solutions did not possess a high degree of accuracy were those in which the chosen stress function had slow convergence. The results demonstrate that if a stress function is chosen having a reasonably rapid convergence, the computer program can be used to obtain the solution to high accuracy with a modest amount of machine computing time.

These results have implications beyond the relatively small number of problems solved here. The point matching technique should be useful as a standard tool in solving any boundary value problem (at least in two dimensions) for which a convergent series solution is known, satisfying the differential equation of the problem.
The technique has been used by a number of scientists to solve a variety of problems. However, apparently, these workers have not realized its widespread usefulness. To the author's knowledge, the problems solved in this dissertation represent the most complicated problems yet solved by the technique. Other, more intricate problems should prove amenable to the attack used here. It is felt that the point matching technique should permit for the first time, sufficiently accurate solutions to the problems involving shells with holes. Of course, there are many problems yet to be solved even in two-dimensional potential theory. This technique should prove equally valuable for solving these problems.
CHAPTER 1

DEVELOPMENT OF THE MATHEMATICAL EQUATIONS OF THE FIRST FUNDAMENTAL PROBLEMS OF PLANE STRAIN AND GENERALIZED PLANE STRESS

We will begin by simply stating the equations of the linear, small deformation theory of elasticity of isotropic bodies. We will then show how the equations of generalized plane stress and plane strain are derived from these equations. The derivation of the three-dimensional equations is given in many books on the theory of elasticity \(1,2\).\(^1\)

**The equations of the three-dimensional theory of elasticity**

The equations of a boundary problem in the theory of elasticity consist of

(a) the equilibrium equations

\[
\begin{align*}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \sigma &= 0, \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \tau_y &= 0, \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \tau_z &= 0
\end{align*}
\]

\(^1\)Numbered references are given at the end of each chapter. A listing of all of the references is given also at the end of the dissertation.

6
(b) the stress-displacement relations

\[
\begin{align*}
\sigma_x &= \lambda \varepsilon_x + 2\mu \frac{\partial \varepsilon_x}{\partial x}, \\
\sigma_y &= \lambda \varepsilon_y + 2\mu \frac{\partial \varepsilon_y}{\partial y}, \\
\sigma_z &= \lambda \varepsilon_z + 2\mu \frac{\partial \varepsilon_z}{\partial z}, \\
\tau_{xy} &= \mu \left( \frac{\partial \varepsilon_y}{\partial x} + \frac{\partial \varepsilon_x}{\partial y} \right), \\
\tau_{yz} &= \mu \left( \frac{\partial \varepsilon_z}{\partial y} + \frac{\partial \varepsilon_y}{\partial z} \right), \\
\tau_{xz} &= \mu \left( \frac{\partial \varepsilon_x}{\partial z} + \frac{\partial \varepsilon_z}{\partial x} \right), \\
\varepsilon_{xy} &= \lambda \varepsilon_x + \mu \varepsilon_y, \\
\varepsilon_{yz} &= \lambda \varepsilon_y + \mu \varepsilon_z, \\
\varepsilon_{xz} &= \lambda \varepsilon_x + \mu \varepsilon_z,
\end{align*}
\]  

(1.2)

where \( \theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \), and \( \lambda, \mu \), are Lamé's coefficients;

and

(c) the boundary conditions (first fundamental problem)

\[
\begin{align*}
\varepsilon_{xx} &= \cos(n_x) \cos(n_y) + \cos(n_y) \cos(n_z) + \cos(n_z) \cos(n_x), \\
\varepsilon_{yy} &= \cos(n_y) \cos(n_z) + \cos(n_z) \cos(n_x), \\
\varepsilon_{zz} &= \cos(n_z) \cos(n_x) + \cos(n_x) \cos(n_y), \\
\tau_{xy} &= \cos(n_x) \cos(n_y), \\
\tau_{yz} &= \cos(n_y) \cos(n_z), \\
\tau_{xz} &= \cos(n_z) \cos(n_x),
\end{align*}
\]

(1.3)

where \( \cos(n_x), \cos(n_y), \cos(n_z) \) are the direction cosines of the outward normal to the boundary at a boundary point, and \( f_x, f_y, f_z \) are the components of the known force vector acting on the surface at the boundary point.

Equations (1.1) and (1.2) hold at every point of the open region of the problem and (1.3) holds at every point of the bounding surface of the problem. It is assumed that the body force components \( X, Y, \) and \( Z \) in (1.1) are completely known (as they usually are).

Two other sets of relations are of fundamental importance in the theory of elasticity. These are the strain-displacement relations

\[
\begin{align*}
\varepsilon_{xx} &= \frac{\partial u}{\partial x}, \\
\varepsilon_{yy} &= \frac{\partial v}{\partial y}, \\
\varepsilon_{zz} &= \frac{\partial w}{\partial z}, \\
\gamma_{xy} &= \frac{\partial u}{\partial y}, \\
\gamma_{yz} &= \frac{\partial v}{\partial z}, \\
\gamma_{xz} &= \frac{\partial w}{\partial x},
\end{align*}
\]

(1.4)
and the strain compatibility equations

\[
\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}, \\
\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = \frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z}, \\
\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} = \frac{\partial^2 \varepsilon_{xz}}{\partial x \partial z}, \\
2 \frac{\partial \varepsilon_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x} \left( -\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{yx}}{\partial z} \right), \\
2 \frac{\partial \varepsilon_{yy}}{\partial x \partial z} = \frac{\partial}{\partial y} \left( -\frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yx}}{\partial z} \right), \\
2 \frac{\partial \varepsilon_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z} \left( -\frac{\partial \tau_{zz}}{\partial z} + \frac{\partial \tau_{yz}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} \right).
\]

In order that these equations be satisfied, it is necessary to assume that the displacements \( u, v, w \) are continuous and have continuous derivatives up to the third order. This condition will be assumed satisfied throughout the dissertation. By (1.2), this implies that the stresses possess continuous derivatives up to the second order.

We will now derive the equations of the plane theory of elasticity from (1.1) through (1.5). We will consider first the problem of plane strain.
A body is said to be in a state of plane strain parallel to the xy-plane if the displacement component \( w \) is identically zero and if the components \( u \) and \( v \) do not depend on \( z \).

With these assumptions, (1.2) becomes

\[
\begin{align*}
\sigma_x &= \lambda \varepsilon + 2\mu \frac{\partial u}{\partial x}, \\
\sigma_y &= \lambda \varepsilon + 2\mu \frac{\partial v}{\partial y}, \\
\sigma_z &= \lambda \varepsilon, \\
\tau_{xy} &= \mu \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right), \\
\tau_{xz} &= \tau_{yz} = 0,
\end{align*}
\]
where \( \sigma = \frac{2u}{\partial x} + \frac{2v}{\partial y} \).

Thus, the stresses are independent of \( z \).

Equations (1.1) then become

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \tau_{yz} = 0 ,
\]

\[
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \gamma = 0 .
\]

(Thus, for plane strain, the body forces must have \( x \)- and \( y \)-components which are independent of the \( z \)-coordinate and vanishing \( z \)-component.)

For cases of plane strain, it is usually assumed that the region consists of a prismatic cylinder whose lateral sides are formed by generators parallel to the \( z \)-axis and whose ends are parallel to the \( xy \)-plane (Fig. 1). In this case, we have for boundary conditions (1.3)

on the lateral surfaces:

\[ \tau_x \cos (\sigma_x) + \tau_{xy} \cos (\tau_{xy}, \gamma) = f_x , \]

\[ \tau_{xy} \cos (\tau_{xy}, \gamma) + \tau_{yz} \cos (\tau_{yz}, \gamma) = f_y , \]

\[ f_z = 0 . \]

on the ends:

\[ f_x = f_y = c , \]

\[ \tau_x \cos (\tau_x, \gamma) = f_z , \]

where in the last relation \( \cos (\gamma, z) = \pm 1 \).
We can thus state the boundary value problem for plane
strain of a prismatic cylinder as follows:

the equilibrium equations

\[ \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + f = 0, \]

\[ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + g = 0; \]

the stress-displacement relations

\[ \varepsilon_x = \lambda \varepsilon + 2\mu \frac{\partial u}{\partial x}, \varepsilon_y = \lambda \varepsilon + 2\mu \frac{\partial v}{\partial y}, \]

\[ \epsilon_{xy} = \mu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right), \]

where \( \varepsilon = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \);

the boundary conditions

\[ \sigma_x \xi_1 (x, y) + \tau_{xy} \omega_2 (x, y) = f_x, \]

\[ \tau_{xy} \xi_1 (x, y) + \sigma_y \omega_2 (x, y) = f_y, \]

where all of the quantities are independent of z. These relations
do not involve \( \sigma_z \). Once the solution for the in-plane stresses is
obtained (independently of \( \sigma_z \)), \( \sigma_z \) is found from the relation

\[ \sigma_z = \lambda \varepsilon = \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right), \]

or, substituting from (1.7),

\[ \sigma_z = \frac{1}{2(\lambda + \mu)} \left( \sigma_x + \sigma_y \right) = \nu (\sigma_x + \sigma_y), \]

where \( \nu \) is Poisson's ratio.
Under the assumption of plane strain, (1.4) and (1.5) are easily found to be

\[ \varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \]

(1.10)

\[ \varepsilon_{zz} = \gamma_{xz} = \gamma_{yz} = 0, \]

\[ \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}. \]

(1.11)

The remaining compatibility equations are satisfied trivially.

It should be emphasized that the plane strain state will not hold unless the surface loads meet the conditions that were imposed on them in the above consideration of the boundary conditions.

These conditions were:

(a) on the lateral surfaces, \( f_z = 0 \), and \( f_x \) and \( f_y \) are independent of \( z \);

(b) on the ends, \( f_x = f_y = 0 \), and \( f_z = \cos(nz) \sigma_z = \gamma(xy) \cos(nz) \).

It was assumed further that the body forces have only components parallel to the \( xy \)-plane and independent of \( z \). Under these conditions, the body will be in a state of plane strain. If these conditions are met, (1.6) through (1.8) represent an exact statement of the problem.

In practice, the loads on the ends of the cylinder are not usually distributed as \( \sigma_z \). However, for long cylinders, the conditions of plane strain will be satisfied quite well in cross
sections of the prism sufficiently far removed from the ends if the resultant force and the moments of the actual force distribution on the ends are equal to the corresponding resultant force and moments of the calculated $\sigma_z$ distribution. If this is not the case, the solution of the actual problem is obtained by superposing the solution of an auxiliary problem on the plane strain solution. This auxiliary problem consists of the cylinder with free lateral surfaces and with the ends loaded with uniform tensions and moments equal to the differences of the resultants for the actual problem and the plane strain solution. The solution of the auxiliary problem is quite simple. The combined solution for the actual problem is accurate away from the ends of the prism although, in general, the prism is no longer in a state of plane strain.

In contrast to the equations for the state of plane strain, the equations for the case of generalized plane stress, considered next, are approximations.

The generalized plane stress state

We will consider a thin plate of thickness 2h. The mid-plane of the plate is taken as the xy-plane. We will assume that the plane faces normal to the z-axis are free of stresses and the stresses acting on the edges are parallel to the xy-plane and symmetrically distributed with respect to this plane. We will assume also that the body forces are parallel to and symmetrically
distributed with respect to the xy-plane. With these assumptions, it is clear that the points of the xy-plane remain in it after the deformation. Since the mid-plane is not bent and since the thickness is small, the displacement component $w$ must be small and the variation in $u$ and $v$ over the thickness of the plate must be negligible.

Now, we have assumed that the stresses on the faces $z = \pm h$ are zero, i.e., that

$$
\sigma_z(x, y, \pm h) = \tau_{x z}(x, y, \pm h) = \tau_{y z}(x, y, \pm h) = 0,
$$

so that

$$
\left(\frac{\partial \tau_{x z}}{\partial x}\right)_{z = \pm h} = \left(\frac{\partial \tau_{x z}}{\partial y}\right)_{z = \pm h} = 0.
$$

Thus, from (1.1), we have

$$
\left(\frac{\partial \sigma_z}{\partial z}\right)_{z = \pm h} = -\left(\frac{\partial \tau_{x z}}{\partial y}\right)_{z = \pm h} - \left(\frac{\partial \tau_{y z}}{\partial x}\right)_{z = \pm h} - Z = 0
$$

($Z$ is zero by assumption). Since $\sigma_z$ and $\frac{\partial \sigma_z}{\partial z}$ are both zero at
\[ z = \pm h, \quad \overline{\sigma_z} \text{ must be very small throughout the thickness of the plate. Therefore, it may be assumed to a good approximation that} \]
\[ \sigma_z = 0 \text{ everywhere. We now consider the remaining equations of} \]
\[ (1.1) \]
\[ \frac{\partial \overline{\epsilon_{xx}}}{\partial x} + \frac{\partial \overline{\epsilon_{xy}}}{\partial y} + \frac{\partial \overline{\epsilon_{xz}}}{\partial z} + \chi = 0, \]
and
\[ \frac{\partial \overline{\epsilon_{yy}}}{\partial x} + \frac{\partial \overline{\epsilon_{yx}}}{\partial y} + \chi = 0. \]

We will take the averages of these equations over the thickness of the plate. We note first that
\[ \frac{\partial \overline{\epsilon_{yz}}}{\partial z} = \frac{1}{\eta} \int_{-h}^{h} \frac{\partial \overline{\epsilon_{yz}}}{\partial z} dz = \frac{1}{\eta} \left[ \overline{\epsilon_{yz}} \right]_{h} = 0, \]
and
\[ \frac{\partial \overline{\epsilon_{xz}}}{\partial z} = \frac{1}{\eta} \int_{-h}^{h} \frac{\partial \overline{\epsilon_{xz}}}{\partial z} dz = \frac{1}{\eta} \left[ \overline{\epsilon_{xz}} \right]_{h} = 0. \]

Thus we have
\[ \frac{\partial \overline{\sigma_{xx}}}{\partial x} + \frac{\partial \overline{\sigma_{xy}}}{\partial y} + \chi = 0, \]
\[ \frac{\partial \overline{\sigma_{yy}}}{\partial x} + \frac{\partial \overline{\sigma_{yx}}}{\partial y} + \chi = 0, \]

where the bar indicates the average over the thickness of the plate.

We now consider from (1.2) the equation
\[ \lambda \varepsilon + 2\mu \frac{\partial^2 \overline{w}}{\partial z^2} = \overline{\tau_z}, \]
which is zero by assumption. Solving for \( \frac{\partial \overline{w}}{\partial z} \), we have
\[ \frac{\partial \overline{w}}{\partial z} = -\frac{\eta}{\lambda + 2\mu} \left( \frac{\partial \overline{\epsilon_{xx}}}{\partial x} + \frac{\partial \overline{\epsilon_{yy}}}{\partial y} \right). \]
Substituting this value in the other stress-displacement relations gives

\[ \sigma_\lambda = \frac{2\lambda\mu}{\lambda + 2\mu} \left( \frac{\partial u}{\partial \lambda} + \frac{\partial v}{\partial \lambda} \right) + 2\mu \frac{\partial^2 u}{\partial \lambda^2} \]

and

\[ \sigma_\gamma = \frac{2\lambda\mu}{\lambda + 2\mu} \left( \frac{\partial u}{\partial \lambda} + \frac{\partial v}{\partial \lambda} \right) + 2\mu \frac{\partial^2 v}{\partial \lambda^2} . \]

Taking the average of these expressions over the thickness of the plate gives

\[ \bar{\sigma}_\lambda = \lambda^* \bar{\varepsilon} + 2\mu \left( \frac{\partial u}{\partial \lambda} \right), \]

\[ \bar{\sigma}_\gamma = \lambda^* \bar{\varepsilon} + 2\mu \left( \frac{\partial v}{\partial \lambda} \right) \]

(1.13)

where \( \lambda^* = \frac{2\lambda\mu}{\lambda + 2\mu} \) and \( \bar{\varepsilon} = \frac{\partial u}{\partial \lambda} + \frac{\partial v}{\partial \lambda} \).

If we take the mean value of the strain-displacement relation,

\[ \varepsilon_{x\gamma} = \varepsilon \left( \frac{\partial u}{\partial \lambda} + \frac{\partial v}{\partial \lambda} \right), \]

and the boundary conditions on the edges of the plate, and write these equations, together with (1.12) and (1.13), we arrive at the equations of generalized plane stress:

the equilibrium equations

\[ \frac{\partial \bar{\sigma}_\lambda}{\partial \lambda} + \frac{\partial \bar{\sigma}_\gamma}{\partial \gamma} + \bar{\kappa} = \bar{\varepsilon}, \]

\[ \frac{\partial \bar{\sigma}_\gamma}{\partial \lambda} + \frac{\partial \bar{\sigma}_\lambda}{\partial \gamma} + \bar{\gamma} = \bar{\varepsilon}; \]

(1.14)

the stress-displacement equations

\[ \bar{\sigma}_\lambda = \lambda^* \bar{\varepsilon} + 2\mu \frac{\partial u}{\partial \lambda}, \]

\[ \bar{\sigma}_\gamma = \lambda^* \bar{\varepsilon} + 2\mu \frac{\partial v}{\partial \lambda} \]

(1.15)

\[ \bar{\varepsilon}_{x\gamma} = \varepsilon \left( \frac{\partial \bar{\varepsilon}}{\partial \lambda} + \frac{\partial \bar{\varepsilon}}{\partial \gamma} \right) , \]
where \( \lambda^* = \frac{2 \lambda \mu}{\lambda + 2 \mu} \) and \( \bar{\varepsilon} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \); the boundary conditions

\[
\begin{align*}
\bar{\delta}_x \cos \psi (u, x) + \bar{\epsilon}_{x \psi} \cos \psi (u, \psi) &= \bar{f}_x,
\bar{\delta}_y \cos \psi (u, x) + \bar{\epsilon}_{y \psi} \cos \psi (u, \psi) &= \bar{f}_y.
\end{align*}
\] (1.16)

Comparing these equations with (1.6) through (1.8) shows that the equations for plane strain are identical with the equations of generalized plane stress, except that the constant \( \lambda \) in (1.7) is changed to \( \lambda^* = \frac{2 \lambda \mu}{\lambda + 2 \mu} \) in (1.15). We will show later that in certain cases, if the equations are written in terms of stresses only, the elastic constants do not appear. In these cases, the stress distributions for plane strain and generalized plane stress will be identical, although the displacements will be different.

We note that in considering averages of the stresses and strains over the thickness of the plate, it was necessary only to assume that \( \sigma_z = 0 \). The remaining quantities \( \tau_{xz}, \tau_{yz}, \) and \( w \) were eliminated from the mean value formulas with no further assumptions. It was noted earlier that the displacements \( u \) and \( v \) have insignificant variation over the thickness of the plate. Thus, the average value of the displacements should be a very good representation of the actual displacements of the plate.

By (1.2), the average stresses should also be good approximations to the stresses in the plate except for the error introduced by the assumption that \( \sigma_z = 0 \) everywhere.
The equations of plane stress are derived by assuming that
\[ \sigma_z = \tau_{xz} = \tau_{yz} = \tau_{xy} = 0 \] and that \( \sigma_x, \tau_y, \tau_{xy} \) are independent of \( z \). This leads to the same set of equations as were derived above. However, it is felt that generalized plane stress is somewhat more meaningful for thin plates in that it indicates the approximate nature of the stress equations. For a discussion of the errors introduced by the assumption of the vanishing stresses mentioned above, see Timoshenko (2), page 243.

Since no assumptions were made regarding the strains or displacements, the strain-displacement and strain compatibility relations (1.4) and (1.5) still hold. However, of these relations, only those involving the in-plane strains are needed in the problems of generalized plane stress. Taking the mean value of these expressions over the thickness of the plate gives

\[ \bar{\varepsilon}_{xx} = \frac{\partial \bar{u}}{\partial x}, \quad \bar{\varepsilon}_{yy} = \frac{\partial \bar{v}}{\partial y}, \quad \bar{\gamma}_{xy} = \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}, \]

(1.17)

\[ \frac{\partial^2 \bar{\varepsilon}_{xx}}{\partial x^2} + \frac{\partial^2 \bar{\varepsilon}_{yy}}{\partial y^2} = \frac{\partial^2 \bar{\gamma}_{xy}}{\partial x \partial y}. \]

(1.18)

These expressions have the same form as (1.10) and (1.11) for the plane strain case.

Thus, the equations for the average stresses and strains for the generalized plane stress case are completely identical to the equations of the plane strain case except for the elastic
constant \( \lambda^* \) in the stress-displacement relation, (1.15). In the remainder of this dissertation we will not make any distinction between the two cases. We will take (1.6) through (1.8), (1.10) and (1.11) for the plane strain case as the basic equations. It is understood that if the particular problem involves generalized plane stress, the constant \( \lambda \) must be replaced by \( \lambda^* \) and that the components of stress and displacement must be replaced by their mean values over the thickness of the plate.

Equations of the plane problems in terms of stresses

We wish now to express the equations of the plane theory of elasticity in terms of the stresses. For ease of reference, we will recopy these equations

\[
\begin{align*}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \sigma &= 0, \\
\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \gamma &= 0,
\end{align*}
\]

\[\begin{align*}
\sigma_x &= \lambda \varepsilon_x + 2\mu \varepsilon_{xy}, \quad \sigma_y = \lambda \varepsilon_y + 2\mu \varepsilon_{xy}, \\
\tau_{xy} &= \mu \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right), \\
\varepsilon_x &= \frac{1}{E} \sigma_x, \quad \varepsilon_y = \frac{1}{E} \sigma_y, \quad \gamma = \frac{1}{G} \tau_{xy},
\end{align*}\]

\[\begin{align*}
\varepsilon_{xx} &= \frac{1}{E} \sigma_x = \frac{1}{E} \frac{\partial u}{\partial x}, \\
\varepsilon_{yy} &= \frac{1}{E} \sigma_y = \frac{1}{E} \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{1}{G} \tau_{xy} - \frac{1}{G} \frac{\partial u}{\partial y} + \frac{1}{G} \frac{\partial v}{\partial x},
\end{align*}\]
We will not consider \( \sigma_z \) as part of the plane problem since in generalized plane stress, it is assumed to be zero, and in plane strain, it is easily calculated from (1.9) after the in-plane stresses have been calculated.

In order to express the equations in terms of the stresses, it is necessary to eliminate the displacements from (1.20). However, this must be done in such a way that the strain compatibility relation (1.23) is satisfied. We proceed as follows: first substitute for the displacements in (1.20) using (1.22) to obtain

\[
\begin{align*}
\sigma_x &= \lambda (\varepsilon_{xx} + \varepsilon_{yy}) + \mu \varepsilon_{xx}, \\
\sigma_y &= \lambda (\varepsilon_{xx} + \varepsilon_{yy}) + \mu \varepsilon_{yy}, \\
\varepsilon_{xy} &= \nu \varepsilon_{xy}.
\end{align*}
\]

We wish to solve these equations for the strains. Adding the first two equations gives

\[
\sigma_x + \sigma_y = 2(\lambda + \mu)\left(\varepsilon_{xx} + \varepsilon_{yy}\right).
\]

Substituting for \((\varepsilon_{xx} + \varepsilon_{yy})\) in the above equations, and solving for the strains gives

\[
\begin{align*}
\varepsilon_{xx} &= \frac{1}{2\mu} \left[\nu \sigma_x - \frac{\lambda}{2(\lambda + \mu)} (\sigma_x + \sigma_y)\right], \\
\varepsilon_{yy} &= \frac{1}{2\mu} \left[\nu \sigma_y - \frac{\lambda}{2(\lambda + \mu)} (\sigma_x + \sigma_y)\right], \\
\varepsilon_{xy} &= \frac{1}{\mu} \varepsilon_{xy}.
\end{align*}
\]

\[ (1.24) \]
We will convert the elastic constants to the Modulus of Elasticity, \( E \), and Poisson's ratio, \( \nu \). These are related to \( \mu \) and \( \lambda \) by the formulas
\[
E = \frac{\mu (3\lambda + 2\mu)}{\lambda + \mu}, \quad \nu = \frac{\lambda}{2(\lambda + \mu)}.
\]

We rewrite, \( E \), as
\[
E = \mu \left( 2 + 2\nu \right),
\]
or
\[
\mu = \frac{E}{2(1+\nu)}.
\]

Thus (1.24) becomes
\[
\epsilon_{xx} = \frac{1+\nu}{E} \left[ \sigma_x - \nu (\sigma_y + \sigma_z) \right], \quad \epsilon_{yy} = \frac{1+\nu}{E} \left[ \sigma_y - \nu (\sigma_x + \sigma_z) \right],
\]
\[
\gamma_{xy} = \frac{2(1+\nu)}{E} \epsilon_{xy}.
\]
Substituting these expressions in (1.23) gives
\[
\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} - \nu \nabla^2 (\sigma_x + \sigma_y) - 2 \frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} = 0. \tag{1.26}
\]

This expression may be simplified by using (1.19). Differentiating the first with respect to \( x \) and the second with respect to \( y \), and adding the results to (1.26) gives
\[
(1 - \nu) \nabla^2 (\sigma_x + \sigma_y) + \left( \frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_y}{\partial y} \right) = 0,
\]
or
\[
\nabla^2 (\sigma_x + \sigma_y) + \frac{1}{1-\nu} \left( \frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_y}{\partial y} \right) = 0. \tag{1.27}
\]
Equations (1.19), (1.21), and (1.27) together constitute the statement of the first fundamental problem of the theory of elasticity for the state of plane strain. For the state of plane stress, the quantity \( \frac{1}{1-\nu} \) is replaced by \( (1 + \nu) \). For simply connected regions (and under certain conditions for multiply connected regions) if the body forces vanish from (1.27), the stresses will not depend on the elastic constants. In this case, the stresses for the generalized plane stress and plane strain problems will be identical. A plate or prism loaded by gravity forces in the y-direction is an example of this type. In this case, \( Y \) is taken as \(-g\rho\), where \( \rho \) is the density of the material.

A very important case in which the stresses are independent of the elastic constants involves problems for which the body forces can be derived from a harmonic potential function. (The resultant force on each boundary must also vanish.)

Thus, if a function \( V(x,y) \) exists such that
\[
\begin{align*}
\sigma_x &= -\frac{\partial V}{\partial x}, & \gamma &= -\frac{\partial V}{\partial y},
\end{align*}
\]
we have from (1.22)
\[
\nabla^2 (\sigma_x + \gamma) - \frac{1}{1-\nu} \nabla^2 V = \sigma,
\]
and if \( \nabla^2 V = 0 \) the second term drops out. The static problem of thermal stresses for the case in which there is no heat generated in the body can be reduced to this form.

A complete mathematical treatment of the boundary value problems of elasticity theory would include also a proof of the
existence and uniqueness of the solutions. We will not go into the proof of the existence of the solutions, although such proofs have been obtained. A discussion of proofs of existence theorems for plane problems is included in Muskhelishili's book (1). Existence proofs for the three-dimensional case have been carried out by many authors, some of which are noted by Muskhelishvili (Refs. 3-8). A proof of the uniqueness of the solutions can be carried out if it is assumed that the displacements are single valued and have continuous derivatives up to the third order. This proof is also given by Muskhelishvili (1). From (1.20), it is clear that if the displacements are assumed to be single valued, the stresses must be single valued also.

Now, we will be primarily concerned in this dissertation with solving elasticity problems that are posed in terms of (1.19), (1.21), and (1.27). We will assume throughout this dissertation that the stresses are single valued and have continuous derivatives up through the second order. However, we must then consider whether this assumption is sufficient to prove that the displacements are single valued. The answer is, of course, that for multiply connected regions, the displacements may be multivalued even though the stresses are single valued. If we wish to solve problems for multiply connected regions that are stated in terms of the stress equations, it will be necessary to impose additional conditions on the problem to guarantee that the displacements are single valued and that the solution to the problem is unique.
These relations are developed in the next article. It will be shown incidentally, as a result of these derivations, that the assumption of single-valued stresses is sufficient to guarantee single-valued displacements in simply connected regions.

**Displacements in multiply connected regions. Michell's equations**

We wish to consider the first fundamental problem in terms of stresses. In the last article, the stress equations were found to be

\[
\frac{\partial \tau_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \sigma = 0, \quad (1.28)
\]

\[
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_y}{\partial x} + \sigma = 0, \quad (1.29)
\]

\[
\nabla^2 (\sigma_x + \sigma_y) + \frac{1}{1-\nu} \left( \frac{\partial \tau_x}{\partial x} + \frac{\partial \tau_y}{\partial y} \right) = 0, \quad (1.30)
\]

\[
\sigma_x \psi_x(m, x) + \tau_{xy}(m, y) = f_x, \quad (1.30)
\]

\[
\tau_{xy} \psi_x(m, x) + \tau_y \psi_x(m, y) = f_y. \quad (1.30)
\]

It will always be assumed that the stresses are everywhere single valued and continuous. However, in multiply connected regions, this is not sufficient to guarantee the single valuedness of the displacements.
For simplicity, we will consider first the problem for the doubly connected region shown in Fig. 3. In order to study the question of the multivaluedness of the displacements, we will first convert the ring into a simply connected region by making some cut AB connecting the inner and outer boundaries, as shown in Fig. 3. Let one side of this cut be designated as + and the other side -.

It is assumed that the stresses acting on the faces of the cut are identical with the internal stresses acting in the uncut ring along the arc AB and that the two sides of the cut remain in contact. We will designate the values of the stresses...
and displacements on the + and - sides of the cut by the superscript + or -, respectively (i.e. $\sigma_x^+$ or $\sigma_x^-$, etc.). We now wish to find the relative values of the stresses and displacements on either side of the cut. In order to make these evaluations, we must calculate the changes in the values of the stresses and displacements for a passage from one side of the cut completely around the ring and back to the same point on the opposite side of the cut. If the values of a function on opposite sides of every cut taken through the ring are equal, the function is said to be single valued in the complete ring. If the values differ, the function is to be said to be multivalued in the ring. The single or multivaluedness of a function in a multiply connected region must be established in this manner [i.e., by evaluation of the change (if any) in the value of the function for a complete passage around a hole and a return to the same point].

Now we are assuming the stresses to be single valued. This implies that on the two sides of AB we have

$$\sigma_x^+ = \sigma_x^-, \sigma_y^+ = \sigma_y^-, \tau_{xy}^+ = \tau_{xy}^-.$$

If we were to assume that the displacements were also single valued in the ring, we would also have $u^+ = u^-$, $v^+ = v^-$ along the cut AB. However, it is our purpose to define the first fundamental problem entirely in terms of the stresses. Instead
of assuming that the displacements are single valued, we will
determine what restrictions are imposed on the differences
\[ \delta u = u^+ - u^- \text{ and } \delta v = v^+ - v^- \]
by the assumption of single valuedness of the stresses.

In order to express \( \delta u \) and \( \delta v \) in terms of the stresses,
we will use the stress-displacement relations (1.20) which are
repeated here
\[
\begin{align*}
\sigma_x &= \lambda \varepsilon_x + 2 \mu \frac{\partial u}{\partial x}, \\
\tau_{xy} &= \lambda \varepsilon_y + \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \\
\sigma_y &= \lambda \varepsilon_y + \mu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right),
\end{align*}
\]
where \( \Theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \).

Adding the first two of these equations gives
\[ \sigma_x + \tau_{xy} = 2(\lambda + \mu) \varepsilon_x. \]

Substituting for \( \varepsilon \) in the first two equations and solving for
\[ \frac{\partial u}{\partial x} \text{ and } \frac{\partial u}{\partial y} \]
gives
\[
\begin{align*}
\frac{\partial u}{\partial x} &= \frac{1}{2 \mu} \left[ \sigma_x - \nu (\sigma_y + \tau_{xy}) \right] \quad \text{(1.32)} \\
\frac{\partial v}{\partial y} &= \frac{1}{2 \mu} \left[ \sigma_y - \nu (\sigma_x + \tau_{xy}) \right] \quad \text{(1.33)}
\end{align*}
\]
where we have used the relation \( \nu = \frac{\lambda}{2(\lambda + \mu)} \).
Now since the stresses are single valued by assumption, so are
the quantities $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$, that is,

$$\frac{\partial u^+}{\partial x} - \frac{\partial u^-}{\partial x} = 0 = \frac{\partial (\delta u)}{\partial x},$$

and

$$\frac{\partial v^+}{\partial y} - \frac{\partial v^-}{\partial y} = 0 = \frac{\partial (\delta v)}{\partial y}.$$

From this it follows that $\delta u$ and $\delta v$ must be of the
form

$$\delta u = u + f(y),$$
$$\delta v = v + g(x).$$

But from the third equation of (1.31)

$$\frac{1}{\mu} \left( \tau_{x^+}^{x_2} - \tau_{x^2}^{-x_2} \right) = 0 = \frac{\partial (\delta u)}{\partial y} + \frac{\partial (\delta v)}{\partial x}.$$

Thus

$$f'(y) + g'(x) = 0.$$

This can only be true if $f'(y)$ and $g'(x)$ are constant. Thus, we
can write

$$f'(y) = -c, \quad g'(x) = c,$$

so that

$$\delta u = u^+ - u^- = a - c y,$$
$$\delta v = v^+ - v^- = b + c x.$$

(1.34)

These expressions have the same form as the equations of
rigid body motion which can be written in general as

$$u = c_1 - ax, \quad v = c_2 + ax,$$

where $\alpha$ represents a rotation about the origin and $c_1$ and $c_2$ are
translations.
Using this fact, we can give a physical interpretation to the quantities $\delta u$ and $\delta v$.

Suppose that we start with the region of Fig. 3, which is assumed to be completely undeformed so that along the cut $AB$ $u^+ = u^-$ and $v^+ = v^-$. Now suppose that along the cut $AB$ we remove the very narrow strip shown in Fig. 4 between the cuts $A'B'$ and $A''B''$.

![Diagram](image)

Fig. 4.--The physical interpretation of dislocations

The cuts $A'B'$ and $A''B''$ must be congruent and disposed in such a way that $A'B'$ may be superimposed on $A''B''$ by the small rigid displacement represented by (1.34).

Now suppose we clamp the side $A''B''$ and move the side $A'B'$ as a rigid unit until it coincides with the side $A''B''$. Then $u^- = v^- = 0$, $u^+ = a - cy$, $v^+ = b + cx$, and we obtain

\[
\begin{align*}
\delta u &= a - cy, \\
\delta v &= b + cx.
\end{align*}
\]
Now if the sides are joined and the "clamps" are released, the ring in general will deform until it reaches equilibrium. However, the adjoining points of the opposite sides of the joined cut now will move together and no further relative displacements can occur between these sides. Thus, the expressions for $\delta u$ and $\delta v$ will not be changed when the clamps are released. The same remarks hold if the ring deforms further under the action of a system of external loads.

We would now like to return to our main theme, which is the definition of the first fundamental problem in terms of the stresses. The main result of our discussion of this article so far is to show that the statement of the problem given by (1.28) through (1.30), together with the assumption of single-valued stresses, is not sufficient to determine the displacements in the multiply connected region. In fact, there is a three-fold indeterminacy in $u$ and $v$ as represented by the constants $a$, $b$, and $c$ in (1.34).

It is therefore necessary to find three more equations which will relate these constants to the stresses. It will be our purpose in the remainder of this article to show one way that these equations can be derived. The first statement of these relations was by J. H. Michell (9) in 1899, and they are called Michell's equations.
We will return to the cut region of Fig. 3. Briefly, what we wish to find are relations that give the changes in $u$ and $v$ in terms of the stresses as we go from the - side of the cut around the hole to the + side of the cut. [Actually what we will find are three relations giving the constants $a$, $b$, and $c$ of (1.34) in terms of the stresses.]

We can get a separate expression for the constant $c$ of (1.34). In fact, by differentiating the first expression with respect to $y$ and the second with respect to $x$, and subtracting the first equation from the second, we obtain

$$c = \frac{\partial (\delta v)}{\partial x} - \frac{\partial (\delta u)}{\partial y}$$

$$= \frac{\partial (v^+)}{\partial x} - \frac{\partial (u^+)}{\partial y} - \frac{\partial (w^-)}{\partial x} + \frac{\partial (u^-)}{\partial y}$$

$$= 2 (\omega^+ - \omega^-)$$

$$= 2 \delta \omega,$$

where

$$\omega = \frac{1}{2} (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})$$

represents rotation of an element about the z-axis (positive in the clockwise sense). We will be seeking relations between $\delta \omega$, $\delta u$, and $\delta v$, and the stresses in the ring. Actually, the only available expressions relating the displacements directly to the stresses, are (1.31) which are given in terms of derivatives of $u$ and $v$. 
In order to make use of these derivative forms, we will form the line integrals
\[ \oint \, du \quad , \quad \oint \, dv \quad , \quad \oint \, d\omega \quad , \]
where the symbol \( \oint \) indicates that the line integral is to be evaluated for a circuit starting with a point \( c^- \) on the - side of AB, passing around the hole in the clockwise direction, and terminating at the corresponding point \( c^+ \) on the + side of AB. Now, from the definition of \( \delta \omega = c \), it must follow that the value
\[ \delta \omega = \oint \, d\omega \]
is independent of the choice of beginning and end points. We note, however, that \( \delta u = \oint \, du \) and \( \delta v = \oint \, dv \) are not independent of the choice of end points unless \( \delta \omega = 0 \). This is seen from (1.34).

For a given choice of end points, we assert that the values of \( \delta u \), \( \delta v \), and \( \delta \omega \) are independent of the paths chosen to evaluate the line integrals. We will verify this assertion after we have obtained expressions for the integrals.

We will begin by rewriting the integrals in terms of \( x \) and \( y \) derivatives of the displacements. Thus
\[ \delta \omega = \oint \, d\omega \quad \text{(1.35)} \]
\[ = \oint \left[ \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \right] \]
\[ = \oint \left\{ \left[ \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} \right] \, dx + \left[ \frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 u}{\partial y \partial x} \right] \, dy \right\} , \]
\[ \delta u = \oint \delta u' \, d \alpha, \]
\[ \delta u = \oint \left[ \frac{\partial u}{\partial x} \, d \alpha + \frac{\partial u}{\partial y} \, d \gamma \right], \quad (1.36) \]
\[ \delta v = \oint \left[ \frac{\partial v}{\partial x} \, d \alpha + \frac{\partial v}{\partial y} \, d \gamma \right], \quad (1.37) \]

Now in these last two integrals we have expressions \( \frac{\partial u}{\partial y} \) and \( \frac{\partial v}{\partial x} \) which cannot be related directly to the stresses (except as the sum \( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \)). We will therefore integrate these integrals by parts in such a way that we obtain the second derivatives of the displacements with respect to \( x \) and \( y \). We will show that these second derivatives are directly expressible in terms of the stresses.

Integrating (1.36) and (1.37) by parts (so that \( dx \) and \( dy \) are integrated), we obtain

\[ \delta u = \left[ x \frac{\partial u}{\partial x} \right]_{c-}^{c+} + \left[ y \frac{\partial u}{\partial y} \right]_{c-}^{c+} - \oint \left[ x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right], \]

and

\[ \delta v = \left[ x \frac{\partial v}{\partial x} \right]_{c-}^{c+} + \left[ y \frac{\partial v}{\partial y} \right]_{c-}^{c+} - \oint \left[ x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} \right], \]

where the symbol \( \left[ \right]_{c-}^{c+} \) indicates that the expression enclosed in brackets is to be evaluated at the beginning and end points of the circuit around the ring. But we have shown that \( \frac{\partial u}{\partial x} \) and \( \frac{\partial v}{\partial y} \) are single valued so that \( \left[ x \frac{\partial u}{\partial x} \right]_{c-}^{c+} = \left[ y \frac{\partial v}{\partial y} \right]_{c-}^{c+} = 0 \). To evaluate the terms \( y \frac{\partial u}{\partial y} \) and \( x \frac{\partial v}{\partial x} \), we use the two equations

\[ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \omega, \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \omega, \]

where \( \omega \) is the angular velocity.

\[ \omega = \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right). \]
From these, we have
\[ \frac{\partial \xi}{\partial \gamma} = \frac{\xi_{x} y}{\partial \mu} - \xi \gamma, \]
\[ \frac{\partial \zeta}{\partial \kappa} = \frac{\xi_{x} y}{\partial \mu} + \omega. \]

Thus, we have
\[ \left[ \gamma \frac{\partial \xi}{\partial \kappa} \right]_{\xi} = \left[ \gamma \frac{\xi_{x} y}{\partial \mu} \right]_{\xi} - \left[ \gamma \omega \right]_{\xi}, \]
\[ = - \gamma \xi \delta \omega, \]
and
\[ \left[ \kappa \frac{\partial \xi}{\partial \gamma} \right]_{\xi} = \kappa \left[ \frac{\xi_{x} y}{\partial \mu} \right]_{\xi} - \left[ \kappa \omega \right]_{\xi}, \]
\[ = \kappa \xi \delta \omega, \]

since \( \xi_{xy} \) is single valued. Thus, we can write for \( \delta u \) and \( \delta v \)
\[ \delta u = - \gamma \xi \delta \omega - \oint \left[ \kappa d \left( \frac{\partial u}{\partial \gamma} \right) + \gamma d \left( \frac{\partial u}{\partial \kappa} \right) \right], \tag{1.38} \]
and
\[ \delta v = \kappa \xi \delta \omega - \oint \left[ \kappa d \left( \frac{\partial v}{\partial \gamma} \right) + \gamma d \left( \frac{\partial v}{\partial \kappa} \right) \right]. \tag{1.39} \]

Comparing (1.38) and (1.39) with (1.34), it is seen that the two integrals are the negatives of a and b. Thus, the values of these integrals must be independent of the choice of end points of the path of integration. We now have \( \delta \omega \), \( \delta u \), and \( \delta v \) expressed
in terms of the second derivatives of the displacements, if in
(1.38) and (1.39) the total differentials are expanded by the
relation \( \text{d}(f) = \frac{\partial f}{\partial x} \, \text{d}x + \frac{\partial f}{\partial y} \, \text{d}y \).

We will now obtain the expressions relating the second
derivatives of \( u \) and \( v \) to the stresses. We note first that \( \frac{\partial^2 u}{\partial x^2} \),
\( \frac{\partial^2 u}{\partial x \partial y} \), \( \frac{\partial^2 v}{\partial y^2} \), and \( \frac{\partial^2 v}{\partial x \partial y} \) can be obtained directly by differentiation
of (1.32) and (1.33). Thus, we need only to find expressions for
\( \frac{\partial^2 u}{\partial y^2} \) and \( \frac{\partial^2 v}{\partial x^2} \). We start with the third of (1.31)

\[
\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad (1.40)
\]

Taking the partial derivative of this with respect to \( y \) gives

\[
\frac{\partial^3 u}{\partial y^3} = \frac{\partial^3 v}{\partial x^2 \partial y} + \frac{\partial^2 u}{\partial y^2} \cdot
\]

Substituting for \( \frac{\partial^2 v}{\partial x \partial y} \) from (1.33) and solving for \( \frac{\partial^2 u}{\partial y^2} \) gives

\[
\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu} \cdot \frac{\partial \varepsilon_{xy}}{\partial y} - \frac{1}{\mu} \cdot \frac{\partial^2}{\partial x} \left[ \sigma_y - \nu (\sigma_x + \sigma_y) \right] \quad (1.41)
\]

Similarly, taking the partial derivative of (1.40) with respect
to \( x \), we obtain

\[
\frac{\partial^3 u}{\partial x \partial y^2} = \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial x^2 \partial y} \cdot
\]
Substituting in this expression from (1.32) for \( \frac{\partial^2 u}{\partial x \partial y} \) and solving for \( \frac{\partial^2 v}{\partial x^2} \), gives

\[
\frac{\partial^2 v}{\partial x^2} = \frac{1}{\epsilon \lambda} \frac{\partial^2 \tau_x}{\partial \gamma} - \frac{1}{\mu} \frac{\partial}{\partial y} \left[ J_x - \gamma (\tau_x + \tau_y) \right]. \tag{1.42}
\]

We now have all of the relations necessary for expressing the quantities \( \delta \omega \), \( \delta u \), and \( \delta v \) in terms of the stresses. We will start with \( \delta \omega \).

Substituting for the second derivatives of \( u \) and \( v \) in (1.35) from (1.32), (1.33), (1.41), and (1.42) gives, after minor simplification

\[
\delta \omega = \frac{1}{2\mu} \int \left\{ \frac{\partial \tau_x}{\partial x} - \frac{\partial}{\partial y} \left[ J_x - \gamma (\tau_x + \tau_y) \right] \right\} d\gamma \tag{1.43}
\]

This expression gives \( \delta \omega \) in terms of the stresses. However, for convenience in evaluating the integrals, we would like to convert this expression to the \((n,s)\) curvilinear coordinate system of the path of integration. We will adopt the convention for the \((n,s)\) coordinates shown in Fig. 5.

We will have need of the following relations, which are easily found from Fig. 5b,

\[
\frac{dx}{dn} = \frac{dz}{ds} \quad \frac{dy}{dn} = -\frac{dx}{ds}. \tag{1.44}
\]
Fig. 5.---The boundary coordinate system

Thus

$$\frac{df}{dn} = \frac{\partial f}{\partial x} \frac{dx}{dn} + \frac{\partial f}{\partial y} \frac{dy}{dn}$$

$$= \frac{\partial f}{\partial x} \frac{dy}{ds} - \frac{\partial f}{\partial y} \frac{dx}{ds} .$$

Applying (1.44) and (1.45) to (1.43) for $\delta \omega$ gives

$$\delta \omega = -\frac{2}{\mu} \int \frac{d}{dn} (v_x + v_y) ds + \frac{1}{2\mu} \int \left[ \frac{\partial^2 r_y}{\partial x} - \frac{\partial r_x}{\partial y} \right] \frac{dy}{ds} ds$$

$$- \frac{1}{2\mu} \int \left[ \frac{\partial^2 r_y}{\partial y} - \frac{\partial r_x}{\partial x} \right] \frac{dx}{ds} ds .$$
This is not a satisfactory form since we still have derivatives in $x$ and $y$. Now, suppose we eliminate $\xi_{xy}$ from this equation by substituting from the equilibrium equations (1.28).

We then obtain

$$\delta \omega = \frac{\gamma}{2 \mu} \int \frac{d}{d\kappa} \left[ (\sigma_x + \sigma_y) d\xi \right] - \frac{i}{\mu} \int \left( \frac{\partial \sigma_y}{\partial x} + \frac{\partial \sigma_x}{\partial y} + Y \right) \frac{d\xi}{\kappa} \, ds$$

$$+ \frac{i}{\mu} \int \left( \frac{\partial \sigma_y}{\partial x} + \frac{\partial \sigma_x}{\partial y} + X \right) \frac{d\xi}{\kappa} \, ds.$$

But by (1.45)

$$\frac{\partial}{\partial x} \left( \sigma_x + \sigma_y \right) \frac{d\xi}{d\kappa} - \frac{\partial}{\partial x} \left( \sigma_x + \sigma_y \right) \frac{d\xi}{d\kappa} = \frac{d}{d\kappa} \left( \sigma_x + \sigma_y \right).$$

Thus

$$\delta \omega = \frac{1 - \gamma}{2 \mu} \int \frac{d}{d\kappa} \left( \sigma_x + \sigma_y \right) d\kappa \, ds + \int \left( X d\eta - Y d\xi \right), \quad (1.46)$$

which is the desired expression for $\delta \omega$. We note that the last integral has a convenient form in the $(n,s)$ coordinates when the body forces are derivable from a potential function such that $X = \frac{\partial V}{\partial x}$ and $Y = \frac{\partial V}{\partial y}$. In this case we have

$$\int \left( X d\eta - Y d\xi \right) = \int \frac{dV}{d\kappa} d\kappa.$$

We will now turn to the evaluation of $\delta u$ in terms of the stresses. We will start with (1.38). Substituting in this
equation from (1.32) and (1.41), we obtain

\[ \delta u = -\frac{4c}{2\mu} \int x \, d \left[ \sigma_x - \nu (\sigma_x + \sigma_y) \right] \]

\[ -\frac{1}{2\mu} \int y \left\{ \frac{2}{c} \frac{\partial \sigma_x}{\partial y} - \frac{2}{c} \left[ \sigma_y - \nu (\sigma_x + \sigma_y) \right] \right\} dy \]

\[ -\frac{1}{2\mu} \int y \frac{\partial \nu}{\partial x} \left[ \sigma_x - \nu (\sigma_x + \sigma_y) \right] dx. \]

Now transform this expression into the \((n,s)\) coordinate system as we did for \(\delta \omega\). This derivation is complicated. However, the basis of the manipulations to be performed is that we must convert the \(x\) and \(y\) derivatives of the stresses into one or the other of the two total differential forms

\[ \frac{df}{du} \, ds = \frac{\partial f}{\partial \sigma_x} \, dy - \frac{\partial f}{\partial \sigma_y} \, dx \quad (1.48) \]

and

\[ df = \frac{df}{ds} \, ds = \frac{\partial f}{\partial \sigma_x} \, dx + \frac{\partial f}{\partial \sigma_y} \, dy. \quad (1.49) \]

Certain of the terms of (1.47) may be transformed directly. Thus, the first integrand is a perfect differential and can be written as \(\frac{df}{ds} \, ds\) by (1.49). The terms involving \((\sigma_x + \sigma_y) \) in the
last two integrals may be combined and transformed by (1.48).

Making these two transformations, we obtain

\[
\delta u = -4e \delta \omega + \frac{\partial}{\partial \mu} \int \left[ x \frac{d}{ds}(\tau_x + \tau_y) - \gamma \frac{d}{dn}(\tau_x + \tau_y) \right] ds \quad (1.50)
\]

\[
- \frac{1}{\beta} \int \frac{d\tau_x}{ds} ds
\]

\[
- \frac{1}{\beta} \int \gamma \left[ (2 \frac{\partial \tau_x}{\partial y} - \frac{\partial \tau_y}{\partial x}) dy + \frac{\partial \tau_y}{\partial y} dx \right].
\]

This last integral is the only one remaining to be transformed. However, its transformation requires some slight-of-hand manipulation. We will make this transformation in two steps.

First, we break up \( \frac{\partial \tau_{xy}}{\partial y} \) and write

\[
(2 \frac{\partial \tau_{xy}}{\partial y} - \frac{\partial \tau_y}{\partial x}) dy + \frac{\partial \tau_y}{\partial x} dx = \frac{\partial \tau_{xy}}{\partial y} dy + \frac{\partial \tau_{xy}}{\partial x} dx
\]

\[
+ \left( \frac{\partial \tau_{xy}}{\partial y} - \frac{\partial \tau_y}{\partial x} \right) dy
\]

\[
- \left( \frac{\partial \tau_{xy}}{\partial x} - \frac{\partial \tau_y}{\partial y} \right) dx.
\]

We have added and subtracted \( \frac{\partial \tau_{xy}}{\partial x} dx \) from the right-hand side of this equation to obtain the perfect differential \( d(\tau_{xy}) \) in the first line. The last two lines of this expression are the same expressions that we had to contend with in transforming \( \delta \omega \).

Substituting again for \( \tau_{xy} \) from (1.28), we have

\[
(2 \frac{\partial \tau_{xy}}{\partial y} - \frac{\partial \tau_y}{\partial x}) dy + \frac{\partial \tau_y}{\partial y} dx = d(\tau_{xy}) - \left( \frac{\partial \tau_x}{\partial x} + \frac{\partial \tau_y}{\partial y} + \chi \right) dy
\]

\[
+ \left( \frac{\partial \tau_x}{\partial x} + \frac{\partial \tau_y}{\partial y} + \gamma \right) dx.
\]
\[
\{ \frac{2}{\varepsilon_{x}} \psi_{y} - \frac{2}{\varepsilon_{y}} \psi_{x} \} d_{y} + \frac{2}{\varepsilon_{y}} \psi_{y} d_{x} = \frac{d}{d\alpha} (\psi_{y} + \nabla_{x}) + \frac{d}{d\beta} (\nabla_{y} + \psi_{y}) \nabla_{x} \varepsilon_{x} d_{y} .
\]

Substituting this in (1.50), we have
\[
\Delta \psi = -\frac{\psi_{x}}{d \varepsilon_{x}} - \frac{\psi_{y}}{d \varepsilon_{y}} + \frac{1}{2 \mu} \int x \frac{d}{ds} (\nabla_{x} + \nabla_{y}) ds + \frac{1}{2 \mu} \int \frac{d}{ds} (\nabla_{x} + \nabla_{y}) ds
\]

We can give this a more symmetric form by adding and subtracting the integral \( \frac{1}{2 \mu} \int x \frac{d\sigma}{ds} ds \), so that
\[
\Delta \psi = -\frac{\psi_{x}}{d \varepsilon_{x}} - \frac{\psi_{y}}{d \varepsilon_{y}} + \frac{1}{2 \mu} \int x \frac{d}{ds} (\nabla_{x} + \nabla_{y}) ds + \frac{1}{2 \mu} \int (x \frac{d\sigma}{ds} - y \frac{d\xi}{ds}) ds
\]

or
\[
\Delta \psi = -\frac{\psi_{x}}{d \varepsilon_{x}} - \frac{\psi_{y}}{d \varepsilon_{y}} + \frac{1}{2 \mu} \int \frac{1}{2 \mu} \int \left( y \frac{d\sigma}{ds} - y \frac{d\xi}{ds} \right) ds (1.51)
\]

We can get a more meaningful expression for this last integral. We integrate by parts first to eliminate the x and y multipliers. Thus
\[
\int (x \frac{d\xi}{ds} + y \frac{d\sigma}{ds}) = \left[ x \psi_{y} \right]_{0}^{t} - \left[ y \psi_{x} \right]_{0}^{t} + \int \left( \psi_{y} \frac{d\sigma}{ds} - \psi_{x} \frac{d\xi}{ds} \right) d_{y}
\]

We can get a more meaningful expression for this last integral. We integrate by parts first to eliminate the x and y multipliers. Thus
\[
\int (x \frac{d\xi}{ds} + y \frac{d\sigma}{ds}) = \left[ x \psi_{y} \right]_{0}^{t} - \left[ y \psi_{x} \right]_{0}^{t} + \int \left( \psi_{y} \frac{d\sigma}{ds} - \psi_{x} \frac{d\xi}{ds} \right) d_{y}
\]
since the stresses are single valued. Now let \( f_{y} \) be the \( y \)-component of the forces acting on the body along the circuit we have chosen (Fig. 3). Then, we have at any point of this circuit

\[
f_{y} = \epsilon y \cos(n, x) + T y \cos(n, y)
\]

or

\[
f_{y} = \epsilon y \frac{dx}{ds} + T y \frac{dy}{ds}
\]

from (1.44). Thus

\[
\int (x d \tau y - y d \epsilon y) = \int (\tau y dx - \epsilon y dy) = -\int f_{y} ds
\]

Substituting this in (1.51) gives finally

\[
\delta u = -\frac{\gamma}{\lambda} \int (x d \tau y - y d \epsilon y) ds - \frac{1}{2 \mu} \int f_{y} ds
\]

This is the desired form of Michell's equation for \( \delta u \).

By carrying out a similar derivation for \( \delta v \), we obtain

\[
\delta v = \kappa \delta \omega - \frac{1 - \nu}{2 \mu} \int [(x d/s + x d/s)(\tau x + \tau y)] ds
\]

We note again that if \( X \) and \( Y \) are derivable from a potential function \( V \), the integrals involving these quantities in (1.53) and (1.54) reduce to simple expressions in the \((n,s)\) coordinate systems.
We note further that $f^\rho_y$ and $f^\rho_x$ are not equal to the boundary loads $f_x$ and $f_y$ given in (1.30) unless we choose a circuit that coincides with the boundary (or unless, as we shall show a little later, $X = Y = 0$).

Now, Michell's equations are to be regarded as conditions to be imposed on the stress distribution from a knowledge of the values of $\delta \omega$, $\delta u$, and $\delta v$. That is, we must know the values of these quantities in order to have a completely determined stress problem.

By far the most important and most useful case is obtained when it is assumed that $\delta \omega = \delta u = \delta v = 0$. Equations (1.46), (1.53), and (1.54) are then the necessary conditions for single-valued displacements in the ring. In fact, Michell derived his equations with the idea in mind that they should be the conditions for single valuedness of displacements, and he assumed from the outset that $\delta \omega = \delta u = \delta v = 0$.

However, it introduced no essential complication in the derivation to carry the non-zero values for $\delta \omega$, etc. (except for the necessity of giving them a physical interpretation).

It should be mentioned in this regard that many others have derived the Michell equations for non-zero values of $\delta \omega$, $\delta u$, and $\delta v$ (10-12). A.E.H. Love (13) proposed the name, "dislocations", for these terms, and we will use this name also.
Now, we indicated earlier that for simply connected regions, the displacements must be single valued if the stresses are. We will use Michell's equations to prove this statement.

We will consider any simple closed curve enclosing a simply connected region \( A \), as shown in Fig. 6.

![Fig. 6.--A line integral circuit around a simply connected region](image)

We will consider first the dislocation \( \delta \omega \) given in (1.46)

\[
\delta \omega = \frac{1-\nu}{2\mu} \oint \frac{d}{dn} (\sigma_x + \sigma_y) ds + \frac{i}{2\mu} \oint (\Gamma dx - \Psi dy). \tag{1.55}
\]

Now, Green's theorem for the plane may be written in the forms

\[
\int \frac{dP}{dn} ds = \iint \nabla^2 P \, dA, \tag{1.56}
\]
and

\[ \oint (Q \, dy + p \, dx) = \iint_{\mathcal{A}} \left( \frac{\partial \omega}{\partial \lambda} - \frac{\partial \phi}{\partial \lambda} \right) \, dA. \quad (1.57) \]

Applying these equations to (1.54), gives

\[ \delta \omega = \frac{i}{\gamma \mu} \iint_{\mathcal{A}} \left[ \left( i - \nu \right) \nabla \cdot (\sigma_x, \sigma_y) \right] \frac{\partial \lambda}{\partial \lambda} + \frac{\partial \gamma}{\partial \gamma} \right] \, dA. \quad (1.58) \]

But (1.29) holds at every point of A. Thus, \( \delta \omega \) is zero for any circuit enclosing a simply connected region. Of course, this proof applies only if A is simply connected. If we consider a region containing a hole, the line integrals must be taken over the entire boundary of A. This leads, as we shall see shortly, to a proof that \( \delta \omega \) on any circuit enclosing a hole is equal to

for a circuit taken on the boundary of the hole.

We will now evaluate \( \delta u \) for a circuit around a simply connected region A. We will use (1.53). However, we will substitute for \( f_{xy} \) its value given in terms of the stresses along the circuit by (1.52). Since \( \delta \omega \) is proved zero over any circuit enclosing a simply connected region, we have

\[ \delta u = \frac{i - \nu}{\gamma \mu} \int [(x \frac{d}{ds} - \gamma \frac{d}{dn})(\sigma_x, \sigma_y)] \, ds \quad (1.59) \]

\[ - \frac{i}{\gamma \mu} \int \gamma (Y \, dx - \lambda \, dy) \]

\[ + \frac{i}{\gamma \mu} \int (\lambda x_y \, dy - \lambda y_x \, dx). \]
Applying Green's theorem (1.57) to each of the terms of this expression, we have

\[ S \omega = \frac{1-\nu}{2\mu} \iint_{R} \left[ \frac{d}{dx}(\sigma_{x}+\sigma_{y}) \cdot \gamma \mathbf{\nabla}^{2}(\sigma_{x}+\sigma_{y}) - \frac{d}{dy}(\sigma_{x}+\sigma_{y}) \right] dA \] (1.60)

\[ + \frac{1}{3\mu} \iint_{R} \left[ \gamma + \frac{\gamma}{\gamma_{y}} \frac{\partial \gamma}{\partial y} + \frac{\gamma}{\gamma_{x}} \frac{\partial \gamma}{\partial x} \right] dA \]

\[ + \frac{1}{2\mu} \iint_{R} \left[ \frac{\partial \mathbf{E}}{\partial x} + \frac{\partial \mathbf{E}}{\partial y} \right] dA \]

\[ = \frac{1}{3\mu} \iint_{R} \left[ \gamma (1-\nu) \mathbf{\nabla}^{2}(\sigma_{x}+\sigma_{y}) + \frac{\partial \mathbf{E}}{\partial x} + \frac{\partial \mathbf{E}}{\partial y} \right] dA \]

\[ + \frac{1}{2\mu} \iint_{R} \left[ \frac{\partial \mathbf{E}}{\partial x} + \frac{\partial \mathbf{E}}{\partial y} + \gamma \right] dA \]

The first line is zero by (1.29). The second integral is zero by the second equation of (1.28). Thus, \( S \omega = 0 \) for any circuit around a simply connected region (but not for a circuit around a hole). \( \xi \nu \) is similarly shown to be zero for a simply connected region. Since the dislocations are zero, the displacements must be single valued in any simply connected region. These results can be used to prove another assertion that was made earlier, namely, that the values of the dislocations, as represented by (1.34), are given by Michell's equations for any path taken in the ring. Essentially, what this means is that for two different circuits taken around the ring (with different beginning and end points), the dislocation \( S \omega \) will be the same on both paths. Since
it is the constant c of (1.34]). The differences between the
dislocations \( \delta u \) and \( \delta v \) on two circuits are given from (1.34)
by the relations

\[
\begin{align*}
\delta u_i - \delta u_j &= (x_{i-1} - x_i) \delta \omega_i - (x_i - x_{i+1}) \delta \omega_j \\
\delta v_i - \delta v_j &= (y_{i-1} - y_i) \delta \omega_i - (y_i - y_{i+1}) \delta \omega_j
\end{align*}
\] (1.61)

where the Subscripts 1 and 2 refer to the two different circuits,
and \((x_{A_i}, y_{A_i})\), \(i = 1, 2\) are the x- and y-coordinates of the
beginning and end point of the i-th circuit.

We will prove that Michell's equations do, in fact, give
these results. Consider any two complete circuits, \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \),
around a hole, as shown in Fig. 7. Let the end points of these
circuits be \( A_1 \) and \( A_2 \).

We will convert the region between the two circuits into
a simply connected region by a cut between \( A_1 \) and \( A_2 \). In this
simply connected region we have that \( \delta \omega, \delta u, \delta v \) are zero by
the preceding analysis leading to (1.58) and (1.60). That is,
Michell's equations must give zero values for any complete circuit
around the boundary of this simply connected region. Suppose we
agree to start such a circuit at point \( A_1 \) and follow the boundary
so that we meet the points \( C_1 B_1 \), etc. in the sequence \( A_1 B_1 C_1 \)
\( A_1 A_2 B_2 C_2 A_2 A_1 \) (Fig. 7). (As a matter of fact, we could choose
any other starting point for the circuit, but we must traverse the
curves in the same directions as indicated so that any other equivalent circuit would meet the points $A_1$, $B_1$, etc. in an order that is a cyclic permutation of the order we have chosen."

Let us consider Michell's equations in detail for the circuit we have chosen. We will start with (1.46) for $\delta \omega$

$$\delta \omega = 0 = \oint \left[ \frac{1 + \nu}{2 \mu} \frac{d}{d\kappa} \left( \sigma_x + \sigma_y \right) ds + \frac{1}{\rho \mu} \left( E dy - Y dx \right) \right],$$
where the integral $\oint$ is to be taken over the complete circuit $A_1 A_2 \cdots A_1$. Now the value of the integral over the arc $A_1 B_1 C_1 A_1$ is $\omega_1$, and the value of the integral over the arc $A_2 B_2 C_2 A_2$ is $-\omega_2$ since the arc is traversed in the counterclockwise direction.

Thus, we have

$$c = \delta \omega_T = \delta \omega_1 - \delta \omega_2$$

$$+ \int_{A_1} ^{A_2} \left[ \frac{l \cdot \gamma}{\alpha} \frac{d}{dn} \left( \sigma_x^+ \gamma_x - \sigma_y^+ \gamma_y \right) ds^+ + \frac{l}{2\mu} (X dy - Y dx) \right]$$

$$+ \int_{A_2} ^{A_1} \left[ \frac{l \cdot \gamma}{\alpha} \frac{d}{dn} \left( \sigma_x^- \gamma_x - \sigma_y^- \gamma_y \right) ds^- - \frac{l}{2\mu} (X dy - Y dx) \right],$$

where the last two integrals are evaluated over the cut $A_1 A_2$.

The orientation of the coordinates $(n^+, s^+)$ and $(n^-, s^-)$ are shown in Fig. 5b. In going from $A_1$ to $A_2$, we must traverse the cut on the + side, and in going from $A_2$ to $A_1$, we must traverse the cut on the - side. Now the stresses and body forces are single valued so that $\sigma_x^+ = \sigma_x^-$, $\sigma_y^+ = \sigma_y^-$, $x^+ = x^-$, and $y^+ = y^-$. Further, $ds^+ = -ds^-$ and $dn^+ = -dn^-$. Thus, the two integrals cancel each other and we have

$$\delta \omega_1 - \delta \omega_2 = c,$$

as we wished to prove. We now consider $\delta u_m$, for which we use (1.59).
We will once again break the line integrals up into integrals over $P_1$, $P_2$, and the arcs $A_1 A_2$ and $A_2 A_1$. Thus

$$0 = \delta u_r = \int_{P_1} \mathcal{f}(n, s)\,ds + \int_{P_2} \mathcal{f}(n', s')\,ds' + \int_{\delta P_1} \mathcal{f}(n, s)\,ds,
$$

where

$$
\mathcal{f}(n, s) = -\frac{l-\varphi}{2\mu} \left[ \chi \frac{d}{ds} \left( \sigma_x + \sigma_y \right) - \frac{d}{ds} \left( \tau_{x r} \tau_{y r} \right) \right]
$$

$$
- \frac{1}{2\mu} \left[ \gamma \left( \tau_{x y} \frac{dx}{ds} - \tau_{y r} \frac{dy}{ds} \right) \right]
$$

$$
+ \frac{1}{2\mu} \left( \sigma_x \frac{dx}{ds} - \sigma_y \frac{dy}{ds} \right).
$$

We can show by the same argument used above for the integrals involved in $\delta \omega$, that

$$
\int_{P_1} \mathcal{f}(n, s)\,ds = -\int_{P_2} \mathcal{f}(n', s')\,ds'.
$$

But

$$
\int_{P_1} \mathcal{f}(n, s)\,ds = \delta u_1 + \delta \omega_1 \delta \omega_1,
$$

and

$$
\int_{P_2} \mathcal{f}(n, s)\,ds = -\delta u_2 - \delta \omega_2 \delta \omega_2,
$$

since $s$ is taken in the counterclockwise direction. Thus, we have

$$
\delta v_r = \delta u_1 + \delta \omega_1 \delta \omega_1 - \delta u_2 - \delta \omega_2 \delta \omega_2,
$$

which is what we wished to prove. We can prove in an identical manner that

$$
\delta v_1 - \delta v_2 = \chi n_1 \delta \omega_1 - \chi n_2 \delta \omega_2.
$$
This completes the proof of our assertion that Michell's equations do indeed give constant values of $\delta \omega$ over any two curves enclosing the same hole, and give the Expressions (1.61) for $\delta u$ and $\delta v$.

The derivations of this section have all been made for a doubly connected region. We will now extend these results to cover multiply connected regions of any connectivity. Such a region is illustrated in Fig. 8.

Fig. 8.—Calculation of dislocations in the general multiply connected region
We can reduce the multiply connected region to a simply connected one by introducing some nonintersecting cuts such as those shown as dotted lines in Fig. 8. The displacements in such a "reduced" region are single valued. If we bring the cut edges back together one at a time, it is clear that Michell's equations, as derived in this article, are valid for the resulting doubly connected regions. Now suppose we bring two of the cuts together, say $A_1 B_1$ and $A_2 B_2$, shown in Fig. 8. Let us assume that the cut $A_1 B_1$ is closed first. Then the dislocations along this cut are calculated by Michell's equations for any circuit enclosing $\Gamma_1$. Now if we bring the cut $A_2 B_2$ together, the joined points of the cut $A_1 B_1$ must move together. That is, closing the cut $A_2 B_2$ does not change the values of the dislocations along the cut $A_1 B_1$.

Further, if we evaluate Michell's integrals for a circuit on the boundary $\Gamma_1$, the closing of cut $A_2 B_2$ does not affect the values of these integrals, since these values depend only on the boundary conditions on $\Gamma_1$. Now suppose we take any other circuit around $\Gamma_1$ that does not enclose $\Gamma_2$ (this circuit must not cross any of the other cuts that remain open). Then, as we have shown above, the values of the integrals involved in $S \omega$, $S u$, and $S v$ are the same as the values of the integrals computed on $\Gamma_1$ itself, so that the values of Michell's equation calculated for any circuit around $\Gamma_1$ are not affected by closing the cut $A_2 B_2$. Thus, Michell's equations for a circuit around $\Gamma_1$ (only) give the
dislocations along the cut $A_1 B_1$ even after we have closed the cut $A_2 B_2$. By closing the cut $A_2 B_2$ first and then closing $A_1 B_1$, we see, by the same argument, that Michell's equations for a circuit enclosing $f'_2$ give the correct values of the dislocations along the cut $A_2 B_2$.

By a continuation of this line of reasoning, it may be shown for the complete multiply connected region, that the dislocations around any hole are given by Michell's equations for a circuit enclosing only that hole.

In the case where the cut is taken between two holes such as $f'_3$ and $f'_4$ of Fig. 8, essentially the same line of reasoning follows as for the pair of cuts $A_1 B_1$ and $A_1 B_2$. In this case, one of the circuits is to be taken around $f'_3$ only and the second circuit is to be taken around both $f'_3$ and $f'_4$. The values of Michell's integrals for this second circuit are the sums of the dislocations along the cuts $A_3 B_3$ and $A_4 B_4$. We could carry out the same argument as before, except that one of the sets of integrals involves the sum of the dislocations for the two holes.

It is noted in this regard that, since the integrals taken around two holes give the sum of the dislocations for each hole, it may happen that the Michell equations for such a circuit may be zero even though there are dislocations associated with each hole separately. For this reason, if the displacements are to be
single valued in a multiply connected region, the Michell equations must give zero values for circuits taken around each hole separately.

In the next chapter we will turn to the derivation of the stress function solution of the stress problems. Once we have obtained this solution, we will show how the Michell integral equations are applied to calculate the stresses arising from the dislocations.
REFERENCES


(The following References 3 through 8 are quoted from Muskhelishvili, op. cit., p. 71.)


In this chapter, we will derive the general form of the stress function for a multiply connected region, as well as the special forms of this stress function satisfying two different types of symmetry conditions. We will first show how the body forces may be eliminated by finding particular solutions of the equilibrium and compatibility conditions.

**Reduction of the plane problem to a problem without body forces**

We will start again with the statement of the first fundamental problem in terms of the stresses. The complete statement of this problem is given by (1.28) through (1.30) and the Michell equations (1.46), (1.53), and (1.54).

We will consider first the equilibrium and compatibility equations (1.28) and (1.29) which we reproduce here as

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \sigma_x = 0, \tag{2.1}
\]

\[
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \tau_{xy} = 0.
\]
\[ \nabla^2 (\sigma_x + \sigma_y) + \frac{1}{\gamma - \nu} \left( \frac{\partial^2 x}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} \right) = 0. \]  
(2.2)

Now let us assume that for the known body forces of a given problem, we can find a particular solution \( \sigma^0_x, \sigma^0_y, \tau^0_{xy} \), satisfying (2.1) and (2.2). If we then write
\[ \sigma'_x = \sigma'_x + \sigma^0_x, \quad \sigma'_y = \tau'_y + \tau^0_y, \quad \tau'_{xy} = \tau'_{xy} + \tau^0_{xy}, \]  
(2.3)

it is easily verified by substituting these expressions into (2.1) and (2.2) that the stresses \( \sigma'_x, \sigma'_y, \) and \( \tau'_{xy} \) will satisfy (2.1) and (2.2) in the absence of body forces
\[ \frac{\partial \sigma'_x}{\partial x} + \frac{\partial \tau'_{xy}}{\partial y} = 0, \]  
\[ \frac{\partial \tau'_{xy}}{\partial x} + \frac{\partial \sigma'_y}{\partial y} = 0, \]  
(2.4)
\[ \nabla^2 (\sigma'_x + \sigma'_y) = 0. \]  
(2.5)

The stress distribution \( \sigma'_x, \sigma'_y, \tau'_{xy} \) must then satisfy the modified boundary conditions
\[ \sigma'_x \cos(n_x) + \tau'_{xy} \cos(n_y) = f_x - \tau^0 \cos(n_x) - \tau^0_{xy} \cos(n_y), \]  
(2.6)
\[ \tau'_{xy} \cos(n_x) + \tau'_y \cos(n_y) = f_y - \tau^0_y \cos(n_x) - \tau^0_{xy} \cos(n_y). \]
and the modified Michell equations\(^1\)

\[
\delta \omega = \frac{1-\nu}{2G} \int \frac{d}{dn} \left( \sigma_x + \sigma_y \right) ds + \frac{1-\nu}{2G} \int \frac{d}{dn} \left( \sigma_x' + \sigma_y' \right) ds \\
+ \frac{i}{2G} \int \left( \mathcal{X} dy - \mathcal{Y} dx \right) ,
\]

\[
\delta u = -\frac{1-\nu}{2G} \delta \omega - \frac{1-\nu}{2G} \int \left( \frac{d}{ds} - \frac{d}{dn} \right) \left( \sigma_x' + \sigma_y' \right) ds
\]

\[
- \frac{1-\nu}{2G} \int \left( \frac{d}{ds} - \frac{d}{dn} \right) \left( \sigma_x'' + \sigma_y'' \right) ds
\]

\[
- \frac{i}{2G} \int \left[ \mathcal{Y} \left( \mathcal{X} dy - \mathcal{Y} dx \right) + \mathcal{F} q \right] ds,
\]

\[
\delta v = \mathcal{X} \delta \omega - \frac{1-\nu}{2G} \int \left( \frac{d}{ds} + \frac{d}{dn} \right) \left( \sigma_x' + \sigma_y' \right) ds
\]

\[
- \frac{1-\nu}{2G} \int \left( \frac{d}{ds} + \frac{d}{dn} \right) \left( \sigma_x'' + \sigma_y'' \right) ds
\]

\[
- \frac{i}{2G} \int \left[ \mathcal{X} \left( \mathcal{Y} dy - \mathcal{X} dx \right) - \mathcal{F} p \right] ds .
\]

For the general set of body forces, the problem of finding a particular solution may be difficult. However, in the important case for which the body forces are derivable from a potential function such that

\[
\mathcal{X} = - \frac{\partial V}{\partial x} , \quad \mathcal{Y} = - \frac{\partial V}{\partial y} ,
\]

\(1\)We will substitute the familiar symbol G for the Lamé constant \(\mu\) in the remainder of this dissertation.
the problem of finding the particular solution is considerably simplified. For this case, we will define a new function $\phi$ by the differential relations

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} + \nabla, \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} + \nabla, \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}. \quad (2.11)$$

Substituting these relations in (2.1) shows that (2.1) is automatically satisfied. Substituting (2.11) in (2.2) gives

$$\nabla^4 \phi = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 \phi = -\frac{\partial^2 \nabla^2 \phi}{\partial \nabla^2 \phi} \cdot \nabla^2 V. \quad (2.12)$$

If we can find any particular solution $\phi_0$, satisfying (2.12), then, if we write $\phi = \phi_0 + \phi'$, it is obvious that $\phi'$ satisfies (2.12) without body forces. Further defining

$$\sigma_x' = \frac{\partial \phi'}{\partial y} + \nabla, \quad \sigma_y' = \frac{\partial \phi'}{\partial x} + \nabla, \quad \tau_{xy}' = -\frac{\partial \phi'}{\partial x \partial y}, \quad (2.13)$$

we have that the stresses $\sigma_x', \sigma_y', \tau_{xy}'$ are given by

$$\sigma_x' = \frac{\partial^2 \phi'}{\partial y^2}, \quad \sigma_y' = \frac{\partial^2 \phi'}{\partial x^2}, \quad \tau_{xy}' = -\frac{\partial^2 \phi'}{\partial x \partial y}, \quad (2.14)$$

where $\phi'$ satisfies the biharmonic equation

$$\nabla^4 \phi' = 0. \quad (2.15)$$
Substituting (2.10), (2.13), and (2.14) in (2.6), we have that \( \phi' \) must satisfy the boundary conditions

\[
\left( \frac{\partial^2 \phi'}{\partial x^2} \right) \cos(n, x) - \frac{\partial^2 \phi'}{\partial x \partial y} \cos(n, y) = f'_x, \tag{2.16}
\]

\[
- \frac{\partial^2 \phi'}{\partial x \partial y} \cos(n, x) + \left( \frac{\partial^2 \phi'}{\partial y^2} \right) \cos(n, y) = f'_y, \tag{2.17}
\]

where \( f'_x \) and \( f'_y \) are given by the relations

\[
f'_x = f_x - x_\alpha \cos(n, x) - x_{x' y} \cos(n, y), \tag{2.18}
\]

\[
f'_y = f_y - x_{x' y} \cos(n, x) - y_{y'} \cos(n, y). \tag{2.19}
\]

We wish also to derive forms of Michell's equations involving only \( \phi' \) and the boundary values \( f'_x \) and \( f'_y \). We will begin with the rotation. Substituting from (2.10), (2.13), and (2.14) in (2.7), we obtain

\[
\delta \omega = \frac{1 - \nu}{2G} \int \frac{d}{dn} \left( \nabla^2 \phi' \right) ds + \frac{1 - \nu}{2G} \int \frac{d}{dn} \left( \nabla^2 \phi^0 \right) ds
- \frac{1}{2G} \int \frac{dV}{dn} ds, \tag{2.20}
\]

where in the last integral we have used the fact that \( X = - \frac{2V}{\partial x} \), \( Y = - \frac{2V}{\partial y} \), so that

\[
X \, dy - Y \, dx = - \frac{\partial V}{\partial x} \, dy + \frac{\partial V}{\partial y} \, dx
= - \frac{dV}{dn} \, ds \quad \text{by (1.45)}. \tag{2.21}
\]
We will now define $\delta \omega'$ by the relation

$$\delta \omega' = \delta \omega - \frac{1 - \nu}{2 \mu} \int \frac{d}{dn} (\nabla \phi) \, ds + \frac{1}{2 \mu} \int \frac{dV}{dn} \, ds , \tag{2.20}$$

so that

$$\delta \omega' = \frac{1 - \nu}{2 \mu} \int \frac{d}{dn} (\nabla \phi') \, ds . \tag{2.21}$$

This is the desired relation. Since (by assumption) we know $\delta \omega$ and $V$, and providing we have found $\phi'$, the quantity $\delta \omega'$ is completely determined.

We wish how to perform a similar operation on $\delta u$.

Substituting in (2.8) from (2.10), (2.13), and (2.14), we obtain

$$\delta u = - \eta \delta \omega - \frac{1 - \nu}{2 \mu} \int (x \frac{d}{ds} - \frac{d}{dn}) \nabla \phi' \, ds$$

$$- \frac{1 - \nu}{2 \mu} \int (x \frac{d}{ds} - \frac{d}{dn}) \nabla \psi \, ds$$

$$- \frac{1}{2 \mu} \int \frac{dV}{dn} \, ds$$

$$+ \frac{1}{2 \mu} \int \sigma \, ds .$$

Now we can write for the y-component of the external forces on the circuit

$$f_{y} = f'_{y} + \tau_{y} \cos (\eta, \chi) + \sigma_{y} \cos (\eta, \gamma) .$$
Substituting from this relation for $f_p$ and from (2.20) for $\delta \omega$
in the above equation for $\delta u$, we obtain

\[ \delta u = - \gamma A \delta \omega' - \gamma A \left[ \frac{1 - \nu}{2 \alpha} \int \left( \frac{d}{ds} \right) (V^2 \phi^0) ds + \frac{1}{2 \alpha} \int \frac{dV}{dn} ds \right] + \frac{1 - \nu}{2 \alpha} \int \left( \frac{c}{ds} - c \frac{d}{dn} \right) \left( V^2 \phi^0 + z \phi' \right) ds \]

\[ - \frac{1}{2 \alpha} \int \frac{dV}{dn} ds \]

\[ + \frac{1}{2 \alpha} \int \left( f_p' + \zeta_g d \frac{d}{ds} \right) ds . \]

We will now define $\delta u'$ by the following relation

\[ \delta u' = \delta u + \gamma A \left[ \frac{1 - \nu}{2 \alpha} \int \left( \frac{d}{ds} \right) (V^2 \phi^0) ds + \frac{1}{2 \alpha} \int \frac{dV}{dn} ds \right] + \frac{1 - \nu}{2 \alpha} \int \left( \frac{c}{ds} - c \frac{d}{dn} \right) \left( V^2 \phi^0 + z \phi' \right) ds \]

\[ + \frac{1}{2 \alpha} \int \frac{dV}{dn} ds \]

\[ - \frac{1}{2 \alpha} \int \left( \zeta_g d \frac{d}{ds} - \zeta_g \frac{d}{ds} \phi' \right) ds , \]

so that by (2.22)

\[ \delta u' = - \gamma A \delta \omega' - \frac{1 - \nu}{2 \alpha} \int \left( \frac{c}{ds} - c \frac{d}{dn} \right) \left( V^2 \phi' \right) ds \]

\[ + \frac{1}{2 \alpha} \int f_p' ds . \]
We can easily show that \( \int_{\Gamma} f'_{r_y} \, ds = \int_{\Gamma} f'_y \, ds \) where \( f'_{r_y} \) is the y-component of the boundary loads given in (2.19) and \( \int_{\Gamma} \) indicates the integral taken over the boundary of the region.

We consider any two circuits \( \Gamma_1 \) and \( \Gamma_2 \) as shown in Fig. 7 of Chapter 1, and make the cut \( A_1 A_2 \) between them.

We replace the y-components of the boundary forces by stresses according to the relation

\[
\begin{align*}
\sigma_y' &= \tau_{xy}' \cos \alpha - \sigma_x' \cos \beta \\
&= \tau_{xy}' \frac{dx}{dy} - \sigma_x' \frac{dy}{dx}.
\end{align*}
\]

Consider the integral over the entire boundary of the cut region

\[
\int_{\Gamma} f'_{r_y} \, ds = \int \left( \tau_{xy}' \frac{dx}{dy} - \sigma_x' \frac{dy}{dx} \right) \, ds
\]

by Green's theorem (1.57). But this last integral is zero by (2.4). We next break the line integral into integrals over \( \Gamma_1, \Gamma_2, A_2 A_1, \) and \( A_1 A_2 \). Then the line integrals over \( A_2 A_1 \) and \( A_1 A_2 \) cancel, and we have \( \int_{\Gamma} f'_{r_y} \, ds = \int_{\Gamma_2} f'_{r_y} \, ds \).

Shrinking \( \Gamma_2 \) until it coincides with the boundary \( B \) gives us the result we wished to prove.

Thus, we can write finally

\[
\delta u' = -\frac{\partial}{\partial y} \delta \omega' - \frac{1}{2\gamma} \int (x \frac{d}{dy} - y \frac{d}{dx}) \nabla^2 \psi' \, ds
+ \frac{1}{2\gamma} \int_0^{\Gamma_2} f'_{r} \, ds.
\]

This is the desired equation. It is seen as before that (2.24) is expressed entirely in terms of \( \psi' \) and its boundary values. The
quantity $\delta u'$ is given entirely in terms of known functions by (2.23). We can simplify (2.23) somewhat. We will substitute for $\gamma_{xy}$ and $\phi^G_y$ from (2.13) and collect terms involving $V$ to obtain

$$\delta u' = \delta u + \gamma_{AR} \left[ \frac{1}{2G} \int \frac{d}{dn} (V^2 \phi^G) ds + \frac{1}{2G} \int \frac{dV}{dn} ds \right] + \frac{1}{G} \int \left( x \frac{d^2 \phi^G}{dn^2} - \frac{d \phi^G}{dn} \right) ds$$

$$+ \frac{1}{2G} \int \left( \frac{dV}{dn} \right)^2 ds$$

$$+ \frac{1}{2G} \int \left( x \frac{dV}{ds} \right) ds$$

$$+ \frac{1}{2G} \int \left( V \frac{dx}{ds} \right) ds$$

$$+ \frac{1}{2G} \int \left( \frac{d^2 \phi^G}{dx^2} \frac{dx}{ds} + \frac{d \phi^G}{dx} \frac{d^2 \phi^G}{dx^2} \frac{dx}{ds} \right) ds.$$

This last integral is evidently $\int d(\frac{d \phi^G}{dx}) = \delta (\frac{d \phi^G}{dx})$. We will also integrate the next to the last integral by parts as follows

$$\int V dx = \left[ xV \right]_{\alpha}^{\beta} - \int x dV = \gamma_{AR} \delta V - \int x dV. \quad (2.26)$$

(The quantity $x_{\alpha} \delta V$ vanishes if $V$ is single valued.)

Substituting (2.26) in (2.25), we obtain finally

$$\delta u' = \delta u + \gamma_{AR} \left[ \frac{1}{2G} \int \frac{d}{dn} (V^2 \phi^G) ds + \frac{1}{2G} \int \frac{dV}{dn} ds \right]$$

$$+ \frac{1}{G} \int \left( x \frac{d^2 \phi^G}{dn^2} - \frac{d \phi^G}{dn} \right) ds$$

$$+ \frac{1}{2G} \int \left( x \frac{dV}{dn} \right)^2 ds$$

$$+ \frac{1}{2G} \int \left( x \frac{dV}{ds} \right) ds$$

$$+ \frac{1}{2G} \int d \left( \frac{d \phi^G}{dx} \right) + \gamma_{AR} \delta V.$$
If we substitute from (2.10), (2.13), and (2.14) in (2.9) and carry out a derivation similar to that used for (2.24) and (2.27), we obtain

\[ \delta V' = x \delta \omega' - \frac{1 - \nu}{2G} \int (\frac{d}{ds} + \kappa \frac{d}{d\gamma})(V \phi') ds \]

\[ - \frac{1}{2G} \int N_\gamma' ds \]

where

\[ \delta V' = \delta V - x_A \left[ \frac{1 - \nu}{2G} \int \frac{d}{d\gamma} (V \phi) ds + \frac{1}{2G} \int \frac{dV}{d\gamma} ds \right] \]

\[ + \frac{1 - \nu}{2G} \int (\frac{d}{ds} + \kappa \frac{d}{d\gamma}) V \phi^0 ds \]

\[ + \frac{1 - \nu}{2G} \int (\frac{d}{ds} + \kappa \frac{d}{d\gamma}) dS \]

\[ + \frac{1}{2G} \int d\left( \frac{\partial \phi^0}{\partial \gamma} \right) \delta V \]

By finding the particular solution \( \phi^0 \) of (2.12), we can thus reduce the problem involving a body force potential to a corresponding problem in which the body forces are absent. The statement of the simpler problem in terms of the stress function \( \phi' \) consists of (2.15), (2.16), (2.17), (2.21), (2.24), and (2.28).

In the remainder of this dissertation, we will consider only the problem of finding stress functions \( \phi' \) which satisfy the homogeneous biharmonic equation.
Specifically, we shall be trying to find stress functions satisfying the following set of equations

\[ \nabla^4 \phi' = 0 , \quad (2.30) \]

\[ f_x' = \frac{\partial^2 \phi'}{\partial y^2} \cos(m, y) - \frac{\partial^2 \phi'}{\partial x \partial y} \cos(m, x) , \quad (2.31) \]

\[ f_y' = -\frac{\partial^2 \phi'}{\partial x^2} \cos(m, x) + \frac{\partial^2 \phi'}{\partial y^2} \cos(m, y) , \quad (2.32) \]

\[ \delta \omega' = \frac{1 - \nu}{2G} \int \frac{d}{ds} (\nabla^2 \phi') ds , \quad (2.33) \]

\[ \delta u' = -\gamma_R \delta \omega' - \frac{1 - \nu}{2G} \int (\frac{d}{ds} - \frac{\partial}{\partial n}) \nabla^2 \phi' ds + \frac{1}{2G} \int f_y' ds , \quad (2.34) \]

\[ \delta v' = \gamma_R \delta \omega' - \frac{1 - \nu}{2G} \int (\frac{\partial}{\partial s} + \frac{d}{dn}) \nabla^2 \phi' ds - \frac{1}{2G} \int f_x' ds . \quad (2.35) \]

The quantities \( f_x', f_y', \delta \omega', \delta u', \delta v' \) will be assumed known. We have, in fact, given explicit formulas expressing these quantities entirely in terms of known quantities in (2.18), (2.19), (2.20), (2.27), and (2.29).

Since we will be referring henceforth only to the quantities involved in (2.30) through (2.34), we will, for simplicity of notation, drop all of the primes.

We will be seeking the solution of this problem in series form. In the remaining sections of this chapter, we will develop various forms of series solutions that are appropriate for different
kinds of regions. We will begin with the simplest series form to be considered, for which all of the terms are expressed in the same coordinate system.

The series solution of the biharmonic equation in polar coordinates

In this article and throughout the remainder of this dissertation, we will be concerned with the series solutions formed from terms satisfying the biharmonic equation in polar coordinates \((r, \theta)\). Substituting for \(x\) and \(y\) in (2.30) from the formulas \(x = r \cos \theta\), \(y = r \sin \theta\), we obtain the equation

\[
\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \phi = 0
\]  

(2.35)

The general series solution of the equation, given by Michell (1) in 1899, is as follows:

\[
\phi(r, \theta) = a_0 \log r + b_0 r^2 + b_0' r^2 \log r + d_0 r^2 \theta + c_0 \theta + a_1 r \sin \theta + (a_1' r^{-1} + b_1 r^3 + b_1' r \log r) \cos \theta + c_1 r \cos \theta + (c_1' r^{-1} + d_1 r^3 + d_1' r \log r) \sin \theta + \sum_{n=2}^{\infty} \left( a_n r^n + c_n r^{-n} + b_n r^{n+2} + d_n r^{-n-2} \right) \cos \theta + \sum_{n=2}^{\infty} \left( e_n r^n + c_n r^{-n} + d_n r^{n+2} + e_n' r^{-n-2} \right) \sin \theta
\]  

(2.36)
It is noted that the sum of all of the terms involving $a_n', c_n', a_{-n}', c_{-n}'$ (except $a_1$ and $c_1$) constitutes the general solution of Laplace's equation $\nabla^2 \phi = 0$. This is of importance in the application of the computer program since it permits essentially the same program to be used for the solution of both harmonic and biharmonic boundary value problems.

The stress function given in (2.36) is adequate for solving a wide range of problems involving simply and doubly connected regions. For more complex problems we will obtain generalizations of (2.36). However, before we turn to these more complicated cases, we will investigate the character of the terms of (2.36) in detail.

We note first that the series (2.36) has a pole at the origin. Since the stresses are the derivatives of the stress function, the series must also give infinite stresses at the origin. If the origin lies in the region, we must set the coefficients equal to zero for those terms that give rise to infinite stresses. In order to find which terms do give infinite stresses at the origin, we will derive the expressions for the stresses.
The stresses in polar coordinates are obtained from the stress function by the following relations

\[ \sigma_\theta = \frac{1}{\lambda} \frac{\partial^2 \phi}{\partial \rho^2} + \frac{\rho}{\lambda^2} \frac{\partial^2 \phi}{\partial \theta^2}, \]

\[ \tau_\rho \theta = \frac{\partial^2 \phi}{\partial \rho \partial \theta}, \]

\[ \tau_\rho \theta = -\frac{\partial}{\rho} \left( \frac{1}{\rho} \frac{\partial \phi}{\partial \theta} \right). \]

Thus

\[ \sigma_\rho = a_0 \rho^{-2} + 2b_0 + b'_0 (2 \log \rho + 1) + 2d_0 \theta \]

\[ + (2a_1 \rho^{-1} - 2a_2 \rho^{-3} + 2d_1 \rho + b_2 \rho^{-1}) \cos \theta \]

\[ - (2e_1 \rho^{-1} + 2e_2 \rho^{-3} - 2d_2 \rho - d_1 \rho^{-1}) \sin \theta \]

\[ + \sum_{n=2}^{\infty} \left[ \frac{-n(n-1) \alpha_n \rho^{n-2} - n(n+1) \xi_n \rho^{-n-2}}{n(n+1)} \cos n \phi \right] \]

\[ + \sum_{n=2}^{\infty} \left[ (1+n)(2-n) b_n \rho^n + (3n)(1-n) b_n \rho^{-n} \right] \cos n \theta \]

\[ + \sum_{n=2}^{\infty} \left[ -n(n-1) \alpha_n \rho^{n-2} + n(n+1) \xi_n \rho^{-n-2} \right] \sin n \phi \]

\[ + \sum_{n=2}^{\infty} \left[ (1+n)(2-n) d_n \rho^n + (3n)(1-n) d_n \rho^{-n} \right] \sin n \theta, \]
\[ \xi_\theta = -a_0 \pi^{-2} \tau^2 b_0 + b_0' (2 \pi \delta \tau + 3) + 2 d_0 \theta \\
+ (2 \pi^{-3} a_{-1} + \xi b_1 + \pi^{-1} b_{-1}) \cos \theta \\
+ (2 \pi^{-3} c_{-1} + \xi d_1 + \pi^{-1} d_{-1}) \sin \theta \\
+ \sum_{n=2}^{\infty} \left[ n(n-1) a_n \pi^{n-2} + n(n+1) a_{-n} \pi^{-n-2} \right] \cos n \theta \\
+ \sum_{n=2}^{\infty} \left[ (n+2)(n+1) b_n \pi^{n} + (2-n)(1-n) b_{-n} \pi^{-n} \right] \cos n \theta \\
+ \sum_{n=2}^{\infty} \left[ n(n-1) c_n \pi^{n-2} + n(n+1) c_{-n} \pi^{-n-2} \right] \sin n \theta \\
+ \sum_{n=2}^{\infty} \left[ (n+2)(n+1) d_n \pi^{n} + 2-n)(1-n) d_{-n} \pi^{-n} \right] \sin n \theta \]

(2.39)

\[ \xi_\theta = -d_0 + c_0 \pi^{-2} - \left( 2 a_{-1} \pi^{-3} - 2 b_1 \pi - b_{-1} \pi^{-1} \right) \sin \theta \\
+ \left( 2 c_{-1} \pi^{-3} - 2 d_1 \pi - d_{-1} \pi^{-1} \right) \cos \theta \\
+ \sum_{n=2}^{\infty} \left[ n(n-1) a_n \pi^{n-2} - n(n+1) a_{-n} \pi^{-n-2} \right] \sin n \theta \\
+ \sum_{n=2}^{\infty} \left[ n(n+1) b_n \pi^{n} - n(n-1) b_{-n} \pi^{-n} \right] \sin n \theta \\
- \sum_{n=2}^{\infty} \left[ n(n-1) c_n \pi^{n-2} - n(n+1) c_{-n} \pi^{-n-2} \right] \cos n \theta \\
- \sum_{n=2}^{\infty} \left[ n(n+1) d_n \pi^{n} - n(n-1) d_{-n} \pi^{-n} \right] \cos n \theta \]

(2.40)
Examination of these stresses shows that the following stress function is applicable to problems for which the origin is contained in the problem region:

\[ \phi_0(r, \theta) = b_0 r^2 + b_1 r^3 \cos \theta + d_1 r^3 \cos \theta + \sum_{n=2}^{\infty} \left[ (a_n r^n + b_n r^{n+1}) \cos n\theta + (c_n r^n + d_n r^{n+1}) \sin n\theta \right]. \]  

(2.41)

We have not included the term \( d_0 r^2 \) here since it leads to multivalued stresses if the origin is in the region or in a hole of the region. Consider the values of the stresses derived from this term. From (2.38), (2.39), and (2.40), we have

\[ \sigma_\tau = \sigma_\phi = 2d_\phi, \]
\[ \tau_\tau \phi = -d_\phi. \]  

(2.42)

Now, if the region completely surrounds the origin, as in Fig. 9, and if we evaluate the stresses given by (2.42) at the beginning and end of the circuit ACBA, we find that the \( \sigma_\tau \) and \( \sigma_\phi \) stresses at A after we have completed the circuit are \( 2\pi d_\phi \) less than when we began the circuit. That is, \( \sigma_\tau \) and \( \tau_\tau \) are multivalued. Since the stresses are always assumed to be single valued, we must choose \( d_\phi = 0 \) whenever the origin lies in, or is surrounded by, the region. However, if the origin is not surrounded by the region, as shown in Fig. 10 a, the terms of (2.42) for points of the region...
Fig. 9.--The circuit for evaluating $d_\circ$.

Fig. 10.--Two cases for which the origin is external to the region.
are single valued. Since no circuit laid out in the region will enclose the origin, \( \theta \), as measured from the origin, will always return to the same values for any closed circuit in the region. Thus, for this case the term \( d_o r^2 \theta \) is a permissible element of the stress function.

From an examination of (2.38) through (2.40), it is clear that all of the terms of the stress function (except \( d_o r^2 \theta \)) give continuous single-valued stresses in both of the cases illustrated by Fig. 10a and 10b where the origin is not a part of the region.

We will consider now the case represented by Fig. 10b in which the origin is in a hole of the region. We will apply Michell's equations (2.32) through (2.34) to the stress function (2.36).

We shall carry out this study in two steps. We will first determine which terms of (2.36) give non-zero values of the integrals

\[
\int f_x \, ds, \quad \int f_y \, ds,
\]

where the symbol \( \int \) indicates, as before, that the line integral is to be evaluated over a closed path which contains the hole of the region. These integrals, as we have shown earlier, are independent of the path taken around the hole.

To evaluate these expressions, we will substitute for the values of \( f_x \) and \( f_y \) from (2.16) and (2.17) to obtain

\[
\int f_x \, ds = \int \left( \frac{\partial \psi}{\partial y} \frac{dx}{dn} - \frac{\partial \phi}{\partial x} \frac{dy}{dn} \right) ds
= \int \left( \frac{\partial \psi}{\partial y} \frac{dx}{ds} + \frac{\partial \phi}{\partial x} \frac{dy}{ds} \right) ds,
\]

\( \psi \) and \( \phi \) being functions of the current line touching the hole.
or
\[ \oint f_x \, ds = \oint \left( \frac{d}{ds} \left( \frac{2\phi}{\partial y} \right) \right) \, ds, \]

and
\[ \oint f_y \, ds = \oint \delta \left( \frac{\partial \phi}{\partial x} \right), \quad (2.43) \]

\[ \oint f_x \, ds = \oint \left( -\frac{\partial^2 \phi}{\partial x \partial y} \frac{dx}{dn} + \frac{\partial^2 \phi}{\partial x^2} \frac{d\theta}{dn} \right) \, ds \]
\[ = -\delta \left( \frac{\partial \phi}{\partial x} \right). \quad (2.44) \]

In evaluating these expressions we will use the relations
\[ \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial \phi}{\partial \theta} \frac{\partial \phi}{\partial \theta} + \frac{\partial \phi}{\partial \phi} \frac{\partial \phi}{\partial \theta} = \frac{\partial \phi}{\partial \phi} \cos \phi - \frac{1}{\rho} \frac{\partial \phi}{\partial \theta} \sin \phi, \quad (2.45) \]
\[ \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial \phi}{\partial \phi} \frac{\partial \phi}{\partial \phi} + \frac{\partial \phi}{\partial \phi} \frac{\partial \phi}{\partial \theta} = \frac{\partial \phi}{\partial \phi} \sin \phi + \frac{1}{\rho} \frac{\partial \phi}{\partial \theta} \cos \phi. \]

We will not write out the entire expression for \( \frac{\partial \phi}{\partial x} \) or \( \frac{\partial \phi}{\partial y} \) from (2.36). It is noted that the multivalued terms of \( \frac{\partial \phi}{\partial x} \) and \( \frac{\partial \phi}{\partial y} \) must contain \( \phi \) as an explicit coefficient. Now for any expressions of the form \( f(r) \cos \rho \phi \) and \( g(r) \sin \rho \phi \) we have by (2.45)
\[ \frac{\partial}{\partial \phi} f(r) \cos \rho \phi = f'(r) \cos \rho \phi - \rho n f'(r) \sin \rho \phi \sin \phi, \]
\[ \frac{\partial}{\partial \phi} f(r) \sin \rho \phi = f'(r) \sin \rho \phi + \rho n f'(r) \cos \rho \phi \sin \phi, \]
\[ \frac{\partial}{\partial \phi} g(r) \sin \rho \phi = g'(r) \sin \rho \phi - \rho n g'(r) \cos \rho \phi \sin \phi, \]
\[ \frac{\partial}{\partial \phi} g(r) \sin \rho \phi = g'(r) \sin \rho \phi + \rho n g'(r) \cos \rho \phi \sin \phi. \]
None of these expressions are multivalued. We need, therefore, to consider only the terms of (2.36) involving the coefficients $c_0$, $a_1$, and $c_1$. For these terms we have

$$
\frac{2\Phi}{\varepsilon_x} = -c_0 n^{-1} \sin \Theta - a_1 \sin \Theta + c_1 (\Theta - \sin \Theta e) \; ,
$$

$$
\frac{2\Phi}{\varepsilon_y} = c_0 n^{-1} \cos \Theta + a_1 (\Theta + \sin \Theta e) + c_1 \cos \Theta \; .
$$

Thus we have simply

$$
\oint f_\lambda ds = \oint \frac{d}{ds} \left( \frac{2\Phi}{\varepsilon_x} \right) ds = \delta \left( \frac{2\Phi}{\varepsilon_x} \right) = -2\pi a, \quad (2.46)
$$

$$
\oint f_\gamma ds = -\oint \frac{d}{ds} \left( \frac{2\Phi}{\varepsilon_y} \right) ds = -\delta \left( \frac{2\Phi}{\varepsilon_y} \right) = 2\pi c, \quad (2.47)
$$

since the circuit is taken in the negative $\Theta$ direction. We can evaluate the resultant moment arising from $\Phi$ just as easily.

The moment $M$ is given by

$$
M = \oint (-\gamma f_\lambda + \kappa f_\gamma) ds
$$

$$
= -\oint \left[ \gamma d \left( \frac{2\Phi}{\varepsilon_x} \right) + \kappa d \left( \frac{2\Phi}{\varepsilon_y} \right) \right] .
$$

Integrating this integral by parts gives

$$
M = -2\delta \left( \gamma \frac{2\Phi}{\varepsilon_x} + \kappa \frac{2\Phi}{\varepsilon_y} \right) + \oint \left( \frac{2\Phi}{\varepsilon_x} d\gamma + \frac{2\Phi}{\varepsilon_y} d\kappa \right) \quad (2.48)
$$

$$
= -2\delta \left( \gamma \frac{2\Phi}{\varepsilon_x} + \kappa \frac{2\Phi}{\varepsilon_y} \right) + \delta \Phi .
$$
From (2.36) we have

\[ \delta \phi = -2\pi \rho_{\phi} - 2\pi \pi \rho \sin \theta - 2\pi \pi \rho \cos \theta . \]  \hspace{1cm} (2.49)

By (2.46) and (2.47) we have

\[ -2\pi \left( \frac{\partial \phi}{\partial y} + x \frac{\partial \phi}{\partial x} \right) = 2\pi \pi \rho \sin \theta + 2\pi \pi \rho \cos \theta . \]

Thus

\[ M = -2\pi \rho_{\phi} . \]  \hspace{1cm} (2.50)

We will now return to evaluating the remaining integrals of the Michell equations (2.32) through (2.34). Specifically, we will evaluate the three integrals

\[ I_1 = \oint \frac{d}{d\alpha} (\nabla^2 \phi) \, ds, \]  \hspace{1cm} (2.51)

\[ I_2 = \oint \left[ x \frac{d}{ds} (\nabla^2 \phi) - \gamma \frac{d}{ds} (\nabla^4 \phi) \right] \, ds, \]  \hspace{1cm} (2.52)

\[ I_3 = \oint \left[ \gamma \frac{d}{ds} (\nabla^2 \phi) + x \frac{d}{ds} (\nabla^4 \phi) \right] \, ds. \]  \hspace{1cm} (2.53)

We note first that, since each of these integrals involve only \( \nabla^2 \phi \), the harmonic terms of the stress function drop out.
Second, by using Green's theorem (1.57), we can show that for a circuit around a simply connected region

\[ \oint_{\partial A} (\nabla \cdot \phi) \, ds = \iint_{A} (\nabla^{2} \phi) \, dA = 0 \]

and

\[ \oint_{\partial A} [x \, \frac{d}{ds} (\nabla \cdot \phi) - \frac{d}{dn} (\nabla^{2} \phi)] \, ds = \iint_{A} x \, \nabla^{2} \phi \, dA = 0 \]

Thus, in the manner of our discussion of Michell's equations in Chapter 1, it follows that the integrals \( I_1 \), \( I_2 \), and \( I_3 \) are independent of the path of integration as long as this path surrounds the hole (Fig. 9), and as long as the path remains in the region of the problem. (The fact that these particular integrals are independent of the integration path is a result of the elimination of the body forces.) We will therefore choose, for simplicity, as the path of integration, a circle with the center at the origin. It may occur in some cases that no such circle may be found that is contained in the region. An example of such a region would be an elliptical ring, with the minor axis of the outer boundary smaller than the major axis of the inner boundary. In this case, we must choose a path composed of circle arcs whose centers are at the origin and segments of radius vectors from the origin. However, we will assume here that we may evaluate the integrals on a single circle. We will take the direction of the normal to the boundary toward the origin as shown in Fig. 11. Thus, we have \( ds = -r \, d\sigma \) and \( dn = -dr \).
Fig. 11.—The circular path for evaluating the dislocations

Thus, the integrals on this circle will become

\[ I_1 = \int_{0}^{2\pi} \kappa \frac{d}{d\kappa} (\nabla^2 \phi) d\theta, \quad (2.54) \]

\[ I_2 = \int_{0}^{2\pi} [\chi \frac{d}{d\theta} (\nabla^2 \phi) - \gamma \kappa \frac{d}{d\kappa} (\nabla^2 \phi)] d\theta, \quad (2.55) \]

\[ I_3 = \int_{0}^{2\pi} [\gamma \frac{d}{d\phi} (\nabla^2 \phi) + \chi \kappa \frac{d}{d\kappa} (\nabla^2 \phi)] d\phi. \quad (2.56) \]

We will remark further that \( \nabla^2 \phi \) is the first invariant of the stresses \( \sigma_x + \sigma_y = \sigma_r + \sigma_\theta \), so that we can take for \( \nabla^2 \phi \) the sum of (2.38) and (2.39).
We will start with the evaluation of $I_1$. We have from (2.38) and (2.39) ($d_0 = 0$)

$$
\pi \frac{d}{d\phi} (\nabla^2 \phi) = 4 b_0' + (-2_{,1} a_{,1} + 8\pi b_{,1} - 2\pi b_{,1}) \cos \theta
$$

$$
+ (2\pi^{-1} e_{,1} + 8\pi d_{,1} - 2\pi d_{,1}) \sin \theta
$$

$$
+ \sum_{n=2}^{\infty} \left[ f_n(\pi) \cos n \theta + g_n(\pi) \sin n \theta \right].
$$

We thus have

$$
I_1 = -8\pi b_0',
$$

since the integrals of all of the trigonometric functions are zero.

To evaluate $I_2$ and $I_3$, we write first from (2.38) and (2.39)

$$
\frac{d}{d\phi} (\alpha + \phi) = -(-2_{,1} a_{,1} + 8\pi b_{,1} + 2\pi b_{,1}) \sin \theta
$$

$$
+ (-2\pi^{-1} e_{,1} + 8\pi d_{,1} + 2\pi d_{,1}) \cos \theta
$$

$$
+ \sum_{n=2}^{\infty} \left[ f'_n(\pi) \sin n \theta + g'_n(\pi) \cos n \theta \right].
$$
Substituting from (2.57) and (2.59) in (2.55) and using
\[ x = r \cos \theta, \quad y = r \sin \theta, \]
we obtain
\[
I_2 = \int_{0}^{2\pi} \int_{0}^{\infty} \frac{16 \pi b_1 \cos \theta \sin \theta \, de \, d\theta}{r} - \int_{0}^{2\pi} 8 b_1 \sin \theta \, d\theta \\
+ \int_{0}^{2\pi} \left( -2 \pi \ell_1 + 8 \pi b_1 + 2 \pi \ell_1 \right) \pi \cos \theta \, d\theta \\
- \int_{0}^{2\pi} \left( 2 \pi \ell_1 + 8 \pi b_1 - 2 \pi \ell_1 \right) \pi \sin^2 \theta \, d\theta \\
+ \int_{0}^{2\pi} \sum_{n=0}^{\infty} \left[ f_n'(n) \sin n \theta + g_n'(n) \cos n \theta \right] \pi \cos \theta \, d\theta \\
+ \int_{0}^{2\pi} \sum_{n=0}^{\infty} \left[ f_n(n) \cos n \theta + g_n(n) \sin n \theta \right] \pi \cos \theta \, d\theta .
\]

All of these integrals vanish except those involving \( \cos^2 \theta \) and \( \sin^2 \theta \). Adding these two integrals we have
\[
I_2 = \int_{0}^{2\pi} \left( -2 \pi \ell_1 + 2 \pi \ell_1 \right) d\theta + \int_{0}^{2\pi} 8 \pi b_1 \left( \cos^2 \theta - \sin^2 \theta \right) d\theta .
\]

This second integral vanishes since \( \cos^2 \theta - \sin^2 \theta = \cos 2\theta \).

Thus
\[
I_2 = 4 \pi \ell_1 - 4 \pi \ell_1 .
\]

By substituting (2.57) and (2.59) into (2.56) and integrating the result, we find
\[
I_3 = 4 \pi \alpha, + 4 \pi b_1 .
\]
Substituting (2.46), (2.47), (2.50), (2.60), and (2.61) in (2.32) through (2.34) we have

\[ \delta \alpha = \frac{1-\nu}{2G} I_1 = -4\pi b_0 \frac{1-\nu}{G}, \]  

(2.62)

\[ \delta u = \frac{2\pi(1-\nu)}{G} \left[ 2\chi b'_0 + d_2 - \frac{1-2\nu}{2(1-\nu)} \xi_1 \right], \]  

(2.63)

and

\[ \delta v = -\frac{2\pi(1-\nu)}{G} \left[ 2\chi b'_0 + b_1 + \frac{1-2\nu}{2(1-\nu)} \xi_1 \right]. \]  

(2.64)

Equations (2.46), (2.47), (2.50), (2.62), (2.63), and (2.64) give explicit relations between the terms of the stress function \( \phi \) and the force resultants and the dislocations on a hole of the region, if the stress function is expanded about a point in the hole.

We will define a function \( P(r, \theta) \) by the equation

\[ P(r, \theta) = b'_0 r^2 \log r + d_\theta r^2 \theta + \xi_0 \theta + a_n \cos \theta 
+ b_n \log r \cos \theta + c_n \cos \theta + d_n \log r \sin \theta. \]  

(2.65)

Since we are assuming that the stress resultants and dislocations are known, the terms of \( P \) are known quantities if the origin is located in a hole of the region. In fact, on rearranging (2.46), (2.47), (2.50), and (2.60) through (2.64) and using the
condition of single valuedness of the stresses, we have for the
constant coefficients of \( P \)
\[
\begin{align*}
\sigma_0' &= \sigma, \\
n_1 &= \cfrac{-\int f_x \, ds}{2 \pi r} = \cfrac{1}{2 \pi} \int f_x \, ds, \\
n_2 &= \cfrac{-\int f_y \, ds}{2 \pi r} = \cfrac{1}{2 \pi} \int f_y \, ds, \\
n_3 &= \cfrac{-M}{2 \pi r}, \\
b_0' &= \cfrac{-G \delta \omega}{\nu r(1-\nu)}, \\
d_{-1} &= \cfrac{G \delta \omega}{2 \pi (1-\nu)} - 2 \nu b_0' + \cfrac{1-2\nu}{2(1-\nu)} c_1, \\
b_{-1} &= \cfrac{-G \delta \omega}{2 \pi (1-\nu)} + 2 \nu b_0' - \cfrac{1-2\nu}{2(1-\nu)} a_1.
\end{align*}
\]

We return to the case for which the origin is outside the
region as shown in Fig. 10a. A review of the above derivations
for the case where the origin is surrounded by the region shows
that the multivaluedness of the various expressions derived from
\( P \) resulted in every case from terms involving \( \theta \) as an explicit
coefficient which, for a complete circuit around the origin,
changed in value by \( 2 \pi \). But for a complete circuit in the
region of Fig. 10a, \( \theta \) returns to its original value, since the
circuit does not pass around the origin. Thus, for the case of
Fig. 10a, \( P \) gives single-valued stresses and displacements in the
region.
We now define a function \( \varphi^* = \varphi - \varphi_o - P \). Comparing (2.36) with (2.41) and (2.65) shows that

\[
\varphi^*(r, \theta) = a_0 + \sum_{n=1}^{\infty} r^{-n} \left( a_n \sin n \theta + b_n \cos n \theta \right)
\]

(2.67)

\( \varphi^* \) has an essential singularity at the origin. Moreover, from the derivations of this article, it may be seen that \( \varphi^* \) gives single-valued and continuous stresses and displacements in any region not containing the origin.

We have thus separated the stress function \( \varphi \) of (2.36) into the three separate stress functions \( \varphi_o, \varphi^*, \) and \( P \) which have the following special properties.

(a) \( \varphi_o \), defined by (2.41), has no pole in the finite plane. It is single valued and continuous and gives single-valued and continuous stresses and displacements everywhere in the finite plane.

(b) \( \varphi^* \), defined by (2.67), has an isolated pole at the origin. However, it is single valued and continuous and gives single-valued and continuous stresses and displacements everywhere except at the origin.

(c) The function \( P \) gives single-valued and continuous stresses and displacements if the origin is not surrounded by the region of the problem. However, when the origin is within a hole of the region, \( P \) is related to the force resultants and dislocations on the hole through (2.66).
These special properties of the three functions will have bearing on the way in which the stress function is chosen for multiply connected regions considered in the next article.

The stress function in multiply connected regions

We now wish to generalize the stress function (2.36) to make it applicable to multiply connected regions such as that shown in Fig. 7 of Chapter 1. In the case of noncircular holes, the choice of the proper form for the stress function is extremely difficult. However, for the case (shown in Fig. 12) involving M circular holes of arbitrary size and location, A. E. Green (2,3) gave the form of the stress function as

\[ \phi(n, \theta) = \phi_c(n, \theta) + \sum_{k=1}^{M-1} p_k(n, \theta) + \sum_{k=1}^{M-1} \phi_k(n, \theta), \]  

(2.68)

where

\[ \phi_c(n, \theta) = b_1 n^2 + b_3 n^3 \cos \theta + d_1 n^3 \sin \theta, \]  

(2.69)

\[ + \sum_{n=2}^{\infty} n^n (a_n \cos n \theta + c_n \sin n \theta), \]

\[ + \sum_{n=2}^{\infty} n^{n+2} (b_n \cos n \theta + d_n \sin n \theta) \]
Fig. 12.—A region containing an arbitrary number of arbitrary circular holes

\[ \phi_k^r(r_k, \theta_k) = a_0^k \log r_k \]

\[ + \sum_{n=1}^{\infty} r_k^{-n} (a_n^k \cos n \theta_k + b_n^k \sin n \theta_k) \]

\[ + \sum_{k=2}^{\infty} r_k^{-M+2} (b_n^k \cos n \theta_k + d_n^k \sin n \theta_k) \]
\[ P_k(r_k, \theta_k) = c_0^k \theta_k + b_1^k r_k^2 \cos \theta_k \\
+ r_k \theta_k (a_1^k \sin \theta_k + c_1^k \cos \theta_k) \\
+ r_k \theta_k^2 (b_1^k \cos \theta_k + d_1^k \sin \theta_k). \]

The coordinates \((r, \theta)\) are measured from the origin of the coordinate system. The coordinates \((r_k, \theta_k)\) are measured from the center of the \(k\)-th circle (Fig. 12).

The function \(\phi^* = \sum_{k=1}^{M-1} \phi_k^* (r_k, \theta_k)\) has poles at the center of each hole and gives single-valued stresses in the region of the problem. The function \(P_k(r_k, \theta_k)\) is calculated from the known values of the resultant of the boundary forces and the dislocations for \(k\)-th hole according to (2.66). \(d_0\) is set equal to zero in (2.71) since the origin of each \(P_k\) is in a hole of the region.

Since it is necessary to have only one function with a singularity at \(\infty\), the function \(\phi_0(r, \theta)\) is not repeated. In fact, the function

\[ \phi_0'(r_k, \theta_k) = b_c^k r_k^2 \cos \theta_k + b_l^k r_k^3 \cos \theta_k + d_1^k r_k^3 \sin \theta_k \\
+ \sum_{n=2}^{\infty} r_k^n (a_n^k \cos n \theta_k + c_n^k \sin n \theta_k) \\
+ \sum_{n=2}^{\infty} r_k^{n+1} (b_n^k \cos n \theta_k + d_n^k \sin n \theta_k). \]
may always be reduced to the form \( \varphi_0 (r, \phi) \) given in (2.69) by the transformation of variables

\[
\begin{align*}
\varphi_k \cos \phi_k & = r \cos \phi + r' \cos \phi', \\
\varphi_k \sin \phi_k & = r \sin \phi + r' \sin \phi',
\end{align*}
\]

where \((r', \phi')\) are the coordinates of the origin of the \((r_k, \phi_k)\) coordinate system measured from the origin.

Once the functions \( P_k (r', \phi') \) have been calculated, the problem is reduced to the determination of the functions \( \varphi_0^+ \) and \( \varphi'^* \). Writing

\[
\varphi^{**} = \varphi_0^+ + \varphi'^* = \varphi - \sum_{k=1}^{N-1} P_k (r_k, \phi_k),
\]

it is evident that \( \varphi^{**} \) is single valued and continuous everywhere in the region. The unknown constants of \( \varphi_0^+ \) and \( \varphi'^* \) must be chosen so that the boundary conditions (2.31) are satisfied. Substituting (2.68) in (2.31), it is evident that \( \varphi^{**} \) must satisfy the boundary conditions

\[
\begin{align*}
\frac{\partial^2 \varphi^{**}}{\partial x^2} \cos (n, x) & - \frac{\partial^2 \varphi^{**}}{\partial y^2} \cos (n, y) = f_k - \frac{\partial^2 \varphi}{\partial x^2} \cos (n, x) - \frac{\partial^2 \varphi}{\partial y^2} \cos (n, y), \\
- \frac{\partial^2 \varphi^{**}}{\partial x \partial y} \cos (n, x) + \frac{\partial^2 \varphi^{**}}{\partial x \partial y} \cos (n, y) & = f_j - \frac{\partial^2 \varphi}{\partial x \partial y} \cos (n, x) - \frac{\partial^2 \varphi}{\partial x \partial y} \cos (n, y).
\end{align*}
\]

We will be concerned with finding the solutions for \( \varphi^{**} \) by using the point matching technique. This technique will be discussed in Chapter 3.

As mentioned before, the problem of selecting the proper form of the stress function for the general case involving non-circular holes is difficult. However, for such problems, it is
still necessary to have only one function \( \Phi_0 (r, \theta) \). It is also sufficient to have only one function \( P_k (r_k, \theta_k) \) for each hole to represent the resultants of the external forces and the dislocations associated with that hole. However, it is reasonable to assume that the stress function may have more than one pole of the type represented by (2.70) in a noncircular hole. It has been suggested by some authors \(^{4,5}\) that, for noncircular holes, an appropriate form for (2.70) would be certain series involving the inverse powers of the product of the complex variables \( (z - z_k) f_k \), where \( (z - z_k) = \Omega(f_k) \) is the mapping function of the \( k\)-th boundary onto the circle.

However, this would require a different computer program to solve problems involving each different hole shape. Instead, we will use combinations of series of the form (2.70) to generate multipoled stress functions for noncircular holes. That is, for the stress function of a multiply connected problem involving noncircular holes, we will assume a function of the form

\[
\phi(\rho, \theta) = \phi_0 (\rho, \theta) + \sum_{k=1}^{N-1} P_k (\lambda_k, \rho_k) \sum_{i=1}^{N_k} \sum_{j=1}^{N_{i}} \phi_{i_k} (l, \rho_j, \theta_j) \quad (2.72)
\]
where $\phi_o$ is given in (2.69), $P_k(r_k, \theta_k)$ is given by (2.71), and $\phi_{1k}$ is given by the relation

$$
\phi_{1k}(r_{ik}, \theta_{ik}) = A_{ik} r^{\alpha_k} e^{i\beta_k \theta_k} + \sum_{n=1}^{\infty} \frac{1}{n} \left( a_{ik} \cos n \theta_{ik} + \frac{i}{n} \sin n \theta_{ik} \right) + \sum_{n=2}^{\infty} \frac{1}{n} \left( b_{ik} \cos n \theta_{ik} + \frac{i}{n} \sin n \theta_{ik} \right),
$$

where $(r_{ik}, \theta_{ik})$ are the coordinates of a point of the region measured from the $i$-th pole of the $k$-th hole. It may be noted that the set of poles on which the functions $P_k$ are constructed, may or may not be chosen from the set for $\phi_{1k}$.

No definitive answer has been found as yet to the question of choosing the locations of the poles represented by the functions $\phi_{1k}$. Since this problem is intimately bound up with some of the questions involved in the point matching approach, we will defer its discussion until Chapter 3. However, it may be mentioned that the computer program was written to permit a completely arbitrary choice of both the number and location of the poles to be located in any hole of the problem. Before turning to a discussion of the point matching approach, we will derive the special forms of the stress function applicable to symmetric problems.
The symmetric stress functions for multiply connected regions

In the case where the problem is symmetric about one or more lines it is extremely desirable to construct, a priori, a stress function with the same symmetry, rather than using the general stress function and forcing it into a symmetric form in satisfying the boundary conditions of the problem. If one starts out with an appropriately symmetric stress function and satisfies the boundary conditions in one symmetry element, the boundary conditions are automatically satisfied on the rest of the boundary.

We will consider two types of symmetry in this dissertation. (We will always assume that the x-axis is a symmetry line unless the problem has no symmetry.) The first type is called rotational or cylindrical symmetry and is exemplified by a set of equally sized holes whose centers lie at equally spaced points on the circumference of another circle (Fig. 13). We will use the term "n-fold symmetry" to denote the case for which there are n-symmetry lines. Thus, Fig. 13 shows a case with 6-fold symmetry. The basic symmetry element is taken in this case to be the 30° wedge XOA.

The second type of symmetry, called translational symmetry, is illustrated by the infinite plate with an infinite row of holes (Fig. 14). In this case, the basic symmetry elements are
Fig. 13. -- A group of holes with cylindrical symmetry

Fig. 14. -- A row of holes with translational symmetry
strips formed by lines normal to the y-axis and passing through the centers of the holes and through points of the y-axis midway between the holes. The y-axis may also be an axis of symmetry (as it is here). However, we will not consider the case of the doubly infinite, doubly periodic array of holes.

The derivation of the form of the symmetric stress function is generally attributed to Howland (6), although Asaba (7) solved the infinite plate with an infinite row of holes in 1928 (seven years before Howland).

We will consider the case of cylindrical symmetry first. We note that a function having M-fold symmetry must satisfy two criteria:

(a) it must be periodic with period $\frac{2\pi}{M}$, and

(b) it must be even with respect to $\Theta$ (since the x-axis is a symmetry line).

We will first construct a stress function which has rotational periodicity of $\frac{2\pi}{M}$. We start with the equally spaced set of points on the circle $z = r_0 e^{i\phi}$ (Fig. 15)

$$Z_j = r_0 e^{i\phi} \left[ e^{i(j/r_j \beta)} \right], \quad j = 0, 1, \ldots, M - 1, \quad (2.74)$$

where $\alpha = 2\pi/M$. [We note that it is usual to consider the case for which $z_0$ lies on the x-axis and $\beta = 0$. However, we shall have need of the more general case represented by (2.74).]
Following Radkowski (9), we consider the function
\[
\omega_c = - \log \left( \bar{z}^M - \bar{z}_o^M \right) = - \sum_{j=0}^{M-1} \log \left( \bar{z} - \bar{z}_j \right),
\tag{2.75}
\]
where \(z_j\) is given by (2.74). This function is evidently harmonic and has poles at each \(z_j\). This function is not periodic under the rotation \(\alpha = 2\pi/M\), although its real part
\[
\Re \left[ \omega_c \right] = - \sum_{j=0}^{M-1} \log |z - \bar{z}_j|
\]
is periodic.
By considering the derivatives \( w_n = \frac{r^n}{(n-1)!} \left[ \frac{\partial w_0}{\partial r} \right]_0 \), other stress functions are obtained which are periodic under the rotation \( \frac{d \phi}{M} \). Thus,

\[
\omega_1 = \rho_0 \sum_{j=0}^{M-1} \left[ z - \rho_0 e^{i(\delta x + \beta)} \right]^{-1} e^{i(\delta x + \beta)} = \sum_{j=0}^{M-1} \frac{z_j}{(z - z_j)}
\]

\[
\omega_2 = \rho_0^2 \frac{\partial \omega_1}{\partial \rho_0} = \sum_{j=0}^{M-1} \left( \frac{z_j}{z - z_j} \right)^2.
\]

Similarly

\[
\omega_n = \sum_{j=0}^{M-1} \left( \frac{z_j}{z - z_j} \right)^n.
\] (2.76)

Now \((z - z_j) = r_je^{i\theta_j}\), where \((r_j, \theta_j)\) are the polar coordinates of the point \(z\), measured as before from the point \(z_j\).

Thus

\[
\omega_n = \sum_{j=0}^{M-1} \rho_0^n e^{i\theta_j} \left[ i n (i \alpha + \beta) \right] = \sum_{j=0}^{M-1} \left( \frac{\rho_0}{\rho_0} \right)^n e^{i\theta_j} \left[ i n (i \alpha + \beta - \theta_j) \right].
\] (2.77)

The stress-function expansion in complex form is then given as

\[
\phi = a_0 \omega_0 + a_1 \omega_1 + \sum_{n=2}^{\infty} \left( a_n' + b_n' \lambda^2 \right) \omega_n.
\] (2.78)
where \( a_0 \) is an arbitrary real constant and \( a_n, a_n', b_n, \) are arbitrary complex constants. We will rearrange this expression.

We write

\[
\hat{z} = \hat{z}_j + (\hat{z} - \hat{z}_j) = \hat{\kappa}_0 \exp \left( i(\alpha + \beta) \right) + \hat{\kappa}_j \exp \left( i \theta \right).
\]

Then

\[
\hat{\eta} = \hat{z} \hat{\bar{z}} = \hat{\kappa}_0^2 + \hat{\kappa}_j^2 + \hat{\kappa}_j \left\{ \exp \left[ i \left( \theta_j - \gamma \right) \sqrt{1 + \alpha^2} \right] \exp \left[ -i \left( \theta_j - \alpha \right) \sqrt{1 + \alpha^2} \right] \right\}.
\]

Thus

\[
\hat{n}^2 \omega_{N} = \sum_{j=0}^{M-1} \hat{n}^2 \left( \frac{\hat{\kappa}_j^2}{\hat{\kappa}_0} \right)^j \exp \left[ i \eta \left( \alpha + \beta - \theta_j \right) \right]
\]

\[
= \sum_{j=0}^{M-1} \left( \frac{\hat{\kappa}_j^2}{\hat{\kappa}_0} \right)^j \left( \hat{\kappa}_0^2 + \hat{\kappa}_j^2 \right) \exp \left[ i \eta \left( \alpha + \beta - \theta_j \right) \right]
\]

\[
+ \sum_{j=0}^{M-1} \left( \frac{\hat{\kappa}_j^2}{\hat{\kappa}_0} \right)^j \hat{\kappa}_0 \hat{\kappa}_j \exp \left[ i \left( \eta - 1 \right) \left( \alpha + \beta - \theta_j \right) \right]
\]

\[
+ \sum_{j=0}^{M-1} \left( \frac{\hat{\kappa}_j^2}{\hat{\kappa}_0} \right)^j \hat{\kappa}_0 \eta \hat{\kappa}_j \exp \left[ i \left( \eta + 1 \right) \left( \alpha + \beta - \theta_j \right) \right]
\]

\[
= \hat{\kappa}_0^2 \sum_{j=0}^{M-1} \left[ \left( \frac{\hat{\kappa}_j^2}{\hat{\kappa}_0} \right)^j + \left( \frac{\hat{\kappa}_j^2}{\hat{\kappa}_0} \right)^j \right] \exp \left[ i \eta \left( \alpha + \beta - \theta_j \right) \right]
\]

\[
+ \hat{\kappa}_0^2 \sum_{j=0}^{M-1} \left( \frac{\hat{\kappa}_j^2}{\hat{\kappa}_0} \right)^j \exp \left[ i \left( \eta - 1 \right) \left( \alpha + \beta - \theta_j \right) \right] + \exp \left[ i \left( \eta + 1 \right) \left( \alpha + \beta - \theta_j \right) \right],
\]

or

\[
\hat{n}^2 \omega_{N} = \hat{\kappa}_0^2 \left( \omega_{N}^2 + \omega_{N}^{*2} + \omega_{N-1}^2 + \omega_{N-1}^{*2} \right),
\]

(2.79)
where
\[
\mu_{m, n}^* = \sum_{j=0}^{M-1} \left( \frac{n_j}{\kappa_0} \right)^{-m+2} e^{j\beta} \left[ i n (j \alpha + \beta - \psi_j) \right] .
\] (2.80)

Substituting (2.79) in (2.78) gives
\[
\Phi = a_0 \mu_{0} + a_1 \mu_{1} + \sum_{n=1}^{\infty} \left( a_n \mu_{n} + b_n \mu_{n}^* \right)
\]
where \( w_0, w_n, \) and \( w^*_n \) are given by (2.74), (2.77), and (2.80) respectively, \( a_0 \) is a real constant, \( a_n \) and \( b_n \) are arbitrary complex constants.

Taking \( \Phi(\psi) \), we have finally
\[
\Phi = a_0 \mu_{0} \mu_{0} \mu_{0} + \sum_{n=1}^{\infty} \left[ a_n \sum_{j=0}^{M-1} \left( \frac{n_j}{\kappa_0} \right)^{-n} \cos n (\theta_j - \beta - j\alpha) \right]
\]
\[
+ \sum_{n=1}^{\infty} b_n \sum_{j=0}^{M-1} \left( \frac{n_j}{\kappa_0} \right)^{-n} \sin n (\theta_j - \beta - j\alpha)
\]
\[
+ \sum_{n=1}^{\infty} d_n \sum_{j=0}^{M-1} \left( \frac{n_j}{\kappa_0} \right)^{-n+2} \sin n (\theta_j - \beta - j\alpha)
\] (2.81)

The notation for the real constants \( a_n \) etc.) was taken to conform with the notation used in (2.67). It is noted that this stress function is the periodic form of \( \phi^* \), given in (2.67).

We now wish to obtain a symmetric function from (2.81).

At this point, we need to distinguish two cases. In the first case, \( z_0 \) lies on the x-axis and \( \beta = 0 \). To obtain an even function in \( \Phi \) for this case, we need only set \( c_n = d_n = 0 \) for all \( n \).
In the second case, the points \( z_k \) do not lie on symmetry lines. In this case, the points \( z_k \), considered above, represent only one-half of the set of symmetric poles. These are represented by circles in Fig. 16.

It is seen that the set of points \( z_0 \) through \( z_5 \) are, in fact, a complete periodic set. However, the complete set of symmetric points must include the points marked with asterisks. As indicated in Fig. 16, the second set of points may be obtained from the first by a reflection in the \( x \)-axis. Thus

\[
\overline{z_k^*} = \overline{z_k} = r_0 \exp \left[ -i (\beta + i \alpha) \right].
\]

---

Fig. 16.--A set of points with cylindrical symmetry
By analogy with (2.81), the periodic stress function \( \phi^* \), constructed on the points \( z^*_k \), is

\[
\phi^* = a_c^* \sum_{j=0}^{M-1} \varepsilon_j^* \xi_j^* \tag{2.82}
\]

\[
+ \sum_{n=1}^{\infty} \sum_{j=0}^{M-1} \left( \frac{\alpha_j^*}{\pi c} \right)^{-n} \left\{ a_{-n}^* \cos n(\xi_j^* + \beta r j\pi) + \xi_{-n}^* \sin n(\xi_j^* + \beta r j\pi) \right\}
\]

\[
+ \sum_{n=1}^{\infty} \sum_{j=0}^{M-1} \left( \frac{\alpha_j^*}{\pi c} \right)^{-n+1} \left\{ b_{n}^* \cos n(\xi_j^* + \beta r j\pi) + \xi_{n}^* \sin n(\xi_j^* + \beta r j\pi) \right\}
\]

where we have used the notation \( z - z_j^* = r_j^* e^{i\theta_j^*} \). (\( a_n^* \), etc. are real.)

We must combine (2.81) and (2.82) in such a way as to obtain a function that is symmetric with respect to the x-axis. If the combined stress function is designated as \( \phi^M \) we must have

\[
\phi^M(z) = \phi^M(\overline{z})
\]

Now,

\[
\overline{z} - z_j^* = (\overline{z} - z_j^*) = r_j^* \xi_j^* e^{i\phi} (-i \xi_j^*) ,
\]

\[
\overline{z} - z_j^* = (\overline{z} - z_j) = r_j \xi_j e^{i\phi} (-i \xi_j ) .
\]

If we substitute \( r_j^* \) and \( -\xi_j^* \) for \( r_j \) and \( \xi_j \) in (2.81), and substitute \( r_j \) and \( -\xi_j \) for \( r_j^* \) and \( \xi_j^* \) in (2.82), we find that

\[
\phi(\overline{z}) = \phi^*(z) ,
\]
and
\[ \phi^*(z) = \phi(z), \]
if \( a_{-n}^* = a_{-n}, \ b_{-n}^* = b_{-n}, \ c_{-n}^* = c_{-n} \), and \( d_{-n}^* = d_{-n} \). With these conditions, if we write \( \phi^M = \phi + \phi^* \) we have \( \phi^M(z) = \phi^M(\bar{z}) \).

We thus obtain the symmetric stress function
\[
\phi^M = a_o \sum_{j=0}^{M-1} \left( \log \tau_j + \log \tau_j^* \right) \tag{2.83}
\]
\[
+ \sum_{n=1}^{\infty} \sum_{j=0}^{M-1} \left[ \left( \frac{\eta_j^2}{\eta_0^2} \right)^{-1} \cos \eta_j (\theta_j - \beta_j \alpha) + \left( \frac{\eta_j^*}{\eta_0^*} \right)^{-1} \cos \eta_j^* (\theta_j^* + \beta_j^* \alpha) \right]
\]
\[
+ \sum_{n=1}^{\infty} \sum_{j=0}^{M-1} \left[ \left( \frac{\eta_j^2}{\eta_0^2} \right)^{-2} \sin \eta_j (\theta_j - \beta_j \alpha) - \left( \frac{\eta_j^*}{\eta_0^*} \right)^{-2} \sin \eta_j^* (\theta_j^* + \beta_j^* \alpha) \right]
\]
\[
+ \sum_{n=1}^{\infty} \sum_{j=0}^{M-1} \left[ \left( \frac{\eta_j^2}{\eta_0^2} \right)^{-2} \cos \eta_j^* (\theta_j^* + \beta_j^* \alpha) + \left( \frac{\eta_j}{\eta_0} \right)^{-2} \cos \eta_j (\theta_j - \beta_j \alpha) \right]
\]
\[
+ \sum_{n=1}^{\infty} \sum_{j=0}^{M-1} \left[ \left( \frac{\eta_j^2}{\eta_0^2} \right)^{-2} \sin \eta_j^* (\theta_j^* + \beta_j^* \alpha) - \left( \frac{\eta_j}{\eta_0} \right)^{-2} \sin \eta_j (\theta_j - \beta_j \alpha) \right].
\]

This is the symmetric analogue of \( \phi^* \) given in (2.67).

We wish now to derive the analogous form of the functions \( P \) given in (2.65). We will obtain these functions in a slightly different manner than that used to obtain (2.83).
We will start with the set of points $z_j$ given by \((2.74)\) and shown in Fig. 15. We will set up the polar coordinates of an arbitrary point $z$ measured from the points $z_j$, as shown in Fig. 17.

Fig. 17.--The angular functions $\psi_j$ for cylindrical symmetry
We will consider the set of angles $\psi_j$, shown in Fig. 17. $\psi_j$ is related to $\theta_j$ by

$$\psi_j = \theta_j - \beta - j\pi.$$  \hspace{1cm} (2.84)

We now consider the sum $S = \sum_{j=0}^{M-1} \psi_j$, and investigate its behavior as we proceed from the point $z$ to the point $z'$, where $z' = ze^{1\pi}$ (Fig. 16). We will distinguish two cases. In the first case, we take a path from $z$ to $z'$ which passes around the outside of $z_\perp$. Then for this case, we have

$$\psi_i' = \psi_o + 2\pi, \quad \psi_o' = \psi_5', \quad \psi_j' = \psi_{j-1}', \quad j = 2, 3, 4, 5',$$

so that $S' = S + 2\pi$. In the second case, we proceed from $z$ to $z'$ along a path which passes between the origin and $z_\perp$. In this case,

$$\psi_i' = \psi_o, \quad \psi_o' = \psi_5', \quad \psi_j' = \psi_{j-1}', \quad j = 2, 3, 4, 5',$$

so that $S' = S$. The difference between these two paths is equivalent to a complete circuit around $z_\perp$. In general, for a passage from the point $z$ to the point $z' = ze^{1\pi}$, the function $S$ will undergo an increase of $2\pi n$, where $n$ is the number of the points of $z_k$ traversed on the outside (or equivalently $n$ is the number of dotted lines crossed in Fig. 17).
We now consider the function

\[
P = \sum_{j=0}^{M-1} \left( c_0 \psi_j + d_c r_j^\ell \psi_j + b'_c r_j^\ell \log r_j \right)
\]

(2.86)

\[
+ \sum_{j=0}^{M-1} r_j \psi_j \left( a_j \sin \psi_j + c_j \cos \psi_j \right)
\]

\[
+ \sum_{j=0}^{M-1} r_j^\ell \delta_{j} \left( b_{\gamma_j} \cos \psi_j + d_{\gamma_j} \sin \psi_j \right)
\]

In view of the above remarks concerning \( \psi_j \), it is apparent that \( P \) is a periodic function for a region not surrounding the points \( z_j \). In the case for which the points \( z_k \) are located in holes of the region, the function \( P \) will give the periodic stress distribution \( (d_0 = 0) \) for the ring of holes arising from a periodic set of dislocations and stress resultants.

In this case, the coefficients of the stress function are related to the stress resultants and dislocations on the \( j \)-th hole in terms of the rotated coordinate system \( (x'_j, y'_j) \), where \( x'_j = r_j \cos (\psi_j), y'_j = r_j \sin (\psi_j) \). That is,

\[
d_c = c,
\]

(2.87)

\[
a_j = - \frac{F_{x'_j}}{2\pi} = \frac{i}{2\pi} \left[ F_{x'_j} \cos (\beta j x) + F_{y'_j} \sin (\beta j x) \right],
\]

\[
c_j = - \frac{F_{y'_j}}{2\pi} = \frac{i}{2\pi} \left[ -F_{x'_j} \sin (\beta j x) + F_{y'_j} \cos (\beta j x) \right],
\]

\[
\ell_c = - \frac{M}{2\pi},
\]

\[
b'_c = - \ell_c \omega / \sqrt{\pi (1-\nu)}
\]
As a somewhat more graphic illustration of what these equations mean, it is noted that a non-zero positive value of $a_1$ would represent a force resultant on each hole directed along the dotted lines of Fig. 17 and inward toward the origin. Similarly, $\delta u'$ is a dislocation measured in the direction of the dotted lines.

We wish now to set up the quasi-symmetric form of $P$. We start with the set of coordinates $(r_j^*, \phi_j^*)$ measured from the point set $\tilde{z}_j$ (Fig. 18), where $\phi_j^* = - (\theta_j^* + \beta + j\alpha)$. Forming the analogue of (2.86) for the $\tilde{z}_j$ point set and adding it to (2.86), we obtain

\begin{equation}
P^M = \sum_{j=0}^{M-1} \left[ c_0 (\psi_j^* + \psi_j^*) + d_0 (\eta_j^2 \psi_j^* + \eta_j^{*2} \psi_j^*) \right] + \sum_{j=0}^{M-1} b_j' (\eta_j^{*2} \log \eta_j^* + \eta_j^{*2} \log \eta_j^*) + \sum_{j=0}^{M-1} a_j (\eta_j^* \psi_j^* \sin \psi_j^* + \eta_j^{*2} \psi_j^* \sin \psi_j^* ) + \cdots
\end{equation}
\[ + \sum_{d=0}^{M-1} a_d (r_d \psi_d \cos \psi_d + r_d^* \psi_d^* \cos \psi_d^*) \]

\[ + \sum_{d=0}^{M-1} b_d (r_d \log r_d \cos \psi_d + r_d^* \log r_d^* \cos \psi_d^*) \]

\[ + \sum_{d=0}^{M-1} c_d (r_d \log r_d \sin \psi_d + r_d^* \log r_d^* \sin \psi_d^*) \]

where \( \psi_j = \theta_j - \beta - j \alpha \) and \( \psi_j^* = - (\theta_j^* + \beta + j \alpha) \).

Fig. 18.--The functions \( \psi^* \) for cylindrical symmetry
This is the required quasi-symmetric form of $P$. It is noted again that $P^M$ is multivalued if the point sets $z_j$ and $\bar{z}_j$, $j = 0, 1, \cdots, M-1$, are in holes of the region. However, the stresses and displacements obtained from (2.88) will be single valued and continuous in any region not surrounding these points. One result of this is that $P^M$ may be used to set up a system of external forces. For instance, the terms involving $a_1$ and $c_1$ will be used (in Chapter 3) to set up the problems with round plates having a ring of holes and loaded on the edge by concentrated forces.

If the points $z_j$ are in holes of the region, then the relations between the constants and the dislocations and stress resultants for each hole are given in the rotated coordinate systems of each hole in the manner of (2.87).

It may be noted that the symmetric form of $\Phi_k$, given in (2.84), could have been derived from the angle functions $\psi_j'$ and $\psi_j^*$ in the same manner as that used in deriving (2.88).

If there are more than one set of symmetric holes, stress functions of the type (2.84) and (2.88) are written for each hole and combined with the symmetric form of $\Phi_0$ from (2.69) to give the complete stress function. Finally, the remarks made earlier, concerning the generalization of the stress function to the case of non-circular holes, holds here also. Thus, the stress
function, for the case of \( N \) rings of \( M \) holes arranged in a pattern

with \( M \)-fold symmetry, is written as follows:

\[
\Phi(\eta, \psi) = \Phi^M_0(\eta, \psi) \pm \sum_{k=1}^{N_M} \sum_{j=1}^{N_k} \Phi^M_{kj},
\]

(2.89)

where

\[
\Phi^M_0(\eta, \psi) = \left( \frac{\eta}{\epsilon_0} \right)^M \left( a_{c}^{M} \cos \psi + e_{c}^{M} \sin \theta \psi \right)
\]

(2.90)

\[
+ \sum_{n=1}^{\infty} \left( \frac{t}{\epsilon_n} \right)^M \left( b_{c}^{M} \cos \psi + d_{c}^{M} \sin \theta \psi \right),
\]

(2.91)

\[
P^M_{kj} = \sum_{j=0}^{M-1} \left[ c_{c}^{M} (\psi_{k, j} + \psi_{k, j}^{*}) + d_{c}^{M} (\psi_{k, j} + \psi_{k, j}^{*}) \right]
\]

(2.92)
and
\[
\Phi_{ik}^M = a_{ik} \sum_{j=0}^{M-1} \left( k \cos \lambda \eta_{ik}^j \eta_{ik}^j + k \sin \lambda \eta_{ik}^j \eta_{ik}^j \right) \tag{2.92}
\]
\[
\Gamma \sum_{n=1}^{\infty} \sum_{j=0}^{M-1} \left[ \left( \frac{\eta_{ik}^j}{\lambda_{nk}} \right)^{\lambda_{nn}^j} \cos \lambda (\theta_{ik}^j, \eta_{ik}^j - j\eta) + \left( \frac{\lambda_{kk}^j}{\lambda_{kk}^j} \right)^{\lambda_{nn}^j} \cos \lambda (\theta_{ik}^j, \eta_{ik}^j + j\eta) \right]
\]
\[
\Gamma \sum_{n=1}^{\infty} \sum_{j=0}^{M-1} \left[ \left( \frac{\eta_{ik}^j}{\lambda_{nk}} \right)^{\lambda_{nn}^j} \sin \lambda (\theta_{ik}^j, \eta_{ik}^j - j\eta) - \left( \frac{\lambda_{kk}^j}{\lambda_{kk}^j} \right)^{\lambda_{nn}^j} \sin \lambda (\theta_{ik}^j, \eta_{ik}^j + j\eta) \right]
\]
\[
\Gamma \sum_{n=1}^{\infty} \sum_{j=0}^{M-1} \left[ \left( \frac{\eta_{ik}^j}{\lambda_{nk}} \right)^{\lambda_{nn}^j} \cos \lambda (\theta_{ik}^j, \eta_{ik}^j - j\eta) + \left( \frac{\lambda_{kk}^j}{\lambda_{kk}^j} \right)^{\lambda_{nn}^j} \cos \lambda (\theta_{ik}^j, \eta_{ik}^j + j\eta) \right]
\]
\[
\Gamma \sum_{n=1}^{\infty} \sum_{j=0}^{M-1} \left[ \left( \frac{\eta_{ik}^j}{\lambda_{nk}} \right)^{\lambda_{nn}^j} \sin \lambda (\theta_{ik}^j, \eta_{ik}^j - j\eta) - \left( \frac{\lambda_{kk}^j}{\lambda_{kk}^j} \right)^{\lambda_{nn}^j} \sin \lambda (\theta_{ik}^j, \eta_{ik}^j + j\eta) \right].
\]

The functions \( \Phi_{ik}^M \) are based on a set of points \( z_{ik} \), one to a hole. The functions \( \Phi_{ik}^M \) are based on the set of points \( z_{ik} \). The function \( \Phi_{ik}^M \) is expanded about the origin, which is taken at the axis of symmetry. In order to obtain \( M \)-fold symmetry in \( \Phi_{ik}^M \), it is necessary to use only trigonometric terms involving multiples of \( M \). If there is a hole surrounding the origin and it is necessary to set up a pole at the origin using a function of the type \( \Phi_{ik}^M \) given in (2.67), it is also necessary to take only terms involving multiples of \( M \) in this expansion.
The symmetric expression (2.88) or (2.91) of \( P \) is apparently new. The symmetric form of the function \( \varphi_k \) for poles not on the symmetry axes is also apparently new. Most problems with cylindrical symmetry that have been solved here-to-fore involved only circular holes for which the poles were on the symmetry axes.

We will next turn to the consideration of the translational symmetry as illustrated by Fig. 14. Howland (6) derived a stress function for this case by considering the function

\[ \mu_c = \log \sin \left( \frac{\pi a}{\pi} \right) \]

and its derivatives

\[ \mu_{c_k} = \frac{a^k}{(n-1)!} \left( \frac{d^n \psi}{d z^n} \right) \].

However, we will develop a set of \( \psi_j \) angle functions in terms of which the stress functions can be written directly. We will assume that the problem has translational periodicity with period \( 2a \). We will consider the set of points \( z_j \), where

\[ z_j = \ell c + \overline{z} j \cdot \alpha, \quad j = -\infty, \ldots, -1, 0, 1, \ldots, \infty \]

as shown in Fig. 14. Since the period elements are congruent under translation without rotation, we select the angle functions as

\[ \psi_j = C_j. \]

We then consider the sum

\[ \sum_{j=-\infty}^{\infty} \psi_j. \]
This function is quasi-periodic as follows: We consider the two points \( z \) and \( z' \) shown in Fig. 19, where \( z = z + 2 \alpha i \). If we follow a path from \( z \) to \( z' \) which passes to the right of \( z_0 \), we have

\[
\psi_j(z') = \psi_{j-1}(z), \quad j = -\infty, \ldots, -2, -1, 1, 2, \ldots, \infty,
\]
and
\[ \psi_o(z') = \psi_{-1}(z) + 2\pi, \]
so that
\[ \sum_{j=-\infty}^{\infty} \psi_j(z') = 2\pi + \sum_{j=-\infty}^{\infty} \psi_j(z). \]

If we take a path which passes to the left of \( z_o \) on going from \( z \) to \( z' \), we have \( \psi_o(z') = \psi_{-1}(z) \) and
\[ \sum_{j=-\infty}^{\infty} \psi_j(z') = \sum_{j=-\infty}^{\infty} \psi_{j+1}(z) + \sum_{j=-\infty}^{\infty} \psi_j(z). \]

The conjugate functions \( \psi_j^* \) measured from the points \( \bar{z}_j \) are given by
\[ \psi_j^* = -a_j^* . \tag{2.93} \]

Now a comparison of (2.65) and (2.67) with (2.88) and (2.83) respectively, shows that the latter two equations are formally obtained by substituting \( \psi_j \) and \( \psi_j^* \) for cylindrical symmetry into (2.65) and (2.67), summing over j, and adding the sum for \( \psi_j \) to the sum for \( \psi_j^* \) in each case. Repeating this formal operation for the case now being considered, we obtain for \( P \) and \( \varphi \)
\[ P = \sum_{j=-\infty}^{\infty} \left[ L_e \left( \psi_j + \psi_j^* \right) \right] \]
\[ + \sum_{j=-\infty}^{\infty} \left[ h_e \left( \alpha_j^* \cos \theta_j + \alpha_j \cos \theta_j^* \right) \right] \]
\[ + \sum_{j=-\infty}^{\infty} \left[ a_j \left( \alpha_j \cos \theta_j \sin \theta_j + \alpha_j^* \cos \theta_j \sin \theta_j^* \right) + \right. \]
\[
+ \sum_{j=1}^{\infty} \left( \eta_j \theta_j \cos \theta_j - \eta_j^* \theta_j^* \cos \theta_j^* \right) \\
+ \sum_{j=1}^{\infty} \left( \eta_j \log \eta_j \cos \theta_j + \eta_j^* \log \eta_j^* \cos \theta_j^* \right) \\
+ \sum_{j=1}^{\infty} \left( \eta_j \log \eta_j \sin \theta_j - \eta_j^* \log \eta_j^* \cos \theta_j^* \right) 
\]

and

\[
\phi = \sum_{j=1}^{\infty} a_j \left[ \log \left( \frac{\eta_j}{\eta_0} \right) + \log \left( \frac{\eta_j^*}{\eta_0} \right) \right] \\
+ \sum_{n=2}^{\infty} a_n \sum_{j=1}^{\infty} \left[ \left( \frac{\eta_j}{\eta_0} \right)^{-n} \cos n \theta_j + \left( \frac{\eta_j^*}{\eta_0} \right)^{-n} \cos n \theta_j^* \right] \\
+ \sum_{n=2}^{\infty} b_n \sum_{j=1}^{\infty} \left[ \left( \frac{\eta_j}{\eta_0} \right)^{-n+2} \sin n \theta_j - \left( \frac{\eta_j^*}{\eta_0} \right)^{-n+2} \cos n \theta_j^* \right] \\
+ \sum_{n=2}^{\infty} c_n \sum_{j=1}^{\infty} \left[ \left( \frac{\eta_j}{\eta_0} \right)^{-n+2} \cos n \theta_j + \left( \frac{\eta_j^*}{\eta_0} \right)^{-n+2} \cos n \theta_j^* \right] \\
+ \sum_{n=2}^{\infty} \sum_{j=1}^{\infty} \left[ \left( \frac{\eta_j}{\eta_0} \right)^{-n+2} \sin n \theta_j - \left( \frac{\eta_j^*}{\eta_0} \right)^{-n+2} \sin n \theta_j^* \right] 
\]

The constant \( r_0 \), introduced here, is some typical length such as the width of the period strip.
We should, of course, verify that $P$ and $\phi$, given by (2.94) and (2.95), have the appropriate properties. It is not hard to show that $\phi$ is even about the $x$-axis and periodic in $y$ of period $2a$. The function $P$ is periodic and gives periodic stresses and displacements if the region lies on the left side of the $y$-axis (or at least does not close around the poles on the right). However, if the region is such that it is possible to pass from one period strip to the next to the right of the poles, $P$ will give multivalued stresses and displacements. In this case, $d_0$ must be set to zero and the remaining terms of $P$ determined by the values of the stress resultants and dislocations.

The stress function for $N$ rows of noncircular holes, arranged in an array having translational symmetry is similar to (2.88). We have

$$\phi(\zeta, \theta) = \phi_0(\zeta, \theta) + \sum_{k=1}^{N} p_k + \sum_{k=1}^{N} \sum_{i=1}^{N_k} \phi_{ik}$$

(2.96)

where

$$\phi_0(\zeta, \theta) = b_c \zeta^2 + b_{1c} \zeta^3 \cos \alpha + d_{1c} \zeta^2 \sin \theta + a_{2c} \zeta \cos 2\theta + \alpha_{2c} \sin 2\theta$$

(2.97)

$P_k$ and $\phi_{ik}$ are obtained from $P_k^M$ and $\phi_{ik}^M$, given in (2.90) and (2.91), by setting $\alpha = 0$ in these equations and taking each summation on $j$ from $j = -\infty$ to $+\infty$. 
We note that the terms of $\phi_o$ in this case are determined from the conditions at infinity and are the only terms that give valid stresses at infinity. The same is true for $\phi^M_o$ for the case of an infinite region with cylindrical symmetry.

We will not consider the case of doubly periodic arrays of holes. However, it will be possible, using the stress functions derived above, to solve problems with several rows of holes. An approximation to the case of a doubly periodic array might be obtained in this way.

This completes the derivation of the various forms of the stress function. We will now turn to the problem of satisfying the boundary conditions.


We will consider, in this chapter, a technique for the numerical solution of elasticity problems using the stress functions developed in Chapter 2. Briefly, this requires that for a given problem, the coefficients of the stress function be calculated so that the boundary conditions of the problem are met everywhere on the boundary of the region.

It is well known that the satisfaction of the boundary conditions is usually a very difficult part of solving any boundary value problem. It is for this reason that only a very few numerical solutions are known, even for those classes of problems for which both the general solution of the differential equation and the boundary conditions are known.

Prior to the development of high-speed computers in the last decade, the number of boundary value problems for which numerical solutions had been obtained was probably an order of magnitude smaller than the number of problems for which a formal solution was known. Thus, A. E. Green (1) could display the
formal solution to the problem of an infinite plate with an arbitrary number of circular holes of arbitrary size and location. However, the problem for which he obtained a numerical solution was the very simple case of three holes in a line. Even for this case, the computations that Green and his co-workers performed were prodigious. Howland (2) also exhibited the formal solution of many problems, although he solved numerically "only" the problems involving a single row of holes. Again, the numerical calculations performed to get the solutions to even these simple problems were enormous.

We now use computers to perform these more or less routine calculations as a matter of course. However, the initial effort and expense involved in writing and "debugging" a computer program is many times as great as the time and effort involved in processing a problem after the program is debugged. Thus, it is desirable to write a program that may be used to solve a large number of problems to amortise the initial cost of writing the program. In other words, it is desirable to make a computer program as general as it can be made without unduly complicating the problem of writing the program.

The optimum numerical procedures, then, are those which are both of wide applicability and simple to program.

---

1 The term "debug" is commonly used to describe the process of eliminating all of the errors contained in the first draft of the computing program. The debugging process usually involves intensive study of the program coupled with trial computer runs with test problems for which the final answers and intermediate results have been hand calculated.
One of the most general techniques for obtaining approximate solutions to boundary value problems is the finite-difference technique. This approach has been exploited more than any other in obtaining numerical solutions of boundary problems. In fact, it can and has been used to obtain approximate solutions for problems for which the differential equation has no known analytic solution.

However, the finite-difference approach suffers from a number of deficiencies. It is inherently an approximate method. In fact, the only way that the accuracy of the finite-difference approach can be increased is by taking smaller mesh sizes and increasing the number of difference equations. It is not uncommon in using this method, therefore, to be solving (or attempting to solve) simultaneous equation systems having of the order of 500 to 1000 equations. This, in itself, is a difficult problem, even on large machines such as the IBM 7090.

The point matching approach, to be discussed in the next article, has most of the advantages and practically none of the disadvantages of the finite-difference approach. Its only disadvantage in comparison with the finite-difference approach is that it may be used only to solve problems for which the differential equation has a known solution in series.
The point matching approach

Briefly, the point matching approach consists of using a truncated series solution of the differential equation to satisfy the boundary conditions at a selected finite set of boundary points of the region. It is hoped that the solution thus found will also approximate satisfactorily the boundary conditions at boundary points between those of the selected set. Depending in some measure on how well the boundary conditions are met at the intermediate boundary points, the solution found by this technique will be a more or less accurate solution of the boundary value problem.

This approach seems to be straightforward and, in a very real sense, a natural way to obtain the approximate solution of boundary problems. That is, since in most problems it is beyond our capabilities to obtain a solution that satisfies the boundary conditions everywhere on the boundary, we will satisfy the boundary conditions only at some finite number of points of the boundary.

Because this technique does seem so very natural, it is no surprise that it has occurred to a number of people independently. This also means that it is hopeless to try to assemble any kind of a complete bibliography on the subject, since an author may casually use the approach in getting some approximate solution to a particular boundary problem. The earliest paper known to this author is by
J. C. Slater (3), who treats the approach in just such a casual way. In his paper, Slater obtained an approximate solution to the problem of the calculation of the electronic energy bonds in metals. It is interesting to note that this was a three-dimensional problem, and that the boundary conditions were met at a finite number of points of a bounding (spherical) surface. The earliest paper (known to the author), applying the point matching method to problems in mechanics, is a short note by J. Barta (4), written in 1937. Barta obtained approximate solutions to the transverse deflections of a uniformly loaded clamped square plate and to the torsion of a square prism.

H. D. Conway also has used the technique to obtain approximate solutions to a number of simple problems in plate bending and torsion theory (5,6). Also, Koronev (7) used the technique to solve a number of simple two-dimensional problems. Fend, Baroody, and Bell (8) used the point matching technique in 1950 to obtain an approximate solution to the two-dimensional steady-state temperature distribution in an infinite solid heated by uniform internal heat generation and cooled by a doubly infinite array of cooling holes arranged on a triangular lattice (Fig. 20).

In all of these papers, the authors used only a very few terms of the appropriate series solutions and performed the necessary calculations by hand.
If more accurate solutions are desired to problems such as those mentioned or if it is desired to solve more complicated problems, the computational labor becomes very large and it becomes necessary to perform these calculations by computer.

In order to illustrate this point, let us assume that we pick a stress function having, say, $N$ terms and that we wish to find the unknown coefficients of this stress function in such a way as to satisfy the boundary conditions at a set of boundary points. Then each boundary condition that we write is a linear equation in the $N$ unknown constant coefficients of the stress function. Since it takes $N$ equations to determine the values of
the unknown constants, we have to solve a set of $N$ by $N$ simultaneous equations. If $N$ is large, this is a tedious operation if performed by hand but may be easily done on a computer.

In attempting to use this technique for solving fairly complicated problems, a difficulty arises that might be best explained in the light of the author's own experience.

At the beginning of the research program for this dissertation, the boundary point set was chosen so that the number of boundary conditions was equal to the number of unknown constants in the stress function. The resulting set of simultaneous equations was solved to obtain a stress function that satisfied the boundary conditions exactly (to the 8 significant digit accuracy of the computing machine) at each member of the selected set of boundary points. However, it soon appeared that the overall accuracy of the solution depended very critically on the choice of boundary point locations. By changing the locations of the boundary points, the values of the maximum errors in boundary conditions calculated from the stress-function "solution" could be made to vary over many orders of magnitude at boundary points between the points used to match boundary conditions. (The process of varying the position of the boundary points to obtain a good solution will be called tuning.)
The reason for this phenomenon is that the values of the terms of the stress function and its derivatives will vary considerably along the boundary arc. The higher order terms of the series will attain a number of maximum and minimum values along the boundary. The same sort of variation will occur in the contribution of these terms to the boundary conditions of the problem.

When the boundary point set is not chosen at or near the location of all of the maxima and minima of the terms of the boundary conditions, the solution will not satisfy the boundary conditions at boundary points between the chosen boundary point set.

This particular phenomenon did not bother the authors mentioned above, either because they were merely lucky, or because with just a few boundary points and using only low order terms of the series solution, it was easy to avoid this pitfall. However, in complicated problems, it requires a careful study of the problem in order to tune the boundary point set properly.

This same phenomena was observed by workers trying to extend Slater's calculations to obtain more precise results. S. L. Altmann (9) in 1957 proposed a modification of the point matching method that eliminated the need for tuning the boundary point set. Using a Ferranti Mark I computer, he was able to obtain much more precise results for the electronic band problem.
Very briefly, Altmann's extension of the point matching technique consists of writing many more boundary equations than there are unknown constants in the stress function. The resulting set of simultaneous equations in the unknown constants is satisfied in the least squares sense.

Suppose we write \( M \) boundary conditions for a stress function with \( N \) unknown constants where \( M > N \). In matrix notation, this system of equations may be written as

\[
A_{MN} X_N = R_N,
\]

where \( X_N \) is the \( N \)-fold vector of unknown coefficients of the stress function, \( R_M \) is the \( M \)-fold vector of constant right-hand sides of the boundary conditions and \( A_{MN} \) is the computed coefficient matrix of the equation system. \( A_{MN} \) has \( M \) rows and \( N \) columns.

We seek a solution \( X_N^* \) of this equation system that minimizes the sum of the squares of the residuals \( v_M \), where

\[
v_M = A_{MN} X_N^* - R_M.
\]

That is, we seek a solution \( X_N^* \) that minimizes \( v_M^2 \). We write the dot product

\[
v_M^2 = (A_{MN} X_N^* - R_M)^T (A_{MN} X_N^* - R_M).
\]

Then the solution \( X_N^* \) that minimizes \( v_M^2 \) is the solution of the \( N \) equations

\[
\frac{\partial (v_M^2)}{\partial X_N^T} = 0 = A_{NM}^T A_{MN} X_N^* - A_{NM}^T R_M,
\]
Independently of Altmann, Lo (10) used the least squares extension of the point matching technique to obtain an approximate solution for the deflections of a square plate clamped on all four sides. However, Lo was not aware of the matrix form (3.2) of the normal equations. W. E. Clausen (11) used this matrix equation in applying the method to the approximate solution of the end problem of rectangular strips.

The present author used the results of Lo and Clausen in modifying the computing program to incorporate the least squares extension of the point matching technique, and was not aware of Altmann's work until this dissertation was almost completed. (The least squares extension of the point matching technique will be called, for brevity, "the extended point matching approach".)

By applying this modification in the present research, the necessity of tuning the boundary points was completely eliminated. This made it possible to focus attention on choosing an appropriate stress function for a given problem without having to worry about choosing the boundary point set. During the course of the research, checks were made from time to time to insure that the boundary points chosen were sufficiently close together.

\[
A_{NM}^T A_{MN} \hat{x}_N = A_{NM}^T R_M .
\]
This check usually involved doubling the boundary point set (in halving the interval between the points) and comparing results. In most cases, the solutions differed only infinitesimally.

The experience gained in the course of the research program for this dissertation as well as the experience of others amply demonstrates that the extended point matching approach is one of the best tools yet devised for solving boundary value problems for which the series solution is known.

We can cite some of the advantages possessed by the extended point matching approach.

(a) Generality.

We have mentioned before that the extended point matching approach can be used to obtain numerical solutions for almost any type of problem for which the solution of the differential equation is known. This generality is also demonstrated by the various kinds of problems for which the technique has been used.

(b) Simplicity.

The fundamental point matching technique is obviously very simple. The addition of the least squares fitting procedure adds very little complexity to the procedure.

(c) Adaptability to Computer Machine Calculation.

This is one of the outstanding features of the point matching technique. Not only is the technique simple but it works the same way for every problem. Essentially three steps are required: calculate the matrix of boundary point equations; solve this matrix for the values of the unknown constants of the series; calculate the values of the stresses at any desired points of the problem region.
It may be mentioned that, in contrast to the finite difference technique, the matrix equations obtained in using this technique are of low enough order that they may be solved with standard Gaussian elimination subroutines.

In addition to these attributes, the generality of the approach is a significant advantage in writing and using the computing program. In this connection, it may be mentioned that essentially the same computing program was used in solving all of the problems to be discussed in the next chapter. Part of the reason for this was that we deliberately developed all of the stress functions in Chapter 2 in terms of the polar coordinate solutions of the biharmonic equations. A side benefit of being able to use a program to solve many different problems is that extra time may be justifiably spent in writing the program in such a way that the work of making up each individual problem is reduced to a minimum. This was done in writing the computer program for this dissertation. It takes somewhere between a half hour and a half-day to make up and key punch all of the necessary input for running problems with this computing program.

(d) Accuracy.

The question of the accuracy of the solutions will be discussed in connection with the numerical results given in the next chapter. However, it may be noted here that the point matching technique will give the exact solution if there is an exact solution in the coordinate system used as a basis for the program. As an illustration of this, the computer program properly found the exact solution for the torsion of an elliptical bar. However, the solution for a pressurized elliptical hole in an infinite plate has an exact solution in elliptical coordinates but not in the polar coordinates used in the program. Thus, the program could find only an approximate solution in this case.
One interesting example of the accuracy of the technique is presented by Heller et al. (12), who considered the stress concentration near rectangular holes with rounded corners in an infinite plate. The stresses were calculated by the complex variable technique. However, the conformal map of the rectangle onto the circle was found by a point matching technique. These authors found that point matching with a given number of terms gave much better results than taking the same number of terms of the exact infinite series representing the Schwartz-Christoffel transformation of the square onto a circle.

It is felt that this is a more or less general property of the point matching process. That is, suppose we know the exact solution in infinite series of a given problem. Then it is usual to calculate values of the solution by summing some finite number of these terms. Now suppose that arbitrary constants are substituted for the explicit coefficients of the series solution and the values of these constants are determined by point matching. The resulting solution will usually be more accurate than the truncated form of the exact series.

Many of the numerical solutions that we will discuss in Chapter 4 were deliberately chosen to correspond to solutions that have appeared in the literature. Most of the solutions obtained for multiply connected regions have been found by what we will call the "classical approach" for want of a better term.
The classical approach for satisfying boundary conditions

This approach consists of transforming the series of the differential equation onto the (n,s) coordinate system of the boundary. The transformed series then depends on the arc length coordinate s measured along the boundary. When the boundary values are also expressed in terms of s, it is usually possible to obtain equations relating the coefficients of the series to the boundary values.

Suppose, for instance, that the boundary of the region is a circle and the stress function is given by (2.36). Then the boundary values of the stress function on the circle will reduce to Fourier series in the arc coordinate θ. If the values of the boundary loads are expanded in Fourier series in θ, relations can be found between these boundary values and the coefficients of the stress function by equating the coefficients of like trigonometric terms.

For a simply connected region the Kolosov-Muskhelishvili method is actually a technique for transforming the stress function (2.36) into the coordinate system for the noncircular boundary of the problem. The necessary formulas have been developed for relating the coefficients of the transformed stress function to the boundary loads expressed in terms of the arc coordinates.
Since problems involving the circular ring are also solvable by the polar coordinate stress function (2.36), the complex variable technique may also be used to transform (2.36) so that the stress function is expressed explicitly in terms of the arc length coordinate measured on both boundaries of a doubly connected region. If this can be done, it is once again possible to develop formulas relating the constants of the stress function to the boundary forces expressed in terms of the boundary coordinates. It is usually much more difficult to solve problems for doubly connected regions than for simply connected regions.

If one is considering triply or more highly connected regions, there is no region like the ring for which boundaries all coincide with coordinate lines obtained by setting the same coordinate equal to a constant. (In the circular ring, the inner and outer boundaries are obtained by setting \( r = C_i \) and \( r = C_o \), respectively, where \( C_i < C_o \).) In these cases, it is tremendously difficult to obtain the solution even of problems involving circular holes. This may be readily verified by reading the papers by A. E. Green (1) and R.C.J. Howland (2).

In spite of the difficulties involved in obtaining solutions to problems by the classical technique, it has been used to obtain approximate numerical solutions to several problems involving multiply connected regions. (The approximations
involved in the application of the classical approach stem from
the fact that the infinite system of equations that is developed
must be solved by some successive approximation technique.)

The numerical solutions obtained with the classical
technique for a number of problems will be included in Chapter 4
and compared with the solutions that we have obtained by using
the extended point matching approach.
REFERENCES


NUMERICAL RESULTS

We will present in this chapter the numerical solutions obtained by applying the computing program to a variety of stress problems.

These problems are broken down into the following major classes:

1. circular holes arranged with cylindrical symmetry in an infinite plate,
2. circular holes arranged with cylindrical symmetry in a finite circular plate,
3. problems involving noncircular holes,
4. problems involving infinite rows of holes in the infinite plate, and
5. torsion problems.

Many of the problems considered here have been solved by others by the classical methods and a few of the problems have closed form solutions in coordinate systems other than the polar coordinates. In these problems we will compare the results obtained by the present approach with the results obtained by others.
Several of the problems, to our knowledge, have not been solved before. In these problems, we will present a reasonably complete picture of the stresses calculated.

The method of presentation of the results will be essentially the same for each problem discussed. We will give the form of the series solution used for the problem. Each series will be written with numerical upper limits showing the point at which the series is truncated. In addition, we will give the maximum value of the error found in satisfying the boundary conditions by the calculated stress function. It is felt that in most problems this maximum error indicates the magnitude of the errors in the calculated solution.

We will present the calculated values of the normal stresses in the direction tangential to the boundary in the form of tables and graphs. In those cases for which the boundaries are arcs of circles, the stresses are presented at intervals of about $10^\circ$ in the angle subtended at the center of the circle. However, it will be seen that the angular positions at which the stresses are given are not exactly at $10^\circ$ intervals. The reason for this is that the locations of the boundary points were specified by giving their x-coordinate. Of course, it would have been possible to calculate the precise x-coordinates of the boundary points at $10^\circ$ intervals on the boundary. However, it considerably simplified the key punching of the input data to give the x-coordinates of the
boundary points to as few digits as possible. Since the stresses in most problems were obtained with an accuracy of four or more significant digits, we will report the angular locations of the boundary points to the nearest second of arc.

In certain of the problems, the stresses were also calculated at interior points of the region. These stresses are given in tabular form as functions of the x- and y-coordinates of the points.

The numerical computations were performed on the computers of the Numerical Computation Laboratory of The Ohio State University. The early work was performed on an IBM 709. During the course of the research, the 709 was replaced by an IBM 7090, and the remaining problems were processed on the 7090. Unfortunately, a complete record was not kept of the machine running times for problems that were solved. Consequently, only a few actual machine running times will be noted in the discussion of the problems. However, a rule-of-thumb estimate 7090 running times can be obtained for problems with cylindrical symmetry by adding one minute of machine time for each hole boundary and two minutes of machine time for the exterior boundary if the region is finite. The cases involving infinite rows of holes took considerably more machine time. These times will be given when the problems are discussed. The torsion problems required about 1/3 to 1/2 of the computing time necessary for the comparable stress problems, since torsion problems involve harmonic functions.
Circular holes in an infinite plate arranged with cylindrical symmetry

We will discuss in this section the solutions obtained for a number of problems involving circular holes in an infinite plate where the holes are arranged in patterns having cylindrical symmetry. The boundary forces also have the appropriate symmetry. In most of the problems the loading will be taken as uniform pressure in the holes. However, for the cases of three holes in a line we will take the loading to be uniaxial tension at infinity to correspond to the problem solved by A. E. Green (1).

Three circular holes in a line. We considered three problems of this type. In each of the problems the loading was taken to be uniaxial tension at infinity in a direction normal to the line of centers of the hole (Fig. 21). If the two outside holes are equal, this problem has symmetry about both the $x$- and $y$-axes (i.e., it has two-fold cylindrical symmetry).

Green solved the problem for which the three circles had equal radii and the spacing was taken to be such that $b/a = 0.15$. In order to check with his results we solved a problem with the same spacing. If $b$ is taken to be one, $a$, in this case, is $6-2/3$. In his article, Green compared his results with some photoelastic measurements made by Capper (2) on a plate with three holes and a spacing $b/a = 1/6$. In order to determine whether the difference
in spacing made a significant difference in the stresses, we also solved the problem with this spacing. It turns out that the difference in the stresses for these two cases is minor as will be seen by a comparison of Tables 1 and 2 given later.

![Diagram](image)

Fig. 21.—Three holes in uniaxial tension

Finally, we solved the problem for which b/a = 1/3. This represents the case for which the minimum distance between the
hole boundaries was equal to the radius. This problem was solved mainly as a check on the effect of the hole spacing on the convergence of the stress-function series. In his paper, Green mentions that he could prove convergence of the various series that he used only if \( b/a \) was less than .255. Although .33 does not seem very much larger than .255, it represents a case for which the web between the holes is reduced by about half.

In all three problems, the stress function used was

\[
\phi = \frac{i}{2} \mu \left( r^2 (1 + \cos 2\theta) + a_0 \log r + a'_0 \left( \log r_1 + \log r_2 \right) \right) + \sum_{n=1}^{10} \left( a_{-2n} \left( r^{-2n} \cos 2n\theta + b_{-2n} r^{-2n+2} \cos 2n\theta \right) + a'_{-2n} \left( r^{-2n} \cos 2n\theta + r^{-2n+2} \cos 2n\theta \right) \right)
\]

where \((r, \theta)\) is measured from the origin \((r_1, \theta_1)\) is measured from the point \((a, 0)\) and \((r_2, \theta_2)\) is measured from the point \((-a, 0)\) (Fig. 21).

The maximum boundary residuals obtained with this stress function for the three problems was

\[
\begin{align*}
\text{b/a} & = .15 & e_{\text{max}} & = 2 \times 10^{-6} \\
\text{b/a} & = .167 & e_{\text{max}} & = 7 \times 10^{-7} \\
\text{b/a} & = .33 & e_{\text{max}} & = 5 \times 10^{-5}
\end{align*}
\]
The difference in the residuals of the first two problems is not significant. However, the third problem does have somewhat larger residuals. This indicates that the convergence of the stress function is slower for the closer hole spacing. However, the solution is still sufficiently accurate for all practical purposes and it was not felt necessary in this case to take more terms in the stress function. No attempt was made to solve a corresponding problem with a smaller spacing. However, several of problems will be considered later in which the web thickness between boundaries is smaller than the circle radii. It was found that the rate of convergence is slowed considerably as the web diminishes.

Tables 1 and 2 give the values of the boundary stresses $\sigma_\theta/p$ as a function of the angle $\alpha = \pi - \theta_1$ for the three spacings. As was mentioned in the introduction to this chapter, the boundary points are specified by giving the x-coordinate of the boundary point as input to the computing program. Although the boundary points were chosen in most cases at approximately equal intervals in $\theta$, no special attempt was made to obtain stresses at nicely even values of $\theta$. We will, therefore, report the stresses that were calculated and record the value of the angle $\alpha$ to the nearest minute of arc.
TABLE 1.—Boundary values of $\sigma_0/p$ on the outside hole of three holes in a line in an infinite plate loaded in uniaxial tension normal to the line of centers of the holes.

<table>
<thead>
<tr>
<th>Degrees</th>
<th>Minutes</th>
<th>$\sigma_0/p$</th>
<th>$b/a = .15$</th>
<th>$b/a = .167$</th>
<th>$b/a = .333$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>00</td>
<td>2.996</td>
<td>2.997</td>
<td>3.355</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>56</td>
<td>2.885</td>
<td>2.889</td>
<td>3.273</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>57</td>
<td>2.560</td>
<td>2.571</td>
<td>3.007</td>
<td></td>
</tr>
<tr>
<td>29</td>
<td>32</td>
<td>2.147</td>
<td>2.098</td>
<td>2.559</td>
<td></td>
</tr>
<tr>
<td>39</td>
<td>39</td>
<td>1.451</td>
<td>1.475</td>
<td>1.901</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>12</td>
<td>0.738</td>
<td>0.764</td>
<td>1.099</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>00</td>
<td>0.107</td>
<td>0.133</td>
<td>0.368</td>
<td></td>
</tr>
<tr>
<td>70</td>
<td>07</td>
<td>-0.431</td>
<td>-0.410</td>
<td>-0.623</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>13</td>
<td>-0.787</td>
<td>-0.769</td>
<td>-0.681</td>
<td></td>
</tr>
<tr>
<td>90</td>
<td>00</td>
<td>-0.914</td>
<td>-0.901</td>
<td>-0.842</td>
<td></td>
</tr>
<tr>
<td>99</td>
<td>47</td>
<td>-0.812</td>
<td>-0.802</td>
<td>-0.749</td>
<td></td>
</tr>
<tr>
<td>109</td>
<td>53</td>
<td>-0.479</td>
<td>-0.470</td>
<td>-0.407</td>
<td></td>
</tr>
<tr>
<td>120</td>
<td>00</td>
<td>0.471</td>
<td>0.541</td>
<td>0.137</td>
<td></td>
</tr>
<tr>
<td>129</td>
<td>48</td>
<td>0.676</td>
<td>0.683</td>
<td>0.788</td>
<td></td>
</tr>
<tr>
<td>140</td>
<td>21</td>
<td>1.402</td>
<td>1.408</td>
<td>1.539</td>
<td></td>
</tr>
<tr>
<td>150</td>
<td>28</td>
<td>2.052</td>
<td>2.059</td>
<td>2.210</td>
<td></td>
</tr>
<tr>
<td>160</td>
<td>03</td>
<td>2.556</td>
<td>2.562</td>
<td>2.729</td>
<td></td>
</tr>
<tr>
<td>170</td>
<td>04</td>
<td>2.900</td>
<td>2.907</td>
<td>3.084</td>
<td></td>
</tr>
<tr>
<td>180</td>
<td>00</td>
<td>3.019</td>
<td>3.026</td>
<td>3.205</td>
<td></td>
</tr>
</tbody>
</table>
TABLE 2.--Boundary values of $\sigma_0/p$ on the center hole of three holes in a line in the infinite plate loaded in uniaxial tension normal to the line of centers of the holes

<table>
<thead>
<tr>
<th>Degrees</th>
<th>Minutes</th>
<th>$\sigma_0/p$</th>
<th>$b/a = .15$</th>
<th>$b/a = .167$</th>
<th>$b/a = .33$</th>
</tr>
</thead>
<tbody>
<tr>
<td>90</td>
<td>00</td>
<td>-0.863</td>
<td>-0.842</td>
<td>-0.786</td>
<td></td>
</tr>
<tr>
<td>99</td>
<td>47</td>
<td>-0.748</td>
<td>-0.726</td>
<td>-0.650</td>
<td></td>
</tr>
<tr>
<td>109</td>
<td>53</td>
<td>-0.404</td>
<td>-0.380</td>
<td>-0.243</td>
<td></td>
</tr>
<tr>
<td>120</td>
<td>00</td>
<td>0.126</td>
<td>0.153</td>
<td>0.385</td>
<td></td>
</tr>
<tr>
<td>129</td>
<td>48</td>
<td>0.751</td>
<td>0.779</td>
<td>1.123</td>
<td></td>
</tr>
<tr>
<td>140</td>
<td>21</td>
<td>1.462</td>
<td>1.488</td>
<td>1.938</td>
<td></td>
</tr>
<tr>
<td>150</td>
<td>28</td>
<td>2.091</td>
<td>2.112</td>
<td>2.614</td>
<td></td>
</tr>
<tr>
<td>160</td>
<td>03</td>
<td>2.572</td>
<td>2.586</td>
<td>3.082</td>
<td></td>
</tr>
<tr>
<td>170</td>
<td>04</td>
<td>2.898</td>
<td>2.905</td>
<td>3.366</td>
<td></td>
</tr>
<tr>
<td>180</td>
<td>00</td>
<td>3.009</td>
<td>3.014</td>
<td>3.455</td>
<td></td>
</tr>
</tbody>
</table>
Table 3 gives the values of $\sigma_x$ and $\sigma_y$ at a few points on the x-axis between the center and outside holes for the three spacings. It is seen that the double maximum in $\sigma_x$ for the wider spacing (which was noted by Green) becomes a single maximum for the closer spacing.

The boundary values of $\sigma_\phi / p$ for the spacings $b/a = .15$ and $b/a = .33$ are plotted in Figs. 22 and 23, respectively. The values of $\sigma_x$ and $\sigma_y$ along the y-axis between holes for these two spacings are plotted in Figs. 24 and 25.

The stresses for the spacing $b/a = .167$ are not plotted since the values of these stresses were so near those for the spacing $b/a = .15$.

Two, four, and six pressurized holes evenly spaced on a circle in the infinite plate. These three problems are grouped together for two reasons. The first reason is that we wished to show how the stress distributions differ between the three cases. The second (more frivolous) reason is that the input to the computing program was identical for these three cases except for one number which specified the number of symmetry elements in each problem.

We considered the case for which the circles were of unit radius and the centers were located on a circle having a radius of
TABLE 3.—Values of $\sigma_x/p$ and $\sigma_y/p$ on the x-axis between the center and outside holes of three holes in a line in the infinite plate loaded in uniaxial tension normal to the line of centers of the holes.

<table>
<thead>
<tr>
<th>$a/b = .333$</th>
<th>$a/b = .167$</th>
<th>$a/b = .15$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$\sigma_x/p$</td>
<td>$\sigma_y/p$</td>
</tr>
<tr>
<td>1.000</td>
<td>0.000</td>
<td>3.455</td>
</tr>
<tr>
<td>1.125</td>
<td>0.311</td>
<td>2.844</td>
</tr>
<tr>
<td>1.250</td>
<td>0.465</td>
<td>2.496</td>
</tr>
<tr>
<td>1.375</td>
<td>0.534</td>
<td>2.313</td>
</tr>
<tr>
<td>1.500</td>
<td>0.553</td>
<td>2.252</td>
</tr>
<tr>
<td>1.625</td>
<td>0.531</td>
<td>2.296</td>
</tr>
<tr>
<td>1.750</td>
<td>0.459</td>
<td>2.459</td>
</tr>
<tr>
<td>1.875</td>
<td>0.305</td>
<td>2.783</td>
</tr>
<tr>
<td>2.000</td>
<td>0.000</td>
<td>3.355</td>
</tr>
</tbody>
</table>
Fig. 22.—Boundary values of $\sigma_\theta/p$ for three equal colinear holes in an infinite plate with uniaxial tension, $b/a = .15$
Fig. 23.—Boundary values of $\sigma_\theta/p$ for three equal colinear holes in an infinite plate with uniaxial tension, $b/a = .33$
Fig. 24.--Values of $\sigma_x/p$ and $\sigma_y/p$ on the x-axis between the holes for three equal colinear holes in an infinite plate with uniaxial tension, $b/a = .15$.

Fig. 25.--Values of $\sigma_x/p$ and $\sigma_y/p$ on the x-axis between the holes for three equal colinear holes in an infinite plate with uniaxial tension, $b/a = .83$. 
three (Figs. 26-28). For the case of the problem with six holes, this gave a minimum distance between the hole boundaries equal to the radius of the holes (Fig. 28).

The stress function that was used for these three problems was

\[ \phi = a_0 \sum_{j=0}^{M-1} \log \lambda_j + \sum_{n=1}^{q} a_n \sum_{j=0}^{M-1} \lambda_j^{-n} \cos n(\theta_j - \frac{2\pi j}{M}) + \sum_{n=2}^{q} b_n \sum_{j=0}^{M-1} \lambda_j^{-n+2} \cos n(\theta_j - \frac{2\pi j}{M}) \]

where \( M \) is the number of holes in each problem and \((r_j, \theta_j)\) are the coordinates of a point measured from the \( j \)-th hole. (The holes are numbered from 0 through \( M-1 \), starting with the hole on the positive x-axis and counting counterclockwise.)

The basic symmetry element of the two-hole problem is the first quadrant in Fig. 26. The basic symmetry elements in the four- and six-hole problems are bounded by the x-axis and the lines \( \theta = \pi/4 \) and \( \pi/6 \), respectively, shown as dotted lines in Figs. 27 and 28.

Once again, since \( \phi \) in (4.2) has the appropriate symmetry, it is necessary to satisfy the boundary conditions only on the
Fig. 26.--The two-hole problem

Fig. 27.--The four-hole problem
semicircle included in the basic symmetry elements of these problems. Using the appropriate form of (4.2), we obtained solutions to the three problems with the following maximum residuals

2 holes \[ e_{\text{max}} = 3 \times 10^{-7} \]
4 holes \[ e_{\text{max}} = 1 \times 10^{-6} \]
6 holes \[ e_{\text{max}} = 1 \times 10^{-4} \]
These results again show the effect on the convergence of the decreased spacing between the holes.

Table 4 gives the values of the boundary loads $\sigma_0/p$ as a function of the angle $\alpha = \pi - \theta_0$ on the hole for the three problems.

**TABLE 4.**--Boundary values of $\sigma_0/p$ on the semi-circle of the basic symmetry element for two, four, and six holes symmetrically arranged along the arc of a circle in the infinite plate.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Degrees</th>
<th>Minutes</th>
<th>2 Holes</th>
<th>4 Holes</th>
<th>6 Holes</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>00</td>
<td></td>
<td>1.155</td>
<td>0.921</td>
<td>0.425</td>
</tr>
<tr>
<td>9</td>
<td>56</td>
<td></td>
<td>1.141</td>
<td>0.947</td>
<td>0.491</td>
</tr>
<tr>
<td>19</td>
<td>57</td>
<td></td>
<td>1.103</td>
<td>1.015</td>
<td>0.686</td>
</tr>
<tr>
<td>29</td>
<td>32</td>
<td></td>
<td>1.053</td>
<td>1.093</td>
<td>0.967</td>
</tr>
<tr>
<td>39</td>
<td>39</td>
<td></td>
<td>0.997</td>
<td>1.150</td>
<td>1.297</td>
</tr>
<tr>
<td>50</td>
<td>12</td>
<td></td>
<td>0.947</td>
<td>1.152</td>
<td>1.570</td>
</tr>
<tr>
<td>60</td>
<td>00</td>
<td></td>
<td>0.914</td>
<td>1.097</td>
<td>1.658</td>
</tr>
<tr>
<td>70</td>
<td>07</td>
<td></td>
<td>0.896</td>
<td>1.005</td>
<td>1.548</td>
</tr>
<tr>
<td>80</td>
<td>13</td>
<td></td>
<td>0.894</td>
<td>0.911</td>
<td>1.289</td>
</tr>
<tr>
<td>90</td>
<td>00</td>
<td></td>
<td>0.905</td>
<td>0.846</td>
<td>1.005</td>
</tr>
<tr>
<td>99</td>
<td>47</td>
<td></td>
<td>0.925</td>
<td>0.820</td>
<td>0.789</td>
</tr>
<tr>
<td>109</td>
<td>53</td>
<td></td>
<td>0.951</td>
<td>0.833</td>
<td>0.687</td>
</tr>
<tr>
<td>120</td>
<td>00</td>
<td></td>
<td>0.979</td>
<td>0.877</td>
<td>0.701</td>
</tr>
<tr>
<td>129</td>
<td>48</td>
<td></td>
<td>1.006</td>
<td>0.938</td>
<td>0.789</td>
</tr>
<tr>
<td>140</td>
<td>21</td>
<td></td>
<td>1.032</td>
<td>1.009</td>
<td>0.920</td>
</tr>
<tr>
<td>150</td>
<td>28</td>
<td></td>
<td>1.053</td>
<td>1.072</td>
<td>1.047</td>
</tr>
<tr>
<td>160</td>
<td>03</td>
<td></td>
<td>1.068</td>
<td>1.119</td>
<td>1.147</td>
</tr>
<tr>
<td>170</td>
<td>04</td>
<td></td>
<td>1.077</td>
<td>1.150</td>
<td>1.214</td>
</tr>
<tr>
<td>180</td>
<td>00</td>
<td></td>
<td>1.080</td>
<td>1.161</td>
<td>1.237</td>
</tr>
</tbody>
</table>

The same stresses are plotted in Figs. 29 through 31. It is seen that the maximum hoop stress is attained at the point of minimum distance between the holes, as would be expected. It is
Fig. 29. -- Boundary values of $\frac{\sigma_\theta}{p}$ for two holes in an infinite plate loaded by internal pressure
Fig. 30.—Boundary values of $\sigma_\theta /p$ for four holes in an infinite plate loaded by internal pressure.
Fig. 31. -- Boundary values of $\sigma_\theta/p$ for six holes in an infinite plate loaded by internal pressure
also seen that the value of the maximum stress increases with a
decrease in distance between the holes. An interesting point to
note is that the second maximum attained in the four-hole case at
$\alpha = 180^\circ$ was as large as the maximum on the line between the hole
centers. The value of $\sigma_0/p$ at $\alpha = 45^\circ$ was 1.160 and at $\alpha = 180^\circ$, it was 1.161.

The two-hole problem has an exact solution in bipolar
coordinates. Ling (3) calculated the stresses at the points
corresponding to $\alpha = 0$ and $\alpha = 90^\circ$ for a number of circle spacings
with biaxial tension at infinity. For the spacing considered
here, Ling found that $\sigma_\theta/p = 2.155$ at $\alpha = 0$ and $\sigma_\phi/p = 2.080$
at $\alpha = 180^\circ$. In order to convert the stresses for a problem
with hydrostatic tension $p$ at infinity to the corresponding
problem with uniform pressure $p$ in the holes, it is necessary
only to subtract $p$ from the normal stresses calculated at each
point of the region. Thus, Ling's results for uniform pressure
in the holes are $\sigma_\theta/p = 1.155$ at $\alpha = 0$ and $\sigma_\phi/p = 1.080$ at
$\alpha = 180^\circ$, which are identical with our results. No attempt was
made to calculate more points than these (for the exact solution)
since the calculations made for the solution in bipolar coordinates
are somewhat complicated.

The six-hole problem was reportedly solved also by Ling in
a paper (4) referred to by Radkowski (5). Since this paper is
unavailable for study, it is not known whether Ling obtained any numerical solutions or simply derived the formal solution. It is noted, incidentally, that Radkowski derived the formal solution to the seven-hole problem, considered next, but did not attempt a numerical solution.

Seven and nineteen pressurized holes in the infinite plate.

We will consider in this section the numerical solution of two problems. The first problem is concerned with a region bounded by a unit circle with center at the origin, surrounded by a ring of six circular holes equally spaced on the circle \( r = 3 \). The second problem is concerned with an array of nineteen circular holes formed by adding twelve more equal holes in a hexagonal pattern around the seven holes of the first problem. The arrangements for the holes of these two problems are shown in Figs. 32 and 33. The spacing is such that the center to center distance between all closest neighbor holes is three times the radius of the holes. It is noted that aside from being interesting problems in themselves, these two problems represent the first and second approximations to the very practical problem of the infinite tube sheet with a ligament efficiency of 50 per cent.\(^1\) In fact, we will show that the stress distribution in the neighborhood of the center hole in the nineteen-hole problem is a fairly accurate approximation to the stress distribution in the infinite tube sheet.

\(^1\)The ligament efficiency of a tube sheet equals the ratio of the minimum distance between adjacent hole boundaries to the diameter of the holes.
Fig. 32.—The seven-hole problem

Fig. 33.—The nineteen-hole problem
The stress function for the seven-hole problem is

\[ \phi = a_0 \log R + \sum_{n=1}^{3} \left( a_{-n} R^{-n} \cos n\theta + b_{-n} R^{-n} \cos (n\theta) \right) + \sum_{j=0}^{5} a_j \log R_{1,j} + \sum_{n=1}^{9} a_n \sum_{j=0}^{5} R_{1,j}^{-n} \cos \left( \theta_{1,j} - \frac{n\pi}{3} \right) \]

\[ + \sum_{n=2}^{9} b_n \sum_{j=0}^{5} R_{1,j}^{-n} \cos \left( \theta_{1,j} - \frac{n\pi}{3} \right), \]

where \((r, \theta)\) is measured from the origin and \((r_{1,j}, \theta_{1,j})\) are measured from centers of the outside holes. These holes are numbered from 0 through 5 in the counterclockwise direction starting with the hole on the positive axis.

For the nineteen-hole problem the stress function was chosen as

\[ \phi = a_0 \log R + \sum_{n=1}^{3} \left( a_{-n} R^{-n} \cos n\theta + b_{-n} R^{-n} \cos (n\theta) \right) + \sum_{k=1}^{2} \sum_{j=0}^{5} a_{k,j} \log R_{k,j} + \sum_{n=1}^{9} a_n \sum_{j=0}^{5} R_{k,j}^{-n} \cos \left( \theta_{k,j} - \frac{n\pi}{3} \right) \]

\[ + \sum_{n=2}^{9} b_n \sum_{j=0}^{5} R_{k,j}^{-n} \cos \left( \theta_{k,j} - \frac{n\pi}{3} \right) + \sum_{j=0}^{5} a_{3,j} \log R_{3,j} + \sum_{n=1}^{9} a_n \sum_{j=0}^{5} R_{3,j}^{-n} \cos \left[ \theta_{3,j} - (2j+1) \frac{\pi}{6} \right] \]

\[ + \sum_{n=1}^{9} \sum_{j=0}^{5} b_n R_{3,j}^{-n} \cos \left[ \theta_{3,j} - (2j+1) \frac{\pi}{6} \right]. \]
The coordinates \((r, \theta)\) are measured from the origin. The coordinates \((r_1, \theta_1, \phi_1)\) are measured from the point \((3,0)\) and points symmetric to \((3,0)\). The coordinates \((r_2, \theta_2, \phi_2)\) are measured from the point \((6,0)\) and points symmetric to \((6,0)\). The coordinates \((r_3, \theta_3, \phi_3)\) are measured from the point \((4.5,1.5\sqrt{3})\) and the points symmetric to it. [The point \((4.5,1.5\sqrt{3})\) is at the center of boundary \(\Gamma_{3,0}\) of Fig. 33.]

The numerical solution obtained for the seven-hole problem satisfied the boundary conditions with a maximum residual of \(7 \times 10^{-5}\). The maximum residual in the nineteen-hole problem was \(6 \times 10^{-6}\). This latter figure is quite remarkable. It indicates that very little round-off error occurred in the machine calculations in spite of the fact that the coefficient matrix was of the order \((97, 132)\). (There were 132 boundary conditions and 97 undetermined constants in the stress function.) As a matter of fact, there seemed to be little evidence of round-off error in any of the calculations made for problems involving infinite plates. It is also interesting to note that the solution of the nineteen-hole problem took only 22 minutes on the IBM 7090!

Table 5 gives the boundary values of \(\sigma_\theta /p\) on the center hole \(\Gamma_0\) for both problems as a function of \(\alpha = \pi - \theta\). Table 6 gives the boundary values of \(\sigma_\phi /p\) on the hole \(\Gamma_{1,0}\) with center at \((3,0)\) for both problems as a function of \(\alpha = \pi - \theta_{1,0}\).
TABLE 5.—Boundary values of $\sigma_0/p$ on $n_0$ for seven and nineteen pressurized holes in the infinite plate

<table>
<thead>
<tr>
<th>Degrees</th>
<th>Minutes</th>
<th>$\sigma_0/p$</th>
<th>7 Holes</th>
<th>19 Holes</th>
</tr>
</thead>
<tbody>
<tr>
<td>150</td>
<td>28</td>
<td>0.935</td>
<td>0.877</td>
<td></td>
</tr>
<tr>
<td>155</td>
<td>30</td>
<td>0.961</td>
<td>0.902</td>
<td></td>
</tr>
<tr>
<td>160</td>
<td>03</td>
<td>1.014</td>
<td>0.954</td>
<td></td>
</tr>
<tr>
<td>165</td>
<td>56</td>
<td>1.108</td>
<td>1.045</td>
<td></td>
</tr>
<tr>
<td>170</td>
<td>04</td>
<td>1.172</td>
<td>1.107</td>
<td></td>
</tr>
<tr>
<td>174</td>
<td>52</td>
<td>1.227</td>
<td>1.161</td>
<td></td>
</tr>
<tr>
<td>180</td>
<td>00</td>
<td>1.249</td>
<td>1.182</td>
<td></td>
</tr>
</tbody>
</table>
### TABLE 6

Boundary values of $\tau_o/p$ on $\phi_o$ for seven and nineteen pressurized holes in the infinite plate

<table>
<thead>
<tr>
<th>Degrees</th>
<th>Minutes</th>
<th>7 Holes</th>
<th>19 Holes</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>00</td>
<td>1.249</td>
<td>1.182</td>
</tr>
<tr>
<td>9</td>
<td>56</td>
<td>1.171</td>
<td>1.107</td>
</tr>
<tr>
<td>19</td>
<td>57</td>
<td>1.014</td>
<td>0.954</td>
</tr>
<tr>
<td>29</td>
<td>32</td>
<td>0.937</td>
<td>0.877</td>
</tr>
<tr>
<td>39</td>
<td>39</td>
<td>1.015</td>
<td>0.948</td>
</tr>
<tr>
<td>50</td>
<td>12</td>
<td>1.192</td>
<td>1.107</td>
</tr>
<tr>
<td>60</td>
<td>00</td>
<td>1.286</td>
<td>1.179</td>
</tr>
<tr>
<td>70</td>
<td>07</td>
<td>1.209</td>
<td>1.102</td>
</tr>
<tr>
<td>80</td>
<td>13</td>
<td>0.986</td>
<td>0.955</td>
</tr>
<tr>
<td>90</td>
<td>00</td>
<td>0.743</td>
<td>0.898</td>
</tr>
<tr>
<td>99</td>
<td>47</td>
<td>0.579</td>
<td>0.996</td>
</tr>
<tr>
<td>109</td>
<td>53</td>
<td>0.541</td>
<td>1.176</td>
</tr>
<tr>
<td>120</td>
<td>00</td>
<td>0.622</td>
<td>1.273</td>
</tr>
<tr>
<td>129</td>
<td>48</td>
<td>0.775</td>
<td>1.201</td>
</tr>
<tr>
<td>140</td>
<td>21</td>
<td>0.968</td>
<td>1.022</td>
</tr>
<tr>
<td>150</td>
<td>28</td>
<td>1.143</td>
<td>0.917</td>
</tr>
<tr>
<td>160</td>
<td>03</td>
<td>1.275</td>
<td>0.960</td>
</tr>
<tr>
<td>170</td>
<td>04</td>
<td>1.364</td>
<td>1.086</td>
</tr>
<tr>
<td>180</td>
<td>00</td>
<td>1.393</td>
<td>1.151</td>
</tr>
</tbody>
</table>
Table 7 gives the values of \( \sigma_q / p \) on the second nearest neighbor holes \( R_{2,0} \) and \( R_{3,0} \) of the nineteen-hole problem. On \( R_{2,0} \), the stresses are given as a function of \( \alpha = \pi - \theta_{2,0} \). On \( R_{3,0} \), the stresses are given as a function of \( \alpha = \theta_{3,0} - \pi \). The change in the way \( \alpha \) is measured in this last case is because the lower half of \( R_{3,0} \) is in the basic symmetry element. As is seen from Fig. 33, the stresses on \( R_{3,0} \) are symmetric about the diameter \( \alpha = 30 \), \( \alpha = 210 \). Thus, the stresses are given on \( R_{3,0} \) for \( \alpha = 30 \) through \( \alpha = 210 \).

The boundary stresses given in Tables 5 through 7 are plotted in Figs. 34 through 37.

As mentioned earlier, the stress distributions in the neighborhood of the center circle are of particular interest since they represent approximations to the stress distribution in the infinite tube sheet with the same hole spacing.

Fig. 38 shows a portion of the infinite tube sheet with holes in a triangular array and having a 50 per cent ligament efficiency. The basic symmetry element of this figure is the darkened area. A comparison of Fig. 38 with Figs. 32 and 33 shows that the seven- and nineteen-hole problems both have symmetry about the lines \( \theta = 0 \) and \( \theta = \pi / 6 \). The accuracy with which the solutions to these two problems approximate the solution to the infinite tube sheet of Fig. 38 then depends on how well the solutions of the
TABLE 7.—Boundary values of $\sigma_0/p$ on $\Gamma_{2,0}$ and $\Gamma_{3,0}$ of nineteen pressurized holes in the infinite plate

<table>
<thead>
<tr>
<th>Degrees</th>
<th>Minutes</th>
<th>$\sigma_0/p$</th>
<th>$\Gamma_{2,0}$</th>
<th>$\Gamma_{3,0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>00</td>
<td>1.440</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>56</td>
<td>1.356</td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>57</td>
<td>1.181</td>
<td></td>
<td></td>
</tr>
<tr>
<td>29</td>
<td>32</td>
<td>1.074</td>
<td>0.870</td>
<td></td>
</tr>
<tr>
<td>39</td>
<td>39</td>
<td>1.101</td>
<td>0.935</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>02</td>
<td>1.192</td>
<td>1.076</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>00</td>
<td>1.186</td>
<td>1.128</td>
<td></td>
</tr>
<tr>
<td>70</td>
<td>07</td>
<td>1.020</td>
<td>1.030</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>13</td>
<td>0.753</td>
<td>0.847</td>
<td></td>
</tr>
<tr>
<td>90</td>
<td>00</td>
<td>0.513</td>
<td>0.797</td>
<td></td>
</tr>
<tr>
<td>99</td>
<td>47</td>
<td>0.383</td>
<td>0.911</td>
<td></td>
</tr>
<tr>
<td>109</td>
<td>53</td>
<td>0.400</td>
<td>1.153</td>
<td></td>
</tr>
<tr>
<td>120</td>
<td>00</td>
<td>0.550</td>
<td>1.351</td>
<td></td>
</tr>
<tr>
<td>129</td>
<td>48</td>
<td>0.776</td>
<td>1.367</td>
<td></td>
</tr>
<tr>
<td>140</td>
<td>21</td>
<td>1.050</td>
<td>1.193</td>
<td></td>
</tr>
<tr>
<td>150</td>
<td>28</td>
<td>1.296</td>
<td>0.960</td>
<td></td>
</tr>
<tr>
<td>160</td>
<td>03</td>
<td>1.483</td>
<td>0.794</td>
<td></td>
</tr>
<tr>
<td>170</td>
<td>04</td>
<td>1.609</td>
<td>0.732</td>
<td></td>
</tr>
<tr>
<td>180</td>
<td>00</td>
<td>1.652</td>
<td>0.763</td>
<td></td>
</tr>
<tr>
<td>189</td>
<td>56</td>
<td>0.837</td>
<td></td>
<td></td>
</tr>
<tr>
<td>199</td>
<td>57</td>
<td>0.905</td>
<td></td>
<td></td>
</tr>
<tr>
<td>209</td>
<td>32</td>
<td>0.932</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Fig. 34. -- Boundary values of $\sigma_\theta/p$ for seven holes in an infinite plate loaded by internal pressure.
Fig. 35.--Boundary values of $\sigma_\theta/p$ on boundaries $r$ and $r_{1,0}$ of nineteen pressurized holes in the infinite plate.
Fig. 36.—Boundary values of $\sigma_\theta/p$ on boundary $\Pi_{2,0}$ of nineteen pressurized holes in the infinite plate.
Fig. 37.--Boundary values of $\frac{\sigma_\theta}{p}$ on boundary $\Pi_{3,0}$ of nineteen pressurized holes in the infinite plate.
two problems meet the symmetry condition on the line $x = 1.5$. Some
evidence has already been presented on this point. If the stresses
are to be symmetric about the line $x = 1.5$, it is necessary that
the boundary stresses on $\mathcal{F}_{1,0}$ over the range $0 \leq \alpha \leq \pi/6$ be equal
to the boundary stresses on $\mathcal{F}_{6,0}$ over the range $0 \leq \theta \leq \pi/6$.
Comparison of Tables 5 and 6 shows that these stresses are equal
to within the accuracy with which they are given, except for one
point for the seven-hole problem. Actually, the differences in
the boundary stresses at comparable points were an order of magnitude
smaller for the nineteen-hole problem than for the seven-hole
problem.
From Fig. 38, it is seen that the stresses on $\mathcal{P}_{1,0}$ should also be symmetric about the line $\alpha = \pi/6$. An examination of Table 6 shows that for $\alpha = 0^\circ$ and $60^\circ$, the values of $\sigma_{\theta}/p$ differ by about 4 per cent for the seven-hole problem and by about 0.3 per cent for the nineteen-hole problem. The positions $90^\circ 56'$ and $50^\circ 12'$ or $19^\circ 57'$ and $39^\circ 39'$ are not symmetrically located with respect to $\alpha = \pi/6$ so the values of $\sigma_{\theta}/p$ cannot be compared for symmetry at these points.

The values of the stresses were calculated at some interior points in the neighborhood of the center circle for the two problems. Tables 8 and 9 give the values of $\sigma_{x}/p$, $\sigma_{y}/p$, $\tau_{xy}/p$, $\sigma_{\text{max}}/p$, $\sigma_{\text{min}}/p$, $\sigma_{\text{max}}/p$, and the angle between the direction of the maximum normal stress and the x-axis calculated at these points.

The values of $\sigma_{x}/p$, $\sigma_{y}/p$, and $\tau_{xy}/p$ are given to five decimals to show the precision with which the stresses are symmetric about $y = 1.5$. It may be seen from a comparison of these tables that the symmetry is satisfied almost uniformly an order of magnitude more closely for the nineteen-hole problem than for the seven-hole problem. An examination of the various stresses given in Tables 5, 6, 8, and 9 shows that the maximum difference in the stress distributions for the two problems is 6 per cent of the maximum stress. It is felt, therefore, that the stress distributions in the neighborhood of the center circle of the nineteen-hole problem represent to within 1 per cent the corresponding stresses in the infinite tube sheet.
### TABLE 8.—Values of the stresses calculated at selected interior points of the seven-hole problem

<table>
<thead>
<tr>
<th>y</th>
<th>x</th>
<th>$\sigma_x/p$</th>
<th>$\tau_y/p$</th>
<th>$\tau_{xy}/p$</th>
<th>$\tau_{max}/p$</th>
<th>$\tau_{min}/p$</th>
<th>$\tau_{max}/p$</th>
<th>Angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.00</td>
<td>-1.00000</td>
<td>1.24930</td>
<td>0.00000</td>
<td>1.249</td>
<td>-1.000</td>
<td>1.125</td>
<td>1.571</td>
</tr>
<tr>
<td></td>
<td>1.25</td>
<td>-0.68342</td>
<td>0.82000</td>
<td>0.00000</td>
<td>0.820</td>
<td>-0.683</td>
<td>0.752</td>
<td>1.571</td>
</tr>
<tr>
<td></td>
<td>1.50</td>
<td>-0.60554</td>
<td>0.72293</td>
<td>0.00000</td>
<td>0.723</td>
<td>-0.606</td>
<td>0.664</td>
<td>1.571</td>
</tr>
<tr>
<td></td>
<td>1.75</td>
<td>-0.68341</td>
<td>0.81986</td>
<td>0.00000</td>
<td>0.820</td>
<td>-0.683</td>
<td>0.752</td>
<td>1.571</td>
</tr>
<tr>
<td></td>
<td>2.00</td>
<td>-0.99994</td>
<td>1.24997</td>
<td>0.00000</td>
<td>1.249</td>
<td>-1.000</td>
<td>1.124</td>
<td>1.571</td>
</tr>
<tr>
<td>0.25</td>
<td>1.00</td>
<td>-0.88343</td>
<td>0.93895</td>
<td>-0.44434</td>
<td>1.044</td>
<td>-0.939</td>
<td>0.991</td>
<td>1.803</td>
</tr>
<tr>
<td></td>
<td>1.25</td>
<td>-0.61515</td>
<td>0.72229</td>
<td>-0.15694</td>
<td>0.740</td>
<td>-0.633</td>
<td>0.687</td>
<td>1.686</td>
</tr>
<tr>
<td></td>
<td>1.50</td>
<td>-0.54640</td>
<td>0.65003</td>
<td>0.00000</td>
<td>0.650</td>
<td>-0.546</td>
<td>0.598</td>
<td>1.571</td>
</tr>
<tr>
<td></td>
<td>1.75</td>
<td>-0.61504</td>
<td>0.72213</td>
<td>0.15696</td>
<td>0.740</td>
<td>-0.633</td>
<td>0.687</td>
<td>1.456</td>
</tr>
<tr>
<td></td>
<td>2.00</td>
<td>-0.83351</td>
<td>0.93852</td>
<td>0.44430</td>
<td>1.044</td>
<td>-0.939</td>
<td>0.991</td>
<td>1.338</td>
</tr>
<tr>
<td>0.50</td>
<td>1.00</td>
<td>-0.48351</td>
<td>0.49952</td>
<td>-0.61449</td>
<td>0.795</td>
<td>-0.779</td>
<td>0.787</td>
<td>2.019</td>
</tr>
<tr>
<td></td>
<td>1.25</td>
<td>-0.40176</td>
<td>0.47370</td>
<td>-0.26505</td>
<td>0.548</td>
<td>-0.476</td>
<td>0.512</td>
<td>1.843</td>
</tr>
<tr>
<td></td>
<td>1.50</td>
<td>-0.36580</td>
<td>0.44778</td>
<td>-0.00024</td>
<td>0.444</td>
<td>-0.366</td>
<td>0.407</td>
<td>1.571</td>
</tr>
<tr>
<td></td>
<td>1.75</td>
<td>-0.40105</td>
<td>0.47321</td>
<td>0.26481</td>
<td>0.548</td>
<td>-0.475</td>
<td>0.511</td>
<td>1.299</td>
</tr>
<tr>
<td></td>
<td>2.00</td>
<td>-0.48240</td>
<td>0.49929</td>
<td>0.61500</td>
<td>0.795</td>
<td>-0.778</td>
<td>0.787</td>
<td>1.112</td>
</tr>
<tr>
<td>0.75</td>
<td>1.00</td>
<td>-0.06933</td>
<td>0.12854</td>
<td>-0.60436</td>
<td>0.642</td>
<td>-0.583</td>
<td>0.612</td>
<td>2.275</td>
</tr>
<tr>
<td></td>
<td>1.25</td>
<td>-0.10035</td>
<td>0.16994</td>
<td>-0.29400</td>
<td>0.358</td>
<td>-0.289</td>
<td>0.324</td>
<td>2.141</td>
</tr>
<tr>
<td></td>
<td>1.50</td>
<td>-0.08288</td>
<td>0.17202</td>
<td>-0.00063</td>
<td>0.172</td>
<td>-0.098</td>
<td>0.135</td>
<td>1.573</td>
</tr>
<tr>
<td></td>
<td>1.75</td>
<td>-0.09482</td>
<td>0.16913</td>
<td>0.29258</td>
<td>0.357</td>
<td>-0.287</td>
<td>0.322</td>
<td>1.001</td>
</tr>
<tr>
<td></td>
<td>2.00</td>
<td>-0.06614</td>
<td>0.12670</td>
<td>0.60342</td>
<td>0.641</td>
<td>-0.581</td>
<td>0.611</td>
<td>0.865</td>
</tr>
<tr>
<td>1.00</td>
<td>1.00</td>
<td>0.21993</td>
<td>-0.12857</td>
<td>-0.582170</td>
<td>0.596</td>
<td>-0.504</td>
<td>0.550</td>
<td>2.517</td>
</tr>
<tr>
<td></td>
<td>1.25</td>
<td>0.19134</td>
<td>-0.11375</td>
<td>-0.28442</td>
<td>0.362</td>
<td>-0.284</td>
<td>0.323</td>
<td>2.602</td>
</tr>
<tr>
<td></td>
<td>1.50</td>
<td>0.18192</td>
<td>-0.11900</td>
<td>-0.00051</td>
<td>0.191</td>
<td>-0.119</td>
<td>0.155</td>
<td>3.140</td>
</tr>
<tr>
<td></td>
<td>1.75</td>
<td>0.19023</td>
<td>-0.11352</td>
<td>0.28235</td>
<td>0.359</td>
<td>-0.282</td>
<td>0.321</td>
<td>0.539</td>
</tr>
<tr>
<td></td>
<td>2.00</td>
<td>0.21930</td>
<td>-0.13042</td>
<td>0.51751</td>
<td>0.591</td>
<td>-0.502</td>
<td>0.546</td>
<td>0.623</td>
</tr>
<tr>
<td>x</td>
<td>$\sigma_x$/p</td>
<td>$\tau_y$/p</td>
<td>$\tau_{xy}$/p</td>
<td>$\sigma_{xy}$/p</td>
<td>$\tau_{\text{max}}$/p</td>
<td>$\tau_{\text{min}}$/p</td>
<td>$\tau_{\text{max}}$/p</td>
<td>Angle</td>
</tr>
<tr>
<td>----</td>
<td>-------------</td>
<td>------------</td>
<td>--------------</td>
<td>---------------</td>
<td>----------------</td>
<td>----------------</td>
<td>----------------</td>
<td>-------</td>
</tr>
<tr>
<td>0.0</td>
<td>1.00</td>
<td>-1.00000</td>
<td>1.18224</td>
<td>0.00000</td>
<td>1.182</td>
<td>-1.000</td>
<td>1.091</td>
<td>1.571</td>
</tr>
<tr>
<td>1.25</td>
<td>-0.69291</td>
<td>0.76571</td>
<td>0.00000</td>
<td>0.766</td>
<td>-0.693</td>
<td>0.729</td>
<td>1.571</td>
<td></td>
</tr>
<tr>
<td>1.50</td>
<td>-0.61740</td>
<td>0.67159</td>
<td>0.00000</td>
<td>0.671</td>
<td>-0.617</td>
<td>0.644</td>
<td>1.571</td>
<td></td>
</tr>
<tr>
<td>1.75</td>
<td>-0.69291</td>
<td>0.76571</td>
<td>0.00000</td>
<td>0.766</td>
<td>-0.693</td>
<td>0.729</td>
<td>1.571</td>
<td></td>
</tr>
<tr>
<td>2.00</td>
<td>-1.00000</td>
<td>1.18228</td>
<td>0.00000</td>
<td>1.182</td>
<td>-1.000</td>
<td>1.091</td>
<td>1.571</td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>1.00</td>
<td>-0.83841</td>
<td>0.88105</td>
<td>-0.43105</td>
<td>0.983</td>
<td>-0.940</td>
<td>0.962</td>
<td>1.803</td>
</tr>
<tr>
<td>1.25</td>
<td>-0.62673</td>
<td>0.67093</td>
<td>-0.15218</td>
<td>0.689</td>
<td>-0.644</td>
<td>0.666</td>
<td>1.686</td>
<td></td>
</tr>
<tr>
<td>1.50</td>
<td>-0.56012</td>
<td>0.60091</td>
<td>0.00000</td>
<td>0.601</td>
<td>-0.560</td>
<td>0.581</td>
<td>1.571</td>
<td></td>
</tr>
<tr>
<td>1.75</td>
<td>-0.62673</td>
<td>0.67093</td>
<td>0.15218</td>
<td>0.689</td>
<td>-0.644</td>
<td>0.666</td>
<td>1.456</td>
<td></td>
</tr>
<tr>
<td>2.00</td>
<td>-0.83841</td>
<td>0.88106</td>
<td>0.43105</td>
<td>0.983</td>
<td>-0.940</td>
<td>0.962</td>
<td>1.338</td>
<td></td>
</tr>
<tr>
<td>0.50</td>
<td>1.00</td>
<td>-0.49898</td>
<td>0.43467</td>
<td>-0.59607</td>
<td>0.741</td>
<td>-0.785</td>
<td>0.763</td>
<td>2.019</td>
</tr>
<tr>
<td>1.25</td>
<td>-0.41970</td>
<td>0.42971</td>
<td>-0.25696</td>
<td>0.501</td>
<td>-0.491</td>
<td>0.496</td>
<td>1.843</td>
<td></td>
</tr>
<tr>
<td>1.50</td>
<td>-0.38504</td>
<td>0.40476</td>
<td>0.00003</td>
<td>0.405</td>
<td>-0.385</td>
<td>0.395</td>
<td>1.571</td>
<td></td>
</tr>
<tr>
<td>1.75</td>
<td>-0.41977</td>
<td>0.42975</td>
<td>0.25699</td>
<td>0.501</td>
<td>-0.491</td>
<td>0.496</td>
<td>1.299</td>
<td></td>
</tr>
<tr>
<td>2.00</td>
<td>-0.49908</td>
<td>0.45468</td>
<td>0.59602</td>
<td>0.741</td>
<td>-0.785</td>
<td>0.763</td>
<td>1.123</td>
<td></td>
</tr>
<tr>
<td>0.75</td>
<td>1.00</td>
<td>-0.09704</td>
<td>0.09469</td>
<td>-0.58626</td>
<td>0.593</td>
<td>-0.595</td>
<td>0.594</td>
<td>2.275</td>
</tr>
<tr>
<td>1.25</td>
<td>-0.12708</td>
<td>0.13481</td>
<td>-0.28499</td>
<td>0.318</td>
<td>-0.310</td>
<td>0.314</td>
<td>2.141</td>
<td></td>
</tr>
<tr>
<td>1.50</td>
<td>-0.12517</td>
<td>0.13696</td>
<td>0.00006</td>
<td>0.137</td>
<td>-0.125</td>
<td>0.131</td>
<td>1.571</td>
<td></td>
</tr>
<tr>
<td>1.75</td>
<td>-0.12715</td>
<td>0.13489</td>
<td>0.28512</td>
<td>0.318</td>
<td>-0.310</td>
<td>0.314</td>
<td>1.001</td>
<td></td>
</tr>
<tr>
<td>2.00</td>
<td>-0.09731</td>
<td>0.09486</td>
<td>0.58634</td>
<td>0.593</td>
<td>-0.595</td>
<td>0.594</td>
<td>0.867</td>
<td></td>
</tr>
<tr>
<td>1.00</td>
<td>1.00</td>
<td>0.18356</td>
<td>-0.15479</td>
<td>-0.50618</td>
<td>0.548</td>
<td>-0.519</td>
<td>0.534</td>
<td>2.518</td>
</tr>
<tr>
<td>1.25</td>
<td>0.15616</td>
<td>-0.14070</td>
<td>-0.27592</td>
<td>0.321</td>
<td>-0.306</td>
<td>0.313</td>
<td>2.603</td>
<td></td>
</tr>
<tr>
<td>1.50</td>
<td>0.15731</td>
<td>-0.14623</td>
<td>0.00005</td>
<td>0.157</td>
<td>-0.146</td>
<td>0.152</td>
<td>0.000</td>
<td></td>
</tr>
<tr>
<td>1.75</td>
<td>0.15627</td>
<td>-0.14069</td>
<td>0.27610</td>
<td>0.321</td>
<td>-0.306</td>
<td>0.313</td>
<td>0.539</td>
<td></td>
</tr>
<tr>
<td>2.00</td>
<td>0.18374</td>
<td>-0.15459</td>
<td>0.50651</td>
<td>0.549</td>
<td>-0.519</td>
<td>0.534</td>
<td>0.624</td>
<td></td>
</tr>
</tbody>
</table>
Finally, it will be noted that a complete study of a problem such as this would include figures giving the curves of equal principal stresses and curves of the stress trajectories. The computer program may be used to calculate stresses similar to those of Tables 8 and 9 at as many points of the region as desired. The problem of making up the curves is thus simply one of plotting. This has not been done as yet because of time limitations. It is anticipated that a paper may be published in the near future giving complete results for this problem as well as solutions for other ligament efficiencies.

Circular holes arranged with cylindrical symmetry in a finite circular plate

We will discuss in this section the solutions obtained for a number of problems involving circular holes in a circular plate. These problems range in complexity from the problem of a single eccentric hole through problems involving one central hole and six eccentric holes equally spaced around a circle in the circular plate to give six-fold symmetry. (The arrangement of the holes for this last problem is the same as that shown in Fig. 32 for the seven-hole problem in the infinite plate.)

It may be noted that, in general, the solutions to these problems showed slower convergence in the series solutions than the
solutions discussed above for the infinite plate. The reason for this, apparently, is that the function $\phi_o$, of (2.90), enters into the solution of a finite plate problem. The boundary values of $\phi$ will vary considerably over the arcs of the eccentric holes and the series $\phi_{ik}$ giving singularities in these holes must adjust for the effect of $\phi_o$. There also seemed to be some evidence that the matrices were less well conditioned for these problems than the infinite plate problems and that there was a certain amount of round-off error. The evidence of round-off error was obtained by noting that a change in the scale factor $r_o$ of (2.83) changed the value of the solution somewhat. However, this question was not investigated fully since the round-off error seemed to affect no more than the fifth or sixth significant digit in most of the problems studied. We will not include $r_o$ in the formulas presented below for the stress functions, since in most cases it was set equal to one.

A single eccentric hole in the circular plate. This problem has an exact solution in bipolar coordinates. Thus, it made a good problem with which to test the accuracy of the technique used here.

We considered the case of a circle of unit radius with a hole having a radius of $1/2$ (Fig. 39).
Fig. 39.—The eccentric hole problem

Problems were considered with an eccentricity \( d, \) of \( 0.1, \)
\( 0.2, \) \( 0.3 \) and \( 0.4. \) The loading in each case was uniform pressure
in the hole. The stress function used was

\[
\phi = a_o \log r + b_o r^2 \\
+ \sum_{n=1}^{k} (a_n r^{-n} \cos n \theta + b_n r^{n+2} \cos n \theta) \\
+ \sum_{n=2}^{\infty} (a_n r^n \cos n \theta + b_n r^{-n+2} \cos n \theta)
\]  

(4.5)

The origin in each case was taken at the center of the hole.
The convergence of the series solution (4.5), as expected, varied greatly with the eccentricity, \( d \). The maximum residuals found for the four cases are

\[
\begin{align*}
\text{d} = 0.1 & \quad e_{\text{max}} = 1 \times 10^{-6}, \\
\text{d} = 0.2 & \quad e_{\text{max}} = .001, \\
\text{d} = 0.3 & \quad e_{\text{max}} = .009, \\
\text{d} = 0.4 & \quad e_{\text{max}} = .03.
\end{align*}
\]

Savin (6) gives formulas for calculating \( \sigma_0/p \) on both boundaries according to the exact solution. The values of \( \sigma_0/p \) are most easily calculated for the points of intersection of the line of centers with the inner and outer boundaries. (Points A, B, C, and D of Fig. 39.) Table 10 gives the values of the exact solution and the approximation (4.5) at the four points for each of the four problems considered.

The solution found for the case \( d = 0.4 \) is unacceptable. A solution was also tried for which the sums of (4.5) were taken through \( n = 12 \). This gave a maximum residual of .02 and gave values of \( \sigma_0/p \) at A, B, C, and D of 6.61, .05, 5.13, and 1.45. No further attempt was made to include more terms in the series solution, since it appeared to be converging slowly.

Stress functions were also tried for the case \( d = 0.4 \) in which the position of the singularity of the stress function was varied along the x-axis between \( x = -0.3 \) and \( x = +0.4 \). However,
TABLE 10.—Boundary values of $v_0/p$ on the intersection of the x-axis with the inner and outer boundaries of the eccentric hole problems

<table>
<thead>
<tr>
<th>Eccentricity $e$</th>
<th>Angle $\theta$</th>
<th>$v_0/p$</th>
<th>Inner Boundary</th>
<th>Outer Boundary</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Exact Solution</td>
<td>Equation (4.5)</td>
<td>Exact Solution</td>
</tr>
<tr>
<td>0.1</td>
<td>0</td>
<td>1.55</td>
<td>1.55</td>
<td>0.43</td>
</tr>
<tr>
<td></td>
<td>$\pi$</td>
<td>1.85</td>
<td>1.85</td>
<td>1.03</td>
</tr>
<tr>
<td>0.2</td>
<td>0</td>
<td>1.48</td>
<td>1.48</td>
<td>0.27</td>
</tr>
<tr>
<td></td>
<td>$\pi$</td>
<td>2.17</td>
<td>2.17</td>
<td>1.65</td>
</tr>
<tr>
<td>0.3</td>
<td>0</td>
<td>1.48</td>
<td>1.47</td>
<td>0.16</td>
</tr>
<tr>
<td></td>
<td>$\pi$</td>
<td>2.84</td>
<td>2.97</td>
<td>2.82</td>
</tr>
<tr>
<td>0.4</td>
<td>0</td>
<td>1.24</td>
<td>1.79</td>
<td>+0.07</td>
</tr>
<tr>
<td></td>
<td>$\pi$</td>
<td>4.82</td>
<td>5.68</td>
<td>6.87</td>
</tr>
</tbody>
</table>
the stress function with the pole at the origin gave the best results. It is seen from the results that the convergence of the series is very much slowed as the distance between the inner circle and outer boundary is diminished. However, even in the case for which \( d = 0.4 \), the series solution apparently still converges.

**Three and four holes in a circular plate.** In a recent paper (7), H. Kraus used the classical approach to solve a number of problems involving three and four holes symmetrically arranged in a circular plate (Figs. 40 and 41). In each case, the holes were equally spaced around a circle whose radius was half that of the outer circular boundary of the plate. The loading consisted of uniform pressure in the holes. Kraus considered a number of cases in which the radii of the holes were varied from \( \frac{R}{20} \) to \( \frac{3R}{20} \) in the case of three holes, and from \( \frac{R}{20} \) to \( \frac{R}{4} \) in the case of four holes. (\( R \) is the radius of the plate.)

We considered one case of three holes and two cases involving four holes. In the case of three holes, the ratio of circle radius to plate radius was 0.20. In the two four-hole problems, the ratios considered were 0.2 and 0.25.

The stress function used for the three-hole problem was

\[
\phi = a_0 \log \rho + \sum_{m=1}^{\infty} \left( a_{3n} \rho^{3n} \cos 3n \theta + b_{3n} \rho^{3n+2} \cos 3n \phi \right) + a_1' \sum_{j=0}^{15} \sum_{n=1}^{2} \lambda_{1,j}^{-n} \cos n \left( \theta - \frac{3j \pi}{3} \right) + b_1' \sum_{n=2}^{n=\infty} \sum_{j=0}^{2} \lambda_{1,j}^{-n} \cos \left( \theta - \frac{3j \pi}{3} \right).
\] (4.6)
Fig. 40.--Three circular holes in a circular plate

Fig. 41.--Four circular holes in a circular plate
The stress function used for the four-hole problems was

\[ \phi = a_0 \log r + \sum_{n=1}^{L} \left( a_{n} \sum_{j=0}^{3} \sum_{i=0}^{3} n_{i,j} \cos n_{i,j} \cos \theta \right) + \sum_{n=1}^{L} \sum_{j=0}^{3} a_{n} \sum_{i=0}^{3} n_{i,j} \cos n_{i,j} \left( \theta_{i,j} - \frac{i \pi}{2} \right) + \sum_{n=1}^{L} b_{n} \sum_{j=0}^{3} n_{i,j} \cos n_{i,j} \left( \theta_{i,j} - \frac{i \pi}{2} \right) . \]  

The maximum residuals in the solutions with these stress functions were found to be

- For 3 holes, \( r_h/r_p = 0.20 \) and \( e_{\text{max}} = 2.6 \times 10^{-5} \).
- For 4 holes, \( r_h/r_p = 0.20 \) and \( e_{\text{max}} = 7.7 \times 10^{-6} \).
- For 4 holes, \( r_h/r_p = 0.25 \) and \( e_{\text{max}} = 1.7 \times 10^{-5} \).

The boundary stresses \( \tau_0/p \) on the hole boundaries and the exterior boundaries are given in Tables 11 through 13 as functions of \( \alpha = \pi - \theta_{1,0} \) for the holes and \( \alpha = \pi - \theta \) for the exterior boundaries. Table 12 gives the values of \( \tau_0/p \) on the hole boundary of the four-hole problem with \( r_h/r_p = 0.25 \) separately from the other two problems since the values of \( \alpha \) are slightly different for this problem.

The values calculated check as closely as may be measured with the graphs given by Kraus for the stresses on the hole boundaries. Kraus did not calculate stresses on the exterior boundaries.
TABLE 11.--Boundary values of $\psi / p$ on the hole boundaries for three and four holes in a circular plate with ratio of hole radius to exterior radius of 0.20

<table>
<thead>
<tr>
<th>Degrees</th>
<th>Minutes</th>
<th>$\psi / p$ 3 Hole</th>
<th>$\psi / p$ 4 Holes</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>00</td>
<td>1.338</td>
<td>1.155</td>
</tr>
<tr>
<td>9</td>
<td>56</td>
<td>1.349</td>
<td>1.214</td>
</tr>
<tr>
<td>19</td>
<td>57</td>
<td>1.370</td>
<td>1.368</td>
</tr>
<tr>
<td>29</td>
<td>32</td>
<td>1.375</td>
<td>1.546</td>
</tr>
<tr>
<td>39</td>
<td>39</td>
<td>1.347</td>
<td>1.678</td>
</tr>
<tr>
<td>50</td>
<td>35</td>
<td>1.284</td>
<td>1.686</td>
</tr>
<tr>
<td>60</td>
<td>00</td>
<td>1.224</td>
<td>1.578</td>
</tr>
<tr>
<td>70</td>
<td>07</td>
<td>1.179</td>
<td>1.401</td>
</tr>
<tr>
<td>80</td>
<td>13</td>
<td>1.168</td>
<td>1.241</td>
</tr>
<tr>
<td>90</td>
<td>00</td>
<td>1.189</td>
<td>1.150</td>
</tr>
<tr>
<td>99</td>
<td>47</td>
<td>1.230</td>
<td>1.136</td>
</tr>
<tr>
<td>109</td>
<td>53</td>
<td>1.281</td>
<td>1.185</td>
</tr>
<tr>
<td>120</td>
<td>00</td>
<td>1.329</td>
<td>1.269</td>
</tr>
<tr>
<td>129</td>
<td>48</td>
<td>1.365</td>
<td>1.359</td>
</tr>
<tr>
<td>140</td>
<td>21</td>
<td>1.390</td>
<td>1.447</td>
</tr>
<tr>
<td>150</td>
<td>28</td>
<td>1.403</td>
<td>1.515</td>
</tr>
<tr>
<td>160</td>
<td>03</td>
<td>1.407</td>
<td>1.561</td>
</tr>
<tr>
<td>170</td>
<td>04</td>
<td>1.408</td>
<td>1.589</td>
</tr>
<tr>
<td>180</td>
<td>00</td>
<td>1.408</td>
<td>1.598</td>
</tr>
</tbody>
</table>
TABLE 12.--Boundary values of $\sigma_0/p$ for four holes in a circular plate with ratio of hole radius to exterior boundary radius of 0.25

<table>
<thead>
<tr>
<th>Degrees</th>
<th>Minutes</th>
<th>$\sigma_0/p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>00</td>
<td>1.169</td>
</tr>
<tr>
<td>10</td>
<td>00</td>
<td>1.311</td>
</tr>
<tr>
<td>19</td>
<td>57</td>
<td>1.684</td>
</tr>
<tr>
<td>29</td>
<td>57</td>
<td>2.134</td>
</tr>
<tr>
<td>39</td>
<td>36</td>
<td>2.434</td>
</tr>
<tr>
<td>50</td>
<td>12</td>
<td>2.446</td>
</tr>
<tr>
<td>60</td>
<td>00</td>
<td>2.163</td>
</tr>
<tr>
<td>70</td>
<td>02</td>
<td>1.725</td>
</tr>
<tr>
<td>80</td>
<td>11</td>
<td>1.341</td>
</tr>
<tr>
<td>90</td>
<td>00</td>
<td>1.139</td>
</tr>
<tr>
<td>99</td>
<td>49</td>
<td>1.119</td>
</tr>
<tr>
<td>109</td>
<td>58</td>
<td>1.232</td>
</tr>
<tr>
<td>120</td>
<td>00</td>
<td>1.414</td>
</tr>
<tr>
<td>129</td>
<td>48</td>
<td>1.604</td>
</tr>
<tr>
<td>140</td>
<td>24</td>
<td>1.796</td>
</tr>
<tr>
<td>150</td>
<td>03</td>
<td>1.944</td>
</tr>
<tr>
<td>160</td>
<td>03</td>
<td>2.063</td>
</tr>
<tr>
<td>170</td>
<td>00</td>
<td>2.138</td>
</tr>
<tr>
<td>180</td>
<td>00</td>
<td>2.164</td>
</tr>
</tbody>
</table>
TABLE 13.--Boundary values of the $\sigma_\theta/p$ on the exterior boundary of problems involving three and four holes in a circular plate

<table>
<thead>
<tr>
<th>Degrees</th>
<th>Minutes</th>
<th>$r_h/r_p = 0.2$</th>
<th>3 Holes</th>
<th>$r_h/r_p = 0.2$</th>
<th>4 Holes</th>
</tr>
</thead>
<tbody>
<tr>
<td>120</td>
<td>00'</td>
<td>0.057</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>129</td>
<td>48</td>
<td>0.073</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>135</td>
<td>00</td>
<td>0.095</td>
<td>0.190</td>
<td>0.381</td>
<td></td>
</tr>
<tr>
<td>140</td>
<td>21</td>
<td>0.129</td>
<td>0.201</td>
<td>0.402</td>
<td></td>
</tr>
<tr>
<td>150</td>
<td>00</td>
<td>0.224</td>
<td>0.276</td>
<td>0.531</td>
<td></td>
</tr>
<tr>
<td>160</td>
<td>03</td>
<td>0.373</td>
<td>0.410</td>
<td>0.736</td>
<td></td>
</tr>
<tr>
<td>170</td>
<td>04</td>
<td>0.536</td>
<td>0.546</td>
<td>0.922</td>
<td></td>
</tr>
<tr>
<td>180°</td>
<td>00'</td>
<td>0.613</td>
<td>0.606</td>
<td>1.001</td>
<td></td>
</tr>
</tbody>
</table>
Seven holes in a circular plate. We considered two problems involving seven circular holes in a circular plate. In the first problem, all seven holes were pressurized. In the second problem, only the central hole was pressurized. The geometric arrangement of the holes was the same in the two problems (Fig. 42).

Fig. 42.—Seven circular holes in a circular plate

The radii of all of the holes are equal and are 1/5 the radius of the plate. The centers of the six eccentric holes are equally spaced around a circle having a radius equal to 3/5 the radius of the plate. It is thus seen that the minimum distances between the holes and between the holes and the boundary of the plate is equal to the radius of the hole.
The stress function used in each problem was

\[
\phi = a_0^* \log r + \sum_{n=1}^{3} \left( a_{in}^* r^n \cos \nu \theta + a_{in}^* r^n \cos (\nu \theta + \frac{\pi}{3}) \right) \\
+ \sum_{n=1}^{3} \left( b_{in}^* r^n \cos \nu \theta + b_{in}^* r^n \cos \nu \theta + \frac{\pi}{3} \right)
\]

The difference in the two problems is only in the boundary conditions in that the normal stress, \(\sigma_N/p\) on the boundaries of the eccentric holes, is set equal to -1.0 in the first problem and equal to 0 in the second problem.

The maximum residuals were found to be

all holes pressurized \(e_{\text{max}} = 5 \times 10^{-5}\)

center hole pressurized \(e_{\text{max}} = 8 \times 10^{-5}\).

The boundary values of \(\sigma_0/p\) on each of the boundaries are given in Tables 14 through 16 for the two problems as functions of \(\alpha = r - \theta\). The same stresses are plotted in Figs. 43 through 46.

The most interesting feature of these stresses is the distribution of the stresses on the eccentric holes. In the case for which all of the holes are pressurized, the hoop stress is
TABLE 14.—Stresses on the central hole of problems involving seven circular holes in a circular plate

<table>
<thead>
<tr>
<th>α</th>
<th>( \sigma_{\theta}/p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Degrees</td>
<td>Minutes</td>
</tr>
<tr>
<td>150</td>
<td>28</td>
</tr>
<tr>
<td>160</td>
<td>03</td>
</tr>
<tr>
<td>170</td>
<td>04</td>
</tr>
<tr>
<td>180</td>
<td>00</td>
</tr>
</tbody>
</table>

TABLE 15.—Stresses on the exterior boundary of problems involving seven circular holes in a circular plate

<table>
<thead>
<tr>
<th>α</th>
<th>( \tau_\theta/p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Degrees</td>
<td>Minutes</td>
</tr>
<tr>
<td>150</td>
<td>00</td>
</tr>
<tr>
<td>160</td>
<td>03</td>
</tr>
<tr>
<td>170</td>
<td>04</td>
</tr>
<tr>
<td>180</td>
<td>00</td>
</tr>
<tr>
<td>α</td>
<td>θφ/p</td>
</tr>
<tr>
<td>--------</td>
<td>------</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>0 00</td>
<td>2.113</td>
</tr>
<tr>
<td>9 56</td>
<td>2.006</td>
</tr>
<tr>
<td>19 57</td>
<td>1.788</td>
</tr>
<tr>
<td>29 32</td>
<td>1.681</td>
</tr>
<tr>
<td>39 39</td>
<td>1.788</td>
</tr>
<tr>
<td>50 12</td>
<td>2.032</td>
</tr>
<tr>
<td>60 00</td>
<td>2.159</td>
</tr>
<tr>
<td>70 07</td>
<td>2.054</td>
</tr>
<tr>
<td>80 13</td>
<td>1.759</td>
</tr>
<tr>
<td>90 00</td>
<td>1.452</td>
</tr>
<tr>
<td>99 47</td>
<td>1.265</td>
</tr>
<tr>
<td>109 53</td>
<td>1.250</td>
</tr>
<tr>
<td>120 00</td>
<td>1.380</td>
</tr>
<tr>
<td>129 48</td>
<td>1.580</td>
</tr>
<tr>
<td>140 21</td>
<td>1.810</td>
</tr>
<tr>
<td>150 28</td>
<td>2.006</td>
</tr>
<tr>
<td>160 03</td>
<td>2.150</td>
</tr>
<tr>
<td>170 04</td>
<td>2.245</td>
</tr>
<tr>
<td>180 00</td>
<td>2.278</td>
</tr>
</tbody>
</table>
Fig. 43.--Boundary values of $\sigma_0/p$ on the center hole and exterior boundary of a circular plate with seven pressurized holes.
Fig. 44.--Boundary values of $\sigma_{\theta}/p$ on the center hole and exterior boundary of a circular plate with seven holes with pressure in the central hole only.
Fig. 45.--Boundary values of $\sigma_\theta/p$ on eccentric holes of a circular plate with seven pressurized holes
Fig. 46.—Boundary values of $\sigma_\theta/p$ on eccentric hole in a circular plate with seven holes: pressure in the central hole only
everywhere tensile and has maxima at each point of minimum section. In the case for which the center hole only is pressurized the hoop stress on the eccentric holes falls off rapidly from the point nearest the center hole and actually becomes compressive along part of the arc. As far as is known, these solutions have not been obtained before.

Six circular holes in a finite circular plate loaded by six concentrated loads on the external boundary. Buivol (8) conducted a series of photoelastic measurements of the stresses in circular plates having six symmetrically located circular holes and loaded by a pair of diametrically opposed concentrated loads. He considered a number of plates having several different sized holes and web thicknesses between the hole boundaries and the outside boundary of the plate. He also considered two different locations for the points of application of the loads. In the one case, the loads were applied at the point of minimum thickness of the web and in the other case the loads were applied halfway between the holes. The points of application for the two cases are the points $A_0, A_3$ and $B_0, B_3$ of Fig. 47.

The stress distributions for the problems involving six concentrated loads equally spaced around the circle ($A_0$ through $A_5$ or $B_0$ through $B_5$ of Fig. 47) are obtained from the corresponding
stress distributions involving two loads by adding the stresses found for two loads to the same stress distributions obtained after each of two successive rotations of 120°. In other words, the stress distribution for six concentrated loads is simply the sum of the stress distributions for each of three pairs of concentrated loads.

Buivol made this calculation and shows curves of the boundary values of $\sigma /\rho$ for six concentrated loads at the points $A_0$ through $A_5$ or $B_0$ through $B_5$ in his paper.
We attempted to obtain the solutions for three problems for which the experimental values of the stresses were reported by Buivol in his paper (8). In a subsequent paper (9), Buivol attempted a theoretical analysis by the classical approach of the stresses in one plate with a slightly different spacing and compared his theoretical solution with some new experimental measurements. We also solved this problem with our computing program.

The essential geometrical features of these four problems are as follows:

Problem 1. The loads were applied at the points $B_j$; the radius of the holes was $1/5$ of the radius of the plate; the centers of the holes lay on a circle having a radius $13/20$ the radius of the plate.

Problem 2. The loads were applied at $B_j$; the radius of the holes was $1/4$ of the radius of the plate; the centers of the holes lay on a circle having a radius $7/10$ the radius of the plate.

Problem 3. Identical to Problem 1 except that the loads were applied at the points $A_j$ instead of $B_j$.

Problem 4. The loads were applied at $A_j$; the radius of the holes was $2/10$ of the radius of the plate; the centers of the holes lay on a circle having a radius $6/10$ the radius of the plate. This is the problem for which Buivol attempted his theoretical solution.
It is noted that in Problem 4, the minimum distance between the holes and the outside boundary was equal to the hole radius. In Problems 2 and 3, this distance was $3/4$ the hole radius, while in Problem 2, it was only $1/5$ the hole radius.

In our discussion of the problems involving an eccentric hole in a circular plate, we remarked that the minimum web thickness relative to the hole radius had a considerable influence on the rate of convergence of the series solution. On this basis, it would be expected that the series solution to Problem 2 would show slower convergence than the series in the solution to Problems 1 and 3. This did occur.

The stress function used in Problem 1 was

$$
\phi = b_0 r^2 + \sum_{n=1}^{3} \left( a_n r^n \cos \theta + b_n r^{n+2} \cos 2\theta \right)
+ a_0 \sum_{j=0}^{5} \log \sqrt{r_j^2 + \sum_{j=0}^{5} \left( a_n r^{n-2} \cos \theta + b_n r^{n+2} \cos 2\theta \right)}
+ \sum_{j=0}^{5} b_n \sum_{j=0}^{5} \log \sqrt{r_j^2 + \sum_{j=0}^{5} \left( a_n r^{n-2} \cos \theta + b_n r^{n+2} \cos 2\theta \right)}
+ \frac{1}{r} \sum_{j=0}^{5} \left[ \theta_{2,j} - (2j+1) \frac{\pi}{6} \right] \sin \left[ \theta_j - (2j+1) \frac{\pi}{6} \right].
$$

The coordinates $(r, \theta)$ are measured from the origin, the coordinates $(r_1, \theta_1, \theta_1, \theta_1)$ are measured from the centers of the holes and the coordinates $(r_2, \theta_2, \theta_2, \theta_2)$ are measured from the points $B_j$. The last summation of (4.9) represents the main effects of six concentrated loads.
For the stress function used in Problem 2, the upper limit on \( n \) was increased from 3 to 5 for the sums of (4.9) involving \( a_6^0 \) and \( b_6^0 \) while the upper limits on \( n \) in the sums involving \( a_{-n}^' \) and \( b_{-n}^' \) were increased from 15 to 24.

The stress function used for Problems 3 and 4 was

\[
\phi = b_2 r^2 + \sum_{n=1}^{5} \left( a_{6n}^o \frac{r^{6n} \cos n \theta}{16} + b_{6n}^o \frac{r^{6n+2} \cos n \theta}{16} \right)
\]

\[
+ a_{12}^' \sum_{j=0}^{5} \sum_{i=0}^{5} \frac{\lambda_{i,j}^2}{16} \cos n \theta_{ij} \left( \theta_{ij} - \frac{j \pi}{3} \right)
\]

\[
+ \sum_{n=2}^{15} \sum_{j=0}^{5} \frac{b_{2n}^o}{16} \frac{r^{2n} \cos n \theta_{ij} \left( \theta_{ij} - \frac{j \pi}{3} \right)}{16} \sin \left( \frac{2n}{3} \right) \sin \left( \frac{2n}{3} \right),
\]

where the coordinates \((r_{2,j}, \theta_{2,j})\) are measured from the points \( A_j \) of Fig. 47. [The extra term \( \pi/6 \) in the angles \( \theta_{2,j} - (2j+1)\pi/6 \) of (4.5) is the angle \( \beta \) discussed in Chapter 2. In Problems 3 and 4 \( \beta = 0 \) since \( A_0 \) lies on the x-axis.]

The maximum residuals found for the three problems were

- Problem 1: \( e_{\text{max}} = 1 \times 10^{-4} \)
- Problem 2: \( e_{\text{max}} = 2 \times 10^{-2} \)
- Problem 3: \( e_{\text{max}} = 6 \times 10^{-3} \)
- Problem 4: \( e_{\text{max}} = 5 \times 10^{-4} \)

The boundary values of \( \sigma_{\theta} R/\pi \) on the hole boundaries calculated for the four problems are given in Table 17 as functions of
\( \alpha = \pi - \theta_{1,0} \). The factor \( R \), which is the radius of the plate, is necessary in this case since the boundary loads are concentrated forces rather than distributed loads.

These stresses are plotted in Figs. 48 through 51 along with the experimental curves obtained by Buivol. Figure 51 also shows the theoretical results calculated by Buivol for this case. It should be noted that all of the results reported by Buivol were given only in the form of curves plotted on the circumference of the circle. These are similar to the lower figure that we have been showing in each plot of the boundary stresses. This type of graph gives a very clear qualitative picture of the stress distribution on the boundary. However, it is both difficult to plot and to interpret with any accuracy. Thus, we do not know just how accurately we have plotted Buivol's results, although we measured the curves given by Buivol as carefully as we could. The experimental curves are reported as the dotted lines of the figures.

It is noted that the stresses are reported for the case of concentrated tensile forces so that the values of the boundary stresses would be mainly tensile. This aided in plotting the curves on the circle arcs in the lower graphs in each case.

It may be seen that the results are in good qualitative agreement for Problems 1 and 2. However, the Buivol's results for Problems 3 and 4 show a region of compressive stresses on the
TABLE 17.--Boundary values of $R \sigma_p / p$ for problems involving six circular holes in a circular plate loaded by six concentrated tensile forces

<table>
<thead>
<tr>
<th>Degrees</th>
<th>Minutes</th>
<th>$\sigma_p / p$ Problem 1</th>
<th>$\sigma_p / p$ Problem 2</th>
<th>$\sigma_p / p$ Problem 3</th>
<th>$\sigma_p / p$ Problem 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>00</td>
<td>2.393</td>
<td>2.630</td>
<td>1.151</td>
<td>1.252</td>
</tr>
<tr>
<td>9</td>
<td>56</td>
<td>2.484</td>
<td>2.782</td>
<td>1.191</td>
<td>1.309</td>
</tr>
<tr>
<td>19</td>
<td>57</td>
<td>2.754</td>
<td>3.228</td>
<td>1.301</td>
<td>1.479</td>
</tr>
<tr>
<td>29</td>
<td>32</td>
<td>3.148</td>
<td>3.924</td>
<td>1.447</td>
<td>1.717</td>
</tr>
<tr>
<td>39</td>
<td>39</td>
<td>3.629</td>
<td>4.697</td>
<td>1.584</td>
<td>1.982</td>
</tr>
<tr>
<td>50</td>
<td>12</td>
<td>4.060</td>
<td>5.426</td>
<td>1.634</td>
<td>2.156</td>
</tr>
<tr>
<td>60</td>
<td>00</td>
<td>4.198</td>
<td>5.754</td>
<td>1.593</td>
<td>2.134</td>
</tr>
<tr>
<td>70</td>
<td>07</td>
<td>3.830</td>
<td>5.440</td>
<td>1.636</td>
<td>2.004</td>
</tr>
<tr>
<td>80</td>
<td>13</td>
<td>2.754</td>
<td>4.041</td>
<td>2.105</td>
<td>2.072</td>
</tr>
<tr>
<td>90</td>
<td>00</td>
<td>1.128</td>
<td>1.591</td>
<td>3.214</td>
<td>2.668</td>
</tr>
<tr>
<td>99</td>
<td>47</td>
<td>-0.566</td>
<td>-0.975</td>
<td>4.920</td>
<td>3.841</td>
</tr>
<tr>
<td>109</td>
<td>53</td>
<td>-1.638</td>
<td>-2.297</td>
<td>6.955</td>
<td>5.369</td>
</tr>
<tr>
<td>120</td>
<td>00</td>
<td>-1.583</td>
<td>-2.129</td>
<td>8.802</td>
<td>6.745</td>
</tr>
<tr>
<td>129</td>
<td>48</td>
<td>-0.623</td>
<td>-1.514</td>
<td>9.909</td>
<td>7.459</td>
</tr>
<tr>
<td>140</td>
<td>21</td>
<td>0.866</td>
<td>-0.788</td>
<td>9.716</td>
<td>7.056</td>
</tr>
<tr>
<td>150</td>
<td>28</td>
<td>2.325</td>
<td>-0.339</td>
<td>7.373</td>
<td>5.082</td>
</tr>
<tr>
<td>160</td>
<td>03</td>
<td>3.533</td>
<td>2.302</td>
<td>2.550</td>
<td>1.685</td>
</tr>
<tr>
<td>170</td>
<td>04</td>
<td>4.423</td>
<td>4.355</td>
<td>-4.292</td>
<td>-2.395</td>
</tr>
<tr>
<td>180</td>
<td>00</td>
<td>4.745</td>
<td>5.236</td>
<td>-7.957</td>
<td>-4.319</td>
</tr>
</tbody>
</table>
Fig. 48.--Theoretical and experimental values of $R \theta / p$ on the hole boundaries of a circular plate having six holes and loaded by six concentrated forces; Problem 1
Fig. 49. -- Theoretical and experimental values of $R \sigma_\theta / p$ on the hole boundaries of a circular plate having six holes and loaded by six concentrated forces; Problem 2
Fig. 50.—Theoretical and experimental values of $R\sigma_0/p$ on the hole boundaries of a circular plate having six holes and loaded by six concentrated forces; Problem 3
Fig. 51. -- Theoretical and experimental values of $R \frac{\sigma_\theta}{p}$ on the hole boundaries of a circular plate having six holes and loaded by six concentrated forces; Problem 4
hole opposite the point of application of the load, and our theoretical solution does not show this. We have not found any reason for this discrepancy. On the basis of the size of the residuals obtained for the calculated solutions, it is felt that these solutions should have an accuracy of better than 1 per cent. It may be noted that a number of other stress functions were tried for Problem 3 involving perturbation of the pole positions and that these solutions all give stresses in essential agreement with solution reported here.

Problems involving noncircular holes

It was originally our intention to try to solve a variety of problems involving noncircular holes using the polar stress functions developed in Chapter 2. Actually, only a modest beginning was made in this direction. The problems that we have studied will be discussed in this section.

The pressurized elliptical hole in the infinite plate. Problems involving a single elliptical hole in the infinite plate seemed particularly well suited for a beginning study of the noncircular holes. This problem has a closed form solution in elliptical coordinates but not in polar coordinates.
We considered ellipses with the three ratios of major to minor axes of 1.1, 2.0, and 4.0. As would be expected, the difficulty of obtaining an accurate solution goes up with the eccentricity. The ellipse with an eccentricity of 1.1 was easily solved with a stress function having a single pole at the origin. In the case of the 2 to 1 ellipse a stress function series with a pole at the origin had marginal convergence while a trial stress function series with a single pole at the origin clearly diverged in the case of the 4 to 1 ellipse. The investigation of these latter two problems consisted of trying stress functions having various numbers of singularities located at various positions in the holes. These ranged from stress functions with a singularity at each focus to stress functions with twenty singularities on the x-axis between the foci. (The major axis of each ellipse was taken along the x-axis.) Stress functions were tried also having singularities not on the x-axis. These were unsatisfactory. However, for the stress functions with strings of singularities on the x-axis there did not appear to be any sharply defined optimum set of singularities. As an example, for the case of the 2 to 1 ellipse, the stress functions (4.11) and (4.12) below both
gave boundary residuals of about $2 \times 10^{-3}$ and values of $\phi_0/p$ everywhere on the boundary of the ellipse to within an error of less than 1 per cent of the maximum value of $\phi_0/p$:

$$\phi = a_0 \log r + \sum_{n=1}^{5} \left( a_{2n} r^{-2n \theta} + b_{2n} r^{-2n+2} \cos 2n \theta \right)$$

$$+ a_i \sum_{j=0}^{1} \log r_{i,j} + a_i \sum_{j=0}^{1} \eta_{i,j} \left( \theta_{i,j} - j \pi \right) \sin \left( \theta_{i,j} - j \pi \right)$$

$$+ b_i \sum_{j=0}^{1} \eta_{i,j} \log \eta_{i,j} \cos \left( \theta_{i,j} - j \pi \right)$$

$$+ \sum_{n=1}^{5} a_n \sum_{j=0}^{1} \eta_{i,j}^{n-1} \cos \left( \theta_{i,j} - j \pi \right)$$

$$+ \sum_{n=2}^{5} b_n \sum_{j=0}^{1} \eta_{i,j}^{n} \cos \left( \theta_{i,j} - j \pi \right),$$

where $(r, \theta)$ is measured from the origin and $(r_{1,0}, \theta_{1,0})$ is measured from the point $(1.6, 0)$ and $(r_{1,1}, \theta_{1,1})$ is measured from $(-1.6, 0)$;

$$\phi = \sum_{i=1}^{5} a_i \sum_{j=0}^{1} \log r_{i,j} + \sum_{i=1}^{5} a_i \sum_{j=0}^{1} \eta_{i,j} \left( \theta_{i,j} - j \pi \right) \sin \left( \theta_{i,j} - j \pi \right)$$

$$+ \sum_{i=1}^{5} b_i \sum_{j=0}^{1} \eta_{i,j} \log \eta_{i,j} \cos \left( \theta_{i,j} - j \pi \right)$$

$$+ \sum_{i=1}^{5} a_i \sum_{j=0}^{1} \eta_{i,j}^{n-1} \cos \left( \theta_{i,j} - j \pi \right),$$

where the coordinates $(r_{i,0}, \theta_{i,0}) 1 = 1, \ldots, 5$ are measured from the points $(0.2, 0), (0.6, 0), (1.0, 0), (1.4, 0)$, and $(1.6, 0)$,
respectively, and the points \((r_{1,1}', \theta_{1,1}')\), \(i = 1, \cdots, 5\), are measured from the image points of \((r_{1,0}', \theta_{1,0}')\) in the y-axis.

Both of these stress functions have twenty unknown coefficients. The stress function (4.11) has only three singularities but has higher order terms than the stress function (4.12) which has ten singularities. It should be noted that the terms involving \(a_{1}^{1}\) and \(b_{-1}^{1}\) (which are actually terms of the function \(P \equiv \text{Chapter 2}\) are permissible functions in this case, since the multivalued parts of the respective terms for \(j = 0\) and \(j = 1\) cancel each other.

In the case of the \(4:1\) ellipse, the best solution found had ten singularities on the x-axis. The stress function used was of the form:

\[
\phi = \sum_{i=1}^{5} a_{o}^{i} \sum_{j=0}^{1} \log \chi_{i,j} + \sum_{i=1}^{5} a_{o}^{i} \sum_{j=0}^{1} \chi_{i,j} \left( \theta_{i,j} - j \pi \right) \sin \left( \theta_{i,j} - j \pi \right) \\
+ \sum_{i=1}^{5} b_{o}^{i} \sum_{j=0}^{1} \chi_{i,j} \left( \theta_{i,j} - j \pi \right) \cos \left( \theta_{i,j} - j \pi \right) \\
+ \sum_{n=1}^{3} \sum_{l=1}^{2} a_{n}^{i} \sum_{j=0}^{1} \chi_{n,j}^{l} \cos n \left( \theta_{i,j} - j \pi \right) \\
+ \sum_{n=1}^{3} \sum_{l=1}^{2} b_{n}^{i} \sum_{j=0}^{1} \chi_{n,j}^{l} \cos n \left( \theta_{i,j} - j \pi \right)
\]

(4.13)

where the coordinates \((r_{1,j}', \theta_{1,j}')\), \(j = 0,1\); \(i = 1,2,\cdots, 5\) are measured from the points \((\pm 0.6,0), (\pm 1.6,0), (\pm 2.6,0), (\pm 3.35,0), \ldots \).
and ($\pm 3.873,0$). The + sign is taken for $j = 0$ and the - sign for $j = 1$. This stress function gave a maximum boundary residual of 0.01 and gave the boundary values of $\sigma_\theta / p$ to an absolute accuracy of 0.02 which is about 0.3 per cent of the maximum stress of 7.0.

A pressurized star-shaped hole in the infinite plate. In a recent paper (10), Wilson calculated the stresses in the infinite plate with a pressurized star-shaped hole for which the portion of the boundary lying in the basic symmetry element is shown in Fig. 52.

![Fig. 52.--The star-shaped hole](image)
The stress function used for this problem was identical in form to (4.13). However, in this problem the coordinates 
\((r_i, \theta_j, \theta_{i,j})\), \(j = 0,1; i = 1,2, \cdots 5\) were measured from the points 
\((\pm 0.28,0), (\pm 0.5,0), (\pm 0.68,0), (\pm 0.8,0), (\pm 0.915,0)\). The maximum boundary residual was found to be 0.01. The boundary values of \(\varphi / p\) check as closely as could be determined with the curves given by Wilson, although a complete comparison was difficult, since Wilson plotted the stresses against some (unspecified) arc coordinate.

Some values of \(\varphi / p\) obtained by our program are listed in Table 18 as a function of \(x\).

<table>
<thead>
<tr>
<th>(x)</th>
<th>(\sigma_\theta / p)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.30</td>
<td>-1.174</td>
</tr>
<tr>
<td>0.40</td>
<td>-1.175</td>
</tr>
<tr>
<td>0.50</td>
<td>-1.176</td>
</tr>
<tr>
<td>0.60</td>
<td>-1.173</td>
</tr>
<tr>
<td>0.70</td>
<td>-1.158</td>
</tr>
<tr>
<td>0.80</td>
<td>-1.121</td>
</tr>
<tr>
<td>0.85</td>
<td>-0.960</td>
</tr>
<tr>
<td>0.904</td>
<td>-0.501</td>
</tr>
<tr>
<td>0.929</td>
<td>+0.490</td>
</tr>
<tr>
<td>0.957</td>
<td>+2.219</td>
</tr>
<tr>
<td>0.969</td>
<td>+3.011</td>
</tr>
<tr>
<td>0.980</td>
<td>+3.870</td>
</tr>
<tr>
<td>0.9886</td>
<td>+4.584</td>
</tr>
<tr>
<td>0.9950</td>
<td>+5.138</td>
</tr>
<tr>
<td>1.0000</td>
<td>+5.584</td>
</tr>
</tbody>
</table>
Actually, it would appear from the first entries in this table that the last digit of these stresses is uncertain. No attempt was made to obtain a more exact solution.

The infinite plate with a hole formed by two overlapping circles with internal pressure. This problem has an exact solution in bipolar coordinates. We considered one case where the distance between the circle centers was equal to the hole radius. Thus, the angle between the radius lines to the points of intersection and the line joining the centers was 60° (Fig. 53).

Fig. 53.--The hole formed by two intersecting circles
The stress function used to solve this problem was

\[
\phi = a_c \sum_{j=0}^{l} A_j \log r_j + d_j \sum_{j=c}^{l} \left( \theta_j - \frac{\pi}{2} \right) \sin(\theta_j - \frac{\pi}{2})
\]

\[
+ b \sum_{j=c}^{l} \log r_j \cos(\theta_j - \frac{\pi}{2})
\]

\[
+ \sum_{n=1}^{N} a_n \sum_{j=0}^{l} \frac{r_j^{-n}}{n} \cos n(\theta_j - \frac{\pi}{2})
\]

\[
+ \sum_{n=2}^{N} b_n \sum_{j=0}^{l} \frac{r_j^{-n}}{n} \cos n(\theta_j - \frac{\pi}{2})
\]

where \((r_0, \theta_0)\) is measured from the point \((0.5, 0)\) and \((r_1, \theta_1)\) is measured from the point \((-0.5, 0)\).

Except at the point of intersection of the two circles, the maximum boundary residual was \(2 \times 10^{-3}\). At the point of intersection, the calculated shear was 0.06.

Ling (3) gave the value of \(\sigma_0 / \mu\) on the boundary at the x-axis as \(1.524\). We obtained a value of 1.49 which is a discrepancy of 2 per cent. No attempt was made to obtain a more accurate solution although it did appear that the series solution was converging satisfactorily.

It would appear from the results of this section that it is possible to obtain approximate solutions to problems involving noncircular holes by using the stress-function series in polar form but with multiple poles in each hole. However, the solutions obtained are not as accurate as the solutions that we have discussed before for problems involving circular holes.
One of the features of the results of this section is that the stress function in each case required a considerable number (20 to 40) terms in order to obtain the solutions. For problems involving a number of noncircular holes the approach used here might easily require a stress function so large that the matrix equation would not fit in the 32,000 word magnetic core storage of the IBM 7090. This would require using magnetic tape and take considerably more machine time.

Because of this, it may actually be desirable to write stress functions in coordinate systems that have coordinate lines more nearly corresponding to the boundaries of the noncircular holes. It should not be necessary that the coordinate lines coincide with the hole boundaries. On the basis of the results obtained so far it is felt some differences between the coordinate lines of the stress function and the boundary curves can be tolerated without seriously slowing the rate of convergence of the stress function. Of course, the use of the extended point matching method eliminates the necessity of having boundary and coordinate curves coincide in order that simple algebraic relations be obtained between the coefficients of the stress function and the boundary conditions.

It should be noted that in order to set up the stress function in some coordinate system other than the polar coordinate system, it would be necessary to perform a considerable amount
of algebraic manipulation to obtain the terms of the stress function, and the derivatives of these terms, in the new coordinate system. It would then be necessary to modify certain subroutines of the computer program to conform to this new stress function. At this point, it seems to be a wide open question as to whether it is more desirable to stick with the polar stress function used in this dissertation or to derive new stress functions more appropriate to a given problem. It is felt that a considerable amount of research will be needed to resolve this question. Fortunately, a majority of the practical problems involving multiply connected regions have circular, or nearly circular, holes for which the polar form of the stress function used here should be adequate.

Problems involving infinite rows of holes

Problems involving an infinite row of holes in an infinite plate were among the first problems to be solved for multiply connected regions. It was mentioned earlier that Asaba obtained the solution to the row of holes problem in 1927 and Howland obtained the solution in 1934. We wrote into the computer program the capability of setting up an approximation to the stress function (2.96). This approximation consisted of taking the sums on j over
a finite interval instead of the infinite interval. The stress function used for the single row of circular holes on the x-axis was

\[ \phi = a_0 \sum_{j=-M}^{M} \log r_j + \sum_{n=1}^{9} a_n \sum_{j=-M}^{M} r_j^{-2n} \cos 2 \theta_j + \sum_{n=1}^{9} b_{2n} \sum_{j=-M}^{M} r_j^{-2n+2} \cos 3 \theta_j. \]  

(4.15)

It was hoped that M could be taken large enough so that each of the sums would represent an accurate approximation to the infinite sums. Actually, the sums \( a_0 \sum \log r_j \) and \( b_{-2} \sum \cos 2 \theta_j \) are divergent, but in solving problems involving stresses, only the second derivatives of (4.15) are considered. The second derivatives of \( \sum \log r_j \) and \( \sum \cos 2 \theta_j \) converge as \( \frac{1}{r_j^2} \). It was found that the sums for these two terms converged so slowly that it would require an excessive amount of machine time to make M large enough so that these two sums were accurately calculated.

In order to overcome this difficulty, the program was modified so that each sum was continued until the terms became smaller than some designated small value. Thus, the series sums involving higher powers of \( \frac{1}{r} \) are truncated after only a few terms while the sums involving the lower order terms in \( \frac{1}{r} \) may actually be carried out over a thousand or more terms.

Two problems were solved in which the sums were carried out until the individual terms were less than \( 1 \times 10^{-6} \). The first problem was a repeat of Howland's problem for a single row of holes.
The spacing between hole centers was four times the hole radius. The boundary condition was taken to be uniform pressure in the holes. It was mentioned earlier that the stress distribution for biaxial tension at infinity is obtained from this case by superimposing on it a hydrostatic tension.

The residual in the boundary conditions on the hole was $4 \times 10^{-5}$. The calculation took 15 minutes to complete on the 7090. However, the accuracy of the solution depends also on the accuracy with which the symmetry condition is met on the line midway between the holes. This symmetry condition depends on the accuracy with which the sums on $j$ are calculated. In order to check the accuracy of these sums one boundary equation was calculated in which the sums were run until the terms were less than $10^{-8}$. It took ten minutes on the 7090 to evaluate the sums of this one boundary equation. Since we were using 36 boundary conditions, it would have taken six hours to complete the calculation of the entire matrix for the system and about 12 hours to complete the solution of the problem. This is clearly too long. However, a comparison of the sums calculated in the one equation with the corresponding sums truncated at terms of $1 \times 10^{-6}$ showed that the two sums involving derivatives of $\log r_k$ and $\cos \theta_k$ were both in error by $2.5 \times 10^{-4}$ times the values of the terms. The remainder of the terms were accurate to from six to eight significant digits. From the way that the terms were calculated, it is felt that the errors
in the sums of the low order terms would be about the same order of magnitude in the remainder of the equations. It is difficult to tell exactly what effect the errors in these two terms have on the accuracy of the solution. However, it is felt that the error should be on the order of 1 per cent or less. The boundary values of the stresses $\sigma_0/p$ calculated are given in Table 19 along with Howland's results (11) for the same problem.

TABLE 19.--Boundary values of $\sigma_0/p$ for an infinite row of pressurized holes as calculated by the computing program and given by Howland

<table>
<thead>
<tr>
<th>Degrees</th>
<th>Minutes</th>
<th>$\sigma_0/p$</th>
<th>Degrees</th>
<th>$\sigma_0/p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>00</td>
<td>0.53</td>
<td>0</td>
<td>0.55</td>
</tr>
<tr>
<td>14</td>
<td>04</td>
<td>0.58</td>
<td>15</td>
<td>0.59</td>
</tr>
<tr>
<td>29</td>
<td>33</td>
<td>0.77</td>
<td>30</td>
<td>0.77</td>
</tr>
<tr>
<td>44</td>
<td>46</td>
<td>1.07</td>
<td>45</td>
<td>1.08</td>
</tr>
<tr>
<td>60</td>
<td>00</td>
<td>1.43</td>
<td>60</td>
<td>1.43</td>
</tr>
<tr>
<td>74</td>
<td>56</td>
<td>1.73</td>
<td>75</td>
<td>1.73</td>
</tr>
<tr>
<td>90</td>
<td>00</td>
<td>1.85</td>
<td>90</td>
<td>1.85</td>
</tr>
</tbody>
</table>

When allowance is made for the slight differences in the angular positions, it is seen that the solutions agree except at lower values of $\theta$ where Howland's solution is slightly higher than the solution we obtained.
The second problem involved two infinite rows of pressurized holes shown in Fig. 54.

![Diagram of two infinite rows of holes](image)

Fig. 54.—Two infinite rows of holes

The spacing between the lines through the centers of the two rows of circles was three times the hole radius. The spacing between the hole centers in each row was also three times the radius.

The stress function for this problem was taken as

\[
\phi = \sum_{k=0}^{1} \sum_{j} \log \sqrt{\pi} \sum_{h=1}^{\infty} \sum_{k=0}^{\infty} \frac{a_{h}}{k} \cos n(\theta_{k,j} - \kappa n) + \frac{\kappa^{2}}{\kappa^{1}} \cos n(\theta_{k,j} - \kappa n) \]  

(4.16)
where the coordinates \((r_{o,j}, \theta_{o,j})\) are measured from the center of the \(j\)-th circle in the row along the line \(x = +1.5 \, r_o\), and the coordinates \((r_{1,j}, \theta_{1,j})\) are measured from the center of the \(j\)-th circle on the line \(x = -1.5 \, r_o\). The argument \(\theta_{k,j} - k \, r\) of the cosine terms was introduced to make the stress function symmetric about the \(y\)-axis. The indicated range of the sums on \(j\) is left indefinite since the sums were carried out until the individual terms in each series were less than \(1 \times 10^{-6}\). The problem took 32 minutes to calculate on the 7090. The maximum residual in the boundary conditions on the holes was \(4 \times 10^{-5}\). Once again, the accuracy of the solution will depend on the accuracy with which the sums were evaluated. We did not run a check on these sums as we did for the single row problem. However, it is felt that the accuracy of the solution should still be on the order of \(1\) per cent. The boundary values of \(\sigma_\theta /p\) are given in Table 20 as a function of \(\alpha = \pi - \theta\) on the circle with center at \((1.5 \, r_o, 0)\). These stresses are also plotted in Fig. 55.

The results obtained for the above two problems are somewhat disappointing in that the machine time necessary to carry out the calculations was felt to be too long. We had hoped to be able to calculate the stresses for problems involving four and even six rows of holes. These problems would have taken approximately four and nine hours, respectively, to compute on the IBM 7090 to the same accuracy that was obtained for the problem involving two rows of holes.
TABLE 20.—Boundary values of $\gamma/\rho$
for two rows of pressurized holes in the infinite plate

<table>
<thead>
<tr>
<th>Degrees</th>
<th>Minutes</th>
<th>$\gamma/\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>00</td>
<td>0.81</td>
</tr>
<tr>
<td>9</td>
<td>56</td>
<td>0.72</td>
</tr>
<tr>
<td>19</td>
<td>57</td>
<td>0.55</td>
</tr>
<tr>
<td>29</td>
<td>32</td>
<td>0.49</td>
</tr>
<tr>
<td>39</td>
<td>39</td>
<td>0.70</td>
</tr>
<tr>
<td>50</td>
<td>12</td>
<td>1.20</td>
</tr>
<tr>
<td>60</td>
<td>00</td>
<td>1.76</td>
</tr>
<tr>
<td>70</td>
<td>07</td>
<td>2.32</td>
</tr>
<tr>
<td>80</td>
<td>13</td>
<td>2.70</td>
</tr>
<tr>
<td>90</td>
<td>00</td>
<td>2.83</td>
</tr>
<tr>
<td>99</td>
<td>47</td>
<td>2.71</td>
</tr>
<tr>
<td>109</td>
<td>53</td>
<td>2.39</td>
</tr>
<tr>
<td>120</td>
<td>00</td>
<td>1.91</td>
</tr>
<tr>
<td>129</td>
<td>48</td>
<td>1.46</td>
</tr>
<tr>
<td>140</td>
<td>21</td>
<td>1.04</td>
</tr>
<tr>
<td>150</td>
<td>28</td>
<td>0.73</td>
</tr>
<tr>
<td>160</td>
<td>03</td>
<td>0.52</td>
</tr>
<tr>
<td>170</td>
<td>04</td>
<td>0.40</td>
</tr>
<tr>
<td>180</td>
<td>00</td>
<td>0.36</td>
</tr>
</tbody>
</table>
Fig. 55.--Boundary values of $\sigma_\theta / p$ for two infinite rows of pressurized holes.
What is clearly needed is a means of improving the rate of convergence of the series involving the low order terms of the stress function. If this can be done, it may still be possible to carry out the desired calculations for multiple rows of holes.

It should be pointed out that, once again, the difficulties met here had to do with constructing the stress function, not in meeting the boundary conditions. One other way that this problem might be solved would be to set up a series solution expanded about the center or centers of the hole or holes in the basic symmetry element and to force this stress function to satisfy symmetry conditions at a finite number of points of the symmetry line midway between the holes. This approach would give a stress function analogous that derived by Howland. Such an approach would be subject to the kind of restrictions on the convergence that Howland found.

The question of the best approach to solving the problems involving rows of holes using the point matching technique will require some further study.

**Torsion problems**

It was mentioned in Chapter 2 that the computing program could be used in solving boundary problems involving the harmonic differential equation. The program was used to solve three torsion
problems. We used the warping function $\psi$ which satisfies the harmonic equation $V^2 \psi = 0$ and the boundary conditions

$$
\left( \frac{\partial \psi}{\partial x} - \gamma \right) \cos (\mu, x) + \left( \frac{\partial \psi}{\partial y} + x \right) \cos (\mu, y) = 0.
$$

(4.17)

The shear stresses are given by

$$
\tau_{xz} = G \theta \left( \frac{\partial \psi}{\partial x} - \gamma \right), \quad \tau_{yz} = G \theta \left( \frac{\partial \psi}{\partial y} + x \right),
$$

(4.18)

where $\theta$ is the angle of twist.

The function $\psi$ is the conjugate harmonic function of $\phi + 1/2(x^2 + y^2)$, where $\phi$ is the Prandtl torsion function. The reason for using $\psi$ is that it satisfies the Laplace equation and is single valued in multiply connected regions.

The first problem solved with the torsion of an ellipse with a $\frac{1}{4}$ to 1 eccentricity. This has an exact solution. In terms of $\phi$ the solution for the torsion of an ellipse is given by

$$
\phi = \frac{M_t}{ab} \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right),
$$

(4.19)

where $M_t$ is the torsional moment, and $a$ and $b$ are the major and minor axes of the ellipse, respectively. In terms of $\psi$ the solution is

$$
\psi = - \left( \frac{a^2 - b^2}{a^2 + b^2} \right) x^2 \phi.
$$
It is noted that the stress function $\psi$ does not depend on the angle of twist $\theta$ or the torque $M_t$. These enter only in the relation (4.18) giving the shear stresses in terms of $\psi$. (The Prandtl function $\phi$ is proportional to $\theta$ or $M_t$ and the relations between $\phi$ and the shear stresses are independent of these quantities.) We note further that our main interest here was in determining how accurately the torsion function $\psi$ could be calculated with the computer program. Therefore, we did not attempt to calculate the torsional rigidity

$$C = \frac{M_t}{\theta}$$

in any problem, although its calculation is necessary for the complete solution to torsion problems.

In applying the computer program to the torsion of the 4 to 1 ellipse, we used the stress function

$$\psi = \sum_{n=1}^N \alpha_n r^{2n} \sin 2n\theta.$$  \hspace{1cm} (4.20)

The function

$$\psi = - \frac{15}{32} \frac{r^2}{2} \sin 2\theta = - \frac{15}{16} \times \gamma$$

is the exact solution to the problem. The computing program found for $a_2$ the value $-0.44117588$ compared with the exact value $0.44117648$. This is an error of six in the seventh significant digit. The coefficients of the other three terms of (4.20) were $1 \times 10^{-7}$ or less.
We also calculated the torsion function $\psi$ for two circular cylinders with longitudinal circular holes. The cross sections of the cylinders are shown in Figs. 56 and 57.

These problems were also solved by Ling (12) using the classical approach.

In both of these problems, the hole radii were equal to $1/5$ the radius of the cylinder. The holes are located so that the minimum distances between neighboring holes and between the holes and the outside surface of the cylinders in both problems are equal to the radius of the holes.

The torsion function $\psi$ for the three-holed problem was taken to be

$$
\psi = \sum_{n=1}^{9} \left( c_{2n}^0 \lambda^{2n} \sin 2n\theta + c_{-2n}^0 \lambda^{-2n} \sin 2n\theta \right) + \sum_{n=1}^{15} \sum_{i=1}^{1} \sum_{j=0}^{m} \lambda_{i,j}^{n-2m} \sin n (\theta_{i,j} - i\pi).$$  \hspace{1cm} (4.21)

For the seven-hole problem, $\psi$ was taken as

$$
\psi = \sum_{n=1}^{7} \left( c_{4n}^0 \lambda^{4n} \sin 4n\theta + c_{-4n}^0 \lambda^{-4n} \sin 4n\theta \right) + \sum_{n=1}^{11} \sum_{i=1}^{1} \sum_{j=0}^{m} \lambda_{i,j}^{n-4m} \sin n (\theta_{i,j} - i\pi).$$  \hspace{1cm} (4.22)
Fig. 56. -- The cross section of a circular cylinder with three longitudinal circular holes.

Fig. 57. -- The cross section of a circular cylinder with five longitudinal circular holes.
where \((r_{1,j}, \theta_{1,j})\) are the coordinates measured from the center of the eccentric holes in each case. The maximum boundary residual for the three-hole problem was \(7 \times 10^{-4}\) and for the seven-hole problem it was \(1 \times 10^{-5}\). The boundary values of the tangential shear stress on the boundary for these problems were given by Ling to four significant digits. The comparable stresses calculated by the program agreed with Ling's results. These stresses are given in Tables 21 through 23.

This completes the presentation of the numerical results. It is felt that the results verify the power of the least squares extension of the point matching approach. If the stress-function series is chosen in a form that has reasonably rapid convergence for the given problem, this approach can be used to give solutions to almost any desired accuracy.
TABLE 21.—Boundary shear stresses on the eccentric holes of circular tubes with longitudinal circular holes as a function of $\lambda = \eta - \theta$

<table>
<thead>
<tr>
<th>Degrees</th>
<th>Minutes</th>
<th>$\frac{C}{N/cR\theta}$</th>
<th>3 Holes</th>
<th>7 Holes</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>00</td>
<td>1.0338</td>
<td>0.6777</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>56</td>
<td>1.1420</td>
<td>0.6597</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>57</td>
<td>0.9585</td>
<td>0.6075</td>
<td></td>
</tr>
<tr>
<td>29</td>
<td>32</td>
<td>0.8754</td>
<td>0.5330</td>
<td></td>
</tr>
<tr>
<td>39</td>
<td>39</td>
<td>0.7601</td>
<td>0.4409</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>12</td>
<td>0.6116</td>
<td>0.3440</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>00</td>
<td>0.4508</td>
<td>0.2592</td>
<td></td>
</tr>
<tr>
<td>70</td>
<td>07</td>
<td>0.2643</td>
<td>0.1691</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>13</td>
<td>0.0621</td>
<td>0.0621</td>
<td></td>
</tr>
<tr>
<td>90</td>
<td>00</td>
<td>-0.1451</td>
<td>-0.0679</td>
<td></td>
</tr>
<tr>
<td>99</td>
<td>47</td>
<td>-0.3580</td>
<td>-0.2228</td>
<td></td>
</tr>
<tr>
<td>109</td>
<td>53</td>
<td>-0.5777</td>
<td>-0.3998</td>
<td></td>
</tr>
<tr>
<td>120</td>
<td>00</td>
<td>-0.7925</td>
<td>-0.5829</td>
<td></td>
</tr>
<tr>
<td>129</td>
<td>48</td>
<td>-0.9885</td>
<td>-0.7542</td>
<td></td>
</tr>
<tr>
<td>140</td>
<td>21</td>
<td>-1.1793</td>
<td>-0.9226</td>
<td></td>
</tr>
<tr>
<td>150</td>
<td>28</td>
<td>-1.3336</td>
<td>-1.0589</td>
<td></td>
</tr>
<tr>
<td>160</td>
<td>03</td>
<td>-1.4463</td>
<td>-1.1582</td>
<td></td>
</tr>
<tr>
<td>170</td>
<td>04</td>
<td>-1.5212</td>
<td>-1.2241</td>
<td></td>
</tr>
<tr>
<td>180</td>
<td>00</td>
<td>-1.5467</td>
<td>-1.2464</td>
<td></td>
</tr>
</tbody>
</table>
TABLE 22.—Boundary shear stresses on the center holes of circular tubes with longitudinal circular holes as a function of

\[ \alpha = \pi - \theta \]

<table>
<thead>
<tr>
<th>Degrees</th>
<th>Minutes</th>
<th>3 Holes</th>
<th>7 Holes</th>
</tr>
</thead>
<tbody>
<tr>
<td>90</td>
<td>00</td>
<td>-0.0258</td>
<td>-</td>
</tr>
<tr>
<td>99</td>
<td>47</td>
<td>-0.0312</td>
<td>-</td>
</tr>
<tr>
<td>109</td>
<td>53</td>
<td>-0.0488</td>
<td>-</td>
</tr>
<tr>
<td>120</td>
<td>00</td>
<td>-0.0807</td>
<td>-</td>
</tr>
<tr>
<td>129</td>
<td>48</td>
<td>-0.1276</td>
<td>-</td>
</tr>
<tr>
<td>140</td>
<td>21</td>
<td>-0.1977</td>
<td>-</td>
</tr>
<tr>
<td>150</td>
<td>28</td>
<td>-0.2810</td>
<td>-0.1859</td>
</tr>
<tr>
<td>160</td>
<td>03</td>
<td>-0.3638</td>
<td>-0.1929</td>
</tr>
<tr>
<td>170</td>
<td>04</td>
<td>-0.4325</td>
<td>-0.2071</td>
</tr>
<tr>
<td>180</td>
<td>00</td>
<td>-0.4588</td>
<td>-0.2142</td>
</tr>
</tbody>
</table>
TABLE 23.--Boundary shear stresses on the exterior boundary of circular tubes with longitudinal circular holes as a function of $\alpha = \pi - \theta$

<table>
<thead>
<tr>
<th>Degrees</th>
<th>Minutes</th>
<th>$\tau / \sqrt{cR\theta}$</th>
<th>3 Holes</th>
<th>7 Holes</th>
</tr>
</thead>
<tbody>
<tr>
<td>90</td>
<td>00</td>
<td>0.9323</td>
<td>..</td>
<td>..</td>
</tr>
<tr>
<td>99</td>
<td>47</td>
<td>0.9319</td>
<td>..</td>
<td>..</td>
</tr>
<tr>
<td>109</td>
<td>53</td>
<td>0.9301</td>
<td>..</td>
<td>..</td>
</tr>
<tr>
<td>120</td>
<td>00</td>
<td>0.9282</td>
<td>..</td>
<td>..</td>
</tr>
<tr>
<td>129</td>
<td>48</td>
<td>0.9278</td>
<td>..</td>
<td>..</td>
</tr>
<tr>
<td>140</td>
<td>21</td>
<td>0.9360</td>
<td>..</td>
<td>..</td>
</tr>
<tr>
<td>150</td>
<td>00</td>
<td>0.9665</td>
<td>0.8972</td>
<td>..</td>
</tr>
<tr>
<td>160</td>
<td>03</td>
<td>1.0528</td>
<td>0.9382</td>
<td>..</td>
</tr>
<tr>
<td>170</td>
<td>04</td>
<td>1.2087</td>
<td>1.0508</td>
<td>..</td>
</tr>
<tr>
<td>180</td>
<td>00</td>
<td>1.3078</td>
<td>1.1272</td>
<td>..</td>
</tr>
</tbody>
</table>
REFERENCES


2. Capper, P. L., (Experimental results quoted by A. E. Green loc. cit.)


CHAPTER 5

THE COMPUTING PROGRAM

This chapter is devoted to the description of the computer program. We will describe in detail the input data for the program and the function of the various programs and subprograms. We will also give complete flow charts for the programs (except for the Gaussian reduction subprogram for solving the set of simultaneous equations). We will not, however, include a listing of the programs since these are quite extensive, even in the FORTRAN source language.

It is felt that this method of presenting the code will be more effective in promoting its use than a simple listing of the programs.

The complete code actually consists of three separate programs "chained" together under FORTRAN Monitor control. These programs will be designated as Programs I, II, and III in the following discussion.

The codes described here were written in FORTRAN and compiled on 32K IBM 709 and 7090 computers. However, the program
uses only one peripheral magnetic tape for intermediate storage of the matrix between Program I and Program II. Thus, the code could be run on smaller machines if the dimension statements are suitably reduced. All of the restrictions on the sizes of the numbers quoted in this section will refer to a 32K core storage facility. It may be mentioned that nearly all of the problems solved in this dissertation could have been solved on a computer with, say, 8K storage.

**Program I**

Program I generates the matrix and right-hand side of the system of simultaneous equations for the undetermined constants of the stress function. The matrix is generated from the card input giving the geometry of the problem and the form of the stress function. The card input to Program I, in the order to be read, is given in Table 24.

**Output of Program I**

I. Output Tape

A. Program I input [less NGCUR, NGCOON, ICUR, CURCO but adding YBA(I), XNBA(I), YNBA(I)]

B. The coefficient matrix and its right-hand side (listed Row-wise).
### TABLE 54. Card input to Program I

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>IC</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>IG</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>MMAX</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>INSYM</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>NØSYM</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>IE</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>PDCYCL</td>
<td>Floating Point</td>
</tr>
<tr>
<td>HCK</td>
<td>Floating Point</td>
</tr>
<tr>
<td>XPøLE(I)</td>
<td>Floating Point</td>
</tr>
<tr>
<td>YPøLE(I)</td>
<td>Floating Point</td>
</tr>
<tr>
<td>IPØS1A(I)</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>MA(I)</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>NA(I)</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>KA(I)</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>IEQNø(I)</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>IBA(I)</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>RTSIDE(I)</td>
<td>Floating Point</td>
</tr>
<tr>
<td>XBA(I)</td>
<td>Floating Point</td>
</tr>
<tr>
<td>NØCUR</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>NØCCøN</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>ICUR(I)</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>CURCøN(I)</td>
<td>Floating Point</td>
</tr>
<tr>
<td>MG(I)</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>NG(I)</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>KG(I)</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>GVALUE(I)</td>
<td>Floating Point</td>
</tr>
</tbody>
</table>
II. Binary Tape

A. The number of rows and columns of the coefficient matrix, the matrix itself and its right-hand side are written on binary tape.

Restrictions

\[ M_{\text{MAX}} \leq IE \leq 100; \ IE \leq IC \leq 200; \]
\[ (M_{\text{MAX}})(N_{\text{SYM}}) \leq 2000; \ N_{\text{CUR}} \leq 10 . \]

Description of the input parameters of Program I

\[ IC - \text{Number of unknown coefficients in the stress function.} \]
\[ IG - \text{Number of known coefficients in the stress function.} \]
\[ M_{\text{MAX}} - \text{Number of poles in one symmetry element (or periodic element).} \]
\[ N_{\text{SYM}} - \text{Number of periodic elements (= symmetry elements/2).} \]
\[ IE - \text{Number of boundary conditions to be written.} \]

In either type of symmetry, one of the symmetry lines must coincide with the x-axis.
PDCYCL - The width of the period strip in symmetry Type 3. This is the distance between locations of periodic poles.

HCK - A constant used as the standard length in scaling the dimensions of the problem region. The nondimensional radius was taken to be \( r/HCK \).

XPÔLE - x- and y-coordinates of the fundamental poles.
YPÔLE \((I = 1, MMAX)\)

IPOS1A(I) - Code for each symmetric set of poles. If the I-th fundamental pole is on a symmetry line or if the problem has periodicity but not symmetry, \( IPOS1A(I) = 0 \). If the I-th fundamental pole is not on a symmetry line, \( IPOS1A(I) = 1 \). \((I = 1, MMAX)\)

MA(I) - Identifiers for the I-th term of the stress function.
NA(I) - NA and KA taken together identify the order and type of the term in the polar expansion about the MA-th fundamental pole. Table 26 gives the form of the term of the polar expansion associated with each NA-KA pair. \((I = 1, IC)\)

KA(I) -

IEQN0(I) - Code for the type of boundary condition written as the I-th equation. \((I = 1, IE)\)

IBA(I) - The number of the boundary point at which the I-th equation is written. The boundary point set is assumed to be numbered sequentially. If there is more than one boundary curve, the boundary points on each curve should be numbered consecutively. For this purpose, arcs of a boundary requiring different parameterization are considered separate boundary curves even though they may be actually part of the same boundary curve of the problem. It is desirable that the entries in the IBA(I) table be monotone increasing. It is necessary that the last entry in the table for each boundary segment refer to the highest numbered point of that segment and that the last entry in the table, IBA(IC), must refer to the highest numbered boundary point of the problem.

The boundary point number is entered in the IBA vector for each equation written at a boundary point. If boundary equations \( j \) through \( j + k \) are written at the n-th boundary point, then IBA(j) through IBA(j + k) will all be n. \((I = 1, IC)\)
RTSIDE(I) - The value of the right side of the I-th boundary equation. (I = 1, IC)

The three tables IEQN∅(I), IBA(I), and RTSIDE(I) must be compatible. That is, the I-th element of the IEQN∅ and RTSIDE vectors must be appropriate to the IBA(I)-th boundary point.

XBA(I) - The x-coordinate of the I-th numbered boundary point. This table has only one entry for each boundary point. If a segment of any boundary curve consists of the line x = constant, then the y-coordinates of the boundary points lying on this line should be inserted in the appropriate elements of the XBA table. [I = 1, IBA(IC)]

N∅CUR - The number of boundary curve equations needed to define the boundary of the problem. Only those boundary curves within the basic symmetry element are considered.

N∅C∅N - The total number of constants needed in the boundary curve equations.

ICUR(I) - The largest boundary point number of a given sequential subset of boundary point numbers referring to the same boundary. The numbers of this table must be monotone increasing. (I = 1, N∅CUR)

CURC∅N(I) - The set of constants used to parameterize the equations of the boundary curves. The table entries should be grouped in sets each of which is the set of parameters for one of the boundary curves. (I = 1, N∅C∅N)

MG(I) - The M, N, and K identifiers for the terms of the stress function whose values are known a-priori. These KG(I) have the same meaning as the MA, NA, KA identifiers have for the unknown terms. (I = 1, IG)

GVALUE(I) - The value of the I-th known coefficient of the stress function. (I = 1, IG)

(If IG = 0, no input is required for MG, NG, KG, or GVALUE.)
The variables $YBA(I)$, $XNBA(I)$, $YNBA(I)$ are outputted in a table along with $XBA(I)$. $YBA(I)$ is the y-coordinate of the $I$-th boundary point. $XNBA(I)$ and $YNBA(I)$ are the direction cosines of the outer normal to the boundary at the $I$-th point.

Discussion of the main program and subroutines of Program I

Program I consists of the main program and the following subroutines: $BØNDRY$, $BC1$, $BC2$, $BC7$, $DPHIØ$, $DPHIDX$, $DPHIDY$, $DPHDX2$, $DPHDX2$, $DPHDX2$, $DPHDX2$, $ANGLE$, and $RHERR$. Each of these programs will be discussed in turn. All communication between the main program and the various subprograms is through common storage. Program I was broken down as much as possible into subroutines in order to achieve flexibility in applying the codes to various kinds of problems. With the subroutine capability it is possible to change the subroutines to correspond to various problems without having to recompile the whole program.

**Main Program I.** The main program first reads all of the input data and calls the $BØNDRY$ subroutine which sets up the boundary point tables. The main program then writes the input on the B.C.D. output tape. Following this, the program sets up the table of locations of all of the poles of the problem from the tables of locations of the poles in the symmetry element and the designation of the type of symmetry.
A DO loop is then begun which ranges over the equation tables (IEQN0, IRA, RTSIDE). For each entry in these tables, the program calls one of the boundary condition subroutines BC1, ..., BC7 according to whether IEQN0 is 1, ..., 7. The output of the BC subroutine is a row of the coefficient matrix. The program writes this row on binary tape, and increments the index of the DO loop. When the DO is completed all of the boundary equations have been written on tape and the program terminates.

Subroutine BONDRTY. This subroutine was set up to eliminate the labor involved in calculating the coordinates of the boundary points and the direction cosines of the outer normal to the boundary at each boundary point. The procedure adopted here is fairly simple but sufficiently flexible to accommodate most types of problems. This procedure will be illustrated by describing a subroutine which will be applicable to any problem in which the boundary curves are segments on conic sections. A great many practical problems actually are of this type. However, subroutines to handle problems with other types of boundary curves can be written very easily.

We will assume then that all of the boundary curves of the problem may be written in the form:

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 xy + \lambda_4 x + \lambda_5 y + \lambda_6 = 0 \quad (5.1)$$
We will further assume that the locations of the boundary points are to be specified by giving the \( x \)-coordinate of the boundary point. These coordinates are inserted in the XBA table of the input. The equation of the boundary curve is given by specifying the values of the constants \( C_1, C_2, \cdots C_6 \). These constants are read into the CURC\( \Phi \)N table. The value of YBA is then found from the equation

\[
YBA = y = \frac{-(c_3 x + c_5) \pm \sqrt{(c_3 x + c_5)^2 - 4c_2 (c_1 x^2 + c_4 x + c_6)}}{2c_2} \quad \text{--- (5.2)}
\]

If \( C_2 = 0 \),

\[
YBA = -\left( \frac{c_1 x^2 + c_4 x + c_6}{c_3 x + c_5} \right).
\]

A seventh constant \( C_7 = \pm 1 \) is read into the CURC\( \Phi \)N table to indicate which sign is to be given the square root in order to obtain the proper branch of the boundary curve in question. Having the values of \( x \) and \( y \), the subroutine calculates the direction cosines of the outer normal to the boundary from the following formulas:

\[
\cos(\eta, x) = XNA = -\frac{2c_1 x + c_3 y + c_4}{\left[(2c_1 x + c_3 y + c_4)^2 (2c_2 y + c_5 x + c_6)^2 \right]^{1/2}} \quad \text{--- (5.4)}
\]

\[
\cos(\eta, y) = YNA = -\frac{2c_2 y + c_3 x + c_5}{\left[(2c_1 x + c_3 y + c_4)^2 (2c_2 y + c_5 x + c_6)^2 \right]^{1/2}} \quad \text{--- (5.5)}
\]

It will be noted that these equations calculate the direction cosines for the outer normal for a hole in the body. If the boundary
point lies on the exterior boundary of a finite region, the proper values of the direction cosines (and the y-coordinate as well) are obtained by changing the signs of all seven constants $C_1, \ldots, C_7$.

This procedure is applicable to all conic sections except for $x = \text{constant}$. On this curve, it is necessary to read the y-coordinates of the points into the XBA table. This curve might be identified by setting $C_7 = 0$. The subroutine BONDRY checks $C_7$ to see if it is zero. If $C_7 = 0$, the subroutine transfers the XBA values to the YBA table, sets $XNBA$ to $C_4 (\pm 1)$, sets $YNBA = 0$, and sets $XBA = C_6$.

If the boundary of the problem is made up of two or more boundary curves, the seven parameters of the first curve are read into CURC0N(1) through CURC0N(7), the parameters of the second curve are read into CURC0N(8) through CURC0N(14), etc. All of the x-coordinates of the boundary points of the first curve are placed first in the XBA table followed by the x-coordinates of the boundary points of the second curve, etc. The number of the highest numbered boundary point on the first curve is entered into ICUR(1), the number of the highest numbered point on the second curve is entered into ICUR(2), etc.

The subroutine BONDRY calculates the boundary point tables using the parameters of the first curve until it has exhausted the boundary points of the first curve. It then uses the parameters of
the second curve until it has gone through all of the points of
the second curve. This process is continued until the boundary
point tables have been completed. Control is then returned to the
main program.

It is fairly easy to see how subroutines could be written
in various ways to calculate the appropriate boundary point
parameters using the set of input parameters. For instance if the
boundary curves of the problem were all arcs of circles, it might
be convenient to specify the boundary points on each arc by the
angle its radius vector from the center of the circular arc makes
with the x-axis. These angles would be entered in the XBA table.
The CURCØN table might then contain the x- and y-coordinates of
the center of the circular arc and the length of the radius of
the arc. A BØNDRY subroutine could easily be programmed to
calculate the boundary point tables from this input. Other variants
of the BØNDRY subroutine can be designed easily. Because of the
subroutine feature, these variants may each be compiled without
having to recompile the main program each time a different
subroutine is called for. However, it is also advantageous to
write these subroutines in such a way that they may be used on
a variety of problems. The labor of making up one subroutine is
approximately the same as making up the input to one problem.
Subroutines BC1 through BC7. These boundary subroutines are very simple subprograms to call the derivative subroutines DPHI0, DPHDX, ..., DPHDXY and to combine their outputs appropriately to form the boundary condition called for. For instance, if the boundary condition is

\[ \sigma_n^I = R T S I D E (I) \quad (B C 1) \]

the subprogram calls the subroutines DPHDX2, DPHDY2, and DPHDXY. The outputs of these subroutines are tables of the second derivatives of the terms of the stress function. Subroutine BCI then combines these terms according to the formula

\[ \sigma_n = x N B A^2 \phi_y + y N B A^2 \phi_x + l (N B A)(y N B A) \phi_y \]

and stores the values of the resulting terms for the main program. The only reason these routines have been set up as subroutines is to facilitate changing the boundary conditions to conform to the various types of boundary value problems. In this way the main program does not have to be recompiled for each different type of boundary problem. It is eventually intended to have a library of boundary condition subroutines from which the appropriate boundary conditions may be chosen for the problem under consideration.

The derivative subroutines. At the heart of the program are the derivative subroutines. These are the subroutines DPHI0,
D PHIDX, ⋯, D PHDXY. As the names imply, the subroutines calculate the terms of the stress function or its derivatives according to the following:

- $\text{DFH} \phi$ → The function $\phi$
- $\text{DPHIDX}$ → $d\phi/dx$
- $\text{DPHIDY}$ → $d\phi/dy$
- $\text{DPHDXY}$ → $d^2\phi/dx^2$
- $\text{DPHDX2}$ → $d^2\phi/dy^2$
- $\text{DPHDY2}$ → $d^2\phi/dxy$
- $\text{DPHDXY}$ → $d^2\phi/dx^2$

It is usually not necessary to use all six subroutines in a given problem. For instance, in the first fundamental stress problem with stress boundary conditions, only the three second derivative subroutines are needed in the program. In this case, the seven boundary subroutines are set up to call these three derivative subroutines and the other subroutines need not be loaded. This would be an advantage for smaller machines since the program would thus take less storage.

All of the six derivative\(^1\) subroutines are constructed with essentially the same logic structure. This structure is indicated in the flow chart in Fig. 65.

The derivative subroutines are made up of two main DO loops. The first loop traverses the tables of the terms of the stress

\(^1\)Here and in the remainder of this section, the words "derivative" and "derivative subroutine" will be taken to include the 0-th order derivative (i.e., the stress function) and the subroutine DPHI$\phi$, ⋯ respectively, as well as the five derivatives and their subroutines.
function. The second loop, which is contained within the first, traverses that part of the table of pole locations which contains the locations of the set of periodic poles associated with the current term of the stress function.

Then:

for a given boundary point
for a given term of the stress function
for a given pole,

the derivative subroutine evaluates the variable part of the derivative of the symmetric stress function. The program then checks to see if the current pole has an image in the x-axis. If it does, the calculation is repeated with this image pole and the results are added. As the inner loop traverses the subset of poles, the values that have been calculated are added together so that when the inner DO loop is satisfied the result is the variable part of the current term of the stress function or its derivative.

The subroutine then proceeds to the next term of the stress function and evaluates the variable part of its derivative. Each of these terms is stored as an element of a vector in common for use by the boundary subroutines. When all of the terms of the derivative have been so evaluated, the subroutine evaluates the derivative of the known terms of the stress function (if any) and stores the sum of these values so that it may be used in the BC subroutine.
When this operation is completed, the derivative subroutine returns control to the boundary condition subroutine that called it.

Subroutine ANGLE. This subroutine is used to obtain the angle \( 0 \leq \theta \leq 2\pi \) subtended by the x-axis and the radius vector to the point \((x,y)\), assuming that the available library subroutine calculates only \( \theta = \tan^{-1} \left| \frac{y}{x} \right| \), \( 0 \leq \theta \leq \pi/2 \). This is used at various points of the main program and the derivative subroutines.

Subroutine RHERR. This subroutine is an error signal. It is called at various parts of the program if a boundary point is coincident with a pole of the stress function. The subroutine writes an error indication on the output tape and exits to Monitor control.

Program II

This program solves the matrix equation generated by Program I for the undetermined constants of the stress function.

Input to Program II

I. Card Input

None.

II. Binary Tape

A. The number of rows, \( M \), the number of columns, \( N \), of the matrix equation and the matrix equation itself written one row per record.
Output from Program II

I. Output Tape
   A. The solution vector

II. Binary Tape
   A. None. (The solution vector is stored in common between Program II and III.)

Discussion of Program II

Program II tests whether $M = N$. If this is the case (i.e., if the matrix is square), the program proceeds directly to the solution of the matrix equation. If $M > N$, the program writes an error signal on the output tape and exits to Monitor control. If $M > N$, the program calculates $A^T A$ and $A^T R$ where $R$ is the right side of the matrix equation. The resulting system of equations

$$A^T A \times = A^T R$$

is then solved for the undetermined constants of the stress function. The solution of the matrix equation is obtained by a standard algorithm for Gaussian reduction with pivotal condensation. More efficient routines can be set up for the symmetric positive definite matrix $A^T A$. However, if this type of program were used, it would not then have the capability of solving the system of equations where $M = N$ which, in general, are not symmetric.
Program III

Program III calculates the stresses at selected points of the problem region with the stress function that is now completely determined. The card input to Program III is given in Table 25.

Output of Program III

I. Output Tape

A. Program III input (less NOCUR, NOCCON, ICUR, CURCON, but adding YBA, XNBA, YNBA).

B. Stresses $\sigma_x$, $\sigma_y$, $\tau_{xy}$, $\sigma_N$, $\sigma_T$, $\tau_T$ at selected boundary points, if called for.

C. Stresses $\sigma_x$, $\sigma_y$, $\tau_{xy}$, $\sigma_{\text{MAX}}$, $\sigma_{\text{MIN}}$, $\tau_{\text{MAX}}$ and angle between x-axis and principal direction for selected interior points of the region, if these are called for.

II. Binary Tape

A. None.

Description of the input parameters of Program III

The description of parameters 1 through 23 of Table 25 is identical to the description of the same parameters given in Program I, except that IE is the number of boundary points for Program III, not the number of boundary equations. The remainder of the input parameters in Table 25 are described as follows:

IRFTP - Code--0, if the solution vector is to be read from cards; 1, if the solution vector is to be taken from common storage.
TABLE 25.--Card input to Program III

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>IC</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>IG</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>MMAX</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>INSYM</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>NØSYM</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>IE</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>PDCYCL</td>
<td>Floating Point</td>
</tr>
<tr>
<td>HCK</td>
<td>Floating Point</td>
</tr>
<tr>
<td>XPØLE(I)</td>
<td>Floating Point</td>
</tr>
<tr>
<td>XPØLE(I)</td>
<td>Floating Point</td>
</tr>
<tr>
<td>IPOS1A(I)</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>MA(I)</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>NA(I)</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>KA(I)</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>XBA(I)</td>
<td>Floating Point</td>
</tr>
<tr>
<td>NØCUR</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>NGCUR</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>CUR(1)</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>CURCØN(1)</td>
<td>Floating Point</td>
</tr>
<tr>
<td>MG(I)</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>NG(I)</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>KG(I)</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>GVALUE(I)</td>
<td>Floating Point</td>
</tr>
<tr>
<td>IRPTP</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>KEY</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>PHI(I)\textsuperscript{a}</td>
<td>Floating Point</td>
</tr>
<tr>
<td>NUMSX</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>NUMSY</td>
<td>Fixed Point</td>
</tr>
<tr>
<td>XMIN</td>
<td>Floating Point</td>
</tr>
<tr>
<td>YMIN</td>
<td>Floating Point</td>
</tr>
<tr>
<td>DELXB</td>
<td>Floating Point</td>
</tr>
<tr>
<td>DELYB</td>
<td>Floating Point</td>
</tr>
</tbody>
</table>

\textsuperscript{a}If PHI (the solution vector) is read from cards, it is inserted here.
KEY - Switch for control of the calculation of the stresses. This switch is read at several locations. The first time it is read the program will calculate the boundary stresses if the switch is 0. If the switch is 1, the boundary stress calculation is bypassed, the first set of values of the parameters NUMSX, DELXB is read in, and the interior stresses at the indicated points are calculated.

A new value of KEY is read in after the boundary stress calculations are made (if they are made) and after each set of interior stresses is calculated. In this second case, KEY has the following meanings:

KEY = -1, the program is to read in a new set of parameters NUMSX, etc. and calculate the set of interior stresses corresponding to these parameters.

KEY = 0, the program exits to Monitor control.

KEY = 1, the program recalls Program I to begin processing another stress problem.

PHI(I) - The solution vector. This may be read from cards by setting IRPTP = 0. This option is useful if the solution was obtained on a previous pass, and it is desired to calculate stresses at more points than had been obtained in the first pass. (I = 1, IC)

NUMSX - These six numbers are used to set up the calculation of the stresses on the rectangular point lattice which has a corner point (XMIN, YMIN), and which has NUMSX intervals of DELXB spacing in the x-direction, and NUMSY intervals of DELYB spacing in the y-direction. It is noted that either or both DELXB or DELYB may be negative so that (XMIN, YMIN) may refer to any one of the corners of the rectangular lattice. There are no restrictions on any of the lattice parameters or on the number of lattices that can be calculated. However, no lattice point may coincide with a pole of the stress function. If this does happen, the program calls the error subroutine RHERR and terminates the calculation.
The use of the given coefficient tables differs slightly from the way that they are used in Program I. In Program I, the boundary values created by the given coefficients are added to the RTSIDE vector of the boundary equations. The solution to the problem set up in Program I thus takes on boundary values determined by this combined RTSIDE vector. If it is desired to find the solution of the problem that takes on the boundary values equal to the original RTSIDE vector specified by the card input data, it is necessary to set up the given coefficient tables in Program III with the signs of the GVALUE table reversed from the signs used in Program I.

If it is desired that the solution take on boundary values equal to the combined PTSIDE vector, the given coefficient tables are omitted from Program III and IG is set equal to zero.

Discussion of the main program and subroutines of Program III

Program III consists of the main program and the subroutines BDNDY, BDYSIG, SIGINT, DPHIØ, DPHIDX, DPHIDY, DPHDX2, DPHDY2, DPHDXY, ANGLE, and RHERR. It is noted that Program III makes liberal use of the subprogram concept in the manner of Program I to facilitate the modifications necessary for application of the codes to various types of problems.

With the exception of BDYSIG and SIGINT, all of the subroutines of Program III are identical to the subroutines with the same names discussed in Program I. We will, therefore, discuss here only the main program and these two differing subroutines.
Main Program III. The main program reads the input data, uses the BONDY subroutine to set up the boundary point tables, writes the input on the output tape, and sets up the table of pole locations in the same manner as Program I. The program then interrogates KEY and, depending on whether KEY is 0 or 1, will call or skip subroutine BDYSIG which calculates the boundary point stresses. If BDYSIG is skipped, the program calls SIGINT immediately to calculate the interior stresses. If BDYSIG is called, the main program reads a new value for KEY after control is returned from BDYSIG. KEY is then interrogated to determine whether to call SIGINT, to exit to Monitor or to recall the first program to begin another problem.

After each call for SIGINT, the main program reads another value of KEY and tests the new value to determine again whether to call SIGINT, to call exit, or to recall the first program. In this way SIGINT can be called as many times as desired.

Subroutine BDYSIG. Subroutine BDYSIG is used to calculate the boundary stresses for plane stress problems. However, similar subroutines can easily be written to calculate the boundary values for other problems. Here again the subroutine concept has been used to aid in programming the necessary modifications.
For the stress calculations, a DO loop is used which ranges over the boundary point set. For each boundary point the subroutines DPHDX2, DPHDY2, and DPHDXY are called. The inner product of the solution vector with the output of these subroutines is then computed to give $\sigma_y', \sigma_x'$, and $\tau_{xy}'$, at the boundary point. From these stresses, the program calculates $\sigma_N', \sigma_T'$, and $\tau_T'$ at the boundary point. The six stresses are then written on the output tape along with the x- and y-coordinates of the boundary point. When the stresses have been calculated at all of the boundary points, the subroutine returns control to the main Program III. (The boundary point set used here need not be the same as that set up for Program I.)

Subroutine SIGINT. This subroutine is used to calculate the values of the solution of the problem at interior points of the problem region. The point set is a rectangular lattice whose points have the x- and y-coordinates

$$X_{\text{MIN}} + (\text{DELXB})n, \quad n = 0, 1, \ldots \text{NUMSX},$$

$$Y_{\text{MIN}} + (\text{DELYB})m, \quad m = 0, 1, \ldots \text{NUMSY}.$$

For plane elasticity problems, the stresses are calculated at each of these points in a manner similar to that used in BDYSIG. For each point, the subroutines DPHDX2, DPHDY2, DPHDXY are called.
The inner products of their outputs with PHI give \( \sigma_y, \sigma_x, \) and \( \tau_{xy}. \) These stresses are then used to calculate the principal stresses, the maximum shear stress, and the angle between the x-axis and one of the principal directions at the point. The six stresses and this angle are written on output tape together with the coordinates of the point.

After the stresses are calculated at all of the points of the lattice, control is returned to main Program III.
TABLE 2b.--Function table for subroutine DPHI0

<table>
<thead>
<tr>
<th>NA</th>
<th>KA</th>
<th>Functiona</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>log r</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>r^2</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>y^2 log r</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>r^2(\theta-\alpha)</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
<td>\theta - \alpha</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>r(\theta-\alpha) sin (\theta-\alpha)</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>r^3 cos (\theta-\alpha)</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>r^{-1} cos (\theta-\alpha)</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>r log r cos (\theta-\alpha)</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>r(\theta-\alpha) cos (\theta-\alpha)</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>r^3 sin (\theta-\alpha)</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
<td>r^{-1} sin (\theta-\alpha)</td>
</tr>
<tr>
<td>1</td>
<td>8</td>
<td>r log r sin (\theta-\alpha)</td>
</tr>
<tr>
<td>\leq 2</td>
<td>1,2,3,4</td>
<td>r^j cos n(\theta-\alpha)</td>
</tr>
<tr>
<td>\leq 2</td>
<td>5,6,7,8</td>
<td>r^j sin n(\theta-\alpha)</td>
</tr>
</tbody>
</table>

The following notation is used: (r, \theta) are the polar coordinates of the point of the region measured from the expansion point in question. \alpha is the angle of rotation between the fundamental symmetry element and the symmetry element containing the pole in question. This angle is measured counterclockwise from the fundamental symmetry element. \( \alpha \geq 0 \) in case of translational symmetry. \( S = +1 \) if the pole in question is congruent in rotation or translation to the fundamental pole. \( S = -1 \) if the pole in question is the image in the x-axis of a periodic pole. In the terms for which \( N \geq 2 \), j has the following values:

- \( K = 1,5 \) \( J = N \)
- \( K = 2,6 \) \( J = N + 2 \)
- \( K = 3,7 \) \( J = -N \)
- \( K = 4,8 \) \( J = -N + 2 \)
TABLE 27.—Function table for subroutine DPHIDX

<table>
<thead>
<tr>
<th>NA</th>
<th>KA</th>
<th>Function&lt;sup&gt;a&lt;/sup&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>$x/r^2$</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>$2x$</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>$(2 \log r + 1)x$</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>$2x(S_\theta-\alpha) - Sy$</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
<td>$-Syr^{-2}$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>$-y^2r^{-2} \cos \alpha + S(xyr^{-2} - S(S_\theta-\alpha)) \sin \alpha$</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>$(3x^2+y^2) \cos \alpha + 2xyS \sin \alpha$</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>$r^{-4}[(y^2-x^2) \cos \alpha - 2xyS \sin \alpha]$</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>$(x^2r^{-2} + \log r) \cos \alpha + 5xyr^{-2} \sin \alpha$</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>$S[S(S_\theta-\alpha)-xyr^{-2}] \cos \alpha - y^2r^{-2} \sin \alpha$</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>$2Sxy \cos \alpha - (3x^2+y^2) \sin \alpha$</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
<td>$r^{-4}[-2Sxy \cos \alpha + (x^2-y^2) \sin \alpha]$</td>
</tr>
<tr>
<td>1</td>
<td>8</td>
<td>$xyr^{-2} \cos \alpha - (x^2r^{-2} + \log r) \sin \alpha$</td>
</tr>
<tr>
<td>≥2</td>
<td>1,2,3,4</td>
<td>$r^{j-2}[jx \cos N(S_\theta-\alpha) + NyS \sin N(S_\theta-\alpha)]$</td>
</tr>
<tr>
<td>≥2</td>
<td>5,6,7,8</td>
<td>$r^{j-2}[jx \sin N(S_\theta-\alpha) - NyS \cos N(S_\theta-\alpha)]$</td>
</tr>
</tbody>
</table>

<sup>a</sup>$x = r \cos \alpha$, $y = r \sin \alpha$. For the remainder of the notation see footnote to Table 26.
### TABLE 28.--Function table for subroutine DPHIDY

<table>
<thead>
<tr>
<th>NA</th>
<th>KA</th>
<th>Functiona</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>$y/r^2$</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>$2y$</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>$(2 \log r + 1)y$</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>$2y(S\theta - \pi) + Sx$</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
<td>$Sx/r^2$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>$[xyr^{-2} + S(S\theta - \pi)] \cos \alpha - Sx^2 r^{-2} \sin \alpha$</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>$2xy \cos \alpha + S(3y^2 + x^2) \sin \alpha$</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>$r^{-4} [-2xy \cos \alpha + S(x^2 - y^2) \sin \alpha]$</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>$xyr^{-2} \cos \alpha + S(\log r + y^2 r^{-2}) \sin \alpha$</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>$Sx^2 y^{-2} \cos \alpha + [xyr^{-2} + (S\theta - \pi)] \sin \alpha$</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>$S(3y^2 + x^2) \cos \alpha - 2xy \sin \alpha$</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
<td>$r^{-4} [S(x^2 - y^2) \cos \alpha + 2xy \sin \alpha]$</td>
</tr>
<tr>
<td>1</td>
<td>8</td>
<td>$S(y^2 r^{-2} + \log r) \cos \alpha - xyr^{-2} \sin \alpha$</td>
</tr>
<tr>
<td>$\geq 2$</td>
<td>1,2,3,4</td>
<td>$r^{j-2} [jy \cos N(S\theta - \pi) - SNx \sin N(S\theta - \pi)]$</td>
</tr>
<tr>
<td>$\geq 2$</td>
<td>5,6,7,8</td>
<td>$r^{j-2} [jy \sin N(S\theta - \pi) + SNx \cos N(S\theta - \pi)]$</td>
</tr>
</tbody>
</table>

*aSee Tables 26 and 27 for notation.*
TABLE 29.---Function table for subroutine DPHDX2

<table>
<thead>
<tr>
<th>NA</th>
<th>KA</th>
<th>Function^a</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>((y^2 - x^2)r^{-h})</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>(2)</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>(2x^2r^{-2} + 1 + 2 \log r)</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>(2[(S_0 - \alpha) - 3xyr^{-2}])</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
<td>(2Sy/r^{-h})</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>(2y^2r^{-h}[x \cos \alpha + Sy \sin \alpha])</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>(6x \cos \alpha + 2Sy \sin \alpha)</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>(-2r^{-6}[x(3y^2 - x^2) \cos \alpha - Sy(3x^2 - y^2) \sin \alpha])</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>(r^{-h}[x(x^2 + 3y^2) \cos \alpha + Sy(y^2 - x^2) \sin \alpha])</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>(2y^2r^{-h}[-Sy \cos \alpha + x \sin \alpha])</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>(2Sy \cos \alpha - 6x \sin \alpha)</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
<td>(2r^{-6}[Sy(3x^2 - y^2) \cos \alpha + x(3y^2 - x^2) \sin \alpha])</td>
</tr>
<tr>
<td>1</td>
<td>8</td>
<td>(r^{-h}[Sy(y^2 - x^2) \cos \alpha - x(x^2 + 3y^2) \sin \alpha])</td>
</tr>
</tbody>
</table>
| ≥ 2 | 1,2,3,4 | \(r^{j-h}[j(x^2 + j(y^2 - x^2) - N_y^2)] \cos N(S_0 - \alpha)\)  
|     |     | + \(2Snr^{j-h}(j-1)xy \sin N(S_0 - \alpha)\) |
| ≥ 2 | 5,6,7,8 | \(r^{j-h}[j(x^2 + j(y^2 - x^2) - N_y^2)] \sin N(S_0 - \alpha)\)  
|     |     | - \(2Snr^{j-h}(j-1)xy \cos N(S_0 - \alpha)\) |

^aSee Tables 26 and 27 for notation.
TABLE 30.--Function table for subroutine DPHDY2

<table>
<thead>
<tr>
<th>NA</th>
<th>KA</th>
<th>Function^a</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>((x^2-y^2)r^{-4})</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>(2y^2r^{-2} + 1 + 2 \log r)</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>(2[(S\theta-\alpha) + 2Sxyr^{-2}])</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
<td>(-2Sxyr^{-4})</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>(2x^2r^{-4}[x \cos \alpha + Sy \sin \alpha])</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>(2x \cos \alpha + 6 Sy \sin \alpha)</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>(2r^{-6}[x(3y^2-x^2) \cos \alpha - Sy(3x^2-y^2) \sin \alpha])</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>(r^{-4}[x(x^2-y^2) \cos \alpha + Sy(y^2+3x^2) \sin \alpha])</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>(2x^2r^{-4}[-Sy \cos \alpha + x^2 \sin \alpha])</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>(6 Sy \cos \alpha - 2x \sin \alpha)</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
<td>(2r^{-6}[-Sy(3x^2-y^2) \cos \alpha - x(3y^2-x^2) \sin \alpha])</td>
</tr>
<tr>
<td>1</td>
<td>8</td>
<td>(r^{-4}[Sy(y^2+3x^2) \cos \alpha - x(x^2-y^2) \sin \alpha])</td>
</tr>
<tr>
<td>≥ 2</td>
<td>1,2,3,4</td>
<td>(r^j\frac{1}{4}[j^2y^2 + j(x^2-y^2) - N^2x^2] \cos N(S\theta-\alpha))</td>
</tr>
<tr>
<td>≥ 2</td>
<td>1,2,3,4</td>
<td>(-2NSr^{j\frac{1}{4}}(j-1)xy \sin N(S\theta-\alpha))</td>
</tr>
<tr>
<td>≥ 2</td>
<td>5,6,7,8</td>
<td>(r^j\frac{1}{4}[j^2y^2 + j(x^2-y^2) - N^2x^2] \sin N(S\theta-\alpha))</td>
</tr>
<tr>
<td></td>
<td></td>
<td>+ (2NSr^{j\frac{1}{4}}(j-1)xy \cos N(S\theta-\alpha))</td>
</tr>
</tbody>
</table>

^a See Tables 26 and 27 for notation.
TABLE 31.—Function table for subroutine DPHDXY

<table>
<thead>
<tr>
<th>NA</th>
<th>KA</th>
<th>Function&lt;sup&gt;a&lt;/sup&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>$-2xyr^{-4}$</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>$2xyr^{-2}$</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>$S(x^2-y^2)r^{-2}$</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
<td>$S(y^2-x^2)r^{-4}$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>$-2xyr^{-4}(x \cos \alpha + Sy \sin \alpha)$</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>$2y \cos \alpha + 2Sx \sin \alpha$</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>$2r^{-6}[y(3x^2-y^2) \cos \alpha + Sx(3y^2-x^2) \sin \alpha]$</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>$r^{-4}[y(y^2-x^2) \cos \alpha + Sx(y^2-x^2) \sin \alpha]$</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>$2xyr^{-4}[Sy \cos \alpha - x \sin \alpha]$</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>$2Sx \cos \alpha - 2y \sin \alpha$</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
<td>$2r^{-6}[Sx(3y^2-x^2) \cos \alpha - y(3x^2-y^2) \sin \alpha]$</td>
</tr>
<tr>
<td>1</td>
<td>8</td>
<td>$r^{-4}[Sx(x^2-y^2) \cos \alpha + y(x^2-y^2) \sin \alpha]$</td>
</tr>
<tr>
<td>$\geq 2$</td>
<td>$1,2,3,4$</td>
<td>$r^{1-h}(j^2 - 2j + N^2)xy \cos N(S\theta-x)$ + $Snr^{j-h}(j-1)(x^2-y^2) \sin N(S\theta-x)$</td>
</tr>
<tr>
<td>$\geq 2$</td>
<td>$5,6,7,8$</td>
<td>$r^{1-h}(j^2 - 2j + N^2)xy \sin N(S\theta-x)$ + $Snr^{j-h}(j-1)(x^2-y^2) \cos N(S\theta-x)$</td>
</tr>
</tbody>
</table>

<sup>a</sup>See Tables 26 and 27 for notation.
Fig. 58. -- Over-all flow chart for Program I
Fig. 59.—Flow chart for Panel I, Program I: initialization
Fig. 60.—Flow chart for Panel II, Program I: symmetry poles for rotational symmetry
Fig. 61.—Flow chart for Panel III, Program I: symmetry poles for translational symmetry.
Each subroutine gets $\phi_{UT}(1)$ to $\phi_{UT}(IC+1)$

RTSIDE(IR) + $\phi_{UT}(IC+1)$ → $\phi_{UT}(IC+1)$

WRITE B.C.D. TAPE:
$\phi_{UT}(1)$ through $\phi_{UT}(IC+1)$ with index

WRITE BINARY TAPE 9:
IC+1, $\phi_{UT}(1)$ through $\phi_{UT}(IC+1)$

Program II

Fig. 62.--Flow chart for Panel IV, Program I: set up boundary conditions and write each condition on tape.
f_1(x), f_2(x), f_3(x,y), and f_4(x,y) are given by Equations (5.2), (5.3), (5.4), and (5.5), respectively.

Fig. 63.—Flow chart for Subroutine B0NDRY (conic sections)
Each of the boundary condition subroutines BC1 will have similar flow charts in which one or more derivative subroutines are called, following which the OUT vector is constructed by a DO loop.

Fig. 64.--Flow chart for Subroutine BC1
Fig. 65. -- Main flow chart for derivative subroutines
Fig. 66.--Flow chart for Panel I, Derivative subroutines calculate $F_{NK}(XB,YB)$

*F_{OK}, F_{IK}, F_{NK} are given in Tables 26 through 31
Fig. 67.--Flow chart for subroutine ANGLE
I. Read Matrix Tape and check for errors

START

N-M

0

WRITE B.C.D. TAPE: Error signal problem set up wrong

WRITE B.C.D. TAPE: PHI Solution Vector

Gaussian System Solver

PHI Vector

II. Calculate A'A and A'B and overwrite them onto A and B

Call EXIT

Call Program III

Fig. 68.--Over-all flow chart for Program II
Fig. 69.—Flow chart for Panel I, Program II: read matrix tape and check for errors
Fig. 70--Flow chart for Panel II, Program II: calculate $A'A$, $A'B$ and overwrite them on $A$ and $B$
I. Initialization

II. Symmetry poles for rotational symmetry

III. Symmetry poles for translational symmetry

Fig. 7a.--Over-all flow chart for Program III

Panels II and III are identical to Program I
Fig. 72.--Flow chart for Panel I, Program III: initialization
ENTRY

IR = 1, IE

XBA(IR) → XB
YBA(IR) → YB
XNBA(IR) → XNB
YNBA(IR) → YNB

DPHDY2

DPHDX2

DPHDXY

σ_x = σ_y = τ_xy = 0

J = 1, IC

σ_x = PHI(J) * DY2C(J) → σ_x
σ_y = PHI(J) * DX2C(J) → σ_y
τ_xy = PHI(J) * DXYC(J) → τ_xy

σ_x + DY2C(IC + 1) → σ_x
σ_y + DX2C(IC + 1) → σ_y
τ_xy + DXYC(IC + 1) → τ_xy
XN^2 σ_x + YN^2 σ_y + 2(XN * YN) τ_xy → σ_N
YN^2 σ_x + XN^2 σ_y - 2(XN * YN) τ_xy → σ_T
(XN^2 - YN^2) τ_xy + (XN * YN)(σ_y - σ_x) → τ_T

WRITE B.C.D. TAPE:
σ_x, σ_y, τ_xy, σ_N, σ_T, τ_NT, x, y

Fig. 73 -- Flow chart for Subroutine BDYSIG
Fig. 74.—Flow chart for Subroutine SIGINT
BIBLIOGRAPHY


Capper, P. L., (Experimental results quoted by A. E. Green Reference 12.)


Ling, Chih-Bing, "A Ring of Holes in an Infinite Plate under All-Round Tension", Bureau of Aeronautics Research, Chengtu, China, Technical Report No. 6 (June, 1943).


Lo, C. C., "Bending of Rectangular Plates with all Edges Clamped", Master's Thesis, The Ohio State University, Columbus, Ohio (1960).


I, Lewis Eugene Hulbert, was born in Somerton, Ohio, November 15, 1924. I received my secondary-school training in the public schools of Geneva, Ohio and my undergraduate education in Chemistry at Iowa State College which granted me the Bachelor of Science degree in 1947. I received the Master of Science degree from Case Institute of Technology in 1951.

From 1950 to 1952, I was employed as an actuarial consultant. Since 1952, I have been employed as a senior mathematician at the Battelle Memorial Institute.

In 1956, I began studies toward the doctorate at The Ohio State University.