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VIMOKE-SUNTAVA, Bunyut, 1933-
USE OF AN ELECTRICAL RESISTANCE NETWORK IN SOLVING PROBLEMS OF STEADY STATE FLOW OF WATER IN SOIL.

The Ohio State University, Ph.D., 1961
Engineering, agricultural

University Microfilms, Inc., Ann Arbor, Michigan
USE OF AN ELECTRICAL RESISTANCE NETWORK IN SOLVING PROBLEMS OF STEADY STATE FLOW OF WATER IN SOIL

DISSERTATION

Presented in Partial Fulfillment of the requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

Sunyut Vimoke-Suntava, B.S., M. Sc.

* * * * *

The Ohio State University

1961

[Adviser signatures]
ACKNOWLEDGMENTS

The author wishes to express his sincere appreciation to Dr. G. S. Taylor for the supervision and friendly encouragement throughout the study. The writer is also grateful to Dr. G. W. Volk, T. J. Thiel, and T. D. Tyra for their assistance and guidance in the construction of the electrical resistance network.

Sincere thanks are extended to the Ministry of Education of the Thai Government, the Ohio Agricultural Experiment Station and the United States Department of Agriculture, Agricultural Research Service.

This dissertation is dedicated to the author's parents who have made it possible to reach this goal.
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INTRODUCTION

Soil moisture content in a cultivated field bears a definite relationship to the rate of plant growth and the amount of crop yield. Its depletion as well as its excess can cause a harmful effect to plant growth.

When water enters the soil, part of it is retained in the pore space of the soil. The other part will percolate to lower depths. Investigations have been made by numerous soil scientists to study the movement of water through soil. Soil itself is a complex flow medium, however, and the study of water flow can only be done with difficulty.

There are several factors which affect the movement of water in soil. These are siltation by silt- and clay-sized colloidal particles, dispersion of organic colloid due to high salinity content, production of waxy materials by soil microorganism, variation of water viscosity due to changes in temperature, and many others. In general, the study of water flow in soil has to be done so that all except one or two factors are controlled. This is the case when one is interested in the effect of individual factors. It is nearly impossible, however, to control these factors without disturbing the soil structure. To simplify the study, the effects of various factors on water flow are grouped into a unique constant called the hydraulic conductivity $K$. Under saturated flow conditions, the term represents the overall characteristic property of the soil related to the ease that water can move through a particular
type of soil. Its units are expressed as unit length per time. In unsaturated flow, the hydraulic conductivity is dependent on the moisture content of the medium. As the moisture content decreases below saturation, the hydraulic conductivity likewise decreases. Also, as the moisture content decreases, the hydrostatic pressure of the soil increases in the negative direction. In other words, the soil moisture tension is increased. The unsaturated hydraulic conductivity is related, therefore, to both the soil moisture content and the hydrostatic pressure.

In agricultural aspects, excess water in the soil is a limiting factor in plant growth, since the depth of the water table will limit the depth through which plant roots can penetrate the soil. This excess water has to be removed from the soil to obtain maximum production of agricultural land. In soil classification, frequently, a soil is classified as well drained if the water table is low, and, poorly drained if the water table is high. The depth of an impervious soil layer has significant effects on the classification since it often determines the height of a water table after a rain, regardless of how porous the soil above the layer is. If a subsurface drainage system is used, this will reduce the adverse effects of the impervious layer. Thus, under this condition the soil is said to be poorly or well drained mainly due to the soil porosity or to its hydraulic conductivity.

In removal of excess or surplus water, two principle practices are utilized, surface and subsurface drainage. In subsurface drainage, one is concerned with such factors as the depth, size and spacing of
drains, the depth to the impervious layer, and the hydraulic conductivity of the soil. Among these factors, the hydraulic conductivity of the soil plays the most important role in drainage. Generally the soil is composed of different layers or horizons. The hydraulic conductivity of each horizon may be different from the others. Also, the hydraulic conductivity within each horizon changes as the moisture content of the soil decreases below saturation.

Various investigators have tried to solve drainage problems both by laboratory and field experiments. These studies are often seriously limited in regard to accuracy, economics and time. Attempts were also made to solve drainage problems by analytical approaches but these were limited by the complexity of the solutions. However, the flow of water through soil follows the same physical laws as the flow of fluid through other porous media and also the flow of electricity through conducting media. As a consequence, several types of analogs have been developed to study drainage problems. Significant among these are (a) numerical analysis, (b) the flow of electricity through electrical conducting paper, and (c) the electrical resistance network. In the past, the first and the last methods have been utilized but results obtained are found to yield less degree of accuracy and consistency in comparison with the results obtained from theoretical solutions. The second method can be used only to represent an isotropic saturated flow medium.

The objectives of this study are to present (a) a description of an electrical network analog, (b) a method used to calculate and assemble network resistances to represent water flow in soil, and
(c) to present improved theories which will significantly improve the accuracy of the data obtained from network analog study and numerical analysis.
Darcy's Law and the Equation of Continuity

In 1856, Darcy (5) investigated the vertical flow characteristic of water through saturated beds of sand. He showed that the rate of flow of water \( Q \) filtering through sand medium was directly proportional to the cross section area \( A \) of the sand, to the difference \( \Delta \phi \) between the fluid heads at the inlet and the outlet faces of the bed and inversely proportional to the thickness \( L \) of the bed. Expressed analytically,

\[
Q = \frac{KA \Delta \phi}{L}
\]

where \( K \) is a coefficient dependent upon the degree of permeability of the sand.

In utilizing Darcy's law to solve fluid flow problems, the law must be combined with an equation expressing the continuity of flow of fluid in the medium, the result being the so-called equation of continuity. The equation was described by Childs (2) and is written as

\[
\frac{\partial}{\partial x}(k_x \frac{\partial \phi}{\partial x}) + \frac{\partial}{\partial y}(k_y \frac{\partial \phi}{\partial y}) + \frac{\partial}{\partial z}(k_z \frac{\partial \phi}{\partial z}) = \frac{\partial c}{\partial t}
\]

where \( c \) is the volume of water per unit volume of permeable material, \( k_x, k_y, k_z \) are hydraulic conductivity in the \( x, y \) and \( z \) direction, respectively, and \( t \) is time.
When steady state conditions are assumed for a homogeneous and isotropic medium, \( c = \text{constant} \), \( \frac{\partial c}{\partial t} = 0 \) and \( k_x, k_y \) and \( k_z \) are equal, equation [2] becomes

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0
\]

Eq. [3]

Equation [3] is widely known as Laplace's equation. Its significance was pointed out by Slichter (17) in 1899 that the equation applies to the flow of water in saturated soils.

Analytical Solution

It was mentioned at the end of the introduction that attempts have been made by various investigators to solve drainage problems, both by analytical solutions and by laboratory or field experiments. In 1939, Vedernikov (19) derived equations to solve the flow of water into tile drains in the form of elliptic functions. Kirkham (8) in 1940 derived equations to solve the same type of problems in the form of hyperbolic function. Gustaffsson (7), who was apparently unaware of either Vedernikov's or Kirkham's work, also solved these problems in terms of elliptic functions in 1946. In 1958, Vimoke and Taylor (20) derived the approximate equation to solve the same type of problems. However, the solution of water flow problem by analytical method is limited to simplified, idealized specific cases with the flow medium consisting of one or two layers of different hydraulic conductivities. It should be mentioned here, however, that analytical solutions play an important part in the development of various types
of analog studies: For example, the results obtained from the analytical solution are theoretical values and can be used to check the validity and accuracy of analog solutions. The analytical solution used most frequently in drainage studies is Kirkham's equation expressing the flow rate of water passing through saturated, isotropic soil into buried drains equally spaced above an impervious layer. The equation yields the flow rate $Q$ in terms of the soil hydraulic conductivity $K$, drain radius $r$, drain depth $d$, height of ponded water $t$, drain line spacing $a$, and depth to an impermeable layer $h$. The equation is expressed analytically as

$$Q = \frac{2\pi K(d + t - r)}{A + B} \quad \text{Eq. [4]}$$

where

$$A = \log_e \left[ \frac{\tan \frac{\pi}{4h} (2d - r)}{\tan \frac{\pi}{4h} r} \right]$$

and

$$B = \sum_{m=1}^{\infty} \log_e \left[ \frac{\cosh \frac{\pi ma}{2h} + \cos \frac{\pi r}{2h}}{\cosh \frac{\pi ma}{2h} - \cos \frac{\pi r}{2h}} \cdot \frac{\cosh \frac{\pi ma}{2h} - \cos \frac{\pi (2d-r)}{2h}}{\cosh \frac{\pi ma}{2h} + \cos \frac{\pi (2d-r)}{2h}} \right]$$

Equation [4] will be used to check the validity and accuracy of the analogs developed in later discussion.

The Relaxation Method

Aside from the analytical methods, there exists a numerical method which can be used to solve drainage flow problem. This is generally called the relaxation method. It is a rather simple method
but a powerful tool to solve problems too complex to be solved readily by analytical solutions. The method is based on the solution of
Laplace's equation for interior regions and with boundary conditions appropriate to the problem to be solved. In the process, the flow region is divided into numerous square grids. Numerical values are arbitrarily assigned to the potential at each point of the intersection of the grid or node point with the values along the boundaries taken in accordance with the conditions specified for the particular problem. These numbers are then adjusted empirically until the value at each grid point is the arithmetic mean of those at the four adjacent points. This method assumes that the hydraulic head drops linearly from one grid point to the adjacent ones. The method was applied by Luthin and Gaskell (14) to solve steady state drainage flow problems. Although the method is simple and powerful, its greatest limitation lies in the fact that it is laborious and takes considerable time.

With the recent advent of high speed computers, certain problems can be rapidly solved by numerical analysis techniques. Since considerable time is often required to program problems on a computer, numerical analysis cannot conveniently be used to study a large group of flow problems at the present.

**Electrical Conducting Paper and the Network Analog**

Slichter pointed out the analogy of Ohm's law to Darcy's law. Childs (4) has made extensive use of this relationship in his study of a wide variety of ground-water problems. He used an electrical analog
made by soaking sheets of filter paper in graphite. At present a conductive paper invented by Western Union is available commercially under the name of "Teledeltos" paper. Electrical conducting paper can be utilized to solve steady-state flow problems in porous media which are saturated, isotropic and homogeneous with respect to its hydraulic conductivity $K$.

Recently, several investigators (6, 11, 16) have combined the principle of numerical analysis with the idea of the electrical analogy to produce electrical resistance network, sometimes called a voltage analyzer. Such a network produces a solution having the same degree of accuracy as one by numerical analysis. The resistance network possesses the important advantage of instantaneously relaxing the entire net. However, time has to be spent in calculating and setting the resistances for the network. At present, the use of high speed computers and electrical network analogs to solve drainage problems found to be equally, or about equally, effective in solving flow of water in homogeneous, stratified or anisotropic media and, with some modification, unsaturated media. Some of these investigations are reported by Bouwer (1), Luthin and Day (13), and Taylor (18). Some information will also be given in the latter part of this study dealing with different approaches in representing unsaturated soil with a resistance network.

Correspondence between Electrical and Water Flow in a Conducting Medium

As mentioned above concerning the relaxation method, Laplace's equation was solved arithmetically for the hydraulic head $\phi$ until the
value at each grid point is the arithmetic mean of those at the four
adjacent points. If square meshes are drawn as shown in
Figure 1, and values of hydraulic head $\phi$ are assigned
to each corner, it was shown by the relaxation method (14) that for a saturated homo-
geneous and isotropic medium,

$$\phi_0 = \frac{\phi_1 + \phi_2 + \phi_3 + \phi_4}{4} \quad \text{Eq. [5]}$$

The relaxation method can be applied to analogous problems in
electrical conducting paper or in circuit theory. If we have a network
of equal resistance, the potential $V_0$ at a node is equal to one-fourth
the sum of those at the four adjacent nodes. That is,

$$V_0 = \frac{V_1 + V_2 + V_3 + V_4}{4} \quad \text{Eq. [6]}$$

This can be proved by utilizing Kirchoff's and Ohm's laws. First ap-
plying Kirchoff's law to Figure 2, the algebraic sum of the currents $i$
entering and leaving the node whose potential is $V_0$ is equal to zero.
That is,

$$i_1 + i_2 + i_3 + i_4 = 0 \quad \text{Eq. [7]}$$
By utilizing Ohm's law, equation [7] becomes

\[
\frac{V_1 - V_0}{R_1} + \frac{V_2 - V_0}{R_2} + \frac{V_3 - V_0}{R_3} + \frac{V_4 - V_0}{R_4} = 0
\]

Eq. [8]

Fig. 2. -- Portion of resistance network

In the case where all of the values of resistance are equal, i.e., \( R_1 = R_2 = R_3 = R_4 \) (square network representing uniform medium), the expression of the potential \( V_0 \) will be given by equation [6]. Thus the voltage \( V_0 \) is analogous to the potential \( \Phi_0 \) used in the relaxation method.
SATURATED FLOW

Methods and Material

Representing Soil Medium by Electrical Conducting Paper

If one has a square piece of conducting paper with dimension of \( a \) by \( a \), it can be used to represent any square dimension of uniform soil medium having unit thickness (see Appendix I). To illustrate, first select a piece of teledeltos paper of dimension 1" by 1". This piece of paper can represent a block of soil medium one foot in the horizontal direction and one foot in the vertical direction. The same piece of paper can also be used to represent a 10' by 10' block of soil. In both cases, the block of soil is of unit thickness. For example, the block of soil represented in the first case has the dimensions of a cube of one foot, while the block represented in the second case has the dimensions of 10' by 10' by 1'. A piece of conducting paper of dimension 5" by 5" will serve the same purpose.

Representing Soil Medium by a Network of Resistors

Consider the boundaries formed by the four lines which bound the square of resistive paper shown in Figure 3a. If one is not concerned with conditions which exist inside these boundaries, we would obtain the same information from using either resistive paper or a group of resistors. It will now be shown that four resistors can be used to represent each square of resistive paper. Since a block of soil can be
Fig. 3—Ways to represent a square piece of electrical conducting paper by a group of resistors.

Fig. 3

(a) \[ \text{conducting paper} \]

(b) \[ \begin{array}{c} 2R_0 \\ 2R_0 \end{array} \]

(c) \[ \begin{array}{c} R_0 \\ 2R_0 \\ 2R_0 \\ \frac{R_0}{2} \end{array} \]

(d) \[ \begin{array}{c} R_0 \\ R_0 \\ R_0 \\ R_0 \end{array} \]
represented by a square of resistive paper, then it will also follow that the resistors will do likewise.

If groups of \( n \) resistors are used to represent a square block of soil, there are different ways to put them together (see Fig. 3). The arrangement that is most convenient is shown in Figure 3b, and this arrangement of resistors will be used throughout the rest of the discussion.

The justification of using \( n \) resistors to represent a square piece of resistive paper can be shown as follows (see Figs. 4 and 5). Take a square of resistive paper and paint two highly conductive lines on it. Silver paint is usually used, similar to that used in printed circuits. Measure the electrical resistance \( R_0 \) between these lines. The resistance \( R_0 \) is called the "characteristic resistance" of the medium, and its reciprocal yields the electrical conductivity \( G \). Now do the same with the group of \( n \) resistors (in Figs. 4b and 5b) using wires instead of painted lines, which is electrically the same thing. In all cases, the resistance is \( R_0 \) ohms, which is the characteristic resistance of the resistive paper.

It should be observed here that the directions in which the resistances are measured are orthogonal. For a two-dimensional flow problem, we can completely specify the resistance of a portion of the medium by the resistances measured in just two directions. The resistance of the medium in any other direction can be expressed in terms of its horizontal and vertical components. For convenience, one of these directions is chosen to be the vertical, and the other
Figs. 4 and 5.—Verification of using 4 resistors to represent a square soil block or a square conducting paper.
horizontal. This choice is made because many boundaries in drainage
problems are usually either horizontal or vertical and because soils are
principally anisotropic with respect to hydraulic conductivity in these
two directions.

In regard to representing a cross sectional area of soil with
four resistors, the same conclusion can be drawn here as with the square
of conducting paper. The mesh of resistors shown in Figure 3b can
represent a square block of soil of any dimension. It is true, however,
that more information can be obtained from a resistance network if the
blocks of soil represented in this manner are small. This is because
one cannot make electrical measurements inside the simulated block of
soil.

The "Building Block" Approach of
Representing Soil Medium

Two resistors in parallel can always be combined into a single
resistor. The use of a single
resistor is merely an economical
convenience. Two resistors
($R_1$ and $R_2$) in parallel are
equivalent to a single resistor
having a resistance of $R$ ohms
(see Fig. 6).

\[
\frac{R_1}{R} \cap \frac{R_2}{R} = R
\]

\[
\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}
\]

\[
R = \frac{R_1 \cdot R_2}{R_1 + R_2}
\]

Eq. [9]
If both $R_1$ and $R_2$ are equal to $2R_0$ ohms, then $R$ is equal to $R_0$ as shown by equation [9a]:

$$R = \frac{R_1 \cdot R_2}{R_1 + R_2} = \frac{2R_0 \cdot 2R_0}{2R_0 + 2R_0} = R_0$$

Eq. [9a]

In most cases, the magnitude of a single resistor in the network is derived from a consideration of two resistance values originally being in parallel. When considering theoretical problems, it is desirable to think in terms of the original resistance values that compose the building block\(^1\) rather than the values of single resistors that result from parallel combinations. To illustrate further, look at the square grid ABCD (Fig. 7c) with resistors $R_0$. Actually, this is just the end result when the building blocks of Figure 3b have been put together. In this finished product, all parallel resistors have been combined into single resistors. It is important to remember that network is started with the basis building blocks as shown in Figure 7a. If there had been no other resistors in an adjacent mesh to form a parallel combination, it would have been left unaltered; and, its value would have remained at $2R_0$. This is what happens at the boundaries of a resistance network (see Fig. 8). The resistors along the top surface and down the right side had no others with which to combine, so they

---

\(^1\)In construction of the resistance network, each mesh of the network represents a single block of soil. These single blocks are then joined together to form the whole network. We choose to call this individual block of soil and the mesh of four resistors that represent it a "building block."
Fig. 7—Joining square-grid building block
Fig. 8.—Network of resistors showing the value of the resistance along the boundaries.
were left at their original value of $2R_o$. The reasoning is clear from the building block approach why boundary resistors are always twice the value of interior resistors in a resistance network.

**Representing a Rectangular Block of Soil**

Up to this time only square building blocks are considered, these being a specialized case of the more general rectangular shape. To determine the values of the 4 resistors which represent a rectangular block, a slightly modified approach will be introduced: Consider a rectangular block of soil and a piece of resistive paper of the same shape. This block of soil can also be represented by 4 resistors as shown in Figure 9. The problem now is to determine the values of $R_a$ and $R_b$.

![Diagram of resistors](attachment:image.png)

**Fig. 9.** Representing a block of soil by 4 resistors.

To determine $R_a$, visualize the procedure outlined in Figure 10. A piece of resistive paper is cut horizontally into two equal sections. Each of the sections of Figure 10b can be thought of as being represented by a resistor of value $R_a$ (see Figs. 10c, 11a and
Figs. 10 and 11.—Schematic drawing to show representation of a block of soil in horizontal direction by 4 resistors.

11b). This value can be computed by equation [10]:

\[ R_a = \frac{\text{length}}{\text{width}} R_o = \frac{a}{b/2} R_o = \frac{2a}{b} R_o \quad \text{Eq. [10]} \]

The total resistance in the horizontal direction is given by equation [11]:

\[ R_{\|} = R_a \| R_a = \frac{R_a \cdot R_a}{R_a + R_a} = \frac{R_a}{2} = \frac{a}{b} R_o \quad \text{Eq. [11]} \]

To obtain the values for the vertical resistors, the same approach is used. By taking a piece of resistive paper representing the block of soil as shown in Figure 12, this time the paper is cut in the vertical direction. Each of the two resulting equal parts can be thought of as being represented by a resistor of \( R_b \) ohms. If the
Fig. 12.—Schematic drawing to show representation of a block of soil in vertical direction by 4 resistors.
equations [12] and [13] are utilized, the total resistance in the vertical direction $R_v$ can be obtained:

$$R_b = \frac{\text{length}}{\text{width}} R_o = \frac{b}{a/2} R_o \quad \text{Eq. [12]}$$

$$R_v = R_b \parallel R_b = \frac{R_b \cdot R_b}{R_b + R_b} = \frac{R_b}{2} = \frac{b}{a} R_o \quad \text{Eq. [13]}$$

To check the validity of $R_a$ and $R_b$, if $a$ and $b$ are set to be equal, then $R_a = 2R_o$ and $R_b = 2R_o$. The result agrees with the resistance of the square mesh discussed previously. The results of these calculations are summarized in Figure 13.

The proper resistors to represent a rectangular section of soil of any size can now be calculated. It is important to notice that only the ratio of $a$ to $b$ occurs in the final relationships. For example, the same values of resistors would be used to represent a block of soil one foot by four feet, two feet by eight feet, or three inches by twelve inches. Each block has the same ratio of 1:4. In Figure 14 through Figure 18 a number of commonly encountered examples are worked out.

Combining "Building Blocks" of Soil

After determining the values of resistances to represent each block of soil, the next job to do is to join them together. If there is a block of soil of dimension $2'$ by $2'$ which is to be joined to a $2'$ by $4'$ block, the procedure is shown in Figure 19. To join side $AB$ to $CD$, it is essential that $AB$ and $CD$ must have the same dimensions,
Fig. 13.—Final representation of a block of soil with dimension $a$ by $b$ with 4 resistors.
Figs. 14, 15, 16, 17 and 18.—Representation of soil blocks of different dimensions by meshes of resistors.
Fig. 19.—Joining two meshes of resistances representing two rectangular blocks of soil.
i.e., of 2 units each. To combine AB and CD in parallel, the resistance to be used at this boundary is given by equation [14]:

\[
\frac{R_{AC \overline{BD}}}{R_{AB} \parallel R_{CD}} = \frac{2R_0}{2R_0 + R_0} = 2R_0 / 3R_0 = \frac{2}{3} R_0 \quad \text{Eq. [14]}
\]

The physical significance of the combined blocks shown in Figure 19c can be summarized as follows:

1. The two blocks now have an outside dimension of 2 by (2+4) or 2 by 6.

2. Since it now has a dimension of 2 by 6, the total resistance in the horizontal direction (from KL to MN) \( R_H = 3R_0 \) and the total resistance in the vertical direction (from KM to LN) \( R_V = \frac{R_0}{3} \).

3. The outside resistances are unchanged. Each side portion can be joined to another block along the side of the same dimension. For example, \( KA \overline{AC} \) can be joined to another block along the side dimension of 2 and side \( \overline{AC} \overline{CM} \) with the side dimension of 4, et cetera.

There is another way to determine the combined resistance \( AC \overline{BD} \). This is done by a reverse analysis of the circuit. If we redraw Figure 19c by designating resistance \( AC \overline{BD} \) as \( R_l \) (Fig. 20a) and short out points K, \( \overline{AC} \), M and likewise points L, \( \overline{BD} \), N, we will have 3 resistors
Fig. 20

Fig. 20.—"Reverse analysis" of resistance $\overline{AC}$ $\overline{BD}$
joined in parallel as shown in Figure 20b. The overall resistance \( R_Y \) will be equal to \( R_0/3 \) as given by the first expression in equation [15]:

\[
R_Y = \frac{\text{length}}{\text{width}} \frac{R_0}{3} = 2R_0/3
\]

\[
\frac{R_0}{3} = 2R_0/\left( R_1/\left( R_0 \right) \right)
\]

or

\[
\frac{1}{2R_0} + \frac{1}{R_1} + \frac{1}{R_0} = \frac{3}{R_0}
\]

\[ R_1 = \frac{2}{3} R_0 \quad \text{Eq. [15]} \]

To check the value of \( R_H \), short out points KL and JN. The circuit is shown as in Figure 21. Notice that this circuit forms a Wheatstone Bridge so that the potential at points AC and BD are equal and the resistor AC BD can be disregarded. We now have the circuit as shown in Figure 21b and which is analyzed by the expression in equation [16]:

\[
R_H = \frac{(2R_0 + 4R_0)}{(2R_0 + 4R_0)}
\]

\[
\therefore \quad \frac{1}{R_H} = \frac{1}{6R_0} + \frac{1}{6R_0}
\]

\[ R_H = 3R_0 \quad \text{Eq. [16]} \]

The reverse analysis of a circuit is very helpful when one joins blocks of soil along sides which are different in dimension. The situation arises, for example, when one wishes to expand the fine meshes of the resistance network near the drain region to the coarse meshes in the region farther away from the drain. This expansion is done for
Fig. 21.—Reverse analysis to check the value of $R_o$ of the circuit shown in Figure 20a.
both practical and economical reasons since it has been found that in a drainage system most of the potential loss occurs near the drain. Since more detailed information is needed near the drain, a finer mesh is used in this region. In the region farther away from the drain, the head loss is smaller and less information is needed. Thus it is economical to use a coarser mesh in that region.

There is a certain type of building block which will permit the expansion of a fine mesh to a coarse one. The building blocks for this special boundary are not as easy to simulate as one may first think. As shown in Figures 22a and 22b, the network is to be expanded from small blocks of soil (D and E) with dimensions 1 by 1 to larger blocks (F) with dimensions 2 by 6. In such a case, blocks A, B, and C are introduced into the picture to function as boundary building blocks between a fine mesh D or E and a coarse mesh F. In actuality, blocks A, B, and C must be combined to a single block with the right side to be joined to the coarse mesh and the left side to be joined to the fine meshes. The problem arises as to how to join a side dimension of block A and B to the side dimension of block C.

To solve such a problem, one must attack it indirectly. One cannot say that one "joins" them together in the sense that blocks were joined in the preceding section. Rather one must look at the "finished product" and analyze it in reverse. For example, the finished product of the boundary building block is shown in Figure 23a, which can be represented by a network of resistors as shown in Figure 23b. It
Fig. 22—Expansion of fine meshes to coarse meshes
Fig. 23 — Representing boundary building blocks between the fine meshes and coarse meshes in the resistance network.
might be noted that all outside resistors are unaltered from their original value of $2R_0$.

As shown in Figures 23c and 23d, the finished block has an outside dimension of 2 by 3. From equations (11) and (13) the total resistance in the horizontal direction is $R_H = (3/2)R_0$ and that in the vertical direction is $R_V = (2/3)R_0$. With the outside resistances in Figure 23b being $2R_0$, they are ready to be joined to the fine mesh on the left with side dimension of 1 and to the coarse mesh on the right with side dimension of 2. Notice that until now nothing is said about the inside dimensions of GH, EH, and HF of Figure 23a. Now one must determine $R_1$, $R_2$, and $R_3$ in Figure 23b so that the overall resistance in the horizontal direction $R_H$ is equal to $(3/2)R_0$ and the overall resistance in the vertical direction $R_V$ is equal to $(2/3)R_0$, which are the original values of the resistances representing a 2 by 3 block of soil.

In determining $R_1$, $R_2$, and $R_3$ we have only two equations, i.e., the equations for $R_H$ and $R_V$. However it is possible to set $R_1 = R_3$ because of symmetry. One now has the network as shown in Figure 24a. To formulate the equation for $R_V$, one shorts out points AEB and DFC. We will then have the network as shown in Figure 24b. Notice also that the circuit G, AEB, H, DFC forms a Wheatstone Bridge so that the potential at points G and H are equal. Therefore, we can disregard the resistance $R_2$ in the calculation. We now have the circuit as shown
Fig. 24.—Circuit analysis to determine resistance $k_1$ of the boundary network between fine meshes to coarse meshes.

$$R_v = \frac{2}{3} R_0$$
in Figure 24c and the magnitude of $R_1$ is obtained by the following:

$$(2R_o + 2R_o) \parallel (R_1 + R_1) \parallel 2R_o = R_y = \frac{2}{3} R_o$$

or \[ R_1 = \frac{2}{3} R_o \]  \hspace{1cm} \text{Eq. [17]}

To solve for $R_2$, short out points AGD and BC of Figure 24a. The new network is shown in Figure 25a. This network can also be thought of as 2 sets of equivalent resistors AEB and AFB connected in parallel (see Figs. 25b and 25c), so that each set will have a resistance of $2R_H$. Therefore, set AEB can be analyzed by equation [18] (see Figs. 25d and 25e):

$$[(2R_2 + \frac{2}{3} R_o) \parallel 2R_o] + 2R_o = 2R_H = 3R_o$$

\[ \therefore R_2 = \frac{2}{3} R_o \]  \hspace{1cm} \text{Eq. [18]}

An interesting point is observed with regard to node H in the interior of this building block (Fig. 26a). The question arises as to where to plot the value of the voltage measured at this internal node. One's first inclination might be to plot it at the intersection designated by H in Figure 26b. It can be shown, however, that the actual point of correspondence is located at X. Consider the situation shown by Figure 26c. Let us short out AGD and BC and then apply 3 volts at BC while AGD is attached to the ground (i.e., at zero volt). If we measure the potential at E or F, we will have a reading of 1 volt, indicating that both E and F are located one-third of the distance from AD to BC as expected. If we then measure the voltage at H, we
Fig. 25. -- Circuit analysis to determine resistance $R_2$ of the boundary network between fine meshes to coarse meshes.
Fig. 26.—Locating interior point of the boundary block between fine meshes and coarse meshes.
will have a reading of 2/3 volt, indicating that \( G \overline{H} \) (or in reality \( G \overline{X} \) of Fig. 26b) is 2/3 the distance of either \( A \overline{E} \) or \( D \overline{F} \). Therefore, in the analog system, the voltage measured at the node \( H \) in Figure 26a should be plotted at point \( X \) located 2/3 of the distance between \( G \overline{H} \) as shown in Figure 26b. The "finished product" of the 2 by 3 block of soil is shown in Figure 26d and its network of resistors is shown in Figure 26a.

The value of the resistance \( G \overline{H} \) of Figure 26a can be used to locate point \( X \) of Figure 26d. In all cases the resistance \( R_{AE} \) is equal to \( R_{PF} \). The distance \( G \overline{X} \) is related to distance \( A \overline{E} \) as shown in equation [19]:

\[
G \overline{X} = \left( \frac{2G \overline{H}}{R_{AE}} \right) A \overline{E} \quad \text{Eq. [19]}
\]

Further illustrations will be shown in the study of anisotropic soil.

With the information on different types of building blocks discussed above, we are now ready to put them together to form a network representing a homogeneous soil medium. For example, let us represent a section of soil having dimensions of 5 by 20 by 1. The dimensional units are arbitrary, but we will consider the soil to have dimensions of 5 feet by 20 feet by 1 foot. The soil may be represented by the building blocks of soil shown in Figure 27a with side dimensions as indicated. Figure 27b gives the resistances representing the building blocks, while Figure 27c shows the assembled network. The arrangement of resistances shown in Figure 27c will yield the most accurate information on potentials along the left hand boundary where the meshes are smallest. This conclusion follows since potentials in a network can only be measured at the nodes. This arrangement is
Fig. 27.—Representing homogeneous soil sections by resistance network.
particularly useful when a subsurface drain is located in the region of small meshes. Since the potential drop is quite large near a sink (i.e., the drain), greater accuracy is needed in that region.

In a later section the simulation of building blocks in the region near a circular drain will be discussed. It will be shown that those resistances which represent the drain can be inserted at any depth on the left side of the network shown in Figure 27c. If a half drain were placed on the left boundary and constant (but different) potentials applied at the drain and at the top boundary of the network, one would simulate ponded flow into a drain. The left hand side of the network would represent the vertical plane through the center of the drain. The top part of the network would represent the soil surface, and the bottom part, an impervious layer. The right boundary of the network would represent a vertical plane across which flow does not occur and would correspond to the plane between equally spaced drained in level topography (11).

Observe that in Figure 27a, if we use a mesh size of one foot, we will have the impervious layer at 5 feet. If one wishes to locate the impervious layer at 4 feet, the top row of the building block can be removed. With this type of arrangement one can build the network with odd or even depth of the impervious layer by adding or removing the top row accordingly.
Representing a Stratified Soil

So far all the concepts are dealing with homogeneous medium where the entire soil is of the same hydraulic conductivity $K$. In actuality, the soil profile is divided into different horizons. Generally, each horizon has a different conductivity. For simplicity, let us choose a soil profile having three layers of different conductivities, namely, $K_1$, $K_2$, and $K_3$, respectively, from the top layer to the bottom one. Also designate $K_1$, $K_2$, and $K_3$ as 0.24, 0.56 and 0.28 cm/hr. The ratio $K_1 : K_2 : K_3$ will be 6 : 2 : 1, respectively. These particular values are chosen so that $K_1 / K_2 = 4$ and $K_2 / K_3 = 2$. Let the characteristic resistance be 5,000 ohms and designate this value as $R_2$ in order to represent $K_2$. One can see that $R_1$ (representing $K_1$) is $5000/4$ or 1250 ohms and $R_3$ (representing $K_3$) is $5000 \times 2$ or 10,000 ohms. In the $K_1$ region, a square block of soil can be represented by 4 resistors as shown in Figure 28b. Figures 28e and 28g represent a square block of soil in the $K_2$ and $K_3$ regions, respectively. Since 5,000 ohms is used as characteristic resistance of a square mesh, it is more convenient to express both $R_1$ and $R_3$ in terms of $R_2$. Figures 28c and 28h show the transformations. Example of the networks for stratified layers are shown in Figures 29a, b, c and d.

Representing an Anisotropic Soil

In the previous discussion we were dealing with soil that had a different hydraulic conductivity in each layer or horizon. It is sometimes found that within the same layer of soil, the hydraulic
Fig. 28.—Representing square blocks of soil (stratified soil) having different hydraulic conductivity $K$ with meshes of resistors.
Fig. 29.—Representing sections of stratified soil of 3 layers by resistance networks.
Fig. 29. (continued)—Representing sections of stratified soil of 3 layers by resistance networks.
conductivities in the horizontal and in the vertical are also different. This type of soil is called anisotropic with respect to its hydraulic conductivity. To set up a network to represent an anisotropic soil, let us designate a soil having two layers. The upper layer has a hydraulic conductivity in the vertical direction, $K_{v1}$ as 0.12 cm/hr. and in the horizontal direction $K_{h1}$ as 0.03 cm/hr. In the lower layer, $K_{h2} = 0.06$ cm/hr. and $K_{v2} = 0.03$ cm/hr. By designating the conductivity of 0.03 cm/hr. as 1.0 one can see that the ratio of $K_{v1} : K_{h1}$ of the upper layer is 4:1 and of the lower layer as 1:2 (see Fig. 30). By choosing the characteristic resistance of 5,000 ohms ($R_0$) to represent the $K_{h1}$ and $K_{v2}$ (each equalling 0.03 cm/hr.) one can see that resistance $R_{v1}$ representing $K_{v1}$ is 1,250 ohms while the resistance $R_{h2}$ representing $K_{h2}$ is 2,500 ohms. In the upper layer, a square block of soil can be represented by 4 resistors as shown in Figures 30a and 30b and in the lower layer by Figures 30c and 30d. An assembled network of resistors for this soil is shown in Figures 31a, 31b and 31c. The overall distances of the simulated flow medium are arbitrarily selected.

Special attention is called to the location of the internal nodes of the boundary building block between the fine and coarse mesh. As shown in Figure 31a, the internal node of the boundary building block of the upper layer is plotted at the point $11/12$ part of 1 and that of the lower layer at $1/3$ part of 1. These locations must be determined by either calculation or experimentation as shown in the section "Combining Building Blocks of Soil."
Fig. 30.—Representing square blocks of soil having different hydraulic conductivity in the vertical and horizontal direction (anisotropic soil) by meshes of resistors.
Fig. 31.—Representing sections of anisotropic soil of two layers with a resistance network.
Representing a Circular Drain

Building blocks up to this point have always represented rectangular sections of soil. However, the drain is usually circular and therefore the building block concept must be extended to include such curved sections. To approach this problem, first cut from a piece of resistive paper the section drawn with a solid line as shown in Figure 32a. In this figure, \( r \) is the drain radius and \( s \) is the square mesh size around the drain. At the present time it will be convenient to remember that this section is just one-fourth of the entire section as shown in Figure 32a. If the boundaries of the square and the circle are painted with conducting paint as shown in Figure 32b, the resistance \( R \) can then be measured. However, direct measurement is neither convenient nor accurate. This problem has been solved mathematically for electrical transmission lines and wave guides. The resistance \( R \) of Figure 32b is essentially the same as the impedance \( Z_0 \) of a square wave guide with a circular inner conductor as shown in Figure 32c. The only difference between \( R \) and \( Z_0 \) is a conversion of factor \( Z'_0 \) relating the characteristic resistance of free space and the characteristic resistance (resistance per square = \( R_0 \)) of the resistive paper. The relationship is given by equation [20] [see (10), pp. 426-429]:

\[
R = \frac{R_0}{Z'_0} Z_0
\]

Eq. [20]

The expression for \( Z_0 \), a transmission line as shown in Figure 32c, is given by equation [21] [see (22), p. 590].
Fig. 32.—Diagram showing approach used to calculate the resistance ($R_2$) used to represent region around the drain. The value $R_2$ is equal to $8R$ where $R$ is the resistance of the conducting paper shown in Figure 32b. The value of $R$ is calculated using the analogy between $R$ and the impedance $Z_0$ of the transmission line shown in Figure 32c. The relationship between $R$ and $Z_0$ is expressed by equations [20] and [21].
\[ Z_0 \approx 138 \log_{10} \rho + 6.48 - 2.34A - 0.48B - 0.12C \]  
Eq. [21]

where

\[ \rho = \frac{D}{d} \quad \text{(Fig. 32b)} \]

\[ A = \frac{1 + 0.405 \rho^{-4}}{1 - 0.405 \rho^{-4}} \]

\[ B = \frac{1 + 0.163 \rho^{-8}}{1 - 0.163 \rho^{-8}} \]

\[ C = \frac{1 + 0.067 \rho^{-12}}{1 - 0.067 \rho^{-12}} \]

and

\[ Z_0' = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 120\pi = 376.7 \text{ ohms} \]

where

\[ \mu_0 = \text{permeability of free space} \]

\[ \varepsilon_0 = \text{permittivity of free space} \]

In the network, if we are dealing with a square mesh around the drain with mesh size \( s \) and radius of the drain \( r \), we see that \( s = \frac{D}{2} \) and \( r = \frac{d}{2} \). Therefore \( \rho = \frac{D}{d} \). From these expressions, the resistance \( R \) of Figure 32b can be determined.

Returning to the original problem, if we paint conducting lines on the outside and inside boundaries of Figure 32d, and then measure the resistance, we will get \( 4R \) ohms since this portion represents only one of the four quarters which are in parallel as shown in Figure 32a. The resistance of the section shown in Figure 32d can be represented by the resistances shown in Figure 32e. In this particular measurement, the resistors of the value \( R_1 \) are effectively shorted out so that the
circuit reduces to Figure 32f. Taking an overall view of Figure 32d
leads us to the relationship:

\[ hR = \frac{R_2}{2} \quad \text{or} \]
\[ R_2 = 8R = \frac{8Z_0}{Z'_0} \quad \text{Eq. [22]} \]

The next question to ask is what value to pick for the R1
resistors in Figure 32e. In the experiments, the value of R1 = 2R0
is chosen. In every way this value seems reasonable, but there is
no basis to assume that it is exact. It may very well depend on the
diameter of the drain as does R2. However, a number of experiments
using R2 equal to 2R0 have yielded results which are in good agreement
with exact analytical solution of Kirkham which is the equation [4]
shown in this text.

To illustrate the above approach in calculating the resistances
to represent the drain region, let us designate the size of the mesh
as 1 foot, radius of the drain r = 3", R0 = 5,000 ohms.

\[ Z_0 = 138 \log_{10} \phi + 6.48 - 2.34A - 0.48B - 0.12C \]

where
\[ \phi = \frac{12}{3} = 4 \]
\[ A = 1.00317 \]
\[ B \approx 1 \]
\[ C \approx 1 \]
\[ Z_0 = 138 \log_{10}4 + 6.48 - 2.3473 - 0.46 - 0.12 = 86.6169 \]

\[ R_2 = \frac{8Z_0}{Z_0} = \frac{8 \times 86.6169 \times 5000}{376.7} = 9197 \text{ ohms} \]

\[ \frac{R_2}{2} = 4599 \text{ ohms} \]

From the above example, we can see that for a given value of \( \rho \) we can calculate the drain resistance \( R_2 = \frac{8Z_0}{Z_0} R_0 \). If we define the constant \( C_d = \frac{8Z_0}{Z_0} \), the drain resistance is given by \( R_2 = C_d R_0 \).

Table 1 reports a few values of \( C_d \) along with the corresponding values of \( \rho \). The values reported therein will suffice for calculating those values of the drain resistance which are normally encountered in network studies.

We now have the building blocks necessary to represent the area around the drain. In Figure 33 these building blocks are put together to form a network in a homogeneous soil with a square mesh around the drain.

Utilizing the Resistance Network to Study Flow Problems

In network studies, there are three things in which we are primarily interested: (1) the potential distribution from which we can obtain -- among other things -- equipotential lines, (2) the streamlines from which we can learn the direction of water movement through the soil, and (3) the flow rate or, in the case of drainage, the amount of water that we can remove from the soil for a given period of time. All of these data can be obtained directly with the network.
Fig. 33.

Fig. 33.—Diagram showing connection of resistance representing drain region to those representing soil medium.
The function \( \rho \) is equal to the ratio \( s/r \), where \( s \) is the size of the square mesh at the drain and \( r \) is the drain radius. The magnitude of the drain resistance \( R_2 \) as shown in Fig. 33 is given by \( R_2 = C_d R_0 \).

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<td>2.0000</td>
<td>0.95483</td>
<td>( \infty )</td>
<td>( \infty )</td>
</tr>
</tbody>
</table>

As an example, consider the ponded flow case in drainage:

Multiple drains of radius \( r \) are equally spaced in a homogeneous soil at a distance \( a \) and at a depth \( d \). Water is ponded on the level ground surface to a height \( t \). The drains are completely permeable to water
and are running full with no back pressure. An impermeable horizontal layer lies at a depth h below the ground surface. First the network is assembled as shown in Figure 34. To obtain the potential distri-

![Diagram](image)

**Fig. 34**—Application of voltage to the network.
The positive pole of the battery is connected to the top boundary and the ground is connected to the drain. A vacuum tube voltmeter (VTVM) is used to measure voltage distribution at each grid point.

bution, an e.m.f. of 100 volts is then applied to the top boundary. The drain terminal (see Fig. 33) is connected to the ground as shown by the diagram in Figure 34. A voltmeter is used to measure the potential at each node. A vacuum tube voltmeter (VTVM) is recommended since its internal resistance is very high (around 11 megohms) and the potential measurement can be done more accurately. For convenience, the potential difference of 100 volts is applied to the upper network so that the
voltage reading at any node yields the potential directly as a percentage of that at the ground surface. After the potentials are measured at each node, the equipotential lines can be drawn by interpolating between potentials at adjacent nodes.

To obtain the flow line, the boundaries of the network are now "reversed" as shown in Figure 35. This is done by first removing the attachments from the top boundary and the drain terminal. This time an emf is applied to the previously unconnected boundaries as shown in Figure 35. The potentials are again measured at each node, and a new set of equipotential lines are drawn. The new set of equipotential line will be the flow lines, that is, the two sets of equipotential lines will be orthogonal.

Fig. 35.—Reversing the boundary of the network to obtain streamlines.
To measure the drain flow rate, two different procedures can be followed: From equation \[4\] in Appendix I, we see that the applied voltage \(V\) and the hydraulic head potential \(\phi\) are related by a conversion coefficient \(C_0\). The potential at the top boundary \(\phi_n\) is equal to \(d + t\) where \(d\) is the distance between the drain center to the soil surface and \(t\) is the height of ponded water. The potential at the drain \(\phi_0\) is equal to \(r\), the radius of the drain. This boundary condition is obtained from the assumption that the drain is running full with no back pressure. Thus the potential difference from the top boundary to the drain is equal to \((\phi_n - \phi_0)\) or \((d + t - r)\). If we apply \(V\) volts to the upper network boundary, the potential \(V_n\) at the top boundary is \(V\). The potential at the drain terminal \(V_0\) is zero. Thus \((V_n - V_0) = V\) volts; and \(C_0\) can be determined by equation \([23]\):

\[
C_0 = \frac{V}{d + t - r} \quad \text{Eq. [23]}
\]

After \(C_0\) is determined, we then measure the total current \(I\) passing through the circuit. The flow rate \(Q\) is then determined by the equation \([7a]\) in Appendix I.

One disadvantage of the above method is the error and inconvenience involved in measuring the current \(I\). In the first place, the characteristic resistance \(R_o\) should be chosen in the range of 5,000 to 10,000 ohms to minimize errors in voltage readings at the network nodes. This range is chosen so that the resistance of the network will be far less than the internal resistance of the VTVM of 11 megohms and far more than the resistance due to wiring and metal contact (i.e., from
1 to 10 ohms). When the characteristic resistance is chosen in this range, the current passing through the network will be too low for accurate measurement. To increase the current reading, one has to increase the applied voltage. This is somewhat dangerous and inconvenient. Also there is the risk of altering the resistance values in the network due to greater current flow.

These disadvantages are avoided by the following method for determining the flow rate: In this method the total resistance $R_n$ between the top boundary of the network and the drain terminal is measured. This is done by attaching wires A and B of Figure 34 to an ohmmeter rather than to a battery, or better, to a Wheatstone Bridge. The resistance of the network $R_n$ can be measured to an accuracy with 0.5 per cent with a Wheatstone Bridge. The current $I$ is given by $\frac{V}{R_n}$ and from equation [23], $V = C_0(d + t + r)$. Therefore, the current $I$ can be expressed as follows:

$$I = \frac{C_0(d + t + r)}{R_n} \quad \text{Eq. [24]}$$

If the value of $I$ is substituted in equation [7a] of Appendix I, the following relationship will be obtained as shown in equation [25]:

$$Q = 2K(d + t + r) \quad \text{Eq. [25]}$$

The factor 2 shown in equation [22] is to convert the flow rate from that of a half drain to that of a whole drain since $R_n$ is the resistance of only half of the flow region. Also from equation [22] one can see that the drain flow rate $Q$ is independent of the applied
voltage $V$ and the current $I$. All we need to know is the resistance $R_n$ across the network. The general expression for the flow rate can be expressed as:

$$Q = \frac{K(\phi_n - \phi_0)}{R_n} R_o \quad \text{Eq. [26]}$$

where $\phi_n$ is the hydraulic potential at the top boundary and $\phi_0$ is the hydraulic potential at the drain or at the point of discharge.

Construction of the Resistance Network

It was mentioned previously that the resistance networks were constructed and used by various investigators such as DePackh (6), Liebmann (11), and Redshaw (16). These networks were used to solve Laplace's equation, partial differential equation or as electrical potential analyzer. Luthin (12) extended the use of the network to solve problems of irregular boundaries found in drainage and seepage studies. Bouwer and Little (1) reported the use of the network to solve drainage problems of water flow through unsaturated soil.

In July 1959 an electrical resistance network was built at the Agronomy Department (Soil Physics) of The Ohio State University under the supervision of Dr. G. S. Taylor. The funds were made available under cooperative agreement of the Ohio Agricultural Experimentation and the United States Department of Agriculture, Agricultural Research Service. The objectives of the construction were to build the network which incorporates certain features that increase network flexibility, accuracy and ease of operation.
In the construction, the network of resistors is mounted on a 6 by 8 foot piece of 1/4 inch masonite (see Fig. 36). It will accommodate 575 plug-in type resistive units. Each plug-in unit has one or two variable resistive elements, which are wired to two banana plugs (electrical terminology) spaced on 2-inch center and plugs into the banana jacks on the network (see Fig. 37). The units make good electrical contact, are held firmly in place, and are not affected by
Fig. 37.—Close-up view showing a plug-in resistive unit consisting of two potentiometers wired to two banana plugs. The unit is placed in the network by inserting the plugs into two jacks on the board. The two potentiometers are joined together in series or parallel depending upon whether high or low resistances are needed.

normal vibrations encountered in a laboratory. Standard potentiometers are used for this purpose, and they are useful over a 1 to 1/5 range, i.e., from 5,000 to 1,000 for a 5,000 ohm potentiometer. In some cases two potentiometers were mounted on the same plug-in unit. The latter arrangement gives a useful range from 1 to 1/10 if identical potentiometers are used and a range from 1 to 1/25 if one potentiometer has a value 5 times larger than the other. Any desired
resistance can be used in the network by merely selecting a plug-in unit covering an appropriate resistance range, setting it to the desired value, and plugging the unit into the network.

Control Console for the Network

The control console shown at the left in Figure 36 contains all the electrical equipment associated with the analog. It contains a variable AC power supply, variable DC power supply, Wheatstone Bridge to set the resistance of the plug-in units and provisions for measuring both AC and DC currents and voltages. The AC supply consists of a voltage variable transformer (variac) which supplies from 0 to 130 volts. The DC power supply furnishes from 0 to 300 volts. The latter serves as the utility DC supply, especially when stability is needed over an extended period of time. Such stability is needed when accurate experimental data are being obtained. Any precise AC and DC voltage can be obtained by simply varying a ten-turn potentiometer (helipot) connected to the common ground. In addition, all circuits associated with the network are also contained in the control console. A small replica of the network is shown on the left side of the panel. The various boundaries on the network are connected electrically to their counterparts on this replica by means of shielded cables. Any connections made to the replica will be automatically connected to the network. Figure 38 shows the wiring diagram of the control console.
Fig. 38.—Wiring diagram of analog and control console.
Selecting the Characteristic Resistance of the Network

The "characteristic" resistance $R_0$ selected for the network is 5,000 ohms. This means that the interior region of a uniform medium would be represented entirely by 5,000 ohm resistances. If layers of different conductivity are to be used or if certain regions are to be represented by an expanded mesh, then most flexibility can be obtained by selecting a characteristic resistance near this value. A resistance of 5,000 ohms is ideally suited because it lies approximately midway on a logarithmic scale between an upper limiting value of 11 megohms and a lower limiting value of 1 ohm. The lower limiting value represents wiring resistances and collective resistances in the many plug and jack assemblies. Such resistances will result in large errors if the characteristic resistance of the network is too low. The upper limiting value is imposed by the instrument used to measure network voltages. Vacuum tube voltmeters are commonly used to make such measurements, and these instruments usually have an internal resistance of 11 megohms. If the resistance of the circuit being measured is also in the vicinity of 11 megohms, then the network voltage will change significantly when the voltmeter is connected into the circuit and results in erroneous readings. In other words, the internal resistance of the measuring device should be sufficiently higher than the resistance of the circuit being measured so that the measuring device will not disturb the operation of the circuit. The value of 5,000 ohms gives a 20 to 1 ratio of flexibility in either direction while still remaining in the 1%
accuracy range -- that is, 5,000 ohms down to 250 ohms and from 5,000 ohms up to 100,000 ohms. Ratios of 100 to 1 can be used while staying within a 5% accuracy range. Analytical discussion dealing with errors due to choosing the characteristic resistance will be discussed in detail in Appendix III.

Setting the Resistance Units

A Wheatstone bridge is used to set the resistance of the plug-in units. In two arms of the bridge are placed precision resistors (0.05%) of equal magnitude. Resistance decade boxes of 0.05% accuracy are placed in another arm of the bridge covering the range of 0.1 ohm up to 1 megohm. The plug-in unit to be set is plugged into the remaining arm of the bridge. These units can easily be set to within 0.5%. The total error in setting the individual units is about 1% at maximum, and since many of them are used, some of these errors cancel each other. An overall accuracy of 1% can be expected in using the network. The time required to set and assemble the 575 resistive units in the network varies from 3 to 5 hours, while the time required to obtain the voltages at the grid points is approximately 10 minutes. In the actual use of the network, several hours of set-up time can be saved by utilizing fixed resistors of 5,000 ohms ± 0.5% in place of the variable resistance units. A large number of these are utilized in the interior of the fine mesh region. Analytical discussion dealing with percentage due to setting the resistance with Wheatstone Bridge will be discussed in detail in Appendix III.
Voltage and Current Measuring Equipment

If the network is operating within 1% tolerance, it is desirable to measure network voltages to something better than 1%. Unfortunately, most meter-reading instruments found in the laboratory are usually only accurate to within 5% or at best, 3%. This error is too high if the data are to be used for further calculations or verification of theory. Also, the accuracy of the network goes to waste if an inaccurate meter is used to measure voltages and currents. The desired accuracy in the voltage-measuring instrument was obtained by using a digital VTVM. This instrument has an accuracy of 0.2% and is shown on the upper right of the control console in Figure 36. It will read voltages down to one-thousandth of a volt and up to 999 volts. It also has a time-saving feature in that it automatically places the decimal point in the right place. This same instrument is used to determine electrical currents by measuring the voltage drop across a resistance of 1, 10, or 100 ohms which is in series with the circuit in which the current is flowing. Through the use of the variable DC power supply, the ten-turn potentiometer, and the digital voltmeter, experimental data can be obtained which are consistent, reproducible, and accurate.

Switching

Numerous switches are used on the network and the control console as time-saving applications. Switches are used to select AC or DC power sources and to route currents to the network or to the Wheatstone Bridge. A rotary switch selects the various points on the control console at
which voltage or current readings are to be made and automatically connects the meter into the desired circuit. One setting of this switch enables the meter to measure the total current flowing through the analog while another setting permits readings of current flowing into any particular node across the top boundary of the network. By throwing a pair of switches located above each node along the top boundary (see Fig. 36), the current flowing into a particular boundary node can be quickly measured. Similar pairs of switches are also located near the sides and the bottom nodes. By throwing one switch of each pair, along the sides and the bottom, one can "reverse" the boundaries and obtain a different set of potential distribution which can be used to map streamlines. The last setting on the rotary switch enables voltage measurement at all the internal nodes of the network by the use of a test prod connected to the console by a shielded cable.

Results and Discussion

Significance of Resistances Representing Drain Region

The combination of equations [20], [21] and [22] are utilized only recently to calculate the resistances representing the drain region. Previous investigators used the approach reported in the literature (12) and found that significant differences existed between flow rates calculated from network data and those obtained from analytical solution of Kirkham shown in equation [4]. The discrepancy was particularly large when a coarse mesh was used.
The procedure which has been reported in the literature (12) can be explained with the sketch shown in Figure 39. The resistance between A and Q is that given by equation [27]:

$$R_{AQ} = \frac{(s - r)}{r} R_0$$  \hspace{1cm} Eq. [27]

where \(r\) is the radius of the drain, \(s\) is the size of the square mesh, and \(R_0\) is the characteristic resistance of the network. In evaluating
the usefulness of equation [27], one must consider the case when the drain radius \( r \) approaches zero. Physically, the drain outflow would also approach zero. For this to occur, the resistance \( R_{AQ} \) must approach infinite. One can see from equation [27] that as \( r \) approaches zero, \( R_{AQ} \) approaches \( R_0 \). Thus as \( r \) becomes small compared to \( s \), equation [27] cannot be expected to give accurate results.

By utilizing equations [20], [21] and [22], derived in this text, \( R_{AQ} \) can be expressed as

\[
R_{AQ} = \frac{\mu Z_0}{Z_O} R_0 \quad \text{Eq. [26]}
\]

The resistance \( R_{AQ} \) is now associated with \( R_0 \) by two relationships: one as a linear function given by equation [27] and the other as a logarithmic function given by equation [26]. Figure 40 shows a

![Graph showing the ratio \( R_{AQ}/R_0 \) as yielded by equations [27] and [26]. The resistance at the drain and at an interior point is given by \( R_{AQ} \) and \( R_0 \), respectively (see Fig. 39).](image-url)

Fig. 40.—The ratio \( R_{AQ}/R_0 \) as yielded by equations [27] and [26]. The resistance at the drain and at an interior point is given by \( R_{AQ} \) and \( R_0 \), respectively (see Fig. 39).
comparison of $\frac{R_{AQ}}{R_o}$ in relation to the ratio $r/s$ when the former was calculated with equations [27] and [28]. Based on equation [27], the ratio $\frac{R_{AQ}}{R_o}$ varies linearly from zero to one as the drain radius $r$ varies from $s$ to zero. According to equation [26], $R_{AQ}$ will approach zero or infinite, respectively, as $r$ approaches $s$ or zero. If $r$ varies from $r = 0.5s$ (half the mesh size) to $r = s$, the two equations yield values which differ from 0 to 18 per cent. As the ratio $r/s$ becomes much less than 0.3, the deviation becomes quite large.

The validity of equation [27] and [28] was examined by evaluating the drain flow rates for the following situation: Open drains are embedded in a homogeneous soil and are running full with no back pressure. The half spacing between drains ($a/2$) is 72 feet, and an impervious layer ($h$) is 18 feet below the ground surface. The depth to the drain center line ($d$) is variable. The drain diameter ($2r$) is 4 inches. A water table is maintained at the ground surface, and a hydraulic conductivity ($K$) of 0.001 foot/min. is assumed.

Because of symmetry, the network was assembled to represent only half of the flow region. For horizontal distances up to 6 feet from the drain, each linear foot in the soil was represented by one resistive unit (i.e., a 1' by 1' mesh). Between 6 and 8 feet a 2' by 2' mesh was used, while a 2' by 8' mesh was employed for distances greater than 8 feet. For the 2' by 8' mesh, the horizontal distance was 8'. The resistances were calculated by the procedure reported previously in this text except for those resistances at the drain. These were calculated first by equation [27] and the drain flow rates $Q$
obtained by the use of equation [26]. The drain resistances were then calculated by equation [26] and Q evaluated again. Following these analyses, the 1' by 1' mesh was changed to a 2' by 2' mesh and all of the above evaluations were repeated. For the 1' by 1' mesh the ratio r/s was 0.167, while for the 2' by 2' mesh the ratio was 0.083.

The drain flow rates as determined with the network are shown in Table 2 along with those calculated by the analytical solution of Kirkham, equation [4]. Equations [27] and [26] were used to calculate the drain resistance (see text for details).

**TABLE 2.** Drain flow rates Q as determined with the resistance network and with the exact solution of Kirkham, equation [4]. Equations [27] and [26] were used to calculate the drain resistance (see text for details).

<table>
<thead>
<tr>
<th>Drain Depth (ft.)</th>
<th>Drain Flow Rate - Q x 10^3 (ft.³/ft. half drain-min.)</th>
<th>Deviation of Q Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.831</td>
<td>*1.760</td>
</tr>
<tr>
<td></td>
<td>*1.744</td>
<td>2.685</td>
</tr>
<tr>
<td>6</td>
<td>4.205</td>
<td>4.117</td>
</tr>
<tr>
<td></td>
<td>4.104</td>
<td>5.460</td>
</tr>
<tr>
<td>10</td>
<td>6.075</td>
<td>5.986</td>
</tr>
<tr>
<td></td>
<td>5.968</td>
<td>7.547</td>
</tr>
<tr>
<td>14</td>
<td>7.351</td>
<td>7.268</td>
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<tr>
<td></td>
<td>7.231</td>
<td>8.812</td>
</tr>
<tr>
<td>16</td>
<td>7.516</td>
<td>7.426</td>
</tr>
<tr>
<td></td>
<td>7.322</td>
<td>8.772</td>
</tr>
<tr>
<td>18</td>
<td>5.689</td>
<td>5.622</td>
</tr>
<tr>
<td></td>
<td>5.611</td>
<td>7.158</td>
</tr>
</tbody>
</table>

*Upper value for 1' by 1' mesh, lower value for 2' by 2' mesh.

**100 (Q_{28} - Q_{4}) / Q_{4}.

***100 (Q_{27} - Q_{4}) / Q_{4}.
Kirkham, equation [4]. These findings are also reported graphically in Figure 41. From these findings it can be seen that the use of

![Diagram](image)

Fig. 41.—Drain flow rates as determined by the resistance network and by Kirkham's equation [4]. Equations [27] and [28] were used to calculate the drain resistances for the network evaluations. A fine mesh (1' by 1') and a coarse mesh (2' by 2') were used in the region near the drain (see text).

Equation [28] yields values of Q which are close to those obtained from the exact solution. This equation gives smaller values of Q than those of the exact solution, while equation [27] gives larger values. When equation [28] was used, only small changes were obtained in Q when the mesh size near the drain was increased from a 1' by 1' to a
2' by 2' mesh. Under similar conditions and the use of equation [27], the percentage deviation in Q was more than doubled. Also of some significance is the effect of the drain resistances on the potential distribution, and these results are shown in Figure 42. In Figure 42, four potentials are shown at each node. Each potential at a node is obtained by a different method. Consecutively, the uppermost potential is obtained from a network of 1' by 1' meshes around the drain region. Equation [27] is used to calculate the drain resistance. The second potential is also from the use of equation [27], but the mesh size is 2' by 2'. The third and the fourth potentials are obtained by using equation [26] to calculate the drain resistance with mesh sizes of 1' by 1' and 2' by 2', respectively. It can be seen from Figure 42 that when equation [28] was used to calculate the drain resistance, little or no variation was observed when the mesh size was altered. However, when equation [27] was used under similar conditions, the potential distribution in the region near the drain decreased considerably when the mesh size was increased from 1' by 1' to 2' by 2'.

**Flow Net Example**

The dimensions of the flow problem discussed in the preceding section are used to show the first example of the flow net. In this particular example, the depth of the drain is located at 12 feet below ground surface and the 1' by 1' mesh is used in the region near the drain. A potential difference of 100 volts was applied through the network, the potential at each node was measured, and the equipotential lines were evaluated by interpolating the potential between
Fig. 42.—Diagram showing four sets of potential distribution at each node. At a node, the first (uppermost) and the second potentials were obtained from the network data by using equation [27] to calculate the drain resistances. The mesh sizes were 1' by 1' and 2' by 2', respectively. Likewise, the third and the fourth potentials were obtained by using equation [28].
adjacent nodes (see Fig. 43). After reversing the boundaries and again measuring the potentials, the flow lines were determined in a similar manner and are also shown in Figure 43.

The characteristic resistance $R_0$ was chosen to be 5,000 ohms and the resistance across the network $R_n$ for the drain depth at 12' was measured to be 8,797 ohms. Using equation [25], the flow rate $Q$ was determined to be

$$Q = 2 \times 0.001 \left(12 \times 0 - \frac{2}{12}\right) \times \frac{5000}{8797}$$

$$= 0.01345 \text{ (ft.}^3/\text{ft. drain-min.)}$$

In the case of stratified or anisotropic medium, the flow rate $Q$ can be determined the same way as for homogeneous medium. However, we have to keep in mind that the value for the hydraulic conductivity $K$ is that chosen to be represented by the characteristic resistance $R_0$. For an example, consider the case for flow of water through a stratified soil having hydraulic conductivities of 2.24, 0.56 and 0.28 cm/hr., respectively (see section of "Representing a Stratified Soil"). If $R_2$ is 5,000 ohms and represents the conductivity $K_2$ of 0.56 cm/hr., the value of $K_2$ and $R_2$ will appear in place of $K$ and $R_0$, respectively, in equation [25]. The value of $K_1$ and $K_3$ can be ignored since their overall effect is already included in the resistance across the network $R_n$. In Figure 44 a flow net for the flow of water through an anisotropic soil is shown. The method for calculating the resistances was previously given in section of "Representing an Anisotropic Soil."
Fig. 43.—Flow net for a homogeneous isotropic soil during ponded flow into a 4-inch drain.
Fig. 44.—Flow net for an anisotropic layered soil during ponded flow into a 4-inch drain.
Representing an Unsaturated Region of Soil

Under saturated flow conditions, the hydraulic conductivity $K$ in each layer of soil is independent of its location within the layer. In unsaturated flow, the hydraulic conductivity of a particular layer may differ from place to place. For flow problems encountered during drainage, unsaturated flow generally occurs above a nearly horizontal water table while saturated flow occurs below this surface.

To represent a block of soil having variable spatial conductivity with a mesh of resistors, first consider a piece of conducting paper with the side dimension $a$ by $b$ as shown in Figure 45. Cartesian

Fig. 45.—Diagram showing flow region with variable resistivity in the $y$-direction.
coordinates are set up with the x-axis along the lower edge and the y-axis along the left boundary. Let us assume that the resistivity R is only a function of the vertical coordinate, i.e., R = R(y). This condition approximates that of a small block of unsaturated soil in which changes in moisture content (and thus in hydraulic conductivity) in the horizontal direction are quite small compared to those in the vertical. Previous work has shown this condition to be essentially the case for unsaturated flow above a receding water table.

If we use the "building block" approach, we must calculate the overall resistance $R_y$ in the vertical direction and the resistance $R_H$ in the horizontal direction. Since the resistivity varies only in the y direction, let us then consider a small strip with dimension $dy$ by $a$. The incremental resistance $\Delta R_y$ can be expressed as

$$\Delta R_y = \frac{dy}{a} R(y) \quad \text{Eq. [29]}$$

From inspection, one can see that the incremental resistances $\Delta R_y$ are joined in series so that the total resistance in the vertical direction $R_y$ is equal to $\Sigma \Delta R_y$. Therefore,

$$R_y = \frac{1}{a} \int_0^b R(y) \, dy \quad \text{Eq. [30]}$$

To determine $R_H$, consider the same strip in Figure 45. The incremental resistance $\Delta R_H$ is expressed as

$$\Delta R_H = \frac{a}{dy} R(y) \quad \text{Eq. [31]}$$
Since the total resistance $R_H$ is the result of joining in parallel the incremental strips of resistance $\Delta R_H$, direct integration of equation [31] is not valid. However, if we designate the term $\Delta G_H$ as the incremental conductance in the horizontal direction and equal to $\frac{1}{\Delta R_H}$, equation [31] can then be expressed as

$$\Delta G_H = \frac{1}{a} \frac{dy}{R(y)} \quad \text{Eq. [32]}$$

so that

$$G_H = \frac{1}{R_H} = \frac{1}{a} \int_0^b \frac{dy}{R(y)} \quad \text{Eq. [33]}$$

From the procedure given above, two points should be emphasized:

1. Integration of incremental resistances can be made if they are joined together in series.

2. Integration of incremental conductances can be made if they are joined in parallel.

To illustrate the use of equations [30] and [33], first assume that the resistivity $R(y)$ varies linearly in the $y$ direction between $y = 0$ and $y = b$. This assumption introduces only a small error unless the function $R(y)$ is changing rapidly in the vicinity $0 \leq y \leq b$ or the soil block represented is of relatively large dimension, i.e., a coarse mesh is used. Thus,

$$R = R(y) = Ay + B \quad \text{Eq. [34]}$$

where $A$ and $B$ are constants. From Figure 45, designate the resistivity of the lower edge as $R_l$ and of the upper edge as $R_u$ (the determination
of $R_1$ and $R_j$ will be discussed in a later section). By setting first $y = 0$ and $y = b$, the constants $A$ and $B$ can be determined as

$$A = \frac{R_j - R_1}{b} \quad \text{Eq. [35]}$$

$$B = R_1 \quad \text{Eq. [36]}$$

Therefore

$$R = R(y) = \left(\frac{R_j - R_1}{b}\right) y + R_1 \quad \text{Eq. [37]}$$

Substituting $R(y)$ of equation [37] in equations [30] and [33] and carrying out the integrations, we find that

$$R_y = \frac{b}{a} \left(\frac{R_j + R_1}{2}\right) \quad \text{Eq. [38]}$$

$$R_H = \frac{a}{b} \log \frac{R_j}{R_1} \quad \text{Eq. [39]}$$

To test the validity of equations [38] and [39], let us consider the case when the entire flow region is saturated. In such case $R_1$ and $R_j$ will be equal to $R_0$, the characteristic resistance of the square piece of resistive paper having uniform resistivity. We then need to prove that under such condition $R_y$ and $R_H$ will also be equal to $\frac{b}{a} R_0$ and $\frac{a}{b} R_0$, respectively. It is obvious from equation [38] that if $R_j$ and $R_1$ are replaced by $R_0$ then $R_y = \frac{b}{a} R_0$. The same substitution, however, will not yield direct proof in the case of $R_H$. For if $R_j$ and $R_1$ are replaced by $R_0$ in equation [39], we would have the indeterminate form $0/0$. With the help of equations [35] and [36], equation [39] can be
rewritten as

\[ R_H = \frac{a}{\log_e \left( \frac{bA}{B} + 1 \right)} \quad \text{Eq. [40]} \]

From inspection of equations [35] and [36], if \( R_j = R_i = R_0 \), then we would have \( A = 0 \) and \( B = R_0 \). If the values of \( A \) and \( B \) are substituted into equation [40], \( R_H \) will be equal to \( \frac{0}{0} \) and l'Hospital's rule is applicable with respect to \( A \). Therefore, differentiations of the numerator and the denominator of equation [40] with respect to \( A \) yields

\[
R_H = \lim_{A \to 0} \left[ \frac{aA}{\log_e \left( \frac{bA}{B} + 1 \right)} \right] = \lim_{A \to 0} \left[ \frac{a \left( \frac{bA}{B} + 1 \right)}{b} \right] = \frac{aB}{b} = \frac{a}{b} R_0 \quad \text{Eq. [41]}
\]

It should be noticed that the expression \( R_H = \frac{aB}{b} \) in equation [41] is valid both in the saturated and unsaturated zones, as long as \( R_j = R_i = B \). Accordingly in the saturated zone \( B \) is equal to \( R_0 \) while in the unsaturated zone, \( B \) is equal to either \( R_j \) or \( R_i \).

After \( R_Y \) and \( R_H \) are determined, we can now proceed to represent the block of soil or resistive paper shown in Figure 45 with \( 4 \) resistors. This representation is shown in Figures 46a and 46b. The terms \( R_Y \) and \( R_H \) are calculated from equations [38] and [39], respectively.

From equation [38] and [39], which yield the values of \( R_Y \) and \( R_H \), we have two unknowns to determine, namely, \( R_i \) and \( R_j \). These terms represent the resistivity of the lower and upper boundaries, respectively, of the non-uniform conducting paper shown in Figure 45. To evaluate \( R_i \) and \( R_j \) for the study of flow in unsaturated soil, it is
Fig. 46.—Representing variable resistive paper of Figure 46a with 4 resistors in Figure 46b.

necessary to relate the resistivity of the conducting paper $R$ and the hydraulic conductivity of the soil $K$.

In unsaturated flow, the hydraulic conductivity is dependent on the moisture content of the medium. As the moisture content decreases below saturation, the hydraulic conductivity likewise decreases. Also, as the moisture content decreases, the hydrostatic pressure $h$ of the soil increases in the negative direction. In other words, the soil moisture tension is increased. The unsaturated hydraulic conductivity is, in turn, related to the hydrostatic pressure. The relationship between conductivity and hydrostatic pressure can be obtained by methods
described by Childs and Collis-George (4) and by Nielsen et al. (15).

An example of this type of relationship is shown in Figure 47. From the graph one can see that each type of medium has a characteristic relationship between the hydraulic conductivity and the hydrostatic pressure.

To relate the resistivity $R$ to the hydrostatic pressure $h$, we first have the relationship between the resistivity $R$ of the conducting paper and the hydraulic conductivity $K$ of the fluid flow medium which can be written as $K = \frac{C}{R}$, where $C$ is a proportionality constant relating $K$ to $R$. From this relationship we can then write

$$\frac{R_0}{R} = \frac{K}{K_0}$$

Eq. (42)

where $K_0$ and $K$ are the hydraulic conductivities, respectively, of saturated and unsaturated soil; and $R_0$ and $R$ are the electrical resistivities, respectively, of uniform (analogous to saturated soil) and non-uniform conducting medium (analogous to unsaturated soil). Both $K$ and $R$ are space functions in their respective media. Another source of needed information is the equation relating the potential head $\phi$, the elevation head $y$ and the hydrostatic pressure head $h$. These three are related by equation (43) for cases when only hydrostatic pressure and elevation components of potential are significant [see Childs (2)].

$$\phi = y + h$$

Eq. (43)

---

Fig. 47.—The fluid conductivities $K$ of the fine and coarse sands as a function of hydrostatic pressure $h$. The points plotted on the right of the ordinate represent the saturated fluid conductivities of the two sand sizes.
With the information above, we are now able to utilize the network analog to solve unsaturated steady state flow problems. To illustrate, consider a soil section having a dimension of 10' by 10'. On the right and left side of the soil section, the hydraulic heads are kept constant at 8' and 6', respectively. For simplicity, assume that the characteristic relationship between $K/K_0$ and the hydrostatic pressure $h$ is as shown in Figure 4.8. From Figure 4.8 we

Fig. 4.8.—Illustrative curve showing the relationship of $K/K_0$ and hydrostatic pressure $h$.

\(^3\) A similar type of study had been reported by Luthin and Day (13).
have $K/K_0 = 1$ for $h \geq -1.0$, $K/K_0 = 0.54 + 1.5$ for $-2.8 \leq h < -1$, and $K/K_0 = 0.1$ for $h \leq -2.8$. The same problem can be assembled on the network with each mesh of resistors representing a small square block of soil of dimension 1' by 1'. The grid points are shown in Figure 49. Let us first assemble the network to represent a saturated

![Network representation](image)

**Fig. 49.** Network representing a 10' by 10' section of soil. Each square mesh has a dimension of 1' by 1'. Hydraulic heads of 6' and 8' are kept along the left and right boundaries, respectively, which is equivalent to applying potentials of 6 and 8 volts along the left and right boundaries of the network.

and homogeneous soil medium having resistances $R_0$ everywhere in the interior and $2R_0$ along the boundaries. Eight boundary resistors on the right side and six on the left side are shorted out as shown in the figure and potentials of 8 and 6 volts are applied along the right
and left boundaries, respectively. The potential at each grid point is measured and recorded. The magnitude of h at each grid point is then calculated by equation [43]. From the value of h, R_i and R_j can be obtained from the relationship given in Figure 48. The resistances of the unsaturated medium can then be calculated as illustrated by the diagram shown in Figure 50. The section of soil represented in Figure 50 is that shown with a heavy line in the interior of Figure 49.

To calculate the resistances, consider the section in Figure 50a with the potentials Ø recorded at each node. As explained above, the initial values of the potential are obtained by utilizing a homogeneous and saturated flow medium. The next step is to adjust the resistors in accordance with the magnitude of the hydrostatic pressure h. To proceed, first the section in Figure 50a is separated into four small squares (see Fig. 50b). The potential at each corner is also shown in Figure 50b. By knowing the potential Ø and the elevation from the datum plane y, the hydrostatic pressure can be calculated. For example, the potential at the upper left corner of block number 1 is 6.86' and y is 9'. Therefore, the hydrostatic pressure h is 6.86' - 9' = -2.14' as shown in Figure 50c. From the known value of h, the ratio of conductivity K/Ko can be obtained directly from Figure 48 and is found to be 0.43. The same procedure is applicable for obtaining K/Ko when the experimental curve shown in Figure 47 is used in the network study. However, if the same problem is solved by numerical analysis using a high speed electronic computer, the relationship between K/Ko and h must be expressed in an analytical equation, the
Fig. 50.—Diagram illustrating the calculation of the resistance representing unsaturated flow medium. See text for detail discussion.
latter sometimes being somewhat cumbersome to use. This difficulty can be avoided by approximating the curve shown in Figure 47 with a series of broken straight lines for which linear equation can be assigned to each line.

After the value of $K/K_0$ is obtained, the value of $R/R_0$ can be calculated from equation [42] as $1/0.13 = 7.33$ and is shown in Figure 50d. With the same procedure, the value of $R/R_0$ at the upper right corner of block number 1 is calculated to be 2.04 (in Fig. 50d), and the value of the two are averaged to yield 2.19. This value is then set equal to $R_{ij}/R_0$ so that $R_{ij} = 2.19 R_0$ as shown in Figure 50e. The magnitude of $R_{ii}$ of block number 1 can be calculated in the same manner starting from the values of $\varnothing$ at the lower corners. From $R_{ii}$ and $R_{ji}$, the values of $R_{ij}$ and $R_{ii}$ of block number 1 can be calculated using equations [38] and [39] and are found to be 1.62 $R_0$ and 1.55 $R_0$, respectively, as shown in Figure 50f. The network of 4 resistors representing block number 1 is shown in Figure 50g. The same procedure is carried out for blocks numbers 2, 3, and 4 and they are joined together in parallel as shown in Figure 50h. The same procedure is carried on throughout the unsaturated region of soil.

At this point the voltages are then reapplied to the network, and the potential distribution is again recorded. The new values of node potential of the same section are shown in Figure 50i. One can see that the potential distribution of the new set is different from the previous one. The entire procedure is repeated until no significant potential change is observed, and the relaxation is completed. The flow net for this particular problem is shown in Figure 51.
Fig. 51.—Flow net for steady-state flow with water ponded at the left and right boundaries.

The discussion above deals only with an unsaturated medium whose conductivity varies only in the vertical direction. If in some cases, the conductivity varies in both vertical and horizontal directions, the determination of $R_V$ and $R_H$ can only be obtained approximately. It would appear, however, that acceptable accuracy can be obtained by using successive linear approximations of resistivity $R$. On this
basis the expressions $R_V$ and $R_H$ for a block of soil having dimensions $a$ by $b$ can be expressed as

$$R_V = \frac{b}{a} \left( \frac{R_j + R_i}{2} \right) \quad \text{Eq. [44]}$$

$$R_H = \frac{a}{b} \left( \frac{R_j + R_i}{2} \right) \quad \text{Eq. [45]}$$

where $R_i$ and $R_j$ are determined similar to those illustrated in Figure 50. This approximation is based on the assumption that the unsaturated soil is homogeneous with respect to the conductivities in the horizontal and vertical directions.
SUMMARY AND CONCLUSION

An electrical resistance network was used to solve the steady state flow of fluid through porous media, both saturated and unsaturated. The "building block" approach was used to calculate the resistance of the network to represent porous media instead of a "node to node" approach used by previous investigators. The building block approach was easier to understand, helped facilitate the calculation of the resistances used in the network, and correct the error in using network when the expansion from fine meshes to coarse meshes was encountered.

The resistance network analog was built with some improved features in flexibility, accuracy, and ease of operation. The network consists of 575 variable resistance units and is mounted on a 6 by 8 foot board. The resistive units are of the plug-in type, and the mesh size is smaller on one-fourth of the board. The use of a control console, automatic digital voltmeter and numerous switches permits rapid measurement of voltages and current.

Improved equations are presented for calculating network resistances representing flow medium around the drain. These equations utilized a logarithmic expression rather than a linear one as reported by previous investigators. Comparisons are made of drain flow rates in homogeneous medium as evaluated by the network and by analytic solutions of Kirkham. Compared to the analytical solutions, the
network data generally deviate less than two per cent if the logarithmic expression is used to calculate drain resistors; while deviations as high as thirty per cent or more may occur if a linear relationship is utilized.
APPENDIX I

Representing Water Flow in Soils with Electrical Current Flow in Conducting Paper

A square of conducting paper with dimensions a by a can be used to represent any square dimension of uniform soil which has unit thickness. For example, consider the block of saturated and homogeneous soil of hydraulic conductivity $K$ shown in Figure 1a. Assume that constant potentials $\phi_n$ and $\phi_0$, respectively, are applied to the upper and lower surfaces. A steady flow will be established, and the quantity of water $Q$ leaving the lower surface per unit time will be given by equation [1],

$$Q = v \cdot A \cdot l = -K \frac{d\phi}{dy} A$$

where $v$ is the average flow velocity per unit and $A$ is the dimension of the square block in the $X$-direction. Similarly, the current $I$ leaving the lower edge of the conducting paper shown in Figure 1b will be given by the following relationship:

$$I = i \cdot a = -G \frac{dV}{dy} a$$

where $G$ is the electrical conductivity, $i$ is the current flux per unit length in the $X'$-direction, and $V$ is the voltage.

Since both media are homogeneous with respect to their conductivity, the functions $\phi$ and $V$ will be related in the following manner:

$$C_0 (\phi_{j+1} - \phi_j) = (V_{j+1} - V_j)$$

Eq. [3]
Fig. 1.—Diagram showing representation of flow through blocks of soil in 1-dimension (Fig. 1a) and 2-dimension (Fig. 1c) by conducting papers in Figs. 1b and 1d, respectively.
where \( C_0 \) is a proportionality constant. By evaluating equation [3] for \((j + 1) = n\) and \(J = o\), it can be seen that the voltage \( V \) is related to \( \phi \) by equations [4] and [5].

\[
C_0 = \frac{(V_n - V_o)}{(\phi_n - \phi_o)} \quad \text{Eq. [4]}
\]

\[
V(x', y') = C_0 \phi(x, y) + C_1 \quad \text{Eq. [5]}
\]

where

\[
x' = \frac{x - A}{A}
\]

\[
y' = \frac{y - A}{A}
\]

\[
C_1 = (V_o - C_0 \phi_o)
\]

For the examples shown in Figures 1a and 1b, it can be seen by inspection that both \( \phi \) and \( V \) vary only in the vertical direction. Thus if equation [5] is differentiated with respect to \( y \), the following results:

\[
\frac{dV}{dy'}(\frac{A}{A}) = C_0 \frac{d\phi}{dy} \quad \text{Eq. [6]}
\]

By substituting equation [6] in [2] and then dividing the resulting expression into equation [1], one obtains the following relationships among the conductivities and flow rates in the two media.

\[
Q = \frac{IK}{C_0 \phi} \quad \text{Eq. [7]}
\]

\[
Q = \frac{IK}{C_0 \phi_0} \quad \text{Eq. [7a]}
\]

where \( R_0 = \frac{1}{\phi_0} \)
From the above analysis, it can be seen that a square of electrical conducting paper can represent one-dimensional flow of water in any square dimension of uniform soil having unit thickness. The potential $\phi$ is related to the voltage $V$ as shown in equation [5], and the flow rate $Q$ is given in terms of $\phi$, $K$, and $I$ by equations [7] and [7a]. While the example presented here is for one-dimensional flow, equations [5] and [7] are equally valid for two-dimensional problems. Examples of the latter are illustrated in Figure 1c and 1d.

An approach similar to that used to derive equations [5] and [7] can be followed to show that any rectangular piece of conducting paper of dimension $a$ by $b$ can be used to represent any rectangular block of soil of dimension $A$ and $B$. Expressions similar to equations [5] and [7] can thus be derived which contain the parameters, $a$, $b$, $A$ and $B$. 
APPENDIX II

Steps to Determine the Potential of a Node
in Numerical Analysis or in a Resistance Network

One of the most diverse cases encountered in determining the potential at a node in saturated, isotropic soil occurs when the node is located (a) at a point on the interface of horizontal layers with different hydraulic conductivity $K$ and (b) is surrounded by "building blocks" of different sizes. Figure 1 shows the location of such a node. From Figure 1a, four blocks of soil have dimensions $a$ by $b$, $b$ by $c$, $a$ by $d$, and $d$ by $c$, respectively. The two blocks at the top have a conductivity $K_T$ and the two blocks at the bottom have a conductivity $K_B$. The ratio of $K_T/K_B$ is given by $m$. If $R_0$ is the characteristic resistance of the upper blocks, the four blocks can be represented by 4 meshes of resistances as shown in Figure 1b. The parallel sides of the mesh can be joined together as shown in Figure 1c and the final product is shown in Figure 1d. In Figure 1d, $R_1$ is the combined resistance of $R_{AB}$ and $R_{CD}$, $R_2$ is that of $R_{DG}$ and $R_{GN}$, et cetera. If $V_1$ is the potential at node AC, $V_2$ is the potential at node MN, et cetera, then $V_0$ is the potential at node BDGE.

Using Kirchoff's law to determine the potential $V_0$, we have:

$$\frac{V_1 - V_0}{R_1} + \frac{V_2 - V_0}{R_2} + \frac{V_3 - V_0}{R_3} + \frac{V_4 - V_0}{R_4} = 0 \quad \text{Eq. [1]}$$

Since

$$R_1 = \frac{2bR_0}{a + c} \quad R_2 = \frac{2mcR_0}{mb + d}$$
$$R_3 = \frac{2mdR_0}{a + c} \quad R_4 = \frac{2maR_0}{mb + d}$$

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Fig. 1.--Building block approach used to calculate the potential $V_0$ in relation to the voltages at adjacent nodes, sizes of blocks surrounded and differences of hydraulic conductivities (see text).
Now substitute $R_1, R_2, R_3, R_4$ into equation [1] and solve for $V_0$.

We then have

$$V_0 = \frac{macd(a+c)V_1 + abd(mb+d)V_2 + abc(a+c)V_3 + bcd(mb+d)V_4}{macd(a+c) + abd(mb+d) + abc(a+c) + bcd(mb+d)} \quad \text{Eq. [2]}$$

If $V_0$ is within the layer where $K_T = K_B$ so that $m = 1$, and also if $a = c$, $b = d$, and $a/b = r$, we will have the following expression:

$$V_0 = \frac{V_1 + V_3}{2(1 + \frac{1}{r^2})} \quad \text{Eq. [3]}$$

which agrees with results previously reported by Kirkham and Gaskell [(9), Eq. [8] p. 39]. Finally, if $a = b$ so that $r = 1$, equation [3] reduces to

$$V_0 = \frac{V_1 + V_2 + V_3 + V_4}{4} \quad \text{Eq. [4]}$$

which is the same as equation [6] in the text.

Potential at the Node Adjacent to the Drain

In general, the building blocks around the drain region are square. Let us consider the case when the drain is buried in a stratified soil at an interface as shown in Figure 2a. The upper half of the drain is in a layer of conductivity $K_B$. The problem can be solved with the same procedure described above. The only difference

Rectangular blocks of soil around the drain region can be handled by proper manipulation of the equation in (22) p. 593, type of line T. This is done by dividing $Z_0$ by 2 and changing the term D to $W/2$. 
Fig. 2.—Diagram showing building blocks around the drain. The diagram is used to determine the potential $V_0$ next to the drain.
is the addition of the term $C_d$ which is reported in Table 1 in the text. The network representing each block of soil is shown in Figure 2b, where $m$ is the ratio $K_p/K_s$. The voltage to be determined is $V_o$ which is shown in the center of Figure 2c. Using the same procedure described above, the voltage $V_o$ can be determined as shown in equation [5].

$$V_o = \frac{2mC_d V_1 + C_d(m+1) V_2 + 2C_dV_3 + 2(m+1) V_h}{(m + 1) (3C_d + 2)}$$

Eq. [5]

In the case when the drain is buried within a layer so that $m = 1$, the expression for the potential adjacent to the drain is given by equation [6]:

$$V_o = \frac{V_1 + V_2 + V_3 + (2/C_d) V_h}{3 + \frac{2}{C_d}}$$

Eq. [6]

This equation is of similar form to Luthin's expression (12), equation [4], where the term $h$ in (12), equation [4] is equivalent to $\frac{C_d}{2}$.

However, Luthin's expression is based on a linear change in the voltage between the drain and an adjacent node; whereas, equations [5] and [6] assume a logarithmic relationship. It has been shown in the text and in (21) that a logarithmic relationship is necessary to give satisfactory agreement between results obtained with the network and analytical solutions.
APPENDIX III

Error Due to Choosing Characteristic Resistance $R_0$

The characteristic resistance $R_0$ was chosen to be 5,000 ohms. This value lies halfway between logarithmic scale of an upper limit of 11 megohms (internal resistance of VTVi) and a lower limit of 1 to 5 ohms (wiring resistances and collective resistances in plug and jack assemblies). That is $\log 11 \text{ meg} \approx 7$ and $\log 2.5 \approx 0.4$ so that $\text{antilog} \frac{7 + 0.4}{2} = 5000$ ohms. One can calculate the errors at the upper and lower limits of $R_0$ as follows:

\[
\text{Per cent error for upper limit} = \frac{100 \ R_0}{11,000,000 + R_0} \%
\]
\[
\text{Per cent error for lower limit} = \frac{250 \ R_0}{R_0} \%
\]

These errors are present at all times. If an $R_0$ of 5000 ohms is chosen, the percentage error of the upper limit is 0.045% and the percentage error of the lower limit is 0.05%. Any deviation of $R_0$ from 5000 ohms will increase one and decrease the other. For example, if an $R_0$ of 200 ohms is chosen, the error of the upper limit is 0.00018% but the error of the lower limit is increased to 1%.

Using Voltmeter to Measure the Current

If one wishes to use a digital voltmeter to measure current flow, this is done by attaching a standard resistor of 1, 10, or 100 ohms in series with the circuit having a resistance of $R$ ohms as
shown in Figure 1. It makes no difference what the magnitude of \( R \) is, just as long as the voltage across it is the voltage desired. The current going through both the standard resistor and the resistance \( R \) will be the same since they are in series. By reading the voltage across a standard resistor, the current \( I \) can be determined as

\[
I = \frac{V}{R_{\text{standard}}}
\]

**Error in Setting Variable Resistor with a Wheatstone Bridge**

In setting the variable resistor with a Wheatstone Bridge, the unknown resistor is put on the arm marked unknown in Figure 2. To
set the resistance, the decade box is set to the desired value then the unknown resistor is adjusted until the voltage $\Delta V$ is zero. However, it is nearly impossible to adjust the unknown resistor until the voltage $\Delta V$ reads exactly zero. In such case, the resistance adjusted will have the error of $\Delta R$. To find the error in setting the resistance, if the two fixed resistances $R$ and the decade box all have the same resistance $R$, then the error is $\frac{\Delta R}{R} = \frac{1}{4} \frac{\Delta V}{V}$ so that the percentage error is

$$\frac{100 \Delta R}{R} = 400 \frac{\Delta V}{V} = \% \text{ error}$$
In setting each resistance, if $\Delta V$ is $\approx 0.025$ volt or less and $V$ is 30 volts

\[ \therefore \text{Per cent error} = \frac{400 \times 0.025}{30} = 0.33\% \]

If two resistors each having 0.33% error are joined in series, the total error will be the total algebraic sum of each resistor. If the two are joined in parallel, the error will be the same as the error of an individual resistor if each is "off" in the same direction and will approach zero if each is "off" in the opposite direction. In network system, these resistors are joined in mesh and the maximum error should not exceed the error of individual resistor. Roughly, the error of

\[ R_n \approx \frac{\sum \Delta R}{n} \text{ where } n \text{ is the number of resistors.} \]


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