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A THEORETICAL ANALYSIS OF TRANSPERSION COOLING IN LAMINAR
AND TURBULENT BOUNDARY LAYERS AND CHANNEL FLOWS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

MICHAEL CHEN-CHIANG FONG, B.S., M.Sc.

* * * * * *

The Ohio State University
1961

Approved by

[Signature]

Adviser
Department of Aeronautical and Astronautical Engineering
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INTRODUCTION

One of the important current problems in the fields of high-speed aerodynamics and rocket propulsion is concerned with the reduction of the detrimental effects due to aerodynamic heating and/or internal high-energy gas flows. Aerodynamic heating is encountered when the kinetic energy of the gas stream is transformed into thermal energy in the region surrounding the body surface. The basic processes under which such an energy conversion takes place are compression and/or viscous dissipation. The former is affected by the steep pressure rise across a shock wave and by the ram compression in the proximity of the stagnation point. The latter occurs in the boundary layer by the action of the viscous shearing stresses. Severe effects of aerodynamic heating arise when the vehicle is traveling at hypersonic speeds. Inside the combustion chamber and the exhaust nozzle of a rocket engine or in a nuclear reactor, the internal gas flows of relatively low velocities may reach a very high temperature due to the conversion from chemical energy into heat energy through combustion processes. The amount of heat energy, and hence the temperature level, depend on the heat of reaction of the propellant. It is evident that under extreme conditions of aerodynamic heating or combustion, the body surfaces may be exposed to temperatures beyond the tolerable limit of the structural material. Therefore, some protective means must be devised in order to prevent serious damages to the vehicle.

One promising means of alleviating the high-temperature effects
is the technique of protective cooling, particularly that involving coolant injection. In view of the immediate practical applications, a variety of injection processes has been investigated in recent years. Most important of the cooling techniques can be listed as follows:

1. Film cooling
2. Transpiration cooling
3. Sublimation cooling
4. Ablation cooling

A film cooling system involves injecting a liquid or gas onto the surface through a number of orifices so that a thin film of the coolant covers the surface. Thus, the heat transfer is reduced in two ways: (1) by the "blocking" effect of the coolant and (2) by direct absorption in the form of heat of vaporization if the injected fluid is a liquid.

A transpiration cooling process is similar to the film cooling process except that the wall surface is made of porous material. Mathematically speaking, it can be regarded that the fluid is injected from an infinite number of cooling orifices. The advantages of the transpiration-cooling process over the film cooling process are that the former eliminates the turbulent mixing with the hot gases and provides greater surface area for heat transfer. Moreover, the injection distribution along the surface can be determined more accurately for transpiration cooling.

A sublimation cooling system makes use of a solid material which sublimes under high temperature conditions. Therefore, the mass transfer process and the heat of sublimation result in heat absorption
and hence the reduction of the heat transfer rate. If, in addition, the surface material decomposes endothermically, the process of heat absorption will be greatly enhanced. For sublimation cooling, the surface injection rate cannot be arbitrarily controlled, since it depends on the heat load, the heat of sublimation, the local surface temperature, and the particular flow system in question.

A more complex process is that of ablation cooling involving surface melting and partial evaporation. A two-phase (gas and liquid) boundary layer exists at the surface; the boundary layer characteristics depend on the heat of fusion, the vapor pressure of the material, the phase transition processes, and the transport properties, and so on. The reduction of heat transfer is essentially the same as that due to sublimation.

In general, the aforementioned methods of coolant injection provide effective protection against the detrimental high-temperature effects. Particularly noteworthy are the advantages of utilizing liquid or ablating (or sublimating) materials as a protective agent over gas injection. Based on the energy considerations, the ablation process which involves an additional heat absorption process due to phase changes is evidently a more effective cooling system. Furthermore, ablation (including sublimation) requires no pumping and other elaborate injection devices. An accurate theoretical description of the ablation process is not yet available because of mathematical complexities. In fact, if a rigorous approach is adopted, the analysis is not amenable to simple solutions even for the simpler cases involving gas-to-gas injections. For example, the effects of dissociation, ionization, the
relaxation phenomena, and the chemical reactions between components in the boundary layer all introduce additional unknowns to the high-order nonlinear governing equations.

In this analysis, emphasis is placed on the theoretical aspect of the fundamental problems involving fluid injection at the surface. A critical review on the mathematical approach will be presented. As a simplification, the discussion is restricted to the case of gas injection. With some modifications, the method of solution of the transpiration-cooling problems presented herein is believed to be valid for cases involving evaporation or sublimation. The problems concerning the design and development of an effective coolant injector are beyond the scope of this analysis and will not be discussed.

In Part I, efforts are devoted to the laminar flow problems. The method of series expansion about a small parameter is employed for the solutions of the velocity and temperature distributions in a boundary layer. The numerical calculations of the first-order approximation for a flat plate and a convergent channel with air-to-air injections in an incompressible flow field have been carried out. The present theory is equally applicable to problems involving foreign gas injection and the effects of pressure gradient and compressibility, but the computation procedure becomes too involved and too tedious without the aid of a digital computer.

In Part II, the discussion deals with the turbulent flows. The basic semi-empirical laws of the incompressible turbulent flow theory are extended to the case involving surface injection. The series-expansion method is also employed for a flat plate and a convergent
channel. However, unlike the laminar flow problems, empirical results must be incorporated in the analysis.

It should be remarked that in both Part I and Part II, the analysis is confined to the studies of the velocity profile and the skin friction as well as the heat transfer characteristics of a transpiration-cooled surface. The integral method, though a useful technique, is not developed herein. Furthermore, the problems of boundary layer instability and the transition phenomena from laminar to turbulent flows in the presence of fluid injection are excluded from the present analysis.
PART I
LAMINAR FLOWS

1. Background

The feasibility of transpiration-cooling as a means of controlling the excessive heat flow to the wall of a rocket engine or alleviating the aerodynamic heating for a high-speed missile has been discussed in the Introduction. In the theoretical aspects, the flow characteristics of a laminar boundary layer with coolant injection can be described completely by the conservation laws of mass, momentum, and energy, together with the equation of state provided that the injected fluid is the same as the primary gas and that the effects of chemical reaction are ignored. With a properly prescribed injection distribution along the porous surface, the so-called "similarity" transformations can be applied and the exact solutions of the boundary layer equations can be obtained. If the injected gas is different from the primary gas, however, an additional diffusion equation describing the conservation of the individual species must be considered simultaneously. To extend this problem further by including dissociation or even ionization, the simplifying assumption of "equilibrium" or "frozen" flow is frequently employed, so that the complexities of the problem may be somewhat reduced.

As mentioned earlier, under certain conditions, "similar" solutions for a laminar boundary layer with transpiration cooling exist. By similarity is meant that with a proper coordinate transformation,
the velocity profile becomes distorted only affinely. One important mathematical feature of the similarity transformation is that in the transformed plane the dependent variables are expressible in terms of one of the similarity coordinates. Thus, the nonlinear partial differential equations describing the boundary layer are now transformed into ordinary differential equations which can be solved numerically in general. When gas injection takes place at the wall surface, the existence of the similar solutions depends on the geometrical contour as well as on the injection distribution along the surface. Without loss of generality, an incompressible wedge flow with porous walls will be used for verifying this statement. The continuity and the boundary layer equations for a wedge flow are these (1):¹

Continuity

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]  

(1)

Boundary Layer

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}
\]

(2)

\[
\frac{\partial p}{\partial y} \approx 0.
\]

(3)

The boundary conditions are

\[
u = u_e, \quad \text{for } y \to \infty
\]

\[
u = 0, \quad v = v_u(x), \quad \text{for } y = 0.
\]

¹Numbers in the parentheses indicate references.
Introducing the following set of transformations

\[
\begin{align*}
\xi &= \frac{y}{x} \sqrt{\frac{u_e}{x'}} \\
\psi &= \sqrt{\frac{u_e}{x'}} \ F(\xi)
\end{align*}
\]  

(5)

where \( \psi \) is the stream function, and utilizing the usual expression for a wedge-type flow

\[
u_e = C x^{\text{Eu}}
\]  

(6)

where \( C \) and \( \text{Eu} \) are constants and \( \text{Eu} = \frac{(-\partial p/\partial x)/(\rho u_e^2)}{\partial p/\partial x} \) is the Euler number, the boundary layer equation in the \((x, \xi)\) plane becomes

\[
\frac{F''}{F'} + \frac{\text{Eu} + 1}{2} \ F' - \text{Eu}(F'^2 - 1) = 0
\]  

(7)

with the boundary conditions

\[
\begin{align*}
F &= F_w \quad \text{and} \quad F' = 0, \quad \text{for} \quad \xi = 0 \\
F' &= 1, \quad \text{for} \quad \xi \to \infty.
\end{align*}
\]  

(8)

From the boundary conditions (4) and (8), it is easily verified that

\[
F_w = \frac{\frac{1-\text{Eu}}{\text{Eu} + 1}}{\sqrt{\frac{x'}{x}}}.
\]  

(9)

Since in the \((x, \xi)\) plane, \( F \) is a function of \( \xi \) alone, the relation between \( v_w(x) \) and \( x \) must satisfy the following condition;

\[
v_w(x) \propto x^{\frac{1-\text{Eu}}{2}}.
\]  

(10)

Therefore, in order that a similar solution for a laminar boundary
layer with fluid injection exists, the following requirements must be fulfilled:

1. The flow over the same body surface with impervious walls must possess similar solutions.

2. The surface injection rate, $v_w(x)$, must vary in a specified manner depending on the particular contour.

Other obvious examples illustrating the similarity requirements are the special cases of wedge flows, such as the flat plate where $v_w \propto 1/\sqrt{x}$ and the stagnation point where $v_w$ = constant. If the compressibility effects are included and/or the injected coolant is different from the primary fluid, greater mathematical complexity will be encountered. However, the basic requirements for admitting similar solutions remain unaltered.

While similar solutions constitute an important class of applications in viscous aerodynamics, they are limited to only a few mathematically possible cases. For cases involving transpiration cooling, the specification of the surface injection distribution in order to fulfill the similarity requirements imposes a further restriction on the problem. According to the literature, theoretical research on transpiration-cooling deals primarily with problems which possess similar solutions. A variety of problems has been studied, including foreign gas injections, compressibility and chemical effects. In Part I of this dissertation, the discussion is centered on the mathematical solution of a more general type of transpiration-cooling problems for which the surface injection distribution is arbitrary.

Since the flow considered herein is non-similar, the solution is
assumed to be represented by a power series in terms of the longitudinal distance with coefficients being functions of a transformed lateral length only. Thus, the partial differential equations describing the velocity and the thermal distributions are replaced by an infinite number of ordinary differential equations. In general, these differential equations can be solved by numerical methods. The extent to which the higher-order solutions are to be carried out will depend on the degree of accuracy desired for the particular problem.

In the ensuing sections, the series solution for an incompressible laminar flow through a two-dimensional transpiration-cooled convergent channel will be discussed first. This solution may be applied to the problem of cooling the convergent portion of a rocket exhaust nozzle where the heat transfer rate is most pronounced. The analysis also includes the general solution for a two-dimensional wedge flow with compressibility effects and foreign gas injection. Numerical solution for the flow characteristics of a special wedge flow (a flat plate) has been carried out in some detail.

2. Incompressible Laminar Flow through a Transpiration-med-Cooled Convergent Channel

For an incompressible laminar flow between two non-parallel impervious walls, a sink- or source-type flow may be assumed for a converging flow and a diverging flow, respectively. This assumption leads to the well-known Jeffery-Hamel solution for the velocity distribution (2) and the Millsaps-Pohlhausen solution for the temperature profile (2). In both cases, the solutions are expressed in terms of elliptic integrals. For a converging flow with small viscosity, boundary layer
assumptions may be employed for which the velocity profile can be described by hyperbolic tangent functions (3).

When fluid injection at the wall is considered, the velocity and the temperature distributions in the flow field will be modified by the coolant flow. In this analysis, the Reynolds number is assumed to be large enough so that the boundary layer assumptions are admissible. The method of solution presented herein is also applicable to the more general problem where the flow field is described by the Navier-Stokes equations. However, aside from its greater mathematical complexity, nothing conceptually new can be derived therefrom. The formulation of the general case applicable to a wide range of Reynolds number is presented in Appendix A, but the boundary layer flow is used to demonstrate the method of solution.

The Prandtl boundary layer equations are assumed to be valid in describing a laminar incompressible flow through a two-dimensional convergent channel with fluid injection. Inasmuch as the flow is of the sink type, it is convenient to express the governing equations in cylindrical coordinates (Fig. 1) as follows:

Continuity \[ \frac{\partial \nu u}{\partial r} + \frac{\partial v}{\partial \theta} = 0 \] (11)

Momentum \[ u \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \theta} = - \frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{\nu}{r^2} \frac{\partial^2 u}{\partial \theta^2} \] (12)

\[ \frac{\partial p}{\partial \theta} \approx 0 \] (13)

Energy \[ c_v (u \frac{\partial T}{\partial r} + \frac{v}{r} \frac{\partial T}{\partial \theta}) = \frac{k}{\rho} \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \nu \left( \frac{1}{r} \frac{\partial u}{\partial \theta} \right)^2 \] (14)
The free-stream velocity, $u_\theta$, based on the sink flow concept, is of the form

$$u_\theta = -C/r$$  \hspace{1cm} (15)

where $C$ is an arbitrary constant. Then from Bernoulli's equation, the pressure gradient can be expressed in terms of the free-stream velocity as follows:

$$-\frac{1}{\rho} \frac{dp}{dr} = u_\theta \frac{du_\theta}{dr} = -\frac{c^2}{r^3}.$$  \hspace{1cm} (16)

It is seen that if the wall surface is impervious, the lateral velocity component is identically zero. Then, from the continuity equation (11), the radial velocity component, $u$, can be represented by

$$u = f(\theta)/r$$  \hspace{1cm} (17)

which leads to the familiar Pohlhausen solution (3).

With the presence of fluid injection, however, the radial velocity is characterized by the distribution in accordance with equation (17) as well as by the tangential velocity component. Although the physical problem appears to be a simple one, an exact solution for arbitrary surface injection distribution, to the knowledge of the author, is not yet available. A possible method of solution is by means of series development from which a set of single-variable functions will result. Thus, the solution of the problem can be reached by successive determinations of these functions.
With the aid of the transformation

\[ \eta = (a - \Theta) \sqrt{c/\nu} \]

(18)

the tangential velocity component, \( v \), is assumed to be represented by a power series

\[ v(r, \eta) = -\sqrt{\nu} c \sum_{n=0}^{\infty} g_n(\eta) r^n. \]

(19)

From the continuity equation (11), the radial velocity component, \( u \), is found to be

\[ u(r, \eta) = -c \left[ \frac{f(\eta)}{r} + \sum_{n=0}^{\infty} \frac{g_n(\eta)}{n+1} r^n \right]. \]

(20)

where the prime denotes differentiation with respect to the argument of the function \( g_n(\eta) \). Substitution of expressions (19) and (20) into equation (12) yields

\[
c^2 \left\{ -\frac{r^2}{r^3} - \frac{f'g_0}{r^2} + \left( \frac{1}{3} g_1^2 f + \frac{g_0 g_1}{2} \right) + \left( \frac{1}{2} g_2^2 f + \frac{2}{3} g_0 g_2 + \frac{f g_1}{3} \right) r + \ldots \right\} \
+ c^2 \left\{ -\frac{f'g_0}{r^2} + \left( \frac{-g_0 g_0 - f'g_1}{r} \right) + \left( -f'g_2 + \frac{g_0 g_1}{2} - g_0 g_1 \right) \right. \\n\left. + \left( -f'g_3 - \frac{1}{3} g_0 g_2 - \frac{1}{2} g_1 g_1 - g_0 g_2 \right) r + \ldots \right\} \
= -\frac{c^2}{r^3} + c^2 \left\{ -\frac{f''}{r^3} - \frac{g_0''}{r^2} - \frac{g_1''}{2} - \frac{g_2''}{3} - \frac{g_3''}{4} r + \ldots \right\} \]

where the number of primes denotes the order of differentiation, and, for reason of brevity, the unknown functions are expressed as \( f, g_0, g_1, \ldots \), which are understood to be dependent on \( \eta \) only.
Since the functions $f$, $g_0$, $g_1$, ..., are independent of $r$, the coefficients of $r$ can be equated to zero in regions where $u$ and $v$ are non-singular. Hence, an infinite set of ordinary differential equations in terms of the functions $f$, $g_0$, $g_1$, ..., is obtained:

$$f^2 - f^2 + 1 = 0$$  \hspace{1cm} (21)

$$g_0^\prime - f g_0^\prime - f g_0 = 0$$  \hspace{1cm} (22)

$$(1/2) g_1^\prime - f^\prime g_1 - g_0 g_0^\prime = 0$$  \hspace{1cm} (23)

$$\frac{1}{3} g_2^\prime + \frac{1}{3} f g_2^\prime - f^\prime g_2 + \left( \frac{g_0 g_1^\prime}{2} - g_0 g_1 - \frac{g_0 g_1^\prime}{2} \right) = 0$$  \hspace{1cm} (24)

$$\frac{g_3^\prime}{4} + \frac{f}{2} g_3^\prime - f^\prime g_3 + \left( -\frac{1}{3} g_0 g_2^\prime + \frac{2}{3} g_0 g_2 + \frac{1}{3} f g_2^\prime \right)$$

$$- g_0 g_2 - \frac{1}{2} g_1 g_1 = 0$$  \hspace{1cm} (25)

e etc.

The boundary conditions for the foregoing system of differential equations are:

1. Non-slip condition

$$u(r,0) = 0, \text{ or } f(0) = g_0(0) = g_1(0) = g_2(0) = ... = 0$$  \hspace{1cm} (26)

2. Free-stream velocity

$$u = u_\infty \text{ as } \gamma \to \infty$$

or $$f(\infty) = 1, \text{ or } g_0(\infty) = g_1(\infty) = g_2(\infty) = ... = 0$$  \hspace{1cm} (27)
3. Surface injection distribution, \( v_w = v_w(r) \)

\[
\text{or } v_w = \sum_{n=0}^{\infty} b_n r^n
\]

where \( b_0, b_1, ..., \) are known constants,

\[
\text{or } g_0(0) = \frac{b_0}{\sqrt{\gamma c}}, \quad g_1(0) = \frac{b_1}{\sqrt{\gamma c}}, \quad g_2(0) = \frac{b_2}{\sqrt{\gamma c}}, ... 
\]

The third boundary condition specifies the surface injection rate, i.e., the distribution of a coolant with a finite injection rate normal to the wall surface. The power series (28) is used merely as a convenient representation for the development of a formal mathematical solution. As will be seen later, a different form of the injection distribution may require different power series for \( u \) and \( v \).

The few representative differential equations (21) through (25) exhibit the property that only the zero-order equation (21) is non-linear. The highest-order approximations, represented by \( g_0, g_1, ... \), can be determined successively by solving the subsequent linear differential equations. In general, numerical methods of solution must be employed, and the calculation procedure may become too tedious without the aid of a digital computer.

The zero-order equation (21) is identical with the boundary layer equation through a convergent channel with impervious walls. Its exact solution is

\[
f(\zeta) = 3 \tanh^2 \left( \frac{\zeta}{\sqrt{2}} \right) + 1.146 - 2.
\]

In this analysis, a constant surface injection distribution is assumed.
for the determination of the higher-order approximations. The numerical method will be discussed in a later section. If suction takes place at the surface instead of injection, the problem remains the same except that \( v_w \) has an opposite sign.

Because of the constant-property assumption, the temperature distribution for a transpiration-cooled convergent channel flow can be determined after the velocity profile has been obtained. In a similar manner, a power series is assumed to represent the temperature distribution as follows:

\[
T(r, \gamma) = \frac{C_1^2}{c_{\text{Pr}}^2} \sum_{n=0}^{\infty} W_n(\gamma) r^n + T_w
\]  (30)

where \( C_1 \) is an arbitrary constant, \( T_w \) represents the isothermal wall temperature, and \( W_n(\gamma) \) depends on \( \gamma \) only. The boundary conditions are

1. \( T = T_w \) for \( \gamma = 0 \)

or \( W_0(0) = W_1(0) = W_2(0) = \ldots = 0 \)  (31)

2. \( W_0'(\infty) = W_1'(\infty) = W_2'(\infty) = \ldots = 0 \)  (32)

Following the same procedure as that for the velocity distribution, the following differential equations are obtained.

\[
W_0'' - 2\text{Pr} (f W_0 - f' W_1^2) = 0
\]  (33)

\[
W_1'' - \text{Pr} f W_1 + \text{Pr} (2f' W_1' - 2W_0 W_1' - g_0 W_0') = 0
\]  (34)

\[
W_2'' + \text{Pr} (g_0^2 + f' g_1' - (g_0 W_1)' - (g_1 W_0)') = 0
\]  (35)
\[ W_3 - \operatorname{Pr} f W_3 + \operatorname{Pr} \left\{ \left( g_0^1 g_1^1 + \frac{2}{3} f g_2^1 \right) \\
- \left( \frac{\partial W_1}{2} + \frac{2}{3} W_0 g_2^1 + g_0 W_2^1 + g_1 W_2^1 \right) \right\} = 0 \quad (36) \]

etc.

where \( \operatorname{Pr} = c_f \mu/k \) is the Prandtl number based on \( c_f \).

Therefore, similar to the velocity distribution, the temperature profile can be determined by successively solving the representative differential equations for \( W_0, W_1, W_2, \ldots \). Since the functions \( f, g_0, g_1, \ldots \) are known functions from the analysis of the velocity distribution, all the differential equations are second-order and linear.

While solutions in closed form of these equations may exist, a straightforward numerical method of solution should prove to be satisfactory for engineering purposes.

From the solutions for the velocity and temperature distributions, the important parameters characterizing the local shearing stress, \( \tau_w \), and the heat transfer rate, \( q_w \), of the flow field, can be determined as follows:

\[ \tau_w = \mu \left( \frac{1}{r} \frac{\partial u}{\partial \eta} \right)_w = \mu \left( \frac{1}{r} \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial \eta} \right)_{\eta=0} \]

\[ = \sqrt{\rho \mu c^3} \left\{ \frac{f'(0)}{r^2} + \sum_{n=0}^{\infty} \frac{g_n^1(0)}{n+1} r^{n-1} \right\} \quad (37) \]

\[ q_w = -k \left( \frac{1}{r} \frac{\partial T}{\partial \eta} \right)_w = -k \left( \frac{\partial T}{\partial \eta} \frac{\partial \eta}{\partial \eta} \right)_{\eta=0} \]

\[ = \sqrt{\gamma c' \rho c F} \sum_{n=0}^{\infty} W_n(0) r^{n-3}. \quad (38) \]
The results of the numerical calculations for the velocity and temperature distributions in a convergent channel with transpiration-cooled walls are illustrated in Figures 2 and 3 and compiled in Tables 1 and 2.

3. Compressible Laminar Boundary Layer with Foreign Gas Injection

In the foregoing section, it has been established that the flow characteristics of an incompressible laminar boundary layer with arbitrary surface injection distribution can be determined by the method of power-series expansion about the streamwise distance. A logical extension of this problem is to include the effects of compressibility and foreign gas injection. In the realm of practical interest, the flow is compressible not only because of Mach number effects but also because of density variations in the presence of large temperature gradients in the flow field. Furthermore, it is sometimes desirable to inject a coolant gas different from the primary gas, such as that with high heat capacity and low thermal conductivity in order to induce greater favorable effects on reducing the heat transfer rate. The inclusion of these effects complicates the mathematical analysis in the following ways:

(1) The momentum and the energy equations become coupled by the viscosity-temperature relation; therefore, it is no longer possible to solve the momentum boundary layer equation without the simultaneous consideration of the energy equation. The influence of temperature on the coefficient of viscosity must be assumed on the basis of the available information on transport properties. In a limited temperature range, a linear relation between the temperature and the viscosity is found to be adequate for many engineering problems (6).
(2) A diffusion equation characterizing the conservation of the different species must be included. This equation depends on the velocity field and the coefficient of mass diffusion. Furthermore, the energy equation is affected by the mass diffusion as well as the thermal diffusion and the pressure diffusion processes, but the contribution of the latter two are of a higher-order magnitude and can be neglected in most analyses.

Under these circumstances, further simplifying assumptions, wherever possible, and a judicious choice of coordinate transformation are desirable in order to reduce the mathematical complexities of the problems of compressible laminar boundary layers with foreign gas injection. In this analysis, attention is directed to the wedge-type flows which admit similar solutions when the wall surfaces are impervious. Although a number of coordinate transformations have been utilized in treating the compressible viscous flow problems, the Mangler and the Dorodnitsyn transformations are used herein, which essentially remove the compressibility effects from some of the terms in the governing equations. The series-expansion technique developed in this analysis is applicable not only to cases which admit similar solutions under the condition of zero injection rates, but also to cases where the free-stream conditions normally do not permit similarity transformations.

The basic equations describing a steady-state compressible laminar boundary layer with arbitrary surface injection distribution of a foreign gas are formulated within the framework of the binary mixture concept, i.e., the primary and the injected gases are regarded as two different species. This assumption holds quite well for helium.
injection into an air boundary layer because of the similarity of the atomic weights and transport properties of oxygen and nitrogen. Without repeating the derivations which have been presented by several authors (5), these equations neglecting the effects due to thermal and pressure diffusions are:

Continuity

\[
\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0
\]

Momentum

\[
\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{\partial}{\partial y} (\mu \frac{\partial u}{\partial y})
\]

Diffusion

\[
\rho u \frac{\partial c_a}{\partial x} + \rho v \frac{\partial c_a}{\partial y} = \frac{\partial}{\partial y} (\rho D_{ab} \frac{\partial c_a}{\partial y})
\]

Energy

\[
\rho u c_p \frac{\partial T}{\partial x} + \rho v c_p \frac{\partial T}{\partial y} = \frac{u dp}{dx} + \mu \left(\frac{\partial u}{\partial y}\right)^2
\]

\[
+ \frac{\partial}{\partial y} (k \frac{\partial T}{\partial y}) + \rho D_{ab} \frac{\partial c_a}{\partial y} \frac{\partial}{\partial y} (h_a - h_b)
\]

where the subscripts a and b indicate the foreign gas and the primary gas, respectively. The nomenclature in accord with the conventional usage is explained in the List of Symbols.

In order to solve the simultaneous partial differential equations (39) through (42), the following transformation coordinates (7) are adopted:

\[
s = \int_0^x \rho u \mu u_e \, dx \quad \text{(Mangler)}
\]
and

\[ \xi = \frac{\rho u_e}{\sqrt{2a'}} \int_{0}^{y} \frac{\rho}{\rho_0} \, dy \]  
(Dorodnitsyn)  

where \( c^* \) is a transformation constant and the subscript \( \sigma \) represents the free-stream conditions. It follows that

\[ \frac{\partial}{\partial x} = c^* \frac{\rho_0 \mu_0}{\epsilon} \frac{u_e}{\partial s} + \frac{\partial}{\partial x} + \frac{\partial}{\partial \xi} \frac{\partial}{\partial \xi} \]  

\[ \frac{\partial}{\partial y} = \frac{\rho u_e}{\sqrt{2a'}} \frac{\partial}{\partial \xi} . \]  

The stream function which satisfies the continuity equation (39) is assumed to be of a power-series form to account for the arbitrary surface injection distribution as follows:

\[ \psi = \sqrt{2a'} \left[ F(\xi) + \sum_{n=0}^{\infty} G_n(\xi) \epsilon^{n+1} \right] \]  

where \( \epsilon = \frac{1-\beta}{(2s)^{\frac{3}{2}}} \) is a length parameter depending on \( s \) and the constant \( \beta \) is functionally related to the Euler number and will be discussed later. From the continuity equation (39) and the expression for \( \psi \) in equation (46), it can be verified that

\[ \frac{u}{u_o} = F' + \sum_{n=0}^{\infty} G_n' \epsilon^{n+1} \]  

and

\[ \frac{v}{u_o} = -\frac{c^* \rho_0 \mu_0}{\rho \sqrt{2a'}} \left[ F + (2-\beta)G_0 \epsilon + (3-2\beta)G_1 \epsilon^2 + (4-3\beta)G_2 \epsilon^3 + \ldots \right] \]

\[ \quad - \frac{\partial}{\partial x} \left[ \frac{\sqrt{2a'}}{\rho u_o} \left[ F' + G_0' \epsilon + G_1' \epsilon^2 + G_2' \epsilon^3 + \ldots \right] \right] \]  

(48)
where the prime denotes total differentiation with respect to the argument and $F$, $G_0$, $G_1$, ..., are understood to be functions of $\xi$ only.

Caution must be exercised in selecting the power series, however, since it must be compatible with the boundary conditions at the surface (Appendix B). Of course, for certain specifically prescribed boundary conditions, similar solutions may be admissible. Then, the power series assumed above will be reduced to one term, namely $F(\xi)$, since by utilizing the similarity transformation the boundary layer equations can be replaced by ordinary differential equations whose dependent variables are functions of $\xi$ alone.

For a general problem involving compressibility effects and foreign gas injection arbitrarily distributed along the surface, all other dependent variables must also be expressible in terms of power series in $\xi$ so as to be compatible with the series representation for the stream function, $\psi$. As a convenience, some of the following functions are expressed in dimensionless form.

$$\bar{T}(\epsilon, \xi) = \frac{T}{T_0} = \bar{T}_0(\xi) + \bar{T}_1(\xi)\epsilon + \bar{T}_2(\xi)\epsilon^2 + \ldots \quad (49)$$

$$N(\epsilon, \xi) = \frac{\rho \mu}{\rho_0 \mu_0} = N_0(\xi) + N_1(\xi)\epsilon + N_2(\xi)\epsilon^2 + \ldots \quad (50)$$

$$\bar{R}(\epsilon, \xi) = \frac{R}{R_0} = \bar{R}_0(\xi) + \bar{R}_1(\xi)\epsilon + \bar{R}_2(\xi)\epsilon^2 + \ldots \quad (51)$$

$$c_s(\epsilon, \xi) = c_{s0}(\xi) + c_{s1}(\xi)\epsilon + c_{s2}(\xi)\epsilon^2 + \ldots \quad (52)$$

Substituting equations (47), (48), (49), (50), and (51) into the momentum equation and after a very tedious rearrangement, the following
system of differential equations for the momentum equation (40) is obtained:

$$
\left( \frac{N_0}{\sigma^*} F_{w} \right)' + F_{w} - \frac{2s}{u_0} \frac{du_0}{ds} \left[ F_{w} - \bar{F}_{w} \bar{T}_0 \right] = 0 \quad (53)
$$

$$
\left[ \frac{1}{c^*} (N_1 F_{w} + N_0 G_{w}) \right]' + F_{G_{w}} - \left[ (1 + \beta) F_{w} \right] G_{w} \left[ (2 - \beta) F_{w} \right] G_{w}
+ \frac{2s}{u_0} \frac{du_0}{ds} \left[ \bar{F}_{w} \bar{T}_0 + \bar{F}_{w} \bar{T}_1 \right] = 0 \quad (54)
$$

$$
\left[ \frac{1}{c^*} (N_2 F_{w} + N_1 G_{w} + N_0 G_{w}) \right]' + F_{G_{1}} - 2F_{w} G_{1} + \left[ (3 - 2\beta) F_{w} \right] G_{1}
- \left[ G_{1} + (2 - \beta) G_{0} G_{1} \right] G_{w} + \frac{2s}{u_0} \frac{du_0}{ds} \left[ \bar{F}_{w} \bar{T}_0 + \bar{F}_{w} \bar{T}_1 + \bar{F}_{w} \bar{T}_2 \right] = 0 \quad (55)
$$

$$
\left[ \frac{1}{c^*} (N_3 F_{w} + N_2 G_{w} + N_1 G_{w} + N_0 G_{w}) \right]' + F_{G_{2}} - \left[ (3 - 3\beta) F_{w} \right] G_{2}
+ \left[ (4 - 3\beta) F_{w} \right] G_{2} - \left[ (3 - 2\beta) G_{1} G_{1} - (3 - 2\beta) G_{0} G_{1} - (2 - \beta) G_{0} G_{1} \right]
+ \frac{2s}{u_0} \frac{du_0}{ds} \left[ \bar{F}_{w} \bar{T}_0 + \bar{F}_{w} \bar{T}_1 + \bar{F}_{w} \bar{T}_2 + \bar{F}_{w} \bar{T}_3 \right] = 0 \quad (56)
$$

etc.

In a similar manner, a system of differential equations describing the diffusion boundary layer in terms of the injected mass fraction components, $C_{a_0}$, $C_{a_1}$, $C_{a_2}$, ..., is found to be as follows:

$$
\left( \frac{N_0}{\sigma^*} C_{a_0}' \right)' + F_{C_{a_0}} = 0 \quad (57)
$$

$$
\left[ \frac{1}{c^*} \left( N_1 C_{a_0}' + N_0 C_{a_1}' \right) \right]' + F_{C_{a_1}} - \left( 1 - \beta \right) F_{a_1} C_{a_1} + \left( 2 - \beta \right) G_{0} C_{a_0}' = 0 \quad (58)
$$
\[
\left[ \frac{1}{c^{*}S_c} \left( N_2C_{a_0} + N_1C_{a_1} + N_0C_{a_2} \right) \right]^{'} + FC_{a_2} = 2(1-\beta)F'C_{a_2} \\
+ \left[ (2-\beta)G_0C_{a_1} - (1-\beta)G_0C_{a_1} + (3-2\beta)G_1C_{a_2} \right] = 0
\] (59)

\[
\left[ \frac{1}{c^{*}S_c} \left( N_2C_{a_0} + N_1C_{a_1} + N_1C_{a_2} + N_0C_{a_3} \right) \right]^{'} + FC_{a_3} = 3(1-\beta)F'C_{a_3} \\
+ \left[ 2(1-\beta)G_0C_{a_2} - 2(1-\beta)G_0C_{a_2} + (3-2\beta)G_1C_{a_1} \right. \\
- (1-\beta)G_1C_{a_1} + (4-3\beta)G_2C_{a_0} \right] = 0
\] (60)

etc.,

where the Schmidt number \( S_c = \gamma / D_{ab} \) is assumed to be a constant.

With the assumption that the specific heat, \( c_p \), is a function of the concentration as well as the temperature, the energy equation (42) can be similarly represented by a system of differential equations. At present, a simple case of vanishing free-stream Mach number is considered. Then, the energy equation becomes

\[
\left\{ \frac{P_o}{c^{*}Pr} \left[ c_{Pb} + (c_{Pa} - c_{Pb})C_{a_0} \right] \right\}^{'} + \frac{N_oZ_o}{c^{*}S_c} - Q_o \left[ c_{Pb} + (c_{Pa} - c_{Pb})C_{a_0} \right] = 0
\] (61)

\[
\frac{1}{c^{*}Pr} \left\{ P_1 \left[ c_{Pb} + (c_{Pa} - c_{Pb})C_{a_0} \right] + (c_{Pa} - c_{Pb})C_{a_1}P_o \right\}^{'} + \frac{1}{c^{*}S_c} \left( Z_1N_o + Z_0N_1 \right) \\
- \left\{ Q_1 \left[ c_{Pb} + (c_{Pa} - c_{Pb})C_{a_0} \right] - Q_o (c_{Pa} - c_{Pb})C_{a_1} \right\} = 0
\] (62)
\[
\frac{1}{c^* Pr} \left\{ P_2 \left[ (c_{p_a} - c_{p_b})C_{a_0} \right] + (c_{p_a} - c_{p_b})(P_1 C_{a_1} + P_0 C_{a_2}) \right\} \\
+ \frac{1}{c^* Sc} \left( Z_2 N_0 + Z_1 N_1 + Z_2 N_0 \right) + \left\{ Q_2 \left[ (c_{p_a} - c_{p_b})C_{a_0} \right] \\
+ Q_1 (c_{p_a} - c_{p_b})C_{a_1} + (c_{p_a} - c_{p_b})C_{a_2} Q_0 \right\} = 0 \quad (63)
\]

etc.,

where \( c_p = C_a(c_{p_a} - c_{p_b}) + c_{p_b} \) is the specific heat of a binary mixture with species a and b, and \( Pr = \mu c_p/k \) is the Prandtl number which is assumed to be a constant. The auxiliary functions \( P_1(\xi) \), \( Q_1(\xi) \), and \( Z_1(\xi) \) in equations (61), (62) and (63) are defined as follows:

\[
P_0 = N_0 T_0' \\
P_1 = N_1 T_0' + N_0 T_1' \\
P_2 = N_2 T_0' + N_1 T_1' + N_0 T_2'
\]

etc.

\[
Q_0 = - F T_0' \\
Q_1 = - F T_1' + (1-\beta) F T_1 - (2-\beta) G_0 T_0' \\
Q_2 = - F T_2' + 2(1-\beta) F T_2 + (1-\beta) G_0 T_1 - (3-2\beta) G_1 T_0 - (2-\beta) G_0 T_1'
\]

etc.

and

\[
Z_0 = (c_{p_a} - c_{p_b})C_{a_0} T_0' \\
Z_1 = (c_{p_a} - c_{p_b})(C_{a_1} T_0' + C_{a_0} T_1') \\
Z_2 = (c_{p_a} - c_{p_b})(C_{a_2} T_0' + C_{a_1} T_1' + C_{a_0} T_2')
\]

etc.
By solving equations (53) through (66), the flow characteristics for a compressible boundary layer with foreign gas injection can be determined completely, provided that appropriate boundary conditions are prescribed. Although the series-expansion method presented herein permits a wide range of applications, unsurmountable mathematical complexities are frequently encountered in carrying out the numerical solutions. The thermodynamic quantities and the transport coefficients, such as $R$, $c_p$, $\mu$, and $k$, are affected by the surface injection distribution. In other words, they are functions of $\varepsilon$ and $\xi$ except by fortuitous circumstances when similarity transformation is admissible. Rigorously speaking, the Prandtl number, $Pr$, and the Schmidt number, $Sc$, also depend on $\varepsilon$ and $\xi$. Otherwise, some of the essential features of the diffusion process are disposed of.

The pressure gradient parameter $(2s/u_\infty)(du_\infty/ds)$ is a function of the streamwise coordinate $\varepsilon$ and depends on the free-stream condition. In the general case, this parameter may be represented by a power series of $\varepsilon$. Then, substitution of this series into the basic conservation equations (40) and (42) yields systems of ordinary differential equations similar to equations (53) through (66). At present, a wedge-type flow field is assumed such that $(2s/u_\infty)(du_\infty/ds)$ may be chosen to be a constant, say $\beta$, i.e.,

$$\beta = \frac{2s}{u_\infty} \frac{du_\infty}{ds} = \text{constant}. \quad (67)$$
If the free-stream Mach number is not vanishingly small, a different form for $\beta$ must be assumed (see Appendix C),

$$\beta = \frac{2s}{u_e} \frac{du_e}{ds} \left(1 + \frac{\gamma - 1}{2} M_e^2\right) = \text{constant}. \quad (68)$$

When the free-stream Mach number is very small, equations (67) and (68) become identical. Then, the following differential equation describes the relation between $u_e$ and $s$:

$$\frac{du_e}{u_e} = \frac{\beta}{2} \frac{ds}{s} \quad (69)$$

from which

$$u_e = a_1 s^{\beta/2} = a_2(2s)^{\beta/2} \quad (70)$$

where $a_1$ is a constant of integration. From equation (63), we have

$$\frac{ds}{dx} = c^* \rho_e/\rho_e u_e = c^* \rho_e/\rho_e a_2(2s)^{\beta/2}.$$  

Integration yields

$$s = a_3 x^{\frac{2}{2-\beta}}. \quad (71)$$

Hence, $u_e$ can be expressed as

$$u = a_4 x^{\frac{\beta}{2-\beta}} \quad (72)$$

where the exponent $\beta/(2-\beta)$ is identical to the Euler number introduced by Falkner and Skan. It can be verified that for $\beta = 0$, $u_e$ = constant represents flows over a flat plate, and for $\beta = 1$, $u_e \propto x$ represents flows in the stagnation-point region.
The boundary conditions for the mass-transfer cooling problem are as follows:

1. \[ F'(0) = G'_0(0) = G'_1(0) = \ldots = 0 \]
2. \[ F'(\infty) = 1, \quad G'_0(\infty) = G'_1(\infty) = \ldots = 0 \]
3. \[ v_w = v_w(\xi) = a_0 + a_1 \xi + a_2 \xi^2 + \ldots \]

(73)

or for a wedge flow where \( u_e = a_4 x^{2-\beta} \) and \( \xi = (2\xi) \), we have, from equation (48),

\[ F(0) = 0, \quad G_0(0) = \frac{-\rho_w a_0}{c^2 \rho_e / \mu_e a_4 (2-\beta)}, \]

\[ G_1(0) = \frac{-\rho_w a_1}{c^2 \rho_e / \mu_e a_4 (3-2\beta)}, \ldots \]

(4) \( C_a(0) = C_{aw}, \quad C_a(\infty) = 0, \quad v_w = -\left( \frac{D_{ab}}{C_b} \frac{\partial C_a}{\partial y} \right)_w \)

(5) \( T(0) = T_w(\xi), \quad T(\infty) = T_e(\xi). \)

It should be remarked that the wedge flows involving fluid injections generally do not admit similar solutions with an arbitrary surface injection distribution as stated in the third boundary condition in equation (73). The requirements for which similar solutions exist for a transpiration-cooled wedge are as follows (5):

1. The mass concentration \( C_a \) (and \( C_b \)) must be dependent on \( \xi \) alone.

2. The free-stream Mach number, \( M_e \), must be vanishingly small or infinitely large. In the former case, the temperature ratio, \( T = T/T_e \), can be regarded as a function of \( \xi \) for an isothermal surface.
It is seen that the free-stream temperature, $T_0$, can now be approximated by the free-stream stagnation temperature, $T_0^* = T_0 + (u^2/2c_p)$, which is constant for adiabatic free-stream flows. If the free-stream Mach number is finite, the Prandtl number, the Schmidt number, and the Lewis number must equal unity (Appendix C). In all these cases, it is necessary that the pressure gradient parameter, $\beta$, given in equation (68) is constant.

3. The third boundary condition in equation (73) must be of the form $v_w \approx x^{\frac{\beta-1}{2}}$. For special cases, we have $v_w \approx x^{-\frac{1}{2}}$ for a flat plate ($\beta = 0$), and $v_w = \text{constant}$ for stagnation point region ($\beta = 1$).

If conditions 1 and/or 2 stated above are not fulfilled, the so-called "local similarity" concept (7) must be resorted to. This means that under certain flow conditions for a particular body shape, the velocity and enthalpy profiles may exhibit similarity properties, at least for an appreciable streamwise distance. The validity of the locally similar condition can only be justified for each particular problem by determining the magnitude of the terms in the governing equations. The local similarity rule is particularly useful for problems involving hypersonic viscous flows over a highly cooled body. By employing this rule the pressure gradient term in the momentum equation can be ignored, since the hypersonic boundary layer is insensitive to its effect. Then, the solution of the momentum equation leads to the Blasius profile. The calculated heat transfer characteristics based on this simplification have reached satisfactory agreement with experimental findings (7).

In the case where all three conditions for similar solutions are satisfied, the series representation of the basic equations under the
condition of vanishing free-stream Mach number reduce to only the zero-order equations (53), (57), and (61), which describe a flow field over an isothermal surface, namely,

\[
\left(\frac{N}{c^*} F^p\right)' + F^p F' - \beta (F'^2 - \frac{K}{T}) = 0
\]

\[
\left(-\frac{N}{c^* S_c} C^a\right)' + FC_a = 0
\]

\[
\left(-\frac{N}{c^* S_c} \frac{c_p T^p}{T}\right)' + \left(-\frac{N}{c^* S_c} c_p F\right)T^p = 0
\]

where the terms involving \(v^2_a\) are neglected for small \(M_a\). These equations are identical with those derived by Baron and Scott (5).

If the injected gas is the same as the primary gas, the system of equations for the diffusion process vanishes, then the differential equations become

\[
\begin{align*}
F^p' + F^p F' - \beta (F'^2 - T_o) &= 0 \\
\left(\frac{1}{Pr}\right)T_o' + F T_o' &= 0
\end{align*}
\]

\[
\begin{align*}
G_o'^p + F C_o'^p &= \left[\left(1+\beta\right)F^p\right]G_o^p + \left[(2-\beta)F^p G_o + \beta T_1\right] = 0 \\
\left(\frac{1}{Pr}\right)T_1'^p + F T_1' &= \left(1-\beta\right)F^p T_1 + \left(2-\beta\right)G_o T_o' = 0
\end{align*}
\]

\[
\begin{align*}
G_1'^p + F C_1'^p &= 2F^p G_1^p + \left[(3-2\beta)F^p\right]G_1 + \left[-G_1'^2 - (2-\beta)G_o G_o - \beta T_2\right] = 0 \\
\left(\frac{1}{Pr}\right)T_2'^p + F T_2' &= 2\left(1-\beta\right)F^p T_2 + \left[(1-\beta)G_o T_1 - (2-\beta)G_o T_o^p\right] \\
&\quad - (3-2\beta)G_1 T_o^p = 0
\end{align*}
\]


\[
\begin{align*}
G'' + FC'' &= \left[ (3-\beta)F' \right] G'' + \left[ (4-3\beta)F' \right] G_2 - \left[ (3-\beta)G'G_1 \right] \\
& \quad - (3-2\beta)G_1 G_0 - (2-\beta)G_0 G_1 - \beta T_3 = 0 \\
(1/Pr)T_3'' + FT_3 &= 3(1-\beta)F'T_3 - 2(1-\beta)G_0 T_2 \\
& \quad + (1-\beta)G_1 T_1 - (3-2\beta)G_1 T_1 - (4-3\beta)G_2 T_0 = 0
\end{align*}
\]

etc.

In formulating the above equations, the Chapman-Rubesin linear relation between temperature and viscosity is assumed, i.e.,

\[
\frac{N}{c^*} = \frac{\rho \mu}{\mu_0 c^*} = \frac{T_0 T G}{T T_0 c^*} = \frac{G}{c^*}
\]

where \(G\) is the Chapman-Rubesin constant. The ratio \((N/c^*)\) has the value unity if \(c^*\) is chosen to be the same as \(G\).

In the case of a flat plate \((\beta = 0)\), the foregoing equations become identical with those for an incompressible flow.

\[
\begin{align*}
F'' + F'F &= 0 \\
(1/Pr)T_0'' + FT_0 &= 0
\end{align*}
\]

\[
\begin{align*}
G''_0 + FC''_0 &= F'G'_0 + 2F'G_0 = 0 \\
(1/Pr)T_0'' + FT_1 &= F'T_1 + 2G_0 T_0 = 0
\end{align*}
\]

\[
\begin{align*}
G''_1 + FC''_1 &= 2F'G'_1 + 2F'G_1 - (G_0^2 - 2G_0 G_1) = 0 \\
(1/Pr)T_0'' + FT_1 &= 2F'T_2 + 2T_0 G_1 - (G_0 T_1 - 2G_0 T_1) = 0
\end{align*}
\]
\[
G^2_2 + FG^2_2 = \mathcal{F}'G^2_2 + \mathcal{G}'G^2_2 - (G^1_0G^1_1 - G^1_1G^1_0 - 2G^1_0G^1_1) = 0
\]
\[
(1/Pr)T^2_3 + FT^1_3 - \mathcal{F}'T^2_3 + \mathcal{T}_0'G^2_2 - (2G^0'T^2_2 + G^1'T^1_1 - X_1T^1_1 - 2G^0'T^2_1) = 0
\]

etc.

From these equations, the simple form of Reynolds' analogy can be established easily for Prandtl number equal to unity. It is seen that in this case there exists a one-to-one correspondence between the functions \( F' \) and \( T_0' \), \( G^1_0 \) and \( T^1_1 \), \( G^1_1 \) and \( T^2_2 \), etc. In other words, the temperature is related linearly to the velocity in a unique manner.

Under special conditions \( Pr = Sc = Le = 1 \) where \( Pr = c_p\mu/k \) is the Prandtl number, \( Sc = \nu/D_a \) is the Schmidt number, and \( Le = (\rho D_a c_p)/k \) is the Lewis number, the calculation procedure can be somewhat simplified by means of the well-known Crocco's relations.

Multiplying the momentum equation (40) by \( u \) and adding to the energy equation (42) we obtain

\[
\rho u \frac{\partial h^0}{\partial x} + \rho v \frac{\partial h^0}{\partial y} = \frac{\partial}{\partial y} \left( \mu \frac{\partial h^0}{\partial y} \right) + \frac{\partial}{\partial y} \left[ \left( \frac{1}{Pr} - 1 \right) k \frac{\partial h}{\partial y} \right] + \frac{\partial}{\partial y} \left[ \rho D_a (1 - \frac{1}{Le})(h_a - h_b) \frac{\partial c_a}{\partial y} \right]
\]

where \( h^0 = c_pT + (u^2/2) \). The diffusion equation (41) can be expressed as

\[
\rho u \frac{\partial c_a}{\partial x} + \rho v \frac{\partial c_a}{\partial y} = \frac{\partial}{\partial y} \left[ \frac{Le}{Pr} / \mu \frac{\partial c_a}{\partial y} \right].
\]
For \( Pr = Le = 1 \), equations (86) and (87) become

\[
\rho u \frac{\partial h^0}{\partial x} + \rho v \frac{\partial h^0}{\partial y} = \frac{\partial}{\partial y}(\mu \frac{\partial h^0}{\partial y}) \quad (88)
\]

\[
\rho u \frac{\partial C_a}{\partial x} + \rho v \frac{\partial C_a}{\partial y} = \frac{\partial}{\partial y}(\mu \frac{\partial C_a}{\partial y}) \quad (89)
\]

A particular solution of equation (88) is that \( h^0 = \) constant which represents the solution for an insulated plate first discovered by Crocco (9). However, irrespective of whatever relation that may exist between the velocity and the enthalpy, the total enthalpy, \( h^0 \), and the mass concentration of the injected gas, \( C_a \), are related linearly under the condition \( Pr = Le = 1 \) provided that their boundary conditions are compatible. This particular solution for equations (88) and (89) is

\[
h^0 = a + bC_a \quad (90)
\]

where \( a \) and \( b \) are arbitrary constants. Upon applying the boundary conditions

\[
C_a = 0, \quad h^0 = h^0_0 \text{ at } y \rightarrow \infty \quad (91)
\]

\[
C_a = C_{aw}, \quad h^0 = h^0_w \text{ at } y = 0
\]

we obtain

\[
h^0 = h^0_0 - (h^0_0 - h^0_w)\left(\frac{C_a}{C_{aw}}\right) \quad (92)
\]

Equation (92) states that if chemical reactions are excluded from the considerations, the stagnation enthalpy varies linearly with the injected mass fraction in a unique manner. Accordingly, for \( Pr = Le = 1 \),
the numerical calculation is simplified by the fact that the simple
equation (92) may replace either the diffusion equation (h1) or the
energy equation (h2).

The important parameters characterizing the local skin friction
and the heat transfer effects are as follows:

1. The skin friction coefficient

\[ c_f = \frac{\mu_w \left( \frac{\partial u}{\partial y} \right)_w}{\frac{1}{2} \rho_w u_w^2} = \frac{N_w/\mu}{\frac{1}{2} \sqrt{2\pi}} \left[ F^w(0) + \sum_{n=1}^{\infty} G_n^w(0) \varepsilon^n \right] \]  

(93)

where

\[ N_w = \frac{\rho_w/\mu_w}{\rho_e/\mu_e} . \]

2. The heat transfer rate which includes the contributions due
to the conductive and the convective components

\[ q_w = - \left[ k \frac{\partial T}{\partial y} + (\rho v) h_a \right] \]  

(94)

However, interest is generally focused on the contribution of the ther-
mal gradient to the temperature increase within the structure. Then, we
have

\[ q_w = - \left( k \frac{\partial T}{\partial y} \right)_w = - k_w \rho_w (2s)^{\frac{\beta-1}{2}} \sum_{n=0}^{\infty} T_n^w(0) \varepsilon^n. \]  

(95)

As stated previously, for Prandtl number of unity, the functions \( F^w, G^w, G_1^w, \ldots \), have one-to-one correspondence with \( T_0^w, T_1^w, \ldots \), respec-
tively, then Reynolds' analogy holds, i.e.,

\[ q_w \sim T_w . \]  

(96)
In conclusion, the general problem for a compressible laminar boundary layer past a transpiration-cooled wedge can be solved, in principle. The systems of ordinary differential equations (53) through (66) with boundary conditions (73) will yield exact solutions. A rigorous approach has been adopted in formulating these equations inasmuch as within the premise of boundary layer assumptions, the only simplifications used are that the Prandtl number and the Schmidt number are constant and the Mach number of the outer edge of the boundary layer is small. As characteristic to most physical problems, mathematical complexities invariably arise when generalised equations are treated. However, in the present case, the calculation procedure for the numerical solutions of equations (53) through (66) should not be too difficult with the aid of digital computer facilities. It should be borne in mind that the only system of nonlinear differential equations is that of the zero-order approximation for which the solution is already known. The remaining systems of differential equations are all linear and can be programmed on a high-speed computer.

If digital computer facilities are not available, excessive labor will be involved in performing the calculations. The tedious numerical procedures cannot be circumvented for problems involving streamwise pressure gradients and diffusion. Only for simple cases, such as a flat plate, could the numerical results be obtained with any reasonable amount of effort. In the present analysis, the problem of a transpiration-cooled flat plate ($\beta = 0$) with $Pr = 1$ was undertaken for numerical analysis. The solution of the zero-order equation (82) is identical with the Blasius solution for an incompressible flow over an impervious
flat plate \((h)\). The solution of the first-order equation \((83)\) was carried out with the aid of a desk calculator. While extensive numerical analysis of this problem was not possible, limited calculation at least achieved its goal by showing the effect of injection on the flow characteristics rather clearly.

4. Method of Solution

Numerical solutions for flows through a convergent channel and over a flat plate were carried out up to the first-order approximation. The zero-order equations which represent flows over an impervious surface have known solutions in both cases. For a flat plate, the exact solution is available in numerical form which is the well-known Blasius solution. For a convergent channel, the solution can be expressed in terms of a hyperbolic tangent of the transformed variable \(\eta\). The temperature distribution for the latter case has been solved by Millsaps and Pohlhausen (2) who treated the exact energy equation without making use of the boundary layer approximations.

The first-order solutions for the velocity distributions for flows through a convergent channel and over a flat plate with uniform injection were obtained by utilizing the Runge-Kutta method. Since the equations are third-order homogeneous linear differential equations, the forward integration procedure involves solving three simultaneous equations for each increment \(\Delta \eta \) (or \(\Delta \xi \)). Furthermore, when two-point boundary conditions are prescribed, i.e., the conditions at 0 and at \(\infty\), the outer-edge condition must be replaced by an estimated surface condition \((g_0^e(0) \text{ or } G_0^e(0))\) in order to proceed with the calculation.
After carrying out the calculations from the surface up to a point far from the surface, it can be predicted whether or not the outer-edge condition will be asymptotically reached. The final numerical solution is achieved after a correct initial estimate is used so that all the boundary conditions are satisfied. Fortunately, because the differential equations are linear, the method of determining unknown coefficients may be used instead of the much involved iteration procedure.

To state this more clearly, the homogeneous differential equation (22) or (83) for the first-order velocity distribution for flows over a transpiration-cooled surface may be written in the following form:

\[ Y'' + P(x)Y'' + Q(x)Y' + R(x)Y = 0 \]  \hspace{1cm} (97)

with boundary conditions

\[
\begin{align*}
Y(0) &= I_0 \\
Y'(0) &= 0 \\
Y'(+\infty) &= 0
\end{align*}
\]  \hspace{1cm} (98)

In order to solve equation (97) numerically, a value for \( Y''(0) \) is assumed in place of \( Y'(+\infty) \). Then, by employing the Runge-Kutta forward integration method, numerical values for each increment \( \Delta x \) can be determined. The solution with the initial estimate, say \( Y_1 \), generally does not satisfy the outer-edge boundary condition \( Y(+\infty) \). The next step is to assume a different \( Y''(0) \) and repeat the numerical calculation,
thus obtaining $Y_2$. Then the boundary conditions (98) require that

\[
\begin{align*}
Y(0) &= Y_0 = AY_1(0) + B Y_2(0) \\
Y'(0) &= 0 = AY_1'(0) + B Y_2'(0) \\
Y'((\infty) &= 0 = A Y_1'(\infty) + B Y_2'(\infty)
\end{align*}
\]  

(99)

where $A$ and $B$ are arbitrary constants. Since the two proper initial conditions $Y(0)$ and $Y'(0)$ are used for obtaining $Y_1$ and $Y_2$, i.e., $Y_1(0) = Y_2(0) = Y_0$, and $Y_1'(0) = Y_2'(0) = 0$, it is obvious that equation (99) now contains only two equations for the unknowns $A$ and $B$.

\[
\begin{align*}
l &= A + B \\
0 &= A Y_1'(\infty) + B Y_2'(\infty)
\end{align*}
\]  

(100)

Therefore, after $A$ and $B$ are determined, $Y = A Y_1 + B Y_2$ is the solution of differential equation (97) since all three boundary conditions are now satisfied.

In a similar manner, the temperature distribution for flows through a convergent channel can be determined by means of the Runge-Kutta method. The calculation procedure is more rapid than that for the velocity profile, since the equation is second-order and of the form

\[
Y'' + P(x)Y + Q(x) = 0
\]  

(101)

with boundary conditions

\[
\begin{align*}
Y(0) &= 0 \\
Y(\infty) &= 0
\end{align*}
\]  

(102)
Again, a value of \( Y'(0) \) is assumed instead of \( Y(\infty) \). However, it is not necessary to estimate for another \( Y'(0) \). Instead, the calculations are carried out by determining the complementary function of equation (101). By utilising the property of a linear differential equation that the solution is the sum of the complementary function and the particular integral, we have

\[
Y = Ay_c + Y_p
\]

(103)

from which \( A \) can be determined by using the outer-edge condition (10) and the problem is solved.

Inasmuch as digital computer facilities were not available, these calculations were carried out by means of a desk calculator. For velocity distributions, an increment of 0.1 was used for solving equations (22) and (83). The calculations became quite tedious since asymptotic results were not reached until \( \gamma \approx 7 \) for a convergent channel and \( \xi \approx 5 \) for a flat plate. The temperature profile for converging flows were calculated by using an increment of 0.2 for the solution of the zero-order equation (33) and an increment of 0.1 for the solution of the first-order equation (34). The accuracy of these results was at most within three significant figures.

The numerical calculations up to the first-order approximations for the flat plate and the convergent channel problems are depicted in Figures 2, 3, 4 and 5, and are compiled in Tables 1, 2, and 3.

An attempt to solve these problems was also made by using the analog computer available in the Aerodynamic Laboratory of The Ohio State University. However, because of the two-point boundary conditions
involved and the sensitivity of the initial boundary conditions on the solution, accurate results could not be ascertained.

In principle, the general problem for a compressible laminar boundary layer with foreign gas injection can be solved by means of the Runge-Kutta method. However, since two other initial approximations in place of the outer-edge conditions for the mass concentration as well as the temperature must be made, the calculation procedure will become prohibitively tedious by using only a desk calculator.

5. Results and Discussion

The method of series expansion about a parameter has found many applications in the field of viscous aerodynamics. One classical example is the series solution for a flow past a circular cylinder investigated by Blasius, Hienenz and Howarth (11). Similar to the present problem, the stream function characterising flows around the cylinder may be assumed to have the form of a power series in x with coefficients being functions of y alone. Then the problem is reduced to the successive determination of these universal functions by solving a system of ordinary differential equations. In essence, the series method can be regarded as a technique for seeking a number of similar solutions joined by an "adjustment" distance. While this method is elementary in principle and straightforward in execution, tedious numerical procedures are frequently encountered, which greatly handicapped the earlier investigators in aerodynamics. However, with the rapid development of high-speed computing devices, this difficulty is largely removed to permit a wide range of applications to this method.
The limitations of the series method to solve differential equations are the possibility of divergence or slow convergence of the power series and the occurrence of a singularity at a point or on a line within the domain of interest. The latter is encountered when the perturbation parameter, such as the transformed variable $r$ or $\varepsilon$ is large, i.e., $r, \varepsilon \gg 1$. In that case, a convergent series is possible only if the universal functions independent of $r$ or $\varepsilon$ vanish asymptotically for higher-order approximations. Therefore, for certain physical problems, the series method may become impractical if a large number of differential equations must be solved.

Under certain circumstances, a singularity may be present in the zero-order solution. Then, the singularity reappears in the higher-order solutions and becomes more severe as the order of solution increases. The power-series method thus breaks down in the vicinity of the singularity. A familiar example is the solution near the leading edge of a flat plate or a wedge. Higher-order perturbations result in infinite shearing force along the surface which is contradictory to physical facts. However, this situation may be remedied by employing the so-called "Poincaré-Lighthill-Kuo" method for which the boundary layer variables are further transformed to allow a uniformly valid solution in the close proximity of the singularity (12). No attempt was made to analyze the leading edge problem in the course of the present study.

Although the numerical calculations have been carried out to a limited extent only, the first-order solution already shows the trend
that fluid injection reduces the wall shearing stress as well as the heat transfer rate.

Figure 2 illustrates the velocity distribution functions $f$, $f'$, $g_0$, $g_0'$ and $g_0^*$ for flows through a convergent channel. The numerical values are also compiled in Table 1. The $f$ functions are the zero-order solutions which are identical with those for impervious channel walls. The $g_0$ functions show the first-order effects of uniform injection along the wall surfaces. For surface injection condition $g_0(0) = 0.1$, the local skin friction coefficient has the value

$$c_f = 2 \sqrt{\nu/C} (1.1547 - 0.05774 r + ...) \quad (104)$$

Because of linearity of the differential equations, an increase in injection rate will result in proportional increase in $g_0$, $g_0'$ and $g_0^*$. Thus, for $g_0(0) = 1.0$, we obtain $g_0^*(0) = 0.5774$ which is ten times the value for $g_0(0) = 0.1$. The local skin friction coefficient then becomes

$$c_f = 2 \sqrt{\nu/C} (1.1547 - 0.5774 r + ...) \quad (105)$$

which shows a greater decrease in the shearing stress. The higher-order solutions, $g_1^*(0)$, $g_2^*(0)$, ..., might have opposite signs to that of $g_0^*(0)$, but for small $r$, their influence would most likely be less important. If the injection rate is sufficiently high, flow separation ($c_f = 0$) may occur at some distance away from the point $r = 0$.

The temperature functions $W_0$, $W_0'$, $W_1$, and $W_1'$ for a converging flow with $g_0(0) = 0.1$ are illustrated in Figure 3 and compiled in Table 2 where the subscript 0 indicates the zero-order solutions (no
injection) and the subscript 1 indicates the first-order solution. Also, in this case, the Prandtl number is defined on the basis of \( c_v \) rather than \( c_p \). The local heat transfer rate is found to be

\[
q_w = \left( -\frac{k \frac{\partial T}{\partial y}}{\rho c_p} \right) - \frac{kn_0^2 \sqrt{\frac{c}{\gamma}}}{\rho c_p} (0.4377 - 0.0334 r + \ldots) \quad (106)
\]

from which it is seen that a reduction in heat transfer rate is effected by fluid injection. It should be remarked that the temperature distribution for the zero-order solution represented by the \( W_0 \)-curve exhibits the same property as that obtained by Millsaps and Pohlhausen (2) who solved the exact energy equation instead of the thermal boundary layer equation. The temperature rapidly increases and reaches a maximum not far from the surface. Thereafter, it decreases asymptotically to a constant value. It should also be mentioned that the accuracy of the temperature distribution functions presented herein is somewhat questionable. The reason is twofold: (1) large increment was used for the step-by-step calculations which may have resulted in significant truncation errors and (2) the calculations were not carried out far enough to reach the proper asymptotic value. The initial slope of the temperature distribution curve may then have been somewhat deviated from the true value. From the experience of the author, a slight deviation of the initial values of a boundary layer problem frequently influences the final result significantly, particularly in the region far from the surface.

For laminar boundary layer over a flat plate, the zero-order solutions \( F, F', \) and \( F'' \) are depicted in Figure 4 (data taken from Ref. 4).
The first-order velocity distribution functions $G_0$, $G_0^1$ and $G_0^2$ which were determined by solving equation (83) are shown in Figure 5 and compiled in Table 3. The local skin friction coefficient has the value

$$c_f = \frac{\mu w (\frac{\partial u}{\partial y})}{\frac{1}{2} \rho e u_e^2} = \frac{K_w / \mu}{\frac{1}{2} \sqrt{2\beta}} \left( 0.4697 - 0.08282 \epsilon + \ldots \right) \quad (107)$$

which shows a slight reduction in $c_f$ when injection is present. It is not necessary to calculate the temperature distribution since it was pointed out in the previous section that for Pr = 1 (based on $C_P$) the temperature and the velocity functions have a one-to-one correspondence.

From the foregoing examples of transpiration–cooling problems in which limited calculations were carried out, the expected reduction in skin friction and heat transfer rate with increasing injection rate has been confirmed. Physically, fluid injection has the same effect as an increase in pressure gradient along the flow direction. As a result, the boundary layer is thickened and the slope of the velocity profile decreases accordingly, thereby causing a reduction in skin friction coefficient and heat transfer rate. While fluid injection has been proven to be an effective method to protect the wall surfaces from excessive heating, it also induces boundary layer instability because of the associated effect of adverse pressure gradient. The instability is referred to the enhancement of flow separation which is more serious if the boundary layer is initially laminar. Thus, the desirable features of the reduction in skin friction and heat transfer rate due to coolant injection are counterbalanced by an appreciable increase in friction
drag under flow separation conditions. However, this paradox does not necessarily imply that transpiration cooling is an impractical technique. Effective protective cooling is generally ascertained for small injection rate without causing flow separation.

As a conclusion, a few remarks pertinent to the series solution of compressible laminar boundary layers with arbitrarily distributed foreign-gas injection should be made. First of all, the systems of ordinary differential equations (53) through (63), and so on, have been formulated on the basis of vanishingly small free-stream Mach number, \( M_e \). This hypothesis permits the omission of several terms containing \( M_e \) in the governing equations. Therefore, the method of series expansion may be employed to separate the variables and the problem is reduced to the determination of a set of universal functions. Practical applications which satisfy the condition of small Mach number are numerous, for example, the flow behind a very strong bow shock wave or a low subsonic boundary layer in which the compressibility effects are induced by density variation including foreign gas injection.

To illustrate this problem further, it is seen from Appendix C that, for the general consideration, the momentum and the energy equations contain the terms \( u_e^2/h_e, \overline{u}_e, \) and \( \beta = \frac{2s}{u_e} \frac{d\overline{u}_e}{ds} \left(1 + \frac{\gamma - 1}{2} M_e^2\right) \) which are generally dependent on the streamwise coordinate \( \xi \) or \( s \). Analytical solutions for a wedge-type flow with surface injection are possible only if \( \beta = \) constant, and \( M_e \rightarrow 0 \), or \( M_e \rightarrow \infty \), or \( Pr = Le = Sc = 1 \), then the above \( M_e \)-dependent terms either vanish from the equations or become constant. However, if \( M_e \) is finite, and \( Pr, Le \) and \( Sc \) are different
from unity, the whole scheme breaks down except for the simple case of constant \( M_0 \). Under this circumstance, the only possible way to avert this difficulty is to make use of the local similarity concept (7 and 8) such that for certain flow problems, these terms may be neglected or treated as constant without causing any serious errors. As mentioned in Section 3, the application of the local similarity rule must be justified a posteriori, since it is not based on theoretical deduction but on physical grounds. On the other hand, if the free-stream flow is not of the wedge type but \( u_0 \) and \( M_0 \) can be approximated by power-series representations, analytical solutions are possible. The expression for the stream function as defined in equation (46) must now be modified to include the free stream condition as well as the surface injection distribution. By so doing, the terms involving \( M_0 \) and \( u_0 \) will be removed from the resulting differential equations.

Another point which deserves some attention is that the viscosity-density parameter, \( N = (\rho \mu)/(\rho_0 \mu_0) \), and the gas constant parameter, \( R = R/R_0 \), expressed in equations (50) and (51), respectively, are also unknown variables. However, they may be expressed in terms of other variables, the former as a function of temperature and concentration, and the latter in terms of concentration alone. Thus, the systems of ordinary differential equations (53) through (63), and so on, consist of three basic unknowns: the velocity function, the mass concentration, and the temperature function (either in static temperature ratio or in stagnation temperature ratio). Then, these equations can be solved when suitable boundary conditions are prescribed.
PART II
TURBULENT FLOWS

1. Background

The fact that the problem of turbulent flow has been subject to intensive research, as reflected in the literature for the past several decades, is indicative of its prominence in the field of fluid mechanics. It is far more than a problem of some academic interest only, since it encompasses the majority of the fluid flow phenomena ranging from the irregular pattern of a rising cigarette smoke to the turbulent wake of a fast-moving aerodynamic body. Challenging though it may be, only a few special cases of turbulent flow phenomena have been treated with some success, while the intricacies of the mechanism of turbulence remain largely incomprehensible. The main difficulty arises from the randomness of the turbulent motion in that the details of the fluctuation quantities cannot be described as a function of time and space coordinates. Fortunately, these irregular motions are accessible to statistical treatment from which easily observable quantities, such as the mean velocity distribution, can be incorporated in the analysis. Within the premises of the semi-empirical analysis, appreciable progress has been made. Mathematical solutions of some turbulent flow problems have been ascertained, and in a broader measure, the similarity concept has been utilized to describe the mean properties of turbulent motion in terms of some characteristic parameters. In this respect, the semi-empirical analysis is invaluable, as it provides a logical extension of
the purely experimental observations when a rational theory of turbulent flow has yet been developed.

The turbulent shear flow, as its designation implies, comprises the turbulent motion and its interrelated effects with shearing stresses, particularly near a fixed wall. Since the mean velocity has a gradient, the turbulent shear flow cannot be isotropic such as typified by the "free turbulence" which exhibits the same statistical features for any direction. No special remarks need be added to stress the importance of the problems of turbulent shear flow since the pipe and channel flows, and the boundary layer phenomena all fall into this category.

Early works concerning turbulent shear flow problems were evolved around the experimental investigations of pipe and channel flows where the streamwise pressure gradient was small. It became known that a large portion of the mean velocity profile for a fully developed turbulent flow is uniform. The steeper velocity gradient at the wall results in greater shear stress than its counterpart for a laminar flow. On the basis of empirical methods, a power law correlating the velocity distribution to a dimensionless length parameter to some power was established. However, the widely used 1/7th power law does not always agree with experimental data. The exponent varies from one-third to one-tenth depending on the particular experimental conditions. As will be explained later, a single-parameter relation is insufficient to describe the velocity distribution in terms of a dimensionless length and the friction coefficient must be incorporated therein.

From a different viewpoint, Prandtl introduced the mixing length concept which correlates the fluctuation quantities to the mean velocity
gradient by means of the momentum transfer theory. Thus, for a "constant-pressure" flow, the turbulent equations of motion can be solved to yield the well-known logarithmic velocity distribution. Without discussing the details underlying the development of the mixing length theory, the essential procedure leading to the logarithmic formula may be summarized below.

The Reynolds equations for a fully developed turbulent flow in Cartesian coordinates are:

\[ \rho \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \nabla^2 u - \rho \left( \frac{\partial u' v'}{\partial x} + \frac{\partial u' w'}{\partial y} + \frac{\partial u' w'}{\partial z} \right) \]  

(108)

\[ \rho \left( \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \nabla^2 v - \rho \left( \frac{\partial u' v'}{\partial x} + \frac{\partial v' w'}{\partial y} + \frac{\partial v' w'}{\partial z} \right) \]  

(109)

\[ \rho \left( \frac{\partial w}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \nabla^2 w - \rho \left( \frac{\partial u' w'}{\partial x} + \frac{\partial v' w'}{\partial y} + \frac{\partial w' w'}{\partial z} \right) \]  

(110)

where \( \nabla^2 \) is the Laplacian operator. All the variables are understood to be mean temporal quantities, and \( u' \), \( v' \), and \( w' \) represent the fluctuation velocity components with the "bar" indicating mean values. The terms in the last parentheses of each equation are called apparent stresses or Reynolds' stresses which are much larger than the viscous shearing stress terms, \( \mu \nabla^2 u \), and so on. Consequently, the latter can generally be ignored in dealing with the turbulent portion of the flow field. For a particular constant-pressure flow where \( u = u(y) \), \( v = w = 0 \), and the apparent stresses are \( y \)-dependent only, the complicated equations (108) to (110) then reduce to a very simple equation showing a constant shear stress, i.e.,
\[ \frac{\partial \rho u_{\tau} v}{\partial y} = 0 \quad \text{or} \quad \tau = -\rho u_{\tau} v = \text{constant.} \quad (111) \]

Following Prandtl's mixing length concept, we have

\[ \tau = \rho \ell^2 \left( \frac{du}{dy} \right) \left| \frac{du}{dy} \right| \quad (112) \]

where \( \ell \) is the mixing length. In the neighborhood of the wall, Prandtl assumed a linear relation between \( \ell \) and \( y \), viz.,

\[ \ell = Ky \quad (113) \]

where \( K \) is the mixing length constant. Thus equation (112) becomes

\[ u_{\tau}^2 = K^2 y^2 \left( \frac{du}{dy} \right)^2 \quad (114) \]

where \( u_{\tau} = \tau/\rho \) is the friction velocity. Integration of equation (114) yields

\[ u = \left( \frac{u_{\tau}}{K} \right) \ln y + C \quad (115) \]

where \( C \) is an integration constant. In terms of the condition at the outer edge of the boundary layer (or at the central line of a channel), the velocity-defect law is obtained:

\[ \frac{u_e - u}{u_{\tau}} = \frac{1}{K} \ln \frac{\delta}{y} \quad (116) \]

where \( u_e \) is the free-stream velocity and \( \delta \) is the boundary layer thickness. On the other hand, if we consider that the turbulent velocity vanishes at the interface between the turbulent portion and the laminar
sublayer, the law of the wall is obtained:

\[
\frac{u}{u_\tau} = \frac{1}{K} \left( \ln \frac{y}{\delta} - \ln \beta \right)
\]

(117)

where \( \beta \) is a constant. In parallel to the Prandtl mixing length theory, Von Kármán obtained a logarithmic velocity distribution of a different form by means of the similarity hypothesis, i.e., the fluctuations differ from point to point only by time and length scale factors, and the mixing length is expressed in a form depending on the velocity gradient only.

The main purpose for reviewing the classical mixing length theory is that the results obtained are essentially the "law of the wall" and the "velocity-defect law" which can be derived on the basis of a more logical and generalized similarity argument. Because of this consideration, the fortuitous agreement between Prandtl's logarithmic velocity profile and experimental results has received a stronger theoretical support. One important feature of the logarithmic velocity distribution is that the mean velocity is a function of both \( y/\delta \) and \( u_\tau \). The latter is closely related to the skin friction coefficient since by definition

\[
u_\tau^2 = \frac{\tau}{\rho} = \frac{(c_f/2)\rho u_\infty^2}{\rho} = (c_f/2) u_\infty^2
\]

or

\[u_\tau = u_\infty (c_f/2)^{1/2}.
\]

(118)

The fact that a constant-pressure turbulent velocity profile contains two parameters, \( y/\delta \) and \( u_\tau \), has no counterpart in laminar flows where
the velocity profile, such as the Blasius flow, can be described by a single-parameter curve. Furthermore, the commonly used displacement thickness, $\delta^*$, and the momentum thickness, $\theta$, have a constant ratio to the boundary layer thickness, $\delta$, for laminar flows. But, as pointed out by Clauser (13), these thickness parameters are affected by the local skin friction coefficient in the turbulent flow field. For these reasons, it is expected that the power law in the simple form $u/u_\infty = (y/\delta)^{-1/n}$ can at best be an approximate representation within a narrow range of Reynolds number even if the effect of wall roughness were ignored.

One apparent universal representation of the turbulent velocity distribution is given in equation (116) where $u_\tau$ is in the denominator of the dimensionless velocity function. In fact, the logarithmic expression does exhibit an almost universal feature. It must be pointed out, however, that since experiments have shown a non-unique variation of $u_\tau$ with the streamwise distance $x$, the velocity distribution cannot be truly universal (13). A more appropriate term "almost universal" is now used to describe the velocity distribution in the form $(u_\infty - u)/u_\tau$, since the experimental scatterings are confined to the close proximity of the precisely universal velocity curve.

A different approach to the turbulent shear flow problem is by means of the similarity concept. For laminar boundary layers, the procedure to obtain "similar" profiles is direct and unequivocal when the similarity requirements are fulfilled. As already discussed in Part I, similarity implies that the velocity profile is a function of one of the length parameters, thereby reducing the independent variables to a
single one. It is a direct consequence of coordinate transformation based on the equations of motion. For turbulent flows, however, the same procedure cannot be applied because of the lack of understanding of the turbulent transport processes. The Newtonian stress law, 
\[ \tau = \mu \left( \frac{\partial u}{\partial y} \right) \], which provides a connecting link between shearing stress and velocity for laminar boundary layers is nonexistent in the realm of turbulent flows. If an eddy viscosity analogous to the dynamic viscosity is introduced, as had been done by Boussinesq, nothing of great importance can be gained directly. The dynamic viscosity is a property of the fluid whereas the eddy viscosity depends on the velocity field, thus limiting the usefulness of the latter quantity to a mere formal analogy with its laminar counterpart. It can be envisaged, therefore, that in contrast to the laminar boundary layer problems a similarity law cannot be established directly from the equations of motion for a turbulent boundary layer, at least at the present status of understanding. To circumvent this difficulty, a dimensional analysis correlating the shearing stress to the various parameters, such as the length, the velocity, the coefficient of viscosity and the density may be made (11). With this technique, two expressions in dimensionless form are derived.

1. The law of the wall attributable to Prandtl is of the form

\[ \frac{u}{u_\tau} = F \left( \frac{u_\tau y}{\nu} \right) \]  

(119)

In the laminar sublayer region, where the shearing stress, \( \tau \), is assumed to be constant, i.e., \( u \) is proportion to \( y \), we obtain

\[ \frac{u}{u_\tau} = \frac{u_\tau y}{\nu} \].  

(120)
2. The von Kármán velocity-defect law can be represented by

\[
\frac{u_u - u}{u_r} = G \left( \frac{y}{\delta} \right) \tag{121}
\]

where \( \delta \) is the boundary layer thickness. This law illustrates the reduction in mean velocity arising from the boundary layer effects.

It should be noted that the two laws given in equations (119) and (121) are deduced from the same origin except that two different boundary conditions are adopted. Their formulation is identical with the special plane flow problem leading to equations (116) and (117). Thus, the law of the wall (119) and the velocity-defect law (121) are coexisting, the former describing the region close to the wall and the latter describing the outer portion of the turbulent boundary layer.

Inasmuch as both laws are valid in the overlapping region, a relationship that exists between them may be determined. Millikan (15) has shown that the functions \( F \) and \( G \) must be logarithmic by the following argument.

The law of the wall and the velocity-defect law may be expressed in the form

\[
\frac{u}{u_r} = F \left( \left( \frac{y}{\delta} \right) \left( \frac{u_r \delta}{u} \right) \right) \tag{122}
\]

\[
\frac{u}{u_r} = \frac{u_u}{u_r} - G \left( \frac{y}{\delta} \right). \tag{123}
\]

Since these equations show that the effect of the multiplication inside a function has the same effect as addition outside of it, \( F \) and \( G \) are
necessarily logarithms. In other words, the two laws can be expressed as

\[ \frac{u}{u_\tau} = A + B \ln \frac{u_\tau y}{y} \]  

(124)

\[ \frac{u_e - u}{u_\tau} = C - A \ln \left( \frac{y}{\delta} \right) \]  

(125)

where \( A, B \) and \( C \) are constant. While these laws were not developed from solving the basic differential equations, the logical reasonings underlying the method of dimensional analysis have engrossed their formulation. The empirical approach is relied upon only insofar as the constants \( A, B, \) and \( C \) are determined. In general, these relations are valid in a wide spectrum of turbulent flow problems, particularly for pipe and plate flows with zero pressure gradient.

A question naturally arises regarding the extent to which the aforementioned laws are applicable. The law of the wall has long been verified by experiments for constant-pressure flows. Ludwig and Tillmann (16) further empirically confirmed its validity for flows involving favorable as well as adverse pressure gradients. This means that the law of the wall is a universally applicable law for any fully developed turbulent flow near a smooth wall.

On the other hand, the outer portion of a turbulent boundary layer, as much as 90 percent of the flow region, is subject to pressure effects. Therefore, the velocity-defect law based on the constant pressure condition must be modified in order to permit a universal applicability. This approach was first explored by Clauser (13 and 17) who empirically extended the velocity-defect concept to equilibrium.
boundary layers. An "equilibrium boundary layer" is one in which the pressure distribution could be adjusted to admit similar boundary layer profiles. It is analogous to the similar flows for laminar boundary layers with pressure gradients in that the same functional relation holds for a given pressure distribution although different relations must be established for other pressure distributions. More recently, Coles (18) has carefully analyzed the existing data on turbulent velocity profiles and proposed the law of the wake which permits a complete description of the velocity distribution irrespective of the pressure gradient. A generalized formula for the turbulent velocity profile may then be represented by

$$u/u_T = F \left( \frac{u_T y}{y^*} \right) + h(x,y)$$ (126)

where $h(x,y)$ is a function describing the outer portion of the profiles. According to Coles, equation (126) may be written as

$$u/u_T = A + B \ln \left( \frac{u_T y}{y^*} \right) + \frac{\pi(x)}{K} W(y/E)$$ (127)

where $A$, $B$, and $K$ are constants ($K$ corresponds to the mixing length constant or the von Kármán constant), $\pi(x)$ is a function depending on the particular profile and pressure gradients, and $W(y/E)$ is a universal function (wake function).

The expression for the mean velocity distribution (127) is an important contribution to turbulent boundary layers since it contains two universally valid functions and the pressure gradient is represented by a function of $x$ only, which must be determined empirically. For
special cases where \( n(x) \) is constant, equation (127) describes the velocity profile of an equilibrium boundary layer. In the case of a flat plate, Coles has found that \( n \approx 0.55 \). The wake function, \( W(y/\delta) \), has a nearly symmetrical S shape and satisfies the normalizing conditions \( W(0) = 1, W(1) = 2 \), and \( \int_0^2 (y/\delta) \, dw = 1 \). The numerical values of Coles' wake function are tabulated in Table 4.

Since the friction velocity, \( u_f \), is defined as \( u_f = u_e(c_f/2)^{1/2} \), a universal resistance formula can be established directly from the general velocity equation (127) as follows.

\[
\sqrt{2/c_f^2} = A + B \ln (Re_\delta \sqrt{c_f/2}) + 2 \frac{n(x)}{k} \tag{128}
\]

where \( Re_\delta = (u_e \delta/\nu) \) is the Reynolds number based on the boundary layer thickness, \( \delta \).

While the pressure effects on the various laws for turbulent flows have been investigated, it is not known whether these laws can be extended to the compressible flow field. However, the von Kármán momentum–integral method has been employed extensively in a variety of turbulent boundary layer problems including the compressibility effects.

If fluid injection or removal takes place at the surface, the mathematical complexities are further enhanced. Although limited research in this phase of the study has been carried out, the lack of sufficient experimental results has prevented the development of a satisfactory theory similar to that involving impervious surfaces. In the ensuing sections, attention will be directed towards the possible extension of the existing turbulent flow theories although experimental verification can be achieved only for a few special cases. The applica-
tion of the general law describing the velocity distribution to cases involving fluid injection will be discussed first. Also, from an entirely different point of view, the perturbation technique will be utilized to solve the Reynolds equations of motion for flows through a two-dimensional channel with and without fluid injection. The mixing length concept will be used as the additional condition required for solving these equations.

2. Velocity Distribution and Resistance

*Formula for Turbulent Boundary Layers over a Transpiration-Cooled Surface*

As mentioned in the preceding section, one obvious shortcoming of the present-day turbulent flow theory is its dependence on empirical results. While useful formulas may be ascertained by means of the semi-empirical analysis, at least for cases involving impervious walls, a rational theory has not been deduced therefrom. For transpiration-cooling problems, the semi-empirical approach is further handicapped by the fact that experimental results are available only for a few special cases. For this reason, some uncertainty invariably exists for the analysis of the problems involving permeable walls even if the semi-empirical theories were carried out with circumspection and logical reasonings.

Among the numerous publications available in recent years, Clarke et al. (19) utilized the mixing length theory to develop an expression for the velocity distribution and a skin friction formula for a turbulent boundary layer over a flat plate with uniform injection.
The Prandtl two-dimensional turbulent boundary layer equation is of the form

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{\rho} \left( \frac{\partial \tau}{\partial y} - \frac{dp}{dx} \right)$$

(129)

and the continuity equation is expressed as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$  

(130)

Because of fluid injection, the wall condition is

$$u = 0, \quad v = v_0 \quad \text{at} \quad y = 0.$$  

(131)

The usual simplification of equation (129) is to assume

$$\frac{\partial u}{\partial y} \gg \frac{\partial u}{\partial x}.$$  

(132)

This assumption is justified experimentally in the region close to the wall surface in the absence of a longitudinal pressure gradient. Thus, equation (129) can be approximated by

$$v_0 \frac{du}{dy} = \frac{1}{\rho} \frac{\partial \tau}{\partial y}.$$  

(133)

Employing the Prandtl mixing length theory

$$\tau = \ell^2 \left( \frac{\partial u}{\partial y} \right)^2$$  

(134)

and

$$\ell = K y$$  

(135)
equation (133) becomes

$$v_0 \frac{du}{d\gamma} = K^2 (d/\gamma^2) \left[ y^2 \left( \frac{du}{d\gamma} \right)^2 \right]. \quad (136)$$

Integration of equation (136) yields

$$\frac{u}{u_T} = A + B \ln \frac{u_T y}{\nu} + \frac{1}{4K^2} \frac{v_w}{u_T} \ln^2 \left( \frac{u_T y}{\nu} \right). \quad (137)$$

It is seen that for the limiting case, $v_w = 0$, equation (137) reduces to the law of the wall for impervious surfaces. With the absence of experimental values, Clarke et al. assumed $A$, $B$ and $K$ to be the same as those for impervious cases to evaluate $u/u_T$, although these constants are generally dependent on the injection rate. Fair experimental verification was achieved with these assumed constants.

An expression for the local skin friction coefficient was also obtained by means of the von Kármán momentum integral equation (19). The integration procedure became quite complicated due to the presence of the injection parameter. Acceptable agreement with the experimental data was reached, however.

From a different approach, Rubesin (20) incorporated the laminar sublayer into the considerations. Thus, the solution of the turbulent portion described by equation (136) and that of the laminar portion must be matched at the interface. The theory developed by Rubesin is applicable to the compressible turbulent boundary layer. The results have obtained fairly good agreement with experiments (21).

Black and Sarnecki (22) developed a bilogarithmic law by
employing the momentum transfer theory. Integration of equation (133) leads to

\[ v_w u = \frac{\tau - \tau_w}{\rho} \]

or

\[ u_t^2 + v_w u = \tau/\rho \] (138)

since \( \tau_w = \rho u_t^2 \) by definition of the friction velocity, \( u_t \). Substitution of equation (138) into equation (134) with the aid of the mixing length expressed in equation (135) yields

\[ u_t^2 + v_w u = K^2 y^2 \left( \frac{\partial u}{\partial y} \right)^2 = \left( \frac{K}{v_w} \right)^2 \left[ \frac{\partial (u_t^2 + v_w u)}{\partial \ln y} \right]^2 \] (139)

or

\[ u_t^2 + v_w u = \left( \frac{v_w}{2K} \ln \frac{Y}{d} \right)^2 \] (140)

where \( d \) is an integration constant. A velocity-defect law may also be obtained from equation (139) in the following form:

\[ \frac{u - u_\infty}{u_t} = \frac{(u_t^2 + v_w u)^{1/2}}{K u_t} \ln \left( \frac{Y}{\delta} \right) + \frac{v_w}{4K^2 u_t} \ln^2 \left( \frac{Y}{\delta} \right) . \] (141)

Although the foregoing theories illustrate that the semi-empirical laws for turbulent boundary layers can be extended to problems involving transpiration cooling, they are formulated on the basis of a simple constant-pressure flow. If a longitudinal pressure gradient is present, the simplifying assumption \( \partial u/\partial y \gg \partial u/\partial x \) leading to the very simple differential equation (133) is less justified.
Therefore, a rigorous mathematical analysis would require solving the turbulent boundary layer equation (129). Even if the mixing length concept were used, the high-order nonlinear differential equation (129) would be extremely difficult to solve. A special method of solution applicable to a particular class of problems will be discussed in the next two sections.

Under these circumstances, it appears that, intuitively, Coles' wake hypothesis may be utilized to analyze the effect of fluid injection (or suction) on the turbulent boundary layer. From the argument that fluid injection has the same effect as the variation in pressure gradient, there is no a priori reason to repudiate the universality of the wake function under the condition of fluid injection. However, in addition to the various constants, the n-function in the expression for the law of the wake (127) is now affected by the fluid injection rate. The problem then is an empirical one, in that a systematic experimental research should be carried out to test the applicability of the wake hypothesis to transpiration-cooling problems. While recognizing the dilemma that a lack of experimental confirmation invariably weakens an empirical theory, the aforementioned argument concerning the universal features of the wake function is nevertheless accepted in this analysis. Particular emphasis will be placed on the effect of fluid injection on the skin friction characteristics.

In accordance with the wake hypothesis, the entire turbulent profile may be amply described without solving the differential equation (129). The modified defect and wall laws may be obtained by adding a wake function to equations (137) and (141). Since the original wake
function is presented in numerical form, an approximate wake function is used in order to facilitate further development. This function is found to be

\[ W(\zeta) = 2 \zeta^2 \exp (1 - \zeta^2) \]  \hspace{1cm} (14.2)

where \( \zeta = y/\delta \). It is seen that this function satisfies the normalizing conditions \( W(0) = 0, W(1) = 2 \), and \( \int_0^2 \zeta \, dw = 0.98 \approx 1 \). A comparison between the values of equation (14.2) and those given by Coles is depicted in Figure 6 and compiled in Table 4. The agreement can be considered satisfactory. When integration is involved, expression (14.2) leads to simple exponential functions and error functions, which can be evaluated from available tables.

The general expression for the velocity distribution with uniform injection rate, \( v_w \), can be written as

\[ \frac{u}{u_t} = A + B \ln \left( \frac{u_t y}{\nu} \right) + \frac{1}{\kappa_k^2} \frac{v_w}{u_t} \ln^2 \left( \frac{u_t y}{\nu} \right) + \frac{2\pi(x,v_w)}{K} \zeta^2 \exp(1 - \zeta^2) \]  \hspace{1cm} (14.3)

where the constants \( A, B \) and \( K \) are probably affected by the injection rate, \( v_w \). The function \( n \) is generally a function of \( x \) and \( v_w \). For an equilibrium boundary layer for which \( n \) is a constant for cases involving impervious walls, it appears reasonable, at the outset, to assume that \( n \) depends on \( v_w \) alone. In other words, the equilibrium properties are assumed to hold under the condition of uniform injection. An expression

Upon completion of this phase of research, it was found that a different approximate wake function \( W(\zeta) = 1 + \sin (\pi/2)(2 \zeta - 1) \) has been used by Hinze (23).
similar to equation (143) has also been obtained by Black and Sarnecki (22) although they did not introduce a wake function in terms of elementary functions.

A resistance formula may be obtained directly from equation (143) by expressing \( u_T \) in terms of the skin friction coefficient, \( c_f \), explicitly. By definition, \( u_T = (\tau/\rho)^{1/2} = u_0 (c_f/2)^{1/2} \) and \( \text{Re} = (u_0 \delta)/\nu \), which is the Reynolds number based on the boundary layer thickness, \( \delta \), equation (143) becomes

\[
\sqrt{\frac{2}{c_f}} = A + B \ln(\text{Re}_\delta \sqrt{\frac{c_f}{2}}) + \frac{1}{\ln^2 \frac{U}{u_0}} \frac{V_w}{u_0} \sqrt{2} \ln^2(\text{Re}_\delta \sqrt{\frac{c_f}{2}}) + \frac{2m}{K}. \quad (144)
\]

Therefore, the skin friction coefficient can be determined from the \( c_f - \text{Re}_\delta \) curves for given values of \( V_w/u_0 \) and \( a \). However, \( \text{Re}_\delta \) is not a well chosen parameter since the boundary layer thickness, \( \delta \), is a somewhat vaguely defined quantity and is difficult to measure directly.

As an alternative, the momentum-loss thickness, \( \Theta \), or the Reynolds number, \( \text{Re}_\Theta \) based on \( \Theta \), is used. Then equation (144) becomes

\[
\sqrt{\frac{2}{c_f}} = A + B \ln(\text{Re}_\Theta \delta \sqrt{\frac{c_f}{2}}) + \frac{1}{\ln^2 \frac{U}{u_0}} \frac{V_w}{u_0} \sqrt{2} \ln^2(\text{Re}_\Theta \delta \sqrt{\frac{c_f}{2}}) + \frac{2m}{K}. \quad (145)
\]

The ratio, \( \delta /\Theta \), may be evaluated by means of the velocity-defect law including fluid injection

\[
\frac{u}{u_0} = 1 + \frac{(u_T^2 + V_w u)^{1/2}}{\frac{K}{u_0}} \ln \zeta + \frac{V_w}{\ln^2 u_T} \ln^2 \zeta + \frac{2m_T}{K u_0} \left[ \zeta^2 \exp(1 - \zeta^2) - 1 \right]. \quad (146)
\]
Accordingly, the factor $\Theta/\xi$ can be determined by the following expression,

$$
\frac{\Theta}{\xi} = \int_0^1 \frac{u}{u_e} (1 - \frac{u}{u_e}) d\xi = 24 \left( \frac{v_w}{ux^2 u_e} \right)^2 - \frac{v_w}{2Kx^2 u_e} + 3.558 \left( \frac{n}{K} \right)^2 \frac{c_f}{2} 
+ (0.984 - 1.896 \frac{v_w}{K x^2 u_e}) \left( \frac{1}{n} \left( \frac{c_f}{2} \right)^2 + \left[ \frac{1}{K} + \frac{ln}{K} \left( \frac{c_f}{2} \right)^2 - \frac{3v_w}{K x^2 u_e} \right] x 
\left( \frac{c_f}{2} + \frac{v_w}{u} \right)^2 \right) + 0.872 \frac{n}{K} \left[ \frac{c_f}{2} \left( \frac{c_f}{2} + \frac{v_w}{u_e} \right)^2 \right]. \quad (147)
$$

From equations (146) and (147), the curves for $c_f/2$ versus $Re_\Theta$ can be obtained. In principle, the skin friction coefficients for a variety of problems including the effects of longitudinal pressure gradients and different injection rates may be determined. It is obvious that the evaluation of the skin friction coefficients requires the knowledge of the relation between $n$ and $v_w$. However, such information is not available at present.

For the simple case of a flat plate where the pressure gradient is absent, sufficient experimental data are available to permit the evaluation of $c_f$ as a function of $Re_\Theta$ for a given injection parameter, $v_w/u_e$. At present, the constants $A$, $B$ and $K$ for impervious flat plates are used, i.e., $A = 5.6$, $B = 2.43$, and $K = 0.4$. The value $n = 0.55$ proposed by Coles is assumed for the case without fluid injection. When fluid injection is present, different values of $n$ must be used. From the limited data available, however, the influence of $v_w$ on $n$ has not yet been established in terms of a functional relationship, although
it has been found that $\pi$ generally increases for increasing $v^*_w$. A few of the representative curves showing $c_f$ as a function of $Re_\theta$ for different $v^*_w/u_\theta$ (and hence $\pi$) are illustrated in Figure 7 and the numerical values are compiled in Table 5. A comparison between the theoretical results and the experimental data obtained by Micklely and Davis (21) shows that good agreement is reached for small injection rates. More significant deviation from the experimental values is evidenced for higher injection rates. One probable cause for this discrepancy is that fluid injection may have created a pressure gradient which ultimately affects the free-stream conditions. Thus, the $\pi$-function no longer behaves like a constant along the streamwise direction but depends on both $x$ and $v^*_w$ even though the pressure effects were absent initially.

It should be mentioned that expressions (11.1) and (11.7) are established for boundary layer flows only. For pipe and channel flows with constant pressure, the wake component is insignificant and $\pi$ is generally regarded as zero. Therefore, if the constant-pressure condition prevailed when fluid injection is involved, the resistance formula would be somewhat simpler than that for a boundary layer, since the flow characteristics were dependent on a single parameter, $v^*_w/u_\theta$ only.
3. **Series-Expansion Method for Turbulent Flows through a Convergent Channel with Impervious Walls**

Although useful formulas covering a large domain of turbulent shear flow problems have been presented in the last section, the basic approach is an empirical one for which no attempt has been made to expound the correlation between the various formulas and the fundamental turbulent boundary layer equations. In this section, a different approach is adopted in that the governing equations are treated directly. However, in view of the lack of understanding on the turbulent transport properties, the present method cannot be totally detached from the semi-empirical considerations.

Insofar as the direct method of solving the turbulent flow equation is concerned, only very limited special problems have been handled in a satisfactory manner. Among these are the flows with constant pressures and with constant pressure gradients and some boundary layer problems (24). However, for the simple problem of a turbulent flow through a convergent or divergent channel for which numerous experiments have been performed, very little theoretical analysis has been made available. The main purpose of this section is to present a power-series method to solve this simple channel flow problem, particularly that of the converging flows. While a satisfactory agreement is attained only within the restrictions imposed on the problem, it appears that this method may be applicable to other turbulent flow problems under different sets of conditions.

For a two-dimensional, incompressible, fully developed turbulent
flow, the continuity and Reynolds equations in cylindrical coordinates can be expressed as (Fig. 1):

Continuity
\[
\frac{\partial u}{\partial r} + \frac{\partial v}{\partial \theta} = 0
\]  
(148)

Reynolds
\[
u \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \theta} - \frac{v^2}{r} = - \frac{1}{\rho} \frac{\partial p}{\partial r} - \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( ru^2 \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( u^2 \right) \right] 
\]
\[+ \nu \left[ \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial^2 u}{\partial \theta^2} - \frac{u}{r^2} - \frac{v^2}{r^2} \frac{\partial v}{\partial \theta} \right]
\]  
(149)

\[
u \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \theta} + \frac{uv}{r} = - \frac{1}{\rho} \frac{\partial p}{\partial r} - \left[ \frac{\partial}{\partial r} \left( ru^2 \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( v^2 \right) \right] 
\]
\[+ \nu \left[ \frac{\partial^2 v}{\partial \theta^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r} \frac{\partial^2 v}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial u}{\partial \theta} - \frac{v}{r^2} \right]
\]  
(150)

The basic assumptions are that the viscous contribution is much less than that due to the turbulent apparent stresses, the angle of convergence is small, and the Prandtl mixing length theory is applicable. Upon introducing the transformation \( \xi = r/R \) and \( \gamma = \theta/a \) where \( R \) is a reference length, \( \Theta \) is the angle between the particular streamline and the channel axis, and \( a \) is the channel half angle, these equations become

\[
\frac{\partial \xi u}{\partial \xi} + \frac{1}{a} \frac{\partial v}{\partial \gamma} = 0
\]  
(151)

\[
u \frac{\partial u}{\partial \xi} + \frac{v}{\xi} \frac{\partial u}{\partial \eta} - \frac{v^2}{\xi} = - \frac{1}{\rho} \frac{\partial p}{\partial \xi} - \left[ \frac{1}{\xi} \frac{\partial}{\partial \xi} \left( ru^2 \right) + \frac{1}{a} \frac{\partial}{\partial \eta} \left( u^2 \right) \right] 
\]
\[+ \nu \left[ \frac{\partial^2 u}{\partial \eta^2} + \frac{1}{\xi} \frac{\partial u}{\partial \xi} + \frac{1}{\xi} \frac{\partial^2 u}{\partial \eta^2} - \frac{u}{\xi^2} - \frac{v^2}{\xi^2} \frac{\partial v}{\partial \eta} \right]
\]  
(152)
At present, primary interest is concerned with the case of converging flows with impervious walls. The effects of fluid injection will be discussed in the next section. In this case, the average flow in a convergent channel can be regarded as purely radial since the tangential velocity component is identically zero. This fact can be illustrated from the differential equations which become

\[ u \frac{\partial v}{\partial \xi} + \frac{v}{\partial \xi} \frac{\partial u}{\partial \eta} - \frac{1}{\rho \xi a} \frac{\partial P}{\partial \eta} - \left( \frac{\partial}{\partial \xi} \frac{u^2}{\xi} + \frac{1}{a \xi} \frac{\partial}{\partial \eta} \frac{v^2}{\xi} + 2 \frac{u^2}{\xi} \right) = 0 \]  

(153)

Therefore, from the continuity equation, it immediately follows that \( u \) varies inversely with \( \xi \), i.e.,

\[ u = \frac{f(\xi, \eta)}{\xi} \]  

(157)

If no additional conditions relating the fluctuation quantities to the temporal mean values are provided, the simultaneous differential equations (154) through (156) cannot be solved. In order to overcome this difficulty, the Prandtl mixing length theory, developed for constant-pressure flows, is rationalized to supplement the requirements. The
mixing length, $l$, assumed to be proportional to the lateral distance can be expressed as

$$l = KR \xi \alpha (1 - \eta) \tag{158}$$

where $K$ is the mixing length constant. In principle, expression (158) is correct only in the proximity of the wall. While this simplifying assumption is open to question, it may lead to satisfactory mathematical solution for certain problems, such as that of a constant-pressure flow through a parallel channel. Besides, it is expected that fairly good results would be ascertained for the region close to the wall surface in general.

On the basis of the momentum transfer concept and the assumed expression for the mixing length (158), the turbulent stresses in terms of the temporal mean velocity parameter, $f$, become

$$\rho \overline{u'\nu'} = - \frac{\rho l^2}{\xi^2} \left( \frac{\partial u}{\partial \xi} \right)^2 = - \rho K^2 (1 - \eta)^2 \frac{f_{12}}{\xi^2} \tag{159}$$

$$\rho \overline{u'^2} = - 2\rho l^2 \left( \frac{1}{\xi^2} \frac{\partial u}{\partial \xi} \right) \left( \frac{\partial u}{\partial \xi} \right) = 2\rho K^2 \alpha (1 - \eta)^2 \frac{f_{12}}{\xi^2} \tag{160}$$

$$\rho \overline{\nu'^2} = - 2\rho l^2 \left( \frac{1}{\xi^2} \frac{\partial u}{\partial \xi} \right) \overline{\nu} = - 2\rho K^2 \alpha (1 - \eta)^2 \frac{f_{12}'}{\xi^2} \tag{161}$$

where the prime denotes differentiation with respect to $\eta$. Substitution of equations (159) through (161) into equations (155) and (156) yields

$$- \frac{f}{\xi^3} = - \frac{1}{\rho} \frac{\partial p}{\partial \xi} + \frac{1}{\alpha \xi^3} \frac{d}{d\eta} \left[ K^2 (1 - \eta)^2 \frac{f_{12}}{\xi^2} \right] \tag{162}$$
0 = - \frac{1}{p} \frac{\partial p}{\partial \eta} + \frac{2a}{\xi^2} \frac{d}{d\eta} \left[ K^2(1 - \eta)^2 f' \right]. \quad (163)

Elimination of $p$ between these two equations results in

$$a\left[f^2 - P(\xi)\right] + \frac{d}{d\xi} \left[ K^2(1 - \eta)^2 f^2 \right] + 4a^2 K^2(1 - \eta)^2 f' f = 0 \quad (164)$$

where $P(\xi)$ is a function representing the pressure gradient effect and in view of equation (164), it is obviously a constant.

Although the governing equations of motion have now been reduced to a single ordinary differential equation (164), it is still very difficult to solve because of its high-order nonlinearity. However, it is seen that since the parameter $a$ is assumed small ($a \ll 1$), the dependent variable and the constant $P$ in equation (164) may be expanded into power series of $a$, viz.,

$$f(a, \eta) = f_0(\eta) + af_1(\eta) + a^2 f_2(\eta) + \ldots \quad (165)$$

$$P = P_0 + aF_1 + a^2P_2 + \ldots \quad (166)$$

Substituting expressions (165) and (166) into equation (164), a system of simultaneous differential equations is obtained as follows:

$$\frac{d}{d\eta} \left[ K^2(1 - \eta)^2 f_0^2 \right] = 0 \quad (167)$$

$$(f_0^2 - P_0) + \left( \frac{d}{d\eta} \right) \left[ 2K^2(1 - \eta)^2 f_0 f_1 \right] = 0 \quad (168)$$

$$2f_1 f_0 + \frac{d}{d\eta} \left[ K^2(1 - \eta)^2 (f_1^2 + 2f_0 f_2) \right] + 4K^2(1 - \eta)^2 f_0 f_2 = 0 \quad (169)$$

\ldots

etc.
Therefore, the only nonlinear differential equation is the zero-order equation (167). The remaining linear equations for \( f_1, f_2, \ldots \), can be solved successively.

It has been found that with the assumptions of small \( a \) and \( d \gamma \) being one order lower than \( d \xi \), a simplified differential equation can be obtained. Comparing the order of magnitude of the terms in equations (155) and (156), several of them can be neglected from which the following equations result.

\[
\frac{u}{\xi} \frac{\partial u}{\partial \xi} = - \frac{1}{\rho} \frac{\partial p}{\partial \xi} - \frac{1}{\xi} \frac{\partial}{\partial \gamma} \frac{\partial}{\partial \xi} \frac{u \nu T}{(170)}
\]

\[
0 = \partial p/\partial \gamma . \quad (171)
\]

Employing the mixing length concept together with the continuity equation, the equation of motion becomes

\[
a(\gamma^2 - P) + (d/d \gamma) \left[ K^2(1 - \gamma)^2 f_1^2 \right] = 0 . \quad (172)
\]

Again assuming the series representations (165) and (166), the following system of equations is obtained.

\[
(d/d \gamma) \left[ K^2(1 - \gamma)^2 f_0^2 \right] = 0 \quad (173)
\]

\[
(f_0^2 - P_0) + (d/d \gamma) \left[ 2K^2(1 - \gamma)^2 f_0 f_1 \right] = 0 \quad (174)
\]

\[
2f_1 f_0 + \frac{d}{d \gamma} \left[ K^2(1 - \gamma)^2 (f_1^2 + 2f_0 f_2) \right] = 0 \quad (175)
\]

etc.
Comparing the systems of equations based on the exact and the approximate equations, it is seen that their zero-order and the first-order equations are identical, and deviation occurs only in the second- and higher-order approximations. From this finding, it can be asserted that for flows through a convergent channel, the approximate solution should yield satisfactory results for a small angle \( \alpha \). If the angle is large, the series method may not be practical. However, it is an experimental fact that the velocity distribution will exhibit a closer resemblance to the boundary-layer type flow pattern as the angle of convergence increases (25). Therefore, it is logical to accept the boundary-layer assumptions for the equations of motion.

It can be recognized that the zero-order approximation yields a logarithmic profile since equation (167) or (173) is the same as that of a parallel flow. This can be verified by introducing a friction velocity function, \( f_T \), so that integration of equation (167) leads to

\[
\frac{f_{o_e} - f_0}{f_T} = - \frac{1}{k} \ln (1 - \eta) \tag{176}
\]

where \( f_{o_e} \) represents the value at the center of the channel. For the present coordinate orientation, the negative sign must be used for the velocity of a converging flow, i.e., \( u = - f/\xi \). The equation (176) can be expressed as

\[
\frac{u_{o_e} - u_0}{u_T/k} = - \ln (1 - \eta) \tag{177}
\]

which is identical with the velocity-defect law obtained by Prandtl.
The solution of the first-order equation (168) or (174) can be determined by integrating it twice whence

\[
2K^2(f_1 - f_0) = (6D - 4f_0^2) \eta - D(1- \eta) \ln(1- \eta) [\ln(1- \eta) - 2] \\
+ 2(D - f_0^2) \ln(1- \eta)(2- \eta) + M [\eta + \ln(1- \eta)]
\]  

(178)

where \( f_1 \) is the value at the center of the channel, \( D = f_T/K \) and \( M = (f_0^2 - P_0)/D \). In terms of velocities, we obtain

\[
\frac{u_1 - u_T}{u_T/K} = \frac{1}{K^2} (3 - 2 \frac{u_0^2}{u_T}) \eta - \frac{1- \eta}{2K^2} \ln(1- \eta) [\ln(1- \eta) - 2] \\
+ \frac{1}{K^2} (1 - \frac{u_0^2}{u_T}) \ln(1- \eta)(2- \eta) + L [\eta + \ln(1- \eta)]
\]  

(179)

where \( L = M/2K^2D \). Therefore, the first-order approximation of the velocity profile for a turbulent flow through a convergent channel with a half angle \( \alpha \) is

\[
\frac{u_e - u_T}{u_T/K} = \frac{u_0^2 - u_T}{u_T/K} + \frac{u_1 - u_T}{u_T/K} a = - \ln(1- \eta) + \left\{ \frac{1}{K^2} (3 - 2 \frac{u_0^2}{u_T}) \eta \\
- \frac{(1- \eta)}{2K^2} \ln(1- \eta) [\ln(1- \eta) - 2] + \frac{1}{K^2} (1 - \frac{u_0^2}{u_T}) x \\
\ln(1- \eta)(2- \eta) + L [\eta + \ln(1- \eta)] \right\} a.
\]  

(180)

Based on the mixing length theory, it is seen that the velocity-defect law presented above contains two parameters, \( u_0^2/K/u_T \) and \( L \), which depend on the friction velocity, \( u_T \), as well as the pressure distribution. Therefore, the velocity profile is similar only for flows having the
same values for these parameters. It should be mentioned that without introducing the laminar sublayer, the boundary condition at the wall cannot be satisfied. Thus, the velocity-defect law primarily correlates the velocity distribution to the outer portion of the boundary layer.

The possible application of equation (180) is that the effect of the channel convergence (or pressure gradient) can be established for given $L$ and $u_0/K/u_\tau$. However, in order to test the validity of equation (180), it is necessary to examine a specific channel flow problem where experimental data are available.

Assuming $K = 0.4$ and $u_0/K/u_\tau = 9.6$ which is based on the data for a straight channel flow, equation (180) can be expressed as

$$\frac{u_e - u}{u_\tau/K} = \frac{u_eK}{u_\tau} (1 - \frac{u_e}{u_\tau}) = - \ln(1 - \gamma) + \Phi \alpha$$ (181)

where

$$\Phi = -101.25 \gamma - 3.12 (1 - \gamma) \ln(1 - \gamma) \left[ \ln(1 - \gamma) - 2 \right]$$
$$- 53.75 \ln(1 - \gamma)(2 - \gamma) + L \left[ \gamma + \ln(1 - \gamma) \right].$$ (182)

The parameter $L$ is determined from Nikuradses' data (25) which show a slight variation in the average velocity (or Reynolds number) for different angles of convergence. The value $L$ is found to be approximately 50.2. The results obtained by evaluating equation (181) for $\alpha = 0^\circ$, $2^\circ$, $4^\circ$ and $8^\circ$ and Nikuradses' experimental data are shown in Figure 8 and tabulated in Table 6. For small angles of convergence, the measured values generally fit the theoretical curve; however, the agreement is poor for $\alpha = 8^\circ$. In all cases except for $\alpha = 0^\circ$, a deviation between the theoretical and the experimental results is evidenced near
the channel midsection. In analysing these results, the limitations of the present method can be stated as follows:

1. The first-order approximation is insufficient to describe the velocity profile for large angles of convergence.

2. The inherent shortcoming of the mixing length theory becomes evident in the channel flow problem. The mixing length, \( \ell \), represented by the linear relationship (158) is apparently not valid for the entire channel region. As a matter of fact, from Nikuradses measurement, \( \ell \) is found to be of a form not easily expressed by a unique function of \( \alpha \) and \( \eta \) throughout the flow field. Thus, the present mixing length assumption limits the applicability of the solution in a region away from the immediate neighborhoods of the wall and the center line.

4. **Series-Expansion Method for Turbulent Flows through Transpiration-Cooled Channels**

   The method of solution for a converging turbulent flow in the last section may be extended to problems involving fluid injections. Because of the presence of the temporal mean velocity components along the lateral direction, the main feature that a converging flow is purely radial is no longer preserved. The differential equations which now involve functions depending on both \( \xi \) and \( \eta \) can be solved by means of the series method similar to that for the laminar transpiration-cooling problems. However, the mathematical procedure will become rather cumbersome since the velocity components must be expanded in power series of \( \alpha \) as well as \( \xi \).
When fluid injection is involved, the approximate turbulent flow equations become

Continuity

$$\frac{\partial \xi}{\partial \xi} u + \frac{1}{a} \frac{\partial v}{\partial \eta} = 0$$  \hspace{1cm} (183)

Momentum

$$u \frac{\partial u}{\partial \xi} + \frac{v}{a} \frac{\partial u}{\partial \eta} = -\frac{1}{\rho} \frac{\partial p}{\partial \xi} - \frac{1}{a^2} \frac{\partial}{\partial \eta} \underbrace{u}^{uv^T}.$$  \hspace{1cm} (184)

First the velocity components are represented by power series in $\xi$ with coefficients as functions of $\eta$. Setting

$$v(\xi, \eta) = -(v_0 + v_1 \xi + v_2 \xi^2 + \ldots)$$  \hspace{1cm} (185)

the continuity equation yields

$$u(\xi, \eta) = -\frac{f}{\xi} + \frac{1}{a} \left( v_0' + \frac{\xi}{2} v_1' + \frac{\xi^2}{3} v_2' + \ldots \right)$$  \hspace{1cm} (186)

where the functions $f$, $v_0$, $v_1$, $\ldots$, are dependent on $\eta$ and $a$ and the prime denotes differentiation with respect to $\eta$. Substitution of $u$ and $v$ expressed in equations (185) and (186) into equation (184) leads to the following system of differential equations:

$$\left( f^2 - P_0 \right) + \left( 1/a \right) \left( d/d \eta \right) \left[ K^2 (1 - \eta)^2 f' f' \right] = 0$$  \hspace{1cm} (187)

$$\left( d/d \eta \right) \left[ a f v_0 + 2K^2 (1 - \eta)^2 f' v_0' \right] = a f(0) v_0'(0)$$  \hspace{1cm} (188)
\[(f'v_1a^2 - v_0v_0a) - (d/d\gamma)[K^2(1 - \gamma)^2(v_0^2 - f'v_1a)] = 0 \quad (189)\]

\[\ldots\]

etc.

It is immediately obvious that the zero-order approximation (187) represents converging flows with impervious channel walls whose solutions have already been discussed. The higher-order approximations showing the effects of fluid injection may be treated in a manner similar to that of the zero-order equation, i.e., by further expanding the variables in terms of power-series in a. The additional power series are

\[
\begin{align*}
  f(a, \gamma) &= f_0(\gamma) + f_1(\gamma)a + f_2(\gamma)a^2 + \\
  v_0(a, \gamma) &= v_{00}(\gamma) + v_{01}(\gamma)a + v_{02}(\gamma)a^2 + \\
  v_1(a, \gamma) &= v_{10}(\gamma) + v_{11}(\gamma)a + v_{12}(\gamma)a^2 + \\
  v_2(a, \gamma) &= v_{20}(\gamma) + v_{21}(\gamma)a + v_{22}(\gamma)a^2 + \\
  \ldots
\end{align*}
\]

\[\text{etc.} \quad (190)\]

From equation (186) it is seen that \(a = 0\) is a singularity which can be removed only if \(v_0' = 0\). In other words, the zero-order velocities, \(v_{00}, v_{10}, v_{20}, \ldots\), must be constant. Substituting the power series (190) into equation (187), the system of differential equations (173), (174) and (175), and so on, will result. The first-order equation is represented by the following equations.

\[
(d/d\gamma)[f_0v_0 + 2K^2(1 - \gamma)^2 f_0'v_0''] = 0 \quad (191)
\]
At present, the case of uniform fluid injection is considered.

Then, the zero-order approximation represents the constant injection rate, i.e., \( v_{00} = v_w \) and \( v_0 = v_2 = \ldots = 0 \). The first-order equation (191) may be solved by integrating it twice, viz.,

\[
\begin{align*}
v'_{01}(\gamma) &= -\frac{1}{2} \frac{w}{K^2} \ln^2(1 - \gamma) + v_{01}^i(0) \tag{193}
\end{align*}
\]

and

\[
\begin{align*}
v_{01}(\gamma) &= \frac{v_w}{4K^2} (1 - \gamma) \{ \ln^2(1 - \gamma) - 2 \left[ \ln(1 - \gamma) - 1 \right] \} - (1 - \gamma)v_{01}^i(0) \tag{194}
\end{align*}
\]

From equation (186) and the previous results for impervious walls, the first-order approximation for the streamwise velocity component becomes

\[
\begin{align*}
u &= -\frac{f_0 + a f_1}{\kappa} + \ldots + \frac{1}{a} (v_{00}^i + a v_{01}^i + \ldots) = -\frac{f_0 + a f_1}{\kappa} + v_{01}^i
\end{align*}
\]

\[
\begin{align*}
&= -\frac{f_{00} + a f_{10}}{\kappa} - \frac{u_T}{K} \left[-\ln(1 - \gamma) + a \phi \right] - \frac{v_w}{4K^2} \ln^2(1 - \gamma) + v_{01}^i(0) \tag{195}
\end{align*}
\]

Replacing the \( f \)-function by the streamwise velocity component, \( u \), equation (195) becomes

\[
\begin{align*}
u &= u^0 - \frac{u_T}{K} \left[-\ln(1 - \gamma) + a \phi \right] - \frac{v_w}{4K^2} \ln^2(1 - \gamma) + v_{01}^i(0) \tag{196}
\end{align*}
\]

where \( u^0 \) represents the velocity at the central line of the channel.
without fluid injection. It is seen that along the channel axis 
\( \gamma = 0 \)

\[
\frac{u_e}{u_e^0} = u_e^0 + v_{01}(0) \quad (197)
\]

where \( v_{01}(0) \) signifies the increase of free-stream velocity due to 
the injected fluid. For boundary layer flows, the term \( v_{01}(0) \) would 
generally be ignored since the free-stream flow is not bounded; there­
fore, the usual assumption is that at the outer edge of the boundary 
layer the free-stream velocity remains unchanged.

In terms of the velocity-defect law, equation (196) may be ex­
pressed as

\[
\frac{u_e - u}{u_\tau} = \frac{1}{K} \left[ - \ln(1-\gamma) + \alpha \bar{\Phi} \right] + \frac{v_w}{4u_\tau^2} \ln^2(1-\gamma). \quad (198)
\]

This equation represents the first-order approximation both in \( \alpha \) and \( \gamma \) 
for turbulent flows through a transpiration-cooled channel. The ve­
locity distribution may be graphically represented in terms of the pa­
rameters \( \alpha, v_w/(u_\tau k^2) \), and \( L \), the last being present in the function 
\( \bar{\Phi} \) defined in equation (182). At present, no experimental data are 
available to check the universal validity of equation (198) in its com­
plete form. However, some experiments have been performed for pipe and 
straight channel flows with fluid injection (26 and 27). In this case, 
equation (198) may be written as

\[
\frac{u}{u_e^0} = \frac{u_e}{u_e^0} - \frac{u_\tau}{u_e^0 k} \left[ - \ln(1-\gamma) \right] - \frac{v_w}{u_e} \left( \frac{u_e}{u_e^0} \right) \frac{1}{4u_\tau^2} \ln^2(1-\gamma) \quad (199)
\]
where \( \frac{u_1}{u_0^*} = 1/9.6 \) and \( K = 0.4 \), and \( u_1/u_0^* \) may be obtained from experimental data for various injection rates. Despite the many non-rigorous assumptions made, the agreement between the theoretical and the experimental results (27) is excellent, as can be seen from Figure 9. The calculated values are also compiled in Table 7.

5. Conclusions

In Part II of this dissertation, limited topics among the vast number of turbulent flow problems have been discussed. Inasmuch as our present knowledge of the mechanism of wall turbulence is not adequate as a basis for a satisfactory theory, semi-empirical methods must be relied upon. The embodiment of the purely empirical wake law and the semi-empirical wall and defect laws represents a major recent contribution to the understanding of the turbulent boundary layer flows. The extension of these well-established laws to problems involving fluid injection is a logical follow-up attempt. By so doing, general success has been achieved, which is indicated by the good agreement between the theoretical and experimental results, as exemplified by the values of the skin friction coefficients for various injection rates. The discrepancy between the theoretical and experimental results for high injection rates is most likely due to a change in the free-stream properties. Since fluid injection results in the creation of a less stable velocity profile and an increase in the boundary layer thickness, stronger turbulence may have been generated. Thus, the assumption that the pressure function, \( w \), depends on the injection rates alone for a flat plate is open to question.
The possibility of employing the series-expansion method for turbulent channel flow problems has been explored. For this class of problems, the channel half angle is found to be a convenient small parameter for the power-series generation. Assuming the validity of the mixing length theory, the first-order solutions for turbulent channel flows with and without fluid injection have been obtained. For converging or diverging flows, the velocity distributions obtained from the velocity-defect law do not exhibit the universal features of those for plane flows with impervious walls. The velocity distributions in non-parallel channels depend on several parameters which are local functions of free-stream velocity and shearing stress. Therefore, families of curves may be plotted for the velocity distribution versus the non-dimensional length parameter, $\gamma$, for various values of these parameters. Experimental verification is generally reached; however, the limitation of the mixing length theory appears to be the main cause for some unsatisfactory results, such as the flows having large pressure gradients.

In conclusion, the lack of experimental information on turbulent flows, particularly those involving both the pressure gradient and fluid injection, was the main handicap in the course of this research. While various methods of approach have been proposed, they can be substantiated by the available data only for a limited number of special cases.
Selected topics concerning laminar and turbulent flows over a plate and through a convergent channel with fluid injection from porous walls have been discussed in this dissertation.

In Part I, laminar flows, particularly laminar boundary layers, with an arbitrarily distributed surface injection have been the chief concern. Inasmuch as "similar" solutions normally do not exist under these circumstances, a method of series expansion about a transformed streamwise distance has been employed for the general treatment of this class of problems. The essential feature of the power-series method is that the governing nonlinear partial differential equations can be transformed into systems of ordinary differential equations in which only the zero-order equations, representing the case with impervious walls, are nonlinear. With appropriate boundary conditions, the effects of fluid injection which are described by the higher-order approximations can be determined by solving the remaining linear differential equations successively.

A two-dimensional incompressible laminar boundary layer through a convergent channel with uniform injection has been analyzed first. Numerical solutions for the velocity and temperature distributions have been obtained. Efforts have also been made to formulate the general transpiration-cooling problems involving compressibility effects and foreign gas injection. With the assumption that the free-stream Mach number is vanishingly small, systems of ordinary differential equations
representing the successive order of approximation have been ascertained. Numerical computation for a simple case of flat plate with uniform injection and Prandtl number unity has been carried out.

In Part II, the important semi-empirical laws concerning turbulent shear flows have been reviewed. It has been found that these laws, i.e., the wall and defect laws together with the wake hypothesis, can be extended to cases with fluid injection. As an example, a fully developed turbulent flow over a flat plate with uniform injection has been analyzed. The wake function was approximated by an exponential function which is more convenient for further mathematical development than that in terms of numerical values. An expression for the skin friction coefficient showing the effect of fluid injection has been developed from which the theoretical result is generally in agreement with experimental values. However, although, in principle, this formula is valid to cases with streamwise pressure gradient, it contains several parameters which must be determined empirically. Because of insufficient experimental information, its general validity to a wide range of applications cannot be proven at this time.

The method of series-expansion about a small parameter has been utilized to solve the two-dimensional incompressible turbulent flow equations. Special attention has been directed towards the problem of fully developed turbulent flows through a convergent channel for which the channel half-angle has been used as a small parameter for the power-series generation. For impervious channel walls, the theoretical result based on this series method together with the mixing length theory has been satisfactory provided that the angle is small. It has been found
that approximate equations similar to the boundary layer equations may be used instead of the exact Reynolds equations since the difference in results based on the two sets of equations arises only in the second- and higher-order equations. If fluid injection is present, a double expansion about the half-angle and a streamwise length parameter must be carried out. Unfortunately, empirical values for some of the parameters contained in the resultant differential equations cannot be averted. The results based on the present method can be verified only for the simple case of a transpiration-cooled straight channel or pipe flow.
# LIST OF SYMBOLS

## Part I - Laminar Flows

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>A, B, C</td>
<td>Constants</td>
</tr>
<tr>
<td>$C_a$</td>
<td>Mass concentration for species a</td>
</tr>
<tr>
<td>$C_b$</td>
<td>Mass concentration for species b</td>
</tr>
<tr>
<td>$c_f$</td>
<td>Skin friction coefficient</td>
</tr>
<tr>
<td>$c_p$</td>
<td>Constant-pressure specific heat</td>
</tr>
<tr>
<td>$c_v$</td>
<td>Constant-volume specific heat</td>
</tr>
<tr>
<td>$c^*$</td>
<td>Transformation constant</td>
</tr>
<tr>
<td>$D_{ab}$</td>
<td>Mutual diffusion coefficient for species a and b</td>
</tr>
<tr>
<td>$E_u$</td>
<td>Euler number</td>
</tr>
<tr>
<td>$F, G, f, g$</td>
<td>Velocity functions</td>
</tr>
<tr>
<td>$h$</td>
<td>Enthalpy</td>
</tr>
<tr>
<td>$\tau = h/h$</td>
<td>Enthalpy ratio</td>
</tr>
<tr>
<td>$h^o = h + (u^2/2)$</td>
<td>Total enthalpy</td>
</tr>
<tr>
<td>$k$</td>
<td>Thermal conductivity or constant</td>
</tr>
<tr>
<td>$L_e$</td>
<td>Lewis number</td>
</tr>
<tr>
<td>$M$</td>
<td>Mach number</td>
</tr>
<tr>
<td>$\Gamma_{e0}$</td>
<td>Mach number function defined in equation (C-5) of Appendix C</td>
</tr>
<tr>
<td>$N = \rho \mu / (\rho_e \mu_e)$</td>
<td>Viscosity-density parameter</td>
</tr>
<tr>
<td>$P_0, P_1, \ldots$</td>
<td>Supplementary functions defined in equation (64)</td>
</tr>
<tr>
<td>$p$</td>
<td>Pressure</td>
</tr>
<tr>
<td>$Pr$</td>
<td>Prandtl number</td>
</tr>
<tr>
<td>$Q_0, Q_1, \ldots$</td>
<td>Supplementary functions defined in equation (65)</td>
</tr>
</tbody>
</table>
Heat transfer rate
Gas constant
Dimensionless gas constant
Radial component of cylindrical coordinates
Schmidt number
Mangler's transformation for two-dimensional flows
Temperature
Dimensionless temperature
Streamwise velocity component
Free-stream velocity
Friction velocity
Lateral velocity component
Temperature function
Coordinate measuring distance along the surface
Coordinate measuring along the lateral distance
Supplementary functions defined in equation (66)
Chapman-Rubesin Constant
Channel half angle
Constant
Transformation coordinate
Dorodnitsyn's transformation \((\rho_e u_e)/\sqrt{2s} \int_0^y (\rho/\rho_e) dy\)
or transformation coordinates \((y/x)\sqrt{u_e x/\nu}\)
Stream function
Transformation coordinate
Angular coordinate
\( \rho \) Density
\( \nu \) Kinematic viscosity
\( \mu \) Dynamic viscosity

Subscripts
0, 1, 2, ... Zero-order, first-order, second-order, ... approximations
\( e \) Free-stream condition
\( w \) Wall condition

Superscript
\( o \) Stagnation condition

Part II - Turbulent Flows

\( c_f \) Skin friction coefficient
\( D = f \sqrt{K} \) Friction velocity parameter
\( f \) Velocity function
\( K \) Mixing length constant
\( L = M/2K^2D \) Velocity parameter
\( \ell \) Mixing length
\( M = (f_{0e}^2 - P_0)/D \) Velocity parameter
\( P_0, P_1, ... \) Pressure functions
\( p \) Pressure
\( R \) Reference length
\( Re_\delta = u_\delta \delta/\nu \) Reynolds number based on the boundary layer thickness \( \delta \)
\( Re_\Theta = u_\Theta \Theta/\nu \) Reynolds number based on the momentum thickness \( \Theta \)
\( r \) Radial component of cylindrical coordinates
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u )</td>
<td>Streamwise velocity component</td>
</tr>
<tr>
<td>( u' )</td>
<td>Streamwise fluctuation velocity component</td>
</tr>
<tr>
<td>( v )</td>
<td>Lateral velocity component</td>
</tr>
<tr>
<td>( v' )</td>
<td>Lateral fluctuation velocity component</td>
</tr>
<tr>
<td>( w(\zeta) )</td>
<td>Coles' wake function</td>
</tr>
<tr>
<td>( w )</td>
<td>Azimuthal velocity component</td>
</tr>
<tr>
<td>( w' )</td>
<td>Azimuthal fluctuation velocity component</td>
</tr>
<tr>
<td>( x )</td>
<td>Coordinate measuring distance along the surface</td>
</tr>
<tr>
<td>( y )</td>
<td>Coordinate measuring along the lateral distance</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>Channel half-angle</td>
</tr>
<tr>
<td>( \delta )</td>
<td>Boundary layer thickness</td>
</tr>
<tr>
<td>( \delta' )</td>
<td>Displacement thickness</td>
</tr>
<tr>
<td>( \tau )</td>
<td>Shearing stress</td>
</tr>
<tr>
<td>( \rho )</td>
<td>Density</td>
</tr>
<tr>
<td>( \Theta )</td>
<td>Angular coordinate or momentum thickness</td>
</tr>
<tr>
<td>( m )</td>
<td>Function indicating pressure gradient</td>
</tr>
<tr>
<td>( \Phi )</td>
<td>Function defined in equation (182)</td>
</tr>
<tr>
<td>( \mu )</td>
<td>Dynamic viscosity</td>
</tr>
<tr>
<td>( \nu )</td>
<td>Kinematic viscosity</td>
</tr>
<tr>
<td>( \zeta = y/\delta )</td>
<td>Dimensionless lateral distance</td>
</tr>
<tr>
<td>( \xi = r/R )</td>
<td>Dimensionless distance measured along surface</td>
</tr>
<tr>
<td>( \gamma = \Theta/\alpha )</td>
<td>Angle variable</td>
</tr>
</tbody>
</table>
Subscripts

$0, 1, 2, \ldots$ Zero-order, first-order, second-order, ... approximations

e Free-stream flow condition

w Wall condition

Superscript

\[ \bar{\cdot} \] Temporal mean value
APPENDIX A

POWER-SERIES REPRESENTATION OF FLOWS THROUGH A CONVERGENT CHANNEL

The velocity and temperature distribution for a steady, incompressible, laminar flow through a two-dimensional symmetrical convergent channel having half angle, \( \alpha \), and with arbitrary coolant injection at the porous walls can be described by the Navier-Stokes equations of motion, the continuity equation and the energy equation. These equations in cylindrical coordinates are of the following form (see Fig. 1).

**Continuity**

\[
\frac{\partial (ru)}{\partial r} + \frac{\partial v}{\partial \phi} = 0
\]  

(A-1)

**Navier-Stokes**

\[
u \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \phi} - \frac{u^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \gamma \left( \nabla^2 u - \frac{2}{r^2} \frac{\partial v}{\partial \phi} - \frac{u}{r^2} \right)
\]

(A-2)

\[
u \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \phi} - \frac{uv}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial \phi} + \gamma \left( \nabla^2 v + \frac{2}{r^2} \frac{\partial u}{\partial \phi} - \frac{v}{r^2} \right)
\]

(A-3)

where

\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}
\]

**Energy**

\[
c_v \left( u \frac{\partial T}{\partial r} + \frac{v}{r} \frac{\partial T}{\partial \phi} \right) = \frac{k}{\rho} \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \phi^2} \right) + \gamma \left[ 2 \left( \frac{\partial u}{\partial r} \right)^2 + 2 \left( \frac{\partial v}{\partial \phi} + \frac{u}{r} \right)^2 + \left( \frac{1}{r} \frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial r} - \frac{v}{r^2} \right)^2 \right]
\]

(A-4)
The boundary conditions for uniform fluid injection are

\[
\begin{align*}
\nu(\pm a) &= \nu_w(r) = \pm \sum_{n=0}^{\infty} a_n r^n \\
u(0) &= 0 \\
n(0) &= 0 \\
T(\pm a) &= T_w = \text{constant} \\
T'(0) &= 0
\end{align*}
\]  

(A-5)

The surface injection distribution is expressed formally in terms of a power series in \( r \), but for practical purposes, a polynomial should result in sufficiently accurate results. Similar to the approach presented in Section I, a stream function is not introduced. Instead, the velocity components, \( u \) and \( v \), are assumed to be represented by power series. We assume

\[
v = v(r, \theta) = \sum_{n=0}^{\infty} g_n(\theta) r^n \quad 0 < r < 1 \quad (A-6)
\]

From the continuity equation (A-1), the streamwise velocity component must be of the form

\[
u = u(r, \theta) = \frac{f(\theta)}{r} - \sum_{n=0}^{\infty} \frac{g_n'(\theta)}{n+1} r^n . \quad (A-7)
\]

The velocity field may be determined by the solution of the Navier-Stokes equations (A-2) and (A-3) without making use of the energy equation (A-4). Substituting equations (A-6) and (A-7) into equations (A-2) and (A-3) and equating the coefficients of the resulting
power series to zero, the following system of differential equations is obtained.

\[ f'' + 2ff' + 4\varphi' = 0 \quad \text{(A-8)} \]
\[ \left[ s_0f + (s_0' + s_0) \right]'' + \left[ -s_0f + \varphi(s_0' + s_0) \right] = 0 \quad \text{(A-9)} \]
\[ \sqrt{2} \left( s_1'' + s_1' \right) + f'g_1 - (s_0'g_0 + s_1^2) = 0 \quad \text{(A-10)} \]

etc.

The corresponding boundary conditions are

\[ \begin{align*}
  f(\pm a) &= 0, \quad f'(0) = 0 \\
  g_i(\pm a) &= \pm a_i, \quad i = 0, 1, 2, \ldots \quad \text{(A-11)} \\
  g_i(0) &= 0, \quad g_i'(\pm a) = 0
\end{align*} \]

From the system of differential equations shown above, the limiting case is that of no injection represented by the zero-order solution (A-8). Its solution is expressible in terms of elliptical integrals first developed by Hamel and Jeffery (2). The effects of fluid injection are represented by \( s_0, s_1, \ldots \) which are determined by solving the subsequent linear differential equations (A-9), (A-10), and so on. The numerical procedure is exactly the same as that discussed previously although the calculations are complicated by the presence of some additional terms.

The exact solutions of the temperature distribution for this problem can be determined with the aid of the transformation

\[ T = T(\theta, r) = \frac{1}{r^2} \sum_{n=0}^{\infty} W_n(\theta) r^n + T_w \quad \text{(A-12)} \]
where $T_w$ is assumed to be the same constant temperature for both walls. Substitution of equations (A-6), (A-7) and (A-12) into the energy equation (A-4) leads to the system of ordinary differential equations for the temperature distribution as follows:

\[
\frac{k}{\rho} \left( w_0^2 + h w_0 \right) + \frac{?c_v w_0 f}{\sqrt{(f')^2 + h f^2}} = 0 \quad (A-13)
\]

\[
\frac{k}{\rho} \left( w_1^2 + w_1 \right) + c_v w_1 f - c_v \left( 2 w_0 g_0' + g_0 w_0' \right) - 2 \sqrt{g_0'^2 + g_0 f} f = 0 \quad (A-14)
\]

\[
\frac{k}{\rho} \left( w_2^2 - c_v \left( w_0 g_1 + w_1 g_0 \right)' + \sqrt{g_0'^2 + (g_0'^2 + g_0)^2} - g_1 f \right) = 0 \quad (A-15)
\]

etc.

with the boundary conditions

\[
W_i(\pm a) = 0 \quad \text{and} \quad W_i(0) = 0, \quad i = 0, 1, 2, \ldots \quad (A-16)
\]

The zero-order equation (A-13) representing the case for impervious walls has been worked out by Millsaps and Pohlhausen (2). The solution of this equation, which is a generalized Lamé equation, may be expressed in terms of the principal Jacobian functions. However, it is more expedient to employ the numerical method instead of the general analytical solution to calculate the various temperature distributions for channel flows. The effect of fluid injection can then be determined by solving the successive differential equations numerically.

The governing equations formulated herein are also valid for flows through a divergent channel. With a slight modification of the
boundary conditions, i.e., an opposite sign for $v_w$, the "suction" problem can be solved in exactly the same manner. However, if compressibility effects are included, the exact Navier-Stokes equations are too complicated to solve without resorting to any simplifying assumptions.
APPENDIX B

COMPATIBILITY CONDITIONS FOR THE POWER-SERIES METHOD

It is not at all infrequent to workers in aerodynamics to encounter problems for which the general solutions of the governing differential equations are not known. However, if a solution for the basic equation is known to exist, or at least is assumed to exist, the method of power-series expansion about a small parameter as discussed in Part I may be employed to obtain numerical results.

In this analysis power-series representations are assumed since the flow field involving an arbitrary injection distribution generally does not admit "similar" solutions. A question arises regarding the uniqueness of the assumed power series under a set of given boundary conditions. In other words, is the assumed power series the only one compatible with the prescribed boundary conditions? At the outset, this problem appears to be a trivial one, since the system of differential equations obtained from an improper power-series representation will automatically lead to zero values for some of the universal functions and/or no solutions for some others. However, in view of the tedious calculation procedure generally involved, it is desirable to predetermine a properly set power series compatible with the boundary conditions so as to avoid excessive labor unnecessarily.

In this appendix, the problem of a converging flow with fluid injection is singled out for detailed analysis but the argument contained herein is equally valid to problems involving other body con-
tours. As discussed in Section 2, the tangential velocity component for an incompressible laminar boundary layer flow through a two-dimensional, transpiration-cooled, symmetrical, convergent channel can be represented by

\[ v = -\sqrt{\gamma c'} \sum_{n=0}^{\infty} g_n(\eta) r^n \]  

(B-1)

provided that the surface injection distribution is prescribed as

\[ v_w = - \sum_{n=0}^{\infty} a_n r^n \]  

(B-2)

where \( a_n \)'s are given constants. In order to prove that the power series stated in equation (B-1) is the proper one for the problem, a more general power series is assumed as follows:

\[ v = -\sqrt{\gamma c'} \sum_{n=0}^{\infty} g_n(\eta)(r')^n \]  

(B-3)

where \( r' \) is an undetermined positive constant. From the continuity equation (1), the radial velocity component is found to be

\[ u = -c \left[ \frac{f(\eta)}{r} + \sum_{n=0}^{\infty} \frac{g_n(\eta)}{n \ell + 1} (r')^n \right]. \]  

(B-4)

Substitution of equations (B-3) and (B-4) into the momentum equation (2) in Section 2 yields
\[
\frac{1}{r^3} (r^n - r^2 + 1) + \frac{1}{r^2} \left( g_0^m - f g_0' - f' g_0 \right) - \frac{g_0 g_0''}{r} + \sum_{k=1}^{\infty} \left[ \frac{g_k'''}{k^2 + 1} \right. \\
- \frac{1 - k}{1 + k} \left. f g_k' - f' g_k \right] r^{k/2 - 2} + \sum_{n=1}^{\infty} \left\{ \sum_{m=0}^{n} \left[ \frac{(n-m)\ell}{(n-m)^2 + 1} \right] g_m g_{n-m} \right. \\
- \frac{1}{n-m+1} g_m g_{n-m} \right\} r^{n-1} = 0 \quad (B-5)
\]

It is seen that the universal functions \( f, g_0, g_1, \ldots \), are all independent of \( r \); therefore, the identity (B-5) requires that the coefficients of \( r^{-3} \), \( r^{-2} \), ..., be zero. Since \( \ell \) is positive and \( k, n \) are positive integers, the coefficients of \( r^{-3} \) and \( r^{-2} \) are not affected by the values of \( \ell \), i.e., the zero-order and first-order solutions are independent of the choice of \( \ell \). For \( \ell = 1 \), the system of differential equations obtained in Section 2 will result. For \( \ell = 0 \), it is a special case which implies the existence of similar solutions. For the particular boundary condition (B-2), this is true only if \( a_0 \neq 0 \), \( a_1 = a_2 = \ldots = 0 \), which is the case of a flow in the stagnation-point region with uniform injection. Therefore, the uniqueness of the power-series representation (B-1) for this problem can be proved by examining the coefficients of \( r^{-1} \), \( r^{k/2-2} \) and \( r^{n-1} \) for \( \ell \neq 1 \). As a convenience, the following cases are discussed separately.

**Case 1. \( \ell = \text{integer} > 1 \)**

In this case, \( g_0 g_0'' \) is the only coefficient for \( r^{-1} \), since \( k/2 = 2 \) and \( n \ell - 1 \) cannot be equal to \(-1\) for \( \ell > 1 \) and \( n, k = 1, 2, \ldots \). Therefore, the only solution compatible to the two differential equations
is that \( g_0 = 0 \). This is possible only if \( a_0 = 0 \) is prescribed for the boundary condition.

**Case 2. \( 0 < \ell = 1/\text{integer} < 1 \)**

In this case, it is possible to have coefficients of \( r^s \) where \(-2 < s < -1\). It is evident that \( r^{n'/1} \) does not satisfy the condition \( s = n'/1 < -1 \) for \( \ell > 0 \). Therefore, the only possible differential equations in the range \(-2 < s = k' \ell - 2 < -1\) are of the form

\[
\frac{g_k''}{k' \ell + 1} - \frac{1-k'}{1+k'} fg_k' - f'g_k = 0 .
\]  

(B-7)

From the boundary conditions (B-2) and those given in Section 2, we must have

\[
g_k(0) = g_k'(0) = g_k'(\infty) = 0 .
\]  

(B-8)

The power-series method may be employed to solve the homogeneous differential equation (B-7) with the boundary condition (B-8). A power series which automatically satisfies \( g_k(0) = g_k'(0) = 0 \) is

\[
g_k(\gamma) = a_2 \gamma^2 + a_3 \gamma^3 + a_4 \gamma^4 + \ldots .
\]  

(B-9)

The function \( f \) may also be expressed in terms of a power series:

\[
f = 3 \tanh^2 \left( \gamma / \sqrt{2} + 1.1h6 \right) - 2
\]

\[
= \beta_0 + \beta_1 \gamma + \beta_2 \gamma^2 + \ldots .
\]  

(B-10)
Substitution of equations (B-9) and (B-10) into the differential equation (B-7) leads to

\[ g_k(\gamma) = \alpha_2(\gamma^2 + \gamma_{\ell}^1 + \gamma_{\ell}^6 + \ldots) \tag{B-11} \]

from which the boundary condition \( g_k^{(\infty)} = 0 \) leads to the condition \( \alpha_2 = 0 \) and thus the trivial solution \( g_k = 0 \). Therefore, for any \( k \) satisfying \(-2 < k\ell - 2 < -1\), the corresponding differential equations admit only the trivial solution \( g_k = 0 \).

For \( k\ell - 2 = -1 \), the differential equation is of the form

\[ g_k^{\prime/2} - f^2 g_k - g_0 g^{\prime} = 0 \tag{B-12} \]

which is identical with equation (23) for \( \ell = 1 \) obtained in Section 2. This non-homogeneous differential equation possesses nontrivial solutions which can be solved by numerical integrations.

In the region \( 0 < k\ell - 2 = n\ell - 1 > -1 \), the differential equations are of the form

\[
\frac{1-k\ell}{1+k\ell} f g_k^\prime + g_k^\prime - \frac{g_k^{\prime/2}}{k\ell + 1} - \sum_{m=0}^{n} \left\{ \frac{(n-m)\ell}{[(n-m)\ell + 1](m+1)} g_m^\prime \right\} = 0.
\tag{B-13}
\]

Following the argument presented above, it was found that again these differential equations have only the trivial solution \( g_k = 0 \) except for \( k\ell - 2 = n\ell - 1 = 0 \) which results in equation (24) in Section 2. Proceeding the same way for \( k\ell - 2 > 0, n\ell - 1 > 0 \), it can be shown that all
the $g_s$-functions are zero except those corresponding to $r$, $r^2$, $r^3$, ... .

In other words, the power series is of the form with $\ell = 1$ as given in equation (B-1).

**Case 3.** $\ell = \text{rational number } \neq 1/\text{integer}$

It is easily seen that $k\ell - 2 = n\ell - 1 = -1$ is impossible since this condition requires that $k \ell = 1$, and $n \ell = 0$ which contradicts the assumptions $\ell \neq 1/k$ and $n \neq 0$. Therefore, the coefficient of $r^{-1}$ in equation (B-5) leads to the differential equation $g_0 g_0^* = 0$. This is a contradiction in view of the given boundary condition (B-2).

**Case 4.** $\ell = \text{irrational number}$

In this case, $k\ell - 2$ and $n\ell - 1$ cannot be integers. Therefore, $g_0 g_0^* = 0$ is again a differential equation which leads to incompatible conditions as discussed above.

In conclusion, if the boundary condition is prescribed by a power series given in equation (B-2), the only possible power-series representation for the tangential velocity component is of the form given by equation (B-1). Furthermore, the power series (B-1) is valid for the case of uniform injection where $a_0 \neq 0$, $a_1 = a_2 = \ldots = 0$ since the differential equations as a result of equation (B-2) all have appropriate boundary conditions leading to nontrivial solutions.
APPENDIX C

DERIVATION OF THE TERMS CONTAINING $u_e$ OR $M_e$ IN THE GOVERNING EQUATIONS

The density ratio, $\rho_e/\rho$, or $\overline{RT} = RT/R_e T_e$ which appears in the momentum equation can be expressed in terms of the stagnation temperature ratio, $\overline{T^0} = T^0/T_e^0$. Assuming constant free-stream stagnation enthalpy, the density parameter becomes

$$\overline{RT} = \frac{\overline{R}}{T_e} \frac{T^0}{T_e} = \overline{R} \left(1 + \frac{\gamma_e-1}{2} \frac{M_e^2}{T_e^0} \right) \frac{T}{T_e^0}. \quad (C-1)$$

From the definition of the free-stream stagnation enthalpy and the velocity functions, the following is obtained

$$\overline{h^0} = \frac{h^0}{h_e^0} = \frac{c_p T^0}{h_e^0} = (h/h_e^0) + \left(\frac{u^2}{2 h_e^0}\right)$$

$$= (h/h_e^0) + \left(F' + \sum_{n=0}^{\infty} G_n (p+1)^2 \frac{u_e^2}{2 h_e h_e^0}\right) \frac{h_e}{h_e^0}$$

$$= \frac{c_p}{T_e^0} + \left(F' + \sum_{n=0}^{\infty} G_n (p+1)^2 \frac{\gamma_e-1}{2} \frac{M_e^2}{h_e^0} (1 + \frac{\gamma_e-1}{2} \frac{M_e^2}{T_e^0})^{-1}\right) \quad (C-2)$$

where $\overline{c_p} = \frac{c_p}{c_p e} = \frac{c_p}{c_p b}$.

Substitution of equation (C-2) into equation (C-1) yields

$$\overline{RT} = \left(1 + \frac{\gamma_e-1}{2} \frac{M_e^2}{T_e^0}\right) \left[\overline{R} \overline{T^0} - \frac{\overline{R}}{c_p} \left(F' + \sum_{n=0}^{\infty} G_n (p+1)^2 \frac{\gamma_e-1}{2} \frac{M_e^2}{h_e^0} (1 + \frac{\gamma_e-1}{2} \frac{M_e^2}{T_e^0})^{-1}\right) \right] \quad (C-3)$$
where $\overline{T_0} = \frac{T_0}{T_0}$. The ratio $\overline{R/c_p}$ may be expanded in terms of power series in $\varepsilon$ as follows,

$$\overline{R/c_p} = (\overline{R_0} + \overline{R_1} \varepsilon + \ldots) c_{p_b} \left[ (c_{a_0} + c_{a_1} \varepsilon + \ldots)(c_{p_a} - c_{p_b}) + c_{p_b} \right]^{-1}$$

$$\approx \overline{R_0} (1 - \frac{c_{p_a} - c_{p_b}}{c_{p_b}}) c_{a_0} + \ldots$$

since $c_p = c_a (c_{p_a} - c_{p_b}) + c_{p_b}$, and

$$\left( \overline{c_p} \right)^{-1} = \frac{c_{p_b}}{c_a (c_{p_a} - c_{p_b}) + c_{p_b}} \approx 1 - c_a \frac{c_{p_a} - c_{p_b}}{c_{p_b}}$$

$$= (1 - c_{a_0} \frac{c_{p_a} - c_{p_b}}{c_{p_b}}) + \frac{c_{p_a} - c_{p_b}}{c_{p_b}} c_{a_1} \varepsilon + \ldots$$

It should be remarked that this simplified form for $\left( \overline{c_p} \right)^{-1}$ is adopted purely for illustration purposes. Otherwise, more terms will be involved in the power-series representation. Finally, all the variables in equation (C-3) are expressed in terms of power series in $\varepsilon$ except $M_e^2$ which remains in its present form as if it were a constant. Then, the zero approximation for equation (C-3) in terms of the stagnation temperature ratio becomes

$$\overline{R_0 \overline{T_0}} - F^{12} = (1 + \frac{\Gamma e^{-1}}{2} M_e^2) (\overline{R_0 \overline{T_0}} - F^{12} \overline{M_e}) \quad (C-4)$$

where $\overline{M_e} = \left[ 1 + (1 - c_{a_0} \frac{c_{p_a} - c_{p_b}}{c_{p_b}}) \frac{\Gamma e^{-1}}{2} M_e^2 \right] (1 + \frac{\Gamma e^{-1}}{2} M_e^2)^{-1} \quad (C-5)$

Accordingly, the zero-order approximation for the momentum equation (53) becomes
The higher-order equations (54), (55), (56), and so on, can be expressed in terms of the stagnation temperature ratio in a manner similar to equation (C-6).

It can be shown that in the energy equation \( u_e \) is present in the dissipation and the pressure gradient terms, viz.,

\[
\mu \left( \frac{\partial u_e}{\partial y} \right)^2 = \frac{\rho_e u_e^2}{2s} \frac{\mu u_e^2 h_0}{\frac{N}{c_s}} \frac{u_e^2}{h_0} (F^\prime + 2F^\prime G_0^\prime \epsilon + \ldots) \quad (C-7)
\]

and

\[
u \frac{dp}{dx} = -\frac{\rho_e u_e^2}{2s} \frac{\mu u_e^2 h_0}{\frac{N}{c_s}} \frac{u_e^2}{h_0} \frac{\rho_e}{p} \frac{u_e^2}{h_0} (F^\prime + G_0^\prime \epsilon + \ldots) \quad (C-8)
\]

where \( \rho_e/\rho \) may be expressed in terms of \( T^\circ \) as derived previously. The factor outside the brackets in expressions (C-7) and (C-8) is superfluous as its presence is due to the coordinate transformation and will be cancelled out from the consideration of energy balance.

For an arbitrary free-stream Mach number, \( M_0 \), the corresponding terms in expressions (C-7) and (C-8) must be added to the energy equations (61) through (63), and so on. If \( M_0 \) and \( u_e^2/h_0 \) are vanishingly small, these terms can be neglected. For \( M_0 \rightarrow \infty \), \( u_e^2/h_0 \) approaches the value 2 since

\[
\frac{u_e^2}{h_0} = \left( \frac{\gamma_e-1}{2} \frac{M_0^2}{1 + \frac{\gamma_e-1}{2} \frac{M_0^2}{h_0}} \right) \rightarrow 2 \quad \text{for} \quad M_0 \rightarrow \infty
\]
Furthermore, from expression (C-5), we have

\[ \tilde{M}_e \rightarrow 1 - C_{a_0} \frac{c_{p_a} - c_{p_b}}{c_{p_b}} \quad \text{for } M_e \rightarrow \infty. \]  

(C-10)

Under this condition, the power-series solution can be obtained provided that \( \beta \) is assumed to be constant. This assumption together with the adiabatic free-stream condition lead to a specific relation between \( M_e \) and the streamwise coordinate, \( s \), as follows:

\[ \beta = \frac{2s}{u_e} \frac{du_e}{ds} (1 + \frac{\gamma_e-1}{2} M_e^2) = \frac{2s}{M_e} \frac{dM_e}{ds} = \text{constant} \]  

(C-11)

whence \( M_e \propto (2s)^{\beta/2} \).

(C-12)

Expression (C-12) indicates that at a distance far from the stagnation point, \( M_e \) instead of \( u_e \) is used to represent the wedge-type freestream gas flow.

For a finite \( M_e \), the power-series solution can be obtained only if the conditions \( \beta = \text{constant}, \tilde{M}_e = \text{constant}, \) and \( \text{Pr} = \text{Le} = \text{Sc} = 1 \) are satisfied. Based on Crocco's relation discussed in Part I, the energy equation can be replaced by the linear relation between the mass concentration and the stagnation enthalpy for \( \text{Pr} = \text{Le} = \text{Sc} = 1 \). Thus, the terms depending on the streamwise coordinate, such as \( u_e^2/n_e^0 \), do not appear in the governing equations.
REFERENCES


## Table 1

**Velocity Distribution Functions for Incompressible Laminar Flows through a Transpiration-Cooled Convergent Channel**

<table>
<thead>
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<th>( \eta )</th>
<th>( a_0 )</th>
<th>( a'_0 )</th>
<th>( a''_0 )</th>
<th>( f )</th>
<th>( f' )</th>
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**TABLE 1 (cont.)**
### TABLE 1 (contd.)

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**Remarks:** Surface injection rate = $e_0(0) = 0.1$

\[
f(\zeta) = 3 \tanh^2 \left[ \left( \frac{\zeta}{\sqrt{2}} \right) + 1.146 \right] - 2
\]

\[
f'(\zeta) = 3 \sqrt{2} \tanh \left[ \left( \frac{\zeta}{\sqrt{2}} \right) + 1.146 \right] \times \sec^2 \left[ \left( \frac{\zeta}{\sqrt{2}} \right) + 1.146 \right]
\]
### TABLE 2

**Temperature Distribution Functions for Incompressible Laminar Flows through a Transpiration-Cooled Convergent Channel**

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Remark: \( Pr = \mu e^2/\kappa = 1 \)
### TABLE 3
First-Order Velocity Distribution Functions for Laminar Boundary Layers over a Transpiration-Cooled Flat Plate

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Remark: Surface injection rate -

$G_0(0) = -0.1$
TABLE 4
Comparison between Coles' Wake Function and Approximate Wake Function

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<th>$\zeta$</th>
<th>$W(\zeta)$</th>
<th>$W_c(\zeta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>0.054</td>
<td>0.029</td>
</tr>
<tr>
<td>0.2</td>
<td>0.209</td>
<td>0.168</td>
</tr>
<tr>
<td>0.3</td>
<td>0.447</td>
<td>0.396</td>
</tr>
<tr>
<td>0.4</td>
<td>0.740</td>
<td>0.685</td>
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<tr>
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<td>1.055</td>
<td>0.994</td>
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<td>1.364</td>
<td>1.307</td>
</tr>
<tr>
<td>0.7</td>
<td>1.630</td>
<td>1.600</td>
</tr>
<tr>
<td>0.8</td>
<td>1.833</td>
<td>1.840</td>
</tr>
<tr>
<td>0.9</td>
<td>1.960</td>
<td>1.980</td>
</tr>
<tr>
<td>1.0</td>
<td>2.000</td>
<td>2.000</td>
</tr>
</tbody>
</table>

Notes: $W(\zeta) = 2 \zeta^2 \exp(1 - \zeta^2)$

$W_c(\zeta) = \text{Coles' Wake Function}$
### TABLE 5

Calculated Local Skin Friction Coefficients for Turbulent Flows over a Flat Plate with Fluid Injection

<table>
<thead>
<tr>
<th>$v_w/v_0$</th>
<th>$n$</th>
<th>$C_f x 10^{-3}$</th>
<th>$Re_0 x 10^{-2}$</th>
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<tbody>
<tr>
<td>0</td>
<td>0.55</td>
<td>3.0</td>
<td>3.3</td>
</tr>
<tr>
<td>0</td>
<td>0.55</td>
<td>2.0</td>
<td>16.8</td>
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<tr>
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<td>0.55</td>
<td>2.2</td>
<td>11.2</td>
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<tr>
<td>0</td>
<td>0.55</td>
<td>1.5</td>
<td>46.5</td>
</tr>
<tr>
<td>0.001</td>
<td>1.70</td>
<td>1.8</td>
<td>5.1</td>
</tr>
<tr>
<td>0.001</td>
<td>1.70</td>
<td>1.6</td>
<td>8.5</td>
</tr>
<tr>
<td>0.001</td>
<td>1.70</td>
<td>1.5</td>
<td>10.9</td>
</tr>
<tr>
<td>0.001</td>
<td>1.70</td>
<td>1.3</td>
<td>21.6</td>
</tr>
<tr>
<td>0.001</td>
<td>1.70</td>
<td>1.1</td>
<td>44.6</td>
</tr>
<tr>
<td>0.002</td>
<td>3.25</td>
<td>0.7</td>
<td>44.7</td>
</tr>
<tr>
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<td>3.25</td>
<td>0.86</td>
<td>17.3</td>
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<td>3.25</td>
<td>0.9</td>
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<td>0.45</td>
<td>24.5</td>
</tr>
<tr>
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<td>5.57</td>
<td>0.50</td>
<td>15.8</td>
</tr>
<tr>
<td>$1-\eta$</td>
<td>$\alpha = 0^\circ$</td>
<td>$\alpha = 2^\circ$</td>
<td>$\alpha = 4^\circ$</td>
</tr>
<tr>
<td>---------</td>
<td>------------------</td>
<td>------------------</td>
<td>------------------</td>
</tr>
<tr>
<td></td>
<td>$\frac{u}{u_e}$</td>
<td>$\frac{u_e-u}{u_e}$</td>
<td>$\frac{u}{u_e}$</td>
</tr>
<tr>
<td>0.995</td>
<td>0.475</td>
<td>5.311</td>
<td>0.555</td>
</tr>
<tr>
<td>0.99</td>
<td>0.520</td>
<td>4.605</td>
<td>0.620</td>
</tr>
<tr>
<td>0.98</td>
<td>0.593</td>
<td>3.912</td>
<td>0.696</td>
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<tr>
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<td>0.688</td>
<td>2.996</td>
<td>0.795</td>
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<tr>
<td>0.90</td>
<td>0.760</td>
<td>2.303</td>
<td>0.857</td>
</tr>
<tr>
<td>0.80</td>
<td>0.832</td>
<td>1.610</td>
<td>0.905</td>
</tr>
<tr>
<td>0.60</td>
<td>0.904</td>
<td>0.916</td>
<td>0.951</td>
</tr>
</tbody>
</table>
TABLE 7

Velocity Distribution for Turbulent Flows through a Straight Channel with Fluid Injection

<table>
<thead>
<tr>
<th>$v_r/u_0$</th>
<th>$\xi$</th>
<th>$u_r/u_0^0$</th>
<th>$u/u_0^0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1.000</td>
</tr>
<tr>
<td>0</td>
<td>0.2</td>
<td>1</td>
<td>0.9768</td>
</tr>
<tr>
<td>0</td>
<td>0.4</td>
<td>1</td>
<td>0.9468</td>
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<tr>
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<td>0.6</td>
<td>1</td>
<td>0.9004</td>
</tr>
<tr>
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<td>0.8</td>
<td>1</td>
<td>0.832</td>
</tr>
<tr>
<td>0</td>
<td>0.9</td>
<td>1</td>
<td>0.760</td>
</tr>
<tr>
<td>0.00485</td>
<td>0</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
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<td>0.2</td>
<td>1.06</td>
<td>1.0372</td>
</tr>
<tr>
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<td>0.4</td>
<td>1.06</td>
<td>1.0089</td>
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<tr>
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<td>0.6</td>
<td>1.06</td>
<td>0.9712</td>
</tr>
<tr>
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<td>0.8</td>
<td>1.06</td>
<td>0.9265</td>
</tr>
<tr>
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<td>0.9</td>
<td>1.06</td>
<td>0.8628</td>
</tr>
<tr>
<td>0.00837</td>
<td>0</td>
<td>1.14</td>
<td>1.14</td>
</tr>
<tr>
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<td>0.2</td>
<td>1.14</td>
<td>1.1175</td>
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<tr>
<td>0.00837</td>
<td>0.4</td>
<td>1.14</td>
<td>1.0907</td>
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<td>0.6</td>
<td>1.14</td>
<td>1.0569</td>
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<td>0.00837</td>
<td>0.8</td>
<td>1.14</td>
<td>1.0116</td>
</tr>
<tr>
<td>0.00837</td>
<td>0.9</td>
<td>1.14</td>
<td>0.9793</td>
</tr>
</tbody>
</table>
FIG. 1. COORDINATE SYSTEM FOR FLOWS THROUGH A CONVERGENT CHANNEL WITH FLUID INJECTION.
FIG. 2. VELOCITY DISTRIBUTION FUNCTIONS FOR FLOWS THROUGH A CONVERGENT CHANNEL WITH FLUID INJECTION.
FIG. 3. TEMPERATURE DISTRIBUTION FUNCTIONS FOR FLOWS THROUGH A CONVERGENT CHANNEL WITH FLUID INJECTION.

\[ T = \frac{C_i^2}{\nu^2} \sum_{n=0}^{\infty} W(n) r^n + T_w \]
FIG. 4. VELOCITY DISTRIBUTION FUNCTIONS FOR A LAMINAR BOUNDARY LAYER OVER A FLAT PLATE.

\[ F(\xi) = \sum_{n=0}^{\infty} G_n(\xi) \xi^{-n} \]
FIG. 5. FIRST-ORDER VELOCITY DISTRIBUTION FUNCTIONS FOR FLOWS OVER A FLAT PLATE WITH FLUID INJECTION.

\[ F(\xi) = F'(\xi) + \sum_{n=0}^{\infty} G_n(\xi) \xi^{n+1} \]
FIG. 6. COMPARISON BETWEEN COLES' WAKE FUNCTION AND APPROXIMATE WAKE FUNCTION.
FIG. 7. LOCAL FRICTION COEFFICIENT VERSUS REYNOLDS NUMBER BASED ON THE MOMENTUM THICKNESS.
FIG. 8. VELOCITY DISTRIBUTIONS FOR TURBULENT FLOWS THROUGH VARIOUS CONVERGENT CHANNELS.
Fig. 9. Effect of fluid injection on velocity distribution for turbulent flows between parallel plates or through straight pipes.

1.0
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1.0
1.2

$\frac{v_w}{u_e} = 0.00837$

$0.00485$

$0$

$u_e$ — Maximum velocity for impervious walls

$\frac{u}{u_e}$
I, Michael Chen-Chiang Fong, was born in Peiping, China, August 3, 1928. I received my secondary-school education in Peiping, China, and some undergraduate training at Yenching University and National Tsinghua University, both in Peiping, China. In 1948, I was awarded a scholarship from Bowling Green State University, Ohio, where I majored in mathematics and from which I received the Bachelor of Science degree in 1950. I enrolled in the Graduate School of The Ohio State University in 1950, specializing in mathematics and later in aeronautical engineering. From the latter department I received the Master of Science degree in 1953. From June 1952 to October 1961, I held the position as Research Assistant and Research Associate at the Rocket Research Laboratory, Department of Aeronautical Engineering, The Ohio State University. At present I am a senior engineer with the Aero-Thermodynamics Development Group at the North American Aviation, Inc., Columbus Division.