CONTRIBUTIONS TO LAMINAR BOUNDARY LAYER THEORY FOR GASES

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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Summary

The paper investigates some of the effects of the variation of fluid properties on boundary layer solutions when the pressure varies along the surface.

The correspondence of boundary layer solutions for fluids having variable properties with solutions for fluids having constant properties is extended in an approximate manner so as to include the effect of Prandtl number different from unity.

It is shown that power series solutions may be obtained for general viscosity-temperature relations.

A new approximate treatment of skin friction is presented for cases in which there exists a large difference between the stagnation temperature of the gas and the surface temperature.

A 91 page Appendix includes background information on the laminar boundary layer.
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1. Introduction.¹

1.1 Prandtl's simplifications; the boundary layer.

Professor Prandtl (Ref. 1) took a tremendously important step toward understanding high Reynolds number flow past a body when he pointed out that in many cases the effects of viscosity are confined to a thin layer near the body, and to a wake consisting of fluid which has passed through this region. The fluid velocity relative to the body changes from zero at the body surface to a velocity of the same order of magnitude as the free stream velocity at the outer edge of the thin layer. The resulting high rate of shear in the thin boundary layer causes viscous effects to be important there. The rotational fluid motion is later diffused in the wake by viscous action. Prandtl recognized that the flow outside of this thin boundary layer and wake is very nearly irrotational and can be treated satisfactorily by the classical methods developed for irrotational flow of a nonviscous fluid. Prandtl pictured the velocity field as consisting of two parts. First, the irro-

¹Readers who desire a more thorough introduction than that given here may read Appendix A, which includes some of the fundamentals of fluid dynamics, as well as background information on the laminar boundary layer.
tational flow field for a very slightly thickened body in a nonviscous fluid, and second, boundary layer velocity field fitted in between the actual body and the irrotational flow field for the fictitious, slightly thickened body.

With this picture in mind, solutions for the velocity field are obtained as follows. One assumes that the boundary layer is so thin that its presence does not affect the irrotational flow. The tangential velocity at the body as obtained from the irrotational flow solution (or as deduced from a measured pressure\(^1\) distribution using nonviscous fluid theory) is used as a boundary condition at the "outer edge" of the boundary layer in the solution for the velocity field in the boundary layer. The boundary conditions at the body are the conditions of no slip and no penetration. Arguments based on the picture of a thin boundary layer for flows at high Reynolds number lead to the conclusion that certain terms in the basic differential equations of momentum and continuity are of much smaller order of magnitude than others. The equations are simplified\(^2\) by neglecting

\(^1\)To the approximation of boundary layer theory, the pressure at the "outer edge" of the boundary layer is the same as the normal stress at the corresponding point on the body surface.

\(^2\)The simplifications in the differential equations are also due to Prandtl.
these small terms. The first skin friction calculations based on this boundary layer theory were performed under Prandtl's direction by H. Blasius (Ref. 2), Hiemenz (Ref. 3), and Töpfer (Ref. 4). In these calculations, the fluid's density and viscosity coefficient were regarded as constant. The calculated values agree well with experimental ones.

1.2 The temperature boundary layer.

The flow conditions which lead to a thin boundary layer for the temperature field may be understood by imagining a warm body immersed in a cooler fluid. If the fluid is at rest with respect to the body, a hand placed in the fluid at some distance from the body may feel the warmth, but if the fluid is flowing, the hand must be placed closer to the body or its wake in order to detect the warmth. The region which is noticeably warmed by the body (i.e., the temperature boundary layer) becomes thinner as the flow velocity increases.

The boundary layer theory for the temperature field is analogous to that for the velocity field.

A relation in addition to the equations of continuity and momentum is needed when thermal effects are added to the boundary layer problem. This relation is furnished by the principle of conservation of energy. An analysis of the order of magnitude of the terms in the differential
equation which expresses this principle shows that some of
the terms are small compared with others. In boundary layer
theory, the smaller terms are neglected.

The temperature at the "outer edge" of the boundary
layer is assumed to be related to the velocity at the "outer
edge" by the theory for nonviscous, non-conductory fluids.
The temperature of the fluid adjacent to the body's surface
has the same temperature as the body's surface. E. Pohlhausen
(Ref. 5) was the first to perform calculations based on this
boundary layer theory. Pohlhausen treated the density, vis-
cosity coefficient, specific heat, and thermal conductivity
as constants in his analysis of temperature boundary layer
problems for a thin, plane, sharp-edged plate aligned with
the free stream. The velocity field for this problem is
of course that obtained by Blasius in Reference 2.

1.3 Effect of fluid property variation in the zero
pressure gradient case.

Since a real fluid's density, viscosity, specific heat,
and thermal conductivity all vary with temperature, a logical
next step was that of investigating the effects of the varia-
tion of these properties. For gases obeying $p = \rho RT$, several
investigations of these effects in the zero pressure gradient
case have been reported in References 6-15. A. Busemann
(Ref. 6) was the first to report results considering variable
viscosity. The work of Crocco (Ref. 7) is outstanding for its completeness.

One interesting result is that, if the product of density and viscosity coefficient is constant the Blasius formula is an exact relation between skin friction coefficient and Reynolds number, regardless of Mach number, Prandtl number, or the heat transfer rate. The Blasius formula is

$$c_{fw} \sqrt{R_w} = 0.664$$

where $c_{fw}$ = skin friction coefficient, and $R_w$ is Reynolds number.

The hypothesis that the fluid's Prandtl number is unity leads to a simple enthalpy-velocity relation for the boundary layer which greatly facilitates theoretical investigations. Calculations by Brainerd and Emmons and by Crocco show that for air there is only a slight difference in skin friction values for insulated plates calculated assuming a Prandtl number of unity and those calculated assuming the approximately correct value, 0.73.

On the other hand, the calculations are more sensitive to assumptions regarding the viscosity-temperature relation. With an approximately correct viscosity-temperature formula for air, such as $\mu \alpha T^{0.768}$, $c_{fw} \sqrt{R_w}$ for insulated surfaces increases slightly with increasing free stream Mach number. Thus, the effect of the assumption $\mu \alpha T$ (which
results in the Blasius formula) is to under-estimate $\frac{c_{f_w}}{\sqrt{\frac{R_w}{w}}}$ for insulated surfaces at high Mach numbers.

The constant property formulas of Blasius for skin friction coefficients and of E. Pohlhausen for Nusselt numbers and insulated surface temperatures are useful even in cases where fluid properties are known to vary providing values of the fluid properties at the surface are used in the application of these formulas. This usefulness of constant property formulas was emphasized in References 16 and 17.

1.4 Cases having non-zero pressure-gradient.

Falkner and Skan (Ref. 18) examined several problems in which the distribution of velocity outside the boundary layer $u_1(x)$, is such as to allow a simple solution of the velocity boundary layer equations when density and viscosity coefficient are constant. The "wedge" solutions for $u_1 \propto x^m$ with $m = \text{constant}$ appear to be the most important of these cases. A corresponding problem for the temperature boundary layer has been analyzed by Tifford (Ref. 19), who anticipated the usefulness of the results for compressible gases (even though the density was treated as constant) if values of fluid properties at the surface are used in the formulas.

The velocity boundary layer when $u_1$ is expressed as a power series in $x$, the distance from a stagnation point, has been examined by Howarth (Ref. 20), who treated density
and viscosity coefficient as constants and obtained power series solutions applicable to arbitrary values of the coefficients in the series for \( u_1(x) \).

Only very incomplete information regarding the effects of variable fluid properties is available for cases in which there is a pressure gradient along the surface.

C. R. Illingworth (Ref. 10), K. Stewartson (Ref. 21), and S. Christensen, (Ref. 22), have independently demonstrated, subject to the approximations of boundary layer theory, that for every velocity boundary layer problem in the flow of a gas along an insulated surface, there is a corresponding problem in the flow of a constant property fluid if the gas

(a) obeys the perfect gas equation of state \( p = \rho RT \),
(b) has constant specific heats,
(c) has a Prandtl number of unity, and
(d) has a viscosity coefficient proportional to absolute temperature.

Thus, subject to these hypotheses (or approximations), the solution of the velocity boundary layer problem for a gas flowing along an insulated surface can be obtained from the solution to the corresponding boundary layer problem for a constant-property fluid.

The research reported in this dissertation resulted...
from an attempt to gain more information about the effects of variable fluid properties on boundary layer solutions when there is a pressure gradient along the surface.

2. Notation.

2.1 List of symbols.

$c = \text{constant. (See equation 3.112, page 16).}$

\[ c_{fw} = \frac{\tau_{w}}{\frac{1}{2} \rho_{w} u_{1}^{2}} = \text{local skin friction coefficient based on density at wall.} \]

\[ c_{f1} = \frac{\tau_{w}}{\frac{1}{2} \rho_{1} u_{1}^{2}} = \text{local skin friction coefficient based on density just outside the boundary layer.} \]

\[ c_{fw} = 2c_{fw} \]

\[ c_{f1} = 2c_{f1} \]

\[ c_{p} = \text{specific heat at constant pressure.} \]

\[ c_{v} = \text{specific heat at constant volume.} \]

\[ E = \text{internal energy per unit mass.} \]

\[ F = \text{body force per unit mass.} \]

\[ \mathbf{i}, \mathbf{j}, \mathbf{k} = \text{unit vectors in the direction of increasing } x, y, \text{ and } z. \]

\[ I = E + \frac{p}{\rho} = \text{enthalpy per unit mass.} \]

\[ k = \text{thermal conductivity = energy per unit area per unit time per unit temperature gradient conducted in the direction of } \mathbf{- \text{grad } T}. \]
\(\mathbf{n}\) = unit vector normal to a surface, taken as positive when pointing outward from a closed surface.

\(-p(x,y,z,t)\) = one-third of the sum of the normal components of stress on any three mutually perpendicular planes at point \((x,y,z)\) and at time \(t\).

\(R\) = gas constant in the equation of state \(p = \rho RT\).

\(R_1 = \frac{u_1}{\nu_1}\) = Reynolds number based on kinematic viscosity just outside the boundary layer.

\(R_w = \frac{u_1}{\nu_w}\) = Reynolds number based on kinematic viscosity at wall.

\(s\) = entropy per unit mass.

\(t\) = time.

\(T\) = absolute temperature.

\(u\) = \(x\)-component of velocity vector.

\(U_1 = \frac{u_1}{\sqrt{2I_0}}\)

\(\mathbf{v}\) = velocity vector.

\(V = \frac{u}{u_1}\) except in Appendix A.

\(w\) = \(z\)-component of velocity vector.

\(x\) = curvilinear coordinate measured along a solid surface, except as noted below.

\(y\) = perpendicular distance from solid surface, except as noted below.

Note: In the discussion of fundamentals in Appendix A, \(x,y,z\) denote rectangular position coordinates. See also abbreviated notation below.
\[ \mu = \text{viscosity coefficient.} \]
\[ \nu = \frac{\mu}{\rho} = \text{kinematic viscosity.} \]
\[ \rho = \text{density.} \]
\[ \sigma = \text{Prandtl number} = \frac{\mu c_p}{k} \]
\[ \tau_w = \text{shear stress at wall.} \]
\[ \Psi = \text{stress dyadic, in terms of which the force per unit area exerted by the material situated on the } +n \text{ side of a surface having unit normal vector } \mathbf{n}, \text{ on the material situated on the } -n \text{ side, is } \mathbf{n} \cdot \Psi. \text{ (Notice that the "dot product" of a vector with the dyadic } \Psi \text{ yields a vector.)} \]

Abbreviated notation.

In the abbreviated notation of Section 1.7 of Appendix A, \( x,y,z \) are replaced by \( x_1,x_2,x_3; \) \( u,v,w \) by \( u_1,u_2,u_3; \) and \( i,j,k \) by \( i_1,i_2,i_3. \) Also

\[
e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \text{rate-of-strain component.} \\
x_{ij} = \text{\( j \)-th component of the force per unit area which the material on the } +i \text{ side of a surface having its normal in the direction indicated by } +i \text{ exerts on the material on the } -i \text{ side.} \\

2.2 \text{ Explanatory remarks.} \]

To be consistent, one must use the same system of units throughout a calculation. All energy quantities are here assumed to be expressed in mechanical units, so that the mechanical equivalent of heat does not appear in any of the
equations. Attention is also invited to the fact that internal energies, enthalpies, and specific heats are not referred to unit weight as is done in some engineering texts, but rather they are here referred to unit mass.

In the boundary layer theory, the subscript \( l \) refers to conditions just outside the boundary layer, and the subscript \( w \) refers to conditions adjacent to a wall or surface.

The name "pressure" is avoided in defining the symbol \( p \) above. When no shear stresses are present, as in a static fluid, the normal component of stress on any plane passing through a given point is independent of the orientation of the plane, and with a sign convention that regards tension as positive, this normal stress is the negative of the pressure, as classically defined. Thus, the symbol \( p \) as defined above, stands for the pressure in the special case of no shear stress. It would be better if the term "pressure" were not used when shear stresses are present, but unfortunately it is often used as a name for the quantity represented by the symbol \( p \) above. To avoid confusion, two distinctions must be emphasized. One is the distinction between the states of stress treated by classical thermodynamics in which the stress on any surface is entirely normal to the surface and the more general states of stress in which shear is present. The other is the distinction (when shear is
present) between the normal component of stress and the quantity represented by the above-defined symbol $p$.

3. Some new analyses for the velocity field in the laminar boundary layer of a perfect gas flowing along an insulated surface.

3.1 **Approximate solution assuming $\mu \alpha T$: correlation with solutions which assume constant $p$ and $\mu$.**

In this section, the equations of continuity and momentum for boundary layer flow of a perfect gas having $\mu \alpha T$ are combined with an approximate relation (3.111) between the enthalpy field and the velocity field to obtain a partial differential equation (3.113) which can be transformed to a corresponding equation (3.114) for boundary layer flow of a constant property fluid. The same transformation, (3.115) and (3.116), also transforms the boundary conditions for the gas to the boundary conditions for the constant property fluid. Hence, solutions for the perfect gas can be obtained from already known solutions for fluids having constant properties. As mentioned in the introduction, this correspondence between solutions for gases and solutions for fluids having constant properties has already been demonstrated (Refs. 10, 21, 22) for the special case in which the gas has a Prandtl number of unity and constant specific heat; the correspondence is here extended in an approximate manner.
to $\sigma \neq 1$ and variable $C_p$.

The basic fluid dynamic equations of continuity, momentum, and energy may be simplified for boundary layer flow of a variable property fluid along surfaces of relatively small curvature by arguments analogous to those originally used by Prandtl. The resulting approximate equations are given as (3.101) through (3.103) below.

$$\frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) = 0 \quad (3.101)$$

$$p = p(x).$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) \quad (3.102)$$

$$\rho u \frac{\partial T}{\partial x} + \rho v \frac{\partial T}{\partial y} - u \frac{\partial p}{\partial x} = \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) + \mu \left( \frac{\partial u}{\partial y} \right)^2 \quad (3.103)$$

Attention is invited to the fact that $x$ and $y$ in these boundary layer equations are curvilinear coordinates, not necessarily rectangular coordinates.

Since $p$ and $u$, are assumed to depend only on $x$, and effects of viscosity outside the boundary layer may be disregarded, we write

$$\frac{dp}{dx} = -\rho_1 u_1 \frac{du_1}{dx} \quad (3.104)$$

where the subscript 1 denotes a value "just outside the boundary layer." Since the effects of conductivity outside the boundary layer may also be disregarded, we have

$$I_1 / I_0 = 1 - U_1^2 \quad (3.105)$$

where $U_1^2 = u_1^2/2I_0$. 
In the following development, it is convenient to use the independent variables \((x,V)\) where \(V = u/u_1\) and to choose \(w\), defined by (3.106) below, as the dependent variable.

\[
w = \frac{p_w}{\rho} \sqrt{\frac{1}{\rho_w u_1^2}} (\frac{\partial u}{\partial y})_x \tag{3.106}
\]

By using the equation of state of a perfect gas,

\[
p = \rho RT, \tag{3.107}
\]

it may be shown from (3.101), (3.102), and (3.104) that \(w(x,V)\) must satisfy the following partial differential equation (3.108), in which the independent variables are \((x,V)\).

\[
w^2 \frac{\partial^2}{\partial V^2} \left( \frac{L_{wT}}{\mu_{wT}} w \right) + \frac{1}{2} V w - x V \frac{\partial w}{\partial x} - \frac{1}{2} x V w \frac{d}{dV} \left( \frac{T_w}{T_w} \right) = \frac{x}{U_1} \frac{dU_1}{dx} \frac{T_0}{T_1} \left[ \left( \frac{T}{T_o} - \frac{T_1}{T_1} v^2 \right) \frac{\partial w}{\partial V} - w \frac{\partial}{\partial V} \left( \frac{T}{T_o} \right) \right] \right] \cdot \tag{3.108}
\]

Appropriate boundary conditions on \(w(x,V)\) are given by (3.109) and (3.110).

\[
\left[ w \frac{\partial}{\partial V} \left( \frac{L_{wT}}{\mu_{wT}} w \right) \right]_V = 0, \quad \frac{T_w}{T_1} \frac{x}{U_1} \frac{dU_1}{dx} \cdot \tag{3.109}
\]

\[
w(x,1) = 0. \tag{3.110}
\]

As an approximate relation between the enthalpy field and the velocity field in the boundary layer along insulated

\[1^{\text{In the derivation it is convenient to transform first to independent variables }}(x,u)\text{ as in Appendix A, Section 2.5.}]
surfaces, let us adopt

\[
\frac{I}{I_0} = 1 + U_1^2 \left[ (1-V^2) B - 1 \right].
\]

where \( B = \frac{1}{4} \beta (\sigma) \).

(3.111)

As justification for (3.111) we may remark that

(a) if \( \sigma = 1 \), then the approximate energy equation (3.103) is exactly satisfied for arbitrary \( \frac{dp}{dx} \) whenever (3.111) holds;

(b) if \( \frac{dp}{dx} = 0 \), \( \rho \mu = \) constant, \( \sigma = \) constant, and \( I \) depends only on \( T \), then values of \( \frac{I_w}{I_0} \) (for insulated surfaces) obtained from an exact analysis of the boundary layer equations (3.101) through (3.103) are identical with values of \( \frac{I_w}{I_0} \) obtained from (3.111);

(c) equation (3.109) gives the correct relation

\( \frac{I_1}{I_0} = 1 - U_1^2 \) at the outer edge of the boundary layer for all values of \( \sigma \) and \( \frac{dp}{dx} \);

(d) even for \( \frac{dp}{dx} \neq 0 \) and \( \sigma \neq 1 \), values of \( I_w \) from (3.111) are indistinguishable from values of insulated surface temperatures obtained from an analysis by A. N. Tifford.

\(^1\)The remarks (a) and (b) may be justified by reference to Sections 2.62 and 2.712 of Appendix A.

\(^2\)Hitherto unpublished; Reference 46 will report this analysis.
using constant $\rho$, $\mu$, $C_p$, and $\sigma$.  

A relation between temperature and enthalpy is also needed. Let us write

$$\frac{T}{T_0} = 1 - c(1 - I/I_0), \quad (3.112)$$

where the value of the parameter $c$ is chosen in such a way as to make a best approximation over the temperature range of interest. Any one of several reasonable criteria may be adopted for choosing this best approximation. If $(3.112)$ is considered as the first two terms of a Taylor's series for $T/T_0$ in powers of $(1 - I/I_0)$, then $c = I_0/C_{p_0} T_0$; but other values of $c$ may conceivably result in a better representation for some purposes. The case $C_p = \text{constant}$ is obtained by setting $c = 1$.

By making use of the hypothesis $\mu \propto T$, and by using $(3.111)$ and $(3.112)$, the differential equation and boundary conditions for $w(x,v)$ become

$$w^2 \frac{\partial^2 w}{\partial v^2} + \frac{1}{2} \frac{d}{dx} \frac{\partial w}{\partial x} - x \frac{\partial w}{\partial x} = \frac{x}{U_1} \frac{dU_1}{dx} \left( \frac{1}{1 - cU_1^2} \right) \left[ (1 - v^2) \left( 1 - cU_1^2(1 - B) \right) \frac{\partial w}{\partial v} \right]$$

$$+ \frac{3}{2} \frac{d}{dx} \left( \frac{I_0}{RT_0} - \frac{3}{2} \frac{v}{2} \right) U_1^2 v w \right] \tag{3.113}$$

$$\left( \frac{w}{V w} v \right) = \frac{1}{U_1} \frac{dU_1}{dx} \frac{1 - cU_1^2(1 - B)}{1 - cU_1^2} \right]$$

$$w(x,1) = 0 \quad \text{when } \sigma = 1 \text{ and } dp/dx \neq 0, \text{ the effect of changing the hypotheses on } \rho \text{ and } \mu \text{ to } p = \rho RT \text{ and } \mu \propto T \text{ is not known; however, for insulated surfaces, it follows from remarks (a) and (b) above that there is no effect on } I_w \text{ when } \sigma = 1 \text{ or when } dp/dx = 0.$$

1When $\sigma = 1$ and $dp/dx \neq 0$, the effect of changing the hypotheses on $\rho$ and $\mu$ to $p = \rho RT$ and $\mu \propto T$ is not known; however, for insulated surfaces, it follows from remarks (a) and (b) above that there is no effect on $I_w$ when $\sigma = 1$ or when $dp/dx = 0.$
For a fluid having $\rho$ and $\mu$ constant, the system corresponding to (3.113) is

$$w_1 \frac{\partial^2 w_1}{\partial x_1^2} + \frac{1}{2} V w_1 - x_1 V \frac{\partial w_1}{\partial x_1}$$

$$= \frac{x_1}{U_1} \frac{dU_1}{dx_1} \left[ (1 - V^2) \frac{\partial w_1}{\partial V} + \frac{3}{2} V w_1 \right]. \quad (3.114)$$

$$w_1 (x, 1) = 0$$

If we can set up a correspondence between solutions of (3.113) and solutions of (3.114), then solutions of (3.113) may be obtained by expressing them in terms of known solutions of (3.114). With the objective of finding such a correspondence, let us seek a transformation which transforms the system (3.114) into (3.113). If we set

$$\frac{dx}{dx_1} = f(x) \quad (a)$$

$$w_1 = g(x) w \quad (b) \quad (3.115)$$

$$U_1 = h(x) U_1 \quad (c)$$

then the desired transformation is specified by (3.115) and the following equations (3.116).

$$f(x) = (1 - cU_1^2)^{-\frac{3}{2} \sqrt{\frac{I_o}{cRT_0}}} \quad (a)$$

$$g(x) = \sqrt{\frac{x_1}{x}} f(x) \quad (b) \quad (3.116)$$

$$h(x) = (1 - cU_1^2)^{-B/2} \quad (c)$$
The remainder of this section is a derivation of (3.116).

Substitution of (3.115 into (3.114) leads to the following equations (3.117), since

\[ \frac{\partial u_i}{\partial x_i} = \frac{dx_i}{dx} \frac{\partial}{\partial x} (q u) = f \left( g \frac{\partial u}{\partial x} + q' u \right) \]

and

\[ \frac{d u_i}{d x_i} = \frac{dx_i}{dx} \frac{d}{dx} (q u) = f (-q u' + u_i u'). \]

\[ q^2 u_{i,2} \frac{\partial^2 u_i}{\partial V^2} + \frac{1}{2} V g u_i - x_i V f (q \frac{\partial u}{\partial x} + q' u) \]
\[ = x_i f \left( \frac{u_i'}{u_i} + \frac{g'}{g} \right) \left[ (1-v^2) q \frac{\partial u}{\partial V} + \frac{3}{2} V g u_i \right]. \]

\[ \left\{ \begin{array}{l}
q^2 \left( u_i \frac{\partial u_i}{\partial V} \right)_{V=0} = -x_i f \left( \frac{u_i'}{u_i} + \frac{g'}{g} \right). \\

w(x, 1) = 0.
\end{array} \right. \] (3.117)

The differential equation in (3.117) may be multiplied by \( (x/x_1 g f) \), and the equation expressing the boundary condition at \( V = 0 \) may be divided by \( g^2 \) to obtain the equivalent system (3.118).
\[
\frac{\alpha}{\chi} \frac{\partial^2 w}{\partial x^2} = \frac{1}{\chi} \left[ \frac{3}{2} \frac{U_i'}{U_i} - \frac{3}{2} \chi \left( \frac{U_i'}{U_i} + \frac{h'}{h} \right) \right]
\]

We now note that (3.119) and (3.113) are equivalent if \( f(x), \ g(x), \) and \( h(x) \) satisfy the three conditions enumerated below as (3.119), (3.120), and (3.121).

\[
\frac{\alpha}{\chi} \frac{\partial^2 w}{\partial x^2} = \chi \left( \frac{U_i'}{U_i} + \frac{h'}{h} \right) \left( 1 - \frac{\alpha}{\chi} \frac{\partial u}{\partial x} \right) \frac{\partial w}{\partial x}.
\]

We now note that (3.119) and (3.113) are equivalent if \( f(x), \ g(x), \) and \( h(x) \) satisfy the three conditions enumerated below as (3.119), (3.120), and (3.121).

\[
\frac{\alpha}{\chi} \frac{\partial^2 w}{\partial x^2} = \frac{1 - c U_i^2 (1 - B)}{1 - c U_i^2} \frac{\alpha}{\chi} \frac{U_i'}{U_i}.
\]

(3.119)

\[
\frac{\alpha}{\chi} \frac{\partial^2 w}{\partial x^2} = \frac{1}{\chi} \left[ \frac{3}{2} \frac{U_i'}{U_i} + \frac{h'}{h} \right]
\]

(3.120)

\[
\frac{\alpha}{\chi} \frac{\partial^2 w}{\partial x^2} = \frac{1}{\chi} \left[ \frac{\frac{3}{2} + \left( \frac{R_i}{R_i} - \frac{3c}{2} + 2c B \right) U_i^2}{1 - c U_i^2} \right].
\]

(3.121)
Fortunately, these equations are consistent; it is also fortunate that the two conditions (3.119) and (3.120) serve the three purposes of

1. making the boundary condition at \( V = 0 \) the same in (3.118) as in (3.113);
2. making the coefficient of \( w^2 \frac{\partial^2 w}{\partial v^2} \) the same in (3.118) as in (3.113);
3. making the coefficient of \( (1 - V^2) \frac{\partial w}{\partial v} \) the same in (3.118) as in (3.113).

One may solve for \( f(x) \), \( g(x) \), and \( h(x) \) by the following procedure. Equations (3.119) and (3.120) may be combined to yield

\[
x \left( \frac{U_1'}{U_1} + \frac{h'}{h} \right) = x \frac{U_1'}{U_1} \frac{1 - c_1 U_1^2 (1 - B)}{1 - cU_1^2}
\]

whence

\[
\frac{h'}{h} = \frac{cB U_1 U_1'}{1 - cU_1^2}
\]

and

\[
h = (1 - c U_1^2)^{-B/2}.
\]

One may use (3.120) and (3.115) to show that

\[
\frac{g'}{g} = \frac{g'}{g^2} = \frac{d}{dx} \left( \frac{1}{2x} \frac{x_1 f'}{f} \right) = \frac{1}{2} \frac{f'}{f} + \frac{1}{2x_1 f} - \frac{1}{2x}.
\]

Substitution of the preceding expressions for \( g'/g \) and \( h'/h \) into (3.121) yields an equation which may be simplified to
\[
\frac{f'}{f} = -\left(\frac{B}{2} + \frac{I_0}{cRT_0}\right) \frac{2cU_1U_1'}{1 - cU_1^2}
\]

from which one may obtain

\[
f = (1 - cU_1^2) \left(-\left(\frac{B}{2} + \frac{I_0}{cRT_0}\right)\right).
\]

Equation (3.116b) follows immediately from (3.120).

For \(\sigma = 1\) and \(C_p = \text{constant}\), we have \(B = 1\), \(c = 1\), and

\[
(I_0/cRT_0) = (C_p/R) - \frac{\gamma}{\gamma - 1}
\]

where \(\gamma = C_p/C_v\). The transformation specified by (3.115) and (3.116) is in that case equivalent to the one obtained independently by Illingworth (Ref. 10), Stewartson (Ref. 21), and Christensen (Ref. 22).

In Chapter 5, an example will be presented to contrast the skin friction calculated assuming

1. \(\mu \propto T; \ \sigma = 1; \ C_p = \text{constant}, \ c = 1.000\)
2. \(\mu \propto T; \ \sigma = 0.7; \ C_p = \text{constant}, \ c = 1.000\)
3. \(\mu \propto T; \ \sigma = 1; \ c = 0.985\)
4. \(\mu \propto T; \ \sigma = 0.7; \ c = 0.985\)

3.2 Series solutions with

\[
\frac{\mu}{\mu_0} = 1 - \sum_{n=1}^{\infty} b_n \left(1 - \frac{T}{T_0}\right)^n
\]

where the \(b_n\)'s are arbitrary parameters.

3.21 "Universal functions" for the case \(\sigma = 1\), \(C_p = \text{constant}\), and \(u_1^{\alpha x^m}\).

Crocco (Ref. 7 or Appendix A, Section 2.62) has shown that the relation

\[1 + \frac{1}{2}u^2 = I_0\]
applies in the boundary layer of a fluid whose Prandtl number is 1 when flowing along an insulated surface. Hence

$$\frac{I}{I_0} = 1 - U_1^2 V^2 \quad (3.2101)$$

where $U_1 = u_1/c_0$ with $c_0^2 = 2I_0$ and where $V = u/u_1$. If $C_p$ is constant, then $I/I_0 = T/T_0$, and the temperature is the following known function of $U_1$ and $V$.

$$\frac{T}{T_0} = 1 - U_1^2 V^2. \quad (3.2102)$$

By virtue of the isentropic relation

$$\frac{p_1}{p_0} = \left(\frac{T_1}{T_0}\right)^{\gamma - 1}$$

and the boundary layer hypothesis that $p$ depends only on $x$, one finds that $p$ is a known function of $U_1$. The fluid properties $\mu$ and $\rho$ are related to $T$ and $p$ and hence may also be considered as known functions of $U_1$ and $V$. These considerations suggest that $(U_1, V)$ may be a convenient pair of independent variables when $\sigma = 1$ and the surface is insulated.

If one chooses $W(U_1, V) \text{ defined by}$

$$g = \left( \frac{\partial u}{\partial y} \right)_x = \sqrt{\frac{\rho_1 u_1^3}{\mu_0 x}} W(U_1, V) \quad (3.2013)$$

as the dependent variable, one may obtain the partial differential equation and boundary conditions for $W(U_1, V)$ by a procedure analogous to that used in the derivation of (3.113).
The result is

\[ 0 = (1 - U_1^2 V^2) W^2 \frac{\partial^2 W}{\partial V^2} \left( \frac{U_1}{U_0} \right) + \frac{1}{2}(1 - U_1^2) VW \]

\[ - \frac{x}{U_1} \frac{dU_1}{dx} \left[ (1 - V^2) \frac{\partial W}{\partial V} + \frac{\gamma+1}{2(\gamma-1)} U_1^2 V W + \frac{2}{2} U_1 (1 - U_1^2) V \frac{\partial W}{\partial U_1} \right] \]

\[ \left( W \frac{\partial W}{\partial V} \right) + \frac{x}{U_1} \frac{dU_1}{dx} = 0 \]

\[ V = 0 \]

\[ (3.2104) \]

\[ W(U_1,1) = 0. \]

The hypotheses (in addition to the usual boundary layer relations of Appendix A, Section 2.2) which lead to (3.2104) may be briefly restated as (a) \( p = \rho RT \), (b) \( C_p \) a constant, (c) \( \sigma = 1 \), and (d) insulated surface. The symbols \( U_1 \) and \( V \) have the same significance as in Section 3.1, namely

\[ U_1 = u_1 / c_0 \text{ with } c_0^2 = 2I_0 \]

and \( V = u / u_1 \).

Since \( \mu / \mu_0 \) may be regarded as a function of \( U_1 V \), it is consistent to consider \( W \) as a function of \( U_1 \) and \( V \) if \( x \frac{dU_1}{dx} \) is expressed as a function of \( U_1 \), or if it is constant, so that \( x \) does not appear explicitly in either the differential equation for \( W \) or in the boundary conditions.

The system corresponding to (3.2104) for a fluid having constant properties may be obtained by substituting
\( \rho = \rho_0, \mu = \mu_0, \frac{dp}{dx} = -\rho_0 u_1 \frac{du_1}{dx}, \quad g = \sqrt{\frac{\rho_0 u_1^3}{\mu_0 x}} W(U_1, V), \)

into equation (2.505) of Appendix A and changing to independent variables \((U_1, V)\). The result is the same as the equation obtained from (3.2104) by setting \(\mu/\mu_0 = 1\) and neglecting the terms multiplied by \(U_1^2\) or \(U_1^3\). This system for a fluid with constant \(\rho\) and \(\mu\) offers the possibility of solutions in which \(W\) depends only on \(V\) if \(1 \frac{dU_1}{U_1 dx} = m = \text{constant}\).

These solutions in which \(W\) depends only on \(V\) are the Falkner and Skan solutions discussed (with different dependent and independent variables) in Appendix A, Section 2.412.

Since \(W\) depends only on \(V\) for a constant property fluid if \(1 \frac{dU_1}{U_1 dx}\) is constant, the variation of \(W\) with \(U_1\) in the variable fluid property case can be attributed solely to the variation of density and viscosity coefficient if \(1 \frac{dU_1}{U_1 dx}\) is constant.

It seems reasonable to assume that the viscosity-temperature relation for a gas may be expressed in the form

\[ \frac{\mu/\mu_0}{\mu_0} = 1 - \sum_{n=1}^{\infty} b_n (1 - T/T_0)^n \quad (3.2105) \]

if \(|1 - T/T_0|\) is not too large. The coefficients \(b_1, b_2, \ldots\) may depend on the nature of the gas and on the parameter \(T_0\), but not on \(T\). That is, they are constants for a given gas and a given \(T_0\). It follows from our assumptions of an
insulated surface, \( \sigma = 1 \), and \( C_p = \) constant that
\[
1 - \frac{T}{T_0} = U^2V^2. 
\]  
(3.2102a)

Hence \( \mu / \mu_0 \) can be expanded in a series of powers of \( U^2V^2 \).

Thus,
\[
\mu / \mu_0 = 1 - \sum_{n=1}^{\infty} b_n U^{2n} V^{2n}. 
\]  
(3.2105a)

For example, if Sutherland's formula is used, we have

\[
\frac{\mu}{\mu_0} = \left( \frac{T}{T_0} \right)^{3/2} \frac{1 + \frac{T_0}{T}}{\left( 1 + \frac{T_0}{T} \right) - \left( 1 - \frac{T}{T_0} \right)} = \frac{\left( \frac{T}{T_0} \right)^{3/2}}{1 - b \left( 1 - \frac{T}{T_0} \right)}
\]

(3.2106)

where \( b = \frac{T_0}{T_0 + \frac{T}{T_0}} \leq 1 \).

Expansion of \( \mu / \mu_0 \) from Sutherland's formula in powers of \( (1 - T/T_0) \), i.e. in powers of \( U^2V^2 \), is valid if \( |1 - T/T_0| < 1 \).

With \( \sigma = 1 \) and an insulated surface, we always have

\( 0 < T \leq T_0 \) so that the expansion is always valid for the range of \( T \) which is physically significant. In terms of the parameter \( b \) in (3.2106), we have

\[
b_1 = \frac{3}{2} - b
\]

\[
b_2 = -\frac{3}{8} + \frac{3b}{2} - b^2
\]  
(3.2107)

\[
b_3 = -\frac{1}{16} - \frac{3b}{8} + \frac{3b^2}{2} - b^3
\]

where \( b_1, b_2, b_3, \ldots \) are the coefficients in (3.2105).

Let us seek a solution of (3.2104) for the case
\[
\frac{dU_1}{dx} = m - \text{constant in which } W(U_1,V) \text{ is expressed as a series of powers of } U_1^2 \text{ whose coefficients are functions of } V. \text{ That is, we set }
\]
\[
W(U_1,V) = w_0(V) + U_1^2w_1(V) + U_1^4w_2(V) + \cdots \tag{3.2109}
\]

One may substitute (3.2109), (3.2108), and (3.2105a) into (3.2104) and equate the coefficients of like powers of \( U_1 \) to zero. There results the following ordinary differential equations and boundary conditions for \( w_0(V), w_1(V), \) and \( w_2(V). \)
\[ \mu_0^2 \dddot{\phi} + m(1-V^2) \dddot{\phi} - \frac{3m-1}{2} \dot{V} \dot{\phi} = 0. \]
\[ \phi(0) = -m, \quad \phi(1) = 0. \] 

\[ \begin{aligned}
\mu_0^2 \dddot{\phi} - m(1-V^2) \dddot{\phi} &= \left( \frac{3m-1}{2} \dot{V} \right) \frac{d}{dV} \left( V^\frac{3m-1}{2} \right) \\
&= \left( \frac{3m-1}{2} \right) \left( \frac{1}{2} m + \frac{1}{2} \right) V \phi \phi' - \frac{1}{2} m^2 V^2 \phi' \phi'' + \frac{3m-1}{2} \mu_0^2 \frac{d^2}{dV^2} (V^\frac{3m-1}{2}) \\
&= \left[ \frac{3m-1}{2} \right] \left[ \frac{1}{2} m + \frac{1}{2} \right] V \phi \phi' - \frac{1}{2} m^2 V^2 \phi' \phi'' + \frac{3m-1}{2} \mu_0^2 \frac{d^2}{dV^2} (V^\frac{3m-1}{2}) \\
&= \left[ \frac{3m-1}{2} \right] \left[ \frac{1}{2} m + \frac{1}{2} \right] V \phi \phi' - \frac{1}{2} m^2 V^2 \phi' \phi'' + \frac{3m-1}{2} \mu_0^2 \frac{d^2}{dV^2} (V^\frac{3m-1}{2}) \\
&\quad + \mu_0^2 \left[ \frac{d}{dV} \left( V^\frac{3m-1}{2} \right) \right]^2 \\
&\quad + \frac{3m-1}{2} \mu_0^2 \frac{d^2}{dV^2} (V^\frac{3m-1}{2}) \right].
\end{aligned} \]

\[ \left[ \frac{d}{dV} (\mu_0^2 \phi) \right]_{V=0} = -\phi'(0) \phi'(0). \] 

\[ \phi'(1) = 0. \]
The function \( w_0 \) does not depend on any of the coefficients \( b_1, b_2, \ldots \) in the viscosity temperature relation (3.2104); but \( w_1 \) depends on \( b_1 \); \( w_2 \) depends on both \( b_1 \) and \( b_2 \); etc. However, we may write

\[
\begin{align*}
  w_1 &= w_{10} + b_1 w_{11} \\
  w_2 &= w_{20} + b_1 w_{21} + b_2 w_{22} + b_1^2 w_{23}
\end{align*}
\tag{3.2113}
\]

in which \( w_{10}, w_{11}, w_{20}, w_{21}, w_{22}, w_{23} \) are functions of \( \nu \) which are independent of the coefficients \( b_1, b_2, \ldots \) in (3.2104). To show explicitly the dependence of \( w_3 \) on \( b_1, b_2, \) and \( b_3 \), the nine additional functions in the relation

\[
\begin{align*}
  w_3 &= w_{30} + b_1 w_{31} + b_2 w_{32} + b_3 w_{33} + b_1^2 w_{34} \\
  &+ b_1 b_2 w_{35} + b_1^2 w_{36} + b_1 b_2 w_{37} + b_2^2 w_{38}
\end{align*}
\]

would be required. The number of functions required for such a "universal" representation of \( w_n \) increases rapidly with \( n \).
The differential equations and boundary conditions satisfied by the functions \( w_{10}, w_{11}, w_{20}, \ldots, w_{23} \) follow. Let \( L_n \) denote the linear operator,

\[
L_n = \mu_0^2 \frac{d^2}{dV^2} - \alpha \eta (1-V^2) \frac{d}{dV} - \left( \frac{3(n+2)n-1}{2} \right) V - 2 \mu_0^2 \tilde{m}_0^2.
\]

(3.2114)

\[
L_n \mu_{10} = \left( \frac{1}{2} + n \right) \mu_{10} + V \mu_{10}^2 \tilde{m}_0^2. 
\]

\[
\left[ \frac{d}{dV} (\mu_0 \mu_{10}) \right]_{V=0} = 0. 
\]

(3.2115)

\[
\mu_{10}(1) = 0. 
\]

\[
L_n \mu_{20} = \mu_0^2 \frac{d^2}{dV^2} (V^2 \mu_{20}). 
\]

\[
\left[ \frac{d}{dV} (\mu_0 \mu_{20}) \right]_{V=0} = 0. 
\]

(3.2116)

\[
\mu_{20}(1) = 0. 
\]

\[
L_2 \mu_{20} = \left( \frac{1}{2} + n \right) \mu_{20} - \mu_0^2 \mu_{10}^2 \mu_{10} \tilde{m}_0^2 - 2 \mu_0^2 \mu_{10} \mu_{10} \tilde{m}_0^2 
+ V^2 (\mu_0^2 \mu_{10}^2 + 2 \mu_0^2 \mu_{10} \mu_{10} \tilde{m}_0^2). 
\]

\[
\left[ \frac{d}{dV} (\mu_0 \mu_{10}) \right]_{V=0} = - \mu_{10} (0) \mu_{10} (0). 
\]

(3.2117)

\[
\mu_{20}(1) = 0. 
\]
\[ L_2 \mu_{x1} = \left( \frac{1}{2} + m \right) V_{\mu n} - 2 \mu_{\nu} \left[ \mu_{\nu} \delta_{\mu n} + \mu_{\nu} \left( \frac{d \rho_{\eta}}{dV^2} \right) - V^2 \mu_{\nu} \mu_{\mu n} \right] - 2 \mu_{\nu} \mu_{\nu} \mu_{\mu n} \]

\[ + \mu_{\nu}^2 \frac{d^2}{dV^2} (V^2 \mu_{\nu}) + V^2 \mu_{\nu}^2 \left[ \mu_{\mu n} - \frac{d}{dV^2} (V^2 \mu_{\nu}) \right]. \]

\[ \left[ \frac{d}{dV} (\mu_{\nu} \mu_{x1}) \right]_{V=0} = -\left[ \frac{d}{dV} (\mu_{\nu} \mu_{n}) \right]_{V=0} \]

\[ \mu_{x1} (1) = 0. \]

\[ L_2 \mu_{x2} = \mu_{\nu}^2 \frac{d^2}{dV^2} (V^4 \mu_{\nu}). \]

\[ \left[ \frac{d}{dV} (\mu_{\nu} \mu_{x2}) \right]_{V=0} = 0. \]

\[ \mu_{x2} (1) = 0. \]

\[ L_2 \mu_{x3} = \mu_{\nu}^2 \frac{d^2}{dV^2} (V^2 \mu_{\nu}) - \mu_{\nu} \mu_{\mu n}^2 \]

\[ - 2 \mu_{\nu} \mu_{\mu n} \left[ \mu_{\mu n} - \frac{d^2}{dV^2} (V^2 \mu_{\nu}) \right]. \]

\[ \left[ \frac{d}{dV} (\mu_{\nu} \mu_{x3}) \right]_{V=0} = -\mu_{\mu n} (0) \mu_{\mu n} (0). \]

\[ \mu_{x3} (1) = 0. \]
Because of the large number of functions involved, it hardly seems practical to continue the solution beyond the third term in (3.2109) when the completely universal viscosity formula (3.2105) is used.

Since the experimental data for air viscosity are satisfactorily reproduced by Sutherland's formula over a wide range of temperature, there is no practical loss of generality from an aerodynamicist's point of view, in using Sutherland's formula instead of (3.2105). When Sutherland's formula is used, $w_n(V)$ can be expressed in terms of $(n+1)$ functions which are universal in the sense that they are independent of the parameter $b$ which appears in Sutherland's formula. This is a great reduction in the number of functions required for a universal representation from the number required when (3.2105) is used.

When Sutherland's formula is used, the polynomials in $b$ from (3.2107) are to be substituted for the $b_n$'s appearing in the ordinary differential equations for the $w_n$'s. One may then write

$$w_1 = h_{10} + bh_{11}$$
$$w_2 = h_{20} + bh_{21} + b^2h_{22}$$
$$w_3 = h_{30} + bh_{31} + b^2h_{32} + b^3h_{33}$$

in which the $h$'s are functions that do not depend on $b$. 
The differential equations and boundary conditions satisfied by \( h_{10}, h_{11}, h_{20}, h_{21}, h_{22}, h_{30}, h_{31}, h_{32}, \) and \( h_{33} \) follow.
The following abbreviations are used in the differential equations for $h_{10}, \ldots, h_{33}$.

\[ A = \frac{d^2}{dv^2}(V^2w_0) \]
\[ B = \frac{d^2}{dv^2}(V^2h_{10}) \]
\[ C = \frac{d^2}{dv^2}(V^2w_c) \]
\[ D = \frac{d^2}{dv^2}(V^2h_{11}) + C \]
\[ E = \frac{d^2}{dv^2}(V^2w_c) \]
\[ F = \frac{d^2}{dv^2}(V^2h_{20}) \]
\[ J = \frac{d^2}{dv^2}(V^2h_{10}) \]
\[ K = \frac{d^2}{dv^2}(V^2h_{11}) + J \]
\[ L = \frac{d^2}{dv^2}(V^2h_{11}) + E \]
\[ N = \frac{d^2}{dv^2}(V^2h_{20}) + L \]

The operators $I_1$, $I_2$, and $I_3$ are defined by (3.5.114).

The functions $h_{10}$, $h_{11}$, $h_{20}$, $h_{41}$, $h_{42}$ are related to the $w_i$'s in the following way:

\[ h_{10} = w_{10} + \frac{3}{2} w_{11} \]
\[ h_{11} = - w_{11} \]
\[ h_{20} = w_{20} + \frac{3}{2} w_{21} - \frac{3}{8} w_{22} + \frac{9}{4} w_{23} \]
\[ h_{41} = - w_{21} + \frac{3}{2} w_{22} - 3 w_{23} \]
\[ h_{42} = - w_{22} + w_{23} \]
\[ 
u_0^2 \dddot{u}_0 - m(1 - \nu^2) \dddot{u}_0 - \frac{3\alpha n - 1}{2} \nu \dddot{u}_0 = 0. \]
\[ \nu_0^2 \dddot{u}_0(0) = -m. \]
\[ \dddot{u}_0(1) = 0. \]

\[ L_1 h_{10} = \nu_0 \left( \frac{1}{2} + 3m \right) + V^2 \nu_0^2 \dddot{u}_0 + \frac{3}{2} A \nu_0^2. \]
\[ \left[ \frac{d}{dV} (\nu_0 \dddot{h}_{10}) \right]_{V=0} = 0. \]
\[ h_{10}(1) = 0. \]

\[ L_1 h_{11} = -\nu_0^2 A. \]
\[ \left[ \frac{d}{dV} (\nu_0 \dddot{h}_{11}) \right]_{V=0} = 0. \]
\[ h_{11}(1) = 0. \]

\[ L_2 h_{20} = \left( \frac{1}{2} + \frac{3m}{2} \right) V \dddot{h}_{10} - \nu_0^2 \dddot{h}_{10} + 2V^2 \nu_0^2 \dddot{u}_0 \dddot{h}_{10} + (\dddot{h}_{10} - \frac{3}{2} A)(V^2 \nu_0^2 - 2 \nu_0 \dddot{h}_{10}) + \nu_0^2 \left( \frac{3}{2} B - \frac{3}{2} C \right). \]
\[ \left[ \frac{d}{dV} (\nu_0 \dddot{h}_{20}) \right]_{V=0} = -h_{10}(0) \dddot{h}_{10}(0). \]
\[ h_{20}(1) = 0. \]
\[ L_2 \dot{h}_{21} = \left( \frac{1}{2} + m \right) V \dot{h}_{11} + 2 \dot{u}_0 \dot{h}_{11} \left( \dot{u}_0 V^2 - \dot{h}_{10} \right) - 2 u_0 \dot{h}_{11} \left( \ddot{h}_{10} - \frac{3}{2} A \right) + \left( \ddot{h}_{11} + A \right) \left( V \dot{u}_0^2 - 2 \dot{u}_0 \dot{h}_{10} \right) - u_0^2 \left( B - \frac{3}{2} D \right). \]
\[
\left[ \frac{d}{dV} \left( u_0^2 \dot{h}_{21} \right) \right]_{V=0} = - \left[ \frac{d}{dV} \left( \dot{h}_{11} + h_{11} \right) \right]_{V=0}.
\]
\[ \dot{h}_{21} (1) = 0. \]

\[ L_2 \dot{h}_{22} = - \dot{u}_0 \dot{h}_{11} - 2 u_0 \dot{h}_{11} \left( \ddot{h}_{11} + A \right) - u_0^2 D. \]
\[
\left[ \frac{d}{dV} \left( u_0 \dot{h}_{22} \right) \right]_{V=0} = - \dot{h}_{11} (1) \dot{h}_{21} (1). \]
\[ \dot{h}_{22} (1) = 0. \]

\[ L_3 \dot{h}_{30} = \left( \frac{1}{2} - m \right) V \dot{h}_{20} - 2 u_0 \left[ \dot{h}_{11} \dot{h}_{20} + \dot{h}_{10} \left( \ddot{h}_{10} - \frac{3}{2} A \right) \right] + 3 u_0 \dot{h}_{11} \left( B - \frac{3}{4} C \right) + u_0^2 \left( \frac{3}{2} F - \frac{3}{4} L - \frac{1}{6} K \right) \]
\[ - \dot{h}_{10} \left( \dddot{h}_{10} - \frac{3}{2} A \right) + u_0 \left[ V^2 \left( \ddot{h}_{11}^2 + 2 \dot{h}_{11} \dot{h}_{20} \right) - 2 \dot{h}_{11} \dot{h}_{10} \dot{h}_{20} \right] + V^2 \left( \dot{u}_0^2 \left( \dddot{h}_{10} - \frac{3}{2} \left( B - \frac{1}{4} C \right) \right) + 2 u_0 \dot{h}_{10} \left( \dddot{h}_{10} - \frac{3}{2} A \right) \right]. \]
\[
\left[ \frac{d}{dV} \left( u_0 \dot{h}_{30} \right) \right]_{V=0} = - \left[ \frac{d}{dV} \left( \dot{h}_{10} \dot{h}_{20} \right) \right]_{V=0}.
\]
\[ \dot{h}_{30} (1) = 0. \]
\[ L_3 h_{31} = \left( \frac{1}{2} - m \right) V h_{21} - 2 \frac{\mu_0^2}{2} \left[ h_{11} h_{11} + h_{11} h_{10} - V^2 (h_{10} h_{11} + \mu_0 h_{31}) \right] - \mu_0^2 \left[ k - \frac{3}{2} + \frac{3}{2} L \right] - 2 \mu_0 \left[ h_{10} \left( \frac{3}{2} h_{11} + B - \frac{3}{2} D \right) + \frac{3}{2} (h_{11} + A) \right] - \frac{3}{2} h_{10} (h_{11} + A) + 2 \mu_0 h_{10} (h_{10} - \frac{3}{2} A) + 2 \mu_0 (h_{11} + A) + 2 \mu_0 (h_{10} - \frac{3}{2} A).
\]

\[ \left[ \frac{d}{dV} \left( u_0 h_{31} \right) \right]_{V=0} = - \left[ \frac{d}{dV} \left( h_{10} h_{11} + h_{11} h_{10} \right) \right]_{V=0}, \quad \hat{h}_{31} (1) = 0. \]

\[ L_3 h_{32} = \left( \frac{1}{2} - m \right) V h_{22} - 2 \frac{\mu_0^2}{2} \left[ h_{10} h_{12} + h_{12} h_{10} - V^2 (h_{10} h_{12} + \mu_0 h_{32}) \right] - \mu_0^2 \left[ k - \frac{3}{2} + \frac{3}{2} L \right] - 2 \mu_0 \left[ h_{10} \left( \frac{3}{2} h_{12} + D \right) + \frac{3}{2} (h_{12} + A) \right] - \frac{3}{2} h_{10} h_{12} (h_{12} + A) + 2 \mu_0 h_{12} (h_{12} + D),
\]

\[ \left[ \frac{d}{dV} \left( u_0 h_{32} \right) \right]_{V=0} = - \left[ \frac{d}{dV} \left( h_{10} h_{12} + h_{12} h_{10} \right) \right]_{V=0}, \quad \hat{h}_{32} (1) = 0. \]

\[ L_3 h_{33} = - \mu_0^2 \left[ k - \frac{3}{2} + \frac{3}{2} L \right] - 2 \mu_0 \left[ h_{10} \left( h_{11} + A \right) \right] - (h_{11} + A) + 2 \mu_0 (h_{11} + A) h_{12}.
\]

\[ \left[ \frac{d}{dV} \left( u_0 h_{33} \right) \right]_{V=0} = - \left[ \frac{d}{dV} \left( h_{10} h_{12} + h_{12} h_{10} \right) \right]_{V=0}, \quad \hat{h}_{33} (1) = 0. \]
It was already mentioned that the variation of \( W \) with \( U_1 \) in this problem can be attributed solely to the variation of density and viscosity coefficient. Therefore, that part of \( W \) which does not vary with \( U_1 \), namely \( w_0(V) \), must be obtainable from the Falkner and Skan solutions for a fluid with constant \( \rho \) and \( \mu \). The transformation

\[
Y = \sqrt{\frac{m + 1}{2}} \int_0^V \frac{dV}{w_0}
\]

\[
F(Y) = \int_0^Y VdY
\]

transforms the differential equation for \( w_0 \) to

\[
F' = \frac{d}{dY} \left[ \frac{2m}{m + 1} \left( \frac{F'}{F''} - 1 \right) - F'' \right]
\]

where primes denote differentiation with respect to \( Y \).

Equation (3.2123) may be immediately integrated; and with the aid of the boundary conditions, we find

\[
F(0) = F'(0) = 0.
\]

\[
F'(Y) \rightarrow 1 \text{ as } Y \rightarrow \infty.
\]

The system (3.2124) is identically the system (2.412) of Appendix A, which Hartree (Ref. 40) studied in connection with the Falkner and Skan solutions. The relation between \( w_0(V) \) from (3.2110) and \( F(Y) \) from Appendix A, equation (2.412),
is expressed in equation (3.2125) below.

\[ w_0(V) = \sqrt{\frac{m + 1}{2}} F''(\gamma) \]
\[ v = F'(\gamma) \]  \hspace{1cm} (3.2125)

Note also that for \( m = 0 \),

\[ w_0(V) = w(V), \]

where \( w(V) \) is given by equation (2.716) in Appendix A. From the constant property solutions already in the literature, one knows what the initial values of \( w_0(V) \) should be, for various values of \( m \); knowledge of these initial values enables one to check the accuracy of integration of (3.2110). Values of \( w_0(o) \) for various values of \( m \) may be read from figure 2.412 in Appendix A, or, they may be read from various tables in the literature. In particular, for

\[ m = 0, \quad w_0(0) = 0.33206 \]

and for \( m = 1, \quad w_0(0) = 1.23258766 \)

3.22 Extension to more general functions \( u_1(x) \).

Equation (3.2104) shows that it is consistent to treat the dependent variable \( W \) as a function of \( (U_1V) \) if \( \frac{x}{U_1} \frac{dU_1}{dx} \) can be expressed as a function of \( U_1 \). With \( \mu/\mu_0 \) given by a power series in \( U_1 \) (by equation (3.2105a)), one may obtain solutions of (3.2104) which are power series in \( U_1 \) with coefficients depending on \( V \), whenever \( \frac{x}{U_1} \frac{dU_1}{dx} \) is a power series.
in $U_1$. If one sets

$$\frac{d U_1}{U_1} = m \alpha_1 U_1 + \alpha_2 U_1^2 + \alpha_3 U_1^3 \ldots \quad (3.221)$$

and

$$W(U_1, v) = g_0(v) + U_1 g_1(v) + U_1^2 g_2(v) \quad (3.222)$$

$$+ U_1^3 g_3(v) + U_1^4 g_4(v) + \ldots,$$

then differential equations and boundary conditions for $g_0(v), g_1(v), \ldots$ may be obtained by substitution of (3.222), (3.221) and (3.2105a) into (3.2104) and equation the coefficients of like powers of $U_1$ to zero. One finds that $g_1(v)$ depends on the parameter $\alpha_1$; but if one sets $g_1(v) = \alpha_1 p(v)$, then $p(v)$ is a (universal) function independent of $\alpha_1$; in this way, the dependence on $\alpha_1$ is made explicit.

Similarly, $g_2(v)$ depends on $\alpha_1, \alpha_2,$ and $b_1$; but by setting $g_2(v) = w_{10} + b w_{11} + \alpha_1^2 w_{12} + \alpha_1 \alpha_2 w_{13}$ one finds differential equations and boundary conditions for the functions $w_{10} \ldots w_{13}$ which do not involve the parameters $\alpha_1, \alpha_2$, or $b_1$, and hence $w_{10}, w_{11}, w_{12}$ and $w_{13}$ are independent of these parameters. In this manner, one may continue the solution in terms of functions which are universal in the sense that they do not depend on the parameters $\alpha_n$ or $b_n$. However, the functions do depend on $m$, just as they did in Section 3.21. Some of the functions which occur in this more general case are identical with those defined in Section 3.21, as may be anticipated from the fact that (3.221) includes the simpler
case \(\frac{\text{d}U_1}{U_1 \text{d}x}\) as a special case. For example,

\[ g_0 = w_0, \]

and \(w_{10}\) and \(w_{11}\) are the same in this section as in Section 3.21.

The notation adopted for expressing \(g_1, g_2, g_3,\) and \(g_4\) in terms of universal functions is defined by the following identities in which \(p, r_n,\) and the \(w^s\) are independent of the parameters \(\alpha_n\) and \(b_n.\)

\[ g_1 = \alpha_1 p \]

\[ g_2 = w_{10} + b_1 w_{11} + \alpha_1^2 w_{12} + \alpha_1^2 w_{13} \]

\[ g_3 = \alpha_1 r_1 + b_1 \alpha_1 r_2 + \alpha_1^3 r_3 + \alpha_1^2 r_4 + \alpha_3 r_5 \quad (3.223) \]

\[ g_4 = w_{20} + b_1 w_{21} + b_2 w_{22} + b_1^2 w_{23} + \alpha_2^2 w_{24} + \alpha_2^2 w_{25} \]

\[ + b_1 \alpha_1^2 w_{26} + b_1 \alpha_2^2 w_{27} + \alpha_1^2 \alpha_2^2 w_{28} + \alpha_1 \alpha_3 w_{29} \]

\[ + \alpha_2 w_{2,10} + \alpha_2 w_{2,11} + \alpha_4 w_{2,12}. \]

For correlation with Section 3.21, we note that identical functions are defined by identical symbols.

The differential equations and boundary conditions satisfied by \(p,\) the \(r^s,\) and the \(w^s\) are listed below. The definition \((3.211^4)\) of the operator \(L_n\) is retained; but \(n\) can now take on half-integral values.
\[
0 = (1) \frac{e_{11}}{\text{gap}}
\]
\[
\gamma = \frac{\Delta P}{P}
\]
\[
\varphi = (1) \frac{\varphi}{\text{gap}}
\]
\[
0 = (1) \frac{\varphi}{\text{gap}}
\]
\[
\frac{\varphi}{\text{gap}} = (1) \frac{\varphi}{\text{gap}}
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\[
0 = (1) \frac{\varphi}{\text{gap}}
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\[
\frac{\varphi}{\text{gap}} = (1) \frac{\varphi}{\text{gap}}
\]
\[ L_1 \gamma = -2 \mu_0^2 \varphi \frac{d^2}{d V^2} + 2 \mu_0 \varphi \frac{d^2}{d V^2} \left( \frac{1}{V^2} \right) \]

\[ + \left( \frac{1}{2} + \frac{2}{3} \right) \frac{d}{d V} \left( \frac{1}{V^2} \right) + V^2 \left( \mu_0^2 + 2 \mu_0 \varphi \right) 
+ \left( 1 - \frac{3}{2} \right) \frac{d^2}{d V^2} \left( \frac{1}{V^2} \right) + \frac{2}{3} \frac{d}{d V} \left( \frac{1}{V^2} \right) \]

\[ L_2 \gamma = -2 \mu_0^2 \frac{d^2}{d V^2} (\frac{1}{V^2}) + 2 \mu_0 \varphi \frac{d^2}{d V^2} (\frac{1}{V^2}) - 2 \mu_0 \varphi \gamma \frac{d}{d V} \left( \frac{1}{V^2} \right) 
+ \left( 1 - \frac{3}{2} \right) \gamma \frac{d^2}{d V^2} \left( \frac{1}{V^2} \right) + \frac{2}{3} \frac{d}{d V} \left( \frac{1}{V^2} \right) \]

\[ L_3 \gamma = -2 \mu_0^2 \varphi \frac{d^2}{d V^2} \left( \frac{1}{V^2} \right) - 2 \mu_0 \varphi \frac{d^2}{d V^2} \left( \frac{1}{V^2} \right) 
+ \left( 1 - \frac{3}{2} \right) \frac{d}{d V} \left( \frac{1}{V^2} \right) + \frac{2}{3} \frac{d}{d V} \left( \frac{1}{V^2} \right) \]

\[ \gamma_1 (1) = 0. \]

\[ \gamma_2 (1) = 0. \]

\[ \gamma_3 (1) = 0. \]
\[ L_{\frac{3}{2}} R_4 = -2 \mu_5 \mu \tilde{u}_3 - 2 \mu_5 \mu \tilde{u}_3 - 2 \tilde{u}_5 \mu \mu \tilde{u}_3 \\
+ (1 - \nu^2) \mu \tilde{u}_3 + \frac{5}{2} V \mu + (1 - \nu^2) \tilde{u}_3 + \frac{7}{2} V \mu \tilde{u}_3. \]
\[
\left[ \frac{d}{dV} (\mu \tilde{u}_3) \right]_{V=0} = - \left[ \frac{d}{dV} (\mu \tilde{u}_3) \right]_{V=0}.
\]
\[ R_4 (1) = 0. \]

\[ L_{\frac{3}{2}} R_5 = (1 - \nu^2) \tilde{u}_5^{\nu} + \frac{3}{2} V \mu \tilde{u}_5^{\nu}.
\]
\[
\left[ \frac{d}{dV} (\mu \tilde{u}_5^{\nu}) \right]_{V=0} = -1.
\]
\[ R_5 (1) = 0. \]

\[ L_5 \mu_0^{\nu} = -2 \mu_5 \mu_0^{\nu} \tilde{u}_0^{\nu} - \tilde{u}_5 \mu_0^{\nu} - \frac{1}{2} (\nu + \mu) V \mu_0^{\nu} \\
+ V^2 \left( \mu_0^{\nu} \tilde{u}_0^{\nu} + 2 \mu_5 \mu_0^{\nu} \tilde{u}_0^{\nu} \right).
\]
\[
\left[ \frac{d}{dV} (\mu_0^{\nu} \tilde{u}_0^{\nu}) \right]_{V=0} = - \mu_0^{\nu} (0) \tilde{u}_0^{\nu} (0).
\]
\[ \mu_0^{\nu} (1) = 0. \]

\[ L_5 \mu_2^{\nu} = \mu_5^{\nu} \frac{d^2}{dV^2} (V^2 \mu_0^{\nu}) + 2 \mu_5^{\nu} \mu_0^{\nu} \frac{d^2}{dV^2} (V^2 \mu_0^{\nu}) \\
- 2 \mu_5^{\nu} \mu_0^{\nu} \tilde{u}_0^{\nu} - 2 \mu_5^{\nu} \mu_0^{\nu} \tilde{u}_0^{\nu} + \frac{1}{2} (\nu + \mu) V \mu_0^{\nu} \\
+ V^2 [\mu_0^{\nu} \tilde{u}_0^{\nu} + 2 \mu_0^{\nu} \tilde{u}_0^{\nu} \tilde{u}_0^{\nu} - \mu_0^{\nu} \frac{d^2}{dV^2} (V^2 \mu_0^{\nu})].
\]
\[
\left[ \frac{d}{dV} (\mu_0^{\nu} \mu_2^{\nu}) \right]_{V=0} = - \left[ \frac{d}{dV} (\mu_0^{\nu} \mu_0^{\nu}) \right]_{V=0}.
\]
\[ \mu_2^{\nu} (1) = 0. \]
\begin{align*}
L_2 \mu_{22} &= \omega_0^2 \frac{d^2}{dv^2} (V^2 \mu_0), \\
\left[ \frac{d}{dv} (\omega_0 \mu_{22}) \right]_{v=0} &= 0. \\
\mu_{22} (l) &= 0.
\end{align*}

\begin{align*}
L_2 \mu_{23} &= \omega_0^2 \frac{d^2}{dv^2} (V^2 \mu_0) + 2 \omega_0 \mu_0 \frac{d^2}{dv^2} (V^2 \mu_0) \\
-2 \omega_0 \mu_0 \mu_{12} - i \omega_0 \mu_0^2. \\
\left[ \frac{d}{dv} (\omega_0 \mu_{23}) \right]_{v=0} &= -\mu_{12} (0) \hat{\mu}_{12} (0). \\
\mu_{23} (l) &= 0.
\end{align*}

\begin{align*}
L_2 \mu_{24} &= -2 \omega_0 \mu_0 \mu_{12} - i \omega_0 (\mu_0^2 + 2 \omega_0 \mu_{12}) - 2 \omega_0 \mu_{12} \hat{\mu}_{12} \\
-2 \mu_0 \mu_{12} - 2 \omega_0 \mu_0 \mu_{12} - 2 \mu_0 \mu_{12} + \frac{1}{2} \mu_0 \mu_{12} \mu_2 + \frac{1}{2} \mu_0 \mu_{12} \mu_2 \\
+ (\frac{1}{2} + \frac{1}{2}) V \mu_{12} + V^2 \mu_{12} + 2 \mu_0 \mu_{12} \mu_2 + \mu_0 \mu_2 + 2 \mu_0 \mu_{12} \mu_2 \\
+ (1-V^2) \mu_{12} + \frac{9}{4} V \mu_{12} + 2 V \mu_{12}. \\
\left[ \frac{d}{dv} (\omega_0 \mu_{24}) \right]_{v=0} &= -\left[ \frac{d}{dv} (\mu_0 \mu_{12} + \omega_0 \mu_{12}) \right]_{v=0} \\
\mu_{24} (l) &= 0.
\end{align*}

\begin{align*}
L_2 \mu_{25} &= -2 \mu_0 \mu_{12} - i \omega_0 (\mu_0^2 + 2 \omega_0 \mu_{12}) - 2 \mu_0 \mu_{12} \\
-2 \mu_0 \mu_{12} - 2 \omega_0 \mu_0 \mu_{12} - 2 \mu_0 \mu_{12} + \frac{1}{2} \mu_0 \mu_{12} \mu_2 + \frac{1}{2} \mu_0 \mu_{12} \mu_2 \\
+ (\frac{1}{2} + \frac{1}{2}) V \mu_{12} + V^2 \mu_{12} + 2 \mu_0 \mu_{12} \mu_2 + \mu_0 \mu_2 + 2 \mu_0 \mu_{12} \mu_2 \\
+ (1-V^2) \mu_{12} + \frac{9}{4} V \mu_{12} + 2 V \mu_{12}. \\
\left[ \frac{d}{dv} (\omega_0 \mu_{25}) \right]_{v=0} &= -\left[ \frac{d}{dv} (\mu_0 \mu_{12}) \right]_{v=0} - \mu_{12} (0) \hat{\mu}_{12} (0). \\
\mu_{25} (l) &= 0.
\end{align*}
\[ L_{12} \mathbf{w}_{26} = \mu_0 \frac{d^2}{dV^2} (V^2 \mathbf{w}_{26}) + 2 \mu_0^2 \frac{d^2}{dV^2} (V^2 \mathbf{w}_{26}) - 2 \mu_0 \mu_1 \mathbf{w}_{12} + \left( -\mathbf{w}_{12} \right) \mathbf{w}_{12} \]

\[ -2 \mu_0 \left( \mathbf{w}_{12} \mathbf{w}_{12} - \mathbf{j} \left( 2 \mu_0 \mathbf{w}_{12} + 2 \mu_2 \mathbf{r}_{12} \right) \right) \]

\[ -i \mathbf{w}_{12} \left( 2 \mu_0 \mathbf{w}_{12} + 2 \mu_2 \mathbf{r}_{12} \right) + \left( 1 - \nu^2 \right) \mathbf{r}_{12} + \frac{7}{2} V \mathbf{r}_{12} \]

\[ \left[ \frac{d}{dV} \left( \mu_0^2 \mathbf{w}_{26} \right) \right]_{V=0} = -\left[ \frac{d}{dV} \left( \mu_0 \mathbf{w}_{12} + \mathbf{w}_{12} \mathbf{w}_{12} \right) \right]_{V=0}. \]

\[ \mathbf{w}_{26} (1) = 0. \]

\[ L_{12} \mathbf{w}_{27} = \mu_0 \frac{d^2}{dV^2} (V^2 \mathbf{w}_{27}) + 2 \mu_0 \mu_1 \mathbf{w}_{13} \frac{d^2}{dV^2} (V^2 \mathbf{w}_{27}) \]

\[ -2 \mu_0 \left( \mathbf{w}_{13} \mathbf{w}_{13} + \mathbf{j} \left( 2 \mathbf{w}_{13} \mathbf{w}_{13} + (1 - \nu^2) \right) \mathbf{r}_{13} + \frac{7}{2} V \mathbf{r}_{13} \right) \]

\[ \left[ \frac{d}{dV} \left( \mu_0 \mathbf{w}_{27} \right) \right]_{V=0} = -\left[ \frac{d}{dV} \left( \mu_0 \mathbf{w}_{13} + \mathbf{w}_{13} \mathbf{w}_{13} \right) \right]_{V=0}. \]

\[ \mathbf{w}_{27} (1) = 0. \]

\[ L_{12} \mathbf{w}_{28} = -2 \mu_0 \left( \mathbf{w}_{13} \mathbf{w}_{13} + \mathbf{j} \left( 2 \nu^2 + 2 \mu_0 \mathbf{w}_{13} \right) \right) - 2 \mu_0 \mu_1 \mathbf{w}_{12} \mathbf{w}_{13} \]

\[ -\mathbf{j} \left( 2 \mu_0 \mathbf{w}_{13} + 2 \mu_2 \mathbf{r}_{13} \right) - i \mathbf{w}_{13} \left( 2 \mu_0 \mathbf{w}_{13} + 2 \mu_2 \mathbf{r}_{13} \right) \]

\[ + \left( 1 - \nu^2 \right) \mathbf{w}_{13} + \frac{7}{2} V \mathbf{w}_{13} + \left( 1 - \nu^2 \right) \mathbf{r}_{13} + \frac{9}{2} V \mathbf{r}_{13} \]

\[ \left[ \frac{d}{dV} \left( \mu_0 \mathbf{w}_{28} \right) \right]_{V=0} = -\left[ \frac{d}{dV} \left( \mu_0 \mathbf{w}_{13} + \mathbf{w}_{13} \mathbf{w}_{13} \right) \right]_{V=0}. \]

\[ \mathbf{w}_{28} (1) = 0. \]
\[ L_2 \omega_{2,1} = -2 \omega_0 \dot{\omega}_5 - 2 \omega_0 \ddot{\omega}_5 - 2 \dot{\omega}_0 \ddot{\omega}_5 + (1-V^2) \dot{\omega}_5 + \frac{3}{2} \dot{V} \dot{\omega}_5. \]

\[
\left[ \frac{d}{dV} (\omega_0 \omega_{2,1}) \right]_{V=0} = - \left[ \frac{d}{dV} (\omega_0 \omega_5) \right]_{V=0}.
\]

\[ \omega_{2,1} (1) = 0. \]

\[ L_2 \omega_{2,10} = -2 \omega_0 \omega_3 \ddot{\omega}_3 - i \dot{\omega}_0 \omega_3^2 + (1-V^2) \dot{\omega}_3 + \frac{3}{2} \dot{V} \omega_3. \]

\[
\left[ \frac{d}{dV} (\omega_0 \omega_{2,10}) \right]_{V=0} = - \omega_{13} (0) \dot{\omega}_3 (0).
\]

\[ \omega_{2,10} (1) = 0. \]

\[ L_2 \omega_{2,11} = -2 \omega_0 (\dot{\omega}_{10} \ddot{\omega}_3 + \dot{\omega}_{10} \omega_3) - 2 \dot{\omega}_5 \omega_{10} \omega_3 + \left( \frac{i}{2} + m \right) \dot{V} \omega_3 + \dot{V}^2 (\omega_0^2 \ddot{\omega}_3 + 2 \omega_0 \dddot{\omega}_3) + (1-V^2) \dot{\omega}_{10} + \frac{3}{2} \dot{V} \dot{\omega}_{10} + 3 \dot{V} \omega_5. \]

\[
\left[ \frac{d}{dV} (\omega_0 \omega_{2,11}) \right]_{V=0} = - \left[ \frac{d}{dV} (\omega_0 \omega_{13}) \right]_{V=0}.
\]

\[ \omega_{2,11} (1) = 0. \]

\[ L_2 \omega_{2,12} = (1-V^2) \dot{\omega}_5 + \frac{3}{2} \dot{V} \dot{\omega}_5. \]

\[
\left[ \frac{d}{dV} (\omega_0 \omega_{2,12}) \right]_{V=0} = -1.
\]

\[ \omega_{2,12} (1) = 0. \]
3.23 Extension to variable specific heat.

If one is willing to introduce enough additional "universal" functions, the effect of variable specific heat can be accounted for in any of the series solutions discussed in Sections 3.21 and 3.22.

With \( W(U_1,V) \) defined by (3.2103), one may transform (2.505) to

\[
\frac{T}{T_0} W^2 \frac{\partial^2}{\partial V^2} \left( \frac{\rho}{\rho_0} W \right) + \frac{T_1}{T_0} VW - \frac{x}{U_1} \frac{dU_1}{dx} \left\{ \left( \frac{T}{T_0} - v^2 \frac{T_1}{T_0} \right) \frac{\partial W}{\partial V} \right\} = 0
\]

(3.231)

with the same boundary conditions as for (3.2104). In obtaining (3.231), from (2.505), it has been assumed that

\[
p = \rho RT,
\]

\[
dp/dx = - \rho_1 u_1 du_1/dx,
\]

and

\[
I_1/I_0 = 1 - U_1^2.
\]

Recall that \( U_1 = u_1/\sqrt{2I_0} \), and \( V = u/u_1 \).

Since the surface is insulated and since \( \sigma = 1 \), equation (3.2101) applies, so that

\[
1 - I/I_0 = U_1^2 V^2
\]

(3.2101a)

An expression for \( T/T_0 \) for use in (3.231) and in (3.2105) may be obtained by expanding \( T/T_0 \) in a Taylor's series of
powers of \((1 - I/I_0)\). Thus,
\[
\frac{T}{T_0} = 1 - \sum_{n=1}^{\infty} c_n (1 - \frac{I}{I_0})^n
\]
(3.232)

which by virtue of (3.2101a), becomes
\[
\frac{T}{T_0} = 1 - \sum_{n=1}^{\infty} c_n U_1^{2n} V_{2n}^n
\]
(3.233)

The coefficients \(c_n\) will depend on the nature of the gas and on \(T_0\), but are constants for a given problem if \(I\) depends only on \(T\); in this case,
\[
c_1 = \frac{I_0}{C_{P_0} T_0}
\]
\[
c_2 = \frac{I_0^2}{2C_{P_0}^3 T_0} \left( \frac{dC_{P_0}}{dT} \right)_{T=T_0}
\]
(3.234)
\[
c_3 = \frac{I_0^3}{6C_{P_0}^5 T_0} \left[ 3 \frac{dC_{P_0}}{dT} - C_{P_0} \frac{d^2C_{P_0}}{dT^2} \right]_{T=T_0}
\]

Substitution of the series for \(W\), for \(\mu/\mu_0\), for \(T/T_0\), and for \(x \frac{dU_1}{U_1 dx}\) into (3.231) and equating the coefficients of like powers of \(U_1\) to zero leads, as with \(C_p\) constant, to differential equations for the coefficients in the series for \(W\). The difference is that additional parameters now appear in the differential equations. However, the dependence of the coefficients in the series for \(W\) on these additional parameters may be expressed explicitly by defining additional universal functions. For example, in case \(x \frac{dU_1}{U_1 dx} = m = \) constant, and \(W = w_0(V) + U_1^2 w_1(V) + U_1^4 w_2(V) + \cdots\), it turns
out that $w_0(V)$ is given by the same function as in Section 3.21, and that we may write

$$w_1 = c_1(w_{10} + b_1 w_{11} + \epsilon w_{14})$$

where $w_{10}$ and $w_{11}$ are the same functions as in Section 3.21, where $\epsilon = (c_{p_0}/R - 3.5)$, and where $w_{14}$ is an additional "universal" function satisfying (3.235) below.

$$L_1 w_{14} = m V w_0$$

$$\left( \frac{d}{dV} (w_0 w_{14}) \right) \bigg|_{V=0} = 0 \quad (3.235)$$

$$w_{14}(1) = 0$$

In (3.235), $L_1$ is the operator defined by (3.2114), page 29.
3.24 **Approximate extension to** $\sigma \neq 1$.

The analyses in Sections 3.21, 3.22, and 3.23 are restricted to $\sigma = 1$ and insulated surfaces because the $\sigma = 1$, insulated surface case is the only one for which we know an "exact" expression for the enthalpy involving only integral powers of $U_1$. Whenever one has available such an expression in other cases, then the analysis is capable of extension to those cases.

For example, if one adopts (3.111), the analysis may be carried out by simply replacing (3.2101) by (3.111). The Prandtl number, $\sigma$, or $\beta (\sigma)$, now appears as an additional parameter, and this greatly increases the number of functions required for a universal representation.

If $\frac{\frac{x}{U_1}}{dU_1} = m = \text{constant}$, and if we again use (3.2105), (3.2109), and (3.232), as well as (3.111) in (3.2311), then $w_0$ is the same function as in Section 3.2; but for a universal representation we must now write

\[ w_1 = c_1 \left[ w_{10} + b_1 w_{11} + \epsilon w_{14} (1 - \frac{\beta}{4}) w_{15} + b_1 (1 - \frac{\beta}{4}) w_{16} \right] \]

where $w_{10}$, $w_{11}$, $w_{14}$ are as in Section 3.23, and $w_{15}$ and $w_{16}$ are additional functions satisfying (3.241) and (3.242) in which $L_1$ is still defined by (3.2114),

\[ L_1 w_{15} = (1 - v^2) w_{0}^2 w_0 - m(1 - v^2) w_0 - 2mv w_0 . \]

\[ \left[ \frac{d}{dv} \left( w_0 w_{15} \right) \right] = 0 . \quad (3.214) \]

\[ w_{15} (1) = 0 . \]
\[ L_1 w_{16} = w_0^2 \frac{d^2}{dv^2} \left[ (1 - v^2) w_0 \right]. \]
\[
\left[ \frac{d}{dv} \left( w_0 w_{16} \right) \right]_{v=0} = 0. \quad (3.242)
\]

Thus, five functions are required for a completely universal representation of \( w_1 \). This number of functions required for a universal representation of \( w_n \) increases very rapidly with \( n \).
4. Analyses for non-insulated surfaces.

4.1 Analysis in series.

Solutions for non-insulated surfaces may be obtained by expressing both the non-dimensional shear stress and the enthalpy ratio as power series in $U_1$ whose coefficients are functions of $V$, but these coefficient functions lack the universal characteristics of the functions for insulated surfaces.

It would be possible to obtain a solution with $\mu / \mu_0$ expressed as a polynomial in $(1 - T/T_0)$, but the general infinite series representation (3.2105) cannot be used here, because each of the b's would enter each term of the series expression for $\mu / \mu_0$ in powers of $U_1$. For the purpose of illustrating how a solution may be obtained, we shall here make use of the approximation $\mu / \mu_0 \approx \omega T^m$ with $\omega = \text{constant}$.

The relation (3.112) connecting $I$ and $T$ may be used, or $T/T_0$ may be expressed as a polynomial in $(1 - I/I_0)$; for illustrative purposes, we here consider only the constant specific heat relation $T/T_0 = I/I_0$.

The pressure gradient parameter, $\frac{x}{U_1} \frac{dU_1}{dx}$, must again be expressed in a form containing only integral powers of $U_1$. We shall here consider the case $\frac{x}{U_1} \frac{dU_1}{dx} = m = \text{constant}$.
One may choose \( w \) defined by (3.106) as one dependent variable and \( I \) as another, and may again use independent variables \((x,V)\) where \( V = u/U_1 \). Then, by using the relations (3.101) through (3.107), it may be shown\(^1\) that the pair of partial differential equations to be satisfied are (3.108) and the following equation (4.11).

\[
x V \frac{\partial I}{\partial x} + \frac{x}{U_1} \frac{\partial U_1}{\partial x} \frac{T_0}{T_1} \left[ 2 \frac{T_0}{T_1} T_1 U_1^2 V + \left( \frac{T_0}{T_1} - \frac{T_0 T_1 V^2}{T_1} \right) \frac{\partial I}{\partial V} \right]
- \frac{1}{\alpha} \frac{\partial I}{\partial V} \frac{\partial}{\partial V} \left[ \frac{T_{W_1}}{T_{W_1}} \right] = \frac{4 T_{W_1}^2}{\mu_{W_1}} \left[ 2 I_0 U_1^2 + \frac{1}{\beta} \frac{\partial I}{\partial V} \right].
\]

The boundary conditions are (3.109), (3.110), the condition
\[
I(x,1) = 1 - U_1^2,
\]
and an additional condition specifying either \( I(x,0) \) or \( \partial I/\partial V \) at \( V = 0 \).

A change of independent variables in these equations from \((x,V)\) to \((U_1,V)\) results in the following equations (4.13 and (4.14).

\(^1\)In the derivation, it is convenient to transform first to independent variables \((x,u)\) as in Section 2.5 of Appendix A.
The substitutions indicated by the following equations (4.15) will be made.

\[
\begin{align*}
    w(U_1, V) &= w_0(V) + U_1w_1(V) + U_1^2w_2(V) \\
    I/I_0 &= h_0(V) + U_1h_1(V) + U_1^2h_2(V) \\
    \frac{\kappa}{U_1} \frac{dU_1}{dx} &= m = \text{constant.} \\
    C_p &= \text{constant; } \gamma = C_p/C_v = 1.400 \\
    \sigma &= \text{constant.}
\end{align*}
\]
The expression for $I/I_o$ in (4.15) demands that the surface enthalpy distribution be expressible as a power series in $U_1$:

$$\frac{I_w}{I_o} = a_0 + U_1a_1 + U_1^2a_2 + \cdots$$  \hspace{1cm} (4.16)

An isothermal surface is included as a special case, of course.

By means of (4.15) and (4.16), all of the dependent variables appearing in (4.13) and (4.14) may be expressed as power series in $U_1$ whose coefficients are combinations of the functions $w_n$ and $h_n$, $n = 0, 1, 2, \cdots$. By substituting these power series into (4.13) and (4.14) and equating the coefficients of like powers of $U_1$ to zero, one may obtain pairs of ordinary differential equations and boundary conditions for the functions $(w_n, h_n)$. The first pair (obtained from the coefficient of $U_1^0$) involves only $w_0$ and $h_0$ as dependent variables. The next pair involves the two additional dependent variables $w_1$ and $h_1$; the next pair the two additional variables $w_2$ and $h_2$; etc. Thus, if the first pair can be solved for $w_0$ and $h_0$, then the second pair involves only two remaining unknowns, $w_1$ and $h_1$. Once $w_1$ and $h_1$ are found, the only unknown functions in the third pair are $w_2$ and $h_2$. Thus, one may hope to obtain a solution by solving the successive pairs of simultaneous differential
equations.

If $a_1 = 0$, both $w_1$ and $h_1$ are identically zero. Only even powers of $U_1$ are involved in the solution if only even powers appear in (4.16).

The first two pairs of differential equations and their boundary conditions follow.
\[
\left( \frac{1}{a_0} \right)^{\omega-1} \frac{d^2}{dV^2} \left( \omega_0^2 \cdot \frac{d^2}{dV^2} \left( \omega_0^2 \cdot \frac{d^2}{dV^2} \right) \right) + \frac{1}{a_0} \nu_0 = m \left[ (h_0 - V^2) \nu_0 - \nu_0 h_0 + \frac{3}{2} \nu_0 \right].
\]

\[
m \left( h_0 - V^2 \right) \nu_0 = \frac{1}{a_0} \left[ (1-\omega) \frac{d}{dV} \left( \omega_0^2 h_0 \nu_0 \right) + \omega_0^2 h_0 \nu_0 \right].
\]

\[
\nu_0 (0) \nu_0 (0) = -m a_0 - \left( \frac{1}{a_0} \right) \left( \omega_0^2 h_0 \nu_0 (0) \right) h_0 (0) \nu_0 (0).
\]

\[
\nu_0 (1) = 0.
\]

\[
h_0 (0) = a_0.
\]

\[
h_0 (1) = 1.
\]

\[
\left( \frac{1}{a_0} \right)^{\omega-1} \left[ 2 \omega_0^2 h_0 \frac{d^2}{dV^2} \left( \omega_0^2 h_0 \nu_0 ^{\omega-1} \right) + \omega_0^2 \frac{d^2}{dV^2} \left( \omega_0^2 h_0 \nu_0 ^{\omega-1} \right) \right] + \frac{1}{a_0} \nu_0 = m \left[ h_0 \nu_0 ^{\omega-1} + (h_0 - V^2) \nu_0 - \nu_0 h_0 + \nu_0 - \frac{3}{2} (\omega - V \frac{a_0}{a_0} \nu_0) \right].
\]

\[
m \left[ (h_0 - V^2) \nu_0 + h_0 \nu_0 + h_0 \right] = \frac{1}{a_0} \left[ (1-\omega) \frac{d}{dV} \left( \omega_0^2 h_0 \nu_0 ^{\omega-1} \right) + \omega_0^2 \frac{d}{dV} \left( \omega_0^2 h_0 \nu_0 ^{\omega-1} \right) \right] + \frac{1}{a_0} \frac{d}{dV} \left( \omega_0^2 h_0 \nu_0 ^{\omega-1} \right) - \frac{2}{2} \omega_0^2 h_0 \nu_0 ^{\omega-1} \nu_0 \nu_0 \nu_0.
\]

\[
\left[ \frac{d}{dV} \left( \omega_0^2 \nu_0 \right) \right] _{V=0} = -m a_0 - \left( \frac{1}{a_0} \right) \left[ \omega_0 ^{\omega-1} \nu_0 ^{\omega-2} \right] _{V=0} - \frac{1}{2} \left( \omega - V \frac{a_0}{a_0} \nu_0 \right) \nu_0 (0) \nu_0 (0) h_0 (0) h_0 (0) h_0 (0).
\]

\[
\nu_0 (1) = 0.
\]

\[
h_0 (0) = a_1.
\]

\[
h_0 (1) = 0.
\]
4.2 Approximate treatment of skin friction for an incompressible fluid neglecting dissipation.

If one neglects $udp/dx$ and $\left(\mu \frac{\partial u}{\partial y}\right)^2$ in (3.103), and then transforms to new independent variables $(x,u)^1$, the following equations (4.20) and (4.21) result from (3.101), (3.102), and (3.103).

\[
\begin{align*}
\text{4.20} & \quad u \frac{\partial g}{\partial x} - ug \frac{\partial g}{\partial x} - \frac{\partial g}{\partial u} \frac{dp}{dx} = g^2 \frac{\partial^2}{\partial u^2} (\mu g) \quad \text{(4.20)} \\
\text{4.21} & \quad \frac{1}{\mu} \left[u \frac{\partial u}{\partial x} - \frac{dp}{dx} \frac{\partial I}{\partial u} - \frac{1}{\alpha} g \frac{\partial}{\partial u} (\mu g) \frac{\partial I}{\partial u} \right] - \mu g^2 \frac{\partial}{\partial u} \left(\frac{1}{\alpha} \frac{\partial I}{\partial u}\right) = 0 \quad \text{(4.21)}
\end{align*}
\]

The dependent variable $g$ in (4.20) and (4.21) is defined by

\[
g = \left(\frac{\partial u}{\partial y}\right)_x
\]

By transforming (4.20) and (4.21) from the independent variables $(x,u)$ to new independent variables $(x,V)$ where $V = u/u_1$, and then setting

\[
g = \frac{1}{\mu} \sqrt{\frac{u_1^3}{x} \rho w F \rho w G} \quad \frac{x}{U_1} \frac{dU}{dx} = m = \text{constant} \quad \text{(4.22)}
\]

there results differential equations and boundary conditions.

---

1 The same transformation is carried out in Section 2.5 of Appendix A including the $udp/dx$ term and the $(\mu \frac{\partial u}{\partial y})^2$ term. The result is equations (2.505) and (2.508).
for \( G \) and \( I \) in which one may consistently treat both \( G \) and \( I \) as functions only of \( V \) if the enthalpy value outside the boundary layer, \( I_1 \), and the value at the wall, \( I_w \), are constants. The differential equation and boundary conditions for \( G(V) \) are

\[
g^2 G'' + \frac{3}{2} \frac{\rho_{w}}{\rho_1} \frac{\rho_{w}}{\rho_1} V G = m \frac{\rho_{l}}{\rho_{w}} \left( 1 - \frac{\rho_{w}}{\rho_1} V^2 \right) G' + \frac{3}{2} \frac{\rho_{w}}{\rho_1} V G + G \left( \frac{\rho_{w}}{\rho_1} \frac{d}{dv} \ln \frac{\rho_{w}}{\rho_1} - \frac{\rho_{w}}{\rho_1} \frac{d}{dv} \ln \frac{\rho_{w}}{\rho_1} \right) \quad (4.23)\]

\[
G(0)G'(0) = - m \frac{\rho_{l}}{\rho_{w}} \quad \{ \}
\]

\[
G(1) = 0.
\]

In this section, we use the approximation that \( G(0) \) in (4.23) is nearly equal to \( G(0) \) in (4.27). Before proceeding, the line of thought which suggested this approximation will be indicated briefly.

If one retains the expression

\[
dp/dx = - (\rho_{l} u_{l} du_{l}/dx)
\]

but otherwise replaces \( \rho \) by \( \rho_{w} = \) constant \( \rho_{l} \), and at the same time assumes \( \mu = \mu_{w} = \) constant, there results

\[
g^2 G'' + \frac{3}{2} V G = m \frac{\rho_{l}}{\rho_{w}} \left( 1 - \frac{\rho_{w}}{\rho_1} V^2 \right) G' + \frac{3}{2} \frac{\rho_{w}}{\rho_1} V G \quad (4.24)\]

\[
G(0)G'(0) = - m \frac{\rho_{l}}{\rho_{w}} \quad \{ \}
\]

\[
G(1) = 0
\]

instead of (4.23),
whereas if one also assumes $\rho_1 = \rho_w$, there results

$$g^2 G'' + \frac{3}{2} VG = m \left[ (1 - V^2)G' + \frac{3}{2} VG \right]$$

$$G(0)G'(0) = -m$$

$$G(1) = 0.$$  \hspace{1cm} (4.25)

Solutions of (4.25) correspond to the Falkner and Skan solutions; with $w_0(V)$ defined as in Section 3.21, we have $G(V) = w_0(V)$, and hence equations (3.2125) page 38 relate $G(V)$ to the $F(Y)$ in Hartree's system (3.2124). These constant-property solutions are useful even when fluid properties are known to vary, as mentioned in the introduction. Since the correct expression for $\frac{dp}{dx}$ is used in (4.24), one might expect solutions of (4.24) to be even more useful than the constant-property solutions of (4.25) when $\rho_1$ is significantly different from $\rho_w$. A comparison of the coefficient of $G'$ in (4.24) and (4.23) shows that (4.24) has the correct value, $m \frac{\rho_1}{\rho_w}$ at $V = 0$; however, the correct value at $V = 1$ is zero, whereas (4.24) gives a value different from zero unless $\frac{\rho_1}{\rho_w} = 1$ or $m = 0$. This discrepancy in the coefficient of $G'(V)$ at $V = 1$ could conceivably lead to serious error because $G'(V) \rightarrow \infty$ as $V \rightarrow 1$. Thus, one might expect solutions of
to be better approximations to solutions of (4.23) than are solutions of (4.24). The fact that the product VG vanishes at both \( V = 0 \) and \( V = 1 \) suggests that the value of the coefficient of the VG term in (4.26) may have only a minor effect on the solution. If the value of this coefficient does have only a slight influence, then solutions of (4.26) may be approximated by solutions of

\[
G''(1) + \frac{1}{2}VG = m \frac{\rho_1}{\rho_w} \int (1 - V^2) G \frac{3}{2} \frac{\rho_w}{\rho_1} VG
\]

\[
G(0)G'(0) = - m \frac{\rho_1}{\rho_w}. 
\]  

(4.26) \]

\[
G(1) = 0.
\]

The choice of (4.27) as an approximation to (4.26) and hence as an approximation to (4.23) is motivated by the fact that \( G(V) \) in (4.27), like \( G(V) \) in (4.25), is easily related to the \( F(Y) \) in Hartree's system (2.412). With this approximation, no new integrations need to be performed in order to make skin friction calculations. The line of thought which led to this approximation that \( G(0) \) in (4.23) equals \( G(0) \) in (4.27) does not in any sense demonstrate the
validity of the approximation. The approximation can be tested by comparison with available solutions.
5. Results.

The only new numerical results which can be given at this time use the solutions of Section 3.1 for insulated surfaces and Section 4.2 for non-insulated surfaces.

5.1 Skin friction on insulated surfaces with $\alpha T$.

The velocity outside the boundary layer for the following examples is taken as

$$u_1 \propto x.$$

The results of skin friction calculations for four cases are presented in Table 5.1.

<table>
<thead>
<tr>
<th>Case</th>
<th>Value of $\sigma$</th>
<th>Value of $c$ in (3.112)</th>
<th>Value of $w(x,0)$ for $U_1^2 = 0.338624$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>1</td>
<td>1</td>
<td>1.37</td>
</tr>
<tr>
<td>II</td>
<td>0.7</td>
<td>1</td>
<td>1.31</td>
</tr>
<tr>
<td>III</td>
<td>1</td>
<td>0.985</td>
<td>1.36</td>
</tr>
<tr>
<td>IV</td>
<td>0.7</td>
<td>0.985</td>
<td>1.31</td>
</tr>
</tbody>
</table>

Appendix B gives the detailed calculations, which are based on the analysis of Section 3.1.

In Appendix B, $w(x,0)$, which equals $\frac{2}{3} \sigma R_0^{\frac{1}{2}}$, is obtained as a Taylor's series. Only the first four terms of the series are obtainable with functions currently available. Evaluation of $w(x,0)$ from these four terms gives the results presented in Table 5.1. The value $U_1^2 = 0.338624$ corresponds to Mach
number $M_1 = 1.6$. For more details, the reader may refer to Appendix B.
5.2 **Approximate skin friction neglecting compressibility and dissipation.**

This section compares skin friction coefficients calculated using the approximation of Section 4.2 with skin friction coefficients calculated using the more explicit assumptions of Reference 23.

Values of \( G(o) = \frac{1}{2} \sigma_w \sqrt{R_w} \) resulting from calculations based on neglecting the terms \( \frac{\partial p}{\partial x} \) and \( \mu \left( \frac{\partial u}{\partial y} \right)^2 \) in (3.103) and based on the assumptions

\[
\begin{align*}
\rho &\propto T^{-1} \\
\mu &\propto T^{0.7} \\
k &\propto T^{0.85} \\
C_p &\propto T^{0.19} \\
\sigma_w &= 0.7
\end{align*}
\]

are tabulated in Reference 23 for various values of \( m \). A comparison of values from Reference 23 with values obtained using the approximation of this section is shown in Table 5.2. For exact agreement of the results of Reference 23 with the results of the approximation described in Section 4.2, the values appearing in the same horizontal row of Table 5.2 would have to be equal. The approximation is poorer for \( m < 0 \) than for \( m > 0 \).
Table 5.2

Comparison of Values of $G(o)$ when $G(V)$ satisfies Equation (4.27) with values when $G(V)$ satisfies (4.23) and the approximations of Reference 23 are used.

<table>
<thead>
<tr>
<th>$M = m^2 / R_w$</th>
<th>$G(o)$ from (4.27)</th>
<th>$G(o) = \frac{f_{fw}}{\rho_w} \sqrt{R_w}$ from Reference 23</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\frac{p_1}{\rho_w} = \frac{1}{4}$</td>
<td>$\frac{p_1}{\rho_w} = \frac{1}{2}$</td>
</tr>
<tr>
<td></td>
<td>$m = 4M$</td>
<td>$m = 2M$</td>
</tr>
<tr>
<td>0.00</td>
<td>0.332</td>
<td>0.2874</td>
</tr>
<tr>
<td>0.125</td>
<td>0.53</td>
<td>0.5367</td>
</tr>
<tr>
<td>0.25</td>
<td>0.675</td>
<td>0.6854</td>
</tr>
<tr>
<td>0.50</td>
<td>0.900</td>
<td></td>
</tr>
<tr>
<td>1.00</td>
<td>1.2326</td>
<td></td>
</tr>
<tr>
<td>2.00</td>
<td>1.68</td>
<td></td>
</tr>
<tr>
<td>4.00</td>
<td>2.28</td>
<td></td>
</tr>
</tbody>
</table>
6. Discussion.

The results of Section 5.1 show the importance of using the correct value of Prandtl number in precise skin friction calculations at supersonic Mach numbers when the pressure varies along the surface. These results are based on the assumption $\mu \alpha T$, which leads to a skin friction coefficient independent of Prandtl number when there is no pressure variation.

The effect of assuming

$$1 - \frac{T}{T_0} = 0.985 \left(1 - \frac{I}{I_0}\right) \quad (6.1)$$

instead of

$$\frac{T}{T_0} = \frac{I}{I_0} \quad (6.2)$$

is much less pronounced than the Prandtl number effect.

Equation (6.2) is the relation commonly associated with constant specific heats. But $dI/dT$, which equals $C_p$, is also constant in (6.1) or in (3.112). Thus, (3.112) does not really allow $C_p$ to vary. Rather, the use of (3.112) instead of (6.2) makes it possible to use a $C_p$ which is a good average over the temperature range of interest instead of the average over the range from 0 to $T_0$.

No numerical results pertaining to the analyses of Section 3.2 can be presented because the integrations required to obtain the universal functions have not yet been satisfactorily performed.
The Flight Research Laboratory's computation laboratory at Wright-Patterson Air Force Base attempted to solve the differential equations (3.2115) and (3.2116) for \( w_{10}(V) \) and \( w_{11}(V) \) with an electronic analog computer. Some of the terms which need to be generated, such as \( \dot{w}_0(V) \) and \( (V/w_0) \), increase without bound as \( V \to 1 \), and this unbounded increase leads to the practical difficulty that certain components of the computer overload before \( V = 1 \) is reached. The inability of the computer components to reproduce these unbounded terms accurately in the neighborhood of \( V = 1 \) causes inaccuracies in the solution curve for values of \( V \) greater than that value for which overloading sets in. Now the initial values of our unknown functions are determined as those initial values which cause the functions to be zero at \( V = 1 \). If the "solution curves" do not really represent accurate solutions at \( V = 1 \), then the correct initial values remain uncertain. The values obtained by the electronic analog integration are

\[
\begin{align*}
\text{for } m = 0, \quad w_{10}(0) &= -0.112 \\
\text{} \quad w_{11}(0) &= -0.05 \\
\text{and for } m = 1, \quad w_{10}(0) &= -0.209 \\
\text{} \quad w_{11}(0) &= -0.068.
\end{align*}
\]

The accuracy of these values is uncertain for the reasons just mentioned.
The correct values of \( w_1(o) \) for \( m = 0 \) and \( m = 1 \), for the special case in which \( \alpha = \tau \), may be obtained from analyses already in the literature. Now \( \alpha = \tau \) corresponds to \( b_1 = 1 \) and all other \( b_n = 0 \) in (3.2105). From (3.2113), we see that \( w_1 = w_{10} + w_{11} \) for this special case, and hence the correct sum \( w_{10}(o) + w_{11}(o) \) may be obtained. It is shown in Appendix E that the correct sum of initial values is

\[
\begin{align*}
\text{for } m = 0, & \quad w_{10}(o) + w_{11}(o) = -0.16603 \\
\text{and for } m = 1, & \quad w_{10}(o) + w_{11}(o) = -0.3367.
\end{align*}
\]

Thus, for \( m = 0 \), the sum of \( w_{10}(o) \) and \( w_{11}(o) \) as obtained by adding the individual values obtained from the computer, is in error by the value .004, and for \( m = 1 \), the corresponding sum is in error by the value .06. In the \( m = 1 \) case, this is an error comparable to the value \( w_{11}(o) \) itself! It is not known how this error is distributed between \( w_{10}(o) \) and \( w_{11}(o) \).

When sufficiently accurate values of \( w_{10}(o), \ldots, w_{23}(o) \) are available, the analysis of Section 3.21 will make possible calculations showing the effect of various hypotheses regarding \( \alpha(T) \) on the skin friction due to a gas flowing along an insulated surface.
The extensions to more general functions $u_1(x)$, to variable $C_p$, and to $\sigma \frac{1}{\gamma} l$ in Sections (3.22) through (3.24) will enable additional interesting computations to be made when the required integrations have been performed.

One incentive for investigating the more general case in which $u_1(x)$ is specified by (3.221) is that it includes the case in which $u_1(x)$ is the power series in $x$, $u_1(x) = \sum_{n=1}^{\infty} a_n x^n$, by setting

$$m = 1$$

$$\alpha_1 = \frac{a_2}{a_1}$$

$$\alpha_2 = \frac{2}{3} \left( a_3 - \frac{a_2^2}{a_1} \right)$$

... Another case included by (3.221) which may be of interest is obtained by setting $m = 1$, $\alpha_1 = 1/a_0$, and all other $\alpha_n = 0$. The result is

$$u_1 = \frac{a_0 x}{L + x}$$

where $L$ is an arbitrary constant having the dimensions of length. What is the explanation for the result that $W$ appears to be the same function of $(U_1, V)$ in this case, regardless of the value assigned to $L$?

The case

$$\frac{x}{U_1} \frac{dU_1}{dx} = 1 - \frac{a_0}{U_1} \quad (6.3)$$
is not included by (3.221), but this case may be of interest because it leads to

\[ U_1 = a_o - \frac{x}{L} \tag{6.4} \]

where again \( L \) is an arbitrary constant having the dimensions of length. Here again it appears that \( W \) will be the same function of \((U_1, V)\), regardless of the value assigned to \( L \).

In treating this case, however, we find that all of the "universal" functions depend on the parameter \( a_o \) in a manner analogous to that in which they depended on \( m \) in the cases previously discussed. The differential equations for the "universal functions" in this case are not presented.

Probably the most general case which can be treated by a power series analysis analogous to the one for the case (3.221) is

\[ \frac{x}{U_1} \frac{dU_1}{dx} = U_1^p \sum_{n=0}^{\infty} \theta_n U_1^n \tag{6.5} \]

where \( p \) is any integer. If \( p \) is a non-negative integer, this case is already included by (3.221). If \( p \) is any integer, equation (3.2104) shows that the function \( W(U_1, V) \) satisfies

\[ 0 = (1 - U_1^2 V^2) W^2 \frac{\partial^2}{\partial V^2} \left( \frac{m}{\lambda_0} \right) + \frac{1}{2}(1 - U_1^2) V W \]

\[ - U_1^p (\beta_1 + \beta_1 U_1 + \cdots) \left[ (1 - V^2)^2 \frac{\partial W}{\partial V} + \frac{3}{2} V W + U_1 (1 - U_1^2) V \frac{\partial W}{\partial U_1} \right] \]

\[ + \frac{\chi + 1}{2(\xi - 1)} U_1^2 V \]
\[
\left( \frac{w \cdot \partial w}{\partial v} \right)_{v = 0} = -u_1^0 (\beta_0 + \beta_1 u_1 + \beta_2 u_1^2 + \cdots)
\]

Because of the boundary condition at \( v = 0 \), a natural substitution is

\[
W(U_1, v) = (\beta_0 u_1^0)^{\frac{3}{2}} F(U_1, v).
\]

Then \( \frac{\partial W}{\partial U_1} = (\beta_0 u_1^0)^{\frac{3}{2}} \frac{\partial F}{\partial U_1} + \frac{p}{2U_1} (\beta_0 u_1^0)^{\frac{3}{2}} F \), and the differential equation becomes

\[
(\beta_0 u_1^0)^{\frac{3}{2}} (1 - u_1^2 v^2) F^2 \frac{\partial^2 F}{\partial v^2} \left( U_0 - \frac{1}{2} \beta_0 u_1^0 \right) + \frac{1}{2} (\beta_0 u_1^0)^{\frac{3}{2}} (1 - u_1^2)VF
\]

\[
- u_1^0 (\beta_0 u_1^0)^{\frac{1}{2}} (\beta_0 + \beta_1 u_1 + \cdots) \left[ (1 - v^2) \frac{\partial F}{\partial v} + \left( \frac{3}{2} + \frac{1}{2} p \right) V F + U_1 V \frac{\partial F}{\partial U_1} \right]
\]

\[
+ \left( \frac{d + 1}{2(d - 1)} - \frac{1}{2} p \right) U_1^2 V F - U_1^3 V \frac{\partial F}{\partial U_1} = 0.
\]

If \( p \) is an integer, one may divide the equation by \( (\beta_0 u_1^0)^{\frac{3}{2}} \) to obtain an equation in which only integral powers of \( U_1 \) occur, and this suggests that it is possible to expand \( F(U_1, v) \) in a power series in \( U_1 \). These possible solutions will not be pursued further here.

Before concluding this discussion of the extension to more general functions \( u_1(x) \), it should be mentioned that the same functions defined in the solution for \( u_1 \alpha x^m \) may be used to obtain a solution in which \( u_1 \neq 0 \) at \( x = 0 \) by changing the definition of \( W(U_1, v) \) from (3.2103) to
\[ g = \left( \frac{3u}{3y} \right)_x \sqrt{\frac{\rho_1}{\mu_0} \frac{u_1^3}{(\frac{a}{b} + x)^m}} W(U_1, V). \quad (6.6) \]

Then, if \( U_1 \propto (a + bx)^m \), we have that
\[
\frac{a}{b} + x \frac{dU_1}{dV} = m.
\]

The differential equation and boundary conditions satisfied by \( W(U_1, V) \) are then exactly the same as when \( W \) is defined by (3.2103) and \( u_1 \propto x^m \).

From a fundamental point of view, no new problem has been solved by this procedure, which amounts merely to translating the origin for \( x \) from the stagnation point to such a position that the stagnation point is at \( x = -\frac{a}{b} \).

Possible applications of the distribution \( U_1 \propto (a + bx)^m \) are to certain thin curved plates with leading edge at \( x = 0 \) in subsonic flow, or to certain bodies with sharp leading edges at \( x = 0 \) in supersonic flow when the shock wave is attached, or to a wedge-like body which has its boundary layer completely removed by suction at some point \( x = 0 \) along its surface. (Cf the last three paragraphs of Ref. 24).

In such applications of (6.6.), one may expect the calculated values of skin friction to be correct when \( x \gg \frac{a}{b} \); but since we expect the skin friction coefficient
to be inversely proportional to the square root of Reynolds number $\frac{u_1 x}{v}$ in these cases, a solution for $W(U_1, V)$ in which the definition (3.2103) is retained would be expected to yield more accurate values near $x = 0$. However, until experimental evidence is more conclusive, we must continue to question the validity of all calculations which make use of the boundary layer approximations when Reynolds number is small.

For particular $\mu(T)$ formulas, one may retain the definition of $W$ in (3.2103) and still obtain solutions in which $U_1 = a_0 \neq 0$ at $x = 0$ by expanding $W$ in powers of $(U_1 - a_0)$ or by considering $W$ as a function of $(x, V)$, and expanding in powers of $x$; but these solutions lack the universal characteristics of the solutions in powers of $U_1$.

The number of functions required for a universal\(^1\) representation of skin friction when specific heat is variable and/or $\sigma = 1$, is discouragingly large. This number of functions can be reduced (at some sacrifice of universality), (1) by using Sutherland's formula (3.2106) instead of the more general (3.2105),

\(^1\)A universal representation in the sense that the various parameters involved appear only explicitly as coefficients; the functions are universal in the sense that they do not depend on the parameters at all.
(2) by using the approximate temperature-enthalpy relation (3.111) instead of the more general (3.232),

(3) by solving the problem only for a specified value of Prandtl number \( \sigma \).

The first of these artifices causes no essential loss of universality; the third certainly causes the solution to be less universal than it would otherwise be.

A series solution allowing arbitrary values of all the parameters appears to be very impractical today; but such a solution is far more general than necessary if one's primary interest is in air flow, because these parameters do not vary independently, but (for a given gas, such as air) depend only on the single parameter \( I_0 \) (or \( T_0 \)). Thus, rather than solve 20 or 30 differential equations to obtain only the first three terms of a series solution applicable to arbitrary values of the parameters, it is probably more efficient to solve the problem for the particular values of the parameters which correspond to some particular values of \( T_0 \) which are of interest. The solution for other values of \( T_0 \) can then be approximated by graphical interpolation.

For non-insulated surfaces, a solution including the effects of variable properties, expressed in terms of uni-
versal functions, appears to be a very remote possibility. The solution of practical problems under these circumstances will probably be accomplished by obtaining particular numerical solutions using the particular values of the parameters involved. When a sufficient number of particular solutions are available, it will be possible to obtain solutions for other values of the parameters by interpolation.
APPENDIX A. Basic Relations and Background Information.

1. Basic Relations of Fluid Dynamics.

1.1 Notation.

Any notation not defined in Section 2 of the dissertation, is defined where first used in this appendix.

1.2 The concept of a continuous medium.

Although the discreteness of matter is now well established, much useful theory treats matter as if it is continuous. By a "fluid particle" in the continuous medium theory, one does not mean a molecule (or other building block) but rather a very small sample of fluid which will generally contain a great number of molecules. We can interpret the density "at a point" in the continuous medium theory to be the average density in a small region surrounding the point; similarly, the velocity "at a point" in the continuous medium theory is the average velocity of the molecules in a small region surrounding the point. In order for the continuum theory to apply, the dimensions of the small regions over which the averages are taken should be such that the values of the averages do not depend on these dimensions. They should also be small enough, compared with other significant dimensions of the problem, to be treated as infinitesimals. If such regions do not exist in a particular problem, then it is likely that a continuum theory
will be unsuccessful in treating that problem. For additional discussion, the reader may refer to Chapter 1 of Reference 25.

1.3 The time derivative of properties of a fluid body.

Consider a body of fluid bounded by some arbitrary closed surface which moves with the fluid so that the body is always composed of the same fluid particles. Any additive properties attributable to the body are expressed by integrals over the volume of the body. For example, if \( \rho, \mathbf{v}, s, \) and \( E \) denote density, velocity, entropy per unit mass, and internal energy per unit mass respectively, and if \( V \) denotes the volume of the body, then

\[
\int_V \rho \, dv, \quad \int_V \rho \mathbf{v} \, dv, \quad \int_V \frac{1}{2} \rho \mathbf{v}^2 \, dv, \quad \int_V \rho s \, dv, \quad \int_V \rho E \, dv
\]

are expressions for the body's mass, momentum, kinetic energy, entropy, and internal energy, respectively. Physical relationships involving the time derivatives of these properties of the body are sometimes useful in discussions of general relationships in fluid dynamics, and it is therefore useful to be able to express such a time derivative as a volume integral.

Let every fluid particle be identified by its position \((x_0, y_0, z_0)\) at some initial time \(t_0\). The position of
any given particle at some other time, \( t \), is a function of the time, \( t \); but to identify the particle, its initial coordinates must also be given. Thus, for the position at time \( t \), we write
\[
\begin{align*}
x &= x(t, x_0, y_0, z_0) \\
y &= y(t, x_0, y_0, z_0) \\
z &= z(t, x_0, y_0, z_0)
\end{align*}
\quad (1.31)
\]

where it is understood that
\[
\begin{align*}
x_0 &= x(t_0, x_0, y_0, z_0) \\
y_0 &= y(t_0, x_0, y_0, z_0) \\
z_0 &= z(t_0, x_0, y_0, z_0).
\end{align*}
\quad (1.32)
\]

The fluid which occupies the region \( V_0 \) at time \( t_0 \), occupies a region \( V \) at time \( t \). Each point \((x,y,z)\) of \( V \) at time \( t \) can be associated with a point \((x_0,y_0,z_0)\) of \( V_0 \) by virtue of the inverses of equations (1.31) which may be regarded as a transformation of points from \( V_0 \) to \( V \). This association of points in \( V \) and in \( V_0 \) with each other is easily described: Each point \((x_0, y_0, z_0)\) of \( V_0 \) gives the position at time \( t_0 \) of a fluid particle; the point \((x,y,z)\) in \( V \) which is associated with \((x_0, y_0, z_0)\) is the position at time \( t \) of the particle which was at \((x_0, y_0, z_0)\) at time \( t_0 \). In order to be consistent with this physical interpretation, let us hypothesize that the transformation of points
from $V_0$ to $V$ is bi-unique and bi-continuous. We will also hypothesize that the nine partial derivatives $\frac{\partial x}{\partial x_0}, \frac{\partial x}{\partial y_0}, \frac{\partial x}{\partial z_0}, \frac{\partial y}{\partial x_0}, \frac{\partial y}{\partial y_0}, \frac{\partial y}{\partial z_0}, \frac{\partial z}{\partial x_0}, \frac{\partial z}{\partial y_0}, \frac{\partial z}{\partial z_0}$, exist and are continuous functions of $(t, x_0, y_0, z_0)$. It follows from the bi-continuous nature of the transformation from $(t, x_0, y_0, z_0)$ to $(t, x, y, z)$ that continuous functions of $(t, x, y, z)$ are continuous functions of $(t, x_0, y_0, z_0)$, and conversely, continuous functions of $(t, x_0, y_0, z_0)$ are continuous functions of $(t, x, y, z)$.

If $F(t,x,y,z)$ is a continuous scalar function having continuous partial derivatives with respect to all four of its variables, if $v(t,x,y,z)$ is continuous, and has continuous partial derivatives with respect to $(x,y,z)$, if $V$ denotes the (moving) region occupied by a body of fluid at time $t$, and if the above hypotheses on the transformation of points from an initial region $V_0$ to the region $V$ are satisfied, then it will be proved that

$$\frac{d}{dt} \int_V F dV = \int_V \left( \frac{\partial F}{\partial t} + \text{div} \ F_v \right) dV. \quad (1.33)$$

The usefulness of formula (1.33) may be illustrated by using it in a derivation of the continuity equation. The principle that the body's mass is conserved tells us that

$$\frac{d}{dt} \int_V \rho \ dV = 0.$$
Application of formula (1.33) then yields
\[ \int_V \left( \frac{\partial \rho}{\partial t} + \text{div} \, \rho \mathbf{v} \right) \, dV = 0, \]
and since this must hold for any arbitrary volume \( V \), we have immediately the differential form of the continuity equation,
\[ \frac{\partial \rho}{\partial t} + \text{div} \, \rho \mathbf{v} = 0, \tag{1.34} \]
which must hold at each point.

It is worth noting that by virtue of the continuity equation (1.34) and the formula (1.33), we may assert that
\[
\frac{d}{dt} \int_V \rho \mathbf{F} \, dV = \int_V \left( \frac{\partial}{\partial t} (\rho \mathbf{F}) + \text{div} \, (\rho \mathbf{F} \mathbf{v}) \right) \, dV
\]
\[
= \int_V \left[ \mathbf{F} \left( \frac{\partial \rho}{\partial t} + \text{div} \, \rho \mathbf{v} \right) + \rho \left( \frac{\partial \mathbf{F}}{\partial t} + \mathbf{v} \cdot \text{grad} \, \mathbf{F} \right) \right] \, dV
\]
\[
= \int_V \rho \frac{d\mathbf{F}}{dt} \, dV.
\]

This last formula
\[ \frac{d}{dt} \int_V \rho \mathbf{F} \, dV = \int_V \rho \frac{d\mathbf{F}}{dt} \, dV \tag{1.35} \]
is often more convenient than (1.33) since many fluid properties are referred to unit mass rather than to unit volume. Sir Horace Lamb apparently made use of (1.35) when he wrote \( \iiint \rho \frac{d\mathbf{u}}{dt} \, dx \, dy \, dz \) for the rate of change of \( x \)-component of momentum of a moving body of fluid in Section 10a,
page 10, of Lamb's Hydrodynamics (Reference 26). However, a mathematically rigorous proof of (1.35) or of (1.33) or of (1.36) below, which is equivalent to (1.33) when the divergence theorem applies, appears not to have been given heretofore. The convincing arguments for (1.35) based on the concept that $\rho dV$ is a constant since it represents the mass of a fluid particle lack rigor because the interchange of limiting processes involved in (1.35) remains unjustified.

By means of the divergence theorem, (1.33) becomes

$$\frac{d}{dt} \int_V FdV = \int_V \frac{\partial F}{\partial t} dV + \int_S Fv \cdot n ds, \quad (1.36)$$

where $S$ is the closed surface bounding $V$. In (1.36),

$$\int_V \frac{\partial F}{\partial t} dV$$

gives the rate of change of $\int_V FdV$ if the volume $V$ were fixed; $\int_S Fv \cdot n ds$ gives the rate of change due to the motion of the surface bounding $V$. We comment here that in view of (1.36), the momentum flow rate relations, often used in fluid mechanics to evaluate a fluid body's rate of momentum change, can be placed on a rigorous mathematical basis, once (1.33) is established. Convincing kinematic arguments (Cf. Reference 27, Section 3.40, page 72, and Reference 25, Section 100, pp. 233-238), can be used to arrive at (1.36) without using (1.33), but again these arguments lack the rigor of the following proof.
The following derivation of formula (1.33) was suggested by the treatment of rate of expansion in O. D. Kellogg's *Foundations of Potential Theory*, pp. 35-36. I am indebted to Professor H. H. Alden for his comment that Kellogg's treatment of rate of expansion can easily be generalized to prove formula (1.33).

The independent variables \((t, x, y, z)\) in the scalar field \(F(t, x, y, z)\) can be changed to \((t, x_0, y_0, z_0)\) by virtue of equations (1.31) and the identity \(t = t\). Thus, it is possible to change the variables of integration \((x, y, z)\) in a volume integral to \((x_0, y_0, z_0)\). For integration over a given fluid body, the region of integration \(V\) for the case in which the variables of integration are \((x, y, z)\) depends on the time \(t\), whereas if the variables of integration are \((x_0, y_0, z_0)\), then the region of integration is the fixed volume \(V_0\), which does not depend on \(t\). Thus, it may be anticipated that the difficulties which arise when one tries to time-differentiate an integral over a moving region of integration may be avoided by changing the variables of integration to \((x_0, y_0, z_0)\).

If we define

\[
\phi(t) = \iiint_{V} F(t, x, y, z) \, dx \, dy \, dz,
\]

then according to the rule for changing the variables of
integration in a multiple integral,

\[ \phi(t) = \iiint_{V_0} f(t, x_0, y_0, z_0) J \left( \frac{x, y, z}{x_0, y_0, z_0} \right) \, dx_0 \, dy_0 \, dz_0 \quad (1.38) \]

where

\[ J \left( \frac{x, y, z}{x_0, y_0, z_0} \right) = \begin{vmatrix}
\frac{\partial x}{\partial x_0} & \frac{\partial y}{\partial x_0} & \frac{\partial z}{\partial x_0} \\
\frac{\partial x}{\partial y_0} & \frac{\partial y}{\partial y_0} & \frac{\partial z}{\partial y_0} \\
\frac{\partial x}{\partial z_0} & \frac{\partial y}{\partial z_0} & \frac{\partial z}{\partial z_0}
\end{vmatrix} \]

is the Jacobian of \((x,y,z)\) with respect to \((x_0,y_0,z_0)\) and

where

\[ f(t, x_0, y_0, z_0) = F(t, x, y, z) \]

with \((t,x,y,z)\) and \((t,x_0,y_0,z_0)\) related by (1.31) and (1.32), and the identity \( t = t \).

Since the region of integration is independent of \( t \) in the right member of (1.38), we may write, subject to appropriate restrictions on the function \( f \) and the elements of \( J \),

\[ \frac{d\phi}{dt} = \iiint_{V_0} \frac{\partial}{\partial t} (fJ) \, dx_0 \, dy_0 \, dz_0, \quad (1.39) \]

where \( f \) and \( J \) are functions of the independent variables \((t,x_0,y_0,z_0)\). Let us note here that

\[ \frac{\partial}{\partial t} f(t, x_0, y_0, z_0) = \frac{\partial}{\partial t} F \left[ t, x(t, x_0, y_0, z_0), y(t, x_0, y_0, z_0), z(t, x_0, y_0, z_0) \right] \]

\[ = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial t} = \frac{\partial F}{\partial t} + v \cdot \text{grad} F = \frac{df}{dt}. \]

Thus, the partial derivative with respect to time when \((t,x_0,y_0,z_0)\) are the independent variables equals the total
time derivative, or "derivative following the fluid" when 
\((t,x,y,z)\) are the independent variables. In forming \(\frac{dF}{dt}\), we must understand that the differentiated property 
\(F(t,x,y,z)\) is a property of a particular fluid particle, namely the particle whose time-dependent coordinates are 
\((x,y,z)\), for without this understanding, a total derivative of \(F(t,x,y,z)\) makes no sense. We digress to comment that 
the symbol \(\frac{D}{Dt}\) introduced by Stokes (Ref. 28) and continued 
by Lamb and others has this same meaning; it is a total 
derivative, and as such it needs to be distinguished from 
the partial derivative \(\frac{\partial}{\partial t}\), but it need not be distinguished 
from \(\frac{d}{dt}\). In Stokes' time, there was considerable justifi-
cation for the introduction of a special symbol, because of 
the objectionable English custom of using the same symbol 
\(\frac{d}{dt}\) for both partial and total differentiation.

Now \(\frac{\partial}{\partial t} \left( fJ \right) = J \frac{\partial f}{\partial t} + f \frac{\partial J}{\partial t} \). In its determinantal form, 
\(J\) is the sum of six terms, each of which is a product of 
three factors. Application of the rule for differentiating 
products results in 

\[
\frac{\partial J}{\partial t} = \begin{vmatrix}
\frac{\partial^2 x}{\partial t \partial x_0} & \frac{\partial^2 y}{\partial t \partial x_0} & \frac{\partial^2 z}{\partial t \partial x_0} \\
\frac{\partial x}{\partial x_0} & \frac{\partial y}{\partial x_0} & \frac{\partial z}{\partial x_0} \\
\frac{\partial x}{\partial z_0} & \frac{\partial y}{\partial z_0} & \frac{\partial z}{\partial z_0}
\end{vmatrix} + \begin{vmatrix}
\frac{\partial^2 x}{\partial t \partial y_0} & \frac{\partial^2 y}{\partial t \partial y_0} & \frac{\partial^2 z}{\partial t \partial y_0} \\
\frac{\partial x}{\partial y_0} & \frac{\partial y}{\partial y_0} & \frac{\partial z}{\partial y_0} \\
\frac{\partial x}{\partial z_0} & \frac{\partial y}{\partial z_0} & \frac{\partial z}{\partial z_0}
\end{vmatrix} + \begin{vmatrix}
\frac{\partial^2 x}{\partial t \partial z_0} & \frac{\partial^2 y}{\partial t \partial z_0} & \frac{\partial^2 z}{\partial t \partial z_0} \\
\frac{\partial x}{\partial z_0} & \frac{\partial y}{\partial z_0} & \frac{\partial z}{\partial z_0}
\end{vmatrix} \]
This expression for $\frac{\partial J}{\partial t}$ is easily evaluated at $t = t_0$, and since $t_0$ can in principle be taken as any instant, the results will be general. From (1.32) it is clear that the following relations apply at $t = t_0$: $\frac{\partial x}{\partial x_0} = 1$, $\frac{\partial y}{\partial y_0} = 0$, $\frac{\partial z}{\partial z_0} = 0$, $\frac{\partial x}{\partial x_0} = 0$, $\frac{\partial y}{\partial y_0} = 0$, $\frac{\partial z}{\partial z_0} = 1$. Also, we note that $\frac{\partial^2 x}{\partial t \partial x_0} = -\frac{\partial}{\partial x_0} \left( \frac{\partial x}{\partial t} \right) = \frac{\partial u}{\partial x_0}$, and similarly $\frac{\partial^2 y}{\partial t \partial y_0} = \frac{\partial v}{\partial y_0}$ and $\frac{\partial^2 z}{\partial t \partial z_0} = \frac{\partial w}{\partial z_0}$, where $u$, $v$, and $w$ are the $x$, $y$, and $z$ components of fluid velocity, $\mathbf{v}$. By substitution we find that $\left(\frac{\partial J}{\partial t}\right)_{t = t_0} = (\text{div } \mathbf{v})_{t = t_0}$ and that at $t = t_0$, $J = 1$.

Finally, from (1.39),
\[
\left(\frac{d\phi}{dt}\right)_{t = t_0} = \iiint_{V_0} \left[ \frac{\partial f(t, x_0, y_0, z_0)}{\partial t} + \mathbf{v} \cdot \text{div } \mathbf{v} \right] d\xi d\eta d\zeta,
\]
or, in view of (1.32) and the fact that $V$ and $V_0$ coincide at $t = t_0$,
\[
\left(\frac{d\phi}{dt}\right)_{t = t_0} = \iiint_{V} \left( \frac{\partial f}{\partial t} + \mathbf{v} \cdot \text{grad } f + \mathbf{F} \cdot \text{div } \mathbf{v} \right) dx dy dz.
\]
But $\frac{d\mathbf{F}}{dt} + \mathbf{F} \cdot \text{div } \mathbf{v} = \frac{\partial \mathbf{F}}{\partial t} + \mathbf{v} \cdot \text{grad } \mathbf{F} + \mathbf{F} \cdot \text{div } \mathbf{v} = \frac{\partial \mathbf{F}}{\partial t} + \text{div } \mathbf{F} \cdot \mathbf{v}$. Finally, since $t_0$ may be any instant, we have
\[
\frac{d}{dt} \int_{V} F dV = \int_{V} \left( \frac{\partial F}{\partial t} + \text{div } F \cdot \mathbf{v} \right) dV, \quad (1.33)
\]
which is the formula (1.33) we set out to prove. In words, formula (1.33) states that the time rate of change of the volume integral of a scalar field $F$ integrated over the
moving volume enclosed by a surface which moves with the fluid is at any instant equal to the volume integral of \[ \frac{\partial F}{\partial t} + \text{div } Fv \] integrated over the volume at that instant. This equality is subject to some restrictions which will now be discussed.

The indicated differentiation of the product \((fJ)\) is certainly valid if \(f\) and the elements of \(J\) are continuous functions of \((t, x_0, y_0, z_0)\) and have continuous time derivatives. Continuity of the time derivatives of the elements of \(J\) is also sufficient to justify the interchange of the order of differentiation involved in relations like

\[ \frac{\partial^2 x}{\partial t \partial x_0} = \frac{\partial u}{\partial x_0}. \]

The function \(f(t, x_0, y_0, z_0)\) and the elements of \(J\) will satisfy these conditions, and the change of variables of integration in going from (1.37) to (1.38) will be justified if

1. the transformation (1.31) is bi-unique and bi-continuous with \(x, y,\) and \(z\) having continuous first partial derivatives with respect to \(t, x_0, y_0,\) and \(z_0.\)

2. the velocity components, considered as functions of \((t, x, y, z)\) have continuous first partial derivatives with respect to \(x, y,\) and \(z.\)

3. the function \(F(t, x, y, z)\) has continuous first partial derivatives with respect to all four of its variables.
Summary of useful relations involving the stress dyadic.

The following relations will be used in the derivations of basic equations in the following sections.

(a) \( \mathbf{n} \cdot \mathbf{\gamma} \) is the stress vector on a surface whose unit normal is \( \mathbf{n} \), with sign convention as stated under Notation; that is, this stress acts on the material situated on the \(-\mathbf{n}\) side of the surface.

(b) The stress dyadic \( \mathbf{\gamma} \) is symmetric. This statement is equivalent to \( \mathbf{\gamma}_{ij} = \mathbf{\gamma}_{ji} \), where \( \mathbf{\gamma}_{ij} \) is the \( j \)-component of stress on a surface whose normal is in the direction designated by \( i \).

(c) \( \mathbf{A} \cdot \mathbf{\gamma} = \mathbf{\gamma} \cdot \mathbf{A} \) for any vector \( \mathbf{A} \). This commutative property follows from the symmetry of the stress dyadic; it is true for all symmetric dyadics, but not true for dyadics in general.

(d) We can formally write \( \mathbf{\gamma} = \mathbf{1}(\mathbf{1} \cdot \mathbf{\gamma}) \) and \( \mathbf{k}(\mathbf{k} \cdot \mathbf{\gamma}) \) and \( \text{div} \mathbf{\gamma} = \mathbf{\nabla} \cdot \mathbf{\gamma} = \frac{\partial}{\partial x} (\mathbf{1} \cdot \mathbf{\gamma}) + \frac{\partial}{\partial y} (\mathbf{1} \cdot \mathbf{\gamma}) + \frac{\partial}{\partial z} (\mathbf{k} \cdot \mathbf{\gamma}) \). Note that \( \text{div} \mathbf{\gamma} \) is a vector.

The physical significance of \( \text{div} \mathbf{\gamma} \) can be seen by examining the force exerted on a small parallelepiped by the surrounding material through the action of stress. For example, the material immediately to the left of the parallelepiped sketched in Fig. 1.4 exerts on it, the force \( (-\mathbf{1} \cdot \mathbf{\gamma}) \, \text{dxdydz} \) according to relation (a), and the material immediately
\[ \text{Resultant Force} = \text{div} \Psi \, dx \, dy \, dz + \text{higher order terms} \]

**Figure 1.4** Force on a small parallelepiped due to stress acting across the bounding faces.
to the right of the parallelepiped exerts on it the force
\[ \int \left( \mathbf{\nabla} \cdot \mathbf{\Psi} + \frac{\partial}{\partial x} (\mathbf{1} \cdot \mathbf{\Psi}) dx \right) dy dz \] plus higher order terms. Similar considerations for the faces perpendicular to the y and z axes shows that

resultant force on parallelepiped due to stress on its bounding surfaces
\[ = \int \left( \frac{\partial}{\partial x} (\mathbf{1} \cdot \mathbf{\Psi}) + \frac{\partial}{\partial y} (\mathbf{1} \cdot \mathbf{\Psi}) + \frac{\partial}{\partial z} (\mathbf{k} \cdot \mathbf{\Psi}) \right) dx dy dz \] + higher order terms
\[ = \text{div} \mathbf{\Psi} dV + \text{higher order terms}. \]

In addition to this force due to stress there may also be a body force acting on the parallelepiped. Thus, \( \text{div} \mathbf{\Psi} \) equals the force per unit volume which is exerted on an infinitesimally small volume due to the stress acting across the surface bounding the volume.

\[(e) \quad \int_S \mathbf{n} \cdot \mathbf{\Psi} dS = \int_V \text{div} \mathbf{\Psi} dV \]
where \( S \) is the closed surface bounding \( V \). This is the "divergence theorem for dyadics." It can be made to follow from "Green's theorem in space" or from the ordinary divergence theorem by proper definition of some auxiliary quantities.

The above equality has the following physical interpretation. According to relation \( (a) \), the left number gives the force which acts on the material inside the closed surface \( S \) as a result of stress across the bounding surface \( S \). From the physical interpretation of \( \text{div} \mathbf{\Psi} \) under \( (d) \) above and from Newton's 3rd Law, it can be shown that the right
member is simply another expression for this same force; so of course the equality must hold. Conversely, the physical significance of \( \text{div} \Psi \) as "force per unit volume due to stress" can follow from (e).

\[(f) \ (n \cdot \Psi) \cdot v - n \cdot (\Psi \cdot v) = \text{a scalar.}\]

\[(g) \ \text{div} (\Psi \cdot v) = v \cdot \text{div} \Psi + (\mathbf{1} \cdot \Psi) \cdot \text{grad} u + (\mathbf{1} \cdot \Psi) \cdot \text{grad} v + (\mathbf{k} \cdot \Psi) \cdot \text{grad} w.\]
1.5 General equation of motion for a continuous medium.

In any continuous, deformable material, consider a closed surface which moves with the material; that is, each point of the surface has the velocity of the medium at that point. The material inside such a surface is a body for which we may write

\[ \text{resultant force} = \text{time rate of change of body's momentum}. \] \hspace{1cm} (1.51)

The resultant force is \( \int_V \rho \mathbf{F} dV + \int_S n \cdot \mathbf{\Psi} \, ds \); that is, it is the body force, plus the force due to stress across the bounding surface. The time rate of change of momentum is

\[ \frac{d}{dt} \int_V \rho \mathbf{v} dV = \int_V \rho \frac{d\mathbf{v}}{dt} dV \] \hspace{1cm} (1.52)

where (1.52) follows from (1.35) by letting the \( F \) in (1.35) equal \( u, v, w \) in turn to obtain three scalar equations which may be combined as (1.52). The equation of motion is therefore

\[ \int_V \rho \mathbf{F} dV + \int_S n \cdot \mathbf{\Psi} \, ds = \int_V \rho \frac{d\mathbf{v}}{dt} dV. \] \hspace{1cm} (1.53)

Then, using the divergence theorem for dyadics (1.4e) to transform the surface integral to a volume integral, and noting that the equation must hold for any arbitrary volume \( V \), we may conclude that at each point in the medium,

\[ \rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{F} + \text{div} \mathbf{\Psi}. \] \hspace{1cm} (1.53)
Equation (1.53) is quite general; it applies for any deformable, continuous medium.
1.6 Energy equation.

Let us again consider the body occupying the volume $V$ inside an arbitrary closed surface $S$ which moves with the medium. An energy balance for this body states that the net rate at which it gains energy equals its net rate of increase of kinetic and internal energy. If the body gains energy only by having work done on it and by heat conduction, then

\[
\text{Rate at which work is done on body by body forces} + \text{Rate at which work is done on body by surface forces} + \text{Rate at which body gains energy due to heat conduction across its bounding surface} = \text{Rate of increase of kinetic energy of body} + \text{Rate of increase of internal energy of body}. \quad (1.61)
\]

If the body should gain energy by means other than those enumerated in the left member of the preceding equation (by radiation, for example), then appropriate terms must be added to the left member.

This energy balance will now be expressed using mathematical symbols. Each of the five terms in the following equation (1.62) equals the corresponding term in the above verbal equation (1.61).

\[
\int_V \rho \mathbf{F} \cdot \mathbf{v} \, dV + \int_S (\mathbf{n} \cdot \mathbf{F}) \cdot \mathbf{v} \, dS + \int_S (k \, \text{grad} \, T) \cdot \mathbf{n} \, dS = \frac{d}{dt} \int_V \frac{1}{2} \rho \mathbf{v}^2 \, dV + \frac{d}{dt} \int_V \rho \mathbf{E} \mathbf{v} \, dV \quad (1.62)
\]
The two surface integrals in this energy balance may be transformed to volume integrals as follows. From the dyadic reactions 1.4 (f) and 1.4 (e),
\[ \int_S (\mathbf{n} \cdot \mathbf{\Psi}) \cdot \mathbf{v} \, dS = \int_S \mathbf{n} \cdot (\mathbf{\Psi} \cdot \mathbf{v}) \, dS = \int_V \text{div}(\mathbf{\Psi} \cdot \mathbf{v}) \, dV. \]

Also, by the divergence theorem for vectors,
\[ \int_S (k \, \text{grad} \, T) \cdot \mathbf{n} \, dS = \int_V \text{div} (k \, \text{grad} \, T) \, dV \]

It follows from (1.35) that the right member of (1.62) may be written as
\[ \int_V \rho \frac{d}{dt} \left( \frac{1}{2} \mathbf{v}^2 \right) \, dV + \int_V \rho \frac{dE}{dT} \, dV. \]

Finally, since the energy balance must hold for any arbitrary volume, we may assert that at each point in the medium,
\[ \rho \mathbf{F} \cdot \mathbf{v} + \text{div}(\rho \mathbf{v}) + \text{div}(k \, \text{grad} \, T) = \rho \frac{d}{dt} \left( \frac{1}{2} \mathbf{v}^2 \right) + \rho \frac{dE}{dT}. \] (1.63)

In this equation (1.63) each term has the dimensions of time rate of change of energy per unit volume, and the individual terms represent contributions due to body forces, surface forces (i.e., stresses), heat conduction, kinetic energy change, and internal energy change. Equation (1.63) is a general equation applicable to any continuous medium as long as all energy transfer takes place through the mechanisms of work and heat conduction.
Taking the scalar product of the momentum equation (1.53) with \( \mathbf{v} \) gives

\[
\rho \mathbf{F} \cdot \mathbf{v} + \nabla \cdot \mathbf{v} = \rho \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} - \rho \frac{d}{dt} \left( \frac{1}{2} \mathbf{v}^2 \right) .
\]  

(1.64)

Notice that (1.64) is also an energy equation in the sense that each term has the dimensions of time rate of change of energy per unit volume.

By using dyadic relation 1.4 (g) in (1.63), and then subtracting (1.64), one arrives at

\[
(\mathbf{1} \cdot \mathbf{\Psi}) \cdot \nabla u + (\mathbf{1} \cdot \mathbf{\Psi}) \cdot \nabla v + (k \cdot \mathbf{\Psi}) \cdot \nabla w + \nabla \cdot (k \nabla T) = \rho \frac{dE}{dt} .
\]

(1.65)

Equation (1.65) is also a general equation applicable to any continuous medium.

The term \( \nabla \cdot (\mathbf{\Psi} \cdot \mathbf{v}) \), which represents power per unit volume added to the fluid through the action of stress, has been split up into two parts: \( \mathbf{v} \cdot \nabla \mathbf{\Psi} \) and \( \left[ (\mathbf{1} \cdot \mathbf{\Psi}) \cdot \nabla u + (\mathbf{1} \cdot \mathbf{\Psi}) \cdot \nabla v + (k \cdot \mathbf{\Psi}) \cdot \nabla w \right] \). By virtue of the momentum equation (from which equation (1.64) was derived), the \( \mathbf{v} \cdot \nabla \mathbf{\Psi} \) part is inter-related only with the rate of doing work by body forces and the rate of kinetic energy change; it does not contribute to any dissipation of mechanical energy. The other part is inter-related only with the heat conduction and the rate of internal energy change. Thus, the \( \mathbf{v} \cdot \nabla \mathbf{\Psi} \) part gives a rate of doing work on an element of fluid as a whole and hence serves to increase its kinetic
energy or overcome body forces, whereas the other part gives the rate at which work is being done in changing the volume and shape of an element. This separation of the rate at which stresses do work into these two parts, and the physical significance of each part is more easily visualized when there is no shear, in which case \( \nabla \cdot \mathbf{v} \) reduces to \( \nabla(-p \mathbf{v}) \), and the two parts referred to above are \(-\mathbf{v} \cdot \nabla p\) and \(-p \nabla \cdot \mathbf{v}\).
1.7 Special case of a viscous, isotropic fluid.

The abbreviated notation used in this Section (1.7) is defined near the end of Section (2.1), page 10.

1.7.1 Relations between stress and rate of strain.

The principles of dynamics require that $\tau_{ij} = \tau_{ji}$ at each point in any continuous material medium, regardless of its state of motion. The state of stress at a point is completely characterized by the six independent quantities $\tau_{ij}$; which are called the stress components.

By Taylor's series,

$$u_1(x_1 + \delta x_1, x_2 + \delta x_2, x_3 + \delta x_3) = u_1(x_1, x_2, x_3) + \sum_{j=1}^{3} \frac{\partial u_1}{\partial x_j} \delta x_j + \text{terms of order } \delta x_j^2 \text{ and higher.}$$

Now

$$\sum_{j=1}^{3} \frac{\partial u_1}{\partial x_j} \delta x_j = \sum_{j=1}^{3} \frac{1}{2} \left( \frac{\partial u_1}{\partial x_j} - \frac{\partial u_1}{\partial x_1} \right) \delta x_j + \sum_{j=1}^{3} \frac{1}{2} \left( \frac{\partial u_1}{\partial x_j} + \frac{\partial u_1}{\partial x_1} \right) \delta x_j$$

where the first sum on the right describes the effect on $u_1(x_1 + \delta x_1, x_2 + \delta x_2, x_3 + \delta x_3)$ of a rigid rotation about an axis through $(x_1, x_2, x_3)$. The velocity field in a small neighborhood of a point $(x_1, x_2, x_3)$ can be thought of as the sum of the three parts just written down:

1. a rigid translation with velocity components $u_1(x_1, x_2, x_3)$;
(2) a rigid rotation about \((x_1, x_2, x_3)\) with angular velocity components \(\omega_i = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)\) where \(i, j, k\) are to be cyclically permuted;

(3) the remaining part, which is called a pure rate of strain, and is characterized by the six independent rate of strain components

\[ e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = e_{ji}. \]

For the substances classified as fluids, observations show that if all six of the \(e_{ij}\)'s are zero, the stress on any plane is wholly normal, and consequently of magnitude independent of the orientation of the plane. Therefore, it is reasonable to assume that for all fluids, the difference of the state of stress from a state in which the stress is purely normal on any surface, is a function of the rates of strain. This assumption leads to

\[ T_{ij} - (-g \delta_{ij}) = f_{ij} \left( e_{11}, e_{12}, e_{13}, e_{22}, e_{23}, e_{33} \right), \quad (1.71) \]

where \(\delta_{ij}\) is the Kronecker delta, where \(g\) is independent of the \(e_{ij}\)'s and where the \(f_{ij}\)'s are functions which would logically be determined for any given fluid in such a way as to make the results of calculations agree with experiments.

If one assumes

(a) that the functions \(f_{ij}\) in (1.71) are linear,
homogeneous functions of the rate of strain components,

(b) that the fluid is isotropic even when in a general state of stress, then it can be shown to follow\(^1\) that

\[ \tau_{ij} = 2\lambda e_{ij} + (\lambda \text{div} \mathbf{v} - q) \delta_{ij}, \]  

(1.712)

and hence

\[ \tau_{11} + \tau_{22} + \tau_{33} = (3\lambda + 2\mu) \text{div} \mathbf{v} - 3q. \]  

Note that \( \text{div} \mathbf{v} = e_{11} + e_{22} + e_{33} \). The coefficients \( \lambda, \mu, \) and \( q \) in (1.712) and (1.713) are independent of the \( e_{ij} \)'s, but may depend on other parameters, namely (for a given fluid) density and temperature. Let us assume

(c) that for a given fluid, the state of stress at a point depends only on \( \rho, T, \) and the \( e_{ij} \)'s.

It follows from (1.713) that in the static case, \( q \) is the pressure; and since by hypothesis \( q \) is independent of the \( e_{ij} \)'s, it also follows (using assumption (c)) that for any given fluid at a given density and temperature, \( q \) equals the pressure of the static fluid at that density and temperature. Then, for any given fluid at a given density and temperature, \( \lambda \) and \( \mu \) remain as experimentally determinable quantities. It is for experiment, also, to test the validity of the assumptions of isotropy and linearity used in arriving

\(^1\)Cf. the analogous derivation in Reference 29, page 65.
at (1.712) and (1.713).

The classical relations (1.716) between stress and rate of strain for viscous fluids are obtained by setting

\[ \lambda = -\frac{2}{3}\mu. \quad (1.714) \]

Equations (1.714) and (1.713) imply

\[ q = -\frac{1}{3}(\tau_{11}+\tau_{22}+\tau_{33}) = p \quad (1.715) \]

In view of (1.713), we may regard (1.714) and (1.715) as equivalent assumptions, either of which is equivalent to the assumption

(d) that for a given fluid at a given density and temperature, the sum \((\tau_{11}+\tau_{22}+\tau_{33})\) is independent of the rate of expansion. With this additional assumption, then, (1.712) becomes

\[ \tau_{11} = 2\mu e_{11} - \left(\frac{2}{3}\mu \text{div} \tau + p\right)\delta_{11} \quad (1.716) \]

If we make the assumptions of this section and the additional assumption that changes in the fluid’s thermo-dynamic coordinates occur quasi-statically, then we are at liberty to replace the pressure by \(p\) in the fluid’s pressure-density-temperature relation.

1.72 Equations of momentum and energy.

We may write

\[ \text{div} \, \mathbf{p} = \sum_{k=1}^{3} i_{k} \sum_{j=1}^{3} \frac{\partial \tau_{11k}}{\partial x_{j}} \]

which becomes, using (1.76) and the definitions of the \(e_{ij}\)’s,
\[
\text{div } \mathbf{\Psi} = \sum_{k=1}^{3} \frac{1}{i_k} \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \left[ k \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right) \right] - \left( \frac{2}{3} \mu \text{div } \mathbf{v} + p \right) \delta_{jk}.
\]

(1.721)

In vector notation (1.721) may be written

\[
\text{div } \mathbf{\Psi} = - \text{grad } p - \frac{2}{3} \text{grad } (\mu \text{div } \mathbf{v}) + \text{curl } (\mu \text{curl } \mathbf{v})
\]

+ 2 \text{div } (\mu \text{grad } \mathbf{v}),
\]

(1.722)

where \text{grad } \mathbf{v} is a dyadic. We may note here that \text{div } (\mu \text{grad } \mathbf{v})

is the vector whose 1-th component is \text{div } (\mu \text{grad } u_1). Substitution from (1.722) into the general mementum equation (1.53) yields

\[
\frac{d \mathbf{v}}{dt} = \rho \mathbf{F} - \text{grad } p - \frac{2}{3} \text{grad } (\mu \text{div } \mathbf{v}) + \text{curl } (\mu \text{curl } \mathbf{v})
\]

+ 2 \text{div } (\mu \text{grad } \mathbf{v}),
\]

(1.723)

The quantity \((\mathbf{1} \cdot \mathbf{\Psi}) \cdot \text{grad } \mathbf{u} + (\mathbf{1} \cdot \mathbf{\Psi}) \cdot \text{grad } \mathbf{v}

+ (\mathbf{k} \cdot \mathbf{\Psi}) \cdot \text{grad } \mathbf{w}\), which appears in the general energy equation (1.65), is \(\sum_{j=1}^{3} (\mathbf{1}_j \cdot \mathbf{\Psi}) \cdot \text{grad } u_j\) in the abbreviated notation. Now \(\mathbf{1}_j \cdot \mathbf{\Psi} = \sum_{k=1}^{3} \mathbf{1}_k \tau_{jk}\). Hence, using (1.716),

\[
\mathbf{1}_j \cdot \mathbf{\Psi} = \sum_{k=1}^{3} \mathbf{1}_k \left[ k \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right) \right] - \left( \frac{2}{3} \mu \text{div } \mathbf{v} + p \right) \delta_{jk}.
\]

(1.724)

Thus, for a viscous, isotropic fluid,

\[
\sum_{j=1}^{3} (\mathbf{1}_j \cdot \mathbf{\Psi}) \cdot \text{grad } u_j = \sum_{j=1}^{3} \sum_{k=1}^{3} \mathbf{1}_j \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right) \frac{\partial u_i}{\partial x_k} - \frac{2}{3} \mu (\text{div } \mathbf{v})^2 - p \text{div } \mathbf{v}.
\]

(1.725)
If we define
\[ \phi = -\frac{2}{3} \mu (\text{div } \mathbf{v})^2 + \sum_{j=1}^{3} \sum_{k=1}^{3} \mu \frac{\partial u_j}{\partial x_k} \left( \frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right) \quad (1.726) \]
then the energy equation (1.65) becomes
\[ \rho \frac{dE}{dt} + p \text{ div } \mathbf{v} = \text{div} (k \text{ grad } T) + \phi, \quad (1.727) \]
subject, of course, to the assumptions involved in equations (1.716).

We may define the enthalpy, I, by
\[ I = E + \frac{p}{\rho}. \quad (1.728) \]
Then, since the continuity equation (1.32) implies
\[ \text{div } \mathbf{v} = -\frac{1}{\rho} \frac{d\rho}{dt}, \]
we see that
\[ \rho \frac{dE}{dt} + p \text{ div } \mathbf{v} = \rho \frac{d}{dt} \left( I - \frac{p}{\rho} \right) - \frac{p}{\rho} \frac{d\rho}{dt} = \rho \frac{dI}{dt} - \frac{dp}{dt}. \]

Thus, an alternative form of (1.727) is
\[ \frac{dI}{dt} - \frac{dp}{dt} = \text{div} (k \text{ grad } T) + \phi. \quad (1.729) \]

With I defined by (1.728), it appears that (1.729) is no less general than (1.727); and neither I nor E is restricted to be a function of the temperature only in either equation. The preceding sentence contradicts statements by Cope and Hartree (Ref. 30) to the effect that E must be a function only of T in order for (1.727) to apply and that both I and E must be functions only of T in order for (1.729) to apply.

On the other hand, if E were defined as \[ \int_{0}^{T} C_{V}(T) dT \]
and I by $\int_0^T C_p(T)dT$, then the statements by Cope and Hartree would be correct.
1.73 The dissipation function, \( \phi \).

The name "dissipation function" is justified in Section 1.92.

1.731 Proof that \( \phi \geq 0 \).

Purely algebraic manipulations show that if \( \phi \) is defined by (1.726), we may write

\[
\phi = \frac{2}{3} \mu \left[ (e_{22} - e_{33})^2 + (e_{33} - e_{11})^2 + (e_{11} - e_{22})^2 \right] \\
+ 4 \mu \left( e_{23}^2 + e_{13}^2 + e_{12}^2 \right).
\]  

Equation (1.731) shows explicitly that \( \phi \) is always \( \geq 0 \), and that it can be zero only if \( e_{12} = e_{13} = e_{23} = 0 \) and \( e_{11} = e_{22} = e_{33} \). Thus, \( \phi \) is zero only if the distortion consists of an expansion or contraction which is the same in all directions.

1.732 Invariant form of \( \phi \).

The fact that \( \phi \) is a scalar invariant follows from its definition (1.726); but the definition can be directly applied only in a rectangular cartesian coordinate system. The correct expression for \( \phi \) if the velocity components belong to some other coordinate system may be obtained by transformation from rectangular coordinates. As an alternative, one may make use of

\[
\phi = - \frac{2}{3} \mu (\text{div} \ \mathbf{v})^2 + \mu \left[ (\text{div} \ \text{grad} \ (\mathbf{v}^2)) \\
- 2 \mathbf{v} \cdot (\text{grad} \ \text{div} \ \mathbf{v} - \text{curl} \ \text{curl} \ \mathbf{v}) \\
- (\text{curl} \ \mathbf{v})^2 \right].
\]  

(1.732)
In (1.732), $\phi$ has been expressed as a sum of quantities which have a known significance independent of the coordinate system used. (With properly defined $e_{ij}$'s, this is also true of (1.731)).
1.8 Boundary conditions.

Although it was recognized very early that a fluid does not ordinarily slip freely past a solid body, careful experimentation is required to establish the exact boundary conditions which should be applied at the surface along which a fluid is in contact with a solid body. For an account of the evidence supporting the presently accepted hypothesis of no slip (except in cases in which the concept of a continuous fluid medium becomes suspect), one may refer to the "Note on the Conditions at the Surface of Contact of a Fluid with a Solid Body" in Ref. 31, Vol. 2, pp. 676-680.

The accepted boundary condition for the temperature field in a continuous fluid medium is that the fluid adjacent to a solid surface has the same temperature as the surface. Discontinuities in the temperature between a surface and an adjacent gas have been observed only when the gas is so rare that it can no longer be properly treated as a continuous medium.
1.9 Remarks regarding thermodynamic relations.

1.91 General remarks.

It was noted at the end of Section 1.71 that the pressure may be replaced by \( p \) in the fluid's pressure-density-temperature relation subject to the assumptions (a) through (d) of Section 1.71 and the additional assumption that changes in the fluid's thermodynamic coordinates occur quasi-statically, so that any sufficiently small portion of the fluid is in thermal equilibrium. Experiments must determine the conditions under which these assumptions are valid.

In classical thermodynamics\(^1\), the internal energy function is defined as a function whose increase between any two states equals the work which must be done on the system to change the state adiabatically from one state to the other; the existence of the internal energy function follows from the experimental fact that this work is always the same for the same initial and final states. Since this experimental fact is simply a manifestation of the well-established principle of conservation of energy, one may feel secure in retaining the above definition of internal energy even for bodies in a quite general state of stress.

\(^1\)Thermostatics would be a more descriptive term than thermodynamics for this subject.
It is relevant to ask how many coordinates are required to describe the thermodynamic state of a real fluid when in a general state of motion and stress. Any attempt to answer this question by direct experiment brings one face to face with the difficulty that direct measurements of the thermodynamic coordinates can be made only on macroscopic systems which are in equilibrium or at least very nearly in equilibrium. One thing which can be done in a general case is to make predictions based on certain hypotheses, and then test these hypotheses by comparing the predictions with the results of those measurements which can be made. The simplest plausible hypothesis is that the presence of shear stress or of motion has no effect on the number of independent thermodynamic coordinates. (This hypothesis is not plausible for elastic solids, since for elastic solids the internal energy, as defined in the preceding paragraph, certainly depends on all six independent stress components, and not simply on the average of the normal stresses on three mutually perpendicular planes.)

If there are two independent thermodynamic coordinates, then any three thermodynamic coordinates are related by an equation of state which may be determined experimentally in the static case. Does this same equation of state hold
when the thermodynamic coordinates are changing with time? If not, then there must really be more than two independent thermodynamic coordinates. For example, in the static case, we can certainly write \( E = E(\rho, T) \) for an equation of state relating internal energy, density, and temperature; but in the dynamic case it is conceivable that \( E \) might depend not only on \( \rho \) and \( T \), but also on the first, second, and even higher time derivatives of \( \rho \) and \( T \), or even on the history of \( \rho \) and \( T \).

The preceding discussion suggests the complications which can conceivably arise when thermodynamic coordinates change with time — and such complications definitely do arise when changes are sufficiently rapid. Much theory in gas dynamics proceeds from the assumption that changes in the thermodynamic coordinates occur quasi-statically; that is, although the thermodynamic coordinates of each fluid element change with time, it is assumed that at each instant, the static equations of state apply. Experiment must answer the question, "How rapid may changes occur without voiding this assumption of quasi-static processes?"
1.92 Rate of entropy increase.

If we can replace the pressure by \( p \) in the equations of state \( p = p(\rho, T) \), \( s = s(E, \rho) \), \( E = E(\rho, T) \); etc., and if the same equations of state hold in the dynamic cases as in the static case, then we may investigate changes in entropy, \( s \), by means of the relations

\[
\left( \frac{\partial s}{\partial E} \right)_\rho = \frac{1}{T} \quad \text{and} \quad \left( \frac{\partial s}{\partial \rho} \right)_E = -\frac{\rho}{T E^2}
\]

which follow from the quasi-static relation

\[
\frac{ds}{dt} = \frac{dE}{dt} - \frac{\rho}{E^2} \frac{d\rho}{dt}.
\]

Thus, \( \frac{ds}{dt} = \frac{3s}{E} \frac{dE}{dt} + \frac{3s}{\rho} \frac{d\rho}{dt} = \frac{1}{T} \left( \frac{\partial s}{\partial T} - \frac{\rho}{E^2} \frac{d\rho}{dt} \right) \). \hspace{1cm} (1.91)

The use of \( \text{div } v = -\frac{1}{\rho} \frac{d\rho}{dt} \) in (1.727) results in

\[
\frac{dE}{dt} - \frac{\rho}{E^2} \frac{d\rho}{dt} = \frac{1}{\rho} \left[ \text{div } (k \text{ grad } T) + \phi \right] \hspace{1cm} (1.92)
\]

so that

\[
\frac{ds}{dt} = \frac{1}{T} \left[ \text{div } (k \text{ grad } T) + \phi \right]. \hspace{1cm} (1.93)
\]

Since \( \phi \geq 0 \), equation (1.93) shows explicitly that the function \( \phi \) is related to a rate of increase in entropy. Thus, the name "dissipation function" is really descriptive.

We may re-write (1.93) as

\[
\frac{ds}{dt} = \text{div } \left( \frac{k}{T} \text{ grad } T \right) + \frac{k}{T^2} (\text{ grad } T)^2 + \frac{\phi}{T}. \hspace{1cm} (1.94)
\]

The result of integrating (1.94) over a finite body of fluid, \( \Omega \), may be written
\[
\frac{d}{dt} \int_V \cdot \mathbf{v} \cdot n \, dS + \int_V \frac{k}{T^2} \left( \nabla T \right)^2 \, dv + \int_V \frac{\phi}{T} \, dv
\]  

(1.95)

where \( S \) is the surface bounding the moving body, \( V \). The first term on the right in (1.95) is the rate of entropy change due to heat conduction across the bounding surface \( S \), and may be either positive or negative. The second term on the right represents the rate of entropy change due to the irreversible process of temperature equalization which tends to take place inside the volume \( V \); this term is never negative. The last term represents the rate of entropy change due to dissipation of mechanical energy inside the volume \( V \) through the mechanism of viscosity; since \( \phi \geq 0 \), this term is never negative.
2. Background Information on the Laminar Boundary Layer.

2.1 The concept of the boundary layer.

The great difficulty of obtaining solutions which satisfy both realistic boundary conditions and the complete differential equations for viscous fluids has forced investigators to make approximations in their attempts to answer questions regarding the flow of viscous fluids. If the viscosity coefficient and the density are regarded as constant, then the momentum equation (1.723) and its boundary conditions determine the velocity field; the energy equation may then be treated separately, in order to determine the temperature field. This simplification, resulting from the assumption that $\mu$ and $\rho$ are constants, has not been sufficient to permit investigators to obtain solutions for the velocity field except in a few cases,¹ and additional approximations had to be made. For example, Stokes'² formula, $D = \frac{6\pi\mu}{\rho u}$,

¹Some examples of solutions for the velocity field which are exact in the sense that they satisfy the conditions of no slip or penetration at a solid surface, and equations (1.34) and (1.723) with $\mu$ and $\rho$ constant, are presented in Ref. 32, pp. 61-69, and Ref. 31, Vol. I, pp. 105-113.

²Transactions of the Cambridge Philosophical Society, Vol. IX, 1851. Reprinted in Stokes' Mathematical and Physical Papers, Vol. III, p. 55, Oxford University Press, 1901. Stokes' formula has played an important role in fundamental physical research. A measurement of the maximum constant velocity (i.e., velocity when drag = weight) of a droplet falling through a gas furnishes information regarding droplet size, by virtue of Stokes' formula. The formula has been used in this manner by H. A. Wilson (Ref. 33), J. J. Thomson (Ref. 34), and R. A. Millikan (Ref. 35). (Continued next page.)
for the drag $D$ on a sphere of radius $r$ and velocity $u$ moving slowly through a fluid of constant viscosity $\mu$ results from neglecting the inertia of the fluid. The "slowness" of the motion is indicated by the Reynolds number $\frac{ur}{\mu}$. Stokes' formula agrees with experiment for values of $\frac{ur}{\mu}$ up to about 0.5. Much larger values of an appropriately defined characteristic Reynolds number occur in aeronautical applications. Neglecting the fluid's inertia in flows characterized by these large values of Reynolds number would lead to serious errors. Hence, a different kind of approximation is necessary.

The approximation which has been successful for relatively large Reynolds numbers is called boundary layer theory. L. Prandtl (Ref. 1) first pointed out that in many cases of flow past a body, the effects of viscosity are confined to a thin layer near the body, and to a wake consisting of fluid which has passed near the body. The flow outside of this thin boundary layer and wake is very nearly irrotational and can be treated, to a good approximation, by the methods developed for irrotational flow of a nonviscous fluid. As a first approximation, one may assume that the boundary

However, in R. A. Millikan's famous oil drop experiment for measuring the electronic charge (Ref. 36), Stokes' formula had to be modified. (The molecular mean free path in the air through which the droplets fell was of the same order of magnitude as the droplets' radii. Therefore, any theory based on a continuous fluid medium could hardly be adequate. Ref. 37 is of interest in this connection.)
layer is so thin that its presence does not affect the irrotational flow. The tangential velocity at the body as obtained from the irrotational flow solution (or as deduced from a measured pressure\(^1\) distribution using nonviscous fluid theory) is used as a boundary condition at the "outer edge" of the boundary layer in the solution for the velocity field in the boundary layer. The velocity field in the boundary layer is then assumed to satisfy approximate differential equations which, in the case of laminar flow along a surface of zero curvature, become exact in the limit as an appropriately defined Reynolds number tends to infinity. These approximate boundary layer equations possess solutions in which the velocity component parallel to the body surface tends to a value independent of \(y\) as \(y\), the distance from the surface, tends to infinity. Therefore, the boundary conditions at the "outer edge" of the boundary layer (which are conditions at the body's surface in the irrotational flow solution), may be treated as conditions at an infinite distance from the body in the mathematical boundary layer analysis. The boundary conditions at the body are the conditions of no slip and no penetration. Skin friction calculations based on boundary layer theory have been

\(^1\)To the approximation of boundary layer theory the pressure at the "outer edge" of the boundary layer is the same as the normal stress at the corresponding point on the body surface.
successful in a large number of cases.

The effects of the fluid's thermal conductivity, like the effects of viscosity, are often confined to thin layers. In the case of viscous effects, these layers are those in which there are large rates of shear. In the case of conductivity effects, these layers are those in which large temperature gradients exist. These layers may be expected to occur near solid bodies, or in regions where two portions of fluid having different temperatures come together. E. Pohlhausen (Ref. 5) appears to have been the first to perform calculations based on this "boundary layer theory" for the temperature field. The energy equation is replaced by an approximate equation whose solutions have the property that the temperature tends to a value independent of \( y \) as \( y \), the distance from the surface, tends to infinity. The temperature "just outside the boundary layer" may then be treated as the temperature obtained when \( y \to \infty \) in the boundary layer problem. This temperature, which is needed as a boundary condition in the boundary layer problem, is assumed to be related to the velocity "just outside the boundary layer" by the theory for non-viscous, non-conducting fluids, since, in boundary layer theory, we proceed as if the effects of viscosity and conductivity are wholly confined to the boundary layer. This
temperature is the temperature which would be attributed to the fluid adjacent to the body if calculations were based on the hypotheses that the fluid is nonviscous and non-conducting. If the body's surface temperature is known, the other boundary condition for the temperature boundary layer problem is that the fluid adjacent to the body's surface \((y = 0)\) have the same temperature as the body's surface. If the body's surface temperature is unknown, this condition still applies, but is of no help in determining the temperature field; in this case, information regarding the heat transfer rate between the surface and the fluid will determine the component of the temperature gradient normal to the body, and this specification of the normal component of the temperature gradient serves as the other boundary condition.

Skin temperature and heat transfer calculations based on boundary layer theory have been successful in a large number of cases.

In laminar boundary layer solutions, the velocity and temperature attain their values "outside the boundary layer" only in the limit as \(y \to \infty\), but these solutions have the property that the final values are practically attained for values of \(y\) so small that the thickness of the boundary
layer can, indeed, often be neglected in calculating the velocity outside it by the methods for irrotational flow. If this were not true, one could obtain a second approximation by calculating the tangential velocity as if the body were thickened by an amount sufficient to account for the boundary layer thickness.

When the velocity and temperature fields in the flow past a body are calculated using the approximation of boundary layer theory, one may visualize the solution as consisting of (1) the solution for a very slightly thickened body in a non-viscous, non-conducting fluid, and (2) the solution to the boundary layer problem fitted in between the unthickened body and the irrotational flow solution for the thickened body.

The practical results of boundary layer theory are skin friction coefficients, skin temperatures, and heat transfer coefficients; and the theory's most impressive defense is the agreement of calculated and experimental values.

Boundary layer theory is an approximation adequate to deal with flow problems when large temperature and (or) velocity changes occur across sufficiently thin layers of fluid. Many such problems occur in aeronautics. It should be emphasized that boundary layer theory may not be applied indiscriminately to all flow problems, since the approximations
involved might not be valid.

2.2 Approximate boundary layer equations for two-dimensional flow.

We take \( x \) to be a curvilinear coordinate measured along the surface, and \( y \) the perpendicular distance from the surface. Arguments can be presented to show that in boundary layers along surfaces of relatively small curvature, some terms in the basic equations are of much smaller order of magnitude than the remaining terms. When the terms which are expected to be of small order of magnitude are neglected, there remains, for steady two-dimensional flow, from (1.34)

\[
\frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) = 0 
\]

(2.21)

from (1.723),

\[
p = p(x), 
\]

\[
\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right), 
\]

(2.22)

and from (1.729),

\[
\rho u \frac{\partial T}{\partial x} + \rho v \frac{\partial T}{\partial y} - u \frac{\partial p}{\partial x} = \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} + \lambda \left( \frac{\partial u}{\partial y} \right)^2 \right). 
\]

(2.23)

Attention is invited to the fact that \( x \) and \( y \) are now curvilinear coordinates. In view of the remarks of Section (2.1), we write

\[
\frac{\partial p}{\partial x} = -\rho u_1 du_1/\partial x 
\]

(2.24)

where the subscript 1 denotes a value "outside the boundary layer."
To these approximate equations of continuity, momentum, and energy, there must be added equations of state and some means of expressing the variation of $\mu$ and $k$. The remarks and questions of Sections 1.71 and 1.9 of this appendix are relevant here. One may also ask whether the thermal conductivity, $k$, is the same function of $\rho$ and $T$ when shear stresses are present as when there is no shear, and also whether the static relation for $k(\rho,T)$ holds when $\rho$ and $T$ for a fluid element vary with time. In our work with gases, we shall assume that $k$, $\mu$, $I$, and $E$ are all functions only of the temperature, $T$, and that $p = \rho RT$. It then follows that $I = \int_0^T C_p(T)\,dT$ and $C_p - C_v = R$. More detailed information regarding the validity of these assumptions for air will be given in the next section.
2.3 Properties of air.

Keenan and Kaye (Ref. 38, p. 199) state that for air at 32° F., deviation from the perfect gas equation of state \( p = \rho RT \) is of the order of 1 per cent at 300 psi and 0.1 per cent at atmospheric pressure; at higher temperatures the deviation is smaller, and at lower temperatures, is somewhat larger. Thus, for many applications, the equation of state \( p = \rho RT \) is sufficiently accurate. It follows from \( p = \rho RT \) that \( I \) and \( E \) are functions of the temperature only, given by \( I = \int_0^T C_p(T) dT \) and \( E = \int_0^T C_v(T) dT \). Also, at low pressures, \( k \) and \( \lambda \) are found experimentally to depend only on \( T \). Graphs showing \( C_p, \lambda, k, \gamma = C_p/C_v, \) and \( \sigma = \frac{C_p}{k} \) as functions of the temperature are shown in figures (2.31) through (2.35). Sample "power law" approximations to some of these functions are also indicated. The approximations selected for comparison are the approximations used in Reference 23.

Before the relatively recent emphasis on high speed flight, there was less incentive than there is now for considering the variation of these properties with temperature. A sample calculation easily shows that in a typical low speed case (say 200 ft./sec. at sea level) the percentage changes in density and temperature between a stagnation
point and the free stream are less than two percent in adiabatic flow. Large temperature variations will not occur at low speeds unless a body is artificially heated or cooled.
Figure 2.83

Graph showing $\frac{\rho}{\rho_0} = \left(\frac{T}{T_0}\right)^{\alpha}$, where $\rho_0 = 0.00245$ lbs and $T_0 = 600 \times 10^7$ ft (at sea level).

At $\frac{\rho}{\rho_0} = \frac{\rho}{\rho_0}$, $\frac{T}{T_0} = \frac{T}{T_0} = \frac{T}{T_0}$.

Where $\frac{T}{T_0} = \frac{T}{T_0}$. $\frac{\rho}{\rho_0}$ vs. $\frac{T}{T_0}$.
Thermal conductivity in...
FIGURE 2.35

Graph showing $\alpha(T)$ for air

Measured range. Remainder of curve was obtained by extrapolation. (See Reference 38, page 203.)
The most accurate integration of (2.411) is due to Howarth (Ref. 39). The value of \( f''(o) \) is related to the shear stress at the plate by the formula
\[
\tau_w \sqrt{\frac{u_1 x}{\beta}} = \frac{1}{4} \rho u_1^2 f''(o),
\]
or
\[
\alpha v^2 = \frac{1}{2} f''(o).
\]
Howarth found \( f''(o) = 1.32824 \).

2.412 Solutions by Falkner and Skan for the cases \( u_1 \propto x^m \).

The theory of irrotational flow of a nonviscous fluid furnishes \( u_1 \propto x^m \) for the tangential velocity along a wedge of central angle \( 2\pi \frac{m}{m+1} \) radians at zero angle of attack. Here \( x \) is the distance from the vertex measured along a face of the wedge. The velocity field is determined by

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - u_1 \frac{du_1}{dx} = v \frac{\partial^2 u}{\partial y^2}. \]

with boundary conditions \( u = v = 0 \) at \( y = 0 \), and \( u \to u_1(x) \) as \( y \to \infty \). One may again define \( \psi \) by \( u = \frac{\partial \psi}{\partial y} \) and \( v = -\frac{\partial \psi}{\partial x} \). If one then makes the transformation

\[
\psi = \sqrt{\frac{u_1 x}{\beta (m+1)}} F(Y) \text{ where } Y = \sqrt{\frac{1}{2}(m+1)} \frac{u_1}{\beta} y,
\]

the differential equations and boundary conditions may be expressed as (2.412).

\[
F'' + \frac{2m}{m+1}(F^{'2} - 1) = (2.412)
\]

\( F(o) = F'(o) = 0 \)

\( F'(Y) \to 1 \) as \( Y \to \infty \).
2.4 Available solutions for a fluid with constant $\rho$, $\mu$, $k$, and $C_P$.

2.4.1 Solutions for the velocity boundary layer.

2.4.11 H. Blasius' solution for zero pressure gradient case.

This case approximately applies to flow along a flat plate at zero angle of attack, with the origin of coordinates at the leading edge. The velocity field is determined by the differential equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2},$$

with boundary conditions $u = v = 0$ at $y = 0$, and $u \to u_l = \text{constant as } y \to \infty$. If one defines a stream function $\psi$ by $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$, then the first equation is satisfied by any well-behaved function $\psi$. If one lets

$$\psi = \sqrt{2u_l x} f(\eta) \text{ where } \eta = \frac{1}{2} \sqrt{\frac{u_l}{2x}} y,$$

then the second equation and boundary conditions are satisfied if $f(\eta)$ satisfies

$$f'' + ff'' = 0$$

$$f(0) = f'(0) = 0 \quad (2.411)$$

$$f'(\eta) \to 2 \text{ as } \eta \to \infty.$$
The system (2.412) was studied by Hartree (Ref. 40). The value of \( \sqrt{\frac{m+1}{2}} F''(0) \), which is needed in computations of shear stress, is plotted as a function of \( m \) in figure (2.412).

We note that Blasius' problem is included as a special \((m = 0)\) case, since the transformation \( F(Y) = \sqrt{\frac{1}{2}} f(\eta) \);
\[
\eta = \sqrt{\frac{1}{2}} Y
\]
serves to transform from (2.412) with \( m = 0 \) to (2.411).
2.413 L. HOWARTH'S SERIES SOLUTION FOR CASES

\[ u_1 = \sum_{n=0}^{\infty} a_n x^n. \]

The origin is a stagnation point.

The stream function \( \psi \) (again defined such that \( u = \frac{\partial \psi}{\partial y} \) and \( v = -\frac{\partial \psi}{\partial x} \)) is expanded in a series of powers of \( x \) whose coefficients are functions of \( y \). This method was used by Blasius (ref. 2), but we owe to Howarth (ref. 20) the technique of expressing all the coefficient functions in such a way that the solution is applicable to arbitrary values of the \( a_n \)'s. If one writes \( \psi(x, y) = x f_1(y) + x^2 f_2(y) + x^3 f_3(y) + x^4 f_4(y) + \cdots \), then the functions \( f_1, f_2, f_3, f_4, \cdots \) certainly depend on \( a_1, a_2, a_3, a_4, \cdots \); but if one writes \( \psi(x, y) = \sqrt{a_1} \sum \left[ \frac{x f_1(y)}{a_1} + 3 \frac{a_2}{a_1} x^2 f_2(y) + 4 x^3 \right. \\
\left. \left( \frac{a_3}{a_1} f_3(y) + \frac{a_4}{a_1} f_4(y) + \frac{a_5}{a_1} f_5(y) + \frac{a_6}{a_1} f_6(y) + \cdots \right) \right] \)

where \( \gamma = \sqrt{\frac{a_1}{2}} y \), then the functions \( f_1, f_2, f_3, f_4, f_5, f_6, \cdots \) satisfy differential equations and boundary conditions in which the parameters \( a_1, a_2, a_3, a_4, a_5, \cdots \) do not appear; hence these functions are the same for all values of \( a_1, a_2, a_3, a_4, \cdots \). For a symmetrical body at zero angle of attack, one expects \( u_1(x) = -u_1(-x) \) and hence the series for \( u_1(x) \) contains only odd powers of \( x \); in other words, \( a_n = 0 \) for even \( n \). For this symmetrical case, the functions \( f_2, f_3, f_4, f_5, f_6, f_7, f_8, \cdots \) satisfy the same equations and boundary conditions.
The additional functions $g_7$, $h_7$, $k_7$, $g_9$, $h_9$, $k_9$, $j_9$, $q_9$, $g_{11}$, ..., $n_{11}$, are also universal in the sense that they do not depend on the $a_n$'s. Reference 20 contains tables of the functions $f_1$, $f_3$, $g_5$, $h_5$, $k_7$, $f_2$, $h_3$, $k_4$, and the first two derivatives of these functions. These tables

2. Howarth's notation for the functions $f_l$, $f_3$, $g_5$, $h_5$, ..., in the symmetrical case has been retained here. These functions also occur in the non-symmetrical case; where necessary, the notation in Howarth's treatment (ref. 20) of the non-symmetrical case has been altered in order that the same function may be denoted by the same symbol in both cases. Howarth carried the analysis through $r_0$ in the non-symmetrical case, and $r_0$ in the symmetrical case. The additional terms given here, and the differential equations and boundary conditions which the functions satisfy were obtained at Ohio State Univ., as part of a project supervised by Prof. Arthur N. Tifford.
are reproduced in reference 31, Vol. 1, pp 151-153. With the exception of \( g_9 \), \( g_{11} \), and \( h_{11} \), all the functions which appear explicitly in (2.413) have been obtained from their differential equations and boundary conditions (ref. 41). These values in reference 41 are presumably the most accurate in existence. Initial values of the second derivatives of these functions are needed in skin friction computations. These values from reference 41 are listed below.

\[
\begin{align*}
 f_1''(0) & = 1.232588 & q_9''(0) & = -0.030787 \\
 f_3''(0) & = 0.724447 & k_9''(0) & = 0.074200 \\
 g_5''(0) & = 0.634702 & f_7''(0) & = 0.080554 \\
 h_5''(0) & = 0.119182 & q_9''(0) & = 0.116361 \\
 g_7''(0) & = 0.579202 & m_9''(0) & = -0.179648 \\
 h_7''(0) & = 0.182948 & n_9''(0) & = 0.051561 \\
 k_7''(0) & = 0.007638 & & \\
 h_9''(0) & = 0.151970 & & \\
 k_9''(0) & = 0.057185 & & \\
 j_9''(0) & = 0.060741 & & \\
\end{align*}
\]

The practical usefulness of series solutions obviously depends on the rapidity of convergence.
2.414 SERIES SOLUTION FOR THE CASES

\[ u_1 = \sum_{n=0}^{\infty} a_n x^n. \]

In this case \( u_1 \) need not be zero at the origin. Thin curved plates at zero angle of attack in subsonic flow, and bodies with attached shock waves suggest themselves as cases to which the above distribution may apply. An analysis analogous to that used by L. Howarth in references 20 and 39 enables one to express the solution in terms of functions which are universal in the sense that the functions do not depend on the \( a_n \)'s. The analysis has been carried out by Chi-Chang Chao (reference 42). Chao found

\[
\begin{align*}
\frac{a_2}{a_o} h' + \frac{a_3}{a_o} x + \frac{a_1 a_2}{a_o} h' + \frac{a_1}{a_o} k' + 8 x^2 (a_2 g_2' + \frac{a_1}{a_o} h_2') x^2 \\
+ 8^3 x^3 (a_3 g_3' + \frac{a_1 a_2}{a_o} h_3' + \frac{a_1}{a_o} k_3') + 8^4 x^4 (a_4 g_4' + \frac{a_1 a_3}{a_o} h_4') \\
+ 8^5 x^5 (a_5 g_5' + \frac{a_1 a_4}{a_o} h_5' + \frac{a_1 a_2}{a_o} k_5') \\
+ \cdots
\end{align*}
\]

(2.414)
where the functions $f_0, f_1, g_2, h_2, g_3, \ldots, n_5$, are all functions of $\gamma - \frac{1}{3} \left( \frac{a_0}{y^2} \right)^{\frac{3}{2}}$. The function $f_0$ is identical with the function $f$ which satisfies (2.411.). The functions $f_1, h_2, k_3, q_4$, and $n_5$ are also identical with functions tabulated by Howarth in reference 39. These identities are indicated in the following table.

<table>
<thead>
<tr>
<th>Function in Chao's notation (ref. 42)</th>
<th>Function in Howarth's notation (ref. 39)</th>
<th>Initial value of 2nd. derivation (Howarth's notation)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_0$</td>
<td>$f_0$</td>
<td>$f_0''(0) = 1.328242$</td>
</tr>
<tr>
<td>$f_1$</td>
<td>$f_1$</td>
<td>$f_1''(0) = 1.02054$</td>
</tr>
<tr>
<td>$h_2$</td>
<td>$f_2$</td>
<td>$f_2''(0) = -0.06926$</td>
</tr>
<tr>
<td>$h_3$</td>
<td>$f_3$</td>
<td>$f_3''(0) = 0.0560$</td>
</tr>
<tr>
<td>$q_4$</td>
<td>$f_4$</td>
<td>$f_4''(0) = -0.0372$</td>
</tr>
<tr>
<td>$n_5$</td>
<td>$f_5$</td>
<td>$f_5''(0) = 0.0272$</td>
</tr>
</tbody>
</table>

The integrations necessary for a determination of the other functions have not been performed.

1. An error in sign in Chao's equation (1.1c) causes errors in some of his differential equations.
2.415 **STEP-BY-STEP METHODS.**

Suppose that one has given not only \( u_1(x) \), but also the velocity profile \( u(x_0, y) \) for some particular station \( x_0 \), and that the problem is to find the profile \( u(x, y) \) for some other station \( x \). A brief introduction to some of the currently available methods of handling this problem is given in reference 31, Vol I, page 153.
2.416 THE MOMENTUM INTEGRAL METHOD.

In this method, the solution obtained does not satisfy the differential momentum equation, but rather, an integral momentum equation which may be obtained by integrating the differential momentum equation (2.22) across the boundary layer. The method is of practical importance in skin friction calculations, but is unrelated to the research of this thesis. For details and references, the reader may refer to reference 31, Vol. I., pp. 156-163. The original papers are references 43 and 44.
2.42 SOLUTIONS FOR THE TEMPERATURE BOUNDARY LAYER.
Since \( \rho \) and \( \mu \) are assumed constant, the velocity field obtained by Blasius applies. The energy equation used by Pohlhausen is (from 2.23)

\[
\frac{u}{x} \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \nu \left[ \frac{1}{\sigma} \frac{\partial^2 T}{\partial y^2} + \left( \frac{\partial u}{\partial y} \right)^2 \right],\tag{2.4211}
\]

To obtain the temperature of an insulated flat plate, E. Pohlhausen (ref. 5) set \( \omega(\eta) = \frac{4C_p}{\frac{1}{2} u_1^2} (T-T_i) \) and found that the energy equation reduced to

\[
\omega'' + \sigma \omega' + 2 \sigma \omega''^2 = 0
\]

with the boundary conditions \( \omega'(0) = 0 \) and \( \omega \to 0 \) as \( \eta \to \infty \). The functions \( f \) and \( \eta \) are defined as in (2.411). This differential equation gives no difficulty since it is linear and first order in \( \omega' \). Pohlhausen's result is

\[
\omega(\eta) = \beta(\sigma) - 2\sigma \int_0^\eta e^{-\sigma f(t)} dt \int_0^r f''(s) e^{-2\sigma f(t)} ds dr,
\]

where

\[
\beta(\sigma) = 2\sigma \int_0^\eta e^{-\sigma f(t)} dt \int_0^r f''(s) e^{-2\sigma f(t)} ds dr.
\]

Pohlhausen tabulated values of \( \beta(\sigma) \) for values of \( \sigma \) between 0.6 and 15. It turns out that \( \beta(\sigma) = 4\gamma \sigma \) is a good approximation for \( \sigma \leq 1.1 \). Pohlhausen's values are reproduced in ref. 31, Vol. II., p. 630 where \( \beta(\sigma) \) is denoted by \( \alpha_2(\sigma) \).

It follows from Pohlhausen's analysis that the temperature. By the word "insulated", one means that there is no heat transfer between the fluid and the plate's surface. This implies \( \frac{\partial T}{\partial y} |_{y=0} = 0 \).
ture of an insulated surface, $T_{w_{\text{ins}}}$, is given by

$$T_{w_{\text{ins}}} = T_1 + \frac{1}{2} \beta(\sigma) \frac{u_1^2}{20}.$$  

(2.4212)

In the treatment of the temperature field around a non-insulated but isothermal plate, Pohlhausen neglected the dissipation term $\nu \left( \frac{\partial u}{\partial y} \right)^2$ in the approximate energy equation. With this dissipation term neglected, the substitution $\theta(\eta) = \frac{T - T_w}{T_1 - T_w}$ reduces the energy equation to

$$\theta'' + \sigma \theta' = 0$$

with boundary conditions $\theta(0) = 0$ and $\theta(\eta) \to 1$ as $\eta \to \infty$. The solution is

$$\theta(\eta) = A(\sigma) \int_0^\eta e^{-\sigma \int_0^s f(t) dt} \, ds$$

where

$$A(\sigma) = \left[ \int_0^\infty e^{-\sigma \int_0^s f(t) dt} \, ds \right]^{-1}$$

Pohlhausen tabulated values $A(\sigma)$ for $\sigma$ ranging from $0.6$ to $15$, and noted that $A \approx 0.664 \sigma^{-1/3}$ is a fair approximation. Pohlhausen's values are reproduced in ref. 31, Vol. II., p. 624, where $A$ is denoted by $\alpha_1$. If one defines a Nusselt number, $N_u$, by the formula

$$N_u = \frac{q x}{k(T_1 - T_w)} = \frac{x}{T_1 - T_w} \left( \frac{\partial T}{\partial y} \right)_{y=0},$$

(2.4213)

where $q$ denotes heat transfer per unit time per unit area, it follows that

$$N_u = \frac{x}{\frac{\partial \eta}{\partial y}} \theta'(\sigma) = \frac{1}{2} \sqrt{\frac{u_1 x}{\nu}} A(\sigma).$$

(2.424)

The effect of the dissipation term can easily be included in the isothermal plate problem. Let us set $\phi(\eta) = \frac{T - T_w}{T_1 - T_w}$
when dissipation is included. The energy equation then reduces to

\[ \phi'' + \sigma \phi' = -\frac{\sigma u_1^2}{\frac{C_p(T_1-T_w)}{T_1}} f''^2 \]  

with \( \phi(0) = 0 \) and \( \phi(\infty) \to 1 \) as \( \infty \to \infty \).

The solution satisfying \( \phi(0) = 0 \) is

\[ \phi(\infty) = \phi'(0) + \phi'(0) \int_0^\infty e^{-\sigma \int_0^t f(s)ds} \left( \frac{u_1^2}{C_p(T_1-T_w)} \right) \left[ \omega(\infty) - \rho(\infty) \right] \]

(2.4216)

and the other boundary condition \( \phi(\infty) \to 1 \) as \( \infty \to \infty \) now implies

\[ \phi'(0) = A(\sigma) \left[ T_1 + \left( \frac{1}{4} \right) \frac{\beta(\sigma) u_1}{C_p(T_1-T_w)} \right] = \]

\[ A(\sigma) \left[ \frac{T_{\text{wins}} - T_w}{T_1 - T_w} \right], \quad (2.4217) \]

where \( A(\sigma) \) and \( \beta(\sigma) \) are the same as in E. Pohlhausen's analysis.

Let us define an "effective fluid temperature causing heat transfer", \( T_e \), by the condition that with dissipation included the Nusselt number based on \( T_e - T_w \) be the same as the Nusselt number based on \( T_1 - T_w \) when dissipation is neglected, that is, our \( T_e \) is defined by

\[ \frac{q_x}{k(T_e - T_w)} = \frac{1}{A(\sigma)} \left( \frac{u_1 x}{\nu} \right)^{1/2} \]

(2.4218)

It follows easily that \( T_e = T_{\text{wins}} \) for this case of an isothermal surface and zero pressure gradient. Another way of stating this same fact is to say that a Nusselt number,
$N$, defined by $N = \frac{\alpha x}{k(T_w - T_w)}$ \hspace{1cm} (2.4219)

when dissipation is included, is equal to the $\text{Nu}$ defined by (2.4213) when dissipation is neglected. Thus, the functions $A(\sigma)$ and $\beta(\sigma)$ which Pohlhausen calculated, are adequate to treat the problem of heat through transfer from an isothermal plate, even when dissipation is included, subject, of course, to the assumptions that $\rho$ and $\mu$ are constant, and to the boundary layer approximations.
Since $\mu$ and $\rho$ are treated as constants, the velocity boundary layer problem is the same as one already treated (section 2.412). With $k$, $C_p$, $\mu$ and $\rho$ all constant (2.23) can be written

$$\frac{u}{\rho} \cdot \frac{\partial T}{\partial x} + v \cdot \frac{\partial T}{\partial y} = \frac{1}{C_p} \cdot \frac{d \rho}{d \mu} \cdot \left( \frac{1}{\sigma} \cdot \frac{\partial^2 T}{\partial y^2} + \frac{1}{C_p} \cdot \left( \frac{\partial u}{\partial y} \right)^2 \right). \tag{2.4221}$$

Tifford's solutions (ref. 19) satisfy (2.4221) with $T=T_w=$ Constant at $y=0$ and $T_1(x)$ as $y \to \infty$. $T_1(x)$ is the function $T_1(x)-T_0=\frac{u_1}{2C_p}$ corresponding to isentropic flow of a gas outside the boundary layer. Tifford defines $T$. The treatment of $\rho$ as constant in the velocity boundary layer problem should not inhibit one from including the effect of compressibility on the temperature outside the boundary layer. Strictly speaking, if compressibility, conductivity, and viscosity are completely neglected, it follows from (1.727) that $\frac{dE}{dt}=0$ and hence $T=$constant if internal energy depends only on the temperature. On the other hand, if conductivity and viscosity are neglected, (1.729) yields $\frac{dI}{dt}=\frac{1}{\rho} \cdot \frac{d \rho}{d t}$ independent of any assumption regarding $\frac{dI}{dt}$ compressibility. It then follows from the assumption that $I$ depends only on $T$ that $C_p \cdot \frac{dT}{d \rho}=0$ so that $T$ must change whenever $\rho$ changes. Thus the behavior of $T$ in a strictly incompressible fluid having $E$ dependent only on $T$ differs from the behavior of $T$ in a fluid for which $I$ depends only on $T$. This difference in behavior manifests the fact that the assumption of constant $\rho$ is inconsistent with the assumption that both $E$ and $I$ depend only on $T$.

The value of Tifford's analysis is greatly increased by his having correctly accounted for the effect of compressibility on $T_1$, instead of assuming $T_1=$ constant, as was done by Squire and by Fage and Falkner (ref. 31)(Vol. II, pp. 631-636).
Using \( Y \) and \( F(Y) \) as in section 2.412, and independent variables \((Y, \xi)\) equation (2.4221) becomes

\[
\begin{align*}
\frac{m+1}{2m} & \left( \xi F_\xi + \frac{\xi}{2} \frac{\partial^2 \xi}{\partial Y^2} \right) + F \left( 1 + 2 \xi \right) F' \left( \frac{\partial \xi}{\partial Y} \right) + \frac{F'}{\xi} \frac{F''}{F'} = m - \frac{1}{2} F''.
\end{align*}
\]

(2.4222)

The boundary conditions on \( \Theta(Y, \xi) \) are

\[
\Theta(0, \xi) = 0
\]

and \( \Theta(Y, \xi) \to 1 \) as \( Y \to \infty \). (2.4223)

A result of Tifford's analysis is that (2.4222) is satisfied by

\[
\begin{align*}
\Theta(Y, \xi) &= \frac{T_1 - \frac{u_1}{2c_p} - T_w}{T_1 - T_w} \left[ \int_0^Y \xi - \frac{\partial}{\partial t} \int_0^t F(s) \, ds \, dt \right] + \frac{\bar{Y}(Y) - \frac{1}{2}}{\xi}
\end{align*}
\]

(2.4224)

where \( \bar{Y}(Y) \) satisfies

\[
\frac{1}{\partial} \bar{Y}'' + F \bar{Y}' - \frac{4m}{m+1} F' \bar{Y} = -F''.
\]

(2.4225)

The boundary conditions (2.4223) will be satisfied if

\[
\bar{Y}(Y) \to 0 \quad \text{as} \quad Y \to \infty.
\]

One may now define an effective fluid temperature causing heat transfer, \( T_e \), in a manner analogous to (2.4218). If we choose to define \( T_e \) by

\[
\frac{qX}{k(T_e - T_w)} = \sqrt{\frac{m+1}{2}} \sqrt{\frac{u_1 X}{\nu}} B(\alpha).
\]

(2.4226)
where \[ B(\sigma) = \left[ \int_{0}^{\infty} e^{-\sigma t} F(s) ds \right]^{-1} \]

then \[ T_e = T_l + \frac{u_1^2}{20p} \left( 1 + \frac{2 \bar{\varphi}'(\sigma)}{B(\sigma)} \right) - T_w \quad (2.4227) \]

Note that for \( m=0 \), \( B(\sigma) = \int_{0}^{1} A(\sigma) \) so that \( \frac{1}{2} A(\sigma) \) in (2.4214) equals \( \sqrt{2} B(\sigma) \) in (2.4226) when \( m=0 \). Also, it is readily shown that for \( m=0, 1 + \frac{2 \bar{\varphi}'(\sigma)}{B(\sigma)} = \frac{1}{2} B(\sigma) \), in agreement with the results presented in section 2.421. The function \( B(\sigma) \) describes the effect of the fluid's Prandtl number on the Nusselt number when the Nusselt number is based on \( (T_e - T_w) \). When dissipation and compressibility are completely neglected (Squire's problem, ref. 31, Vol.II, p. 631), this same function arises in the expression for Nusselt number based on \( (T_l - T_w) \). Values of \( B(\sigma) \) for \( m=1 \) are available from Squire's work. In ref. 31, Vol II, p. 632, \( B \) is denoted by \( \alpha_3(\sigma) \). For \( m=0 \), \( B \) is obtainable from Pohlhausen's \( A \), as previously mentioned. Values of \( B \) for \( m=0 \) and \( m=1 \) also appear in ref. 19. The calculation of \( T_e \) from (2.4227) requires a knowledge of \( \bar{\varphi}'(\sigma) \), which may be obtained by integrating (2.4225). Tifford furnished a few results for \( T_e \) in ref. 19; additional results appear in ref. 45.

If one treats \( T_l \) as constant and neglects the \( u \frac{dp}{dx} \) term in (2.4221), then \( \Theta \) is given by

\[ \Theta = B(\sigma) \int_{0}^{\infty} e^{-\sigma t} F(s) ds + \frac{2 \bar{\varphi}(\gamma)}{(T_l - T_w)} \left( \frac{u_1^2}{20p} \right) \]

This \( B \) is different from that in Sec 3.1.

(2.4228)
instead of by (2.4224). \( \bar{\psi}(Y) \) now satisfies (2.4225) with the changed boundary conditions.

\[
\bar{\psi}(0) = 0 \\
\bar{\psi}(Y) \to 0 \text{ as } Y \to \infty.
\]
2.43 The Use of Surface-Properties in Constant-Property Formulas.

Analyses in which the properties $\rho$, $\mu$, $C_p$, and $k$ are treated as constants are useful even in cases in which these fluid properties are known to vary, providing the values of these properties at the surface are used in the application of the constant-property formulas. It is clear from ref. 24 that this is the way in which Tifford intended his analysis to be applied. The usefulness of constant property formulas, provided surface properties are used in them, was emphasized in references 16 and 17.

1. For example, $C_f \sqrt{\frac{R}{W}}$ for insulated surfaces undergoes only a slight change with Mach number, whereas the change of $C_f \sqrt{\frac{R}{W}}$ with Mach number is much more noticeable. Compare with section 2.72.
2.5 Boundary layer equations with independent variables \((x,u)\).

Since functions of \((x,y)\) are to be transformed to functions of \((x,u)\), we write

\[
f(x,y) = f[x,u(x,y)]
\]
from which it is clear that

\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial x} + \left(\frac{\partial u}{\partial x}\right) \frac{\partial}{\partial u}
\]

and

\[
\frac{\partial}{\partial y} = \left(\frac{\partial u}{\partial y}\right) \frac{\partial}{\partial u}.
\]

Our next task is to express \(\left(\frac{\partial u}{\partial x}\right)_y\) and \(\left(\frac{\partial u}{\partial y}\right)_x\) in terms of derivatives of \(y(x,u)\). Differentiation of the equality

\[
y = y[x,u(x,y)]
\]
with respect to \(y\) holding \(x\) constant leads to

\[
\left(\frac{\partial y}{\partial y}\right)_x = 1 = \left(\frac{\partial y}{\partial u}\right)_x \left(\frac{\partial u}{\partial y}\right)_x \quad \text{whence} \quad \left(\frac{\partial u}{\partial y}\right)_x = \frac{1}{\left(\frac{\partial y}{\partial u}\right)_x},
\]

whereas differentiation with respect to \(x\) holding \(y\) constant yields

\[
\left(\frac{\partial y}{\partial x}\right)_y = 0 = \left(\frac{\partial y}{\partial x}\right)_u + \left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial y}{\partial u}\right)_x \quad \text{whence} \quad \left(\frac{\partial u}{\partial x}\right)_y = -\frac{\left(\frac{\partial y}{\partial x}\right)_y}{\left(\frac{\partial y}{\partial u}\right)_x}.
\]

Thus, we finally obtain the formulas

\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial x} + \left(\frac{\partial y}{\partial x}\right) \frac{\partial}{\partial u} - \frac{\left(\frac{\partial y}{\partial x}\right)_y \left(\frac{\partial y}{\partial u}\right)_x}{\left(\frac{\partial y}{\partial u}\right)_x} \frac{\partial}{\partial u} \quad (2.500)
\]
As long as \( \frac{\partial u}{\partial y_x} \) is bounded, \( \frac{\partial v}{\partial u} \neq 0 \).

The continuity equation (2.21) transforms into

\[
\nu \frac{\partial v}{\partial u} \frac{\partial e}{\partial x} - \frac{\partial v}{\partial x} \frac{\partial e}{\partial u} (\rho u) + \frac{\partial v}{\partial u} (\rho v) = 0
\]

while the momentum equation (2.22) becomes

\[
- \rho u \frac{\partial v}{\partial x} + \rho v + \frac{\partial v}{\partial u} \frac{\partial p}{\partial x} = \frac{\partial}{\partial u} \left( \frac{\partial e}{\partial u} \right).
\]

Elimination of \( \frac{\partial}{\partial u} (\rho v) \) between (2.502) and the equation obtained by differentiating (2.503) with respect to \( u \) results in

\[
- u \frac{\partial}{\partial u} (\rho \frac{\partial v}{\partial x}) - \rho \frac{\partial v}{\partial x} + \frac{\partial v}{\partial u} (\rho u) - u \frac{\partial}{\partial u} \left( \frac{\partial e}{\partial u} \right) + \frac{\partial^2 v}{\partial u^2} \frac{\partial p}{\partial x} = \frac{\partial}{\partial u} \left( \frac{\partial e}{\partial u} \right),
\]

which may be simplified to

\[
- u \frac{\partial}{\partial x} \rho \frac{\partial v}{\partial u} + \frac{\partial^2 v}{\partial u^2} \frac{\partial p}{\partial x} = \frac{\partial}{\partial u} \left( \frac{\partial e}{\partial u} \right).
\]

If we now let \( \frac{\partial e}{\partial y} = \frac{1}{g} \) so that \( g = \frac{\partial e(x,y)}{\partial y} \), then (2.504) becomes, after multiplication by \( g^2 \),

\[
u \rho \frac{\partial g}{\partial x} + u g \frac{\partial e}{\partial x} - \frac{\partial g}{\partial u} \frac{\partial p}{\partial x} = g^2 \frac{\partial^2 e}{\partial u^2} (\lambda g).
\]

The energy equation (2.23) transforms into

\[
\rho u \frac{\partial v}{\partial u} \frac{\partial e}{\partial x} - \rho u \frac{\partial v}{\partial x} \frac{\partial e}{\partial u} + \rho v \frac{\partial e}{\partial u} - u \frac{\partial p}{\partial x} \frac{\partial e}{\partial u}
\]
Subject to the assumption that $I$ depends only on $T$ for the fluid, we may write $\frac{\partial T}{\partial u} = \frac{1}{\rho} \frac{\partial I}{\partial u}$, and this was done in obtaining (2.506) from (2.23). Substitution of $\rho v$ from (2.503) into (2.506) gives

$$\frac{\rho u - \frac{\partial v}{\partial u}}{\frac{\partial I}{\partial x}} - \frac{\rho u - \frac{\partial v}{\partial u}}{\frac{\partial I}{\partial u} + \left[ \frac{\partial}{\partial u} \left( \frac{\frac{\partial I}{\partial y}}{\frac{\partial I}{\partial u}} \right) - \frac{\partial v}{\partial u} \frac{d\rho}{dx} + \frac{\rho u - \frac{\partial I}{\partial x}}{\frac{\partial I}{\partial u}} \right] \frac{\partial I}{\partial u} \frac{\partial u}{\partial I}$$

$$- \frac{d\rho}{dx} \frac{\partial v}{\partial u} = \frac{\partial}{\partial u} \left( \frac{\frac{\partial I}{\partial y}}{\frac{\partial I}{\partial u}} \right) + \frac{\rho}{\partial v} \frac{\partial I}{\partial u},$$

which may be simplified and re-arranged to

$$\frac{\rho u - \frac{\partial v}{\partial u}}{\frac{\partial I}{\partial x}} - \frac{d\rho}{dx} \frac{\partial v}{\partial u} \left( u + \frac{\partial I}{\partial u} \right) - \frac{1 - \sigma}{\sigma} \frac{\partial I}{\partial u} \frac{\partial u}{\partial v} \left( \frac{\frac{\partial I}{\partial y}}{\frac{\partial I}{\partial u}} \right)$$

$$= \frac{\rho}{\partial v} \left[ 1 + \frac{1}{\sigma} \frac{\partial^2 I}{\partial u^2} + \frac{\partial}{\partial u} \left( \frac{1}{\sigma} \right) \right].$$

Again setting $\frac{\partial v}{\partial u} = 1/g$, (2.507) becomes

$$\frac{1}{g} \left[ \frac{\rho u - \frac{\partial I}{\partial x}}{\frac{\partial I}{\partial u} + \frac{\partial I}{\partial u}} \left( u + \frac{\partial I}{\partial u} \right) - \frac{1 - \sigma}{\sigma} g \frac{\partial}{\partial u} (\mu g) \frac{\partial I}{\partial u} \frac{\partial I}{\partial u} \frac{\partial u}{\partial v} \right.$$  

$$- \mu g^2 \left[ 1 + \frac{1}{\sigma} \frac{\partial^2 I}{\partial u^2} + \frac{\partial I}{\partial u} \frac{\partial u}{\partial v} \left( \frac{1}{\sigma} \right) \right].$$

Equations (2.505) and (2.507), with $\tau/\mu$ written in place of $g(x,u)$, and with $\frac{\partial v}{\partial u} \left( \frac{1}{\sigma} \right) = 0$, were derived by Luigi Crocco (ref. 29).

Since $v = 0$ when $u = 0$ (i.e., at the solid surface), equation (2.503) shows that an appropriate boundary condition
is
\[(\frac{1}{\varepsilon}) \frac{dP}{dx} = \left[ \frac{\partial}{\partial u} (\mu g) \right]_{u=0}. \] (2.509)

Another condition is
\[\lim_{u \to u_1} g(x,u) = 0. \] (2.510)

One boundary condition on \(I(x,u)\) is
\[\lim_{u \to u_1} I(x,u) = I_1(x). \] (2.511)

The remaining boundary condition may be specified either by specifying the surface temperature or by specifying the heat transfer rate between the fluid and the surface. For specified surface temperature, \(I_w(x)\) will be known, and the condition takes the form
\[I(x,0) = I_w(x). \] (2.512)

The other alternative, specification of the heat transfer rate, imposes a condition on \(k \left( \frac{\partial T}{\partial y} \right)_{y=0} = \left( \frac{\partial I}{\partial u} \right)_{u=0}. \) In particular, if there is no heat transfer between the fluid and the surface, then \( (g \frac{\partial I}{\partial u})_{u=0} = 0. \)
2.6 Enthalpy-velocity relations in case Prandtl number \( \sigma = 1 \).

2.6.1 Case of zero pressure gradient and isothermal surface.

If one begins by seeking a solution of (2.508) in which

- (1) \( \sigma = 1 = \text{constant} \),
- (2) \( I \) depends only on \( u \),
- (3) \( \frac{dp}{dx} = 0 \),

then the differential equation for \( I(u) \) is

\[ I'' + 1 = 0 \]

from which

\[ I' + u = c = \text{constant} \] (2.611)

and

\[ I + \frac{1}{2}u^2 = cu + I_w \] (2.612)

where the constant of integration in (2.612) is denoted by \( I_w \) since it has the meaning "enthalpy at the wall". By applying (2.612) just outside the boundary layer, one may determine

\[ c = \frac{I_1 + \frac{1}{2}u_1^2 - I_w}{u_1} = \frac{I_0 - I_w}{u_1} \]

and hence

\[ I + \frac{1}{2}u^2 = (I_0 - I_w) \frac{u}{u_1} + I_w \] (2.613)

where \( I_0 = I_1 + \frac{1}{2}u_1^2 \) = stagnation enthalpy just outside the boundary layer.
Notice, first, that if \( c = 0 \) in 2.611, then (2.508) will be satisfied even if \( \frac{dp}{dx} \neq 0 \). The restriction \( c = 0 \) implies \( I'(o) = 0 \) and hence there is no heat transfer between the fluid and the surface; this condition of no heat transfer is what one means by saying the surface is insulated. Equation (2.612) now reduces to

\[
I + \frac{1}{2}u^2 = I_w
\]  

(2.614),

and we also have

\[
I_o = I_w
\]

(2.615)

where \( I_o = I_1 + \frac{1}{2}u_1^2 \) as before.
2.7 The case of zero pressure gradient

2.7.1 Solution if \( \rho \mu = \text{constant} \).

2.7.1.1 Solution for the velocity boundary layer

For a gas obeying \( p = \rho RT \), we have

\[
\frac{\partial \rho}{\partial x} = \frac{\partial}{\partial x} \left( \frac{p}{RT} \right) = \frac{1}{RT} \frac{\partial p}{\partial x} - \frac{p}{RT^2} \frac{\partial T}{\partial x}.
\]

Thus, if one seeks a solution in which \( \frac{\partial p}{\partial x} = 0 \) and \( T \) depends only on \( u \), (2.505) reduces to

\[
\rho \frac{\partial g}{\partial x} = g^2 \frac{\partial^2}{\partial u^2} (\mu g).
\]

If the dependent variable is now changed from \( g(x,u) \) to \( w(x,u) \) where

\[
\left( \frac{\partial u}{\partial y} \right)_x = g = \frac{1}{\mu} \sqrt{\frac{\rho_0 \mu_0 u_1^3}{x}} w, \quad (2.7111)
\]

the differential equation for \( w(x,u) \) is

\[
u_x \frac{\partial w}{\partial x} - \frac{1}{2} w w = u_1^3 w^2 - \frac{\rho_0 \mu_0}{\mu} \frac{\partial^2 w}{\partial u^2},
\]

with boundary conditions

\[
\left( \frac{\partial w}{\partial u} \right)_{u=0} = 0
\]

\[
w(x,u_1) = 0
\]

We have made use of the hypothesis that \( \mu = \mu(T) \) depends only on \( u \). If the dependent variables are now changed to \( (x,V) \) where \( V = u/u_1 \), one finds that \( w(x,V) \) satisfies

\[
V_x \frac{\partial w}{\partial x} - \frac{1}{2} V w = \frac{\rho_0 \mu_0}{\mu} \frac{\partial^2 w}{\partial V^2}
\]

\[
\left( \frac{\partial w}{\partial V} \right)_{V=0} = 0
\]

\[
\lim_{V \to 1} w(x,V) = 0.
\]

(2.7112)
The system (2.7112) has a solution in which \( w \) depends only on \( V \). In particular, if \( \rho \mu = \text{constant} = \rho_0 \mu_0 \), then \( w \) is determined by

\[
\begin{align*}
\dot{w} + \frac{1}{2} \dot{V} &= 0 \\
\dot{w}(0) &= 0 \\
\dot{w}(1) &= 0 
\end{align*}
\tag{2.7113}
\]

where the dots denote differentiation with respect to \( V \).

Since one also obtains (2.7113) if \( \rho \) and \( \mu \) are individually constant, there must exist a correspondence with the Blasius solution discussed in section 2.411. This correspondence (specified by equation 2.7116 below) will now be developed. It is readily shown that the transformation

\[
\eta = \frac{1}{2} \int_{0}^{\eta} \frac{dV}{w}
\tag{2.7114}
\]

transforms (2.7113) into

\[
f' + \frac{d}{d\eta} \left( \frac{f'''}{f''} \right) = 0
\]

which may be immediately integrated to

\[
f f'' + f''' = 0
\]

with \( f(0) = f'(0) = 0 \) \tag{2.7115}

and \( f'(\eta) \rightarrow 2 \) as \( \eta \rightarrow \frac{1}{2} \int_{0}^{1} \frac{dV}{w(V)} \).

It follows from (2.7111) and the definition \( V = u/u_1 \) that for a fluid having constant density and viscosity coefficient,
\[ y(x, v) = \sqrt{\frac{2x}{\rho u_1}} \int_0^V \frac{dv}{w(v)}. \]

Then the condition that \( y \to \infty \) as \( u \to u_1 \) (cf. section 2.1, page 15) clearly implies that \( \int_0^1 \frac{dv}{w(v)} \) diverges. Hence (2.7115) is identically (2.411). It follows that

\[ w(v) = \frac{1}{2} f''(\eta) \quad v = \frac{1}{2} f'(\eta) \quad (2.7116) \]

The hypothesis that \( T \) depends only on \( u \) is consistent with the boundary conditions and energy equation if the surface is isothermal.

It can be concluded that for zero pressure gradient, isothermal surface, and a gas obeying \( p = \rho RT \) and \( \mu \propto T \), the local skin friction coefficient times square root of Reynolds number is given by

\[ \frac{1}{2}\rho_f \sqrt{R_w} = \left( \frac{\mu}{\rho} \frac{\partial u}{\partial y} \sqrt{\frac{\rho u_1 x}{\mu}} \right)_{y=0} = w(o) = 0.33206, \quad (2.7117) \]

independent of Mach number, Reynolds number, and the heat transfer rate.
2.712 The enthalpy-velocity relation.

If \( w \) from (2.7111) is introduced into (2.508), and if the independent variables are changed from \((x,u)\) to \((x,V)\), there results, for \( \frac{dp}{dx} = 0 \),

\[
xV \frac{dI}{dx} - \frac{1-\sigma}{\rho} \frac{dI}{dV} w \frac{d}{dV} \left( \frac{\kappa}{\rho_0} \frac{w}{w} \right)
= \frac{\kappa}{\rho_0} \frac{w}{w} w^2 \left[ u_1^2 + \frac{\sigma}{\rho} \left( \frac{1}{\sigma} \frac{dI}{dV} \right) \right].
\]  

(2.7121)

For an isothermal surface, (2.7121) has a solution in which \( I \) depends only on \( V \). If we assume \( \kappa \propto T \) so that \( \kappa \rho = \text{constant} = \kappa_0 \rho_0 \), then \( w(V) \) may be considered known from (2.7116), and (2.7121) can be treated as a first order linear differential equation in \( \frac{1}{\sigma} \dot{I}(V) \). This equation is

\[
\frac{d}{dV} \left( \frac{1}{\sigma} \dot{I}(V) \right) + (1-\sigma) \frac{w}{w} \left( \frac{1}{\sigma} \dot{I}(V) \right) = -u_1^2. \tag{2.7122}
\]

Let \( G(V) = e^{\int_0^V (1-\sigma) \frac{w(s)}{w(s)} ds} \), \( \phi(V) = \int_0^V \frac{\sigma}{G(s)} \int_0^s G(t) dt \) \( ds \), and

\[
\lambda(V) = \int_0^V \frac{ds}{G(s)} .
\]

Then (2.7122) is satisfied by

\[
I(V) = I(o) + \dot{I}(o) \lambda(V) - u_1^2 \phi(V). \tag{2.7123}
\]

The condition that \( I(1) = I_o \) - \( u_1^2 \) furnishes the relation

\[
I(o) + \lambda(1) \dot{I}(o) = I_o + u_1^2 \left[ \phi(1) - \frac{3}{5} \right]
\]  

(2.7124)

between the constants of integration \( I(o) \) and \( \dot{I}(o) \). Thus, specification of either \( I(o) \) or \( \dot{I}(o) \) serves to determine \( I(V) \).
It may be shown that if \( \sigma = \text{constant} \), then

\[
G(V) = \left( \frac{W(V)}{W(\sigma)} \right)^{1/\sigma}, \quad \text{and} \quad \lambda(V) = \frac{1}{3A(\sigma)} f''(\sigma) \theta(\eta)
\]

\[
\phi(V) = \frac{1}{6} \left[ \beta(\sigma) - \omega(\eta) \right]
\]

\[
V = \frac{1}{6} f'(\eta)
\]

where \( \theta(\eta), \omega(\eta), A, \) and \( \beta \) are the functions and parameters introduced in section 2.421 of this Appendix in the discussion of E. Pohlhausen's analysis, and \( f(\eta) \) is the Blasius function satisfying (2.7124).

Since \( \omega(\infty) = 0 \), we have \( \phi(1) = \frac{1}{6} \beta(\sigma) \); and since \( \theta(\infty) = 1 \), \( \lambda(1) = \frac{f''(\sigma)}{2A(\sigma)} \). Thus (2.7124) becomes

\[
I(0) + \frac{f''(\sigma)}{2A(\sigma)} \dot{I}(0) = I_0 + \frac{u_1^2}{2} \left[ \frac{1}{2} \beta(\sigma) - \frac{1}{3} \right], \quad (2.7126)
\]

when \( \sigma = \text{constant} \).

For a fluid with density, viscosity, specific heat, and Prandtl number all constant, \( I(V) \) is also given by (2.7123) with the same functions for \( \phi(V) \) and \( \lambda(V) \). Thus, for an isothermal surface in the zero pressure gradient case, the enthalpy function \( I(V) \) is the same function.

1. for a gas having \( \mathcal{K} \rho = \text{constant} \), Prandtl number = constant, and enthalpy a function only of the temperature,

and

2. for a fluid whose density, viscosity, specific heat, and Prandtl number are all constant.

This conclusion is subject, of course, to the approximations involved in the boundary layer equations (2.21) through (2.24).

Most of the relations in this section were given by Crocco in reference 7.
2.72 Other solutions for zero pressure gradient considering variable \( \rho \) and \( \mu \).

The zero pressure gradient case has been the subject of much investigation because of its simplicity as compared with the non-zero pressure gradient cases.

A. Busemann (ref. 6) analyzed the problem subject to the hypotheses \( \sigma = 1 \) and \( \mu \propto T^{0.5} \), \( C_p \) = constant, and \( p = \rho RT \). The assumption \( \sigma = 1 \) enabled him to derive an equation equivalent to (2.613) with \( C_p \) = constant. One of Busemann's results is that for an insulated plate, \( c_{fw} \sqrt{R_w} \) increases only slightly with increasing Mach number. The value corresponding to an infinite Mach number is only 20% larger than 0.664, the value for an incompressible fluid. This finding of Busemann's was probably the first indication of the usefulness of using surface properties in constant property formulas. (cf section 2.43).

Th. von Karman and H. S. Tsien (ref. 8), used the same assumptions as Busemann, except that whereas Busemann had used \( \mu \propto T^{0.5} \), Karman and Tsien took \( \mu \propto T^{0.76} \) in closer agreement with experimental data for air. Using these assumptions, Karman and Tsien calculated the velocity profile, temperature profile, and skin friction coefficient for various values of free stream Mach number for an insulated plate, and also discussed the heat transfer problem. It is interesting to interpret the results of Karman and Tsien in the light of section 2.43. There is only a small
change in $c_{fw} \sqrt{R_w}$ with Mach number; moreover, if the velocity profiles are plotted against $y \sqrt{\frac{u_1}{2w}}$ instead of against $y \sqrt{\frac{u_1}{2T}}$, the profiles for different Mach numbers appear to have nearly the same shape as the Blasius profile.

To determine the relative effect of using different formulas for $\mu(T)$, Brainerd and Emmons (ref. 9) calculated temperature, velocity, shear, and heat transfer rate distributions in the boundary layer of an insulated flat plate with $p = \rho RT$, $\sigma = 0.733$, $C_p = \text{constant}$, and free stream Mach number, $M_1$, equal to 2, for the cases:

(C) $\mu = \text{constant}$

(B) $\mu \propto T^{0.5}$

(K) $\mu \propto T^{0.768}$

(S) $\mu$ given by Sutherland's formula

$$\mu = \mu_1 \left( \frac{T}{T_1} \right)^{3/2} \frac{T_1 + 114}{T + 114}$$

with $T_1 = 175^\circ C$, $273^\circ C$, and $375^\circ C$.

They found that $T_w/T_1 \leq 1.66$ in each case. This corresponds to

$$\text{temperature recovery factor} = \frac{T_w/T_1 - 1}{\sqrt{-1} \frac{2}{M_1} \frac{2}{0.8}} = 0.66 = 0.825$$

in contrast to $\sqrt{\sigma} = \sqrt{0.733} = 0.856$ for an incompressible fluid.

Values of $C_{f1} \sqrt{R_1}$ from ref. 9 appear in table 2.721. The corresponding values of $c_{fw} \sqrt{R_w}$ have been appended, as well as the results for $\mu \propto T$. 
Table 2.721

\[ p = \rho RT \quad \sigma = 0.733 \quad C_p = \text{constant} \]

Effect of \( \mu(T) \) formula for insulated plate at \( M_1 = 2 \).

<table>
<thead>
<tr>
<th>Viscosity Formula</th>
<th>( Cr_1 \sqrt{R_1} )</th>
<th>( Cr_w \sqrt{R_w} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C) ( \mu = \text{constant} )</td>
<td>1.096</td>
<td>1.41</td>
</tr>
<tr>
<td>(B) ( \mu \alpha T^{0.5} )</td>
<td>1.208</td>
<td>1.571</td>
</tr>
<tr>
<td>(K) ( \mu \alpha T^{0.768} )</td>
<td>1.273</td>
<td>1.350</td>
</tr>
<tr>
<td>( \mu \alpha T^T )</td>
<td>1.328</td>
<td>1.328</td>
</tr>
<tr>
<td>(S) 175</td>
<td>1.300</td>
<td>1.379</td>
</tr>
<tr>
<td>(S) 273</td>
<td>1.274</td>
<td>1.351</td>
</tr>
<tr>
<td>(S) 375</td>
<td>1.264</td>
<td>1.341</td>
</tr>
</tbody>
</table>

Brainerd and Emmons selected \( \mu \alpha T^{0.768} \) as giving the most convenient (although not necessarily the most accurate) representation under all conditions. Using \( \mu \alpha T^{0.768} \) they then calculated \( T_w/T_1 \) and \( Cr_1 \sqrt{R_1} \) as functions of Mach number for each of the three Prandtl numbers 0.733, 1.00, and 1.20. The results are shown in table 2.722. Again, the corresponding values of \( Cr_w \sqrt{R_w} \) have been appended. Values of the recovery factor \( \frac{(T_w/T_1)-1}{2 \cdot M_1^2} \) are also shown.
Skin friction and insulated surface temperature as functions of Prandtl number and Mach number calculated in ref. 9 using \( \mu \propto T^{0.768} \), \( p = \rho RT \), and \( C_p = \) constant.

\[
L b = \frac{(\gamma-1)M_1^2}{\mu} M_1 \quad T_w/T_1 \quad C_{f1} \sqrt{R_1} \quad C_{fw} \sqrt{R_w} \quad \frac{T_w}{T_1-1} \frac{1}{\delta_b}
\]

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>( b = (\gamma-1)M_1^2 )</th>
<th>( M_1 )</th>
<th>( T_w/T_1 )</th>
<th>( C_{f1} \sqrt{R_1} )</th>
<th>( C_{fw} \sqrt{R_w} )</th>
<th>( \frac{T_w}{T_1-1} \frac{1}{\delta_b} )</th>
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</tr>
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</tbody>
</table>

These results show that for \( M_1 \) up to 3.17, skin friction coefficients for insulated plates calculated assuming \( \sigma = 0.733 \) are within 1% of those calculated assuming \( \sigma = 1 \),
and the temperature recovery factor is nearly independent of Mach number. The recovery factors are only slightly less than those given by E. Pohlhausen's simplified analysis. However, it should be noted that for $\sigma = 1$, the results of section 2.61 apply, and the correct value of the recovery factor is 1.000. This fact can be used to test the accuracy of some of the calculations reported by Brainerd and Emmons.
2.8 Correspondence between variable-property and constant-property solutions for insulated surfaces when \( \sigma = 1, \mu \propto T, \) and \( C_p = \text{constant}. \)

If one can transform the differential equations and boundary conditions for some problem in the flow of a variable property fluid to the differential equations and boundary conditions for some problem in the flow of a constant property fluid, then by means of the transformation, the solution to the variable property problem can be expressed in terms of the solution to the simpler constant property problem. The transformation which sets up a correspondence between variable property and constant-property solutions for insulated surfaces when \( \sigma = 1, \mu \propto T, \) and \( C_p = \text{constant}, \) has been derived independently by C. R. Illingworth (ref. 10), K. Stewartson (ref. 21), and S. Christensen (ref. 22). A transformation equivalent to the one obtained by Illingworth, Stewartson, and Christensen, may be obtained by setting \( \sigma = 1 \) in the transformation developed in section 3.1 of this dissertation.

K. Stewartson (ref. 21) has also showed how to solve the boundary layer problem for a non-insulated surface, subject to the assumptions that (a) \( \sigma = 1, \) (b) \( \mu \propto T, \) (c) the surface is isothermal, and (d) the "image problem" in the flow of a constant property fluid has the velocity outside the boundary layer given by \( u_{\parallel}(x) = Ax^m. \)
2.9 Some work for the future.

To conclude this section on background information, let us mention some of the work which remains to be done in order to complete a satisfactory theory of the laminar boundary layer.

The conditions under which the classical stress-rate-of-strain relations (1.716) apply should be defined for various substances. It is reasonable to expect that careful experimentation may reveal deviations from (1.716) at high rates of strain, even for many substances classified as viscous fluids. Equally fundamental as, and perhaps more pressing than the questions regarding stress are questions such as those in section 1.9 regarding the thermodynamics of fluids when in a general state of stress and when undergoing rapid changes of state.

Then there are questions regarding the conditions under which the approximations of boundary layer theory are valid. The answers are known -- vaguely. To ask precise questions regarding this point, and to give precise answers is part of the difficult task remaining before investigators who wish to complete a satisfactory boundary layer theory.

Criteria are still lacking for infallibly predicting whether the flow will be laminar or turbulent in a given situation.

One may assert that solutions of the laminar boundary layer equations (2.21), (2.22), and (2.23) for the zero
pressure gradient case are fairly well in hand, but practically the only available solutions in the non-zero pressure
gradient case are those which assume either

\begin{align*}
(1) \quad & \rho \text{ and } \mu \text{ constant} \\
\text{or} \quad & (2) \quad \sigma = 1, \ p = \rho RT, \ \mu \alpha T, \ C_p = \text{constant, and insulated surface; and even in these cases, the accuracy is often limited by the fact that only a finite number of terms can be evaluated in the series solutions, or by the limitations of the approximate momentum integral method. Of course it is desirable to fill in the gaps in our knowledge of solutions by obtaining solutions for } \\
& \sigma \neq 1, \ C_p \neq \text{constant, and for } \mu(T) \text{ formulas, other than } \\
& \mu \propto T, \text{ for both insulated and non-insulated surfaces. Sections 3 and 4 of this thesis are a step in the direction of filling in these gaps, but much more needs to be done. For air, the extension to pressure-density-temperature relations other than } \rho = \rho RT \text{ seems somewhat less pressing than the extension to } \\
& \sigma \neq 1, \ C_p \neq \text{constant, and } \mu(T) \text{ formulas other than } \mu \propto T. \\
& \text{More information is needed in cases in which the pressure gradient parameter } \frac{x}{u_1} \frac{du_1}{dx} \text{ changes rapidly with } x. \text{ More solutions of the problem posed in section 2.415 will be helpful for this, and obviously more convenient methods of solution of this problem are desirable.}
\end{align*}

Finally, the importance of careful experimental work against which the results of theory may be tested should be emphasized.
APPENDIX B. Skin Friction Calculations for Insulated Surfaces assuming $\mu \propto T$.

In this appendix, the analysis of section 3.1 is applied to skin friction calculations for the four cases indicated below.

<table>
<thead>
<tr>
<th>Case</th>
<th>Value of $c$ in (3.112)</th>
<th>Value of $c$ in (3.112)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>2</td>
<td>0.7</td>
<td>1.0</td>
</tr>
<tr>
<td>3</td>
<td>1.0</td>
<td>0.985</td>
</tr>
<tr>
<td>4</td>
<td>0.7</td>
<td>0.985</td>
</tr>
</tbody>
</table>

The function $u_1(x)$ is chosen to be $u_1 = \sqrt{2I_0} \beta x$ so that

$$U_1 = \beta x$$

in each case.

To apply the analysis of Section 3.1, we first use (3.115) and (3.116) to find the function $U_{11}(x_1)$ which corresponds to the given function $U_1(x)$. Since $U_1 = \beta x$, we obtain

$$U_{11} = \beta x (1 - c \beta^2 x^2)^{-B/2} \quad (1)$$

and

$$\frac{dx_1}{dx} = (1 - c \beta^2 x^2) (B/2 + I_0/cRT_0) \quad (2)$$

The first steps are directed towards expressing $U_{11}$ as a power series in $x_1$. We first obtain $dx_1/dx$ as a power series in $x$, which is integrated to express $x_1$ as a power series in $x$. By reversion of this series, we express $x$ as a power series in $x_1$. The above expression (1) is then expanded to yield $U_{11}$ as a power series in $x$. Substitution for $x$ in terms of $x_1$ then leads to the desired series for $U_{11}$ in powers of $x_1$. We proceed.
By the binomial theorem,

\[(1 - c \beta^2 x^2)^K = 1 - Kc\beta^2 x^2 + \frac{K(K-1)}{2} (c \beta^2 x^2)^2 - \frac{K(K-1)(K-2)}{3!} (c \beta^2 x^2)^3 + \ldots \]

Equations (2) and (3) show that \(dx_1/dx\) for this case is obtained from (3) by setting \(K = (B/2 + I_0/cRT_0)\). By integration with respect to \(x\), one obtains

\[\beta x_1 = \beta x - \frac{Kc}{3}(\beta x)^3 + \frac{K(K-1)}{10}(c^2 \beta x)^5 - \frac{K(K-1)(K-2)}{42}(\beta x)^7 + \ldots\]

where \(K = B/2 + I_0/cRT_0\).

Values of \(K\) for the four cases are indicated in the following table. The values of \(I_0\) used in calculating \(I_0/cRT_0\) for the cases with \(c = 0.985\) correspond to air at temperature \(T_0 = 800^\circ\)F abs.

<table>
<thead>
<tr>
<th>Case</th>
<th>(c)</th>
<th>(B)</th>
<th>(I_0/cRT_0)</th>
<th>(K = B/2 + I_0/cRT_0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0</td>
<td>1.000</td>
<td>1.000</td>
<td>3.5000</td>
</tr>
<tr>
<td>2</td>
<td>0.7</td>
<td>1.000</td>
<td>0.835</td>
<td>3.5000</td>
</tr>
<tr>
<td>3</td>
<td>1.0</td>
<td>0.985</td>
<td>1.000</td>
<td>3.5507</td>
</tr>
<tr>
<td>4</td>
<td>0.7</td>
<td>0.985</td>
<td>0.835</td>
<td>3.5507</td>
</tr>
</tbody>
</table>

If we write

\[x/x_1 = 1 + k_3(\beta x)^2 + k_5(\beta x)^4 + k_7(\beta x)^6 + \ldots\]

then clearly

\[k_3 = -Kc/3; \]
\[k_5 = K(K-1)c^2/10; \]
\[k_7 = -K(K-1)(K-2)c^3/42.\]

Let us define constants \(m_3, m_5, m_7\) by the relation

\[x/x_1 = 1 + m_3(\beta x_1)^2 + m_5(\beta x_1)^4 + m_7(\beta x_1)^6 + \ldots.\]

It follows easily from Appendix C that
\[m_3 = -k_3;\]
\[m_5 = -k_5 + 3k_3^2;\]
\[m_7 = -k_7 + 8k_3k_5 - 12k_3^3.\]

It follows from equation (1) that if we define constants \(n_3, n_5, n_7\) by the relation
\[U_{11} = \beta x \left[ 1 + n_3(\beta x)^2 + n_5(\beta x)^4 + n_7(\beta x)^6 + \cdots \right],\]
then
\[n_3 = \frac{1}{2} B c;\]
\[n_5 = \frac{1}{4} B (\frac{B}{2} - 1)c^2;\]
\[n_7 = \frac{1}{12} B (\frac{B}{2} - 1)(\frac{B}{2} - 2)c^3.\]

Let us also define constants \(p_1, p_2, p_3\) by the relations
\[(\beta x)^3 = (\beta x_1)^3 + p_1(\beta x_1)^5 + p_2(\beta x_1)^7 + \cdots\]
\[(\beta x)^5 = (\beta x_1)^5 + p_3(\beta x_1)^7 + \cdots\]

It is easily shown that
\[p_1 = 3m_3;\]
\[p_2 = 3m_5 + 3m_3;\]
\[p_3 = 5m_3.\]

The desired series for \(U_{11}\) in powers of \(x_1\) is obtained by substituting for \((\beta x), (\beta x)^5\), and \((\beta x)^7\) in terms of \((\beta x_1)\).

If we define constants \(\alpha_3, \alpha_5, \alpha_7\) by the relation
\[U_{11} = \beta x_1 + \alpha_3(\beta x_1)^3 + \alpha_5(\beta x_1)^5 + \alpha_7(\beta x_1)^7 + \cdots \quad (4)\]
then
\[\alpha_3 = n_3 + m_3;\]
\[\alpha_5 = n_5 + n_3p_1 + m_5;\]
\[\alpha_7 = n_7 + n_5p_3 + n_3p_2 + m_7.\]
Values of $\alpha_3, \alpha_5, \alpha_7$ for the four cases under discussion follow.

<table>
<thead>
<tr>
<th>Case</th>
<th>$\alpha_3$</th>
<th>$\alpha_5$</th>
<th>$\alpha_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.833</td>
<td>6.508</td>
<td>27.895</td>
</tr>
<tr>
<td>2</td>
<td>1.723</td>
<td>5.904</td>
<td>24.584</td>
</tr>
<tr>
<td>3</td>
<td>1.822</td>
<td>6.436</td>
<td>27.451</td>
</tr>
<tr>
<td>4</td>
<td>1.714</td>
<td>5.844</td>
<td>24.250</td>
</tr>
</tbody>
</table>

With $U_{i1}(x_i)$ specified by equation (4), L. Howarth's analysis (Ref. 20; or section 2.413 of Appendix A) can be used to obtain $w_i(x_i, 0)$. However, in order to be consistent with our present notation, the subscript $i$ must be added to some of the symbols in Reference 20 or in (2.413). With this change in the notation of (2.413), we obtain (from 2.413)

$$\frac{\partial u_i}{\partial y_1} = \frac{a_1}{a_{11}} \frac{\partial^2 \psi}{\partial \eta^2} = \frac{a_1^{3/2} x_i}{a_{11}} \left[ f_1' + \frac{a_3}{a_1} x_i^2 f'' + \frac{a_3^2}{a_1^2} x_i^2 + \frac{a_3^3}{a_1^3} x_i^2 + \cdots \right]$$

and hence

$$w_i(x_i, 0) = \left(\frac{a_1}{a_{11}} \frac{\partial u_i}{\partial y_1}\right)_0 = \left(\frac{a_1 x_i}{a_{11}} \right)^{3/2} \left[ f_1' + \frac{a_3}{a_1} x_i^2 f'' + \frac{a_3^2}{a_1^2} x_i^2 + \frac{a_3^3}{a_1^3} x_i^2 + \cdots \right]_{y=0}.$$

Now in the notation of section 2.413 (as amended by adding the subscript $i$), we have, for the symmetric case,

$$u_{i1} = a_1 x_i + a_3 x_i^3 + a_5 x_i^5 + a_7 x_i^7 + \cdots$$

and hence

$$\frac{u_{i1}}{a_1 x_i} = 1 + \frac{a_3}{a_1} x_i^2 + \frac{a_5}{a_1} x_i^4 + \frac{a_7}{a_1} x_i^6 + \cdots.$$
By setting $u_{11}/a_1x_1 = U_{11}/\beta x_1$, we see that 
\[
\begin{align*}
a_3/a_1 &= \alpha_3 / \beta^2 \\
1/a_1 &= \alpha_5 / \beta^4 \\
a_7/a_1 &= \alpha_7 / \beta^6
\end{align*}
\]
and hence 
\[
\left( \frac{u_{11}}{a_1x_1} \right) = 1 + \alpha_3 x_1^2 / \beta^2 + \alpha_5 x_1^4 / \beta^4 + \alpha_7 x_1^6 / \beta^6 + \cdots
\]
and 
\[
\frac{a_1x_1}{u_{11}} \sqrt[3/2]{3} = 1 - \frac{3}{2} \alpha_3 x_1^2 / \beta^2 + \left( \frac{15}{8} \alpha_3^2 - \frac{3}{4} \alpha_5 \right) x_1^4 / \beta^4
\]
\[
- \left( \frac{35}{16} \alpha_3^3 - \frac{15}{4} \alpha_3 \alpha_5 + \frac{3}{2} \alpha_7 \right) x_1^6 / \beta^6 + \cdots,
\]
by Appendix D. Finally, 
\[
w_i(x_1,0) = \left\{ f_1^2 + \beta x_1^2 \alpha_3 (4f_3'' - 3f_1^2) + \beta x_1^4 \left[ \alpha_3 (6g_5'' - 3f_1^2) + \alpha_5 (6h_5 - 3f_1^2) + \alpha_7 (8g_7 - 3f_1^2) \\
+ \alpha_3 \alpha_5 (8h''_7 - 9g''_5 - 6f''_3 + \frac{15f''_1}{4}) + \alpha_3^3 (8k''_7 - 9h''_5 + \frac{15f''_3}{2} - \frac{35f''_1}{16}) \right] + \cdots \right\} \eta = 0
\]
(5)
Let us define functions $f_{21}, f_{41}, f_{42}, f_{61}, f_{62}, f_{63}$ as follows.
\[
\begin{align*}
f_{21} &= 4f_3 - \frac{3}{2}f_1 \\
f_{41} &= 6g_5 - 3f_1 \\
f_{42} &= 6h_5 - 6f_3 + \frac{15}{8}f_1 \\
f_{61} &= 8g_7 - 3f_1 \\
f_{62} &= 8h_7 - 9g_5 - 6f_3 + \frac{15}{4}f_1 \\
f_{63} &= 8k_7 - 9h_5 + \frac{15}{2}f_3 - \frac{35}{16}f_1
\end{align*}
\]
(6)
With the definitions (6), equation (5) becomes 
\[
w_i(x_1,0) = \left\{ f_1'' + \beta x_1^2 \alpha_3 f_{21}'' + \beta x_1^4 \left[ \alpha_3 f_{41}'' + \alpha_5 f_{42}'' \right]
\right. \left. + \beta x_1^6 \left( \alpha_3 f_{61}'' + \alpha_5 f_{62}'' + \alpha_7 f_{63}'' \right) + \cdots \right\} \eta = 0
\]
(7)
By using the values of $f_1''(0), f_3''(0), g_2''(0)$, etc. listed on page 134, and the definitions (6), one obtains

\begin{align*}
f_2(0) &= 1.048906 \\
f_4(0) &= 1.959330 \\
f_2(0) &= -1.320488 \\
f_6(0) &= 2.784734 \\
f_2(0) &= -3.973211 \\
f_6(0) &= 1.725533
\end{align*}

Let us define constants $F_0, F_2, F_4, F_6$ by the equation

$$w_1(x_1,0) = F_0 + F_2 F_2 + F_4 F_4 + F_6 F_6 + \cdots (8)$$

Comparison of (7) and (8) shows how $F_0, F_2, F_4,$ and $F_6$ may be expressed in terms of other quantities already determined. A determination of values of $F_0, F_2, F_4,$ and $F_6$ for use in equation (8) completes the determination of $w_1(x_1,0)$. The remaining portion of our skin friction calculation consists of using $w_1(x_1,0)$ from (8) and the transformation (3.115) and (3.116) to obtain $w(x,0)$.

Let us write

$$w_i(x_1,0) = w_i(x,0) = F_0 + (\beta x)^2 F_2 + (\beta x)^4 G_4 + (\beta x)^6 G_6 + \cdots (9)$$

If we also write

\begin{align*}
(\beta x_1)^2 &= (\beta x)^2 + p_4 (\beta x)^4 + p_5 (\beta x)^6 + \cdots \\
(\beta x_1)^4 &= (\beta x)^4 + p_6 (\beta x)^6 + \cdots
\end{align*}

then

\begin{align*}
G_4 &= F_4 + p_4 F_2 \\
G_6 &= F_6 + p_6 F_4 + p_5 F_2
\end{align*}

The constants $p_4, p_5, p_6$ may be expressed in terms of $k_3, k_5, k_7$ defined on page 169. In particular,
\[ p_4 = 2k_3; \]
\[ p_5 = 2k_7 + 2k_3k_5; \]
\[ p_6 = 4k_3; \]

Since \( w = (x/x_1)^{\frac{3}{2}}(1-U_1)^2 w_1 \), let us next express \( x/x_1 \) as a power series in \( x \). If the result is denoted by

\[ (x/x_1)^{\frac{3}{2}} = 1 + q_3(\beta x)^2 + q_5(\beta x)^4 + q_7(\beta x)^6 + \cdots \]  

then, by Appendix D and the definitions of \( k_3, k_5, k_7 \) on page 4,

\[ q_3 = -\frac{1}{2}k_3 \]
\[ q_5 = -\frac{1}{2}k_5 + \frac{3}{8}k^2_3 \]
\[ q_7 = -\frac{1}{2}k_7 + \frac{3}{4}k_3k_5 - \frac{5}{16}k^3_3 \]

Now let constants \( s_2, s_4, s_6 \) be defined by the equation

\[ (x/x_1)^{\frac{3}{2}}w_1(x,0) = F_0 + (\beta x)^2s_2 + (\beta x)^4s_4 + (\beta x)^6s_6 + \cdots \]  

From (9) and (10), we find,

\[ s_2 = F_0q_3 + F_2; \]
\[ s_4 = F_0q_5 + F_2q_3 + G_4; \]
\[ s_6 = F_0q_7 + F_2q_5 + G_4q_3 + G_6. \]

Finally, if we define \( A_2, A_4, A_6 \) by

\[ w(x,0) = F_0 + A_2U_1^2 + A_4U_1^4 + A_6U_1^6 + \cdots \]  

then

\[ A_2 = s_2 - 2F_0 \]
\[ A_4 = s_4 - 2A_2 + F_0 \]
\[ A_6 = s_6 - 2A_4 + F_2 \]

The following table presents values of \( A_2, A_4, A_6 \), for the four cases.
The value of \( F_0 \) is the same for all four cases, namely
\[
F_0 = 1.232588.
\]

The first four terms of the series on the right in equation (12) are probably sufficient to allow \( w(x, \theta) \) to be calculated within one percent for values of \( U_1 \) up to about 0.58. If \( \gamma = C_p/C_v = 1.400 \) is constant, this value of \( U_1 \) corresponds to a Mach number of about 1.6. The following table shows the values of the second, third, and fourth terms in (12), and the value of the sum of the first four terms for \( U_1^2 = 0.3386 \). The symbol \( S \) means \( F_0 + A_2 U_1^2 + A_4 U_1^4 + A_6 U_1^6 \).

<table>
<thead>
<tr>
<th>Case</th>
<th>( A_2 )</th>
<th>( A_4 )</th>
<th>( A_6 )</th>
<th>( S )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.280</td>
<td>0.293</td>
<td>0.282</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.147</td>
<td>0.197</td>
<td>0.206</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.266</td>
<td>0.260</td>
<td>0.237</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.136</td>
<td>0.168</td>
<td>0.246</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Case</th>
<th>( A_2 U_1^2 )</th>
<th>( A_4 U_1^4 )</th>
<th>( A_6 U_1^6 )</th>
<th>( S )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0947</td>
<td>0.0336</td>
<td>0.0110</td>
<td>1.372</td>
</tr>
<tr>
<td>2</td>
<td>0.0493</td>
<td>0.0226</td>
<td>0.0080</td>
<td>1.313</td>
</tr>
<tr>
<td>3</td>
<td>0.0901</td>
<td>0.0299</td>
<td>0.0092</td>
<td>1.362</td>
</tr>
<tr>
<td>4</td>
<td>0.0480</td>
<td>0.0193</td>
<td>0.0096</td>
<td>1.307</td>
</tr>
</tbody>
</table>
APPENDIX G. Reversion of a Power Series.

Suppose a series for \( x(t) \) is given as \( x = \sum_{n=0}^{\infty} b_n t^n \).
It is desired to revert this series to obtain a series for \( t \)
in powers of \( x \), which series will be denoted by \( t = \sum_{n=1}^{\infty} a_n x^n \).

Now \( \left( \frac{d^n x}{dt^n} \right)_{t=0} = n!b_n \) and \( \left( \frac{d^n t}{dx^n} \right)_{x=0} = n!a_n \) for all \( n \). Thus, any
\( a_n \) may be expressed in terms of the \( b \)'s if \( d^n t/dx^n \) is ex­
pressed in terms of derivatives of \( x \) with respect to \( t \). Re­
peated application of the formula

\[
d/dx = \frac{d/dt}{dx/dt}
\]

enables us to so express \( d^n/dx^n \). Since the process becomes
tedious for the higher \( d \)erivatives, it is desirable to pre­
serve the results. Hence, the expressions for \( a_1 \) through \( a_7 \)
are listed below.

\[
a_1 = 1/b_1.
\]
\[
a_2 = -b_2/b_1^3.
\]
\[
\begin{align*}
b_1 a_3 &= -b_3 b_1 + 2b_2^2.
\end{align*}
\]
\[
\begin{align*}
b_1 a_4 &= -b_4 b_1^2 + 5b_3 b_2 b_1 - 5b_2^3.
\end{align*}
\]
\[
\begin{align*}
b_1 a_5 &= -b_5 b_1^3 + 6b_4 b_2 b_1^2 + 3b_2 b_1^4 - 21b_3 b_2^2 b_1 + 14b_2^4.
\end{align*}
\]
\[
\begin{align*}
b_1 a_6 &= -b_6 b_1^4 + 7b_5 b_2 b_1^3 + 7b_4 b_3 b_1^2 - 28b_4 b_2 b_1^2 - 28b_3 b_2^2 b_1
\end{align*}
\]
\[
\begin{align*}
+ 84b_3 b_2 b_1^3 - 42b_2^5.
\end{align*}
\]
\[
\begin{align*}
b_1 a_7 &= -b_7 b_1^5 + 8b_6 b_2 b_1^4 + 8b_5 b_3 b_1^4 - 36b_5 b_2 b_1^3 + 4b_1 b_4^2
\end{align*}
\]
\[
\begin{align*}
- 72b_4 b_3 b_2 b_1^3 + 120b_4 b_2 b_1^2 - 12b_3 b_1^3 + 45b_3 b_2^2 b_1 - 330b_3 b_2 b_1^2 + 132b_2^6.
\end{align*}
\]
APPENDIX D. Power Series for a Power Series Raised to an Arbitrary Power.

Suppose we have given

\[ x = \sum_{n=0}^{\infty} b_n t^n. \]  

(1)

It is desired to express \( x^s \) as a power series in \( t \). The Taylor's series for expansion of a function \( u(t) \) in powers of \( t \) is

\[ u(t) = u(0) + t u'(0) + \frac{t^2}{2} u''(0) + \frac{t^3}{3!} u'''(0) \cdots. \]  

(2)

A power series for \( x^s \) may be obtained by setting

\[ u = x^s. \]  

(3)

The required derivatives of \( u \) may be obtained from (3) and (1). Thus,

\[ u(0) = b^s. \]

\[ u'(0) = s b_1 b^{s-1}. \]

\[ u''(0) = 2 s b_2 b^{s-1} + s(s-1) b_1 b^{s-2}. \]

\[ u'''(0) = 6 s b_3 b^{s-1} + 6 s(s-1) b_2 b_1 b^{s-2} + s(s-1)(s-2) b_1^2 b^{s-3}. \]
APPENDIX E. The Series for \( W(U_1,0) \) when \( \sigma = 1 \) and \( \lambda \alpha T \).

Recall that

\[
W(U_1,0) = \sqrt{\frac{k_0 x}{\rho_1 u_1^3}} \left( \frac{\partial u}{\partial y} \right)_x
\]

and that

\[
w(x,V) = \sqrt{\frac{k w e w x}{\rho^3 u_1^3}} \left( \frac{\partial u}{\partial y} \right)_x.
\]

Hence,

\[
W(U_1,0) = \sqrt{\frac{\rho w k_0}{\rho_1^3 u_1^3}} w(x,0).
\]

Now \( \rho_w/\rho_1 = T_1/T_w \). And for an insulated surface with \( \sigma = 1 \), we have \( T_w = T_0 \). Finally,

\[
W(U_1,0) = (T_1/T_0)^{\frac{1}{2}} w(x,0) = (1-U_1^2)^{\frac{1}{2}} w(x,0).
\]

In Appendix B, \( w(x,0) \) is expanded in powers of \( U_1^2 \):

\[
w(x,0) = w(U_1,0) = F_0 + A_2 U_1^2 + A_4 U_1^4 + A_6 U_1^6 + \cdots.
\]

With this notation, we obtain

\[
W(U_1,0) = F_0 + U_1^2 (A_2 - \frac{1}{2} F_0) + U_1^4 (A_4 - \frac{1}{8} A_2 - \frac{1}{8} F_0) + U_1^6 (A_6 - \frac{1}{2} A_4 - \frac{1}{8} A_2 - \frac{1}{16} F_0) + \cdots
\]

For \( m = 0 \), \( F_0 = 0.33206 \) and if \( \lambda \alpha T \), \( A_2 = 0 = A_4 = A_6 = \cdots \), as shown in section 2.711 of Appendix A.

For \( m = 1 \), \( F_0 = 1.232588 \). Values of \( A_2 \), \( A_4 \), and \( A_6 \) for \( m=1 \) can be obtained from the table on page 175 of Appendix B.

(The \( \sigma = 1 \) constant specific heat case is there designated by case 1). Using these values of \( F_0 \) and the \( A \)'s, it is
found that for $m = 0$

$$W(U_1,0) = 0.33206 - 0.16603U_1^2 - 0.04151U_1^4 - 0.02075U_1^6 - \ldots$$

and for $m = 1$,

$$W(U_1,0) = \begin{array}{c} 1.232588 - 0.33675U_1^2 - 0.00080U_1^4 + 0.02378U_1^6 + \ldots \end{array}$$

The series for $W(U_1,V)$ may be obtained explicitly in the $m = 0$ case. In fact, since

$$W(U_1,V) = \frac{\sqrt{\frac{e^2}{e_w}}}{e_w} w(U_1,V),$$

and since

$$\frac{e^2}{e_w} = \sqrt{\frac{T_1}{T_2}} = \sqrt{\frac{1-U_1^2}{(1-U_1U_2)^2}}$$

we obtain

$$W(U_1,V) = 0.33206 \left[ (1-U_1^2) \frac{1}{(1-U_1U_2)^2} \right]$$

$$= 0.33206 \left[ 1 + U_1^2(V^2 - \frac{3}{2}) + U_1^4(V^4 - \frac{5}{2}V^2 - \frac{1}{8}) + \ldots \right]$$
References


References (Cont.)


References (Cont.)


33. Wilson, H. A., "A Determination of the Charge on the


41. To be published soon as a USAF technical report. These results were obtained as part of a research project supported by the O.A.R. and supervised by Professor A. N. Tifford of the Ohio State University, Department of Aeronautical Engineering.

42. Chao, Chi-Chang, "Universal Functions Applicable to Certain Boundary Layer Problems," Master's Thesis, Ohio State University. Investigators using Chao's thesis should be warned that an error in sign in equation 1.1c causes errors in some of the differential equations.


I was born in Kilbourne, Ohio, December 31, 1924. My secondary school education was received at Brown School, in Kilbourne. After one year at Ohio Wesleyan University, I transferred to the Illinois Institute of Technology, which awarded me the degree Bachelor of Science in Mechanical Engineering in 1945. During the school year 1946-47, I taught mathematics in Ashley High School, Ashley, Ohio. I resigned this position to enter the Graduate School at Ohio State University, where I specialized in physics. I received the degree Master of Science in 1948. While in residence at the Ohio State University, I held several part-time positions and the following full-time positions: four quarters as Instructor in the Department of Aeronautical Engineering, one year as University Fellow, one quarter as Research Associate.