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A THEORETICAL AND ANALOG STUDY OF A STEEP DESCENT
COEFFICIENT COMPUTER FOR PROCESS ANALYSIS
AND ADAPTIVE CONTROL

DISSERTATION
Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By
Bruce John Miller, B.S.M.E., M.Sc.

* * * * * *

The Ohio State University
1962

Approved by

L. M. Marco
Adviser
Department of Mechanical Engineering
PREFACE

One of the most difficult problems encountered in process analysis and adaptive control is the identification of system parameters. Although many successful methods have been found for identifying system parameters, most of the methods have application only to rather narrow classes of systems and processes.

The work presented herein is an engineering study of what is believed to be a very general method of identifying system coefficients, with its application to adaptive control processes. Although the specific systems studied throughout this work are linear, the general theory of the coefficient computer is applicable to non-linear systems as well. It also appears that the adaptive control scheme used to force slowly varying coefficient linear systems to behave as constant coefficient linear systems can be used to force some non-linear slowly varying coefficient systems to behave as linear constant coefficient systems. The study of these non-linear systems, however, is not included in this dissertation.

A decimal numbering system is used throughout the dissertation such that equation (paragraph or figure or
table) (2.31) would be the thirty-first equation (paragraph or figure or table) in Chapter II.

In the discussions involving transfer functions it will be noted that either the complex variable $S$ (associated with the Laplace transform) or the operator $D$ (which is defined as $\frac{d}{dt}$) is used. In this dissertation the complex variable $S$ is used only for constant coefficient systems; in linear systems having slowly variable coefficients the Laplace transform cannot be computed for arbitrary variations of the variable coefficients—i.e., the specific time variations of the variable coefficients must be known in order to compute the Laplace transform. For this reason the transfer functions of linear systems having variable coefficients are expressed using the "D" notation.

The author is indebted to Professor S. M. Marco and Dr. E. O. Doebelin of the Mechanical Engineering Department of the Ohio State University; Dr. Doebelin originally introduced the author to the field of automatic controls, and also acted as the author's dissertation co-adviser; Professor S. M. Marco acted as the author's dissertation adviser.

Much of the work reported in this dissertation was performed by the author as part of a research program at North American Aviation, Inc., Columbus Division. The author's indebtedness to Mr. A. B. Clymer and Mr. T. F. Potts of North American Aviation, Inc., Columbus Division,
for their technical contributions and enthusiastic interest is gratefully acknowledged. Without the opportunity to utilize the excellent analog computing facility at North American Aviation, Inc. much of the work reported herein would not have been possible. The author thus wishes to record his special thanks to his former supervisor, Mr. C. E. Knox, whose active support made the research program possible. Special thanks are also due Mr. J. C. Italiano, Mr. C. H. Zimmerman, and other members of the excellent maintenance staff in the Analog Computation Group at North American Aviation, Inc., Columbus, for their rapid and masterly solution of many unusual problems that developed in the course of the author's work on the analog computer.

To my wife, Shirley, a special debt of gratitude is acknowledged; in all non-technical aspects this dissertation is a product of our mutual efforts.

March, 1962

B.J.M.
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CHAPTER I

INTRODUCTION

1.1 Background

The extensive development of feedback control theory during the past three decades has been centered around linear, constant coefficient systems. Excellent treatises have been written on the subject of analyzing, synthesizing, and compensating systems of this type. The development of linear constant coefficient control theory has been described extensively in the literature and will not be treated here.

Certain difficulties in applying linear constant coefficient control theory appeared in the mid 1950's in the aircraft industry. The widespread interest in adaptive controls probably owes much of its popularity to this fact. In the following sections the problem encountered in the aircraft industry will be briefly described, the general characteristics of this class of problem will be discussed, and an adaptive control system which provides a solution to such problems will be proposed. The investigation of this adaptive control system is the subject of this report. Following these discussions, a general scheme for the classification of adaptive control systems will be presented, and the adaptive control system herein proposed will be fitted into this general classification scheme.
1.2 Introduction to a Specific Class of Problems Encountered in the Aircraft Industry

Briefly, the problem which appeared in the aircraft industry in the mid 1950's may be thought of as having two parts, part (1) being the description of the aircraft motion in terms of a set of aerodynamic transfer functions and part (2) being the compensation of the aerodynamic transfer functions by the aircraft flight control system so as to meet certain over-all system performance specifications. The following discussion of aircraft flight control systems is presented to indicate the existence of an important class of practical control problems and not to develop the details of aircraft dynamic analysis. The discussion will be oversimplified and should not be considered as complete or thorough.

1.2.1 Description of the Aircraft Motions in Terms of a Set of Aerodynamic Transfer Functions

The position and orientation of an aircraft in three-dimensional space can be controlled by manipulating the external forces and moments which act upon the aircraft. These forces and moments are either non-aerodynamic or aerodynamic. The non-aerodynamic control effects are produced by reaction controls, jets, rockets, etc.; these are not of interest in this discussion. The aerodynamic
control effects are determined by the positions of the control surfaces (elevators, rudders, ailerons, etc.).

Starting from Newton's second law, a set of differential equations describing the rigid body (six degrees of freedom) motion of an aircraft can be deduced. The complexity of this set of coupled, non-linear, variable coefficient differential equations tremendously handicaps analytical mathematical studies of the character of the aircraft responses. While it is possible to mechanize this complex set of coupled, non-linear, variable coefficient differential equations on an analog computer for analysis purposes, preliminary design studies require more direct and rapid approaches. Such approaches may be realized by use of linearizing procedures leading to transfer functions.

Under suitable assumptions\(^1\) (small perturbation theory, etc.) these variables of the rigid body (six degrees of freedom) equations can be related by a linearized set of coupled differential equations. If the coefficients appearing in this set of differential equations are assumed to be constant (this assumption will be discussed in the next paragraph), the Laplace transformation of each of the differential equations in the set can be taken, and Cramer's rule can be used to uncouple the differential equations. Among the uncoupled equations derivable in this manner, perhaps the equations of most importance in analytical flight
control work are those relating (1) pitch angle to elevator position, (2) yaw angle to vertical stabilizer position, and (3) roll angle to aileron position. The reason for the importance of these equations is that it is through them that the full force of the powerful analytical techniques of linear constant coefficient feedback control theory can be brought to bear on the design and analysis of flight control systems.

The fact that the coefficients appearing in the coupled set of linear differential equations are assumed constant demands special discussion. In reality, the coefficients vary, since the equilibrium operating points about which the perturbations are taken are in general not themselves constant. Thus when the operating points change (due to changes in speed, altitude, angle of attack, etc.) the coefficients must also change. For practical purposes, however, it is very convenient to disregard this strict interpretation of constant coefficients, because, in so doing the powerful and well-developed linear constant coefficient control theory can be applied. In practice, the method often used for analyzing aircraft flight control is as follows:

1) Instead of considering the infinite number of combinations of altitude and Mach number, only a few representative points are chosen from the flight envelope for analysis (see Figure 1.1).
$X = \text{representative points chosen from the flight envelope for analysis}$

**FIGURE 1.1**

FLIGHT ENVELOPE FOR A PICTITIOUS AIRCRAFT
2) The aircraft is considered to be flying at constant altitude and Mach number at each of the chosen points, and the system is analyzed using root locus and/or frequency response methods.

In essence, the complex problem of studying aircraft response as the aircraft accelerates, decelerates, climbs and dives, banks, etc. is replaced by the problem of studying aircraft response under constant altitude and Mach number conditions. It should be mentioned in passing that these analytical studies of aircraft response under constant altitude and Mach number conditions are valid only in describing the dynamics of aircraft flight near equilibrium flight conditions, since the variables describing the position and orientation of the aircraft are restricted to having only small perturbations about an equilibrium flight condition.

That this sort of approximation is useful and valid has been demonstrated repeatedly in application; the reason for the success of this approximation is directly attributable to the fact that altitude and Mach number of aircraft can change only very slowly when compared with the aircraft response. An example will illustrate this fact: the "short period" response of most aircraft to a step or impulse motion of the elevator transpires within a very few seconds, while appreciable changes in Mach number or altitude ordinarily require time durations measured in minutes. The "phugoid"
or long period response of some aircraft may have a period approaching a minute; this long period response can usually be neglected (except for speed control systems), however, since its effect on system performance is almost always of little consequence. Difficulties in applying these approximating techniques are somewhat more prevalent in the missile field where the thrust to mass ratio is much higher (producing much more rapid variations in Mach number and altitude), and where the maneuverability is much more radical (since the "g" limitation on a missile is not limited to the "g" limitations of a human pilot).

To recapitulate then: instead of using a coupled set of nonlinear differential equations (having time varying coefficients) to describe aircraft dynamics, several sets of coupled linear differential equations (having constant coefficients) are used in analytical studies, there being one set of constant coefficient coupled linear differential equations for each point of the flight envelope to be considered (as discussed above and illustrated in Figure 1.1).

The most convenient form in which to study these constant coefficient coupled linear differential equations is to:

1) Uncouple the Laplace transformed equations by using Cramer's rule, and
2) Express each of the position and orientation (pitch angle, etc.) variables as a function of a
control surface variable (elevator position, etc.). These functional relationships are most conveniently expressed as transfer functions. All workers in this field refer to these transfer functions as "aerodynamic transfer functions." By way of illustration, a typical aerodynamic transfer function could have the following form:

\[
\text{Aerodynamic Transfer Function}
\]

\[
\begin{align*}
\delta H & \quad (\text{Elevator Position (degrees)}) \\
\frac{b_1 S + b_0}{a_2 S^2 + a_1 S + 1} & \quad \dot{\phi} \\
\dot{\phi} & \quad (\text{Pitch Rate (degrees/sec.)})
\end{align*}
\]

Figure 1.2. A Typical Aerodynamic Transfer Function

where \(b_1, b_0, a_2, a_1\) are constants, depending upon the aircraft, altitude, and Mach number being described, and where \(S\) is a complex variable associated with the Laplace transformation. The block diagram above represents the linear, constant coefficient differential equation:

\[
a_2 \ddot{\phi} + a_1 \dot{\phi} + \phi = b_1 \delta_H + b_0 \delta_H
\]

Having discussed the description of aircraft motion by a set of aerodynamic transfer functions, the second part of the problem of designing flight control systems can be discussed.
1.2.2 **Description of Compensation of the Aerodynamic Transfer Functions by the Aircraft Flight Control System**

The most important functions of the aircraft flight control system are (1) to force a desirable static and dynamic relationship to exist between the pilot's commands and the aircraft responses and (2) to force the aircraft to exhibit desirable stability characteristics. The methods of achieving (1) and (2) above, given a specific aerodynamic transfer function, are well discussed in the literature; the results of applying the general methods will be illustrated with a simple example.

Assume the aerodynamic transfer function $G_{\text{aero}}$, relating pitch angle to elevator angle of a specific airplane, is known. Assume further that inspection of the flight envelope of the aircraft indicates that only two flight conditions must be examined, sea level flight at Mach 0.5 and flight at 15,000 feet at Mach 0.8. The root locus methods or frequency response methods can be used to find the compensating transfer functions required to yield desirable stability and pilot control characteristics. Let the results of these analyses be presented in block diagram form, where all the transfer functions (T.F.'s) shown are known.
Figure 1.3. Block Diagram of a Flight Control System.

Figure 1.4. Block Diagram of a Flight Control System.
It must be emphasized at this point that the only reason for having more than one analysis is that the aero-
dynamic transfer function is different at each point in the flight envelope. At this stage in the design development
procedure, it is customary to make a compromise. It usually turns out that the $G_{11}$ and $G_{12}$ transfer functions can be
made very similar, as can the $G_{21}$ and $G_{22}$ transfer functions. A single compromise transfer function, denoted by $G_1$ is made
between $G_{11}$ and $G_{12}$ and, similarly, a transfer function $G_2$ compromises $G_{21}$ and $G_{22}$. The advantage, of course, to this method is that the compensating transfer functions are invariant, which makes them grossly simpler to design and build. The penalty paid for the compromises is that the stability and pilot control characteristics are not exactly those desired at either sea level, Mach .5 or at 15,000 feet, Mach.8.

In the simple illustration presented above, it was assumed that a reasonable compromise between $G_{11}$, $G_{12}$ and $G_{21}$, $G_{22}$ can be made for the two flight conditions. In practice it turns out that if the aircraft speed is subsonic and the altitude ceiling relatively low, a reasonable compromise can be made. However, when the flight envelope becomes very broad (for example Mach number variations from 0 to 2.0 and altitude variations from sea level to perhaps 50,000 feet or higher), it is impossible to find one
invariant set of compensating devices that produces (1) the desirable dynamic relationships between the pilot's commands and aircraft responses and (2) the desirable stability characteristics. The fact that invariant compensating networks cannot be used forces attention upon preprogrammed and adaptive systems.

This difficulty then is the difficulty referred to in the second paragraph of this section (p. 1); it was the existence of this difficulty that encouraged the development of much of the work in the adaptive controls field. Other areas of application (process controls, etc.) present similar problems, as will be discussed later.

Before beginning a discussion of the adaptive control techniques that form the subject of this work, it is worth while to review a non-adaptive technique that is used in the aircraft industry to solve this problem. In the cases of many modern high performance aircraft it is possible to predict the variations of the aerodynamic transfer functions. This prediction is a very expensive and time-consuming effort since it involves the collection of extensive aerodynamic wind-tunnel and atmospheric data. However, once the variations of the aerodynamic transfer functions are known, then slowly time varying compensation can be used. All that needs to be done is to measure the altitude and Mach number of the aircraft, to use this information to compute the proper values for coefficients in the
compensating transfer functions, and then to adjust the coefficients in the compensating transfer functions to their proper values. The physical component which performs this job is often referred to as an "air data computer."

Some of the advantages of this method are (1) that the method is straightforward, (2) that the computing equipment in the air data computer is practical to construct, and (3) that, in many aircraft, some form of an air data computer is required for other purposes (such as navigation, etc.) and, hence, the air data computer may be the least expensive method of achieving the desired performance.

Some of the disadvantages are (1) that extensive experimental and theoretical data must be collected, this usually being a time consuming and costly effort; (2) that if any of the collected data is incorrect, there is a possibility of delaying aircraft flight testing and safe flight into those regimes where the proper functioning of the air data computer is required; (3) that as with truly adaptive systems, the air data computer cannot adjust for changes that have not been anticipated and "designed into" the system.

1.3 Extension of this problem and Generalization of its Characteristics

Although this background discussion has been in terms of a problem in the aircraft industry, the essential
characteristics of the problem can be associated with a much more general class of control problems of many industries.

Figure 1.5. Block Diagram of a General Control System Having a Linear Slowly Varying Sub-System.

The predominant characteristic of this type of problem is that a process or system exists which can be described by a linear differential equation having slowly varying coefficients. A coefficient will be said to be "slowly-varying" only if the dynamic effect of the rates of change of the coefficients can be neglected in computing the output response of the system given some usual or common input; another way of saying this is that if the system is excited by its usual inputs, the time variations of the coefficients introduce negligible dynamics into the output of the system, i.e., the system behaves almost as a constant coefficient system over a time interval which is short when compared with the variations of the coefficients, but which is long when compared with the periods of the usual system responses.
Some of the methods of compensating such a time varying process have already been discussed.

Method I - If possible, find a time-invariant compensating system (linear or non-linear) that causes the output/input dynamic relationship to correspond to a desired dynamic relationship.

Method II - If a time-invariant compensating system cannot be found to produce the desired dynamic relationship between input and output, a time varying compensating system may be required. If the instantaneous values of the coefficients can be predicted from some measurable quantities (such as Mach number and air density in the air data computer), it is possible to pre-program a compensating system to yield the desired dynamic relationships between the input and output.

Method III - There have been many so-called "adaptive" techniques proposed to solve problems with time variant systems. These will be briefly discussed in section 1.6.

The presentation of the material in the next section and in the remainder of this dissertation is an attempt to provide the theory underlying an adaptive method of compensating linear, slowly time varying systems of arbitrary order. This method adds significant new features to some
previously proposed schemes so as to result in an adaptive controller of considerable versatility.

1.4 Qualitative Description of an Adaptive Control Scheme

The essential point of view taken in this dissertation is that a general compensation scheme to be applied to a slowly varying linear system can be based upon a knowledge of the instantaneous values of each of the slowly varying coefficients. Thus, the first requirement arbitrarily established here is that it must be possible to describe and specify the operation of a "coefficient computer" which computes the instantaneous values of the coefficients of the slowly varying linear system. Much of this dissertation is devoted to an analytical and analog computer investigation of a powerful method of effecting coefficient computation. Once the successful operation of a coefficient computer has been established, several compensation schemes can be used. In Figure 1.6 a general compensation scheme is presented.

Only one compensation scheme will be investigated. The reasons leading to its choice are as follows:

Since feedback control theory has not been nearly as well developed around linear slowly varying coefficient systems as for linear constant coefficient systems, it appears very desirable to consider methods for forcing
Figure 1.6. Schematic of a Class of Compensated Time Varying Linear Systems.

Linear slowly varying coefficient systems to behave as linear constant coefficient systems. If this can be done, the design and analysis problems associated with linear slowly varying coefficient systems are replaced with those of the usual constant coefficient linear systems.

Many methods for forcing a slowly varying linear system to behave as a constant coefficient system can be described. Only the one shown in Figure 1.7 will be investigated here, however. It should be noted that this scheme is in reality a very general scheme, since it does not preclude the possibility of using feedback loops, other tandem
compensation, etc. The primary function of this scheme is merely to force the varying coefficient system to appear as a constant coefficient system.

The method of forcing a time-varying system $G_1$ to behave as a constant coefficient system is to put another time varying system $G_2/G_1^*$ in series with it in such a way that the effects of the variations of the coefficients in the one system are effectively neutralized by the variations of the coefficients in the other. The symbol $G_1$ represents a linear system having a differential equation with slowly varying coefficients. No knowledge concerning the time histories of the slowly varying coefficients of $G_1$ is assumed, however, the form of $G_1$ must be known. The coefficient computer operates only on the input and output of $G_1$ and computes the instantaneous values of the coefficients.

Figure 1.7. Schematic of an Adaptive System which Forces a Linear Slowly Varying Coefficient System to Appear as a Linear Constant Coefficient System.
The coefficients of \( G_1^* \), are then set equal to these computed coefficients. The symbol \( G_2 \) represents a linear, constant coefficient system. The net effect of the operation of this sort of scheme is that \( x \) and \( z \) appear to be related by a constant coefficient transfer function,

\[
\frac{z(s)}{x(s)} = G_2(s).
\]

as in an ordinary linear system with constant coefficients. The attractiveness of this method of achieving adaptive control is that:

1) In theory, adaptive control can be achieved no matter how complex \( G_1 \) might be. (A detailed discussion of limitations is made in Chapter II.)

2) The compensation required is easy to deduce and analyze, since all that needs to be done is to specify a desired transfer function which relates \( x \) and \( z \). This transfer function is then mechanized as \( G_2 \).

There are disadvantages to this scheme also. The idea of "canceling" undesirable poles and zeros is well known. Some of the major objections to this method are that:

1) \( G_1 \) must have roots with negative real parts if no feedback compensation is used.

2) It is impossible to achieve perfect cancellation.

3) The compensation may be unnecessarily complex.
Objectives (2) and (3) are discussed in Reference 3, pages 303-306. Even though cancellation is imperfect, performance will usually be satisfactory, with one major exception: In canceling a complex pole which has a very low value of damping associated with it, or in canceling a pole lying in the right half S plane, imperfect cancellation results in the fact that a pole always exists in approximately the same location as the pole that was to be canceled; hence it may be necessary to use feedback and other compensation methods to achieve the desirable stability characteristics.

1.5 Review of the General Problem and Its Proposed Solution

The general problem considered is the problem of analyzing or synthesizing control systems which have a linear slowly varying coefficient sub-system somewhere in its control loop. Many processes fall in this category—viz, aircraft flight control systems, and many systems from the chemical and process industries.

The proposed adaptive control solution to this problem evades the question of choosing a criterion for judging over-all system performance, since the coefficient computer allows one to replace the difficult problem of analyzing or synthesizing compensation for a linear varying coefficient system with the much easier problem of analyzing or synthesizing compensation for a linear constant coefficient
system. The proposed adaptive control system is different from many other adaptive control systems in this respect.

1.6 Definition and Classification of Adaptive Control Systems

Definitions and discussions describing adaptive control have been presented in many references2,3,5-13. Adaptive systems are defined in References (5) and (13) as being characterized by the following:

1) A means must be provided for measuring system performance.
2) The measured performance must be converted to a "figure of merit," that is, a quantitative description of the quality of performance.
3) Provisions must be made for controlling system parameters by means of the figure of merit.

Five categories for the classification of adaptive systems are presented in Reference (5) as follows:

Class (1) Passive Adaptation: Systems which achieve adaptation without system parameter changes due to inherent insensitivity to wide variations in environment.
Class (2) Input Signal Adaptations: Systems which adjust their parameters in accordance with input signal characteristics.
Class (3): Extremum Adaptation: Systems which self-adjust for maximum or minimum of some system variable.


Class (5): System-characteristic Adaptation: System which make self-adjustments based on measurement of transfer characteristics.

In the next section the adaptive control system studied in this report will be classified in the above framework.

1.7 Classification of the Adaptive Control System Proposed for Study

The classification of the proposed system as an adaptive control system ultimately rests upon an intuitive idea of what should constitute an adaptive control system. However, it is possible to demonstrate that the proposed system can logically be classed an adaptive control system under the definitions and classifications presented in the previous section. This can be done as follows:

1) A means must be provided for measuring system performance; performance is measured in the sense that the instantaneous values of the variable coefficients are computed.
2) The measured performance must be converted to a figure of merit, i.e., a quantitative description of the quality of performance; the figures of merit can be considered the differences between the coefficients of $G_1$ and $G_1^*$.  

3) Provision must be made for controlling system parameters by means of the figure of merit; the compensator attempts to adjust the values of the coefficients of $G_1^*$ to be equal to the computed values of $G_1$.

A review of the five classifications shows that this mode of achieving adaptive control falls under Class (5), since it is a system which makes self-adjustments based upon measurements to transfer characteristics.

Table I of Reference (5) lists important characteristics of adaptive systems. The table could be extended to cover this system as follows:

**Function of system:** Servomechanism  
**Adapts to What?** Slowly time-varying linear systems are automatically compensated so that the over-all systems of which they are a part appear to have constant transfer characteristics.  
**Criterion for Adjustment:** Parameters in a compensator are changed in accordance with the computed values of the slowly varying coefficients.
1.8 The Use of the Analog Computer as an Analytical Tool

As will be seen later, the practical difficulties of analytically investigating the properties of the solutions of the class of integral equations encountered in the study of the method described above are great. Application of the standard techniques of solving the integral equations which arise quickly displays the fact that the standard techniques produce analytical forms of great complexity; time limitations on this investigation have precluded the possibility of presenting all but the simplest solutions to these equations analytically.

The electronic analog computers in use today are testimony to the practicality and usefulness of performing analyses in this fashion. The objective of this investigation has not been to develop and investigate powerful analytical mathematical techniques (in spite of their unquestioned desirability) for solving integral equations, but rather to present a new, useful, and practical method of achieving adaptive control. Thus, mathematical analysis is considered a tool, and must compete with other methods for practicality and fruitfulness in producing understanding of physical systems. The electronic analog computer has, in fact, yielded considerable insight and understanding to the processes involved, and further has suggested some clues to the paths which should be taken in an analytical investigation.
1.9 **Scope of this Work**

This dissertation is limited to a description of a general form of achieving adaptive control and a presentation of the underlying theory for this form of adaptive control, of the results of an analog computer analysis of the peculiarities of the method, and of experimental (analog computer simulation) results of the performance of various adaptive control systems employing the principles and methods developed herein.
CHAPTER II

CONVERGENCE OF THE COMPUTED COEFFICIENTS TO
THE ACTUAL COEFFICIENTS

2.1 The Coefficient Computer

As discussed in Part I, the success of the compensator which is required to cancel out the effects of time variations of the coefficients in constant and slowly varying linear systems depends upon the accurate calculation of the values of the constant or slowly varying coefficients.

In this chapter, a general method for computing the constant coefficients of systems describable by sets of algebraic or ordinary integro-differential, linear or non-linear equations is presented. The method presented is a review and amplification of the method proposed by Potts, Ornstein and Clymer. The stability analysis of the method and its application to adaptive control systems presented here is however thought to be original.

In the course of performing the stability analysis difficult mathematical problems are encountered. By assuming certain sets of hypotheses, it is possible to provide certain sufficient convergence conditions which, when satis-

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to their proper values. However, in the most commonly expected circumstances, it is found that either the hypotheses or the convergence conditions are difficult to satisfy. By requiring the satisfaction of an obviously necessary condition for coefficient convergence, it is possible to provide an intuitive argument which provides assurance of the proper convergence of the computed coefficients. In analog computer tests, discussed in later chapters, it was found that when the necessary condition just discussed was satisfied, proper convergence of the computed coefficients always resulted (when the necessary condition was not satisfied, the computed coefficients did not converge properly). Analog computer tests and the appeal of the intuitive argument suggest that the necessary condition, and the condition of continuity, may be sufficient conditions guaranteeing the convergence of the coefficients. At present, however, a rigorous mathematical proof as to when the "necessary condition" may be a sufficient condition cannot be provided.

In presenting the theory of the coefficient computer, the most general case will be developed first, followed by its application to constant coefficient linear systems. In these applications to constant coefficient linear systems, the method of steep descent will be described and the reasoning behind the evolution of the "steep descent" equations will be displayed.
2.2 The General Problem of Computing Coefficients

The general problem discussed in what follows is that wherein a set of ordinary integro-differential or algebraic, linear or non-linear equations describing the dynamic performance of a system is of known form, and the "coefficients" appearing in the equation are known to be real valued constants. The problem is how can the constant coefficients of the system be computed if only the system inputs and outputs (and perhaps their time derivatives) are known?

Consider the general system described in Figure 2.1:

![Figure 2.1. A General System](image)

Assume that a set of equations exists such that:

\[
\begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1M-1} & 1 \\
    a_{21} & a_{22} & \cdots & a_{2M-1} & 1 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    a_{L1} & a_{L2} & \cdots & a_{LM-1} & 1
\end{bmatrix}
\begin{bmatrix}
g_1 \\
g_2 \\
\vdots \\
g_M
\end{bmatrix}
\]
where

\[ a_{ij} = \text{constant} \]

\[ f_1 = f_1(x_1, x_1', x_1'', \ldots, x_p, x_p', x_p'', \ldots, y_1, y_1', y_1'', \ldots, t) \]

\[ x_1^{(k)} = \frac{d^k x_1}{dt^k} (t) \]

\[ y_1^{(k)} = \frac{d^k y_1}{dt^k} (t) \]

\[ g_1 = g_1(t) = f_1 \]

Let \( a_{ij}^*(t) \), the computed estimate of \( a_{ij} \), be considered a function of time. The expression defining the variation of \( a_{ij}^*(t) \) will be presented subsequently; the basic idea here is that it is desired to subject \( a_{ij}^*(t) \) to some condition such that \( a_{ij}^*(t) \) is forced to approach \( a_{ij} \) as \( t \to \infty \), regardless of the initial value of \( a_{ij}^*(t) \).

Define \( E_i(t) \) as follows where \( i = 1, 2, \ldots, L \)

\[
\begin{bmatrix}
E_1 \\
E_2 \\
\vdots \\
E_L
\end{bmatrix} =
\begin{bmatrix}
a_{11}^* & a_{12}^* & \ldots & a_{1M-1}^* & 1 & g_1 \\
a_{21}^* & a_{22}^* & \ldots & a_{2M-1}^* & 1 & g_2 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
a_{L1}^* & a_{L2}^* & \ldots & a_{LM-1}^* & 1 & g_M
\end{bmatrix}
\]

\( (2.2) \)
Subtracting (2.1) from (2.2) yields:

\[
\begin{bmatrix}
E_1 \\
E_2 \\
\vdots \\
E_L \\
\end{bmatrix}
= 
\begin{bmatrix}
a_{11}^* - a_{11} & a_{12}^* - a_{12} & \cdots & a_{1M-1}^* - a_{1M-1} & 0 \\
-a_{21}^* & a_{22}^* - a_{22} & \cdots & a_{2M-1}^* - a_{2M-1} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-a_{L1}^* & a_{L2}^* & \cdots & a_{LM-1}^* & a_{LM-1} & 0 \\
\end{bmatrix}
\begin{bmatrix}
g_1 \\
g_2 \\
\vdots \\
g_M \\
\end{bmatrix}
\]  

(2.3)

From (2.3) it is clear that:

\[
E_1 = \sum_{j=1}^{M-1} (a_{1j}^* - a_{1j})g_j 
\]  

(2.4)

Equation (2.4) defines an "error" $E_1$ which will henceforth be called the "equation error." The value of this definition of error can hardly be overstated. Its unique characteristics lie at the very core for the success of this method. (To the author's knowledge, the first suggested use of this error function for these purposes was made by A. B. Clymer,\(^{14}\) although the work of K. K. Graupe\(^{15}\) indicates an independent, parallel utilization of this concept.) A difficulty encountered in the work of Margolis and Leondes\(^{16,17}\) on coefficient computation was that the "model error" partial derivatives, taken with respect to the "computed" coefficients, could not be exactly expressed analytically in terms of simple functions of the input and output variables of the system and the model. This required the use of certain approximations in the computing scheme. The "equation error" defined by equation (2.4) however possesses several
unique characteristics, one being that the partial derivatives taken with respect to the "computed" coefficients have a simple form, i.e., $\frac{\partial E_i}{\partial a_{ij}^*} = g_j(t)$. Since $g_j(t)$ can almost always be mechanized in practical cases, this partial derivative can be formed exactly.

It will be shown later that there are two conditions which must be satisfied in order to force $a_{ij}^* \rightarrow a_{ij}$ (for all $i,j$) as $t \rightarrow \infty$. The general methods of steep descent suggest a method for defining the variation of the variables $a_{ij}^*(t)$. (More will be said about the steep descent equation in the later portions of this chapter.) Many functions $F_k(E_i)$ can be used in setting up the steep descent equations; $F_k(E_j)$ is required to be continuous, possess a unique minimum zero value at $a_{ij}^* = a_{ij}$ (for all $i,j$) and be a monotonically increasing function of $E_i$ as $E_i$ moves away from zero. Several examples of suitable error functions $F_k(E_i)$ are: $E_i^2$, $|E_i|$, $|E_i^3|$, $E_i^{2n}$ (n a positive integer), etc. To exhibit the arguments of the following discussion with maximum clarity, the specific function of the error $F_k(E_i)$ considered will be $E_i^2$; the arguments for the other error functions discussed above are similar, however. [See Potts, Ornstein and Clymer for examples of $F(E_i) = |E_i|$ .] Employing steep descent methods,
define the derivative of $a^*_{ij}(t)$ with time as follows:

$$\frac{da^*_{ij}}{dt} = g_1 \frac{\partial E_i^2}{\partial a^*_{ij}} = -2G_1E_1(t)g_j(t) \quad (2.5)$$

for $i = 1,2,\ldots,L$

$$j = 1,2,\ldots,M-1$$

and where $G_1$ is any positive number, either constant or time varying. It has been assumed by the author in applications of the methods discussed herein that $G_1$ is constant. However, the study of systems where $G_1$ is intentionally varied according to some predetermined plan may reveal advantages in terms of coefficient convergence times.

Equation (2.5) can be written in the form:

$$\begin{bmatrix}
a^*_{11} - a_{11} \\
a^*_{12} - a_{12} \\
\vdots \\
a^*_{1M-1} - a_{1M-1}
\end{bmatrix} = -2G_1 \begin{bmatrix}
\varepsilon_1g_1 & \varepsilon_1g_2 & \cdots & \varepsilon_1g_{M-1} \\
\varepsilon_2g_1 & \varepsilon_2g_2 & \cdots & \varepsilon_2g_{M-1} \\
\vdots \\
\varepsilon_{M-1}g_1 & \varepsilon_{M-1}g_2 & \cdots & \varepsilon_{M-1}g_{M-1}
\end{bmatrix} \begin{bmatrix}
a^*_{11} - a_{11} \\
a^*_{12} - a_{12} \\
\vdots \\
a^*_{1M-1} - a_{1M-1}
\end{bmatrix} \quad (2.5a)$$

Equation (5a) can be written in vector form as:

$$\frac{d\bar{a}_i}{dt}(t) = -2G_1 F(t) \bar{a}_i(t) \quad i = 1,2,\ldots,L. \quad (2.5b)$$

2.2.1 Properties of the Solution

Equations (2.5), (2.5a) and (2.5b) are differential equations, the solutions of which yield the time variations of $a^*_{ij}(t)$. The properties of the solutions are best discussed by determining $a^*_{ij}(t)$ as an explicit function of time. Since the $a^*_{ij}(t)$ are coupled through $E_1(t)$ as defined
In equation (2.4), the first step in solving for $a_{1j}(t)$ is to solve for $E_1(t)$. Having determined an explicit function of time $E_1(t)$, $a_{1j}^*(t)$ can be determined by integrating equation (2.5).

To obtain an integro-differential equation in terms of $E_1(t)$, $a_{1j}(t)$ can be expressed as an integral from equation (2.5). Substitution of these integrals into equation (2.4) yields:

$$E_1(t) = \sum_{j=1}^{M-1} \left( a_{1j}^*(t_0) - a_{1j} - 2G_1 \int_{t_0}^{t} E_1(\tau) g_j(\tau) d\tau \right) g_j(t)$$

(2.6)

where $t_0$ is the initial time.

Equation (2.6) is an integral equation of the second kind with a variable parameter. Of the conventional techniques for solving this integral equation for explicit time functions $E_1(t)$, the methods of successive substitutions\textsuperscript{19} and the method of successive approximations\textsuperscript{19} are perhaps the most powerful techniques to employ.

The successive approximation and successive substitution methods require the evaluation of many integrals for even an approximate solution for $E_1(t)$, however. The convergence characteristics of the computed coefficients could not be deduced in a direct manner from the form of these integral equations for general (non-specific) functions $g_j(t)$. Analysis and a literature search failed to provide a satisfactory method for solving equation (2.6).
Since the form of the time solutions of equations (2.4), (2.5) or (2.6) could not be found, it was desirable to find at least some of the properties of the solution. The most critical question to be answered concerns the convergence of $a^*_i(t)$ to $a_{ij}$. Assuming continuity of the $g_j(t)$, it can be shown that $a^*_i(t)$ has a unique solution (see (1) of the Appendix). An additional theorem of Bellman's (see (2) of the Appendix) can be applied to the convergence question of $a^*_i(t)$ as $t \to \infty$. It was found that the hypotheses of this theorem were often difficult to satisfy in this problem and further, it was found in several specific examples that this theorem could not provide a guarantee of the convergence of $a^*_i(t)$ to $a_{ij}$ as $t \to \infty$, even though analog computer studies verified this convergence. The next approach attempted was an appeal to Lyapunov's Second Method. (A discussion and all of the essential theorems for this development are included in (3) of the Appendix).

Define $R^2_1(t)$ as:

$$R^2_1(t) = \sum_{j=1}^{M-1} (a^*_i(t) - a_{ij})^2 \quad i = 1, 2, \ldots L$$

Taking the derivative of (2.7) with respect to time yields:

$$\frac{dR^2_1}{dt} = -4g_j E_i \sum_{j=1}^{M-1} (a^*_i - a_{ij}) g_j(t) \quad i = 1, 2, \ldots L$$
Substituting (2.4) into (2.8) yields:

\[
\frac{dR_i^2}{dt} = -4G_1E_1^2, \quad i = 1, 2, \ldots, L (2.9)
\]

The set of equation (2.9) can be written in matrix form as:

\[
\frac{d}{dt} \begin{bmatrix} R_1^2 \\ R_2^2 \\ \vdots \\ R_L^2 \\ \end{bmatrix} = -4 \begin{bmatrix} G_1E_1^2 \\ G_2E_2^2 \\ \vdots \\ G_LE_L^2 \\ \end{bmatrix} (2.9a)
\]

From equations (2.7) and (2.9) it is seen that \(R_i^2(t)\) will be a Lyapunov function, provided certain conditions are satisfied. Appealing to Corollary 1.1 of the Kalman and Bertram paper, "uniform stability" of the system (i.e., containment of the \(a_{ij}^*(t)\) within a certain bounded area, where the bound is a function of the initial values of the \(a_{ij}^*(t)\)) is always assured by the nature of the definition of \(R_i^2, E_i^2\), and equation (2.9)—regardless of the specific forms of the continuous functions \(g_j(t)\). However, "uniform asymptotic stability in the large" (i.e., convergence of \(a_{ij}^*(t)\) to \(a_{ij}\) as \(t \to \infty\) for any initial value of \(a_{ij}^*\)) is desired but is not always assured. If the functions \(g_j(t)\) are such that for all time \(t\):

\[
E_1^2 \geq C R_1^2, \quad C > 0 (2.10)
\]

(Functions other than \(R^2\) can be used), then by Theorem 1 uniform asymptotic stability in the large is assured.
The general conclusions from the above discussion are convergence of the coefficients $a^*_i^j$ to $a_{i^j}$ as $t \rightarrow \infty$ will always be assured if the following sufficient conditions are satisfied:

(a) the functions $g_j(t)$ are continuous and (2.11)
(b) inequality (2.10) (or one similar to it (2.12) which satisfies condition (ii) of Theorem 12) is satisfied.

Although in practical applications condition (2.11) is ordinarily satisfied, condition (2.12) above is often difficult to satisfy since in the general case $E^2_1(t)$ is often a continuously varying quantity which takes on positive and negative values—hence $E^2_1$ frequently takes on instantaneous zero values. (Note: if $E^2_1$ takes on instantaneous zero values, then it follows that it is impossible to satisfy condition (2.12) above.) In spite of the fact that condition (2.12) is often impossible to satisfy, convergence of $a^*_i^j$ to $a_{i^j}$ may still be assured in many cases since condition (2.12) is a sufficient but not a necessary condition.

The statement of the necessary and sufficient conditions for $a^*_i^j \rightarrow a_{i^j}$ as $t \rightarrow \infty$ when condition (2.12) cannot be satisfied is difficult. At the present time it appears that the answer to the question of the convergence of $a^*_i^j$ to $a_{i^j}$ must depend upon knowledge of the specific forms of the functions $g_j(t)$. If the specific forms of the
$g_j(t)$ are known, it may be possible to construct a new Lyapunov function which will directly guarantee the convergence of $a_{ij}^*$ to $a_{ij}$. The techniques discussed by G. P. Szego are applicable to this approach. Attempts at finding sufficient conditions for $a_{ij}^* \rightarrow a_{ij}$ when condition (2.12) cannot be satisfied have led to the idea of listing some of the necessary conditions for $a_{ij}^* \rightarrow a_{ij}$ as $t \rightarrow \infty$, and then examining these necessary conditions to determine their possible sufficiency. Proceeding in this direction, an important necessary condition can be deduced as follows:

Consider the consequence of $E_1^2$ being identically zero when some $a_{ij}^* \neq a_{ij}$. This is an equilibrium condition since by equation (2.5) all $a_{ij}^*$ will remain constant, hence convergence of $a_{ij}^*$ to $a_{ij}$ is prevented. Thus, an important necessary condition for convergence of $a_{ij}^*$ to $a_{ij}$ is prevented. Thus, an important necessary condition for convergence of the $a_{ij}^*(t)$ to their correct values is:

$$E_1^2 \text{ is not identically zero for } R_1^2 \neq 0 \quad (2.13)$$

(It is obvious by their definitions that $R_1^2 = 0$ implies $E_1 = 0$.) If it is assumed that (2.11) and (2.13) are satisfied, it is difficult to demonstrate with rigor that $R_1^2 \rightarrow 0$ as $t \rightarrow \infty$. The difficulty encountered here is that it is difficult to relate the behavior of $R_1^2(t)$ to the behavior of $E_1^2(t)$ as $t \rightarrow \infty$. An intuitive argument implying the convergence of $R_1^2$ to zero can be made, however. It can be
shown that $E^2_{1} \rightarrow 0$ as $t \rightarrow \infty$ (see (4) of the Appendix) by assuming that the $g_j(t)$ and their derivatives are continuous bounded functions of time. If conditions (2.11) and (2.13) are satisfied, and if $g^2_j(t)$ has a non-zero average value over any interval (as when $g_j(t)$ is a sinusoidal or randomly varying quantity), then since $E^2_{1} \rightarrow 0$ as $t \rightarrow \infty$, $a^*$ varies at an increasingly slower rate (ref. equation 2.4) and will appear approximately as a constant over "short" intervals of time. Since no constant values of $a^*_{ij}(t)$ other than $a^*_{ij}(t) = a^*_{ij}$ can be at an equilibrium state (condition (2.13)), convergence of $a^*_{ij}(t)$ to $a^*_{ij}$ is expected. Another way of expressing this idea is that since an equilibrium state is always being approached, and since there is only one equilibrium state (where $a^*_{ij} = a^*_{ij}$), the convergence of $a^*_{ij}(t)$ to $a^*_{ij}$ as $t \rightarrow \infty$ is expected. A difficulty with intuitive arguments is that they often avoid resolution of a difficult mathematical question. In the above intuitive argument the following problem is entirely avoided:

Assume that $R^2(t)$ is a decaying exponential (or other monotonically decreasing function) having an asymptote of $C_1$ where $0 < C_1$. It appears conceivable that a function $R^2(t)$ of this sort could satisfy conditions (2.11) and (2.13). Since (2.13) only requires $R^2(t)$ to be zero when $E^2_{1}$ is zero, we simply argue that $E^2_{1}$ may never reach zero,
i.e., that \( \frac{dR_1^2}{dt} \) always has a negative non-zero value, which accounts for the asymptotic behaviour of \( R_1^2(t) \).

Although many interesting mathematical questions are resolved in this study, time limitations have prevented resolution of a few questions, of which the rigorous demonstration of coefficient convergence is the most important. As will be seen in later chapters, the engineering applications of these techniques are very useful. The analog computer studies have unquestionably demonstrated the utility of the method under conditions which resemble those of practical problems. The fact that every application of the method has been successful suggests that the method may be very general—however, final conclusions regarding the generality of the method must be based upon a rigorous mathematical study of the convergence question.

2.3 Application of the Coefficient Computing Technique to Linear Systems

In this section the methods described in Section 2.1 are applied to the general problem of computing the coefficients of linear constant coefficient systems. The applications are made to the problem of computing one unknown, two unknown, and finally \( N \) unknown coefficients of linear systems. The evolution of the "steep descent equation" is described.
2.3.1 The One Coefficient Case

A block diagram of the system considered here is

\[ x(t) \rightarrow \frac{1}{a_1 s + 1} \rightarrow y(t) \]

Figure 2.2. A One-Coefficient System

The system equation is thus:
\[ 0 = a_1 \dot{y} + y - x \quad (2.14) \]

Let \( a_1^* \) represent a computed approximation to \( a_1 \), and define an "equation error" \( E(t) \) as follows:
\[ E(t) = a_1^*(t) \dot{y} + y - x \quad (2.15) \]

Subtracting (2.14) from (2.15),
\[ E(t) = (a_1^*(t) - a_1) \dot{y} \quad (2.16) \]

The function of the error \( f(E) \) to be minimized will be chosen as \( E^2 \). (As discussed in Section 2.1, other forms of \( f(E) \) may be chosen such as \(|E|\), \(|E|^3\), \(E^{2n}\) where \( n \) is a positive integer, etc.) We here choose \( E^2 \) mainly for ease of mathematical manipulation.
\[ f(E) = E^2 = (a_1^*(t) - a_1)^2 \dot{y}^2 \quad (2.17) \]

\[ \frac{df(E)}{dt} = 2E \frac{dE}{dt} = 2(a_1^*(t) - a_1) \dot{y}^2 \frac{da_1^*}{dt} + 2(a_1^*(t) - a_1)^2 \dot{y} \ddot{y} \quad (2.18) \]

Define the time variation of \( a_1^* \) in accordance with steep descent concepts as:
\[ \frac{da_1^*}{dt} = -G \frac{\partial E^2}{\partial a_1^*} = -2GE \dot{y} \quad (2.19) \]
Where $G$ is a positive constant.

Substituting (2.16) into (2.19) and rearranging,

$$\frac{d(a^*_1-a_1)}{(a^*_1-a_1)} = -2\epsilon \int y^2 \, dt$$

Integrating (2.20),

$$\left( \frac{a^*_1(t) - a_1}{a^*_c - a_1} \right) = e^{-2G \int_0^t y^2(\tau) \, d\tau}$$

where $a^*_c$ is the initial value of $a^*_1(t)$.

If $y^2(t)$ has a non-zero average value, then it is seen that $\left( a^*_1(t) - a_1 \right) \to 0$ as $t \to \infty$, hence, $a^*_1(t) \to a_1$.

If $y(t)$ is for example sinusoidal, or a continuous random variable, convergence of $a^*_1(t)$ to $a$ is thus always assured.

A simple integral equation results when the time integral of (2.19) is substituted into (2.16)

$$E(t) = (a^*_1 - a_1) \int_0^t 2G y(\tau) \, d\tau$$

Although not immediately obvious from (2.21), the solution for $E(t)$ is easily written by substituting (2.21) into (2.16). Thus,

$$E(t) = (a^*_1 - a_1) \int \int_0^t y^2(\tau) \, d\tau$$

$$E(t) = (a^*_1 - a_1) \int \int_0^t \frac{y^2(\tau) \, d\tau}{(a^*_1 - a_1)}$$

(2.23)
It is interesting to note that equation (2.22) is an integral equation of the second kind with a variable parameter. Equations of this form are often difficult to solve. In this case fortunately, the solution of the integral equation has a very simple form. In cases involving two or more coefficients, however, such a simple solution has not been found.

Although \( a_1 \) was assumed strictly constant in these calculations, intuition suggests that if \( a_1(t) \) were approximately constant (i.e., varied slowly with time), then \( a_1^*(t) \) would track the value of \( a_1(t) \). The allowable maximum rate of change of \( a_1(t) \) to give a certain tracking accuracy is seen to depend on the exponent of \( e \) in equation (2.21).

Since this can be controlled by varying \( G \), we are not necessarily restricted to very slow variations of \( a_1(t) \). Also, since \( y^2(\tau) \) appears under the integral sign it is seen that if \( y^2(\tau) \) is a slowly varying quantity, the convergence rate will be slow if \( y^2(t) \) has small values in time intervals where \( a_1(t) \) undergoes relatively rapid variations. If \( y^2(t) \) were a rapidly oscillating quantity, however, the difference of \( a_1^*(t) \) and \( a_1(t) \) would quickly begin driving \( a_1^*(t) \) toward \( a_1(t) \). These intuitive suggestions are reinforced in what follows.

Some interesting conclusions are deducible when the system is excited by a single sinusoid of frequency \( \omega \) and
amplitude $A$. By adjusting the initial conditions of $x(t)$ and $y(t)$ so as to make the transient term and the phase angle disappear, $y(t)$ can be expressed as:

$$y(t) = \frac{A}{\sqrt{a_1^2 \omega^2 + 1}} \cos \omega t$$  \hspace{1cm} (2.24)

Substituting (2.24) into equations (2.22) and (2.23) and performing the integration yields:

$$\left( \frac{a^*(t) - a_1}{a^*_{lc} - a_1} \right) = e^{-A^2 \omega^2} e^{\frac{-i \omega}{a_1^2 \omega^2 + 1}} (t - \frac{\sin \omega t}{\omega})$$ \hspace{1cm} (2.25)

$$E(t) = -A \omega (a^*_{lc} - a_1) \sin \omega t \cdot e^{-A^2 \omega^2} (t - \frac{\sin \omega t}{\omega})$$ \hspace{1cm} (2.26)

It is apparent that $a^*(t)$ converges to $a_1$ as $t \to \infty$; it is also apparent that $c$, $A$, and $\omega$ should be as large as possible for rapid convergence.

In this example of the coefficient computation process the simple forms of the equations permitted a direct solution for the instantaneous value of the computed coefficient. When the number of computed coefficients is greater than one, however, difficulties result from the coupling action of the equation error. In these more complex situations, explicit solutions of the equations are difficult, and an examination of the properties of the
solutions of the equations must be based upon other factors. In these cases it was found advantageous to utilize the capabilities of an electronic analog computer.

2.3.2 The Two Coefficient Case

A block diagram of the system considered here is:

\[
\begin{array}{c}
x(t) \rightarrow \\
\frac{K}{a_1 s + 1} \rightarrow y(t)
\end{array}
\]

Figure 2.3. A Two-Coefficient System.

The system equation is thus:

\[0 = a_{\frac{1}{2}} \dot{y} + y - Kx \quad (2.27)\]

Let \(a^*(t)\) and \(K^*(t)\) represent computed approximations to the constant coefficients \(a_{\frac{1}{2}}\) and \(K\), respectively. Define the equation error as follows:

\[E(t) = a^*(t) \dot{y} - y - K^*(t) x \quad (2.28)\]

Subtracting (2.27) from (2.28) yields:

\[E(t) = [a^*(t) - a_{\frac{1}{2}}] \dot{y} - [K^*(t) - K] x \quad (2.29)\]

Obviously when \(a^*(t) = a_{\frac{1}{2}}\) and \(K^*(t) = K\), \(E = 0\).

However, there are many functions \(a^*(t)\) and \(K^*(t)\) different from \(a^*(t) = a_{\frac{1}{2}}, K^*(t) = K\) which produce \(E = 0\), for example,

\[\text{If } a^*(t) = a_{\frac{1}{2}} + x(t) \quad (2.30)\]

\[K^*(t) = K + y(t),\]
then substitution of (2.30) into (2.29) yields:

\[ E = x(t)\dot{y}(t) - \dot{y}(t)x(t) = 0 \]  

(2.31)

Thus the requirement that \( E \to 0 \) is necessary but not sufficient to ensure \( a_1^* \to a_1 \), and \( K^* \to K \).

The steep descent method will be used to define the time derivatives of \( a_1^*(t) \) and \( K^*(t) \). As discussed in Section 2.1, the error function \( f(E) \) to be used here will be \( E^2 \), although there are an infinite number of other possible choices. Some of the basic ideas of the steep descent method which is essentially a "gradient method" will now be presented.

Suppose in equation (2.28) or (2.29) that we choose to consider \( E \) to be a function of \( a_1^*, K^*, t \). Then

\[ E^2 = E^2 (a_1^*, K^*, t) \]  

(2.32)

and

\[ \frac{dE^2}{dt} = \frac{\partial E^2}{\partial a_1^*} \frac{da_1^*}{dt} + \frac{\partial E^2}{\partial K^*} \frac{dK^*}{dt} + \frac{\partial E^2}{\partial t} \]  

(2.33)

Rewriting equation (2.33) and letting \( e_a \) and \( e_k \) be unit vectors,

\[ \frac{dE^2}{dt} = \left( \frac{\partial E^2}{\partial a_1^*} e_a + \frac{\partial E^2}{\partial K^*} e_k \right) \cdot \left( \frac{da_1^*}{dt} e_a + \frac{dK^*}{dt} e_k \right) + \frac{\partial E^2}{\partial t} \]  

(2.34)
The two vectors whose dot product appears in equation (2.34) can be represented geometrically as follows:

Equation (2.34) can also be expressed as:

\[
\frac{dE^2}{dt} = \sqrt{\left(\frac{dE^2}{da_1^*}\right)^2 + \left(\frac{dE^2}{dK^*}\right)^2} \left[\left(\frac{da_1^*}{dt}\right)^2 + \left(\frac{dK^*}{dt}\right)^2\right] \cos \theta + \frac{dE^2}{dt}
\]

(2.35)

No defining equations have yet been given for \(a_1^*(t)\) or \(K^*(t)\). The point of view taken at this stage of the development is that it is desired to somehow force \(a_1^*(t)\) toward \(a_1\) and \(K^*(t)\) toward \(K\). Since \(E(t)\) is easily mechanized in practical cases, it is desirable to perform operations on \(E(t)\) so that this convergence of \(a_1^*(t)\) to \(a_1\) and \(K^*(t)\) to \(K\) is assured. Equation (2.29) indicates that a necessary (but not sufficient) condition for \(K^*(t) = K\) and \(a_1^*(t) = a_1\) is that \(E = 0\). As a first step in forcing
coefficient convergence, then, it appears reasonable to force $E$ (or $E^2$) to zero.

From equation (2.35) it is seen that if the quantity under the radical is large (larger than $\frac{\partial E^2}{\partial t}$) and if $\cos \theta = -1$, then $\frac{dE^2}{dt}$ will have a large negative value, hence $E^2$ will decrease rapidly and monotonically.

The requirement that $\cos \theta = -1$, simply means that the angle $\theta$ between the two vectors is $180^\circ$, i.e., the two vectors of equation (2.34) must be directed in opposite directions, as shown below:

![Figure 2.5. Geometrical Representation of the Vectors Whose Dot Product Appears in Equation (2.34) When $\cos \theta = -1$.]
Another way of stating that these two vectors are oriented in opposite directions is:

\[
\frac{da^*_1}{dt} ea + \frac{dK^*}{dt} ek = - G \left( \frac{\partial E^2}{\partial a^*_1} ea + \frac{\partial E^2}{\partial K^*} ek \right) \tag{2.36}
\]

Where \( G \) is any time varying or constant positive number.

For convenience and simplicity, \( G \) has been assumed constant in this work, although the intentional variation of \( G \) according to some predetermined plan may reveal advantages in terms of coefficient convergence times.

Equating the vector components of (2.36), we obtain the steep descent equations:

\[
\frac{da^*_1}{dt} = - G \frac{\partial E^2}{\partial a^*_1} = - 2GE_y \tag{2.37}
\]

\[
\frac{dK^*}{dt} = - G \frac{\partial E^2}{\partial K^*} = + 2GE_x \tag{2.38}
\]

The above equations have been called "steepest descent" equations, but the existence of the term \( \frac{\partial E^2}{\partial t} \) in equation (2.34) indicates that the most rapid (timewise) rate of decrease of \( E^2(t) \) is not necessarily along the gradient \( \frac{\partial E^2}{\partial a^*_1} ea + \frac{\partial E^2}{\partial K^*} ek \), hence the quantity \( E^2(t) \) is not necessarily proceeding along a path of "steepest descent." Equations (2.37) and (2.38) are thus perhaps more properly called equations of steep descent.
For convenience, the important equations developed in this section are assembled below:

**System equation:**
\[
0 = a_1 \dot{y} + y - Kx \tag{2.27}
\]

**Equation error:**
\[
E = a_1^* \dot{y} + y - K^*x \tag{2.28}
\]

or
\[
E = (a_1^* - a_1) \dot{y} - (K^* - K)x \tag{2.29}
\]

**Equations of steep descent:**
\[
\frac{da_1^*}{dt} = -G \frac{\partial E}{\partial a_1^*} = -2GE\dot{y} \tag{2.37}
\]
\[
\frac{dK^*}{dt} = -G \frac{\partial E}{\partial K^*} = +2GE\dot{x} \tag{2.38}
\]

The integral equation which yields the time variation of \(E(t)\) as a solution can be found by substituting the integrals of equation (2.37) and (2.38) into (2.29):
\[
E(t) = \dot{y}(t) \left[ a_1^* - a_1 - 2G \int_{\tau=0}^{\tau=t} E(\tau)\dot{y}(\tau)d\tau \right] - x(t) \left[ K^* - K + 2G \int_{\tau=0}^{\tau=t} E(\tau)x(\tau)d\tau \right] \tag{2.39}
\]

Equation (2.39) is an integral equation of the second kind with a variable parameter. As discussed in section 2.1, the solution of this integral equation is difficult. The mathematical solution of this integral equation by the methods of successive substitutions or successive approximations is unwieldy, and the simpler method of investigating the properties of the solutions \(a_1^*(t), K^*(t)\) and \(E(t)\) on an
electronic analog computer will be followed. This two coefficient case is investigated in detail in the next chapter.

The question of the convergence of $a_1^*(t)$ to $a_1$ and of $K^*(t)$ to $K$ was discussed in Section 2.1. Similar to that discussion, we can define $R^2$ as (see Fig. 2.6):

$$R^2(t) = (a_1^*(t) - a_1)^2 + (K^*(t) - K)^2$$  \hspace{1cm} (2.40)

From the figure, $R^2$ is clearly the square of the Euclidean radius from the answer point to the coordinates of the moving point, $(K^*(t), a_1^*(t))$.

Taking the time derivative of (2.40) yields

$$\frac{dR^2}{dt} = 2 (a_1^* - a_1) \frac{da_1^*}{dt} + 2(K^* - K) \frac{dK^*}{dt}$$ \hspace{1cm} (2.41)
Substituting (2.37), (2.38) and then (2.29) into equation (2.41) yields:

\[
\frac{dR^2}{dt} = - 4CE^2
\]  

(2.42)

The discussion in Section 2.1 presented an argument indicating that \( R^2(t) \) would approach zero if two conditions (2.11) and (2.13) were satisfied; as interpreted in this two coefficient problem, these two conditions require that:

(a) The functions \( x(t) \) and \( y(t) \) are continuous.

(2.43)

(b) \( E^2 \) is not identically zero for \( R^2 \neq 0 \).  

(2.44)

Requirement (2.43) is easily satisfied in most practical cases; requirement (2.44), however, can be used to define a class of impermissible inputs—i.e., inputs \( x(t) \) which permit \( E^2 \) to be identically zero for \( R^2 \neq 0 \). This class of inputs can be found from the system equation (2.27) and the equation given (2.29) with \( E(t) \) set identically equal to zero, i.e.,

\[
0 = a_1^2 y + y - Kx
\]

(2.27)

\[
0 = (a_1^2 - a_2) y - (K^* - K)x
\]

(2.45)

Since \( E = 0 \), equations (2.37) and (2.38) indicate that \( a_1^2(t) \) and \( K^*(t) \) are constant.

Since \( R^2 \neq 0 \), any of the following conditions may prevail:

(a) \( a_1^2 = a_2^2, K^* \neq K \)  

(2.46)
If condition (2.46) prevails, then \( x(t) = 0 \) by equation (2.45). Similarly if condition (2.47) prevails, then \( y = 0 \). Thus (2.46) and (2.47) yield trivial cases. If condition (2.48) prevails, then from (2.45):

\[
\frac{\dot{y}}{x} = \frac{K^* - K}{a_1^* - a_1} = \text{constant} = c_1
\]  

(2.49)

Assuming \( K \neq 0 \) in equation (2.27) [this problem is trivial if \( K = 0 \), since there is then no relationship between \( x(t) \) and \( y(t) \)], an expression for \( \dot{y} \) may be obtained:

\[
\frac{\dot{y}}{x} = \frac{K}{a_1} - \frac{y}{a_1 x}
\]  

(2.50)

Equating (2.49) and (2.50) yields:

\[
y = (K - c_1 a_1)x
\]  

(2.51)

Substituting (2.51) into (2.27) yields:

\[
a_1 (K - c_1 a_1) \ddot{x} + (-c_1 a_1) x = 0
\]  

(2.52)

Thus, solving (2.52) for \( x \):

\[
x(t) = c_0 e^{\frac{c_1}{c_1 a_1 - K} t}
\]  

(2.53)

If the constant \( \frac{c_1}{c_1 a_1 - K} \) is denoted by \( c_2 \), (2.53) can be written as:

\[
x(t) = c_0 e^{c_2 t}
\]  

(2.54)
Equation (2.54) defines the class of impermissible inputs referred to above. If \( x(t) \) has the form of (2.54) then it is possible for \( E \) to be identically equal to zero over a finite or infinite time interval when \( R^2 \neq 0 \). Note that (2.54) allows the following cases:

\[
\begin{align*}
(a) & \quad x(t) = 0. \\
(b) & \quad x(t) \text{ is a specific decreasing exponential.} \\
(c) & \quad x(t) \text{ is a specific increasing exponential.} \\
(d) & \quad x(t) \text{ is a constant.}
\end{align*}
\]

Thus, the inputs permissible to the system must be restricted so that they are not describable over any finite or infinite time interval by those of (2.55). It is inferred from the discussion above that when the system inputs are random signals, periodic signals, etc., that we expect convergence of \( a^*(t) \) to \( a \), and \( K^*(t) \) to \( K \). Analog computer results, to be discussed later, confirm this expectation.

2.3.3 The Multi-Coefficient Case

A block diagram of the system considered here is:

\[
\begin{align*}
x(t) & \quad \frac{b_m S^m + b_{m-1} S^{m-1} + \ldots + b_1 S + b_0}{a_n S^n + a_{n-1} S^{n-1} + \ldots + a_1 S + 1} \quad y(t)
\end{align*}
\]

Figure 2.7. A General Multi-Coefficient System
The system equation is thus:

\[ 0 = y(t) + \sum_{i=1}^{n} a_i^{(1)} y(t) - \sum_{k=0}^{m} b_k^{(k)} x(t) \]  \hspace{1cm} (2.56)

Let \( a_i^*(t) \) and \( b_k^*(t) \) represent computed approximations to the constant coefficients \( a_i \) and \( b_k \), respectively. Define the equation error as follows:

\[ E(t) = y(t) + \sum_{i=1}^{n} a_i^*(t) y(t) - \sum_{k=0}^{m} b_k^*(t) x(t) \]  \hspace{1cm} (2.57)

Subtracting (2.56) from (2.57) yields:

\[ E(t) = \sum_{i=1}^{n} [a_i^*(t) - a_i] y(t) - \sum_{k=0}^{m} [b_k^*(t) - b_k] x(t) \]  \hspace{1cm} (2.58)

The steep descent equations follow from arguments identical with the argument given for the two coefficient case:

\[ \frac{da_i^*}{dt} = - G \frac{\partial E^2}{\partial a_i^{(1)}} = - 2G E y \]  \hspace{1cm} (2.59)

\[ \frac{db_k^*}{dt} = - G \frac{\partial E^2}{\partial b_k^{(k)}} = + 2G E x \]  \hspace{1cm} (2.60)

The integral equation which yields the time variation of \( E(t) \) as a solution can be found by substituting the integrals of equations (2.59) and (2.60) into equation (2.58).
Equation (2.61) is an integral equation of the second kind with a variable parameter. As mentioned previously, the time solution of \( E(t) \) is difficult to express analytically, and requires the evaluation of many integrals when solved by either the method of successive substitutions or the method of successive approximations.

As discussed in Section 2.1, the question of the convergence of \( a^*_i(t) \) to \( a_i \) and of \( b^*_k(t) \) to \( b_k \) was investigated by defining \( R^2(t) \); in this case, \( R^2(t) \) can be defined as:

\[
R^2(t) = \sum_{i=1}^{n} (a_i^* - a_i)^2 + \sum_{k=0}^{m} (b_k^* - b_k)^2
\]

Taking the time derivative of (2.62), and substituting the equations of (2.59), (2.60) and (2.58) yields:

\[
\frac{dR^2}{dt} = -4GE^2
\]
multi-coefficient problem, these two conditions require that:

(a) The functions \( y \) and \( x \), for all \( i,k \), are continuous.

\[(2.64)\]

(b) \( E^2 \) is not identically zero for \( R^2 \neq 0 \).

\[(2.65)\]

Requirement (2.64) is easily satisfied in most physical problems; requirement (2.65) however can be used to define a class of impermissible inputs—i.e., inputs \( x(t) \) which permit \( E^2 \) to be identically zero for \( R^2 \neq 0 \). In the discussion of the two coefficient case, this entire class of impermissible inputs was defined. In this multi-coefficient case, however, it has not been possible to define the entire class; instead, it has been possible, at present, to identify only one sub-class of these inputs, viz., the class of periodic inputs which will not necessarily produce coefficient convergence. In the following discussion it will be shown that if there are \( N \) unknown coefficients being computed in the system, then \( E^2 \) can be zero when \( R^2 \neq 0 \) if there are less than \( N/2 \) different frequencies in the input; it is more difficult to specify the conditions under which \( R^2 \) is forced to be identically zero when \( E^2 \) is identically zero. A conjecture is presented, however, and an example involving four coefficients is presented which supports the conjecture.
Setting $E = 0$ in equations (2.59) and (2.60) shows that $a_i^*$ and $b_k^*$ are constants. With $E = 0$, equation (2.57) becomes:

$$0 = y + \sum_{i=1}^{n} a_i^* y_i - \sum_{k=0}^{m} b_k^* x_k$$

(2.66)

If $x(t)$ is periodic, containing frequency components $\omega_p$, where $p = 1, 2, \ldots$, then by the principle of superposition, the components of the output $y_p$ and the components of the input $x_p$ at frequency $\omega_p$ have a steady state relationship deducible from (2.66) as follows:

$$\frac{y_p}{x_p} (j\omega_p) = \sum_{k=0}^{m} b_k^* (j\omega_p)^k$$

$$\sum_{i=0}^{n} a_i^* (j\omega_p)^i + 1$$

(2.67)

where $\frac{y_p}{x_p} (j\omega_p)$ is to be interpreted as a sinusoidal transfer function in the usual way. Similarly, equation (2.56) yields:

$$\frac{y_p}{x_p} (j\omega_p) = \sum_{k=0}^{m} b_k (j\omega_p)^k$$

$$\sum_{i=0}^{n} a_i (j\omega_p)^i + 1$$

(2.68)
Equating expressions (2.67) and (2.68) yields:

\[
\sum_{k=0}^{m} b_k(j\omega_p)^k = \sum_{k=0}^{m} b_k^*(j\omega_p)^k
\]

Equation (2.69) may be cross multiplied to yield:

\[
\left[ \sum_{i=1}^{n} \sum_{k=0}^{m} a_i^* b_k (j\omega_p)^{i+k} + \sum_{k=0}^{m} b_k(j\omega_p)^k \right] = \left[ \sum_{i=1}^{n} \sum_{k=0}^{m} a_i b_k^* (j\omega_p)^{i+k} + \sum_{k=0}^{m} b_k^*(j\omega_p)^k \right]
\]

Equation (2.70) can be rewritten as:

\[
\sum_{i=1}^{n} a_i^* \sum_{k=0}^{m} b_k (j\omega_p)^{i+k} - \sum_{k=0}^{m} b_k^* \left( \sum_{i=1}^{n} a_i(j\omega_p)^{i+k} + (j\omega_p)^k \right) = - \sum_{k=0}^{m} b_k (j\omega_p)^k
\]

Although awkward to write in the general case, equation (2.71) yields two equations when the imaginary parts are equated and when the real parts are equated. These two equations can be written in the following matrix form.
Equating real parts:

\[
\begin{bmatrix}
  c_1 p & \ldots & c_{(m+n+1)p}
\end{bmatrix}
\begin{bmatrix}
  a_1^* \\
  a_2^* \\
  \vdots \\
  a_n^* \\
  b_0^* \\
  b_1^* \\
  \vdots \\
  b_m^*
\end{bmatrix}
= c_p
\] (2.72)

Equating imaginary parts:

\[
\begin{bmatrix}
  d_1 p & \ldots & d_{(m+n+1)p}
\end{bmatrix}
\begin{bmatrix}
  a_1^* \\
  a_2^* \\
  \vdots \\
  a_n^* \\
  b_0^* \\
  b_1^* \\
  \vdots \\
  b_m^*
\end{bmatrix}
= d_p
\] (2.73)

where all of the c's and d's are real numbers and

where:

\[
c_{hp} + j d_{hp} = \sum_{k=0}^{m} b_k(j \omega_p)^{h+k}, \text{ for } h = 1, 2, \ldots, n
\]

\[
c_{hp} + j d_{hp} = \sum_{i=1}^{n} a_i(j \omega_p)^{i+h-n-1}(j \omega_p)^{n-h-1}, \text{ for } h = n+1, \ldots, m+n+1
\]

\[
c_p + j d_p = \sum_{k=0}^{m} b_k(j \omega_p)^k
\]
The c's and d's are functions of $\alpha_p$, $a_1$ and $b_k$, and hence, in any specific calculation, would appear as constants.

Equations (2.72) and (2.73) can be combined into the same matrix equation as:

$$
\begin{bmatrix}
  c_{1p} & \cdots & c_{(m+n+1)p} \\
  d_{1p} & \cdots & d_{(m+n+1)p}
\end{bmatrix}
\begin{bmatrix}
  a_1^* \\
  \vdots \\
  a_m^* \\
  b_0^* \\
  \vdots \\
  b_m^*
\end{bmatrix}
= 
\begin{bmatrix}
  c_p \\
  d_p
\end{bmatrix}
$$  \hspace{1cm} (2.74)

If the system were excited by M different frequencies, where $p = 1, 2, \ldots, M$, then equation (2.74) would appear as:

$$
\begin{bmatrix}
  c_{11} & c_{21} & \cdots & c_{(m+n+1)1} \\
  d_{11} & d_{21} & \cdots & d_{(m+n+1)1} \\
  c_{12} & c_{22} & \cdots & c_{(m+n+1)2} \\
  d_{12} & d_{22} & \cdots & d_{(m+n+1)2} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{1M} & c_{2M} & \cdots & c_{(m+n+1)M} \\
  d_{1M} & d_{2M} & \cdots & d_{(m+n+1)M}
\end{bmatrix}
\begin{bmatrix}
  a_1^* \\
  \vdots \\
  a_m^* \\
  b_0^* \\
  \vdots \\
  b_m^*
\end{bmatrix}
= 
\begin{bmatrix}
  c_1 \\
  d_1 \\
  c_2 \\
  d_2 \\
  \vdots \\
  c_M \\
  d_M
\end{bmatrix}
$$  \hspace{1cm} (2.75)
If \( m + n + 1 > 2M \), then \((m + n + 1 - 2M)\) rows of zeros can be added to the rectangular matrix on the left side of (2.58) to form a square matrix.

\[
\begin{bmatrix}
c_{11} & c_{21} & \cdots & c_{(m+n+1)1} \\
\vdots & \vdots & \ddots & \vdots \\
d_{1M} & d_{2M} & \cdots & d_{(m+n+1)M} \\
0 & 0 & \cdots & 0 \\
- & - & \cdots & - \\
0 & 0 & \cdots & 0 \\
\end{bmatrix}
\begin{bmatrix}
a_1^* \\
\vdots \\
a_m^* \\
b_0^* \\
b_m^* \\
\vdots \\
d_M \\
\end{bmatrix} =
\begin{bmatrix}
c_1 \\
d_1 \\
c_2 \\
\vdots \\
\vdots \\
d_M \\
\end{bmatrix}
\tag{2.76}
\]

Since the determinant of the square matrix is zero, there is no unique solution for \( a_i^* \), \( b_k^* \). Thus, when there are \( N \) unknown coefficients, and where there are less than \( N/2 \) frequencies in the input signal, there are many sets of values \( a_i^* \), \( b_k^* \) which satisfy the equations \( E = 0 \), \( R^2 \neq 0 \).

If \( m+n+1 \leq 2M \) then any \((m+n+1)\) rows of the rectangular matrix can be chosen to form a square matrix. The key question at this point is whether the determinant of the square matrix can possibly have a zero value. If the determinant has a non-zero value, then the matrix has an inverse, and we are assured of a unique solution.\(^{23}\) It is obvious in equation (2.69) that the equation is satisfied when \( a_i^* = a_i \) and \( b_k^* = b_k \). Hence, since \( a_i^* = a_i \) and \( b_k^* = b_k \) always satisfies (2.75), it is the unique solution when
\[ m+n+1 \leq 2M. \] This result can be stated more succinctly as:

If the determinant of the square matrix is non-zero, then a minimum of \( N/2 \) different excitation frequencies are required to determine \( N \) unknown parameters.

The only remaining question is to determine what conditions must be imposed to guarantee that the determinant of the square matrix is non-zero.

The following condition is conjectured to be a sufficient condition for guaranteeing that this determinant has a non-zero value:

"The determinant of the square matrix discussed above will be zero if and only if the numerator and denominator of the system transfer function have no common factors, and if no parameters have zero values."

Attempts at proving the conjecture in its most general form have proven very difficult due to notational difficulties. This conjecture has proven to be correct, however, in several investigated particular cases. An example demonstrating the validity of this conjecture in such a particular case follows.

The block diagram of this four coefficient case considered is:

\[
\begin{array}{c}
\text{x(t)} \\
\end{array} \xrightarrow{b_1s+b_0\over a_2s^2+a_1s+1} \begin{array}{c}
\text{y(t)} \\
\end{array}
\]

Figure 2.8. A Four-Coefficient System.
The equation corresponding to equation (2.67) is:

\[
\frac{V(j\omega)}{X} = \frac{b_1 j\omega_p + b_o}{1-a_2^m\omega_p^2 + a_1 j\omega_p} \tag{2.77}
\]

The equation corresponding to equation (2.68) is:

\[
\frac{V(j\omega)}{X} = \frac{b_1 j\omega_p + b_o}{1-a_2^m\omega_p^2 + a_1 j\omega_p} \tag{2.78}
\]

Equating (2.72) and (2.78) yields:

\[
\frac{b_1 j\omega_p + b_o}{1-a_2^m\omega_p + a_1 j\omega_p} = \frac{b_1^* j\omega_p + b_o^*}{1-a_2^m\omega_p^2 + a_1^* j\omega_p} \tag{2.79}
\]

Cross multiplying equation (2.79) and simplifying yields:

\[
\begin{bmatrix}
-a_2^m \omega_p^2 b_o a_1^* \omega_p^2 + b_1^* a_1 \omega_p^2 - b_o^* (1-a_2^m \omega_p^2) \\
-a_2^m a_1 \omega_p b_o - b_1 \omega_p (1-a_2^m \omega_p^2) - b_o^* a_1 \omega_p
\end{bmatrix}
= 
\begin{bmatrix}
-b_o \\
-j b_1 \omega_p
\end{bmatrix} \tag{2.80}
\]

Equation (2.80) yields two separate equations when the imaginary parts are equated and when the real parts are equated. These equations can be written in matrix form as follows:

\[
\begin{bmatrix}
b_1 a_1^* \\
b_1 a_2^* \\
b_o^* \\
b_1^*
\end{bmatrix}
= 
\begin{bmatrix}
b_o \\
b_1 \omega_p
\end{bmatrix} \tag{2.81}
\]
Obviously, if there is only one frequency exciting the system, 2 additional rows of zeros can be added to (2.81) to form a square matrix, and the determinant of the square matrix will obviously be zero. Hence, \(a_1^*, a_2^*, b_0^*, b_1^*\), will not have a unique solution.

If it can be shown that \((a_1^*, a_2^*, b_0^*, b_1^*)\) has some unique solution, then it is apparent that the unique solution must be \((a_1, a_2, b_0, b_1)\), since equation (2.79) is always satisfied for these values.

If the system is excited by at least two different frequencies, \(\omega_1, \omega_2\), the matrix equation resulting from (2.81) by setting \(p = 1, 2\) is:

\[
\begin{bmatrix}
  b_1 \omega_1^2 & b_0 \omega_1^2 & 1 - a_2 \omega_1^2 & -a_1 \omega_1^2 \\
  -b_0 \omega_1 & b_1 \omega_1^3 & a_1 \omega_1 & \omega_1 (1 - a_2 \omega_1^2) \\
  b_1 \omega_2^2 & b_0 \omega_2^2 & 1 - a_2 \omega_2^2 & -a_1 \omega_2^2 \\
  -b_0 \omega_2 & b_1 \omega_2^3 & a_1 \omega_2 & \omega_2 (1 - a_2 \omega_2^2)
\end{bmatrix}
\begin{bmatrix}
a_1^* \\
a_2^* \\
b_0^* \\
b_1^*
\end{bmatrix}
= \begin{bmatrix}
b_0 \\
b_1 \omega_1 \\
b_0 \\
b_1 \omega_2
\end{bmatrix}
\tag{2.82}
\]

or rewriting the above equation is more abbreviated notation:

\[[C] \vec{a} = \vec{b}\]

If it can be demonstrated that the determinant of the square matrix \((c)\) of equation (2.82) does not have a value of zero, then \((a_1^*, a_2^*, b_0^*, b_1^*)\) has a unique solution, as explained above.
The next steps will be to explore the conditions for $\det C = 0$, and to state constraints that preclude these conditions.

Setting $|C| = 0$,

\[
\begin{vmatrix}
 b_1 \omega_1^2 & b_0 \omega_1 & 1 - a_2 \omega_1^2 & -a_1 \omega_1^2 \\
-b_0 \omega_1 & b_1 \omega_1^2 & a_1 \omega_1 & \omega_1(1 - a_2 \omega_1^2) \\
b_1 \omega_2^2 & b_0 \omega_2 & 1 - a_2 \omega_2^2 & -a_1 \omega_2^2 \\
-b_0 \omega_2 & b_1 \omega_2^3 & a_1 \omega_2 & \omega_2(1 - a_2 \omega_2^2)
\end{vmatrix} = 0 \quad (2.83)
\]

Subtracting appropriate rows and simplifying (2.83) yields:

\[
\begin{vmatrix}
 0 & 0 & 1 & 0 \\
 0 & b_0 & 0 & -a_2 \\
b_1 & b_0 & \frac{1}{a_2} & -a_2 \\
b_0 & b_1 \omega_2 & a_1 & 1 - a_2 \omega_2^2
\end{vmatrix} = 0 \quad (2.84)
\]

Expanding (2.84) on its first row:

\[
\begin{vmatrix}
 0 & b_1 \omega_2 & -a_2 \omega_2^2 \\
b_1 & b_0 & -a_1 \\
b_0 & b_1 \omega_2 & 1 - a_2 \omega_2^2
\end{vmatrix} = 0 \quad (2.85)
\]

Expanding (2.85) yields:

\[
a_1 b_0 b_1 - a_2 b_0^2 - b_1^2 = 0 \quad (2.86)
\]
A question arises as to the meaning of equation (2.86). To answer this question, consider the division of the numerator of the transfer function into the denominator of the transfer function, carried out as follows (it is assumed that \( b_1 \neq 0 \)):

\[
\frac{a_2 S + (a_1 - a_2 b_0)}{b_1 S + b_0} = \frac{a_2 S^2 + a_1 S + 1}{a_2 S^2 + \frac{a_2 b_0}{b_1} S} = \frac{(a_1 - \frac{a_2 b_0}{b_1}) S + 1}{(a_1 - \frac{a_2 b_0}{b_1}) S + \frac{b_0}{b_1} (a_1 - \frac{a_2 b_0}{b_1})} = 1 - \frac{b_0}{b_1} (a_1 - \frac{a_2 b_0}{b_1})
\]

If the remainder of the above division is zero, then the numerator exactly cancels into the denominator. Setting the remainder equal to zero, then, yields:

\[
\frac{a_1 b_0}{b_1} - \frac{a_2 b_0^2}{b_1^2} - 1 = 0
\]

(2.87)

\[
a_1 b_0 b_1 - a_2 b_0^2 - b_1^2 = 0
\]

Obviously equation (2.86) and (2.87) are identical. Thus it is seen that if the numerator and denominator of the system transfer function have common factors, there is no unique solution for \((a_1, a_2, b_0, b_1)\). Conversely, if there are no common factors in the numerator and denominator of the
system transfer function, and if no parameter has a zero value there is a unique solution. Furthermore, this unique solution is:

\[(a_1^*, a_2^*, b_0^*, b_1^*) = (a_1, a_2, b_0, b_1).\]

2.3.3.1 Discussion of Noise Inputs

If a continuous random signal \(x(t)\) excited the system, then a qualitative argument for coefficient convergence can be given as follows:

![Figure 2.9. Time History of a Continuous Random Signal.](image)

Over any finite interval (which is large compared to the periods of the dominant sinusoidal frequency components), a Fourier series analysis will yield an approximate frequency and amplitude spectrum of the noise signal \(x(t)\). If it can be shown that during this time interval the noise signal contains \(N/2\) components at different frequencies having relatively large amplitudes, then it appears reasonable to conclude that this noise signal will have practically the same effect as a periodic signal, and hence, would be capable of forcing the \(N\) coefficients \(a_i^*, b_k^*\) to converge
to their proper values. That a noise signal can be used to force coefficient convergence has been effectively demonstrated in analog computer tests, and will be described later.

2.3.3.2 Discussion of Transient Inputs

If \( x(t) \rightarrow 0 \) as \( t \rightarrow \infty \), then \( x(t) \) can be called a transient input. Qualitatively, the argument for coefficient convergence follows the same argument as for noise inputs, with the exception that a condition must be added. The time required for convergence of the coefficients must be sufficiently short relative to the duration of the transient.
CHAPTER III

INVESTIGATION OF THE COEFFICIENT CONVERGENCE PROCESS

In this chapter the character of the coefficient convergence processes is investigated by means of graphical and analog computer techniques. Questions of special interest are:

1) Are there special ranges of excitation frequencies which produce more rapid convergence of the coefficients than others?

2) Since the steep descent gain $G$ is arbitrary, is there a special value or range of values which produces the most rapid coefficient convergence?

3) Do analog computer results confirm all of the theory presented in Chapter II?

3.1 The One Coefficient Case

The block diagram and equations for this case are presented in Section 2.2.1.

It was shown in equation (2.21) that the time variation of $a_+^*(t)$ can be written as:

$$
\frac{a^+_1(t) - a_1}{a^*_1 - a} = e^{-2G \int_0^t y^2(\tau) d\tau} 
$$

(2.21)
Where $a^*_{1c}$ is the initial value of $a^*_1(t)$. It was concluded in the discussion of Section 2.2.1 that $a^*_1(t)$ converged to $a_1$ provided $\dot{y}^2(t)$ had a non-zero average value. Thus, there are many time functions $\dot{y}(t)$ which will produce convergence of $a^*_1(t)$ to $a_1$, e.g., periodic functions, random functions, etc.

The time variation of $a^*_1(t)$ when the system is excited by a sinusoidal frequency yields some interesting conclusions. It was shown in equation (2.25) that when the system is excited by a sinusoid of amplitude $A$ and frequency $\omega$ that the computed coefficient $a^*_1(t)$ has a solution given by:

$$\frac{a^*_1(t) - a_1}{a^*_{1c} - a_1} = e^{-GA^2\omega^2 \left( t - \frac{\sin 2\omega t}{2\omega} \right)} \left( \frac{t}{a^2\omega^2 + 1} \right)^{-1}$$

The conclusions from this equation are that:

1) $G$, $A$, and $\omega$ should be as large as possible for rapid convergence.

2) It is of interest to note that to achieve a specific exponential decay rate, the product $GA^2$ must be held constant. This means that if the amplitude $A$ exciting the system is to be small (which is desirable), the gain $G$ must be large; if the gain $G$ is small, the amplitude driving the system must be large. This simple fact is important in
design work where it is desired to keep the system disturbances small.

Analog computer and/or graphical investigations were not considered necessary in this simple case because of the existence of the analytical solution given by equation (2.21).

3.2 The Two Coefficient Case

The block diagram and equations for this case are presented in Section 2.2.2. This two coefficient case exhibits many interesting characteristics totally absent in the one coefficient case, and has suggested generalizations for the N coefficient case at their most elementary level. As discussed previously, analytical solutions for $a^*_1(t)$ and $K^*(t)$ are difficult to obtain. To investigate the properties of the convergence process for $a^*_1(t)$ and $K^*(t)$, graphical and analog computer techniques are employed.

In the special case when the system is driven by a sinusoid, and initial conditions are chosen so as to eliminate the transient in $y$,

$$x = A \cos \omega t$$  \hspace{1cm} (3.1)

$$y = \frac{AK \cos(\omega t+\theta)}{\sqrt{a_1^2 \omega^2 + 1}}$$  \hspace{1cm} (3.2)

where \( \theta = \tan^{-1} a \omega \)  \hspace{1cm} (3.3)
Substituting the above equations into equations (2.28), (2.37) and (2.38),

\[
E = \left\{ \begin{align*}
    & -a_1^* A K \sin(\omega t + \Theta) + \frac{AK}{\sqrt{a_1^2\omega^2 + 1}} \cos\omega t + \Theta \\
    & -K A \cos\omega t
\end{align*} \right\} \quad (3.4)
\]

\[
\frac{da_1^*}{dt} = \frac{2G E A K \omega}{\sqrt{a_1^2\omega^2 + 1}} \sin(\omega t + \Theta) \quad (3.5)
\]

\[
\frac{dK^*}{dt} = + 2G E A \cos\omega t \quad (3.6)
\]

Taking the ratio of (3.5) and (3.6) yields:

\[
\frac{da_1^*}{dK^*} = \frac{K \cos(\omega t + \Theta)}{\sqrt{a_1^2\omega^2 + 1} \cos\omega t} = \frac{K \omega}{a_1^2\omega^2 + 1}(a_1^* \tan \omega t) \quad (3.7)
\]

Equation (3.7) has an interesting interpretation:

If a plot of \(K^*\) vs. \(a_1^*\) is made (refer to Figure 3.1), the initial values of \(K^*\) and \(a_1^*\) may be plotted \((K_{c^*}, a_{1c}^*)\).

If the steep descent process is allowed to proceed, the moving point \((K^*, a_1^*)\) will trace out some path as shown.

At time \(t_1\), the moving point is located at \([K^*(t_1), a_1^*(t_1)]\) and has an instantaneous velocity vector

\[
V(t_1) = \frac{da_1^*}{dt}(t_1) e_a + \frac{dK^*}{dt}(t_1) e_k.
\]

The velocity vector \(V(t_1)\) must make an angle \(\phi(t_1)\) with the \(K^*\) axis (see Figure 3.1), where:

\[
\phi(t_1) = \tan^{-1} \left( \frac{da_1^*}{dK^*}(t_1) \right) \quad (3.8)
\]
\[ a_i^* \]

FIGURE 3.1

A CONVERGENCE PATH OF THE MOVING POINT \((k^*(t), a_i^*(t))\)
and where
\[ \frac{da^*}{dK^*}(t_1) = \frac{K\omega}{a_1^{2}\omega^2+1} [a_1\omega + \tan\omega t_1] \quad (3.9) \]

By choosing values for K and a_1, a plot of \( \phi(t) \) can be made as a function of time. As an example, let K = 1, a = 1. A plot of \( \phi(t_1) \) at various times is given in Figure 3.2, for various frequencies. These plots illustrate a peculiar characteristic: at low frequencies (\( \omega \approx 0.05 \text{ rad/sec} \)) the angle \( \phi(t) \) is nearly zero for a very large percentage of the time in every cycle; at higher frequencies (\( \omega \approx 20 \text{ rad/sec} \)) the angle \( \phi(t) \) is nearly \(-45^\circ \) for a very large percentage of the time in every cycle; only at intermediate frequencies (on the order of 1 or 2 rad/sec) does \( \phi(t) \) vary substantially during every sub-interval of a cycle. This thus implies that the velocity vector \( \mathbf{v}(t) = \frac{da^*}{dt}(t) \mathbf{e}_a + \frac{dK^*}{dt}(t) \mathbf{e}_K \) has "preferred" directions at low and high frequencies. This implication, upon further investigation, leads to some interesting additional features of the convergence characteristics, as will be discussed in the following section.

3.2.1 The Existence of an Optimum Excitation Frequency

Consider the fixed point \((K^*, a^*)\) in the \(K^*, a^*\) plane. If the moving point \([K^*(t), a^*(t)]\) passes through the point
FIGURE 3.2

PLOTS OF $\phi(t) = \tan^{-1} \frac{da^*}{dt}$, AT VARIOUS FREQUENCIES
\[ (3.10) \]

\[
\mathbf{V} = \frac{\partial \mathbf{a}}{\partial t}(t) \quad \mathbf{V}_a = \frac{\partial \mathbf{K}^*(t)}{\partial t} \quad \mathbf{e}_K
\]

\[ a^*, K^* \]

\[ P^*, P \]

\[ a^* = a^*_P \]

\[ K^* = K^*_P \]

\[ V^* = V^*_P \]

\[ \mathbf{V} \] can be calculated in (3.10) if the time \( t \) is specified, along with the values for \( K \) and \( a \). If the moving point \([K^*(t), a^*(t)]\) is considered to pass through point \([K^*_P, a^*_P] \) at times \((t_1, t_2, t_3, \ldots, t_n)\), then the various values of the velocity vector \( \mathbf{V}(t_1) \) can be computed (for a given \( a, K, \omega \)) from equations (3.4), (3.5), (3.6) and (3.10). These various values of the vectors \( \mathbf{V}(t_1) \) then describe some of the possible velocity vectors which the moving point \([K^*(t), a^*(t)] \) can have as it passes through \([K^*_P, a^*_P] \). By taking the values of \( t_1, t_2, t_3, \ldots, t_n \) very close together, an envelope can be drawn for all possible velocity vectors that the point \([K^*(t), a^*(t)] \) can have as it passes through the fixed point \([K^*_P, a^*_P] \). The points on this envelope can be computed in a conventional manner by calculating values by hand and plotting points; a more efficient way, however, is to automatically make these plots by using an x-y recorder in conjunction with the analog computer. A comparison of the curves obtained by the two methods indicates that they are identical, as expected.
The analog computer mechanization for this problem is shown in Figure 3.3. A plot of the velocity vector envelopes for various choices of the point \((K, a)\) are given in Figures 3.4, 3.5, and 3.6. Since the interpretation of these figures can be easily misunderstood, they will be explained in detail. Referring to Figure 3.4, the velocity vector envelopes located at the point \(K^* = 0, a^* = 1\) will be discussed. To construct the velocity vectors at this point the following procedure can be followed:

1) Choose a value of \((a, K) = (1, 1)\).

2) The origin for the velocity vectors associated with the point \((y^*, a^*_p) = (0, 1)\) is placed at \((0, 1)\).

3) Choosing a frequency of say \(\omega = \text{rad/sec}\), all possible velocities possessed by the moving point \((K^*(t), a^*(t))\) as it passes through the point \((0, 1)\) can be computed from equations (3.4), (3.5), (3.6) and (3.10). When these velocity vectors are plotted, and the tips of their "arrows" are connected, they form the curve marked \(\omega = 1\) (located on Figure 3.4 at \((x^*, a^*) = (0, 1)\)). Since the origin of these vectors is located at \((x^*, a^*_p) = (0, 1)\), the plot of Figure 3.4 reveals the fact that no velocity vector associated a frequency of \(\omega = 1 \text{ rad/sec}\) has radial components pointing away from the point \((K^*, a^*) = (1, 1)\). Velocity vector enveloped at other frequencies and at other points \((K^*_p, a^*_p)\) in the \(K^*, a^*_p\) plane are plotted similarly.
FIGURE 3.3

ANALOG COMPUTER MECHANIZATION OF A TWO COEFFICIENT COMPUTER
FIGURE 3.4

VELOCITY VECTOR ENVELOPES FOR \((k, \alpha) = (1, 1)\) IN THE \(k^*, \alpha^*\) PLANE
FIGURE 3.5

VELOCITY VECTOR ENVELOPES FOR \((k, \eta) = (2,1)\) IN THE \(k^*, \eta^*\) PLANE
Figure 3.6

Velocity vector envelopes for \((k, \omega) = (1, 2)\) in the \(k^*, \omega^*\) plane
Note that since the envelopes of the velocity vectors are tangent to concentric circles centered about \((a, K)\), the moving point \(a^*, K^*\) can never move out of the circle. This vividly illustrates the previously demonstrated fact that the Euclidean radius, \(R = \sqrt{(a^*_1 - a_1)^2 + (K^* - K)^2}\) is a monotonically decreasing function, as is evident from equation (2.42).

Inspection of Figure 3.4 suggests that for specific values of \(\omega, K,\) and \(a\), the "shape" and "orientation" of the velocity vector envelopes do not change at different points in the \(a^*, K^*\) plane, although the "size" of the envelope does change. It has been found in analog computer investigations that the orientation and shape are invariant in the \(a^*, K^*\) plane at any specific set of values of \(a, K, \omega\) and \(G\) (see Figures, 3.4, 3.5 and 3.6). For this reason, it is possible to examine the shape and orientation of the velocity vector envelopes at one point and draw general conclusions regarding all points in the \(K^*, a^*\) plane.

Figures 3.7, 3.8 and 3.9 are plots of the velocity vector envelopes at various frequencies at specific points in the \(K^*, a^*\) plane for various values of \((K, a)\).

It is apparent from Figures 3.7, 3.8, and 3.9 that at high and low frequencies the components of the velocity vectors are very small in directions perpendicular to the "preferred" directions, hence the time required for the
FIGURE 3.7

VELOCITY VECTOR ENVELOPES FOR $(k,a) = (1,1)$ AT $(k^*,a^*) = (0,1)$
FIGURE 3.8

VELOCITY VECTOR ENVELOPES FOR $(k, a) = (2, 1)$ AT
$(k^*, a^*) = (1, 1)$
FIGURE 3.9

VELOCITY VECTOR ENVELOPES FOR \((K,a) = (1,2)\)
AT \((K^*,a^*_k) = (0,2)\)
moving point to move in directions perpendicular to the "preferred" directions is long, while, in contrast, the time required for the moving point to move in the "preferred" directions is short. Since it is desirable for the moving point to converge to the "answer point" as rapidly as possible, regardless of its initial position, it is undesirable to excite the system at excessively high or low frequencies. From the shape of the velocity envelope for $\omega = 1 \text{ rad/sec}$ of Figure 3.7 it is seen that excitations near this frequency permit relatively large velocities in all directions—hence, relatively rapid convergence of the moving point to the answer point is expected regardless of the initial position of the moving point. The frequencies near $\omega = 1 \text{ rad/sec}$ in this example are thus seen to be "optimum" frequencies of excitation. The "optimum" frequency in Figure 3.8 is seen to be in the vicinity of .5 to 1.0 rad/sec, while the optimum frequency of Figure 3.9 is seen to be .5 rad/sec.

It is of interest to note the relationship of the "optimum" frequency to the corner frequency of the system. Obviously, the two frequencies coincide. Although much more substantiation of the statement is required, it has been noted during the course of these investigations that the optimum frequencies of excitation have in all cases been in the vicinity of the corner frequencies. In the four
coefficient cases investigated, where there are as many as three corner frequencies, but where only two excitation frequencies are required for coefficient convergency, good (i.e., rapid) convergence properties of the coefficients could be obtained by placing the two excitation frequencies over the range of the three corner frequencies. When the two excitation frequencies were placed far removed from the corner frequencies, very slow coefficient convergence resulted.

3.2.2 Convergence Paths as a Function of Frequency

Since the Euclidean radius \( R = \sqrt{(a^*_1 - a)^2 + (K^* - K)^2} \) has been shown to be a monotonically decreasing function, the "convergence" paths of the moving point can be deduced.

Figure 3.10 presents the "deduced" convergence paths of \((a^*_1, K^*)\) to \((a, K)\) as a function of frequency \(\omega\). For high frequencies it was demonstrated that the preferred direction of the velocity vector had a slope of minus one. Hence, referring to Figure 3.10, the point \(a^*, K^*\) will move rapidly from an initial point \((a^*_c, K^*_c)\) to point \(Q\) along path \(P_a\), and then very slowly from point \(Q\) to \((1,1)\) along path \(P_{a2}\). The fact that the moving point \((a^*, K^*)\) cannot move beyond \(Q\), to say \(Q'\) is evident when it is noted that such a motion would require the Euclidean radius \(R\) to increase—and, since \(R\) can never increase, the motion is impossible. A similar argument applies to the low frequency convergence path.
Figure 3.10

Deduced convergence paths for \((K^*(t), a^f(t))\) for high and low frequencies, where \((K, a) = (1, 1)\)
Actual analog computer results for these cases are presented for comparison in Figure 3.11. Perfect agreement of the expected behavior and observed behavior is noted. The convergence path at high frequencies proceeds along a line having a unit slope, while convergence at low frequencies proceeds along a vertical line.

3.2.3 The Existence of an Optimum Steep Descent Gain G for a Fixed Frequency of Excitation

Variation of the steep descent gain G revealed the fact that at a given frequency the convergence of the coefficients was most rapid at a unique value of G. Analog computer results are displayed in Figures 3.12 and 3.13 which demonstrate this fact. In Figure 3.12 the optimum value of G is $4 \times 10^{-4}$ and values of G higher or lower produce slower coefficient convergence. It is interesting to note from Figure 3.12 that the coefficients converge exactly to the proper values in one-half period ($\frac{\pi}{\omega}$ seconds) when the optimum value of G is used. (The fact that the time interval is exactly one-half period follows from a study of the slope $\frac{da}{dk^*}$, Figure 3.2). At present the analytical calculation of this optimum gain G has not been possible.
FIGURE 3.11

ACTUAL CONVERGENCE PATHS FOR \((K^*(t), a^*(t))\) AS DETERMINED ON THE ANALOG COMPUTER
FIGURE 3.12 VARIATION OF THE COEFFICIENT CONVERGENCE RATE AS A FUNCTION OF THE STEEP DESCENT GAIN $G$ FOR $(K,a) = (1,1)$

FIGURE 3.13 VARIATION OF THE COEFFICIENT CONVERGENCE RATE AS A FUNCTION OF THE STEEP DESCENT GAIN $G$ FOR $(K,a) = (2,1)$
3.2.4 Example of the Convergence Path of Computed Coefficients when System Coefficients Undergo Step Changes

An analog computer investigation of the computed coefficient convergence paths when the system coefficients undergo step changes is investigated here on the analog computer. This investigation served two purposes: it revealed the nature and time "response" of the coefficients, and also indicated the feasibility of achieving a kind of adaptive control utilizing the computed coefficients in a "compensator." Figure 3.14 shows the block diagram and mechanization of this test. The system coefficients \( a_1 \) and \( K \) were given step changes as follows:

\[
\begin{align*}
  a_1 &= 1.0 & K &= 1.0 \\
  a_1 &= 2.0 & K &= 1.97
\end{align*}
\]

The input \( u(t) \) was made a sinusoid having a frequency of 1.0 rad/sec.

The tests (a) and (b) were run. The results of test (a) are shown in Figures 3.15(a) and 3.16(a), the results of test (b) are shown in Figures 3.15(b) and 3.16(b). Figure 3.15 shows the convergence paths of \([K^*(t), a^*(t)]\), and Figure 3.16 shows the time histories of the variables of interest. Note in Figure 3.16 that the quantity \( E \) is small, which indicates that the time-varying system has practically the same response as the constant coefficient linear
FIGURE 3.14
MECHANIZATION OF THE TWO COEFFICIENT ADAPTIVE CONTROL PROBLEM
FIGURE 3.15(a) CONVERGENCE PATHS FOR $k^*(t), a^*(t)$ WHEN $k(t), a(t)$ UNDERGOES STEP CHANGES

FIGURE 3.15(b) CONVERGENCE PATHS FOR $k^*(t), a^*(t)$ WHEN $k(t), a(t)$ UNDERGOES STEP CHANGES
FIGURE 3.16(a) TIME HISTORIES OF THE TWO COEFFICIENT CASE FOR STEP CHANGES
FIGURE 3.16(b) TIME HISTORIES OF THE TWO COEFFICIENT CASE FOR STEP CHANGES
system \( \frac{1}{D+1} \), in spite of the fact that step changes occur in \( a_1 \) and \( K \). From Figure 3.16 it is seen that the coefficients converge to their proper values in a few cycles or less.

3.3 The Four Coefficient Case

In the multi-coefficient case, described in Section 2.2.3, it was found difficult to define the entire class of impermissible functions \( x(t) \) which would not produce coefficient convergence. Attention was directed at only periodic inputs; for periodic inputs it was shown that at least \( N/2 \) frequencies in the input signal were required to determine \( N \) coefficients. It was thought desirable to give an example which would test the practical aspects of the result. The block diagram of the example is shown in Figure 3.17.

\[
x(t) \xrightarrow{b_1D + b_0 \over a_2D^2 + a_1D + 1} y(t)
\]

Figure 3.17. A Four Coefficient System

The values arbitrarily chosen for the system coefficients were \( b_1 = b_0 = a_2 = a_1 = 1 \).

The equation error and steep descent equations for this case (when the error function \( f(E) \) was chosen as \( E^2 \)) are:

\[
E = a_2 \ddot{y} + a_1 \dot{y} + y - b_1 \dot{x} - b_0 x
\]
\[
\begin{align*}
\frac{da_2^*}{dt} &= -2GE_y^* \\
\frac{da_1^*}{dt} &= -2GE_y^* \\
\frac{db_1^*}{dt} &= +2GE_x^* \\
\frac{db_2^*}{dt} &= +2GE_x^*
\end{align*}
\]

The analog computer mechanization for this example is shown in Figure 3.18. Since the corner frequencies of the system are located at \( \omega = 1 \), the "optimum" frequencies (see Section 3.2.1) were thought to be around \( \omega = 1 \) rad sec. Setting the initial values of \( a_2^*, a_1^*, b_1^*, b_0^* \) at zero, and exciting the system at the various frequencies produced convergence of the coefficients to the coefficient values shown in Table 3.1. (The convergence times--i.e., the time for the error \( E \) to become identically zero for practical purposes, was on the order of two minutes.)

In tests (1) and (2) the "final" values of the computed coefficients were seen to drift slowly with time. In test (3), however, the "final" values of the computed coefficients were always within a few percent of their correct values. These tests help support the conclusion that \( N/2 \) frequencies in the input produce the proper convergence of the computed coefficients.
FIGURE 3.18
MECHANIZATION OF A FOUR COEFFICIENT COMPUTER
TABLE 3.1
FINAL VALUES OF THE FOUR COMPUTED COEFFICIENTS AS A FUNCTION OF EXCITATION FREQUENCIES

<table>
<thead>
<tr>
<th>Input</th>
<th>&quot;Final&quot; Values of the Computed Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( a^*_t )</td>
</tr>
<tr>
<td>Excitation Frequencies</td>
<td></td>
</tr>
<tr>
<td>(1) ( \omega = 1 \text{ rad/sec.} )</td>
<td>1.430</td>
</tr>
<tr>
<td>(2) ( \omega = 2 \text{ rad/sec.} )</td>
<td>.659</td>
</tr>
<tr>
<td>(3) ( \omega = 1 \text{ rad/sec.} ) and ( \omega = 2 \text{ rad/sec.} )</td>
<td>.989</td>
</tr>
</tbody>
</table>

As another example of four coefficient computation, the system was excited by a random noise signal, and system coefficients as follows were arbitrarily chosen:

\[
\begin{align*}
a_0 &= 9.0 \\
 a_1 &= 3.1 \\
 b_1 &= 98.3 \\
 b_0 &= 53.0
\end{align*}
\]

The initial values of all of these coefficients were set at zero. The time histories of the computed coefficients is shown in Figure 3.19. Good convergence of the values of the computed coefficients is noted in about 10 seconds. It is noted that in spite of the small damping value, the coefficient \( a^*_1(t) \) converges to a good approximation of \( a_1 \). Further tests with this system showed that \( a^*_1(t) \) tracked the value of \( a_1 \) well, even when \( a_1 \) was given negative values (i.e., when the system was unstable!).
FIGURE 3.19 TIME HISTORIES OF THE CONVERGENCE OF FOUR COEFFICIENTS
3.4 **Summary**

Discussions and examples presented in this chapter revealed some of the peculiarities of this method of computing coefficients.

In the two coefficient case, the existence of an "optimum" frequency for exciting the system, and an "optimum" steep descent gain $G$ when exciting the system at a particular frequency were demonstrated. In the four coefficient case ($N = 4$) it was demonstrated that at least 2 frequencies ($N/2 = 2$) were required to produce coefficient convergence. These examples serve to support (rather than prove) the characteristics of the convergence process discussed in Chapter II.
CHAPTER IV

ADAPTIVE CONTROL OF LINEAR SLOWLY VARYING SYSTEMS

The coefficient computer permits many different kinds of adaptive control, since the properties of a system can be described if the coefficients of the system are known. It is tacitly assumed that the form of the equations describing the system is known also. However it is not necessary to know the exact form of the equations describing the system. Consider the case where a fourth ordered (constant coefficient) system which is excited by relatively low frequencies can be approximated by a third ordered system—i.e., the coefficient of the $S^4$ term in the denominator of the system transfer function is much smaller than the other coefficients. Then, if a coefficient computer were constructed and hooked up to the system under the assumption that the system was a third ordered system the coefficient computer would be expected to accurately determine the proper values of the approximating third order system. Now, if the coefficient of $S^4$ in the denominator were gradually increased until it produced significant changes in the output of the system, then the computed values of the coefficients would oscillate—the greater the value of the
coefficient of \( S^4 \), the greater the oscillations of the coefficients. In effect the coefficient computer tends to produce time-varying coefficients of a third ordered system which duplicates the output of a fourth ordered system which cannot be approximated well by a constant coefficient third ordered system. This is undesirable, however, since the magnitudes and time variations of the computed coefficients will depend greatly upon the nature of the input to the system.] In Section 1.4, Figures 1.6 and 1.7, general adaptive methods and special adaptive methods, respectively, were presented for compensating systems having a linear slowly varying sub-system somewhere in their control loops.

In Chapter II a general method for computing the coefficient values of systems describable by sets of equations having constant coefficients was presented. It was stated in that chapter that although it was intuitively obvious that this method should be successful for computing the values of coefficients that varied sufficiently slowly, the theoretical difficulties at the present time preclude the possibility of an analytical solution to this problem.

In this chapter the adaptive method of Figure 1.7 is applied to the general case of a linear system having \( N \) slowly varying coefficients. Later in this chapter an analog computer is used to investigate the properties, behavior, and success of this approach in terms of an example
wherein a time-varying linear system is forced to behave as a constant coefficient system.

4.1 An Adaptive Scheme for Forcing a Time-Varying Linear System to Behave as a Constant Coefficient System

A general adaptive scheme for forcing a slowly-varying linear system to behave as a constant coefficient system is presented in Figure 4.1. (The same scheme is applicable to continuous nonlinear systems if the operating point moves along the nonlinear characteristic slowly enough.)
The cancellation process in Figure 4.1 has its disadvantages. If any of the characteristic roots of the slowly varying system $G_1$ have positive real parts, and if there is no feedback compensation, it can be shown that even assuming "perfect cancellation," the system will diverge, with $[y(t) - w(t)]$ approaching infinity as $t$ increases, if the initial values of $y(t)$ and $w(t)$ are not equal. Other important disadvantages are that it is impossible to achieve perfect cancellation and the compensation may be unnecessarily complex. Nonetheless, this cancellation process is practical in many applications.

The equations describing the slowly varying linear system and the coefficient computer are as follows:

**Time varying system equation:**
\[ o^y = \sum_{k=1}^{n} a_k y^{(k)} - \sum_{j=0}^{m} b_j x \]  (4.1)

where $a_k = a_k(t)$, $b_j = b_j(t)$.

**Equation error:**
\[ E = y^{*} - \sum_{k=1}^{n} a_k^{*} y^{(k)} - \sum_{j=0}^{m} b_j^{*} x \]  (4.2)

or
\[ E = \sum_{k=1}^{n} (a_k^{*} - a_k) y^{(k)} - \sum_{j=0}^{m} (b_j^{*} - b_j) x \]  (4.3)

**Steep descent equations:**
\[ \frac{d a_k^*}{d t} = -G \frac{\partial E^2}{\partial a_k^*} = -2GE y^{(k)} \]  (4.4)

\[ \frac{d b_j^*}{d t} = -G \frac{\partial E^2}{\partial b_j^*} = +2GE x^{(j)} \]  (4.5)
As discussed in Chapter II, the convergence of $a_k^*(t)$ and $b_j^*(t)$ is assured if conditions (2.11) and (2.12) are satisfied, and is expected if conditions (2.11) and (2.13) are satisfied. Although intuition suggests that convergence of $a_k^*(t)$ to $a_k(t)$ and $b_j^*(t)$ to $b_j(t)$ will occur if $a_k(t)$ and $b_j(t)$ vary sufficiently slowly, a demonstration of this convergence is desired; it is also desirable to demonstrate the practicality of achieving this method of adaptive control. In the next section an application of this form of adaptive control to a slowly varying linear system is presented.

4.2 Example Application on an Analog Computer

The block diagram for this example is shown in Figure 4.2.

![Block Diagram]

Figure 4.2. Adaptive Scheme for Forcing a Linear Systems Having Two Slowly Varying Coefficients to Behave as a Constant Coefficient System.
The time varying process is represented by the transfer function $G_1$ and its dynamics are cancelled by the compensator $G^*_2$. The desired transfer function relating $y$ and $u$ is arbitrarily chosen as:

$$G_0 = \frac{1}{D+1} \quad (4.6)$$

The equations for this example are:

slowly varying system equation:

$$a_1(t) \dot{y}(t) + y(t) - K(t) x(t) = 0 \quad (4.7)$$

equation error:

$$E = a_1^*(t) \dot{y}(t) + y(t) - K^*(t) x(t) \quad (4.8)$$
or

$$E = (a_1^*(t) - a_1(t)) \dot{y}(t) - (K^*(t) - K(t)) x(t) \quad (4.9)$$

Using the error function $E^*$, the equations of steep descent are:

$$\frac{da_1^*}{dt} = -G \frac{\partial E^2}{\partial a_1^*} = -2CE_y^* \quad (4.10)$$

$$\frac{dK^*}{dt} = -G \frac{\partial E^2}{\partial K^*} = 2GEx \quad (4.11)$$

For purposes of this investigation, both $a_1(t)$ and $K(t)$ are given ten to one variations, where

$$0.5 \leq a_1(t) \leq 2.5$$

$$0.5 \leq K(t) \leq 5.0$$

The analog computer mechanization of the system shown in Figure 4.2 is the same as that of Figure 3.14. In the block diagram shown in Figure 3.14, note that a "model" has been included in the mechanization, and that the difference
of the model output and the time varying system output has been labelled $E_m(t)$. Note that this $E_m(t)$ is not used in the scheme for computing coefficients; this $E_m(t)$ is only used to judge the quality of the adaptive control. The model output $v(t)$ provides a criterion for judging the quality of this form of adaptive control, since for perfect operation of the adaptive controller the output of the slowly varying system $y(t)$ would be identical with the output of the model $v(t)$. Hence, if $E_m(t)$ is either zero or very small under excitations $u(t)$, then the adaptive compensation of the slowly varying system $G$ is performing well. If $E_m(t)$ is large (compared with $v(t)$), then the adaptive compensation would be performing poorly.

4.2.1 The Method of Performing the Tests

A total of eight (8) tests were performed. The only differences in each of the tests were differences in the excitation $u(t)$. Each test was made up of two parts, (a) and (b). In part (a) the coefficients were varied with time so that the point $a_1(t)$, $K(t)$ moved in a path as shown in Figure 4.3; in part (b), the coefficients $a_1(t)$, $K(t)$ were varied with time so that the point $a_1(t)$ moved in a path as shown in Figure 4.4.
Figure 4.3. Path Taken by the Moving Point $a_1(t)$, $K(t)$ in the $a_1(t)$, $K(t)$ Plane for Part (a) of Each Test.

Figure 4.4. Path Taken by the Moving Point $a_1(t)$, $K(t)$ in the $a_1(t)$, $K(t)$ Plane for Part (b) of Each Test.
The method of executing the tests was as follows: the values of \( a_1(t) \), \( K(t) \) were varied at a constant rate on each of the legs of the straight line paths in the coefficient plane. In all cases the time duration for the point to move along one leg required 90 seconds. On reaching a corner of its path, the point \( a_1(t) \), \( K(t) \) was delayed for a reasonable time interval (30 to a few hundred seconds) to permit the point \( a_1^*(t) \), \( K^*(t) \) to converge reasonably close to the "corner" point. The coefficients \( a_1(t) \), \( K(t) \) were then again varied along the next leg of its path.

Table 4.1 lists the eight (8) tests carried out on the system, and lists the excitation \( u(t) \) applied on each test. For convenience of reference, the figure numbers associated with each test are also tabulated in Table 4.1. As indicated, two kinds of data were recorded during each test:

1) The time histories of \( u(t) \), \( v(t) \), \( y(t) \), \( y(t) - u(t) = E_m(t) \), \( a_1(t) \), \( a_1^*(t) \), \( K(t) \) and \( K^*(t) \), and
2) The paths taken by \( a_1(t) \), \( K(t) \) and by \( a_1^*(t) \), \( K^*(t) \) in the coefficient plane.
### TABLE 4.1

**TABULATION OF THE TESTS PERFORMED ON THE EXAMPLE SYSTEM**

<table>
<thead>
<tr>
<th>Test</th>
<th>Excitation</th>
<th>Figures</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$u(t)$</td>
<td>Paths of $a, K$</td>
</tr>
<tr>
<td>1(a)</td>
<td>sinusoidal</td>
<td>4.3</td>
</tr>
<tr>
<td></td>
<td>$\omega = 0.4$ rad/sec</td>
<td>4.4</td>
</tr>
<tr>
<td>2(a)</td>
<td>sinusoidal</td>
<td>4.3</td>
</tr>
<tr>
<td></td>
<td>$\omega = 1.0$ rad/sec</td>
<td>4.4</td>
</tr>
<tr>
<td>3(a)</td>
<td>sinusoidal</td>
<td>4.3</td>
</tr>
<tr>
<td></td>
<td>$\omega = 4.0$ rad/sec</td>
<td>4.4</td>
</tr>
<tr>
<td>4(a)</td>
<td>superposition of three frequencies (equal amplitudes), $\omega_1 = 0.4$, $\omega_2 = 1.0$, $\omega_3 = 4.0$ rad/sec</td>
<td>4.3</td>
</tr>
<tr>
<td>5(a)</td>
<td>white noise passed through the filter $\left(\frac{(S+0.3)^2}{S^2+3}\right) \left(\frac{(S+1.4)^2}{S^2+6}\right)$</td>
<td>4.3</td>
</tr>
<tr>
<td>5(b)</td>
<td>$\left(\frac{(S+0.3)^2}{S^2+3}\right) \left(\frac{(S+1.4)^2}{S^2+6}\right)$</td>
<td>4.4</td>
</tr>
<tr>
<td>6(a)</td>
<td>white noise passed through the filter $\left(\frac{(S+0.7)^2}{S^2}\right) \left(\frac{(S+1.4)^2}{S^2+1.4}\right)$</td>
<td>4.3</td>
</tr>
<tr>
<td>6(b)</td>
<td>$\left(\frac{(S+0.7)^2}{S^2}\right) \left(\frac{(S+1.4)^2}{S^2+1.4}\right)$</td>
<td>4.4</td>
</tr>
<tr>
<td>7(a)</td>
<td>white noise passed through the filter $\left(\frac{6^2}{S^2}\right) \left(\frac{6^2}{S^2}\right)$</td>
<td>4.3</td>
</tr>
<tr>
<td>7(b)</td>
<td>$\left(\frac{6^2}{S^2}\right) \left(\frac{6^2}{S^2}\right)$</td>
<td>4.4</td>
</tr>
<tr>
<td>8(a)</td>
<td>white noise passed through the filter $\frac{10^4}{S^2}$</td>
<td>4.3</td>
</tr>
<tr>
<td>8(b)</td>
<td>$\frac{10^4}{S^2}$</td>
<td>4.4</td>
</tr>
</tbody>
</table>
FIGURE 4.5(a) TIME HISTORIES FOR TEST 1(a)
FIGURE 4.5(a) TIME HISTORIES FOR TEST 1(a) - Continued
FIGURE 4.5(b) TIME HISTORIES FOR TEST 1(b)
4 of slowly varying system 40v/cm

\[ v(t) \] output of model 40v/cm

\[ E(t) = y(t) - v(t) \]

10 sec. 

\[ a^x(t) \] (solid line)

\[ a^y(t) \] (dashed line)

\[ K^x(t) \] (solid line)

\[ K^y(t) \] (dashed line)

FIGURE 4.6(a) TIME HISTORIES FOR TEST 2(a)
FIGURE 4.6(a) TIME HISTORIES FOR TEST 2(a) - Continued
FIGURE 4.6(b) TIME HISTORIES FOR TEST 2(b)
FIGURE 4.6(b)  TIME HISTORIES FOR TEST 2(b) Continued
**FIGURE 4.7(a) TIME HISTORIES FOR TEST 3(a)**

- $u(t)$ input 769/cm
- $y(t)$ output of slowly varying system 169/cm
- $v(t)$ output of model 169/cm
- $e(t) = y(t) - v(t)$ 169/cm
- $x^*(t)$ (solid line)
- $w(t)$ (dashed line)
- $x^*(t)$ (solid line)
- $x(t)$ (dashed line)

**Legend:**
- 1 cm.
- 10 sec.
- 110 sec. removed
- 160 sec. removed
FIGURE 4.7(a) TIME HISTORIES FOR TEST 3(a) - Continued
FIGURE 4.7(b) TIME HISTORIES FOR TEST 3(b)
FIGURE 4.7(b) TIME HISTORIES FOR TEST 3(b) - Continued
FIGURE 4.8(a) TIME HISTORIES FOR TEST 4(a)
FIGURE 4.8(a) TIME HISTORIES FOR TEST 4(a) - Continued
FIGURE 4.8(b) TIME HISTORIES FOR TEST 4(b)
FIGURE 4.9(a) TIME HISTORIES FOR TEST 5(a)
FIGURE 4.9(b) TIME HISTORIES FOR TEST 5(b)
FIGURE 4.9(b) TIME HISTORIES FOR TEST 5(b) - Continued
FIGURE 4.10(a) TIME HISTORIES FOR TEST 6(a)
FIGURE 4.10(a) TIME HISTORIES FOR TEST 6(a) - Continued
FIGURE 4.10(b) TIME HISTORIES FOR TEST 6(b)
FIGURE 4.11(a) TIME HISTORIES FOR TEST 7(a)
FIGURE 4.11(a) TIME HISTORIES FOR TEST 7(a) - Continued
FIGURE 4.11(b) TIME HISTORIES FOR TEST 7(b)
FIGURE 4.11(b) TIME HISTORIES FOR TEST 7(b) - Continued
FIGURE 4.12(a) TIME HISTORIES FOR TEST 8(a)
FIGURE 4.12(a) TIME HISTORIES FOR TEST 8(a) - Continued
FIGURE 4.12(b) TIME HISTORIES FOR TEST 8(b)
FIGURE 4.12(b) TIME HISTORIES FOR TEST 8(b) – Continued
FIGURE 4.13(a)
PATHS IN THE COEFFICIENT
PLANE FOR TEST 1(a)

FIGURE 4.13(b)
PATHS IN THE COEFFICIENT
PLANE FOR TEST 1(b)

FIGURE 4.14(a)
PATHS IN THE COEFFICIENT
PLANE FOR TEST 2(a)

FIGURE 4.14(b)
PATHS IN THE COEFFICIENT
PLANE FOR TEST 2(b)
FIGURE 4.15(a)
PATHS IN THE COEFFICIENT
PLANE FOR TEST 3(a)

FIGURE 4.15(b)
PATHS IN THE COEFFICIENT
PLANE FOR TEST 3(b)

FIGURE 4.16(a)
PATHS IN THE COEFFICIENT
PLANE FOR TEST 4(a)

FIGURE 4.16(b)
PATHS IN THE COEFFICIENT
PLANE FOR TEST 4(b)
Figure 4.17(a) Paths in the coefficient plane for Test 5(a)

Figure 4.17(b) Paths in the coefficient plane for Test 5(b)

Figure 4.18(a) Paths in the coefficient plane for Test 6(a)

Figure 4.18(b) Paths in the coefficient plane for Test 6(b)
FIGURE 4.19(a)
PATHS IN THE COEFFICIENT
PLANE FOR TEST 7(a)

FIGURE 4.19(b)
PATHS IN THE COEFFICIENT
PLANE FOR TEST 7(b)

FIGURE 4.20(a)
PATHS IN THE COEFFICIENT
PLANE FOR TEST 8(a)

FIGURE 4.20(b)
PATHS IN THE COEFFICIENT
PLANE FOR TEST 8(b)
4.2.2 Discussion of Tests (1), (2), (3) and (4)

In this series of tests the excitation \( u(t) \) was either a single sinusoid or a superposition of three sinusoids. Since the value of \( a_1(t) \) varies between 0.25 and 2.5, the corner frequency varies between 0.4 and 4.0 rad/sec. If only one frequency were used to excite the system, it would be expected on the basis of the discussion presented in Section 3.2.1 that the optimum frequency should lie in the range 0.4 rad/sec to 4.0 rad/sec; since an excitation frequency of 1.0 rad/sec lies somewhat in the middle of the optimum frequency range, it would be expected that an excitation frequency of 1.0 rad/sec would produce more rapid over-all coefficient convergence than excitation frequencies of 0.4 rad/sec and 4.0 rad/sec.

An examination of Figures 4.5 (a) and (b); 4.6 (a) and (b), and 4.7 (a) and (b) indicates that an excitation frequency of 1 rad/sec does indeed produce more rapid, higher quality coefficient tracking than excitation frequencies of 0.4 or 4.0 rad/sec. Examination of the paths of the coefficients in coefficient plane in Figures 4.13 (a) and (b), 4.14 (a) and (b) and 4.15 (a) and (b) also indicates that an excitation frequency of 1 rad/sec produces much tighter tracking than do frequencies of 0.4 or 4.0 rad/sec. (A peculiarity of the mechanization of these tests is revealed in these figures of the paths in the coefficient
A "loading" condition in the mechanization has caused the actual maximum value of $a_1(t)$ to appear as 2.65 instead of 2.50. In other tests where the value 2.50 of the coefficient $a_1$ was set on a potentiometer, the coefficient $a_1^*(t)$ converged almost exactly to 2.50. Hence, the apparent "overshoot" of the convergence point for $a^*(t)$ beyond $a_1^* = 2.5$ is believed to be simply a deficiency of the mechanization.

Comparison of Figures 4.5, 4.6, 4.7, 4.8, 4.13, 4.14, 4.15 and 4.16 indicates that the best coefficient convergence occurs for excitation with the single sinusoid $\omega = 1.0 \text{ rad/sec}$; the coefficient convergence corresponding to three superimposed excitations $\omega_1 = .4$, $\omega_2 = 1.0$, $\omega_3 = 4.0 \text{ rad/sec}$ and the convergence corresponding to excitation at the single sinusoid $\omega = .4 \text{ rad/sec}$ are approximately of the same quality, and are superior to the coefficient convergence obtained for sinusoidal excitations of $\omega = 4.0 \text{ rad/sec}$.

Examining the characteristics of the paths in the coefficient plane, Figures 4.13, 4.14, 4.15 and 4.16 reveals that at the higher excitation frequencies there is much more of a tendency for the $a_1^*(t), K^*(t)$ path to deviate from the $a_1(t), K(t)$ path. Comparison of Figures 4.13 and 4.15 reveals that the coefficient $K(t)$ is tracked much more poorly at the higher frequencies, there being a characteristic
"V" shaped deviation for those paths where the gain $K(t)$ is varying. Examination of Figure 4.16 also reveals the same characteristic deviation; tests run without the 4.0 rad/sec excitation frequency always display much better coefficient tracking. Hence, it is concluded that the higher frequency excitations in a system actually do more harm than good as far as coefficient convergence is concerned.

It is noted then, that these results suggest a practical way of avoiding the disadvantages of having high frequency signals passing through the time varying system. An obvious solution to the problem is to employ a low pass filter as shown in Figure 4.21.

![Diagram](image-url)

Figure 4.21. Use of a Filter in the Forward Path to Realize More Rapid Coefficient Convergence.
Since the system is linear, however, identical results can be obtained by filtering the inputs to the coefficient computer directly, leaving the main forward path of the system unaffected, as shown in Figure 4.22.

In this system all high frequency components are severely attenuated from the signal \( x(t), y(t), \dot{y}(t) \) entering coefficient computer. Since it can be shown that the transfer function relating these filtered signals to one another is unchanged, the computed coefficient values of \( K^* \) will converge to their proper values more rapidly. This result has considerable practical significance, since the output of the slowly varying system is unaffected by the filters, yet the coefficient convergence is much more rapid.
4.2.3 Discussion of Tests (5), (6), (7) and (8)

In this series of tests the excitation $u(t)$ was a continuous random noise signal produced by filtering the output of a Gaussian white noise generator. The filter transfer functions used in each test are listed in Table 4.1. Comparison of Figures 4.9, 4.10, 4.11, 4.12, 4.17, 4.18, 4.19 and 4.20 reveals that the best convergence characteristics occur for the low frequency excitations. In Figures 4.18, 4.19, and 4.20 the characteristic "V" deviation of the $a_1^*$, $K^*$ path from the $a_1$, $K$ path is apparent. As discussed in Section 4.1.1.2, this "V" deviation is associated with the high frequency signals present in the input signal $x(t)$. A remedy for the situation where the slowly varying system is excited by high frequency signals (which degrade the rapidity of coefficient convergence) is to pass all signals entering the coefficient computer [$x(t)$, $y(t)$, $\dot{y}(t)$] through identical low pass filters, as described in Section 4.1.1.2; as mentioned before these filters will not disturb the apparent transfer function which the coefficient computer "sees" in calculating the coefficient values $a^*$ and $K^*$. The fact that the filters do not disturb the system plus the fact that it was concluded in Section 4.1.1.2 that frequencies in the 1 rad/sec range produced good convergence of the computed coefficients, implies that the use of a filter which passes only those frequencies in the 1 rad/sec
range will produce the best computed coefficient convergence.

Examination of the values of $E_m(t)$ in Figure 4.9 indicates that the adapted variable coefficient system is indeed behaving almost like a constant coefficient system. In Figures 4.10, 4.11 and 4.12, however, poorer operation of the adapted system is noted due to the poorer convergence of the computed coefficients.

It was noted in additional testing that as the rates of change of the system coefficients increased (or decreased), the tracking capability of the computed coefficients deteriorated (improved), as expected. In the work reported in Section 3.2 it was demonstrated that if the system coefficients were given step changes, then the minimum convergence time of the computed coefficients was one-half period of the excitation frequency. In the tests reported in this chapter attempts were made to adjust the value of $G$ to its optimum value (which would have produced convergence in half a cycle of the dominant frequency in the excitation signal). However, the limitations of the analog computer (specifically the noise problems which result from having several electronic multipliers in a single "loop") required too low a value of $G$ to attain this more desirable and quicker computed coefficient response.
5.1 Summary

In Chapter I a relatively "universal" scheme based upon a coefficient computer was presented for providing adaptive control for slowly varying linear systems. A general scheme for cancelling the dynamics of slowly varying linear (S.V.L.) systems was then presented which can be classed as a special form of the "universal" scheme. The operation of these schemes depended upon a successful method of computing the coefficients of S.V.L. systems.

In Chapter II the theory behind a general method for computing the coefficients of systems describable by sets of ordinary integro-differential or algebraic, linear or non-linear equations having constant coefficients was presented. In this chapter this general scheme was applied to the problem of computing the constant coefficients of linear systems. It was remarked that analysis for the cases of slowly varying coefficients was difficult, so that an analog computer would be used to evaluate the success of the coefficient computation method when applied to S.V.L. systems.
In Chapter III the coefficient convergence process for computing constant coefficients was investigated analytically, graphically and on an analog computer. Peculiarities of the coefficient convergence process were revealed and discussed.

In Chapter IV the application of the method to the adaptive control of S.V.L. systems was presented. Extensive analog computer tests were run on one particular S.V.L. system, and it was shown that if suitable filters were used on the signals entering the coefficient computer, and if the system was suitably excited, the S.V.L. system was forced to behave as a constant coefficient system by this form of adaptive control.

5.2 Discussion and Comparison with Other Adaptive Systems

The adaptive control approach presented here is similar in approach to two other reported approaches.

Margolis and Leondes\textsuperscript{16,17} presented a method of achieving adaptive control of S.V.L. systems by using a "model" of the S.V.L. system. In this scheme the form of the S.V.L. system was known and mechanized in the "model." The coefficients of the "model" were made variable. The method of "computing" the S.V.L. system coefficients used was to subject the model and the S.V.L. system to identical inputs, and to adjust the model coefficients until the model
output was identically equal to the S.V.L. system output. Having reached this condition, it would be asserted that the model coefficients were identical with the S.V.L. system coefficients. [It is noted that special inputs to the model and S.V.L. system can be used so that the model output and S.V.L. system can be exactly equal when the model coefficients are not equal to the S.V.L. system coefficients. Thus, the "model approach" presented by Margolis and Leondes is also limited to special inputs for proper coefficient convergence, as is the case with the method set forth in this dissertation (see Section 2.3)]. A study of the "model" approach seems to reveal certain inherent disadvantages. For example, if the damping of the S.V.L. system had a low value then variations of the model coefficients could be expected to introduce dynamic variations into the model output which would require a long time interval to die out. The computed model coefficients would be forced to oscillate in this time interval, hence long convergence times could be expected; if the S.V.L. system were actually unstable (as for a negative damping value), then, even if the S.V.L. system were in a stable loop, the model output would approach infinity as $t \to \infty$. Hence, the coefficients of the system could not be computed. Note, in contrast, that the methods proposed for computing coefficients in previous chapters does not have this limitation. Since
both the "model approach" and the approach presented in this work must be considered in the development stages, final conclusions of the merit of the one method over that of the other must await further research.

In the Weygandt and Puri paper, a method is presented for computing the coefficients of special kinds of transfer functions—viz., transfer functions describable in Laplace transforms by a constant numerator divided by polynomial of the complex variable $S$. By using the orthogonality properties of sinusoidal and cosinusoidal signals, the authors present a method for computing the values of all coefficients in the denominator of the transfer function. The method requires that the input signals be sinusoidal having known frequencies. Although this method is one of the most general presented to date, it is not a completely general method for computing the coefficients of transfer functions having an arbitrary form with arbitrary inputs. As is the case with the "model approach" of Margolis and Landes, final conclusions of the merits of this scheme over the those presented in previous chapters must await further research. However, as the authors assert, it is clear that the most severe limitation of the Weygandt and Puri approach is that the method is not applicable to arbitrary systems and must be limited in its application to systems describable in Laplace transforms.
by transfer functions having constant numerators and denominators which are polynomials of the complex variable $S$.

5.3 Conclusions

The work presented herein may be regarded as only laying the groundwork for what appears to be a very general method of achieving adaptive control over slowly varying linear systems. In the course of the development of the techniques presented herein a scheme for computing the coefficients of systems describable by ordinary integro-differential or algebraic, linear or non-linear equations is presented. This scheme offers promise as being developed into one of the more powerful methods of identifying process and system parameters.
APPENDIX

1) Theorem 1, page 6, Ref. 20. [A (t) is a matrix and \( \vec{y} \) a vector.]

Let \( A (t) \) be continuous in the interval \((0, t_0)\). Then there exists a unique solution of

\[
\frac{d\vec{y}}{dt} = A (t) \vec{y}
\]

in this interval.

2) Theorem 2, page 36, Ref. 20.

If all solutions of \( \frac{d\vec{y}}{dt} = A\vec{y} \) approach zero as \( t \to \infty \), the same holds for the solutions of

\[
\frac{d\vec{z}}{dt} = (A + B(t))\vec{z}
\]

provided that

\[ ||B(t)|| \leq C_1 \text{ for } t \geq t_0, \text{ where } C_1 \text{ is a constant which depends upon } A. \]

3) A discussion of the Second Method of Lyapunov (Ref. 21):

The objective of the so-called "second method" of Lyapunov is this: To answer questions of stability of differential equations, utilizing the given form of the equations but without explicit knowledge of the solutions. The name "second method" is an unfortunate but well-entrenched misnomer; actually the "second method" is more accurately
described as a point of view, a philosophy of approach, rather than a systematic method. At present, much depends on the ingenuity of the user.

The principal idea of the second method of Lyapunov is contained in the following reasoning:

If the rate of change \( \frac{dE(x)}{dt} \) of the energy \( E(x) \) of an isolated physical system is negative for every possible state \( x \), except for a single equilibrium state \( x_e \), then the energy will continually decrease until it finally assumes its minimum value \( E(x_e) \). In other words, a dissipative system perturbed from its equilibrium state will always return to it; this is the intuitive concept of stability.

In order to develop this idea into a precise mathematical tool, the foregoing reasoning must be made independent of the physical concept of energy. As a rule there is no natural way of defining energy when the equations of motion are given in purely mathematical form.

The main theorems are as follows:

Theorem 1. Consider the continuous-time, free dynamic system

\[
\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}, t)
\]

where \( \vec{f}(\vec{0}, t) = \vec{0} \) for all \( t \).

Suppose there exists a scalar function \( V(\vec{x}, t) \) with continuous first partial derivatives with respect to \( \vec{x} \) and \( t \) such that \( V(\vec{0}, t) = 0 \) and:
1) $V(\vec{x}, t)$ is positive definite; i.e., there exists a continuous non-decreasing scalar function $\alpha$ such that

$$\alpha(0) = 0 \text{ and, for all } t \text{ and all } \vec{x} \neq \vec{0},$$

$$0 \leq \alpha(||\vec{x}||) \leq V(\vec{x}, t)$$

ii) There exists a continuous scalar function $\gamma$ such that $\gamma(0) = 0$ and the derivative $\dot{V}$ of $V$ along the motion starting at $t$, $\vec{x}$ satisfies, for all $t$ and all $\vec{x} \neq \vec{0}$,

$$\dot{V}(\vec{x}, t) - \gamma(||\vec{x}||) \leq 0;$$

iii) There exists a continuous, nondecreasing scalar function such that $\beta(0) = 0$ and, for all $t$,

$$V(\vec{x}, t) \leq \beta(||\vec{x}||);$$

iv) $\alpha(||\vec{x}||) \to \infty$ with $||\vec{x}|| \to \infty$.

Then the equilibrium state $\vec{x}_e = \vec{0}$ is uniformly asymptotically stable in the large; $V(\vec{x}, t)$ is called a Lyapunov function of the system.

Corollary 1.1. The following conditions are sufficient for the various weaker types of stability:

a) Uniform asymptotic stability: (i-iii).

b) Equiasymptotic stability in the large: (i-ii),(iv).

c) Equiasymptotic stability: (i-ii).

d) Uniform stability: (i), (iii), and (ii):

$$\dot{V}(\vec{x}, t) \leq 0 \text{ for all } \vec{x}, t.$$

e) Stability: (i-ii).

f) No finite escape time: (i), (iv), and ii2):

$$\dot{V}(\vec{x}, t) \leq cV(\vec{x}, t) \text{ for all } \vec{x}, t; c \text{ being a positive constant.}$$
4) Proof that $E \to 0$ for the General Multi-Coefficient Case:

From equation (2.58), (2.62) and (2.63) we have:

$$E(t) = \sum_{i=1}^{n} [a_i^*(t) - a_i] y(t) - \sum_{k=0}^{m} [b_k^*(t) - b_k] x(t) \quad (2.58)$$

$$R^2(t) = \sum_{i=1}^{n} [a_i^*(t) - a_i]^2 + \sum_{k=0}^{m} [b_k^* - b_k]^2 \quad (2.62)$$

$$\frac{dR^2}{dt} = -4GE^2 \quad (2.63)$$

Assuming that all input and output variables, $x$, $y$, and their time derivatives are continuous and bounded functions of time, it can be shown that for some $L$ and $M$:

$$|E^2(t)| \leq L \quad (A.1)$$

$$\left| \frac{dE^2(t)}{dt} \right| \leq M \quad (A.2)$$

$R$ is a real variable, hence $R^2$ has a zero lower bound. Since $R^2(t)$ decreases monotonically and has a zero lower bound, $R^2(t)$ must approach a limit as $t \to \infty$. Let $R^2_\infty$ denote this limit, and let $R^2_0$ denote the initial value of $R^2$ at $t = 0$. Then

$$R^2_0 \geq R^2(t) \geq R^2_\infty \geq 0 \quad (A.3)$$

Integrating equation (2.63) yields:

$$R^2(t) - R^2_0 = -4G \int_{\tau=0}^{\tau=t} E^2(\tau)d\tau \quad (A.4)$$
Taking the limit of (A.4) as \( t \to \infty \) yields:

\[
R_\infty^2 - R_0^2 = \lim_{t \to \infty} - 4G \int_{\tau = 0}^{\tau = t} E^2(\tau) d\tau
\]  

(A.5)

Thus,

\[
\lim_{t \to \infty} \int_{\tau = 0}^{\tau = t} E^2(\tau) d\tau = \frac{R_\infty^2 - R_0^2}{4G} = 0
\]

(A.6)

From equation (A.6), for any \( 0 \leq t_1 \leq \infty \),

\[
\lim_{t \to \infty} \int_{\tau = t_1}^{\tau = t} E^2(\tau) d\tau = \frac{R_\infty^2 - R_0^2}{4G} - \int_{\tau = 0}^{\tau = t_1} E^2(\tau) d\tau
\]

(A.7)

Define \( H(t) \) as the monotonically decreasing function:

\[
H(t_1) = \left[ 2 M \lim_{t \to \infty} \int_{\tau = t_1}^{\tau = t} E^2(\tau) d\tau \right]^{1/2}
\]

(A.8)

It is claimed that \( H(t_1) \) is an upper bound of \( E^2(t_1) \).

Proof:

Assume there exists a time \( t_2 \) such that

\[
E^2(t_2) = H(t_2)
\]

(A.9)
Since by (A.2) \( \left| \frac{dE^2}{dt} \right| \leq M \), the rate at which \( E^2(t) \) could approach zero has a lower bound, and hence, there is a minimum value of \( \Delta t \) for which \( E \) could become zero. Since \( E^2 \) cannot lie below line (1), a minimum value of \( \lim_{t \to \infty} \int_{\tau = t}^{\tau = t_2} E^2(\tau) d\tau \) exists, thus

\[
\lim_{t \to \infty} \int_{\tau = t}^{\tau = t_2} E^2(\tau) d\tau = \frac{\Delta t}{2} E^2(t_2) \quad (A.10)
\]

but \( \Delta t = \frac{E^2(t_2)}{M} \). \( (A.11) \)

Thus, substituting (A.11) into (A.10)

\[
\lim_{t \to \infty} \int_{\tau = t}^{\tau = t_2} E^2(\tau) d\tau = \frac{E^4(t_2)}{2M} \quad (A.12)
\]

From (A.9),

\[
\frac{E^4(t_2)}{2M} = \frac{H^2(t_2)}{2M} \quad (A.13)
\]

Substituting (A.12) into (A.13)

\[
\lim_{t \to \infty} \int_{\tau = t}^{\tau = t_2} E^2(\tau) d\tau = \frac{E^4(t_2)}{2M} > \frac{H^2(t_2)}{2M} \quad (A.14)
\]

Substituting (A.8) into (A.14) yields

\[
\lim_{t \to \infty} \int_{\tau = t}^{\tau = t_2} E^2(\tau) d\tau > \lim_{t \to \infty} \int_{\tau = t}^{\tau = t_2} E^2(\tau) d\tau \quad (A.15)
\]
Equation (2.61) is an obvious contradiction, hence $H(t)$ is an upper bound of $E^2(t)$.

It will now be shown that $H(t) \to 0$ as $t \to \infty$.

From equation (A.8),

$$H^2(t_1) = 2M \lim_{t \to \infty} \int_{t=t_1}^{\tau=t} E^2(\omega) \, d\tau$$

(A.16)

Thus,

$$\lim_{t \to \infty} H^2(t_1) = \lim_{t_1 \to \infty} \left[ 2M \lim_{t \to \infty} \int_{\tau=t_1}^{\tau=t} E^2(\tau) \, d\tau \right]$$

(A.17)

Substituting (A.7) into (A.17),

$$\lim_{t_1 \to \infty} H^2(t_1) = \lim_{t_1 \to \infty} \left[ 2M \begin{array}{c} \frac{(R_0^2 - R_\infty^2)}{4G} - \int_{\tau=0}^{\tau=t_1} E^2(\tau) \, d\tau \end{array} \right]$$

(A.18)

(A.18) can be written as:

$$\lim_{t_1 \to \infty} H^2(t_1) = \frac{2M(R_0^2 - R_\infty^2)}{4G} - 2M \lim_{t_1 \to \infty} \int_{\tau=0}^{\tau=t_1} E^2(\tau) \, d\tau$$

(A.19)

Substitution of equation (A.6) into (A.19) yields:

$$\lim_{t_1 \to \infty} H^2(t_1) = 0$$

(A.20)

Since $H(t_1)$ is an upper bound on $E^2(t_1)$, $E^2(t_1)$ must also converge to zero as $t \to \infty$. Q.E.D.
REFERENCES


AUTOBIOGRAPHY

I, Bruce John Miller was born in Cincinnati, Ohio June 9, 1932. My education in the Cincinnati public schools was completed with my graduation from Walnut Hills High School in June, 1950. After attending Miami University and Ohio State University, I worked for about a year with the Jet Engine Department of the General Electric Co. in Evendale, Ohio until being drafted in May, 1953. After a tour of duty in Korea and being discharged May, 1955, I returned to General Electric for the summer and, since September, 1955, have been a full-time student at the Ohio State University. My wife and I were married June, 1956. I have been self-supporting, by teaching at the Mathematics Department at Ohio State University, and through employment at North American Aviation, Inc. in Columbus. I hold memberships in Pi Tau Sigma, Tau Beta Pi, and Sigma Xi. My B.S.M.E. and M.Sc. degrees were conferred June, 1959 and June, 1960, respectively at The Ohio State University.