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NUCLEAR FORCES

DISSERTATION
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School of The Ohio State University

By
Duo-Liang Lin, B. Sc., M. Sc.

The Ohio State University
1961

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I. INTRODUCTION

It is well known that there exists a definite discrepancy between the experimental values of zero-energy scattering lengths for the singlet S-state of np and pp systems. This implies that the assumption of charge independence of the nuclear force is more or less questionable. In order to keep charge independence, Schwinger tried to calculate the effect of anomalous magnetic moment of the nucleons for np and pp systems. He showed that the different contributions of magnetic interactions to np and pp systems would make the nuclear force practically charge independent, provided that a very long-tailed potential of Yukawa type was assumed. However, his result was later corrected by Salpeter who showed that the effect of magnetic interaction would be greatly reduced in the presence of a hard core of radius larger than 0.3 fermi. Since such a model of nuclear potential has been justified both experimentally and theoretically, it is quite reasonable to believe that the nuclear force between a neutron and a proton is a little bit stronger than that between a pair of protons, that is to say, the nuclear force is somehow charge dependent.

More recently Kiazuddin made a calculation similar to that of Schwinger. By taking into account
the finite extension of the charge and current distri-
butions, Kiazuddin found that a sizeable difference
remained between $a_{np}$ and $a_{pp}$ for singlet $S$-state
even without the repulsive core. He also pointed out
that the introduction of a hard core would result in an
even bigger discrepancy. He then calculated the effect
of the pion mass difference on the singlet $S$ scatter-
ing lengths which could be made to agree with the
experimental value by adjusting the different coupling
constants for $\pi^\pm$-nucleon and $\pi^0$-nucleon systems.
Furthermore, Kiazuddin suggests that from a cut-off
theory calculation the renormalized coupling constants
for the neutral pion is larger than that for the charged
one. This reduces the effect of the pion mass
difference of the scattering lengths. Unfortunately,
Greenberger found by analyzing the phase shifts of
pion-nucleon scattering that the charged pion would
have a larger coupling constant. This ambiguity may be
due to the fact that Kiazuddin had neglected the fourth
order contributions which are very important. On the
other hand, the pion-nucleon scattering data are not
quite good enough to yield unambiguous results.

Another effect that is examined below is the
excitation energy of the $T = 1$ multiplets in light
nuclei. If one assumes charge symmetry one can show
that the first excited state of an $A = 2Z$ nucleus
would be exactly the energy difference between $E_{oo}$ of $A = 2Z$ nucleus and $E_{11}$ of $A = 2(Z-1)$ nucleus plus their Coulomb energy difference where the subscripts stand for the isotopic spin $T$ and its $z$-component $T_z$. In this way Wilkinson investigated eighteen self-conjugate nuclei and came to the conclusion that the calculated energy was definitely greater than the empirically known value. Blin-Stoyle and Kearsley thus estimate the pion mass difference effect by using lowest order perturbation theory. A more careful investigation is made below including two-pion exchange effects.

In the present work we first obtain the explicit form of the nuclear potentials up to the fourth order by inserting the pion masses semi-phenomenologically into Brueckner and Watson's charge independent potentials. These potentials are then used to calculate the corrections to the zero-energy $^1S$ scattering lengths for np and pp systems and to the first excited $T = 1$ state of the self-conjugate nucleus $^6\text{Li}$. The three coupling constants together with the hard core radius are treated as open parameters to be fitted by empirical data. It is found that a very small variation in the coupling constants will change the results considerably. The implication of this fact is that the three coupling constants are almost the
same. With these coupling constants we also calculate the singlet $nn$ scattering length which is about one to six per cent less than $a_{pp}$.

In order to see how much the nucleon-nuclear potentials are affected by this pion mass difference, we have also made an approximate estimation. The proton and the neutron potential well depths in optical model are calculated by neglecting all the spin interactions. It is found that they differ from each other only by an amount of $0.02\%$. The spin dependent part of the potentials is therefore expected to afford a significant contribution to the nucleon-nuclear interaction.
II. EXPLICIT FORM OF THE POTENTIALS

In obtaining the nuclear force due to the exchange of pions by nucleons, one usually makes the perturbation expansion in the coupling constants as well as the static approximation in which the recoil of the nucleon is completely neglected. It has been found that with a local Yukawa theory the nuclear two-body force has a strong singularity at the small internucleon distance and that this singularity is aggravated by including higher order terms in the expansion. A phenomenological, infinite hard core is then introduced at small distance to make the problem solvable. This model of potential with repulsive core was first proposed by Jastrow in order to explain the peculiar behavior of pp scattering at energies between 100 and 400 Mev, and its theoretical foundation was given by Levy. Treating the radius of the repulsive core as an adjustable parameter, Brueckner and Watson were able to fit the low energy np data. Taketani found that the best fit to np data was made by an infinite repulsive core. The static approximation has been justified since the nucleon mass is seven times larger than the pion mass and since one is primarily dealing with low energy phenomena in which the nucleon velocity is very small.

With symmetric PS-PS theory we have the interaction
Hamiltonian

\[ H' = i g \bar{\psi} \gamma^\nu \partial_\nu \psi \phi, \]

where \( \psi \) describes the nucleon field and \( \phi \) describes the meson field, the operator \( V \) specifies the interaction between the fields and \( g \) is the coupling constant. Performing a unitary transformation proposed by Dyson, it can be shown that in the nonrelativistic approximation this interaction is equivalent to

\[ H' = \frac{g}{2M} \vec{\sigma} \cdot \nabla (\vec{r} \cdot \vec{\phi}) \rho(\vec{x}) + \frac{g^2}{2M} \phi^2 \rho(\vec{x}) \]

where \( M \) is the mass of the nucleon, \( \rho(\vec{x}) \) is the source function with the property

\[ \int \rho(\vec{x}) \, d\vec{x} = 1. \]

We shall follow Brueckner and Watson to take a \( \delta \)-function distribution, there will be no problem at the origin because we have a hard core. \( \vec{\sigma} \) and \( \vec{r} \) are the spin and isotopic spin operators of the nucleon and their three components are the Pauli matrices:

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}.
\]

The vector \( \vec{\phi} \) had three real components \( (\phi_+, \phi_-, \phi_0) \) corresponding to the three charge states \((+e, -e, 0)\) of the pion fields respectively. The neutral meson \( \pi^0 \)
is described by $\varphi_i$, while $\pi^\pm$ are described by

$$\frac{1}{\sqrt{2}} (\varphi_1 \pm i \varphi_2).$$

The first term in (2) is just the nonrelativistic limit of the psuedovector interaction which is the main part in the nuclear force problem. The second term corresponding to pair formation is suppressed at least in the theory of nuclear force as has been shown by Klein\textsuperscript{14}. He demonstrated that for the nuclear force problem the effective coupling constant for the emission of a pair of pions is the same as the renormalized coupling constant which occurs in the matrix element for pion-nucleon scattering in low energy limit (S-wave scattering) and hence is an order of magnitude smaller than the $P$-wave coupling constant\textsuperscript{15}.

If we replace the coupling constant $\varphi$ by $f^{(0)}$ such that $g^{(2)} / 2\mu = \sqrt{4\pi \mu} f^{(0)}$ and then sum over the two interacting nucleons, we can rewrite the Hamiltonian as

$$H' = \sqrt{4\pi} f^{(0)} \sum_{n \geq 1} \sum_{\lambda=1}^{3} \int d \vec{x} \, \delta(\vec{x} - \vec{x}_n) \, \overline{\varphi}_{\lambda} \cdot \nabla \, \varphi_{\lambda} \, \mathcal{T}^{(n)}_{\lambda}$$

where $\mu$ is the mass of the charged pion, $f^{(0)}$ is the so-called unrenormalized coupling constant which is dimensionless. We have used here and throughout this paper the natural unit such that $\hbar = \mu = c = 1$.

Taking account of the different masses and different coupling constants, we have
\[ H' = \sqrt{4\pi} \sum_n \bar{\sigma}_n \cdot \int d\vec{x} \delta(\vec{x} - \vec{x}_n) \]

\[ \times \left\{ f_1^{(z)} \gamma_1 \nabla \varphi_1 + f_2^{(z)} \gamma_2 \nabla \varphi_2 + \left( f_{\sigma}^{(o)} \frac{1 + \gamma_3}{2} + f_{\sigma}^{(o)} \frac{1 - \gamma_3}{2} \right) \gamma_3^{(m)} \nabla \varphi_3 \right\} \]

For simplicity, we shall keep charge symmetry, i.e.

\[ f_{\sigma}^{(o)} = f_{\sigma}^{(o)} \]

in the evaluation of the potentials, so that we have, for the time being, only two different coupling constants, namely, \( f_{\pi}^{(z)} = f_{\pi}^{(z)} \) for \( \pi^-\) -nucleon and \( f_{\pi}^{(z)} = f_{\pi}^{(z)} \) for \( \pi^0\) -nucleon systems respectively. The interaction Hamiltonian has therefore the simpler form

\[ H' = \sqrt{4\pi} \sum_n \bar{\sigma}_n \cdot \int d\vec{x} \delta(\vec{x} - \vec{x}_n) \sum_\lambda f_\lambda^{(o)} \gamma_\lambda^{(m)} \nabla \varphi_\lambda \] \hspace{1cm} (4)

The different \( f_{\sigma}^{(o)} \) and \( f_{\sigma}^{(o)} \) will not be introduced unless we have the final form of the potentials.

For the meson field operators we take

\[ \varphi_\lambda = (2\pi)^{-\frac{1}{2}} \sum_k \frac{1}{\sqrt{2\omega_\lambda}} \left( a_{\lambda,\vec{k}}^+ + a_{\lambda,-\vec{k}} \right) e^{i\vec{k} \cdot \vec{x}}, \quad \lambda = 1, 2, 3. \hspace{1cm} (5) \]

where \( \omega_\lambda \) is the energy of the pion defined by

\[ \omega_\lambda^2 = \vec{k}^2 + 1, \quad \lambda = 1, 2. \hspace{1cm} (6) \]

\[ = \vec{k}^2 + \alpha^2, \quad \lambda = 3, \quad \alpha = \frac{\mu}{\mu_0} \]

The operators \( a_{\lambda}^+ \) and \( a_{\lambda} \) are the creation and annihilation operators defined by
\[ a^\dagger |N\rangle = \sqrt{N+1} |(N+1)\rangle \]
\[ a |N\rangle = \sqrt{N} |(N-1)\rangle \]

where $N$ is the number of pions in the intermediate states and $\chi, \mathbf{k}$ specify the charge and momentum state of the meson respectively. The commutation relations obeyed by these operators \( a \) are

\[
\begin{align*}
[a_{\chi, \mathbf{k}}, a_{\chi', \mathbf{k}'}^\dagger] &= \delta_{\chi, \chi'} \delta^{\mathbf{k} - \mathbf{k}'} \\
[a_{\chi, \mathbf{k}}, a_{\chi, \mathbf{k}'}] &= 0 \\
[a_{\chi, \mathbf{k}}, a_{\chi', \mathbf{k}'}^\dagger] &= 0
\end{align*}
\]

since the pions are bosons.

Since the Fourier transformation of the source function $\delta(\mathbf{x} - \mathbf{x}_n)$ is just unity, that is,

\[ V(\mathbf{k}) = \int e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}_n)} \delta(\mathbf{x} - \mathbf{x}_n) \, d\mathbf{x} = 1 \]  

(8)

it is much more convenient to work in the momentum space. In what follows we will use the renormalized coupling constant $f$ instead of $f^{(0)}$ in the potentials. This means that we are dealing with the physical nucleons rather than the bare nucleons because an arbitrary number of mesons may exist in the intermediate states.

For instance, if we use $f^{(0)}$ in the second order potential, then this term represents the contribution of a single process in which only a single meson is exchanged.
between the two bare nucleons. A replacement of \( f^{(0)} \) by \( f \) means that the second order potential will include all the terms in which any number of mesons may be emitted and reabsorbed by the same nucleon although one meson is exchanged between the two interacting nucleons.

Figure 1 shows the possible Feynmann diagrams for the nuclear potential for the Hamiltonian (2). The pair terms are completely damped as has been discussed earlier. The one- and two-meson exchange terms give the second and fourth order potential respectively. We see that the charge states of the exchanged mesons depend upon those of the interaction nucleons. We list in Table 1 the possible intermediate mesons for np and pp systems.

Table 1. Possible charge states of the exchanged pions between the two interacting nucleons.

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<th>Nucleon Systems</th>
<th>Exchanged Mesons</th>
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<td>2nd order</td>
</tr>
<tr>
<td><strong>NP</strong></td>
<td>( \pi^+; \pi^-; \pi^0 )</td>
</tr>
<tr>
<td><strong>PP</strong></td>
<td>( \pi^0 )</td>
</tr>
</tbody>
</table>

Making use of the formulas (5)-(8), the evaluation of the potentials is then straightforward. Most of the
Fig. 1. Feynmann diagrams for the nuclear two-body potentials according to the Hamiltonian (2).
integrals are carried out by means of the formulas given in Levy's paper. Since some integrals arising from the exchange of one charged and one neutral pion in the fourth order potential involve the mass difference $\Delta = 1 - \alpha^2$, it seems impossible to evaluate them exactly. We shall summarize in the Appendix the approximation we have made in the calculation.

Using perturbation theory, we write the second order potential as

$$V_2 = - \sum_{\eta, k, \lambda} \frac{\langle 0 | H' | n_{\eta, k} \rangle \langle n_{\eta, k} | H' | 0 \rangle}{\omega_\lambda}$$

where $|N_{\lambda, k}\rangle$ stands for the state vector and $N = 0, 1, 2, \cdots$ is the number of mesons with charge $\lambda$ and momentum $\vec{k}$. If we make use of the relation $\vec{k} = -i \nabla$ so that

$$(\vec{\sigma}_1 \cdot \vec{k})(\vec{\sigma}_2 \cdot \vec{k}) = - (\vec{\sigma}_1 \cdot \nabla)(\vec{\sigma}_2 \cdot \nabla), \quad (9)$$

a simple calculation will yield

$$V_2 = - \frac{3}{4\pi} \sum_\lambda \sum_{f, g} \tau^f_\lambda \tau^g_\lambda \left\{ \frac{\vec{\sigma}_1 \cdot \vec{\sigma}_2}{\lambda} \int_0^\infty dk \frac{k^3 \sin k_\lambda}{k^2 + \mu_\lambda^2} ight. \\
+ 3 S_{\mu_\lambda} \left[ \frac{1}{3\lambda^2} - \frac{1}{\lambda^2} \right] \sin k_\lambda + \frac{k_\lambda}{\lambda^2} \cos k_\lambda \right\} \\
= \sum_\lambda \mu_\lambda^2 \frac{f^2}{f^2 + \mu_\lambda^2} \left\{ \frac{\vec{\sigma}_1 \cdot \vec{\sigma}_2}{3\lambda^2} + \frac{S_{\mu_\lambda}}{3\lambda^2} \left[ (\mu_\lambda^2)^2 + 3\mu_\lambda^2 + 3 \right] \right\}. \quad (10)$$
where

\[ S_{12} = 3 \left( \vec{r}_1 \cdot \vec{\sigma}_1 \right) \left( \vec{r}_2 \cdot \vec{\sigma}_2 \right) - \vec{r}_1 \cdot \vec{r}_2 \]

is the familiar tensor force operator. The fourth order potential is given by

\[ V_4 = -2 \sum \frac{\langle 0 \left| H' \right| \chi \rangle \langle \chi \left| H' \right| \chi \rangle \langle \chi \left| H' \right| \chi \rangle \langle \chi \left| H' \right| \chi \rangle}{\omega_{\chi \kappa} (\omega_{\chi \kappa} + \omega_{\chi' \kappa'})} \]

Upon substitution of the field operator (5) and summing up over the two nucleons, we can reduce \( V_4 \) into the following form:

\[ V_4 = \frac{i}{8 \pi^2} \int \int \frac{d \vec{r}_1 d \vec{r}_2}{\omega_{\lambda \kappa}^3 (\omega_{\lambda \kappa} + \omega_{\lambda' \kappa'})} \left( \sum_{\lambda \lambda'} f_{\lambda}^1 f_{\lambda'}^2 \right) \left[ (\sigma_1 \cdot \vec{r}) (\sigma_2 \cdot \vec{r}) (\sigma_1' \cdot \vec{r}) (\sigma_2' \cdot \vec{r}) \right. \]

\[ \left. + (\sigma_1 \cdot \vec{r}) (\sigma_1' \cdot \vec{r}) (\sigma_2 \cdot \vec{r}) (\sigma_2' \cdot \vec{r}) \right] \]

\[ \left. + \frac{\omega_{\lambda \kappa}}{2 \omega_{\lambda' \kappa'}} \left[ (\sigma_1 \cdot \vec{r}) (\sigma_1' \cdot \vec{r}) (\sigma_2 \cdot \vec{r}) (\sigma_2' \cdot \vec{r}) \right] \right. \]

\[ \left. + \frac{\omega_{\lambda \kappa}}{2 \omega_{\lambda' \kappa'}} \left[ (\sigma_1 \cdot \vec{r}) (\sigma_1' \cdot \vec{r}) (\sigma_2 \cdot \vec{r}) (\sigma_2' \cdot \vec{r}) \right] \right. \]

\[ \left. + (\sigma_2 \cdot \vec{r}) (\sigma_2' \cdot \vec{r}) (\sigma_1 \cdot \vec{r}) (\sigma_1' \cdot \vec{r}) \right] \]

\[ \left. + (\sigma_2 \cdot \vec{r}) (\sigma_2' \cdot \vec{r}) (\sigma_1 \cdot \vec{r}) (\sigma_1' \cdot \vec{r}) \right] \]

\[ \left. + (\sigma_1' \cdot \vec{r}) (\sigma_1 \cdot \vec{r}) (\sigma_2' \cdot \vec{r}) (\sigma_2 \cdot \vec{r}) \right] \]

\[ \left. + (\sigma_1' \cdot \vec{r}) (\sigma_1 \cdot \vec{r}) (\sigma_2' \cdot \vec{r}) (\sigma_2 \cdot \vec{r}) \right] \]

\[ \left. + (\sigma_2' \cdot \vec{r}) (\sigma_2 \cdot \vec{r}) (\sigma_1' \cdot \vec{r}) (\sigma_1 \cdot \vec{r}) \right] \]

\[ \left. + (\sigma_2' \cdot \vec{r}) (\sigma_2 \cdot \vec{r}) (\sigma_1' \cdot \vec{r}) (\sigma_1 \cdot \vec{r}) \right] \]

\[ \left. + (\sigma_1' \cdot \vec{r}) (\sigma_1 \cdot \vec{r}) (\sigma_2' \cdot \vec{r}) (\sigma_2 \cdot \vec{r}) \right] \]

\[ \left. + (\sigma_1' \cdot \vec{r}) (\sigma_1 \cdot \vec{r}) (\sigma_2' \cdot \vec{r}) (\sigma_2 \cdot \vec{r}) \right] \]

(11)
Since the Pauli spin matrices satisfy the following multiplication rules,

\[ \sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1 \]  

\[ \sigma_x \sigma_y = i \sigma_z, \quad \sigma_y \sigma_z = i \sigma_x, \quad \sigma_z \sigma_x = i \sigma_y \]  

it is easy to verify the identity

\[ (\vec{\sigma} \cdot \vec{k})(\vec{\sigma} \cdot \vec{k}') = (\vec{k} \cdot \vec{k}') + i \vec{\sigma} \cdot (\vec{k} \times \vec{k}') \]  

Exactly the same relations hold of course for the isotopic spin operators. By the aid of (12), (13) and (9) together with

\[ \sigma_i \cdot (\vec{k} \times \vec{k}') \sigma_j \cdot (\vec{k} \times \vec{k}') = - \sigma_i \cdot (\vec{k}' \times \vec{\sigma}) \sigma_j \cdot (\vec{k} \times \vec{\sigma}) \]  

we can express the fourth order potential in the form found on the next page.
\[
V_4 = -\frac{f^4}{4\pi^4} \int d^3k d^3k' \frac{e^{i(k + k') \cdot \overrightarrow{r}}}{\partial^3 k \partial^3 k'} \\
\times \left\{ \left( \frac{2}{\partial^3 k} + \frac{2 \gamma^{(2)} \gamma^{(2)}}{\partial^3 k + \partial^3 k'} \right) (k \cdot k') + \left( \frac{2}{\partial^3 k'} + \frac{2 \gamma^{(2)} \gamma^{(2)}}{\partial^3 k + \partial^3 k'} \right) \gamma \cdot (k \times k') \gamma \cdot (k \times k') \right\} \\
- \frac{f^2 f^2}{4\pi^4} \int d^3k d^3k' \frac{e^{i(k + k') \cdot \overrightarrow{r}}}{\partial^3 k \partial^3 k'} \\
\times \left\{ \left( \frac{1}{\partial^3 k'} \right)^2 + \frac{\gamma \cdot (k \times k') \gamma \cdot (k \times k')}{\partial^3 k + \partial^3 k'} \right\} \\
- \frac{f^2 f^2}{4\pi^4} \int d^3k d^3k' \frac{e^{i(k + k') \cdot \overrightarrow{r}}}{\partial^3 k \partial^3 k'} \\
\times \left\{ \left( \frac{1}{\partial^3 k} \right)^2 + \frac{\gamma \cdot (k \times k') \gamma \cdot (k \times k')}{\partial^3 k + \partial^3 k'} \right\} \\
\times \left\{ \left( \frac{2}{\partial^3 k} + \frac{2 \gamma^{(2)} \gamma^{(2)}}{\partial^3 k + \partial^3 k'} \right) (k \cdot k') + \left( \frac{2}{\partial^3 k'} + \frac{2 \gamma^{(2)} \gamma^{(2)}}{\partial^3 k + \partial^3 k'} \right) \gamma \cdot (k \times k') \gamma \cdot (k \times k') \right\}
\]

where \( \gamma_2 = \gamma_1 \pm i \gamma_2 \), so that

\[
\gamma_+^{(1)} \gamma_-^{(2)} + \gamma_-^{(1)} \gamma_+^{(2)} = \gamma_1 \gamma_2^{(1)} + \gamma_2 \gamma_1^{(1)}
\]

with the techniques described above we finally obtain the explicit form of \( V_4 \) as follows:
\[ V_4 = -\frac{2f_4^4}{\pi \lambda^3} \left\{ \varphi_3'' \varphi_3'' \left[ \frac{12 \lambda^2 + 21}{\lambda^2} K_1(2 \lambda) + (2 \lambda^2 + 24) K_0(2 \lambda) \right] \\
+ 2(1 - \tau_3'' \tau_3'') \left[ \frac{4 + 4 \lambda + \lambda^2}{\lambda^2} K_1(\lambda) + (2 + 2 \lambda + \lambda^2) K_0(\lambda) \right] e^{-\lambda} \\
- \frac{2}{3} \bar{\sigma}_1 \cdot \bar{\sigma}_2 \left[ \frac{12 + 8 \lambda^2}{\lambda^2} K_1(2 \lambda) + 12 K_0(2 \lambda) \right] \\
+ \frac{4}{3} \bar{\sigma}_1 \cdot \bar{\sigma}_2 \left( 1 - \tau_3'' \tau_3'' \right) \left[ \frac{2 + 2 \lambda + \lambda^2}{\lambda^2} K_1(\lambda) + (1 + \lambda) K_0(\lambda) \right] e^{-\lambda} \\
+ \frac{1}{3} S_{12} \left[ \frac{30 + 6 \lambda^2}{\lambda^2} K_1(2 \lambda) + 24 K_0(2 \lambda) \\
- 2(1 - \tau_3'' \tau_3'') \left[ \frac{5 + 5 \lambda + \lambda^2}{\lambda^2} K_1(\lambda) + (1 + \lambda) K_0(\lambda) \right] e^{-\lambda} \right] \right\} \\
- \frac{2f_0}{\pi \lambda^3} \left\{ \left[ \frac{4 + 4 \lambda + (\lambda^2)^2}{\lambda^2} K_1(\lambda) + \left( 2 + 2 \lambda + (\lambda^2)^2 \right) K_0(\lambda) \right] e^{-\lambda} \\
- \frac{2}{3} \bar{\sigma}_1 \cdot \bar{\sigma}_2 \left[ \frac{6 + 4 \lambda}{\lambda^2} K_1(2 \lambda) + 6 K_0(2 \lambda) \right] \\
+ \frac{2}{3} \bar{\sigma}_1 \cdot \bar{\sigma}_2 \left( 2 + 2 \lambda + (\lambda^2)^2 \right) \left[ \frac{\lambda}{\lambda^2} K_1(\lambda) + (1 + \lambda) K_0(\lambda) \right] e^{-\lambda} \\
+ \frac{1}{3} S_{12} \left[ \frac{15 + 4 \lambda}{\lambda^2} K_1(2 \lambda) + 12 K_0(2 \lambda) \right] \\
- \frac{1}{3} S_{12} \left[ \frac{5 + 5 \lambda + (\lambda^2)^2}{\lambda^2} K_1(\lambda) + (1 + \lambda) K_0(\lambda) \right] e^{-\lambda} \right\} \right\} \\
\quad \text{(to be continued)} \]
where \( K_p(\mu r) \) is the modified Hankel function of the order \( p \) defined by Basset's formula\(^{17}\):

\[
K_p(\mu r) = \frac{\Gamma(p + \frac{1}{2})}{\Gamma(p + \frac{1}{2})} \left( \frac{2\mu}{r} \right)^p \int_0^\infty \cos kr \frac{dr}{(r^2 + \mu^2)^{p+1/2}}.
\]
The two-body forces for np and pp systems in various spin states are therefore simply the sum of the eigenvalues of $V_2$ and $V_4$ given by (10) and (16) respectively. We are now in the position to take into account the different coupling constants. We introduce two parameters $\beta_p$ and $\beta_n$ defined by

$$f_{\pm N} = \beta_p f_{0p} = \beta_n f_{0n} \tag{18}$$

Throughout the text, energy is in the unit of the rest energy of $\pi^\pm$, and length is in the unit of the Compton wave length of $\pi^\pm$. The potentials thus obtained are listed below.

(I) Potentials for singlet spin states:

$$V^p = -\lambda^2 \beta_p \int \frac{e^{-\lambda \rho}}{\rho}$$

$$- \frac{2f_4}{\pi \lambda^3} \left[ \frac{2\xi \lambda^{2\frac{1}{3}} + 4.7}{\lambda} \left( K_0(2\lambda) + (4 \lambda^2 + 4.7) K_0(2\lambda) \right) \right]$$

$$- \frac{2f_4}{\pi \lambda^3} \beta_p \left\{ \frac{12 + \delta(\Delta \lambda)^2}{\Delta \lambda} K_0(2\Delta \lambda) + 12 K_0(2\Delta \lambda) \right\}$$

$$- \left[ d\lambda K_1(\Delta \lambda) - (\Delta \lambda)^2 K_0(\Delta \lambda) \right] e^{-\lambda \rho} \tag{19} \]
\[ V^m = -\frac{s^2}{\pi} \left[ 2 e^{-\lambda} - \alpha^2 \beta \rho \rho_n e^{-\alpha n} \right] \]

\[ - \frac{2f^4}{\pi \lambda^3} \left\{ \frac{4 \lambda^2 + 1}{\alpha} K_1(2\lambda) + (1 - 4\lambda^2) K_0(2\lambda) \right\} \]

\[ - \left[ 4\lambda K_1(\lambda) - 4\lambda^2 K_0(\lambda) \right] e^{-\lambda} \]

\[ - \frac{2\alpha^2 f^4}{\pi \lambda^3} \beta \rho \rho_n \]

\[ x \left\{ \left[ -\lambda K_1(\lambda) + (\lambda \alpha)^2 K_0(\lambda) \right] e^{-\alpha n} \right. \]

\[ + \frac{12 + 8 (\lambda \alpha)^3}{\alpha} K_1(2\lambda) + 12 K_0(2\lambda) \right\} \]

\[ - \frac{4f^4}{\pi \lambda^3} \beta \rho \rho_n \left\{ \left[ n K_1(n) - n^2 K_0(n) \right] \alpha^2 e^{-\alpha n} \right. \]

\[ + \left[ n K_1(n) - \alpha n^2 K_0(n) \right] \alpha e^{-\lambda} \]

\[ + \frac{12\lambda^2 + 23}{\lambda} K_1(2\lambda) + (4\lambda^2 + 23) K_0(2\lambda) \]

\[ + (1 - \lambda^2) \left[ (2\lambda^3 + \frac{11}{2} \lambda) K_1(2\lambda) + 2\lambda K_0(2\lambda) \right] \}\]

(20)
(11) Pp-potentials for triplet spin states:

\[ 3V_{P}^{PP} = \frac{a^2 x^2 f^2}{3 \lambda^2} e^{-\lambda r} \]

\[ - \frac{2 x^4}{\pi \lambda^3} \left\{ \frac{20 \lambda^2 + 45}{3 \lambda^2} K_1(2 \lambda) + (4 \lambda^2 + 15) K_0(2 \lambda) \right\} \]

\[ - \frac{2 a^2 x^4 f^4}{\pi \lambda^3} \left\{ 16 + 16 \alpha \lambda + 5 (\alpha \lambda)^2 \right\} K_1(\alpha \lambda) \]

\[ + \frac{8 + 8 \alpha \lambda + 3 (\alpha \lambda)^2}{3} K_0(\alpha \lambda) \right\} e^{-\lambda r} \]

\[ - \frac{2}{3} \left\{ \frac{6 + 4 (\alpha \lambda)^2}{\alpha \lambda} K_1(2 \alpha \lambda) + 6 K_0(2 \alpha \lambda) \right\} \}

\[ - \frac{2 x^4}{3 \pi \lambda^3} \left\{ \frac{30 + 8 \lambda^2}{\lambda} K_1(2 \lambda) + 24 K_0(2 \lambda) \right\} \]

\[ - \frac{2 a^2 x^4 f^4}{3 \pi \lambda^3} \left\{ 15 + 4 (\lambda \lambda)^2 \right\} K_1(2 \alpha \lambda) + 2 K_0(2 \alpha \lambda) \]

\[ - \left[ \frac{5 + 5 \alpha \lambda + (\alpha \lambda)^2}{\alpha \lambda} K_1(\alpha \lambda) \right] + (1 + \alpha \lambda) K_0(\alpha \lambda) \right\} e^{-\lambda r} \}

\[ (21) \]
(III) \( N^p \)-potentials for triplet spin states:

\[
\begin{align*}
3V_c^p &= \frac{f^4}{3\lambda^3} \left[ 2e^{-\lambda} - \alpha^2 \beta_p \beta_n e^{-\lambda} \right] \\
-\frac{2f^4}{\pi \lambda^3} &\left\{ -\frac{52\lambda^2 + 93}{3\lambda} K_1(2\lambda) - (4\lambda^2 + 31) K_0(2\lambda) \right. \\
&\left. + \left[ \frac{64 + 64\lambda + 20\lambda^2}{3\lambda} K_1(\lambda) + \frac{32 + 32\lambda + 12\lambda^2}{3} K_0(\lambda) \right] e^{-\lambda} \right\} \\
- \frac{2\alpha^2 \beta_p \beta_n f^4}{\pi \lambda^3} \\
\times &\left\{ -\frac{2}{3} \left[ \frac{6 + 4(\alpha\lambda)^2}{\alpha\lambda} K_1(2\alpha\lambda) + 6 K_0(2\alpha\lambda) \right] \\
&\left. + \left[ \frac{16 + 16\alpha\lambda + 5(\alpha\lambda)^2}{3\alpha\lambda} K_1(\alpha\lambda) + \frac{8 + 8\alpha\lambda + 3(\alpha\lambda)^2}{3} K_0(\alpha\lambda) \right] e^{-\alpha\lambda} \right\} \\
- \frac{4 \beta_p \beta_n f^4}{\pi \lambda^3} \\
\times &\left\{ -\left[ \frac{16 + 16\alpha\lambda + 5(\alpha\lambda)^2}{3\lambda} K_1(\alpha\lambda) + \frac{8 + 8\alpha\lambda + 3(\alpha\lambda)^2}{3} K_0(\alpha\lambda) \right] e^{-\alpha\lambda} \right. \\
&\left. - \left[ \frac{16 + 16\alpha\lambda + 5(\alpha\lambda)^2}{3\lambda} \alpha^2 K_1(\alpha\lambda) + \frac{8 + 8\alpha\lambda + 3(\alpha\lambda)^2}{3} \alpha^2 K_0(\alpha\lambda) \right] e^{-\alpha\lambda} \right\} \\
&\left. + \frac{12\lambda^2 + 23}{\lambda} K_1(2\lambda) + (14\lambda^2 + 23) K_0(2\lambda) \right. \\
&\left. + (1 - \lambda^2) \left[ (2\lambda^3 + \frac{11}{2}\lambda) K_1(2\lambda) + 2\lambda^2 K_0(2\lambda) \right] \right\}
\end{align*}
\] 

\[(23)\]
\[ \begin{align*}
3 V^n_t &= \frac{p^2}{3 \lambda^3} \left\{ 2 e^{-2} (\lambda^2 + 3 \lambda + 3) - \beta_p \beta_n e^{-\alpha \lambda} \left[ (\alpha \lambda)^2 + 3 \alpha \lambda + 3 \right] \right\} \\
- \frac{2 f^4}{3 \pi \lambda^3} \left\{ \frac{30 + 8 \lambda^2}{\lambda} K_i(2 \lambda) + 2 f K_0(2 \lambda) \\
- 4 \left[ \frac{5 + 5 \lambda^2 + \lambda}{\lambda} K_i(\lambda) + (1 + \lambda) K_0(\lambda) \right] e^{-\lambda} \right\} \\
- \frac{2 \alpha^2 \beta_p \beta_n f^4}{3 \pi \lambda^3} \\
x \left\{ \frac{15 + 4 (\alpha \lambda)^2}{\alpha \lambda} K_i(2 \alpha \lambda) + 12 K_0(2 \alpha \lambda) \\
- \left[ \frac{5 + 5 \lambda^2 + (\alpha \lambda)^2}{\alpha \lambda} K_i(\alpha \lambda) + (1 + \alpha \lambda) K_0(\alpha \lambda) \right] e^{-\alpha \lambda} \right\} \\
- \frac{4 \beta_p \beta_n f^4}{3 \pi \lambda^3} \\
x \left\{ \left[ \frac{5 + 5 \lambda^2 + (\alpha \lambda)^2}{\lambda} K_i(\lambda) + (1 + \alpha \lambda) K_0(\lambda) \right] e^{-\lambda} \\
+ \left[ \frac{5 + 5 \lambda^2 + \lambda^2}{\lambda} K_i(\lambda) + (1 + \lambda) \lambda^2 K_0(\lambda) \right] e^{-\lambda} \right\}
\end{align*} \]

The numerical values of the potentials (19) - (24) are calculated by IBM-704 in Table 2, and are plotted in
Table 2. Numerical values of the two-body potentials obtained by IBM 704 at Research Foundation of The Ohio State University.

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<th>( V_{np} )</th>
<th>( V_{cp} )</th>
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Since the deviations from Brueckner and Watson's charge independent potentials are very small, we do not make the proper combinations to give the potentials for even and odd angular momentum states. It is seen that in singlet states the np force is a little bit stronger than the pp force, while in triplet states the pp potential has a deeper well.

At this point, we would like to mention that even if charge symmetry has been destroyed, the isotopic spin $T$ and its $z$-component $T_z$ are still good quantum numbers in our formulation. This can be seen as follows.

Our Hamiltonian has the form

$$H = V_0(\lambda) + \tau_3^{(v)} \tau_3^{(u)} V_1(\lambda) + (\tau_+^{(v)} \tau_-^{(u)} + \tau_-^{(v)} \tau_+^{(u)}) V_2(\lambda).$$

Let $\tau^\uparrow = \tau^{(v)} + \tau^{(u)}$ and $T_3^\uparrow = \tau_3^{(v)} + \tau_3^{(u)}$

then $T_3^\uparrow = 2 + 2 \tau_3^{(v)} \tau_3^{(u)}$, $T^\uparrow = 6 + 2 \tau^{(v)} \cdot \tau^{(u)}$.

Upon substitution into $H$, we obtain

$$H = W_0(\lambda) + T_3^\uparrow W_1(\lambda) + \tau^\uparrow W_2(\lambda)$$

since $\tau_+^{(v)} \tau_-^{(u)} + \tau_-^{(v)} \tau_+^{(u)} = \tau^{(v)} \cdot \tau^{(u)} - \tau_3^{(v)} \tau_3^{(u)}$. If we put in the different coupling constants, we have an additional term $T_3^\uparrow W_3(\lambda)$ in the Hamiltonian. In any
case we can readily show that

\[
[T_x, H] = 0 \quad \text{and} \quad [T^2, H] = 0
\]

because the commutator \([T^2, T^2] = 0\) is always valid. It should be noted that the above proof is only true for a two-body problem.
Fig. 2. The 2nd plus 4th order potentials for singlet spin states. The pair terms are not involved. The energy and length are in unit of $\mu c^1 = 139.67$ Mev and $\lambda_f = 1.413 f$ respectively.
Fig. 3. The central part of the 2nd plus 4th order potentials for triplet spin states. The pair terms are not involved. The energy and length are in unit of \( \mu c^3 = 139.67 \text{ Mev} \) and \( \beta = 1.413 \) fermi respectively.
Fig. 4. The tensor part of the 2\textsuperscript{nd} plus 4\textsuperscript{th} order potentials for triplet spin states. The pair terms are not involved. The energy and length are in unit of $\mu c^2 = 1.99.07$ Mev and $\hbar = 1.413$ fermi respectively.
III. EFFECTS ON THE SCATTERING LENGTH

Following Schwinger's method, Salpeter\textsuperscript{2} calculated the effect of magnetic interaction on the zero-energy singlet $S$ scattering lengths $a_{np}$ and $a_{pp}$ for a point source with Hulthen potential having various hard core radii. On the other hand, Riazuddin\textsuperscript{3} calculated this effect by assuming a finite distribution of nuclear charge and current. We cite their results in Table 3.

It is seen that after the reduction of magnetic interaction there still remains a definite discrepancy.

<table>
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<th>Core Radius</th>
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<th>Salpeter</th>
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<td>$b_0/a_{np}$</td>
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<tr>
<td>$b_0/a_{pp}$</td>
<td>1.308</td>
<td>1.39</td>
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</table>

$(b_0 = \frac{k^2}{\hbar^2} = 2.88 \times 10^{-12} \text{ cm})$

In order to account for this discrepancy, one expects that the pion mass difference plays a role since the meson field is presumably the origin of nuclear
forces according to Yukawa. Thus, Sugie obtained
\[ \frac{b_o}{a_{np}} - \frac{b_o}{a_{pp}} = \sigma \]
by using the potentials of Taketani with a repulsive core of radius 0.54 fermi. He also pointed out that the calculated difference could be made to agree with the experimental value by adjusting the coupling constants. In the following we shall calculate \( \delta (b_o/a) \) owing to the potentials (19) and (20).

According to Jackson and Blatt, a small perturbation \( \delta V \) in the Hamiltonian will yield a change in scattering length given by
\[ \delta a = a^2 M \int_{r_o}^{\infty} \delta V u_o^2 \, dr \]  
(25)
where \( a \) represents the scattering length for unperturbed state, \( M \) the nucleon mass, and \( r_o \) the radius of the repulsive core. In order that the results can be compared with those cited above, we rewrite (25) in the form
\[ \delta (\frac{1}{a}) = \frac{1}{a_{np}} - \frac{1}{a_{pp}} = -M \int_{r_o}^{\infty} \delta V u_o^2 \, dr \]  
(26)
where \( \delta V \) is the difference between the singlet np and pp potentials whose explicit form is found on the next page.
\[ f(t) = \left\{ \begin{array}{ll}
-1 & \text{if } n \leq t < 0 \\
0 & \text{if } 0 \leq t < 1 \\
1 & \text{if } 1 \leq t < 2 \\
0 & \text{if } 2 \leq t < 3 \\
-1 & \text{if } 3 \leq t < 4 \\
0 & \text{if } 4 \leq t < 5 \\
1 & \text{if } 5 \leq t < 6 \\
0 & \text{if } 6 \leq t < 7 \\
-1 & \text{if } 7 \leq t < 8 \\
0 & \text{if } 8 \leq t < 9 \\
1 & \text{if } 9 \leq t < 10 \\
0 & \text{if } 10 \leq t < 11 \\
\end{array} \right. \\
\]
The radial wave function for zero-kinetic energy has the following form:
\[ U_0 = 0, \quad \alpha < \lambda \leq \lambda_* \]
\[ = 1 + \frac{\lambda}{\alpha} - A e^{-b\lambda}, \quad \lambda > \lambda_* \]
where \( \alpha \) and \( b \) are parameters to be determined in the following manner. If we take the field free wave function as
\[ \mathcal{U}_0 = 1 + \frac{\lambda}{\alpha}, \quad \lambda > 0, \]
the effective range for zero kinetic energy is given by
\[ \lambda_e = 2 \int_0^{\infty} (\mathcal{U}_0^2 - U_0) \, d\lambda \]
and the zero of the wave function at the repulsive core leads to
\[ A = \left(1 + \frac{\lambda_*}{\alpha}\right) e^{b\lambda_*} \]
These last two equations fix \( A \) and \( b \) provided that \( A \) and \( \lambda_* \) are known.

In the numerical calculation of the integral (28), we use the following values:
(i) The nucleon mass
\[ M = \frac{1}{2} (M_n + M_p) = 6.724 \]
(ii) The \( ^1S \)-scattering length for zero kinetic energy
\( a = - \frac{1}{2} (a_{np} + a_{pp}) = 14.65 \)

(iii) The singlet effective range for zero kinetic energy

\( r_e = \frac{1}{2} (r_{np} + r_{pp}) = 1.89 \)

(iv) The ratio of \( \pi^0 \)-mass to \( \pi^+ \)-mass

\( \propto = \frac{M_*}{M} = 0.967 \)

(v) The renormalized coupling constant for \( \pi^+ \)-nucleon

\( f_{\pi, N}^2 = f^2 = 0.08 \).

We have taken the average values rather than the individual ones because we are dealing with charge dependent forces. It has been shown that an error in the wave function will result in a higher order correction to \( \delta(1/a) \). The parameters \( \beta_p \) and \( \beta_n \) will be determined later by fitting the empirical data. For the case for which

\[ \beta_p = \beta_n = 1 \]

we list the calculated values for various \( r_o \) in Table 4.

Table 4. Calculated values of \( b_o \delta(1/a) \) for various \( r_o \). The three coupling constant are set to be equal.

<table>
<thead>
<tr>
<th>Core Radius</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta(b_o/a) )</td>
<td>0.4015</td>
<td>0.3358</td>
<td>0.2914</td>
<td>0.24604</td>
</tr>
</tbody>
</table>
IV. EXCITATION ENERGY OF FIRST $T = 1$ STATES OF Li$^6$

Recent investigation of Wilkinson on the isotopic triplet states of light nuclei shows that the average difference between the calculated and experimental values of the lowest $T = 1$ multiplets, after allowing for Coulomb effects and the neutron proton mass difference, is definitely positive but probably not more than 50 kev. This deviation also serves as a measure of charge dependence of the nuclear force. As an example, let us consider the isotopic triplet Be$^6$, Li$^6$, and He$^6$. If the nuclear force is assumed to be charge independent, the first excitations of Li$^6$ and He$^6$ in $T = 1$ states would differ from each other by the amount of Coulomb energy and neutron proton mass difference. The observed energy levels are illustrated in Figure 5.

If $E_{00}$ is taken to be zero, we have

$$E_{10} = E_{11} + \Delta E_m + \Delta E_C \quad (32)$$

where $\Delta E_m$ represents the correction due to neutron proton mass difference and $\Delta E_C$ the Coulomb correction. The quantities in (32) have been observed in nuclear reactions except $\Delta E_C$, which can be calculated from the energy difference between the mirror nuclei Li$^5$ and He$^5$. Since Coulomb energy is inversely proportional to the nuclear radius, we have to multiply $\Delta E_C$ for the
Fig. 5. Observed levels of the isooptopic triplets. The energies are in unit of Mev.
mirror nuclei by the factor \((1 - 1/A)^{1/3}\). By inserting the
values \(E_{11} + \Delta E_m = 2.77\) Mev, and \(\Delta E_c = 0.89(1 - 1/6)^{1/3}
= 0.838\) Mev into (32), we get

\[E_{10} = 2.77 + 0.838 = 3.61\) (Mev)\]

which differs from the observed value 3.57 Mev by 40 kev.

We investigate the possibility that this discrepancy is
due completely to the charge dependent effect of the
nuclear force, and then calculate the energy deviation
caused by the potentials derived in section II.

Since the two nucleons outside the closed core are
in \(1p\)-state coupled to \(L = 0, S = 0\) for both Li\(^6\) and
He\(^6\), we take the \(1p\)-state harmonic oscillator wave
function

\[
\phi_{1p} = \sqrt{\frac{8\nu}{3\pi \nu^2}} \frac{\nu}{n} e^{-\nu r^2} \int_0^\infty (\phi_0 \phi)(33)
\]

where the parameter \(\nu\) is determined by taking the mean
value of the root mean square radius of a nucleus having
four nucleons in the \(1s\) oscillator state and two nucleons
in the \(1p\) state. The root mean square radius is given
by

\[
\langle \lambda^2 \rangle_{n, \ell} = \nu^{-1} \left[ 2(n-1) + \ell + \frac{3}{2} \right] \quad (34)
\]
Thus,

\[
\langle \lambda^2 \rangle = \frac{2}{3} \langle \lambda^2 \rangle_1 + \frac{1}{3} \langle \lambda^2 \rangle_1 = 6 \times 1.2 \times 10^{-13} \]

\[
\therefore \sqrt{V} = 0.6 \times 10^{13} \text{ cm }^{-1} = 0.8478 \left( \frac{x}{\text{m}} \right)^{-1}
\]

According to (33), the two-particle wave function for \( L = 0, \ S = 0 \) state is then

\[
\Psi_{oo}^\text{c} = \frac{8 \nu^{5/2}}{3 \pi^{1/2}} \rho_1 \rho_2 e^{-\frac{\rho}{a} (\lambda_1^2 + \lambda_2^2)} \times \sum C_{1,1} (\lambda, \nu; m, -m) Y_{lm}^{(1)} Y_{1, -m}^{(2)}
\]

where \( C_{kk} (L,M; m, m') \) are the Clebsch-Gordan coefficients in Blatt and Weisskopf's notation. After substituting the coefficients explicitly, formula (35) reduces to

\[
\Psi_{oo} = \frac{8 \nu^{5/2}}{3 \pi^{1/2}} \rho_1 \rho_2 e^{-\frac{\rho}{a} (\lambda_1^2 + \lambda_2^2)} \times \left\{ Y_{1,1}^{(1)} Y_{1,1}^{(2)} - Y_{1,0}^{(1)} Y_{1,0}^{(2)} + Y_{1,1}^{(1)} Y_{1,1}^{(2)} \right\}
\]

The energy correction is therefore simply

\[
\delta E = - \int \Psi_{oo}^* \delta V \Psi_{oo} d\lambda_1 d\lambda_2
\]

where \( r = |\vec{r}_1 - \vec{r}_2| \) is the internucleon distance and \( \delta V \) is the potential difference for singlet states.  


Thus,

\[ \delta V = \nabla^n p - \nabla^p p \]

\[ = - \frac{f^4}{\lambda} \left[ 2 e^{-\lambda} - \lambda^2 \beta_n (\beta_p + \beta_n) e^{-\lambda^2} \right] \]

\[ + \frac{2 f^4}{\pi \lambda^3} \left\{ \frac{24 \lambda^2 + 4 \lambda^2}{\lambda} K_1(2\lambda) + (4\lambda + 8\lambda^2) K_0(2\lambda) \right. \]

\[ + \left[ 4\lambda K_1(\lambda) - 4\lambda^2 K_0(\lambda) \right] e^{-\lambda^2} \}

\[ - \frac{4 f^4}{\pi \lambda^3} \left\{ - \lambda^2 \beta_n (\beta_p - \beta_n) \right. \]

\[ \times \left\{ \frac{12 + 8 (\lambda^2)^2}{\alpha \lambda} K_1(2\alpha \lambda) + 12 K_0(2\alpha \lambda) \right. \]

\[ - \left[ 2\lambda K_1(\lambda) - (\lambda^2)^2 K_0(\lambda) \right] e^{-\lambda^2} \}

\[ - \frac{4 f^4}{\pi \lambda^3} \left\{ - \lambda^2 \beta_n (\beta_p - \beta_n) \right. \]

\[ \times \left\{ [2 K_1(\alpha \lambda) - \alpha^2 K_0(\alpha \lambda)] \alpha^2 e^{-\alpha \lambda^2} \right. \]

\[ + \left[ 2 K_1(\alpha \lambda) - \alpha^2 K_0(\alpha \lambda) \right] \alpha e^{-\lambda^2} \]

\[ + \left[ 12 \lambda^2 + 23 \right] K_1(2\lambda) + (4 \lambda^2 + 23) K_0(2\lambda) \]

\[ + (1 - \alpha^2) \left[ (2 \lambda^3 + \frac{11}{2} \lambda) K_1(2\lambda) + 2 \lambda^2 K_0(2\lambda) \right] \} \].

\[ (38) \]
If we introduce the new variables

\[ \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \quad \text{and} \quad \mathbf{R} = \mathbf{r}_1 + \mathbf{r}_2 \]

the integral (37) is easily transformed to

\[
JE = \frac{2\sqrt{2}}{3} \frac{\nu^3}{\pi\nu^4} \int_0^\infty \delta V e^{-\frac{r^2}{2\nu^2}} \left[ 7 - \frac{2}{3} \nu^2 + \frac{2}{15} (\nu^2)^2 \right] \lambda^2 d\lambda (39)
\]

where we have neglected the incomplete \( \Gamma \) -function contributions for the error is much less than one per cent. In the case in which the three coupling constants are the same, namely, \( f_{\pi} = f_{\pi} = f_{\pi} = 0.08 \), the calculated energy shifts for various hard core radii and nuclear potential widths are listed in the table 5.

Table 5. Calculated energy shifts for the case for which the three coupling constants are set to be the same, i.e., \( f^4 = 0.08 \). Lengths are in fermi and energies are in Mev.

<table>
<thead>
<tr>
<th>Nuclear Size</th>
<th>Core Radius</th>
<th>0.3</th>
<th>0.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 4.2 \ A^{1/3} )</td>
<td>1.055</td>
<td>0.926</td>
<td></td>
</tr>
<tr>
<td>( 1.45 \ A^{1/3} )</td>
<td>0.037</td>
<td>0.501</td>
<td></td>
</tr>
</tbody>
</table>

It is seen that the effect of the fourth order potential is surprisingly large and that the presence of
a repulsive core does not compensate this large effect. This situation is quite different from what Blin-Stoyle and Kearsley have expected. Therefore we can only expect that the different coupling constants will have opposite effect to compensate the mass correction. We shall discuss this in the next section. The calculated results also show that the mass correction depends rather less sensitively upon the hard core radius and that it changes considerably for different values of $\rho$ which is related to the nuclear size by (34). We have set $K = 1.45 a_{\text{F}}^{1/3}$ fermi since the actual nuclear size should be smaller than the potential well width. This value of $K$ corresponds to $\sqrt{\rho} \approx 0.5 \times 10^{-13} \text{cm}^{-1}$ which has been estimated by Blin-Stoyle and Kearsley from the Coulomb energy difference of He$^0$ and Li$^0$. 
V. DETERMINATION OF THE COUPLING CONSTANTS

The Wilkinson calculation can be improved upon if account is taken of the symmetry of the nuclear wave function. This has been done by Fairmairn. In his paper, Fairmairn works out the effect of including symmetry and the corrections to those given by Wilkinson. He finds that the average deviations from the experimental values is no longer definitely positive. For Li, he gets -10 keV instead of 40 keV. In either case, we see from Table 3 that the energy shifts due to the mass correction alone are much too large to account for the expected discrepancies. We therefore wish to compensate the excess by adjusting the coupling constants. If we fix \( \delta E \) and \( \delta (b_0/a) \), we can solve the equations (26) and (37) for \( \beta_p \) and \( \beta_n \). Table 6 shows the relative magnitudes of the three coupling constants obtained in this way.

It seems that the most reasonable result is obtained when \( r_0 = 0.3 \) fermi and \( \sqrt{\nu} = 0.5 \) (fermi)\(^{-1}\) or the potential width \( K = 1.45 \) A\(^{1/3}\) fermi. The interesting thing is that in spite of the large mass correction, a very small variation in coupling constants will reduce it considerably. This implies that the three coupling constants are almost the same. A similar conclusion has also been claimed by Breit et al. They calculate the
high \((l > 2)\) partial wave phase shifts by assuming a one
pion exchange potential \((\text{OPEP})\) and then make an over-all
fit to the two-body scattering data with intermediate
bombarding energies \((9 \text{ MeV} < E < 345 \text{ MeV})\).

Table 0. Relative magnitudes of the coupling constants
obtained by fitting data. Lengths in this table are in
fermi and energies are in kev. \(\delta (b_0/a) = 0.236\).

<table>
<thead>
<tr>
<th>(\sqrt{\nu} )</th>
<th>( r_0 )</th>
<th>( E )</th>
<th>( f_{\pm,N} )</th>
<th>( f_{\text{op}} )</th>
<th>( f_{\text{on}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>40</td>
<td>1</td>
<td>0.997</td>
<td>0.999</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-10</td>
<td>1</td>
<td>0.990</td>
<td>0.9999</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>40</td>
<td>1</td>
<td>0.993</td>
<td>0.0007</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-10</td>
<td>1</td>
<td>0.991</td>
<td>1.0015</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>40</td>
<td>1</td>
<td>0.980</td>
<td>1.014</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-10</td>
<td>1</td>
<td>0.978</td>
<td>1.0113</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>40</td>
<td>1</td>
<td>0.993</td>
<td>1.0008</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-10</td>
<td>1</td>
<td>0.992</td>
<td>1.0013</td>
<td></td>
</tr>
</tbody>
</table>
VI. STRENGTH FUNCTIONS OF A NUCLEON INSIDE A NUCLEUS

The neutron strength function $\frac{P_n}{D}$, as a function of $A$, shows maxima at $A \approx 55$ and $A \approx 170$. Similar maxima for the proton strength function have been predicted by Margolis and Weisskopf and they should occur at $A \approx 68$ and $A \approx 230$. This is mainly because of Coulomb interaction. The neutron and the proton potentials are expected to be very close to each other in depth. A more careful investigation is performed later by Schiffer and Lee. They estimate from the measurement of $(p,n)$ reaction cross section that the potential well depth is 0-6% larger for protons than it is for neutrons. In what follows we shall examine the effect of our charge dependent forces on the potential well depths in optical model.

In a nucleus of intermediate size there are almost half of the nucleons with spin up and the others with spin down. The net contribution of the spin dependent force is therefore taken as negligible. In this approximation the charge dependent nuclear two-body potential takes the form found on the next page.
Let us consider a proton moving in the nucleus of \( N \) nucleons, it is acted by the potential

\[
V_p(\vec{r}) = \sum_{i=1}^{2} V_{pp}(\vec{r}_i - \vec{r}_c) + \sum_{j=1}^{N} V_{np}(\vec{r}_i - \vec{r}_j). \tag{41}
\]

If we assume that the nucleons are uniformly distributed over the nucleus, the nucleon density inside the nucleus is assumed to be
\[ \rho (r) = \text{constant}, \quad r \leq R \]
\[ = 0, \quad r > R \]

where \( R \) is the nuclear size and the constant is determined by normalization condition, thus

\[ 4\pi \int_0^\infty \rho_p (\lambda) \lambda^2 d\lambda = Z, \quad \rho_p = \frac{3Z}{4\pi R^3} \]  \hspace{1cm} (42a)

\[ 4\pi \int_0^\infty \rho_n (\lambda) \lambda^2 d\lambda = N, \quad \rho_n = \frac{3N}{4\pi R^3} \]  \hspace{1cm} (42b)

with \( N + Z = A \).

The proton potential well is then

\[ V_p (\lambda') = \frac{3Z}{4\pi R^3} \int V_{pp} (\lambda' - \lambda_i) d\lambda_i + \frac{3N}{4\pi R^3} \int V_{np} (\lambda' - \lambda_i) d\lambda_i. \]

In order to evaluate this potential, we set \( \vec{r} = \vec{r}' - \vec{r}_1 \) as the independent variable and take the integral over the entire nucleus, we have
where $\tilde{S}_V$ and $\tilde{S}_V^*$ are defined in the following pages.

Exact the same expression for the neutron potential well is obtained by replacing $V_{pp}$ by $V_{pp}$ and $V_{np}$ by $V_{np}$ in (43). The difference is then given by

$$
\Delta V = V_{pp} - V_{np} = \frac{3}{4} \left( \frac{R_+}{R_-} \right)^2.(25V + N) V_{pp} \frac{2}{3} dN
$$

where

$$
V_{pp}(R) = \frac{32}{7R} \int_{R_-}^{R_+} \int_{r_-}^{r_+} \int_{r_-}^{r_+} \frac{\cos z}{2} \left\{ \frac{r^2 + r'^2}{2} - R^2 \right\} dR' dR d\phi
$$
$\delta V_i = V_{\eta \rho} - V_{\nu \rho}$

$$\begin{align*}
&= -\frac{4\pi^3}{\alpha^3} \left\{ 2 e^{-\lambda} \left[ \frac{4 + 4\lambda^2 + \lambda^2}{\lambda} K_\nu(\alpha \lambda) + (2 + 2\lambda + \lambda^2) K_0(\alpha \lambda) \right] \\
&\quad - \left[ \frac{12 + 2\lambda^2 + 2\lambda}{\lambda} K_\nu(2\alpha \lambda) + (4\lambda^2 + 2\lambda) K_0(2\alpha \lambda) \right] \right\} \\
&\quad - \frac{2\pi^3}{\alpha^3} \beta_p (\beta_n - \beta_p) \\
&\quad \times \left\{ \frac{4 + 4\lambda^2 + (\lambda^2)^3}{\alpha \lambda} K_\nu(\alpha \lambda) + [2 + 2\lambda + (\lambda^2)^3] K_0(\alpha \lambda) \right\} e^{-\alpha \lambda} \\
&\quad - \frac{4\pi^3}{\alpha^3} \beta_p \beta_n \\
&\quad \times \left\{ - \left[ \frac{4 + 4\lambda^2 + (\lambda^2)^3}{\alpha \lambda} K_\nu(\alpha \lambda) + [2 + 2\lambda + (\lambda^2)^3] K_0(\alpha \lambda) \right] e^{-\alpha \lambda} \\
&\quad - \left[ \frac{4 + 4\lambda^2 + \lambda^2}{\lambda} K_\nu(2\alpha \lambda) + (2 + 2\lambda + \lambda^2) K_0(2\alpha \lambda) \right] e^{-2\alpha \lambda} \\
&\quad + \left[ \frac{23 + 12\lambda^2 + \lambda^2}{\lambda} K_\nu(2\alpha \lambda) + (23 + 4\lambda^2) K_0(2\alpha \lambda) \right] \\
&\quad -(1-\lambda^2) \left[ (2\lambda^3 + \frac{11}{2} \lambda) K_\nu(2\alpha \lambda) + 2\lambda K_0(2\alpha \lambda) \right] \right\}
\end{align*}$$

\[ (45) \]
\[ \Sigma V = V_{nn} - V_{np} \]

\[ = \frac{4F^4}{\pi \lambda^3} \left\{ 2 e^{-\lambda} \left[ \frac{4 + 4\lambda + \lambda^2}{\lambda} K_1(\lambda) + (2 + 2\lambda + \lambda^2) K_0(\lambda) \right] \right\} \]

\[ - \left\{ \frac{\lambda}{2} K_1(2\lambda) + (4\lambda^2 + 2) K_0(2\lambda) \right\} \}

\[ - \frac{2F^4}{\pi \lambda^3} \beta_p^2 (\beta_n^2 - \beta_p^2) \]

\[ \times \left\{ \frac{4 + 4\lambda + (\lambda^2\alpha^2)}{\alpha \lambda} K_1(\lambda\alpha) + (2 + 2\lambda + (\lambda^2\alpha^2)) K_0(\lambda\alpha) \right\} e^{-\lambda\alpha} \]

\[ + \frac{4F^4}{\pi \lambda^3} \beta_p \beta_n \]

\[ \times \left\{ - \left[ \frac{4 + 4\lambda + (\lambda^2\alpha^2)}{\lambda} K_1(\lambda\alpha) + (2 + 2\lambda + (\lambda^2\alpha^2)) K_0(\lambda\alpha) \right] e^{-\lambda\alpha} \right\} \]

\[ - \left[ \frac{4 + 4\lambda + \lambda^2}{\lambda} \alpha K_1(\lambda) + (2 + 2\lambda + \lambda^2) \alpha^2 K_0(\lambda) \right] e^{-2} \]

\[ + \frac{(2 + 2\lambda^2 + 4\lambda^2 \alpha^2)}{\lambda} K_1(2\lambda) + (2 + 4\lambda^2 \alpha^2) K_0(2\lambda) \]

\[ - (1 - \lambda) \left[ (2\lambda^3 + \frac{1}{2} \lambda^2) K_1(2\lambda) + 2 \lambda^2 K_0(2\lambda) \right] \}

\[ - \quad - \quad - \quad - \quad (46) \]
In the case in which charge independence is assumed, the nuclear two-body potential has the form

\[ V_{nn}(\lambda) = -\frac{f^4}{\pi \alpha^3} \left[ \frac{4 + 4\lambda + \lambda^2}{\lambda} K(\lambda) + \left( 2 + 2\lambda + \lambda^2 \right) K'_{0}(\lambda) \right] e^{-\lambda} \]  

(47)

where we have neglected all the spin-dependent terms as discussed before. The nucleon-nuclear potential is then just

\[ V(\lambda') = \frac{3A}{2R^3} \int_{R-R'}^{R+R'} \left( 1 - \frac{2\lambda + \lambda'}{2\lambda\lambda'} - \frac{R^2}{2\lambda\lambda'} \right) V_{nn} r^2 dr + \frac{3A}{R^3} \int_{\lambda}^{R-\lambda'} V_{nn} r^2 dr \]  

(48)

According to (44) and (48), a numerical calculation for the nucleus \( A = 60 \) \((n_p/2A = 0.225, n_n/2A = 0.275)\) gives the ratio

\[ \frac{\Delta V}{V} = \frac{V_n - V_p}{V} \approx -0.02\% \]

everywhere inside the nucleus. This is much too small to account for any significant difference between the proton and the neutron potential well depths. However, the proton well is slightly deeper than the neutron as has been expected by Schiffer and Lee. We may imagine that more accurate result can be obtained by including the spin interactions. This has been pointed out by Blin-Stoyle and Le Tourneux\(^ {31} \) in their recent letter concerning the difference between the coupling constants for \( \mu \)-decay and \( \beta \)-decay of \( 0^{14} \).
APPENDIX

In the evaluation of the fourth order potentials, we meet some integrals arising from the exchange of one charged and one neutral mesons between a neutron and a proton. They involve the mass difference \( \Delta = 1 - \alpha^4 \), and hence cannot be evaluated exactly. We shall pick a typical term here and work it out in the approximation up to the order of \( \Delta \).

Consider, for instance, the integral

\[
I = I_1 + \Delta I_2 + I_3 - \Delta I_4
\]  

(1)

where

\[
I_1 = \int_0^\infty \frac{k^3 \sin kr \cos (\sqrt{k^2 + \alpha^2} \cdot \Delta)}{(k^2 + 1)^{3/2}} \, dk
\]  

(2)

\[
I_2 = \int_0^\infty \frac{k^3 \sin kr \cos (\sqrt{k^2 + \alpha^2} \cdot \Delta)}{(k^2 + 1)^{3/2}} \, dk
\]  

(3)

\[
I_3 = \int_0^{\sqrt{\Delta}} \frac{k^3 \sin kr \cos (\sqrt{k^2 - \alpha^2} \cdot \Delta)}{(k^2 + \alpha^2)^{3/2}} \, dk + \int_{\sqrt{\Delta}}^\infty \frac{k^3 \sin kr \cos (\sqrt{k^2 - \alpha^2} \cdot \Delta)}{(k^2 + \alpha^2)^{3/2}} \, dk
\]  

(4)

\[
I_4 = \int_0^{\sqrt{\Delta}} \frac{k^3 \sin kr \cos (\sqrt{k^2 - \alpha^2} \cdot \Delta)}{(k^2 + \alpha^2)^{3/2}} \, dk + \int_{\sqrt{\Delta}}^\infty \frac{k^3 \sin kr \cos (\sqrt{k^2 - \alpha^2} \cdot \Delta)}{(k^2 + \alpha^2)^{3/2}} \, dk
\]  

(5)

Since the first terms in \( I_3 \) and \( I_4 \) are of the order of \( \Delta^{3/2} \), which is much smaller than \( \Delta \), they are completely neglected in our calculation. Thus,
\[ I_3 \approx \int_{\sqrt{\Delta}}^{\infty} \frac{k^2 \sin \kappa \cos (\sqrt{k^2 - \Delta} \, \kappa)}{(k^2 + \kappa^2)^{3/2}} \, dk \]  
(4')

\[ I_4 \approx \int_{\sqrt{\Delta}}^{\infty} \frac{k^3 \sin \kappa \cos (\sqrt{k^2 - \Delta} \, \kappa)}{(k^2 + \kappa^2)^{3/2}} \, dk \]  
(5')

If we transform the variable \( k \) in (4') and (5') to a new variable \( k' \) such that

\[ \sqrt{k^2 - \Delta} = k' \]

and write \( k \) for \( k' \) after the transformation, we have

\[ I_3 = \int_{0}^{\infty} \frac{(k^5 + 2\Delta k^3 + \Delta^2 k) \sin (\sqrt{k^2 + \Delta} \, k) \cos k \, dk}{(k^2 + 1)^{3/2}} \]  
(4'')

\[ I_4 = \int_{0}^{\infty} \frac{(k^3 + \Delta k) \sin (\sqrt{k^2 + \Delta} \, k) \cos k \, dk}{(k^2 + 1)^{3/2}} \]  
(5'')

An application of the elementary trigonometric identity

\[ \sin (\alpha \pm \beta) = \sin \alpha \sin \beta \pm \cos \alpha \cos \beta \]

then yields

\[ I = \int_{0}^{\infty} \frac{k^3 \, dk}{(k^2 + \Delta)^{3/2}} \sin (k + \sqrt{k^2 + \Delta}) \kappa + \Delta \int_{0}^{\infty} \frac{k^3 \, dk}{(k^2 + 1)^{3/2}} \sin (k + \sqrt{k^2 + \Delta}) \kappa \]  
(6)

It is still impossible to evaluate (6) exactly. However, we expand \( \sqrt{k^2 + \Delta} \) by means of the binomial theorem

\[ \sqrt{k^2 + \Delta} = k + \frac{\Delta}{2k} - \frac{3}{8} \frac{\Delta^2}{k^3} + \ldots \]  
(6)

To the first order of \( \Delta \), we have approximately
\[ I = \int_0^\infty \frac{k^5 \, dk}{(k^2 + 1)^{\nu}} \left( \frac{\triangle}{2k} \cos 2kr + \sin 2kr \right) \]

\[ + \int_0^\infty \frac{k^3 \, dk}{(k^2 + 1)^{\nu}} \left( \sin kr + \frac{\triangle}{2k} \cos 2kr \right) \]  

(7)

The integrals on the right hand of (7) can then be carried out in the manner summarized in Appendix of Levy's paper\(^{10}\). They are derivatives of different order of the modified Hankel function defined in (17).
12. F. J. Dyson, Phys. Rev. 72 929 (1948). A more detailed work has been carried out by Schweber et al. See, Schweber, Bethe and de Hoffman, Neson and Fields Vol. 1, section 20.
13. For the formulation of the field theory see, for example, G. Wentzel, Quantum Theory of Fields, Inter-science, New York, N. Y. (1949).
16. The masses of pions and nucleons are taken from a lecture notes on *Strange Particle Physics* given by E. d'Espagnat, J. M. Jauch and Y. Yamanachi at CERN (1956--59).


19. We take the average value of the coupling, constant \( \frac{\alpha^2}{4\pi} = 14.65 \) given by Breuckner and Watson. This value of \( \alpha^2 \) is equivalent to \( f^2 = 0.08 \).


22. For the scattering lengths and effective ranges, we take those values given by L. Hulthen in *Handbuch der Physik* XXXII (1957).


26. We use singlet spin states for the two Ip nucleons since they are coupled to form an \( L = 0, S = 0 \) state.


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