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ANALYSIS OF NONHOMOGENEOUS, POLAR-ORTHOTROPIC CIRCULAR DISKS THAT VARY IN THICKNESS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

Charles Wesley Bert III, B. S., M. S.

*****

The Ohio State University
1961

Approved by

[Signature]

Adviser
Department of Engineering Mechanics
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<td>a, b</td>
<td>inside and outside radii</td>
</tr>
<tr>
<td>c</td>
<td>dimensionless shear-modulus coefficient [ \frac{E}{2G} ]</td>
</tr>
<tr>
<td>D</td>
<td>flexural rigidity in the radial direction [ D_r = \frac{1}{12} \frac{Eh^3}{(1 - \nu^2)} ]</td>
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<tr>
<td>D_o</td>
<td>constant appearing in ( D = D_o r^{-n} )</td>
</tr>
<tr>
<td>d</td>
<td>dimensionless shear-rigidity coefficient [ \frac{Gh^3}{6D} ]</td>
</tr>
<tr>
<td>E</td>
<td>modulus of elasticity corresponding to the radial direction [ E_r ]</td>
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<tr>
<td>e</td>
<td>orthotropic ratio [ \frac{E_r}{E_\theta} ]</td>
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<td>F_r, F_\theta</td>
<td>radial and tangential body forces per unit volume</td>
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<td>G</td>
<td>modulus of rigidity [ G_{r\theta} ]</td>
</tr>
<tr>
<td>g</td>
<td>gravitational acceleration</td>
</tr>
<tr>
<td>h</td>
<td>thickness of disk or plate</td>
</tr>
<tr>
<td>h_o</td>
<td>constant appearing in ( h = h_o r^{-n} )</td>
</tr>
<tr>
<td>M</td>
<td>bending or twisting moment per unit length</td>
</tr>
<tr>
<td>N</td>
<td>effective transverse-shear force per unit length</td>
</tr>
<tr>
<td>n</td>
<td>dimensionless constant appearing in ( S = S_o r^n ) (Chapters 2 and 3), ( D = D_o r^{-n} ) (Chapters 4 and 5), or ( h = h_o r^{-n} ) (Appendix)</td>
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1 Only the basic symbols are listed here; additional more specialized symbols are defined in the specific sections in which they are used in the text.
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<td>force in the plane of the disk</td>
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<tr>
<td>( Q )</td>
<td>vertical shear force per unit length</td>
</tr>
<tr>
<td>( r )</td>
<td>radius</td>
</tr>
<tr>
<td>( S )</td>
<td>compliance [1/Eh]</td>
</tr>
<tr>
<td>( S_o )</td>
<td>constant appearing in ( S = S_o r^n )</td>
</tr>
<tr>
<td>( T_o )</td>
<td>mean temperature through the thickness</td>
</tr>
<tr>
<td>( T_{1/2} )</td>
<td>one-half of the temperature difference between the upper and lower surfaces</td>
</tr>
<tr>
<td>( t )</td>
<td>time</td>
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<tr>
<td>( u, v )</td>
<td>radial and tangential displacements</td>
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<td>( V_r, V_\theta )</td>
<td>radial and tangential body-force functions</td>
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<td>( w )</td>
<td>plate deflection</td>
</tr>
<tr>
<td>( z )</td>
<td>distance from the middle surface</td>
</tr>
<tr>
<td>( a )</td>
<td>coefficient of linear thermal expansion in the radial direction ([a_r])</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>specific weight</td>
</tr>
<tr>
<td>( \varepsilon )</td>
<td>strain component</td>
</tr>
<tr>
<td>( \theta )</td>
<td>angular position</td>
</tr>
<tr>
<td>( \nu )</td>
<td>Poisson's ratio in the tangential direction ([= \nu_\theta])</td>
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<td>Symbol</td>
<td>Quantity</td>
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<td>--------</td>
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</tr>
<tr>
<td>$\sigma$</td>
<td>stress vector (no subscripts); stress component (with one or two subscripts)</td>
</tr>
<tr>
<td>$\phi$</td>
<td>stress function</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>rotational velocity</td>
</tr>
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<td>$\omega$</td>
<td>average rotational displacement</td>
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INTRODUCTION

Thin circular disks are widely used components in many important engineering structures and machines. The general theory of the elastic stresses in such elements, when subjected to various kinds of loadings, fortunately has been developed quite extensively for the uniform-thickness case. The general equations governing the elastic behavior of polar-orthotropic \(^1\) circular disks have long been known. Furthermore, exact solutions of many problems of this type have been found and approximate methods of solution with sufficient accuracy for engineering purposes have been developed to treat problems which do not admit of an exact solution.

Because of the very practical reasons of economy of material and reduction in weight, circular disks often are manufactured to have a thickness profile which varies with the radius. Unfortunately, the theory of the elastic behavior of such disks is not nearly as well developed as that for uniform-thickness disks. Although certain solutions for particular problems involving such disks have been known for over

\(^1\)Polar-orthotropic disks are briefly defined as those which have different elastic properties in the radial and tangential directions. For additional information on this topic, the reader is referred to Chapter 1, Section 3.
sixty years, the general equations have not been formulated for the polar-orthotropic case. The primary purpose of the present study is to formulate general equations governing elastic behavior of circular disks with radial variations of thickness and of polar-orthotropic elasticity and subject to various loadings.

The classes of stress problems treated include (1) "disks" experiencing generalized plane stresses (GPS) produced by any arbitrary systems of in-plane boundary forces or couples, of in-plane body forces, or of plane temperature distributions over the entire disk, and (2) "plates" subject to plate bending (PB) due to any arbitrary system of lateral forces or moments at the boundaries or any arbitrary distributions over the entire surface of lateral pressure or thermal gradients linear through the thickness.

The scheme of presentation is to derive all of the necessary fundamental relationships from first principles in Chapter 1, and then to apply them to members subjected to various loadings in succeeding chapters. The general equations applicable to any radial variation in stiffness or flexural rigidity are first presented and then they are applied to and solved for the particular case of an annular disk or plate

---

It is shown later that for GPS problems, the effects of radial variations in thickness and in modulus of elasticity can be combined into one quantity called the stiffness. In analogous fashion, it is shown that for PB, these same quantities can be combined into another quantity known as the flexural rigidity.
with a power-function variation in stiffness or rigidity. Several of the closed-form solutions are applied to particular problems which have not been solved previously.

The importance of closed-form solutions is twofold: (1) They can be obtained for some problems which are of considerable technological importance in themselves and (2) they are useful in evaluating various approximate methods of solution. It should also be mentioned that closed-form solutions usually involve less computational effort than other forms of exact solutions (such as series solutions) and approximate solutions.

Knowledge of the temperature distribution is necessary in order to compute thermal stresses; also, within approximations, the problem of heat transfer in a thin member, like the problem of elastic stresses in the same member, is a two-dimensional boundary-value problem. For these reasons, the theory and application of heat transfer in thin disks is treated in the Appendix.

Where it is possible, reference is made to the work of previous investigators from various nations. However, human limitations necessarily prevent any list of references of this type from being complete.
In the course of the outlined work, which in itself is believed to be novel due to its broad coverage, the following specific original contributions are to be noted:

1. New limitations on the smallest Poisson's ratio of an orthotropic material (Chapter 1, Section 3).

2. First general solution for plane dislocations in polar-orthotropic disks and in varying-stiffness disks (Chapter 2, Section 3).

3. First general solution for GPS in polar-orthotropic varying-stiffness disks subject to arbitrary body forces (Chapter 2, Section 3), exemplified by the first solution for the stresses in a varying-stiffness circular disk rotating about an eccentric normal axis (Chapter 2, Section 4).

4. First general solution for GPS in polar-orthotropic disks subject to plane temperature distributions (Chapter 3, Section 3).

5. First general solution for PB of polar-orthotropic plates with arbitrary lateral-pressure distributions (Chapter 4, Section 3), with the first solution for bending of a slightly-tilted rotating plate of varying flexural rigidity as an example (Chapter 4, Section 4).

6. First general solution for PB of plates with arbitrary distributions of thermal gradients linear through the thickness (Chapter 5, Section 3).

7. First general solution for nonsymmetric heat transfer in varying-thickness disks and for heat transfer in varying-thickness disks with non-uniform local heat-transfer coefficients (Appendix, Section 3).
CHAPTER I

THEORY OF ANISOTROPIC ELASTICITY

1. Analysis of Stress

Fundamental concepts. -- All forces acting on or within a body can be classified as either "applied" or "internal". Applied forces mean all forces exerted on a body by some external agent. These include gravitational, magnetic, and inertial\(^1\) forces and those due to fluid pressure on contact with another body. Internal forces are the reactions of the body to the applied forces.

Applied forces are of two kinds: surface or body. As the name implies, surface forces are those which act on any portion of the surface of the body; they can usually be expressed as a force per unit area. Examples are forces due to contact or fluid pressure. Body forces are distributed forces, acting within the volume of the body, which are generally proportional to the mass density of the body. They are usually expressed in units of force per unit volume. Examples are gravitational, magnetic, and inertial forces.

\(^{1}\)By using d'Alembert's principle of reversed effective force, virtual forces which arise by virtue of the motion of a body can be treated as static forces.
The stress vector, denoted by \( \sigma \), is the intensity of resultant internal force at a given point. Here intensity is taken to mean the limit of the resultant internal force \( P \) per unit area \( A \), as the area becomes infinitesimally small. Thus, mathematically, the stress vector is defined as \( \frac{dP}{dA} \). It is to be emphasized that the stress vector is a point function, not a path, area, or surface function. The direction of the stress vector is the same as the direction of \( dP \).

Now, in general, the stress vector may not be perpendicular to the area on which it acts (Figure 1). Thus, it is convenient to resolve \( \sigma \) into two components: a normal stress perpendicular to the area and a shear stress in the plane of the area. Normal-stress direction is denoted by writing its coordinate direction as a subscript. Thus, in Figure 1, \( \sigma_n \) acts in the direction of the \( n \) axis. However, additional notation is required for shear stress. The first subscript denotes the direction of the normal to the plane area and the second subscript signifies the direction (in the plane) in which the shear stress acts. For example, in Figure 1, \( t \) and \( t' \) are two particular axes in the plane of \( dA \), which has \( n \) as its normal. Then, in plane \( dA \), \( \sigma_{nt} \) and \( \sigma_{nt'} \) are the shear stresses acting in the \( t \) and \( t' \) directions, respectively.

It can easily be shown by consideration of the rotational equilibrium of an infinitesimal element, such as the square one depicted in Figure 2, that \( \sigma_{mn} = \sigma_{nm} \), where \( m \) and \( n \) are orthogonal. No normal-stress components are shown acting because they do not affect rotational
Figure 1. A portion of a general body which has been cut to depict the stress vector and the stress components.
Figure 2. Shear stresses acting on a small element of a body.
equilibrium. Now it is supposed that the shear stress on face 1 is of known magnitude and is acting in the direction shown. Then, in order not to disturb equilibrium of horizontal forces, the shear stress (shown dotted in Figure 2) acting on face 2 must be equal in magnitude and opposite in direction to that on face 1. For rotational equilibrium, the couple produced by the forces on faces 1 and 2 must be exactly balanced by a couple, acting in the opposite sense, due to forces acting as shown on faces 3 and 4. Thus, since the moment arms of all four forces are the same, the magnitudes of all of the forces and of all of the stresses are equal. Now at the lower left-hand corner of the element, shear stress \( \sigma_{nm} \) acts on face 1 and \( \sigma_{mn} \) acts on face 3. The result can be generalized as follows\(^2\): At any point in any two-dimensional stress field, the shear stresses in any two perpendicular directions must be equal.

The geometrical configuration treated throughout this investigation is circular in planform. Therefore, it would seem natural to use circular cylindrical coordinates in this work. However, as discussed in Chapter 2, Section 1, in analysis of thin elements, such as are often encountered in engineering (e.g., turbine disks), it has been found that it is sufficiently accurate to neglect the stresses in the axial

\(^2\)For convenience, the element was taken here to be square. However, since the element is infinitesimal, the geometrical configuration is immaterial, provided that two sides are orthogonal.
direction and the variation, along the axis, of the stresses acting in
the other directions. Thus, a plane polar coordinate system is ade-
quate, provided that the thickness variation is taken into consideration.
Figure 3 shows a typical small disk element with its center at a radius
r from the disk axis and at an angle θ from a reference axis perpendi-
cular to the disk axis.

Any problem in plane elasticity involves at most only three
different stress components: normal stresses in the radial and tangen-
tial directions, respectively, denoted by $\sigma_r$ and $\sigma_\theta$, and shear stresses,
denoted by $\sigma_{r\theta}$. Radial and tangential body forces per unit volume of
material are expressed by $F_r$ and $F_\theta$, respectively. The thickness $h$
of the disk is assumed to vary only with radius.

Summing the forces in the direction of the radius passing
through the center of the element gives, for equilibrium,

$$
2\sigma_r (h + dh)(r + dr) d\theta - 2\sigma_r (h - dh)(r - dr) d\theta
$$

$$
-2 (\sigma_\theta^4 + \sigma_\theta^3)hdr \sin d\theta + 2(\sigma_r^4 - \sigma_r^3)hdr \cos d\theta
$$

$$
+ 4F_r hr dr d\theta = 0.
$$

Here plane elasticity problems are understood to include prob-
lems in both generalized plane stress and plane strain, but not plate-
}\hline
bending problems. In plate bending, there are five different stress
components: $\sigma_r$, $\sigma_\theta$, $\sigma_{r\theta}$, plus two transverse shear stresses $\sigma_{rz}$ and $\sigma_{\theta z}$,
where $z$ is the axial direction.
Figure 3. Disk element used in deriving the equations of equilibrium.
Similarly, summation of the forces in the direction of the tangent at the center of the element results in the following expression:

\[
2 (\sigma_4 - \sigma_3) \cdot h \cdot r \cdot \cos \theta + 2 \sigma_2 (h + dh)(r + dr) \, d\theta \\
-2 \sigma_1 (h - dh)(r - dr) \, d\theta + 2(\sigma_4 + \sigma_3) \cdot h \cdot r \cdot \sin \theta \\
+ 4 F_\theta hr \, dr \, d\theta = 0.
\]

In the above equations,

\[
\sigma_r = \sigma - \frac{\partial \sigma}{\partial r} \, dr \\
\sigma_{r\theta} = \sigma_r - \frac{\partial \sigma_r}{\partial r} \, dr \\
\sigma_{\theta} = \sigma - \frac{\partial \sigma}{\partial \theta} \, d\theta \\
\sigma_{\theta\theta} = \sigma_{\theta} - \frac{\partial \sigma_{\theta}}{\partial \theta} \, d\theta.
\]

Now the elemental angle \(d\theta\) is assumed to be very small so that \(\sin d\theta \approx d\theta\) and \(\cos d\theta \approx 1\). Then, dividing by \(4 \, dr \, d\theta\) and neglecting terms containing the products of differential quantities, the final resulting equations for equilibrium in the radial and tangential directions, respectively, are:

\[
\begin{align*}
\frac{\partial}{\partial r} (rh\sigma_r) - h\sigma_\theta + h \frac{\partial \sigma_{r\theta}}{\partial \theta} + rhF_r &= 0, \\
\frac{1}{r} \frac{\partial}{\partial r} (r^2h\sigma_{r\theta}) + h \frac{\partial \sigma_{\theta \theta}}{\partial \theta} + rhF_\theta &= 0.
\end{align*}
\]
These equations are equivalent to those previously obtained by Kovalenko (1) and later, independently, by Musick (2).

2. **Analysis of Strain**

*Strain-displacement relations.* --In this derivation of the relationships existing among the strains and displacements in polar coordinates, it is assumed that the body is subjected to a two-dimensional stress system. Furthermore, the displacements are taken to be relatively small, so that the usual infinitesimal-deformation theory, rather than finite-deformation theory, is applicable.

Reference is made to Figure 4. There are two elemental gage lines: \( dr \) in the radial direction and \( ds (= r \, d\theta) \) in the tangential direction. Initially, their intersection is at point \( P_1 \), located at coordinates \( r, \theta \). After loading, the intersection of the two gage lines has displaced to point \( P_2 \). The respective radial and tangential displacements are denoted by \( u \) and \( v \). In addition to undergoing displacement of their intersection, the two gage lines have elongated and rotated.

The radial elongation of the radial gage line is \( \frac{\partial u}{\partial r} \, dr \), so that the radial strain \( \varepsilon_r \) is given simply by

\[
\varepsilon_r = \frac{\partial u}{\partial r} .
\]

\[\text{1-2}\]

\[\text{4The numbers appearing in parentheses throughout this manuscript refer to the references listed at the end of each chapter.}\]
Figure 4. Gage lines used in the derivation of the strain-displacement relations.
Because of the increase \( u \) in the radius, there is some tangential elongation \( u \, d\theta \) and due to the tangential displacement, \( v \), there is some additional tangential elongation \( (\partial v/\partial \theta)d\theta \). The total elongation of the tangential gage line \( r \, d\theta \) is equal to the sum of these two displacements so that the tangential strain is expressed by

\[
\varepsilon_\theta = (u/r) + (1/r)(\partial v/\partial \theta) \quad 1-3
\]

As shown in Figure 4, there are angular rotations \((1/r)(\partial u/\partial \theta)\) and \(\partial v/\partial r\) of the tangential and radial gage lines, respectively. However, as can be seen in the figure, a portion \(v/r\) of the latter rotation is due to the tangential displacement \(v\) and thus does not give rise to nonorthogonality of the gage lines after loading. The shear strain is the angular distortion from orthogonality of the gage lines and thus is given by

\[
\varepsilon_{r\theta} = \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \quad 1-4
\]

Equations 1-2, 1-3, and 1-4 are known collectively as the strain-displacement relations in polar coordinates.

As can be observed in Figure 4, the element undergoing deformation has an average rotation \(\omega\), taken as positive in the clock-wise direction, given by

\[
\omega = \frac{1}{2} \left( \frac{\partial v}{\partial r} - \frac{1}{r} \frac{\partial u}{\partial \theta} \right) \quad 1-5
\]
**Compatibility equation.** -- There are three strains expressed as functions of only two independent displacements. Therefore, it is obvious that there must be a relationship among the three strains so that only two of the strains are independent. This relationship is usually known as the compatibility equation and can be derived as follows.

Integration of Equation 1-2 gives the following expression for the radial displacement:

\[ u = \int r \varepsilon_{\theta} \, dr + f(\theta), \]

where \( f(\theta) \) is an arbitrary function of \( \theta \) only.

In similar fashion, integration of Equation 1-3 and substitution of the above expression for \( u \) gives the following expression for the tangential displacement:

\[ v = \int \varepsilon_{\theta} \, d\theta - \int \varepsilon_{r} \, dr \theta - \int f(\theta) \, d\theta + g(r), \]

where \( g(r) \) is an arbitrary function of \( r \) only.

Substituting the above equations for \( u \) and \( v \) into Equation 1-4 gives the following result:

\[ \varepsilon_{r \theta} = \frac{1}{r} \int \varepsilon_{r} \, d\theta + \frac{1}{r} \frac{df(\theta)}{d\theta} + \int \frac{\partial}{\partial r} \left( r \varepsilon_{\theta} \right) d\theta - \int \varepsilon_{r} \, d\theta \]

\[ + \frac{dg(r)}{dr} - \frac{1}{r} \int \varepsilon_{r} \, d\theta + \frac{1}{r} \int \varepsilon_{r} \, dr \theta + \frac{1}{r} \int f(\theta) \, d\theta. \]

\[ ^6 \text{Sometimes called continuity or consistency.} \]
In order to remove the functions $f$ and $g$ from the above expression, it is necessary to multiply by $r$ and differentiate with respect to $r$ and with respect to $\theta$. Then the resulting expression can be simplified to give the general two-dimensional compatibility equation:

$$\frac{\partial}{\partial r} (r \frac{\partial \epsilon_r}{\partial \theta} - r^2 \frac{\partial \epsilon_\theta}{\partial r}) + r \frac{\partial \epsilon_r}{\partial r} - \frac{\partial^2 \epsilon_r}{\partial \theta^2} = 0 . \quad 1-6$$

Although the originator of the general compatibility equation is not known to the present author, it is known that it was derived as early as 1934 by Odqvist (3).

When the stresses and thus the strains are axisymmetric, all strains are independent of $\theta$ so that $r$ is the only independent variable. Therefore, the tangential displacement $v$ is zero and the radial displacement $u$ is independent of $\theta$. Then the first two strain-displacement equations become

$$\epsilon_r = u' \quad \text{and} \quad \epsilon_\theta = u/r ,$$

where the prime denotes an ordinary derivative with respect to $r$.

Rewriting the above equation for $\epsilon_\theta$ and differentiating with respect to $r$ gives

$$u' = (r \epsilon_\theta)' .$$

Therefore, it is now obvious that the compatibility equation for axisymmetric strains is simply

$$(r \epsilon_\theta)' - \epsilon_r = 0 . \quad 1-7$$
It is interesting to note that the axisymmetric compatibility equation is a first-order ordinary differential equation, even though the general plane compatibility equation is a second-order partial differential equation.

3. Stress-Strain Relations

Basic concepts and historical background. The term material is used here in a very broad context to mean any substance of which a loaded body is composed. Thus, composite or nonhomogeneous substances are included.

A material is said to be elastic if all deformation due to external loading disappears when the load is removed. All structural materials exhibit elastic behavior, provided that the loading does not produce any stresses exceeding a certain limit, called the elastic-limit stress.

A material loaded below the elastic-limit stress is said to be within the elastic range; if it is loaded beyond this limit, it is in the plastic range. The present investigation is limited to elastic materials; materials loaded into the plastic range are the subject of a number of books on the theory of plasticity.

The stress-strain relations for a material are, as the name implies, the relationships existing among the various components of

---

7 In this section, the author has drawn upon Timoshenko's historical book (4) for historical information. Therefore, descriptions of the work of previous investigators mentioned in this section, but not alluded to by reference number, can be found in this book.
stress and strain. For a **linear-elastic** material, these relations are linear and are often called the generalized Hooke's law, in honor of Robert Hooke (1635-1703). For most engineering materials loaded within the elastic range, the assumption of linearity is a reasonable one. One notable exception, so far as structural materials are concerned, is stainless steel, which has a decreasing slope in its stress-strain relations, even when loaded well within the elastic range. The scope of the present investigation, as well as that of the classical theory of elasticity, is limited to linear-elastic materials.

The coefficients of proportionality in the linear stress-strain relations are known as the **elastic coefficients** (sometimes called the elastic properties or the elastic constants). If these coefficients are dependent upon location within the body, it is said to be elastically **nonhomogeneous** (sometimes called heterogeneous); if they are really constants (i.e., independent of location), the body is elastically **homogeneous**. Several types of nonhomogeneity are discussed near the end.

---

8. By definition, a material exhibits linear elasticity only when loaded below its proportional-limit stress. However, for most structural materials, the proportional-limit stress is only slightly smaller than the elastic-limit stress and the deviation from linearity at the elastic-limit stress level is exceedingly small.

9. Viscoelastic behavior, in which stress is dependent upon the strain rate as well as the strain itself, is therefore excluded here.
of Section 1 in Chapter 2. In the classical theory of elasticity, it is assumed that the body is homogeneous. However, as described in later chapters, for a smoothly-varying modulus of elasticity (which is merely a combination of certain elastic coefficients), if the elastic coefficients are treated as smoothly-varying functions of the coordinates, the same stress-strain relations serve for this particular type of nonhomogeneity as well as for the homogeneous case.

An elastically **anisotropic** material is one which exhibits different elastic properties when tested in different directions; an elastically **isotropic** material has elastic properties unaffected by orientation. Particular classes of anisotropy are discussed later.

The form of the stress-strain relations for a material with general elastic anisotropy and the minimum number of independent coefficients required in these relations have been quite controversial from the time of Augustin Cauchy (1789-1857). Although Robert Hooke (1635-1703) introduced the concept of an elastic coefficient, he used it to relate load and deformation for a particular structure. M. H. Navier (1785-1836) was first to consider an elastic coefficient as a property of the material; however, he treated only isotropic materials, for which he used, incorrectly only one coefficient.

It was not until Cauchy introduced the concept of stress that the stress-strain relations and what we now term the elastic coefficients came into use. Cauchy used two independent coefficients in the
stress-strain relations for an isotropic material and thirty-six coefficients for a general anisotropic material. His stress-strain relations for the latter case can be written in cartesian coordinates as follows:

\[
\begin{align*}
\sigma_x &= C_{11} \varepsilon_x + C_{12} \varepsilon_y + C_{13} \varepsilon_z + C_{14} \varepsilon_{yz} + C_{15} \varepsilon_{zx} + C_{16} \varepsilon_{xy}, \\
\sigma_y &= C_{21} \varepsilon_x + C_{22} \varepsilon_y + C_{23} \varepsilon_z + C_{24} \varepsilon_{yz} + C_{25} \varepsilon_{zx} + C_{26} \varepsilon_{xy}, \\
\sigma_z &= C_{31} \varepsilon_x + C_{32} \varepsilon_y + C_{33} \varepsilon_z + C_{34} \varepsilon_{yz} + C_{35} \varepsilon_{zx} + C_{36} \varepsilon_{xy}, \\
\sigma_{yz} &= C_{41} \varepsilon_x + C_{42} \varepsilon_y + C_{43} \varepsilon_z + C_{44} \varepsilon_{yz} + C_{45} \varepsilon_{zx} + C_{46} \varepsilon_{xy}, \\
\sigma_{zx} &= C_{51} \varepsilon_x + C_{52} \varepsilon_y + C_{53} \varepsilon_z + C_{54} \varepsilon_{yz} + C_{55} \varepsilon_{zx} + C_{56} \varepsilon_{xy}, \\
\sigma_{xy} &= C_{61} \varepsilon_x + C_{62} \varepsilon_y + C_{63} \varepsilon_z + C_{64} \varepsilon_{yz} + C_{65} \varepsilon_{zx} + C_{66} \varepsilon_{xy},
\end{align*}
\]

Using a theory of elasticity based upon material points and molecular central forces, Cauchy later claimed that the number of independent elastic coefficients in the general anisotropic case was only fifteen.

George Green (1793-1841) introduced a derivation based upon only fundamental principles of solid mechanics. He reasoned that the sum of the internal forces multiplied by the elements (dx, dy, dz) in the directions of the forces must always be an exact differential of some function \( \phi \). Then, using d'Alembert's principle and the principle of virtual displacements \(^{10}\), he arrived at an expression from which the equations could be obtained. Next, assuming small displacements,

\(^{10}\)This is the basis for the better known principle of virtual work.
Green proved that the function $\phi$ must be a homogeneous function of
second degree in the six strain components. Finally, from the prop-
erties of such a homogeneous function, he concluded that twenty-one of
the thirty-six coefficients appearing in Equations 1-8 are independent.

It is interesting to note in passing that Green's reasoning was
equivalent to using the principle of work (Castiglano's first theorem),
which was not even introduced until 1858 and was proved even later by
Alberto Castiglano (1847-1884). In fact, the principle of work, as
used by Wolf (5), for example, affords a shorter way of proving Green's
result. Following Wolf, the strain-energy intensity, i.e., the strain
energy stored in a unit volume of material due to loading, can be formu-
lated as follows:

$$ U_o = \frac{1}{2} \sum_i \sum_j \sigma_{ij} \varepsilon_{ij} \quad (i, j = 1, 2, 3) \quad 1-9 $$
or

$$ U_o = \sum_i \sum_j A_{ij} \varepsilon_{ij} \varepsilon_{ij} \quad (i, j = 1, 2, 3) \quad 1-10 $$

where the $A_{ij}$ are combinations of the $C_{rs}$ appearing in Equation 1-8.

Now the principle of virtual work can be stated as follows:

$$ \sigma_{ij} = \frac{\partial U_o}{\partial \varepsilon_{ij}} \quad 1-11 $$
Equilibrium, of course, requires that $\sigma_{ji} = \sigma_{ij}$. This introduces a symmetry about the main diagonal ($C_{11}$ through $C_{66}$) of Equation 1-8 which requires that

$$C_{sr} = C_{rs} .$$

1-12

This reduces the number of independent coefficients to twenty-one, the same result as that of Green.

Even after Green's significant derivation, there was considerable doubt on the part of many of the scholars of the day (the nineteenth century) as to the number of independent elastic coefficients. This controversy has been called the "rari-constant" controversy, because it involved the rari-constant theory (Cauchy's later work) versus the multi-constant theory (Green's work). It remained for the careful and comprehensive experimental work of Woldemar Voigt (1850-1919), using thin crystals, to quell the remaining doubts that twenty-one, not fifteen, coefficients are required for general anisotropy and that two, not one, are needed in the isotropic case. Further impetus to the multi-constant theory was supplied by Born's work on atomic structure, which discredited the central-force concept used as a basis for Cauchy's rari-constant theory.

Classification of anisotropy. -- Between the two extremes of the most general anisotropic elastic material on one hand and an isotropic elastic material on the other, there are many degrees of anisotropy.
Perhaps the closest to a general anisotropic material as a paratropic material. This is really the same as a general anisotropic material, except that there are certain additional relations, six in number, among the elastic coefficients, so that the number of independent coefficients is reduced to fifteen (6). This is the class which Cauchy in his later years incorrectly contended was the most general case of anisotropy. Nevertheless, certain materials, crystalline in nature, do exhibit this form of anisotropy. Listed in the location in which they appear in Equation 1-8, the fifteen independent coefficients are as follows:

\[
\begin{align*}
C_{11} & \quad C_{12} & \quad C_{13} & \quad C_{14} & \quad C_{15} & \quad C_{16} \\
C_{22} & \quad C_{23} & \quad C_{24} & \quad C_{25} & \quad C_{26} \\
C_{33} & \quad C_{34} & \quad C_{35} & \quad C_{36}
\end{align*}
\]

(It is not implied that these coefficients are the only ones appearing in the stress-strain relations, but rather that they are the only independent ones and that all of the remaining coefficients appearing in Equations 1-8 are repetitious.)

The next lesser degree of anisotropy is possessed by a material with only one plane of elastic symmetry. Within this plane, the elastic properties in any two directions are identical. In this case, the
complete stress-strain relations include the following coefficients arranged in the locations in which they appear in Equations 1-8:

\[
\begin{array}{cccc}
C_{11} & C_{12} & C_{13} & C_{16} \\
C_{12} & C_{22} & C_{23} & C_{26} \\
C_{13} & C_{23} & C_{33} & C_{36} \\
C_{16} & C_{26} & C_{36} & C_{66}
\end{array}
\]

(It is assumed here that the xy-plane is the plane of elastic symmetry, and the independent coefficients are underlined.) It is noted that the total number of coefficients appearing has decreased to twenty, while the number of independent ones is only thirteen.

An orthotropic material is one which has three mutually perpendicular planes of elastic symmetry. Assuming that the xy, yz, and zx planes are parallel to the respective planes of elastic symmetry,
the stress-strain relations contain the following coefficients, the independent ones being underlined:

\[
\begin{bmatrix}
  C_{11} & C_{12} & C_{13} \\
  C_{12} & C_{22} & C_{23} \\
  C_{13} & C_{23} & C_{33}
\end{bmatrix}
\]

\[C_{44}\]

\[C_{55}\]

\[C_{66}\]

Thus for an orthotropic elastic material, there are a total of twelve coefficients, nine of which are independent.

A material is said to be **transversally isotropic** if it possesses a plane in which all directions are elastically equivalent. If the plane of isotropy is the xy-plane, the coefficients appearing in the stress-strain relations are as follows, with the independent ones underlined:

\[
\begin{bmatrix}
  C_{11} & C_{12} & C_{13} \\
  C_{12} & C_{11} & C_{13} \\
  C_{13} & C_{13} & C_{33}
\end{bmatrix}
\]

\[C_{44}\]

\[C_{44}\]

\[\frac{1}{2} (C_{11} - C_{12})\]
Thus, in this case, there a total of twelve coefficients, only five of which are independent.

It is instructive to arrange the elastic coefficients for the isotropic case in the form of Equations 1-8 as follows:

\[
\begin{array}{ccc}
  C_{11} & C_{12} & C_{12} \\
  C_{12} & C_{11} & C_{12} \\
  C_{12} & C_{12} & C_{11} \\
\end{array}
\]

\[
\frac{1}{2}(C_{11} - C_{12})
\]

\[
\frac{1}{2}C_{11} - C_{12}
\]

In all of the classes of anisotropy described above, there was an elastic equivalence of certain parallel directions. If the elastically-equivalent directions form a curvilinear coordinate system, the material is called \textit{curvilinearly anisotropic}. Although there is an infinite number of possible kinds of curvilinear anisotropy, the kind most often encountered in nature and in engineering practice is \textit{circular cylindrical orthotropy}, often called \textit{polar orthotropy} for brevity. In nature, an example is found in the structure of a tree, due to the growth rings. An engineering example is afforded by a circular disk with radial and/or circumferential stiffeners, which provide a constructional orthotropy. This kind of orthotropy is discussed at the end of this chapter and in Chapter 4, Section 1.
Cylindrical coordinates $r$, $\theta$, $z$ are natural for a polar-orthotropic material; they play the same role as that played by the cartesian coordinates $x$, $y$, $z$ in an ordinary orthotropic material.

The present investigation is limited to the study of thin disks and plates; therefore, stress components acting in the direction of the thickness (the $z$ direction) are neglected. Then the only coordinates involved are $r$ and $\theta$, and it is often more convenient to write the polar-orthotropic stress-strain relations in the following form:

$$
\varepsilon_r = \frac{\sigma_r - \nu \sigma_\theta}{E_r}, \quad \varepsilon_\theta = \frac{\sigma_\theta - \nu \sigma_r}{E_\theta}, \quad \varepsilon_{r\theta} = \frac{\sigma_{r\theta}}{G_{r\theta}}, \quad 1-13
$$

where $E_r$ and $E_\theta$ are the moduli of elasticity, $\nu_r$ and $\nu_\theta$ are the Poisson's ratios, and $G_{r\theta}$ is the modulus of rigidity. Now for brevity and convenience, $E \equiv E_r$ and the following dimensionless parameters are introduced: $\nu \equiv \nu_\theta$, $e \equiv E_\theta/E_r$, and $c \equiv E_\theta/(2G_{r\theta})$. Then the stress-strain relations of Equations 1-13 reduce to the following relations:

$$
\varepsilon_r = \frac{e \sigma_r - \nu \sigma_\theta}{e E}, \quad \varepsilon_\theta = \frac{\sigma_\theta - \nu \sigma_r}{e E}, \quad \varepsilon_{r\theta} = \frac{2c \sigma_{r\theta}}{e E}. \quad 1-14
$$
For reference, the coefficients appearing in Equations 1-14 are, in terms of Cauchy's coefficients (defined in Equations 1-8), as follows:

\[ e = \frac{C_{22}}{C_{11}} \]

\[ c = \frac{C_{22} - \left(\frac{C_{12}^2}{C_{11}}\right)}{2C_{66}} \]

\[ \nu = \frac{C_{12}}{C_{11}} \]

\[ E = \frac{C_{11} - \left(\frac{C_{12}^2}{C_{11}}\right)}{1-\nu} \]

Finally, an isotropic elastic material is one in which any plane or surface possesses elastic symmetry. In other words, there is complete elastic symmetry. Then, the orthotropic ratio \( e \) is equal to unity and the coefficient \( c \) is no longer an independent one (now \( c = 1 + \nu \)). Thus, there are only two independent isotropic coefficients, \( E \) and \( \nu \).

Elastic coefficients of orthotropic materials. -- There have been many controversial views regarding the physically realizable values of the elastic coefficients of materials. One of the earliest studies of limitations on an elastic coefficient was that of Kirchhoff (7), who found that in order for the general mixed-boundary-value problem of isotropic, linear-elasticity theory to have a unique solution, Poisson's ratio must lie between -1 and 1/2.

There has especially been considerable controversy in the case of polar-orthotropic materials, since certain solutions for polar-orthotropic elasticity problems seem very different, even qualitatively, from solutions of the corresponding isotropic problems. As an example,
a solid circular disk possessing this kind of elastic orthotropy and subjected to uniform radial pressure $p$ on the edge is considered here. The solution of this problem given by Lekhnitski (8) can be expressed as follows:

$$
\sigma_r = -p \left( \frac{r}{b} \right)^{1/2}, \quad \sigma_\theta = -p \sqrt{\varepsilon} \left( \frac{r}{b} \right)^{1/2}, \quad \sigma_{r\theta} = 0 , \quad 1-16
$$

where $b = $ disk radius. The peculiarity is that for values of the ratio less than unity, the radial and tangential stresses at the origin are unbounded. In order to determine whether such a stress system is conservative, the external work and the internal strain energy will be calculated separately below.

Axisymmetric radial deformation $u$ is given by

$$
u - 0 - \nu \sigma_r
$$

$$
\sigma = \sigma_\theta - \frac{\nu \sigma_r}{b} E
$$

so that at the edge ($r = b$),

$$
\sigma_b = \frac{b}{E} \left( \sigma_\theta - \nu \sigma_r \right)_{r=b} = - \frac{b \sigma}{E} \left( \sqrt{\varepsilon} - \nu \right).
$$

Then, the total work done is given by

$$
- \frac{1}{2} \int_0^{2\pi} \frac{\sigma_\theta}{b} b d\theta = \pi b^2 \frac{E_h^2}{E} \frac{\sqrt{\varepsilon} - \nu}{E}. \quad 1-17
$$

The total strain energy $U$ stored in the disk is given by

$$
U = \frac{1}{2} \int_0^{2\pi} \int_0^{b} \left( \sigma_r \varepsilon_r + \sigma_\theta \varepsilon_\theta + \sigma_{r\theta} \varepsilon_{r\theta} \right) hrdrd\theta.
$$
After substituting for the strains and then for the stresses, integration and some algebraic manipulation yield the following result:

\[ U = \pi b^2 h p^2 \frac{ye - y}{Ee} \]

Equation 1-18

The expression for the potential energy of strain, Equation 1-18, is thus identical to the expression for the external work done, Equation 1-17. Therefore, the energy is conserved without placing any requirements whatsoever on the elastic coefficients. This result, which at first may seem somewhat surprising, can be explained in this way: Although the stresses and the strain-energy intensity\(^{11}\) increase without bound at the disk center when \( e < 1 \), the total strain energy stored in the entire disk (Equation 1-18) nevertheless remains finite.

It is also surprising to note that in the particular case when \( e = \nu^2 \), no work is done and no energy is stored within the disk, regardless of the applied-pressure magnitude. Strangely enough, in this case, the disk can be considered to be perfectly rigid, since it cannot undergo any radial deformation, as shown by the equation preceding Equation 1-17.

Another more fruitful approach to the polar-orthotropic elastic-coefficient problem is a purely geometric one. At the disk center, there can be no distinction geometrically between a radial plane and a

\(^{11}\) Strain-energy intensity is the strain energy stored in a unit volume of material.
tangential plane. Thus, there can be no difference between the elastic moduli in any direction at the center. In other words, at the center only, the disk must be isotropic. This means that all polar-orthotropic elastic bodies must be nonhomogeneous; they must contain a central isotropic core surrounded by polar-orthotropic material\textsuperscript{12}. Referring to a tree trunk as an example of polar orthotropy, this means that the pith at the center of the trunk must be isotropic.

Still another approach is to invoke the requirement that strain-energy intensity physically cannot change sign due to changes in the values of the elastic coefficients. For example, when subjected to hydrostatic pressure, all physical bodies so far as known experience an increase in potential energy, regardless of the values of their elastic coefficients. This approach is applied below and it is equally applicable to the ordinary orthotropic case as well as to the polar-orthotropic one.

Assuming that the axes 1, 2, 3 are the principle-stress directions, the strain-energy intensity is given by

\[ U_o = \frac{1}{2} \sum_{i=1}^{3} \sigma_i \varepsilon_i. \]

\textsuperscript{12} This has been pointed out by Carrier (9).
If it is further assumed that the principal-stress axes coincide with the axes of elastic orthotropy, the stress-strain relations are:

\[ E_i \varepsilon_i = \sigma_i - \nu_{ij} \sigma_j - \nu_{ik} \sigma_k \quad (i, j, k = 1, 2, 3). \]

From the symmetry of the general anisotropic stress-strain relations, it can be shown that

\[ \frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \quad (i, j = 1, 2, 3). \]

Then the expression for the strain-energy intensity becomes

\[ U = \frac{1}{2} \sum_{i=1}^{3} \frac{\sigma_i^2 - 2\nu_{ij} \sigma_i \sigma_j}{E_i} \quad (i, j = 1, 2, 3). \]

Now four orthotropic ratios are defined as follows:

\[ e = E_2/E_1, \quad e' = E_3/E_1, \quad f = \nu_{23}/\nu_{12}, \quad f' = \nu_{31}/\nu_{12}. \]

Also the following dimensionless stress ratios are defined:

\[ a \equiv \sigma_2/\sigma_1, \quad \beta \equiv \sigma_3/\sigma_1. \]

Thus, the energy expression can be rewritten in dimensionless form as follows:

\[ \frac{2E_1 U_o}{\sigma_1^2} = 1 + \frac{a^2}{e} + \frac{\beta^2}{e'} - 2\nu_{12} (a + \frac{f}{e} a\beta + \frac{f'}{e'} \beta). \]

For arbitrary values of \( e, e', f, f' \), a criterion for determining the upper bound of \( \nu_{12} \) is to determine the maximum value of \( \nu_{12} \).
which will prevent the energy expression from changing sign from positive to negative, for any combination of \(a\) and \(\beta\). However, in order to make the final result of general importance, it is advantageous to place limitations on the dimensionless parameters involved so that \(\nu_{12}\) is the smallest of the six Poisson's ratios. Unfortunately, placing the following limitations on the parameters is not sufficient to insure that \(\nu_{12}\) is the smallest: \(e, e', f, f'\) each greater than unity. Two additional relationships are necessary. Here the following relationships, which appear to be reasonable for certain classes of materials, are assumed:

\[
\nu_{ij} = \nu_{ik} \quad (i,j,k = 1,2,3). \tag{1-22}
\]

This implies that \(f = e\) and \(f' = e'\). Using these relations in Equation 1-21 and applying the criterion stated above, it is found that the following must hold:

\[
1 + \frac{d^2}{e} + \frac{\beta^2}{e'} > 2\nu_{12}(a + 2\beta + \beta) \quad \text{or} \quad \nu_{12} \leq \frac{1}{2} \left[ \frac{1 + \frac{d^2}{e} + \frac{\beta^2}{e'}}{a + a\beta + \beta} \right] \min.
\]

Now setting the partial derivatives with respect to the stress ratios equal to zero separately yields these relations which must be satisfied if the expression in brackets above is to be a minimum:

\[
(a + a\beta + \beta) \frac{2a}{e} = (1 + \frac{a^2}{e} + \frac{\beta^2}{e'}) (1 + \beta),
\]

\[
(a + a\beta + \beta) \frac{2\beta}{e} = (1 + \frac{a^2}{e} + \frac{\beta^2}{e'}) (1 + a).
\]
Noting the symmetry between $a^2/e$ and $\beta^2/e'$ in the above relations and substituting back into the expression for $v_{12}^{\text{max}}$ gives the following:

$$v_{12}^{\text{max}} = \frac{1}{2} \left[ \frac{1 + 2 \frac{a^2}{e}}{a(1 + \delta + \delta a)} \right]_{\text{min}}^{\text{max}},$$

1-23

where $\delta \sqrt{e'/e}$. Then setting the first derivative with respect to $a$ of the right-hand side of Equation 1-23 equal to zero gives the following expression for the value of $a$ corresponding to a minimum value of the bracketed quantity in Equation 1-23:

$$a_m = \frac{e \delta}{2(1 + \delta)} + \sqrt{\left[ \frac{e \delta}{2(1 + \delta)} \right]^2 + \frac{e}{2}}. \quad 1-24$$

Substitution of $a_m$ from Equation 1-24 into Equation 1-23 gives the upper bound of the smallest of the six orthotropic Poisson's ratios (assuming that $e$, $e'$, $f$, $f'$ are each greater than unity). The other Poisson's ratios corresponding to this value of the smallest $\nu$ can easily be calculated from Equations 1-20 and 1-22.

Another entirely different approach to the orthotropic elastic-coefficient problem is to use a volume-change criterion. Toward the development of such a criterion, it is noted that the change in volume $\Delta V_o$ per unit volume is equal to the sum of the principal strains. Thus:

$$\frac{\Delta V_o}{V_o} = \frac{3}{\sum_{i=1}^3 \epsilon_i}.$$
Then, substituting the orthotropic stress-strain relations (Equations 1-19), the symmetry relations (Equations 1-20) of the general anisotropic elastic coefficients, and the definitions of the dimensionless ratios, yields the following expression in dimensionless form:

\[
\frac{E_1}{\sigma_1} \frac{\Delta V}{V_0} = 1 + \frac{a}{e} + \frac{\beta}{e'} - \nu_{12} \left(1 + \frac{f'}{e'} + \frac{f}{e} + \frac{a + \alpha + f'\beta + f\beta}{e}\right) . \tag{1-25}
\]

Now applying the relations of Equation 1-22, Equation 1-25 becomes:

\[
\frac{E_1}{\sigma_1} \frac{\Delta V}{V_0} = 1 + \frac{a}{e} + \frac{\beta}{e'} - 2\nu_{12}(1 + \alpha + \beta) . \tag{1-26}
\]

Therefore, in order to avoid having a decrease in volume for a non-compressive stress system \((a > 0, \beta > 0)\), the following expression must define the upper bound of the smallest Poisson's ratio (again assuming \(e, e', f, f'\) are each greater than unity):

\[
\nu_{12}^{\text{max}} = \frac{1}{2} \left[ \frac{1 + \frac{a}{e} + \frac{\beta}{e'}}{1 + \alpha + \beta} \right]_{\text{min}} .
\]

From the symmetry of the above expression, it is observed that the following relationship must hold between the two stress ratios at which the quantity in brackets in the preceding equation is a minimum:

\[
a_m/e = \beta_m/e' .
\]
Substituting the above relation into the expression for $v_{12}^{\text{max}}$ gives

$$
v_{12}^{\text{max}} = \frac{1}{2} \left[ \frac{\alpha_{m}}{1 + \frac{2}{e_{\text{min}}} \left( 1 + \frac{e'}{e} a_{m} \right)} \right]
$$

The above expression takes on a minimum value when $a_{m}$ increases without bound. This minimum value is the upper bound of the smallest Poisson's ratio, provided that $e$, $e'$, $f$, $f'$ are each greater than unity:

$$
v_{12}^{\text{max}} = \frac{1}{e + e'}
$$

1-27

For comparison, Table 1 lists values of $v_{12}^{\text{max}}$ calculated on a basis of energy (Equation 1-23) and on a volume basis (Equation 1-27) for the same values of $e$ and $e'$. With the exception of the isotropic case, in which both criteria give the same result, the volume-change criterion results in a smaller value for the upper bound. However, this does not imply that this criterion is to be preferred. In fact, the energy criterion has a much more fundamental basis.

Although a general polar-orthotropic elastic material is characterized by four independent coefficients ($E$, $v$, $e$, $c$), it is often desirable to estimate the fourth coefficient $c$ in terms of known values of the other three. For this purpose, Wolf (5) suggested the use of the value of $G$ (and thus $c$) corresponding to a plane at 45 degrees to the
<table>
<thead>
<tr>
<th>Orthotropic Ratios</th>
<th>Upper Bound for Least Poisson's Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Energy Basis</td>
</tr>
<tr>
<td>e</td>
<td>e'</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
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<tr>
<td>8</td>
<td>1</td>
</tr>
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<td>1</td>
<td>1</td>
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<td>2</td>
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<td>4</td>
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<td>8</td>
<td>8</td>
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<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
</tr>
</tbody>
</table>
elastic axes. This assumption results in the following expression for the dimensionless coefficient \( c \):

\[
c = \frac{1}{2} \left( 1 + e \right) + \frac{\nu}{e} .
\]

A fresh approach to the estimation of the shear coefficient \( c \) in terms of the other three polar-orthotropic coefficients has been introduced by Lang (10). He used an affine transformation to transform the solution of any isotropic plane-elasticity problem to the solution of the corresponding orthotropic problem. However, in order to accomplish this, it was necessary for the shear modulus \( G \) to be related to the other three elastic coefficients in a particular way. In terms of the dimensionless shear coefficient \( c \), Lang's relation can be expressed as:

\[
c = \sqrt{e} \left( 1 + \nu \sqrt{e} \right) .
\]

Comparison of the actual values of the shear modulus for various real, orthotropic materials with the estimated values obtained from Wolf's and Lang's relations indicated that Wolf's estimate is better for highly-orthotropic materials \((e >> 1)\), while Lang's is better for materials with only a weak orthotropy \((1 < e < 2)\). Both expressions reduce to the familiar one for an isotropic material when \( e = 1 \):

\[ c = 1 + \nu. \]
Elastic coefficients for constructional orthotropy. --Apparently the idea of considering a nonhomogeneous structure constructed of isotropic material as a homogeneous element of orthotropic material was originated by Huber (11) in 1914. He analyzed reinforced-concrete slabs (plates) by considering them as homogeneous plates of orthotropic material. This is an example of what is known as constructional anisotropy. Although there does not appear to be any reason for limiting this concept of analysis to the plate-bending problem, in practice its use appears to be limited almost entirely to this problem. Further examples of this concept are, therefore, discussed in Chapter 4, Section 1.
References


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CHAPTER 2

GENERALIZED PLANE STRESS
WITH ARBITRARY BODY FORCES

1. Introduction and Historical Background

Isotropic disks with axisymmetric loading. -- Very few three-dimensional elasticity problems involving body forces have yet been solved. For example, centrifugal forces in rotating bodies are the most important type of body force, yet, for over seventy years, only two rigorous solutions of this problem have existed: Chree's solutions of the uniform-thickness circular disk (1) (2) and the ellipsoid of revolution (1) (3), both treated first in 1889.

Perhaps the first attempt to calculate the three-dimensional stresses in rotating disks with practical thickness variations, e. g., turbine disks, was that of Cornock (4) in 1931. Tumarkin (5) later noted some errors in his work. Recently Kobayashi and Trumpler (6) developed a general method to compute the three-dimensional stresses in a rotating disk of any arbitrary profile shape.

\[1\] Since, in practice, thickness profiles of circular disks vary only with the radius, all known analyses of circular disks have been confined to such a variation.
In view of the difficulty in obtaining three-dimensional solutions of elasticity problems involving rotating disks, it has been only natural that simplifying assumptions would be used to obtain tractable solutions.

Since, in practice, rotating disks usually are relatively thin, it was reasonable to assume that the axial stress components are negligible and that the remaining stresses are uniformly distributed along the axis. Although these assumptions had been used previously, they were first clearly enunciated in 1903 by Filon (7). He called the two-dimensional state of stress existing under these conditions "generalized plane stress", because he considered it to be a limiting case of plane stress, which is a three-dimensional stress state. Winkler (8), in his analysis of a uniform-thickness circular disk, apparently was first to assume this state of stress in a rotating disk. However, Timoshenko (9) attributes the first correct solution of this problem under this assumption to Maxwell.

The assumption of generalized plane stress was also used in 1903 by Stodola (10) in the first published analysis of a varying-thickness disk. He treated a rotating disk having a general hyperbolic thickness profile, i. e., with a thickness given by \( h = h_o r^{-n} \), where \( r \) is the radius and \( h_o \) and \( n \) are arbitrary constants. Stodola later (11) compared the results of his calculations for a shape approaching an ellipsoid with those computed according to the Chree solution for the rotating ellipsoid. The Stodola solution was unaffected by changes in the
thickness by the same ratio at all radii, while the Chree solution showed a difference of only 5 per cent for ellipsoids ranging from a very flat one to one having a maximum thickness of 25 percent of its outer radius. Since most rotating disks used in practice are quite flat, the two-dimensional assumption was considered to be a valid approximation.

It was noted by Michell (12) in 1899 that, in the absence of body forces, the solution of the generalized-plane-stress problem for a uniform-thickness isotropic disk must satisfy the same partial differential equation, namely the biharmonic equation, which must be satisfied by the solution of the plane-strain problem, i.e., the plane elasticity problem for relatively "long" members. One of the few attempts which have been made to calculate three-dimensional "corrections" to plane elasticity solutions is that of E. Reissner (13).

In his previously-mentioned 1903 paper, Stodola also first disclosed the synthesis performed at the turn of the last century by engineers of the de Laval Company of Sweden for the optimum profile shape to give a uniform stress distribution throughout the face of a circular disk rotating at its design speed. This resulted in an exponential

---

2 Actually, it is not necessary that all body forces be absent; it is necessary only that those which give rise to a body-force function (see Section 2 of this chapter) be absent. In this regard, reference is made to the discussion of Biot's work later in the present section.
profile with thickness expressed by \( ae^{-br^2} \), where \( a \) and \( b \) are constants and \( e \) is the base of the natural logarithm system.

Because of various detailed design considerations, it is not always possible to utilize a disk profile giving a uniform stress distribution. Therefore, various investigators have considered the thickness profile required to give various other desirable distributions of radial stress, for instance. Early investigations of this nature were made by Leon (14), Basch and Leon (15), Holzer (16), and Grammel (17). Perhaps the most extensive study of this type was made by Held (18). Another interesting synthesis of this type was made by Srinivasan (19), who used a radial-stress distribution given by a unit step function. In order to obtain a valid synthesis, the thickness profile must be determined so as to satisfy both equilibrium and compatibility, rather than satisfying only equilibrium, a mistake made by Kumar and Joga Rao (20), for instance.

Table 2 lists some specific profile shapes of axisymmetrically-loaded isotropic disks for which closed-form solutions have been obtained.

It should be mentioned that rotating disks of profile shapes which are not symmetrical about the centroid of their cross section, i. e., which do not have flat midplanes, are subject to circular bending. This bending is due to the offset of the resultant centrifugal force and is in
TABLE 2. List of Profile Shapes of Axisymmetrically-loaded Isotropic Disks for which Closed-form Solutions Have Been Obtained

<table>
<thead>
<tr>
<th>Thickness Function</th>
<th>Remarks</th>
<th>Investigator</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a(1 - br^c)^d$</td>
<td>General case</td>
<td>Fischer (26)</td>
</tr>
<tr>
<td>$c = d = 1$</td>
<td></td>
<td>Martin (27)</td>
</tr>
<tr>
<td>$c = 2, d = 1$</td>
<td></td>
<td>Sen (28)</td>
</tr>
<tr>
<td>$ar^n(1 - br)^m$</td>
<td>General case</td>
<td>Suhara (29)</td>
</tr>
<tr>
<td>$m = 0$, arbitrary $n$</td>
<td></td>
<td>Stodola (10)</td>
</tr>
<tr>
<td>$n = 0$, arbitrary $m$</td>
<td></td>
<td>Suhara and Yasakawa (30)</td>
</tr>
<tr>
<td>$ae^{-br^c}$</td>
<td>General case</td>
<td>Gran Olsson (31)</td>
</tr>
<tr>
<td>$c = 2$</td>
<td></td>
<td>de Laval Company (10)</td>
</tr>
<tr>
<td>$c = 1$</td>
<td></td>
<td>Sen (28)</td>
</tr>
<tr>
<td>$c = 2/3, 4/3$</td>
<td></td>
<td>Malkin (32)</td>
</tr>
<tr>
<td>$cr(1-a)/(b-1)e^{-kr^{1-b}}$</td>
<td>General case</td>
<td>Sen Gupta (33)</td>
</tr>
<tr>
<td>$1/(ar^m + b)$</td>
<td>$m = 2, 2/3$</td>
<td>Adams (34)</td>
</tr>
</tbody>
</table>
addition to the in-plane stretching due to the centrifugal force. This topic is discussed in Chapter 4 on plate bending.

Perhaps the first approximate method of analysis for calculation of stresses in rotating disks of arbitrary thickness profile was a finite-difference method proposed by H. Keller (21) in 1909. Later, in 1912, a somewhat similar approximate method was introduced by Donath (22). Further refinements of this method have been discussed by Tumarkin (5) and by d'Isa and Leone (23). Apparently the first application of the energy or variational method of W. Ritz to the approximate calculation of stresses in rotating disks was made by Pöschl (24) in 1913.

Recently Botto (25) considered a tapered rotating disk of a material with isotropic, homogeneous elasticity but having a density which varies with a power of the radius.

Perhaps the first stress analysis of a rotating disk with a hub, thicker in the axial direction than the disk, was that of Stodola (11) in 1907. This problem has also been treated analytically by d'Isa and Leone (23) and by Meyer (35) and experimentally by Guernsey (36). The effect of a rim, thicker in the axial direction than the disk, has been studied by Rogers (37), Biezeno and Grammel (38), and Kline (39).

Rotating disks with radial blades attached to their faces were treated as early as 1936 by Tumarkin (5). However, he considered only the inertia effect of the blades and neglected their stiffening effect.
Isotropic disks with nonsymmetric loading. -- All of the problems described above possess symmetry about an axis. Among the earliest solutions of plane problems involving nonsymmetric forces are Michell's analyses (40) (41) (1900-1902) of circular disks subject to certain systems of concentrated forces. This work was later generalized by Mindlin (42). Reference is also made to the work of Kohl (43), Tungl (44), Matumoto (45), and Miyao (46), all of whom considered circular disks subjected to various nonsymmetric boundary-force systems.

The first analysis of a varying-thickness member subject to nonsymmetric generalized plane stress but no body forces was probably that of Shepherd (47), who in 1933 treated a straight-tapered rectangular planform disk. Recently Musick (48) and Conway (49) independently considered circular disks of hyperbolic profile subjected to nonsymmetric boundary forces.

One of the first solutions for a circular disk under gravity loading in its plane was that of Michell (40) in 1900. Reference is also made to analyses by Mindlin (50), Sen (51), and Yu (52). Biot (53) in 1935 was perhaps first to note that body forces derivable from potential functions do not appear in the plane equilibrium equations.

\[ \nabla^2 \varphi = 0, \] where \( \nabla^2 \) is the two-dimensional Laplacian operator.
Circular disks with nonsymmetric body forces not derivable from potential functions, but derivable from more general functions, have been analyzed by Mindlin (54), Sen (55), Il'yn (56), and Szelagowski (57), who all treated an eccentrically-rotating, uniform-thickness disk. Also Hodge (58) considered a uniform-thickness circular disk rotating about a diameter. Recently Teodorescu (59) gave a general treatment of the uniform-thickness, plane elasticity problem involving arbitrary body forces.

Apparently varying-thickness disks subjected to nonsymmetric body forces were first treated in 1952 by Vainberg (60), who analyzed a disk rotating about a diameter.

As first shown by Michell (12) in 1899, the problem of a uniform-thickness, isotropic disk with a nonzero force resultant on a boundary requires consideration of the displacements, in order that they may be made single-valued. Similar work relating specifically to a circular annular disk was performed by Timpe (61) in 1905. The possibility that the displacements may not necessarily be continuous and thus single-valued was first noted by Weingarten (62) in 1901. The theory of such discontinuous displacements was developed quite intensively by Volterra (63), who called them "distorsioni" in Italian. They were later called "dislocations" by Love (64) and this latter name has been retained in the English-language literature. Michell's general
study of displacements in multiply-connected plane bodies was extended in 1946 by Mindlin (65) to include dislocations. It is to be noted that although the displacements are not continuous, all of the stress and strain components are continuous in Volterra dislocations. It was pointed out by Mann (66) that in order for the stresses to be single-valued, the most general displacement increments possible on crossing the dislocation barrier are rigid-body motions. It is to be emphasized that even though the motion of the dislocation barrier itself consists of rigid-body displacements only, other portions of the body undergo more general displacements and thus induce stresses, called initial stresses, which do not depend upon subsequent loading.

In recent years, more general dislocations than the Volterra type have been treated by Mann (66), Bogdanoff (67), Goodier and Wilhoit (68), and Ju (69). In these, not only displacements, but also certain strains and stresses, are discontinuous upon crossing the dislocation barrier. It is obvious that in such dislocations, compatibility of strains is not satisfied. Such dislocations are believed to be possible on a microscopic scale and are of importance in solid-state physics of metallic crystals. Although some of the concepts of dislocations in

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A multiply-connected disk is one containing one or more holes; a disk without holes is termed simply-connected. In general, the degree of connectivity of a member is one less than the number of holes it contains.
solid-state physics are based on analyses of Volterra dislocations in
the theory of elasticity, it is cautioned that many of the terms are
declared quite differently.

So far as is known to the present investigator, dislocations in
varying-thickness disks have not previously been considered. As an
original contribution, simple Volterra dislocations in varying-thick-
ness disks are discussed in this chapter, Section 3.

In Section 4 of this chapter, an example problem involving a
doubly-connected disk with nonsymmetric body forces and a nonzero
force resultant on a boundary, but no dislocations, is treated in detail.

Disks with anisotropic and nonhomogeneous elasticity. --Timoshenko (70) gives Voigt credit for the first correct formulation of the
differential equation governing plane problems in uniform-thickness
anisotropic bodies. Additional significant contributions have been
made by Michell (40), Lekhnitski (71), Okubo (72), and Carrier (73).

Anisotropic disks subject to body forces were treated in 1939
by Glushkov (74), who analyzed a uniform-thickness rotating circular
disk of polar-orthotropic material. Later Wilson (75) considered such
a disk under gravity loading, and recently Teodorescu (76) gave a
genreal treatment of the uniform-thickness, orthotropic, plane elas-
ticity problem with arbitrary body forces.
Perhaps the first analysis of the stresses in an anisotropic disk of varying thickness was made by Sen Gupta (33), who in 1949 treated rotating circular disks of hyperbolic and exponential profiles.

So far as known no solutions for varying-thickness anisotropic disks subjected to nonsymmetric body forces have appeared in the literature. A closed-form solution for this class of problem is given in Section 3 of this chapter.

Basically, there are two kinds of elastic nonhomogeneity encountered in engineering applications. The first kind may be termed the discontinuous type, as exemplified by a composite structure made of two or more materials with different elastic coefficients. A specific example of this type involving generalized plane stress is a recent analysis by Szelagowski (77) of a rotating disk with a rigid circular inclusion at its center. Problems involving this kind of nonhomogeneity are usually handled best by considering each homogeneous component separately, then fitting the components together by application of joint boundary conditions. In general, this gives rise to a mixed boundary-value problem.

The other kind of nonhomogeneity may be termed the continuous or smoothly-varying type. Golecki (78) attributed the first analysis which takes into account this form of nonhomogeneity to Hruban in 1944. Hruban considered the special case of an isotropic material in which only the modulus of elasticity varies, Poisson's ratio remaining
constant. Apparently the principal justification for assuming that Poisson's ratio stays constant is that in most problems a large variation in Poisson's ratio has only a small effect on the stresses. In engineering, smoothly-varying nonhomogeneity is found in the study of soil mechanics and in nonuniformly-heated structures of materials with temperature-dependent elastic properties. For examples of the latter application, the reader is referred to the next chapter.

Apparently Kovalenko (79) was first to combine the effect of nonhomogeneity (varying modulus of elasticity only) and varying thickness into a single quantity. As Zyckowski (80) recently stated quite clearly, this means that sufficient generality is obtained by considering either a varying-thickness homogeneous disk or a uniform-thickness nonhomogeneous disk. In other words, varying thickness can be viewed as a particular kind of nonhomogeneity of the disk.

2. Derivation of the General Equation

With the thickness assumed to be a function of the radius only and in the presence of arbitrary body forces, the polar-coordinate
equilibrium equations corresponding to the radial and tangential directions are (from Chapter 1, Section 1):

\[
\begin{align*}
\frac{\partial}{\partial r} (rh\sigma_r) - h\sigma_\theta + h\frac{r\sigma_\theta}{\partial \theta} + rhF_r &= 0, \\
\frac{1}{r} \frac{\partial}{\partial r} (r^2 h\sigma_\theta) + h\frac{\partial \sigma_\theta}{\partial \theta} + rhF_\theta &= 0,
\end{align*}
\]

where \( r, \theta \) are the polar coordinates, \( h \) is the thickness of the disk element, \( \sigma_r \) and \( \sigma_\theta \) are the normal stresses in the radial and tangential directions, \( \sigma_{r\theta} \) is the shear stress, and \( F_r \) and \( F_\theta \) are the radial and tangential body-force components per unit volume.

There are many possible ways of handling Equations 2-1 by an approach analogous to the stress-function method first used by Airy (81) for uniform-thickness plane elasticity problems without body forces. As a guide, in the treatment of varying-thickness rotating disks, A. Föppl (82) used the quantities \( h\sigma_r \) and \( (h\sigma_\theta - rhF_r) \) in plane polar coordinates in a manner analogous to the way \( \sigma_x \) and \( \sigma_y \) were originally used in rectangular coordinates by Airy. Further inspiration is provided by the body-force-function concept used in conjunction with the stress-function approach by Biot (53) for the uniform-thickness plane elasticity problem with body forces, in terms of rectangular coordinates. In that case it happens that a very simple formulation is
obtained by subtracting the same function, called a body-force function because it depends upon the body forces, from both normal stresses and handling the shear stress unchanged.

Although a single body-force function is satisfactory for uniform-thickness disks subjected to various kinds of body forces, experience has shown that a single function is not adequate for varying-thickness disks subjected to certain kinds of nonsymmetric body forces. The concept of using two different body-force functions was originated by Kovalenko (83) in 1955. To apply this concept, it is first advantageous to rewrite Equations 2-1 in the following form:

\[
\begin{align*}
\frac{\partial}{\partial r} \left[ r (h \sigma_r - V_r) \right] - (h \sigma_\theta - V_\theta) + h \frac{\partial^2 \sigma_r}{\partial \theta^2} &= 0, \\
\frac{1}{r} \frac{\partial}{\partial r} (r^2 h \sigma_r) + \frac{\partial}{\partial \theta} (h \sigma_\theta - V_\theta) &= 0,
\end{align*}
\]

where \( V_r \) and \( V_\theta \) are the two body-force functions, as yet undetermined.

Then the relations between the stress components, the body-force functions, and the stress function must be as follows in order to identically satisfy Equations 2-2:

\[
\begin{align*}
\frac{h \sigma_r}{r} &= \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + V_r, \\
\frac{h \sigma_\theta}{r} &= \frac{\partial^2 \phi}{\partial r^2} + V_\theta, \\
\frac{h \sigma_r}{r} &= -\frac{\partial^2 \phi}{\partial \theta \partial \theta} \left( \frac{\phi}{r} \right).
\end{align*}
\]
It can easily be shown that in order for Equations 2-1 to be identically satisfied, the body-force functions in Equations 2-2 must be defined as follows:

\[
\frac{\partial V_\theta}{\partial \theta} = rh F_\theta , \quad \frac{\partial}{\partial r} (rV_r) - V_\theta = rh F_r . \quad 2-4
\]

(The functions \(V_\theta\) and \(V_r\) are particular solutions of Equations 2-4 and thus they can be considered as particular solutions of Equations 2-1.)

As shown in Chapter 1, Section 2, the condition of compatibility of strains requires that the following relation be satisfied:

\[
\frac{\partial}{\partial r} \left( r \frac{\partial \epsilon_r \theta}{\partial \theta} - r^2 \frac{\partial \epsilon_\theta \theta}{\partial r} + r \frac{\partial \epsilon_r}{\partial r} - \frac{\partial^2 \epsilon_\theta}{\partial r^2} \right) = 0 . \quad 2-5
\]

Also, the following stress-strain relations for a polar-orthotropic elastic material, from Chapter 1, Section 3, are used:

\[
\epsilon_r = \frac{e\sigma_r - \nu\sigma_\theta}{eE} , \quad \epsilon_\theta = \frac{\sigma_\theta - \nu\sigma_r}{eE} , \quad \epsilon_{r\theta} = \frac{2c\sigma_r \theta}{eE} . \quad 2-6
\]

However, instead of assuming that the material is homogeneous, as is customary in classical elasticity theory, here the modulus of elasticity is considered to be an arbitrary function of the radius, while the other elastic parameters, \(\nu\), e, and c, remain constant. As mentioned in the preceding section, this form of smoothly varying nonhomogeneity was first considered, for the isotropic case, by Hruban.
Following Kovalenko (79), it is expedient to introduce here the quantity \( S \), called the compliance of the disk and defined as

\[
S = \frac{1}{Eh}.
\]

(The quantity \( Eh \) is referred to as the stiffness.)

Finally, substituting Equations 2-3, 2-6, and 2-7 into Equation 2-5 and simplifying give the following result:

\[
S\left[ \frac{\partial^4 \phi}{\partial r^4} + \frac{2}{r} \frac{\partial^3 \phi}{\partial r^3} - \frac{e}{r^2} \frac{\partial^2 \phi}{\partial r^2} + \frac{e}{r^3} \frac{\partial \phi}{\partial r} + \frac{2(c-v)}{r^2} \frac{\partial^4 \phi}{\partial r^2 \partial \theta^2} \right. \\
- \left. \frac{2(c-v)}{r^3} \frac{\partial^3 \phi}{\partial r \partial \theta^2} + \frac{2(c-v+e)}{r^4} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{e}{r^4} \frac{\partial^4 \phi}{\partial \theta^4} \right]
\]

\[
+ \frac{dS}{dr}\left[ \frac{\partial^3 \phi}{\partial r^3} + \frac{2-v}{r^2} \frac{\partial^2 \phi}{\partial r^2} - \frac{e}{r^2} \frac{\partial \phi}{\partial r} + \frac{2(c-v)}{r^2} \frac{\partial^3 \phi}{\partial r^3 \partial \theta^2} \right. \\
- \left. \frac{2(c-v)+e}{r^3} \frac{\partial^2 \phi}{\partial \theta^2} \right] + \frac{d^2 S}{dr^2} \left[ \frac{\partial^2 \phi}{\partial r^2} - \frac{v}{r} \frac{\partial \phi}{\partial r} - \frac{v}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right]
\]

\[
= \left[ v \frac{\partial^2 \phi}{\partial r^2} - \frac{e}{r} - 2v \frac{\partial \phi}{\partial r} - \frac{e}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right](SV_r)
\]

\[
- \left[ \frac{\partial^2 \phi}{\partial r^2} + \frac{2+v}{r} \frac{\partial \phi}{\partial r} - \frac{v}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right](SV_\theta).
\]

It is believed that Equation 2-8 has not been published previously. However, for the case of isotropy, it reduces to an equation given by Kovalenko (83).

At this point it is interesting to note that with only a slight modification Equation 2-8 is applicable to plane-strain problems in circular
members, either solid or hollow, having polar-orthotropic elasticity and a modulus of elasticity which varies with the radius. For such problems it is necessary only to substitute the quantity \( \nu/(1-\nu) \) for \( \nu \) at every place it appears in Equation 2-8, with the following result:

\[
\frac{1}{E} \left[ \frac{\partial^4 \phi}{\partial r^4} + \frac{2}{r} \frac{\partial^3 \phi}{\partial r^3} - \frac{\nu}{r^2} \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{2(c-c\nu-\nu)}{(1-\nu)r^2} \frac{\partial^2 \phi}{\partial r^2 \partial \theta^2} \right]
\]

\[
- \frac{2(c-c\nu-\nu)}{(1-\nu)r^3} \frac{\partial^3 \phi}{\partial r \partial \theta^2} + \frac{2}{r^4} (c+e - \frac{\nu}{1-\nu}) \frac{\partial^2 \phi}{\partial \theta^2} + \frac{3}{r^4} \frac{\partial^4 \phi}{\partial \theta^4}
\]

\[
+ \frac{d}{dr} \left( \frac{1}{E} \right) \left[ 2 \frac{\partial^3 \phi}{\partial r^3} + \frac{2}{(1-\nu)r} \frac{\partial^2 \phi}{\partial r^2} - \frac{\nu}{r^2} \frac{\partial \phi}{\partial r} + \frac{2(c-c\nu-\nu)}{(1-\nu)r^2} \frac{\partial^3 \phi}{\partial r \partial \theta^2} \right]
\]

\[
- \frac{2(c-c\nu-\nu)}{1-\nu} \left( e \frac{1}{r^3} \frac{\partial^2 \phi}{\partial \theta^2} \right) + \frac{d^2}{dr^2} \left( \frac{1}{E} \right) \left[ \frac{\partial^2 \phi}{\partial r^2} - \frac{1}{(1-\nu)r} \frac{\partial \phi}{\partial r} \right]
\]

\[
- \frac{\nu}{(1-\nu)r^2} \frac{\partial^2 \phi}{\partial \theta^2} = \left[ \frac{\nu}{1-\nu} \frac{\partial^2}{\partial r^2} - \frac{e-2\nu}{(1-\nu)r} \frac{\partial}{\partial r} - \frac{e}{r^2} \frac{\partial^2}{\partial \theta^2} \right] \frac{V_r}{E} \]

\[
- \left[ \frac{\partial^2}{\partial r^2} + \frac{2-\nu}{1-\nu} \frac{\partial}{\partial r} - \frac{\nu}{(1-\nu)r^2} \frac{\partial^2}{\partial \theta^2} \right] \frac{V_\theta}{E} \]. \quad 2-9
\]

(Since plane-strain conditions prevail only in relatively long members, it is not possible to consider any thickness variation in such problems; Thus, varying \( 1/E \) replaces varying \( S \).)

3. General Solution for a Disk with a Power-Function Stiffness Distribution

The entire analysis presented in the preceding section is applicable to circular disks with any arbitrary radial distribution of stiffness \( 1/S \). Now the analysis is specialized to a stiffness function
known as a general power-function distribution, expressed by

\[ \frac{1}{S} = \frac{1}{S_o r^n} \]  

where \( S_o \) is an arbitrary compliance constant and \( n \) is an arbitrary constant which is always positive for practical disk designs. It is noted that the stiffness distribution given above is a generalization of the hyperbolic thickness profile, first used by Stodola.

Substituting the stiffness distribution of Equation 2-10 into Equation 2-8 gives the following result:

\[
\frac{\partial^4 \phi}{\partial r^4} + \frac{2(1+n)}{r} \frac{\partial^3 \phi}{\partial r^3} + \frac{n^2 - vn + n - e}{r^2} \frac{\partial^2 \phi}{\partial r^2} + (e + vn) \frac{1 - n}{r^3} \frac{\partial \phi}{\partial r} + \frac{2(c - v)(1 - n) - e(n - 2) - wn(n - 1)}{r^4} \frac{\partial^2 \phi}{\partial r^2} + \frac{2(c - v)(1 - n) - e(n - 2) - wn(n - 1)}{r^4} \frac{\partial^2 \phi}{\partial r^2} \\
+ \frac{2(c - v)}{r^2} \frac{\partial^4 \phi}{\partial r^2 \partial \theta^2} - 2(c - v) \frac{1 - n}{r^3} \frac{\partial^3 \phi}{\partial r \partial \theta^2} + \frac{2(c - v)(1 - n) - e(n - 2) - wn(n - 1)}{r^4} \frac{\partial^2 \phi}{\partial r^2} + \frac{2(c - v)(1 - n) - e(n - 2) - wn(n - 1)}{r^4} \frac{\partial^2 \phi}{\partial r^2} \\
+ \frac{\partial^2 \phi}{\partial \theta^2} + \frac{2(c - v)}{r^2} \frac{\partial^4 \phi}{\partial \theta^2} = [\frac{\partial^2 \phi}{\partial r^2} + \frac{2n + 1}{r} \frac{\partial \phi}{\partial r} - \frac{\partial^2 \phi}{\partial \theta^2} + \frac{2n + 1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{2n + 1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}](V_r) - [\frac{\partial^2 \phi}{\partial r^2} + \frac{2n + 1}{r} \frac{\partial \phi}{\partial r} - \frac{\partial^2 \phi}{\partial \theta^2} + \frac{2n + 1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{2n + 1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}](V_r). 
\]

\[ 2-11 \]

\[ ^5 \text{In practice, for homogeneous disks, the power-function form cannot be used for solid varying-thickness disks over the entire disk; it can be used over the entire surface only for annular ones. This is because, for a finite thickness at the outer periphery, if } n > 0 \text{ (hyperbolic), the thickness at the center is unbounded and if } n < 0 \text{ (parabolic), the center thickness is zero.} \]
Limitations on the solution due to stress considerations. --

Rather than consider all possible solutions of the compatibility equation, Equation 2-11, it is expedient to determine the nature of the solutions by studying the limitations imposed on the stress function by the physical requirement that all of the stress components be periodic in the angular coordinate.

The approach taken here follows that used by Langhaar (84) in connection with an Airy-stress-function analysis of elastic shells. It will be shown that periodicity of the stress components does not necessarily imply periodicity of the stress function.

Since the shear stress is required to be periodic in θ, it must be capable of being expanded in a Fourier series in θ as follows:

\[
\frac{\delta^2}{\delta r \delta \theta} \left( \frac{\phi}{r} \right) = - h \sigma_r - \frac{a_i(r)}{2} + \sum_{i=1}^{\infty} [a_i(r) \cos \theta + b_i(r) \sin \theta].
\]

---

This study is limited to simple Volterra-type dislocations, which have continuous stress components across the dislocation barrier. The more general type of dislocation, first considered by Mann (66) and described in Section 1 of this chapter, does not have continuous stress components across the barrier and is excluded here.
Then integrating term by term gives the following result,

\[
\frac{\phi}{r} = \frac{\theta}{2} \int_0^r a_0(\tau)d\tau + \sum_{i=1}^\infty \left[ \int_1^r b_i(\tau)d\tau \left( \frac{\sin \theta}{i} - \frac{\cos \theta}{i} \right) \right] + \int_1^r f(\tau)d\tau + g(\theta),
\]

which may be rewritten as follows:

\[
\phi = r \theta F(\tau) + r G(\theta) + \psi(\tau, \theta),
\]

where \( F(\tau) \) and \( G(\theta) \) have continuous second derivatives and \( \psi(\tau, \theta) \) is periodic in \( \theta \) and has continuous second partial derivatives. This represents the necessary and sufficient condition for the shear stress to be periodic.

In view of the periodicity requirement for the tangential stress and its expression in terms of the stress function, Equation 2-3, the body-force function \( V_\theta(\tau, \theta) \) must be periodic. Furthermore, it is necessary that

\[
\frac{d^2}{dr^2} [rF(\tau)] = 0.
\]

Thus,

\[
F(\tau) = C + \frac{D}{r},
\]

where \( C \) and \( D \) are constants.

Therefore, Equation 2-13 now takes the form

\[
\phi = Cr\theta + D\theta + rG(\theta) + \psi(\tau, \theta).
\]
This equation is both necessary and sufficient to assure periodicity of $\sigma_{r\theta}$ and $\sigma_\theta$.

Placing the above expression for the stress function $\phi$ into the expression for the radial stress, Equation 2-3, gives

$$h\sigma_r = \frac{C\theta}{r} + \frac{G(\theta)}{r} + \frac{1}{r} \frac{d^2G(\theta)}{d\theta^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + V_r(r, \theta).$$

Thus, periodicity of $\sigma_r$ requires that $C$ be zero and that

$$V_r(r, \theta) \quad \text{and} \quad \frac{d^2G(\theta)}{d\theta^2} + G(\theta)$$

be periodic. In order for the latter to be periodic in $\theta$, it must be capable of being expanded in a Fourier series in $\theta$ as follows:

$$\frac{d^2G(\theta)}{d\theta^2} + G(\theta) = \frac{A_0}{2} + \sum_{i=1}^{\infty} (A_i \cos i \theta + B_i \sin i \theta),$$

where $A_i$'s and $B_i$'s are constants. The complete solution of the above differential equation is

$$G(\theta) = P \cos \theta + Q \sin \theta + \frac{1}{2} A_0 + \frac{1}{2} A_1 \theta \sin \theta + \frac{1}{2} B_1 \theta \cos \theta + \sum_{i=1}^{\infty} \left( \frac{A_i \cos i \theta + B_i \sin i \theta}{1 - i^2} \right).$$
Now the $P$, $Q$, $A_0$, $A_1$, and $B_i$ terms are periodic and thus can be included in $\psi$. Therefore, the most general form of the stress function which guarantees the periodicity of the stresses is as follows:

$$\phi = C_0 \theta + C_1 r \theta \sin \theta + C_2 r \theta \cos \theta + \psi(r, \theta), \quad 2-15$$

where the $C$'s are constants and $\psi$ is periodic in $\theta$.

Equation 2-15 can be rewritten as follows:

$$\phi = C_0 \theta + C_1 r \theta \sin \theta + C_2 r \theta \cos \theta + \sum_{p=0}^{\infty} R_p(r) \cos p \theta + \sum_{p=1}^{\infty} S_p(r) \sin p \theta, \quad 2-16$$

where $R_p(r)$ and $S_p(r)$ are arbitrary functions of the radius.

In addition to consideration of the periodicity of the stresses, the stresses must be bounded at all points on and within the boundary of the region occupied by the disk, except in the case of a singularity of load. In the case of a load singularity, such as a concentrated force at the center of a disk, for example, the point of load application cannot be considered to be a point of the region, strictly speaking. Then, under this understanding, the member becomes a doubly-connected one, rather than a singly-connected one. For a multiply-connected region, displacements must be considered, as described later. The requirement that the stresses remain bounded on and within the disk boundaries places certain restrictions on the functions $R_p(r)$ and $S_p(r)$ appearing in Equation 2-16.
Complementary solution. Rather than discuss displacement considerations next, it is now desirable to solve the compatibility equation, Equation 2-11, which is a linear partial differential equation. Since it is linear, its general solution is the sum of the complementary solution and a particular solution. The complementary solution is the general solution of the homogeneous equation obtained by setting the right-hand side of Equation 2-11 equal to zero. Here only solutions of the form of the terms in Equation 2-16 will be considered.

Considering first the term \( C_0 \theta \), substitution shows that it satisfies the equation without restriction. Also, by putting this term into Equation 2-3, it is found that it gives rise to only a shear stress, which produces a resultant axial twisting moment.
Next the term $C_1 r \theta \sin \theta$ is considered. Since it satisfies the following integro-differential equation

$$\left[2 \int_0^{2\pi} \left( \frac{\partial \phi}{\partial r} - \frac{\phi}{r} \right) \cos \theta + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \right]_0^{2\pi} \bigg|_{r=a} = \Sigma F_x,$$

it produces a nonzero horizontal-force resultant $\Sigma F_x$ on a boundary $r = a$ of the disk. The term $C_2 r \theta$ analogously gives a vertical-force resultant.

The term $C_1 r \theta \sin \theta$ satisfies the compatibility equation, Equation 2-11, only when $n = 0$ or when $n = 1 - (e/v)$. When neither of these conditions are satisfied, terms of the form $r^{-3} \cos \theta$ remain. This

---

Equation 2-17 is derived as follows. Figure 5 shows a circular disk with its boundary subjected to some radial-pressure distribution $p(\theta)$ acting as shown. At the boundary, the only stresses acting are $\sigma_r$ and $\sigma_r \theta$. Thus, the resultant horizontal force is

$$\Sigma F_x = h_a \phi \int_0^{2\pi} p(\theta) \cos \theta d\theta = h_a \phi \left[ \sigma_r \cos \theta + \sigma_r \theta \sin \theta \right]_0^{2\pi} a d\theta,$$

where $h_a$ = thickness at radius $a$. Putting the expressions for $\sigma_r$ and $\sigma_r \theta$ into this equation gives

$$\phi \left[ \left( \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + V_r \right) \cos \theta - \frac{\partial^2 \phi}{\partial r \partial \theta} (\frac{\phi}{r}) \sin \theta \right]_r=a \ d\theta = \frac{\Sigma F_x}{a}.$$

Then, partial integration and simplification result in Equation 2-17.

In passing, it is worthwhile to note that for the uniform-thickness case, polar-orthotropic or isotropic, a solution of the form

$$r^{-1} + \lambda \theta \cos \theta; \lambda \neq 0,$$

given by Carrier (73), is not a solution of the compatibility equation. Also, it does not lead to periodic stresses.
Figure 5. Stress components and loading acting on the external boundary of a circular disk of radius $a$. 
suggests the possibility of combining with another term leading to a remainder of the same form in such a proportion that the compatibility equation is satisfied by the combination of the two terms. The only other stress-function term which results in periodic stresses and which has this form of remainder is \( D_1 r \ln r \cos \theta \). However, when \( n = 0 \) or when \( n = (1 + e + 2c - 2v)/(1 - v) \), the remainder vanishes. This, of course, means that under these latter conditions \( D_1 r \ln r \cos \theta \) is an independent solution of Equation 2-11. All of the above cases, \( n = 0 \), \( n = 1 - (e/v) \), and \( n = (1 + e + 2c - 2v)/(1 - v) \), are the exceptional cases; when none of them are met, the combination \( C_1 r \theta \sin \theta \) + \( D_1 r \ln r \cos \theta \) satisfies the compatibility equation, provided that

\[
D_1 = \frac{-2(e - v + vn)}{1 + e + 2(c - v) - (1 - v)n} C_1.
\]

2-18

It is important to mention that the term \( r \ln r \cos \theta \), like the term \( r \theta \sin \theta \), gives a nonzero horizontal-force resultant on a boundary, since when it is substituted into the left-hand side of Equation 2-17, a nonzero constant \((2\pi)\) remains. This means that so long as either of these terms appears in the stress function, there is a nonzero horizontal-force resultant on a boundary. (The case of a nonzero vertical-force resultant is handled analogously by the terms \( r \ln r \sin \theta \) and \( r \theta \cos \theta \)). It has already been shown that for all cases of thickness variation (i.e., values of the constant \( n \)), the three exceptional cases and the general case, one (or both) of the terms is present.
Recapitulating, the terms $C_1 r \theta \sin \theta$ and $D_1 r \ln r \cos \theta$ satisfy the compatibility equation for the cases listed, which include all possible values of $n$, in the following manner:

<table>
<thead>
<tr>
<th>Case</th>
<th>Remarks on Solution of Equation 2-11</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 0$</td>
<td>The $C_1$ and $D_1$ terms are both independent solutions, i.e., $C_1$ and $D_1$ are both arbitrary.</td>
</tr>
<tr>
<td>$n = 1 - (e/v)$</td>
<td>The $C_1$ term is an independent solution and the $C_1$ term is not, i.e., $C_1$ is arbitrary and $D_1 = 0$.</td>
</tr>
<tr>
<td>$n = (1 + e + 2c - 2v)/(1 - v)$</td>
<td>The $D_1$ term is an independent solution and the $C_1$ term is not, i.e., $D_1$ is arbitrary and $C_1 = 0$.</td>
</tr>
<tr>
<td>None of the above</td>
<td>The sum of the $C_1$ and $D_1$ terms is an independent solution, provided that $D_1$ satisfies Equation 2-18.</td>
</tr>
</tbody>
</table>

Substitution of the next class of $\phi$ term, a series of the form

$$
\phi = \sum_{p=0}^{\infty} R_p (r) \cos p \theta ,
$$

leads to the same result as that obtained by using the analogous sine series. Thus, for brevity, only the cosine series is written.
Multiplying Equation 2-3 by \( r^4 \), setting its right-hand side equal to zero, and putting in \( \phi \) from Equation 2-19 gives

\[
r^4 \frac{R}{p} + 2(1 = n) r^3 \frac{R'}{p} + [n^2 - vn + n - e - 2(c - v)p^2] \frac{R''}{p} + [e + vn - 2(c - v)p^2] (1 - n) \frac{R'}{p}
\]

\[- [2(c - v)(1 - n) - e(n - 2) - v(n - 1) - ep^2] p^2 \frac{R}{p} = 0, \tag{2-20}
\]

where primes denote differentiations with respect to \( r \).

Since Equation 2-20 is an ordinary linear differential equation of the Euler type, the fundamental solution is of the form \( r^\lambda \), where \( \lambda \) is a constant. Putting this form into the equation leads to the following fourth-degree algebraic equation:

\[
\lambda^4 + 2(n - 2)\lambda^3 + [n^2 - vn - 5(n - 1) - e - 2(c - v)p^2] \lambda^2
\]

\[
+ [-(1 + v)n^2 + (3 + 2v - e)n + 2(e - 1) + 2(2 - n)(c - v)p^2] \lambda
\]

\[
+ [vn^2 + (e - v)n - 2e + ep^2 + 2(c - v)(n - 1)]p^2 = 0. \tag{2-21}
\]

Equation 2-21 is known as the characteristic equation. Following Conway (49), it is tentatively assumed that the characteristic equation is factorable as follows:

\[
(\lambda + \alpha + \beta_p) (\lambda + \alpha - \beta_p) (\lambda + \alpha + \beta_p) (\lambda + \alpha - \beta_p) = 0, \tag{2-22}
\]
where $\alpha, \beta_{p_1},$ and $\beta_{p_2}$ are constants. Multiplying together the above factors gives

$$\lambda^4 + 4\alpha \lambda^3 + [6\alpha^2 - (\beta_{p_1}^2 + \beta_{p_2}^2)] \lambda^2$$

$$+ 2\alpha[2\alpha^2 - (\beta_{p_1}^2 + \beta_{p_2}^2)] \lambda + (\alpha^2 - \beta_{p_1}^2)(\alpha^2 - \beta_{p_2}^2) = 0. \quad 2-23$$

Equating the coefficient of the $\lambda^3$ term in Equation 2-23 with the corresponding coefficient in Equation 2-21 gives

$$\alpha = \frac{(n-2)}{2} . \quad 2-24$$

Substituting this expression for $\alpha$ and equating coefficients of the $\lambda^2$ terms in Equations 2-21 and 2-23 gives

$$\beta_{p_1}^2 + \beta_{p_2}^2 = \frac{1}{2} (n-2)^2 + (1+\nu)n + e - 1 + 2(c-\nu)p^2 . \quad 2-25$$

A similar procedure for coefficients of $\lambda$ terms results in exactly the same relationship. This redundancy verifies the tentative assumption that only three distinct constants are needed in Equation 2-22.

Equating the constant terms in Equations 2-21 and 2-23 yields

$$[\nu n^2 + (e-\nu)n - 2e + \epsilon p^2 + 2(c-\nu)(n-1)] p^2$$

$$= [(\frac{n-2}{2})^2 - \beta_{p_1}^2 ][(\frac{n-2}{2})^2 - \beta_{p_2}^2] . \quad 2-26$$

Although this expression constitutes a second relationship between $\beta_{p_1}^2$ and $\beta_{p_2}^2$, it is not in a form convenient for calculation. Noting that
Equation 2-25 contains the sum of these quantities, it is convenient to develop an expression for their differences. Toward this end, it is noted that

\[(A - B)^2 = A^2 - 2AB + B^2 = (A + B)^2 - 4AB = D^2 - 4C ,\]

where \(A \equiv \beta_{p_1}^2\), \(B \equiv \beta_{p_2}^2\), \(C \equiv AB\), and \(D\) denotes the right-hand side of Equation 2-25. Then, Equation 2-26 can be rewritten as

\[E = (\frac{n - 2}{2})^4 - (\frac{n - 2}{2})^2 D + C ,\]

where \(E\) denotes the left-hand side of Equation 2-26. Substituting the above expression into the one preceding it gives

\[(A - B)^2 = D^2 - (n - 2)^2 D - 4E + (\frac{n - 2}{2})^2 .\]

Now, substituting for the quantities \(D\) and \(E\) in this expression gives

\[\beta_{p_1}^2 - \beta_{p_2}^2)^2 = [(1 + \nu)n + e - 1]^2 - 4np^2[vn + e - \nu + (1 - \nu)(c - \nu) + 4p^2[(e + 1)(c - \nu) + (c - \nu)^2 p^2 + 2e - ep^2].\]

2-27

The four roots of Equation 2-21 are given by

\[\lambda_{p_{1,3}} = -\frac{n - 2}{2} + \beta_{p_1} + \beta_{p_2}, \quad \lambda_{p_{2,4}} = -\frac{n - 2}{2} + \beta_{p_2} .\]

2-28

Then, provided that there are no multiple roots, i. e., repeated roots in the sets 2-28, the general expression for \(R_p\) is as follows:

\[R_p = A_{p_1} r_{p_1} + A_{p_2} r_{p_2} + A_{p_3} r_{p_3} + A_{p_4} r_{p_4} ,\]

2-29

where \(A_{p_1}, \ldots, A_{p_4}\) are arbitrary constants which must be determined from the boundary conditions of the particular problem.
In the case of multiple roots, assuming that \( \lambda_{p_1} = \lambda_{p_2} \) and that
\[
\lambda_{p_3} = \lambda_{p_4},
\]
the expression for \( R \) becomes
\[
R = A r r_{p_1} + A r r_{p_2} \ln r + A r r_{p_3} + A r r_{p_4} \ln r. \quad 2-30
\]

Particular solution. -- In order to obtain a general expression for the particular solution, it is convenient to use the following Fourier power series to approximate the body-force functions
\[
V_r = \sum_{i,j} C_{ij} r^i \cos j \theta, \quad V_\theta = \sum_{s,t} C_{st}^i r^s \cos t \theta \quad 2-31
\]
plus similar sine terms. Then the right-hand side of Equation 2-11 becomes
\[
\sum_{i,j} [v(i - 1) + v(n + 1)(n + 2i)] C_{ij} r^{i-2} \cos j \theta - \sum_{s,t} [s(s + 1 + 2n + v)] C_{st}^i r^{s-2} \cos t \theta. \quad 2-32
\]
Dimensional analysis of Equation 2-11 with expression 2-32 as its right-hand side shows that its particular solution is of the form
\[
\phi_p = \sum_{i,j} B_{ij} C_{ij} r^{2+i} \cos j \theta + \sum_{s,t} B_{st} C_{st}^i r^{2+s} \cos t \theta, \quad 2-33
\]
where the constants \( B_{ij} \) and \( B_{st} \) are determined by substitution to be given by
\[
B_{ij} K_{ij} = v(n^2 + n + 2ni + i + i^2) - e(1 + i - j^2), \quad 2-34
\]
\[
B_{st}^i K_{st} = -n^2 - (1 + v - 2s)n + s^2 + (1 + v)s - vt^2, \quad 2-35
\]
where

\[ K_{ij} = (2 + i)(1 + i)[n^2 + (1 - v + 2i)n + i^2 + i - e] + (e + vn)(1 - n)(2 + i) \]

\[- 2(c - v)[(n + i)(2 + i) + 1 - n]j^2 + [e(n - 2) + vn(n - 1) + ej^2]j^2, \quad 2-36\]

and \( K_{st} \) is given by the same expression with \( s, t \) replacing \( i, j \).

**Limitations on the solution due to displacement considerations.**

Although many degrees of connectivity are possible in a disk, such as in a multi-perforated disk, this study is limited to disks possessing axisymmetry and thus includes only solid disks (which are simply connected) and concentric annuli (which are doubly connected).

For a simply-connected region, the displacements, like the stresses must be single-valued. In this case, an investigation of the limitations imposed on the stress function due to the requirement of periodicity of displacements (in similar fashion to that used for the stresses) shows that all of the solutions of the compatibility equation which lead to bounded stresses at all points on and within the disk would have the required periodicity. Thus, this kind of investigation of displacements does not place any additional limitations on the forms of the general solution.

For a multiply-connected region, a displacement investigation of the kind described above would only guarantee that if the displacements are periodic, they would have the proper periodicity. As previously mentioned, in multiply-connected bodies with continuous stress
and strain components, there exists the possibility of Volterra-type dislocations. The three kinds of plane Volterra-type dislocations possible in a doubly-connected disk are illustrated in Figure 6 for a concentric circular annular disk, referred to hereafter as an annulus for brevity.

A radial dislocation can be thought of as being formed by cutting an annulus radially and then cementing (or welding) the edges of the cut together with an offset as shown in Figure 6 (a). The amount of offset is the magnitude of the dislocation. A tangential dislocation can be considered to be formed by removing a parallel-sided slice from an annulus and then cementing the edges together with no offset, Figure 6 (b). The width of the slice removed is the magnitude of this kind of dislocation. Finally, a rotational dislocation is made by radially cutting an annulus and then cementing the edges together so that there is an angular discontinuity in lines which were originally tangential, Figure 6 (c). The magnitude of the angular discontinuity is considered to be the magnitude of the dislocation.

It is to be mentioned that due to the manner in which dislocations are formed, some material may necessarily have to be removed before the cut is cemented together. Also, certain initial stresses are set up in the annulus. The distribution of these stresses depends only upon the kind, magnitude, and location of the dislocations. The
Figure 6. The three kinds of plane Volterra-type dislocations possible in a doubly-connected body.
statement that a member is free of dislocations is equivalent to stating that it contains no initial stresses.

Using the strain-displacement relations, the stress-strain relations, the relations between the stress components and the stress function, and the definition of the compliance results in the following general expressions for the displacements:

\[
eu = \int (-v \frac{\partial^2 \phi}{\partial r^2} + \frac{e}{r} \frac{\partial \phi}{\partial r} + \frac{e}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}) + eV_r - \nu V_\theta \right) \right) \right) S \right) \right) d \theta + f_1(\theta), 2-37
\]

\[
ev = \int \left( \frac{\partial^2 \phi}{\partial r^2} - \frac{\nu}{r} \frac{\partial \phi}{\partial r} - \frac{\nu}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + V_\theta - \nu V_r \right) d \theta - \int \eu d \theta + g_1(r), 2-38
\]

\[
2e \omega = \frac{\partial (ev)}{\partial r} - \frac{1}{r} \frac{\partial (eu)}{\partial \theta}, 2-39
\]

where \( f_1(\theta) \) and \( g_1(r) \) are functions of integration.

Upon inserting various stress functions \( \phi \) and compliance distributions \( S \) into the preceding equations, inspection of the resulting expressions for the displacements can be used to determine whether or not the displacements are multi-valued. Furthermore, evaluating the displacements given by these equations at two values of \( \theta \) separated by \( 2\pi \) radians and subtracting the displacement corresponding to the smallest value of \( \theta \) from the other gives the magnitudes of the three
kinds of Volterra-type plane dislocations corresponding to any stress function in a circular annulus with any arbitrary radially-varying compliance. Conversely, if the physical conditions stipulate that no initial stresses are present, by setting the left-hand side of these equations equal to zero, they can be used to eliminate possible solutions which give rise to dislocations. Equations 2-37 through 2-39 can be considered to be the varying-thickness version of Timpe's equations (61).

The following expression is obtained by equating the expressions for the shear strain calculated by means of the displacements $u$ and $v$ and via the shear stress:

$$
\frac{1}{r} \frac{\partial (eu)}{\partial \theta} + \frac{\partial (ev)}{\partial r} - \frac{ev}{r} = -2cS \frac{\partial^2 (\phi)}{\partial r \partial \theta} \left( \frac{\phi}{r} \right). \tag{2-40}
$$

This expression is used to obtain differential equations for determining the functions of integration $f_1(\theta)$ and $g_1(r)$, using exactly the same procedure as that used in the case of uniform thickness (85).

Placing the power-function compliance expression into Equations 2-37 through 2-40 and simplifying gives the following results:

$$
eu = -vr^n \frac{\partial \phi}{\partial r} + (e + vn) r^{n-1} \phi + [(e + vn)(1-n)
+ e \frac{\partial^2}{\partial \theta^2} \int r^{n-2} \phi \, dr + \int r^n (eV_r - vV_\theta) \, dr + f(\theta), \tag{2-41}
$$
\[ \frac{ev}{S_o} = \int (r^{n+1} \frac{\partial^2 \phi}{\partial r^2} - \nu r^{n-1} \frac{\partial^2 \phi}{\partial \theta^2} - (e + \nu n)r^{n-1}\phi \]

\[ - [(e + \nu n)(1 - n) + e \frac{\partial^2}{\partial \theta^2}] \int r^{n-2} \phi \, dr + r^{n+1}(V_\theta - \nu V_r) \]

\[ + \int r^n(\nu V_\theta - eV_r) \, d\theta - f(\theta) d\theta + g(r), \quad 2-42 \]

\[ \frac{2e\omega}{S_o} = \frac{\partial (ev/S_o)}{S_o} - \frac{1}{r} \frac{\partial (eu/S_o)}{\partial \theta}, \quad 2-43 \]

\[ \frac{1}{r} \frac{\partial (eu/S_o)}{\partial \theta} + \frac{\partial (ev/S_o)}{\partial r} - \frac{ev/S_o}{r} = -2cr^n \frac{\partial^2}{\partial r \partial \theta} \left( \frac{\phi}{r} \right), \quad 2-44 \]

where \( f(\theta) \equiv f_1(\theta)/S \) and \( g(r) \equiv g_1(r)/S \).

To determine the displacements corresponding to each of the terms of \( \phi \) in Equation 2-16 which are solutions to Equation 2-11, each term is substituted into the above equations. To illustrate the procedure, the first part of Case 1 (\( \phi = \theta \)) for \( n \neq 1,2 \) is treated in detail below.

Use of Equations 2-41 and 2-42 gives the following results:

\[ \frac{eu}{S_o} = f(\theta), \quad \frac{ev}{S_o} = -\int f(\theta) d\theta + g(r). \]

Then, application of Equation 2-44 gives

\[ \frac{1}{r} \frac{df}{d\theta} + \frac{dg}{dr} + \frac{1}{r} \int f(\theta) d\theta - \frac{g}{r} = 2cr^{n-2}. \]
The above expression can easily be separated into functions of \( r \) only and of \( \theta \) only, respectively, to give the following differential equations:

\[
\frac{dg}{dr} - \frac{g}{r} = 2cr^{n-2}, \quad \frac{d^2f}{d\theta^2} + f = 0,
\]

which have the following general solutions:

\[
g = Hr - \frac{2c}{2-n} r^{n-1}, \quad f = F \cos \theta + G \sin \theta.
\]

After inserting the above expressions for \( f(\theta) \) and \( g(r) \) back into the expressions for the displacements \( u \) and \( v \), the results are:

\[
\frac{eu}{S_0} = F \cos \theta + G \sin \theta,
\]

\[
\frac{ev}{S_0} = -\frac{2c}{2-n} r^{n-1} - F \sin \theta + G \cos \theta + Hr.
\]

Now, use of Equation 2-43 gives the following expression for the rotational displacement \( \omega \):

\[
\frac{2e\omega}{S_0} = \frac{1-n}{2-n} 2cr^{n-2} + \frac{1}{r} (F \sin \theta - G \cos \theta) + H.
\]

For completeness, the results for all of the possible stress functions which give rise to stresses which are periodic in \( \theta \) and to
strains which are compatible, including the part of Case 1 discussed above, are presented below:

**Case 1.** \( \phi = \theta \)

\[
\frac{eu}{S_0} = F \cos \theta + G \sin \theta ,
\]

\[
\frac{ev}{S_0} = - \frac{2c}{2-n} r^{n-1} - F \sin \theta + G \cos \theta + Hr , \quad (n \neq 1, 2)
\]

\[
\frac{ev}{S_0} = e + \nu - 2c - F \sin \theta + G \cos \theta + Hr , \quad (n = 1)
\]

\[
\frac{ev}{S_0} = 2c r \ln r - F \sin \theta + G \cos \theta + Hr , \quad (n = 2)
\]

\[
\frac{2ew}{S_0} = \frac{1-n}{2-n} 2c r^{n-2} + \frac{1}{r} (F \sin \theta - G \cos \theta) + H \quad (n \neq 2)
\]

\[
\frac{2ew}{S_0} = 2c (1 + \ln r) + \frac{1}{r} (F \sin \theta - G \cos \theta) + H . \quad (n = 2)
\]

**Case 2.** \( \phi = r(\theta \sin \theta + \beta \ln r \cos \theta) ; \quad \beta = \frac{2(e - \nu + \nu n)}{(1-\nu)n - (1 + e + 2c - 2\nu)} ; \quad n \neq 0^{9} ;
\]

\( n \neq (1 + e + 2c - 2\nu)/(1 - \nu) , \) except when also \( n = 1 - (e/\nu) . \)

\[
\frac{eu}{S_0} = [2e + (e - \nu)\beta] \frac{r^n}{n} \cos \theta + F \cos \theta + G \sin \theta ,
\]

---

\(^{9}\)The uniform-thickness case \((n = 0)\) is given by Timoshenko and Goodier (86).
\[
\frac{ev}{S_o} = \left[2(e + \nu n) + (e + \nu n - \nu - n)\beta\right] \frac{r}{n} \sin \theta \\
- F \sin \theta + G \cos \theta + H ,
\]

\[
\frac{2\omega}{S_o} = \left[2(e - \nu n - n^2) + (e - \nu - \nu n - \nu n + \nu n^2 - n^2)\beta\right] \frac{r}{n} \sin \theta \\
+ \frac{1}{r} (F \sin \theta - G \cos \theta) + H .
\]

**Case 3.** \( \phi = r^\lambda \cos p \theta, \quad (p \neq 0) \)

\[
\frac{eu}{S_o} = \left[\frac{\nu \lambda (1 - \lambda) + e (\lambda - p^2)}{\lambda + n - 1}\right] r^{\lambda + n - 1} \cos p \theta + F \cos \theta + G \sin \theta ,
\]

\[
\frac{ev}{S_o} = \left[\frac{\nu \lambda (1 - \lambda) + e (\lambda - p^2)}{\lambda + n - 1}\right] r^{\lambda + n - 1} \sin p \theta \\
- F \sin \theta + G \cos \theta + H ,
\]

\[
\frac{2\omega}{S_o} = \left[\frac{(\lambda^2 - \lambda + \nu p^2)(\lambda + n - 1) - \nu n \lambda + e (\lambda - p^2)}{\lambda + n - 1}\right] r^{\lambda + n - 2} \sin p \theta \\
+ \frac{1}{r} (F \sin \theta - G \cos \theta) + H .
\]

\[
\frac{eu}{S_o} = \left\{\begin{array}{l}
\left[\frac{\nu^2 (1 - \nu p^2) + n (\nu + \nu p^2 - \nu \epsilon^2 - 2 \epsilon \nu p^2 - 2 \epsilon - 1)}{\epsilon^2 - \nu + \epsilon^2}
+ F \cos \theta + G \sin \theta ,
\end{array}\right.
\end{array}
\]

\[
\lambda = 1-n, \quad (p \neq 1)
\]
\[
\frac{e_v}{S_o} = \left\{ [(e - \nu)(1 - p^2)p^2 + np^2(e + \nu n + 1 - n + 2c - 2\nu)] \frac{1}{1 - p^2} \right. \\
\left. + \left[ ep^2 - (e + \nu n)(1 - n) \right] \ln r \right\} \frac{\sin \theta \theta}{p} - F \sin \theta + G \cos \theta + H, \\
\right. \\
\lambda = 1 - n, \\
p \neq 1
\]

\[
\frac{2e_\omega}{S_o} = \{(e + \nu n)(n - 1) + (\nu - \nu^2 p + e - \nu^2 p + \nu^2 p^2) - (e + \nu n)(1 - n) - \nu^2 p^2 \ln r \} \\
\\cdot \frac{\sin \theta \theta}{p} + \frac{1}{r} (F \sin \theta - G \cos \theta) + H.
\]

\[
\frac{e_u}{S_o} = \frac{n}{2} [(1 - \nu) n - 1 - \epsilon] \sin \theta + F \cos \theta + G \sin \theta, \\
\lambda = 1 - n, \\
p = 1
\]

\[
\frac{e_v}{S_o} = \frac{n}{2} [(1 - \nu) n - 1 - \epsilon] \cos \theta + \frac{n}{2} [(1 + \nu) n - 1 + \epsilon + 2\nu - F] \sin \theta + G \cos \theta + H, \\
\lambda = 1 - n, \\
p = 1
\]

\[
\frac{2e_\omega}{S_o} = -\frac{n}{2} [(1 - \nu) n - 1 - \epsilon] r^{-1} (\theta \cos \theta + \sin \theta) \\
+ \frac{1}{r} (F \cos \theta - G \sin \theta) + H.
\]

**Case 4. \( \phi = r^\lambda \)**

\[
\frac{e_u}{S_o} = \frac{e + \nu n - \nu}{\lambda + n - 1} \lambda^\frac{r + n - 1}{1 - \epsilon} + F \cos \theta + G \sin \theta, \\
\lambda \neq 1 - n
\]

\[
\frac{e_v}{S_o} = (\lambda - 1 - \frac{e + \nu n}{\lambda + n - 1}) \lambda^\frac{r + n - 1}{\epsilon} - F \sin \theta + G \cos \theta + H, \\
\lambda \neq 1 - n
\]

\[
\frac{2e_\omega}{S_o} = [(\lambda - 1)(\lambda + n - 1) - (e + \nu n)] \lambda^\frac{r + n - 2}{\epsilon} \\
+ \frac{1}{r} (F \sin \theta - G \cos \theta) + H.
\]
\frac{\epsilon u}{S_0} = (\nu + n)(n - 1) + F \cos \theta + G \sin \theta, \\
\frac{\epsilon v}{S_0} = -F \cos \theta + G \sin \theta + H r, \\
\frac{2e\omega}{S_0} = \frac{1}{r} (F \sin \theta - G \cos \theta) + H.

Because of the infrequency of occurrence of double roots in practical cases of varying thickness, this contingency is not explored here.

4. Application: Example 1 - Polar-Orthotropic Disk Rotating about an Eccentric Normal Axis

The problem here is to determine the stresses in a circular disk mounted concentrically on a circular shaft rotating at constant speed about an axis parallel with, but eccentric to, the axis of the shaft and disk. This is encountered, for example, when clearances in shaft-support bearings are excessive or when the shaft is sufficiently flexible to permit significant deflection

The published solutions for eccentrically-rotating disks of unit-thickness, discussed on page 50, treated only solid disks and assumed

\textit{Substitution of the root } \lambda = 1 - n \textit{ into the characteristic equation, Equation 2-21, shows that this is a root only if } \nu = -e/n, \textit{ which is not of practical interest.}

\textit{Obviously, if the disk overhangs the bearings or if it is non-symmetrically located with respect to the bearings, the disk also undergoes an angular rotation. This tilting action induces bending stresses and is treated in Chapter 4, Section 4.
that the centripetal force was concentrated at the center of rotation.
Thus, they are more applicable to disks rotating with an intentional eccentricity\(^{12}\) than to the problem considered here.

Figure 7 shows an axial view of the disk and the geometrical parameters involved. The directions of the centrifugal body-force resultant and its components at a typical point on the disk are also shown in the figure. The resultant force per unit volume is

\[ F = \gamma \Omega^2 r_c / g , \]

where \( \gamma \) = specific weight of disk material, \( g \) = gravitational acceleration, \( \Omega \) = rotational velocity, and \( r_c \) = distance from center of rotation.

Then, after some geometrical manipulations, the respective radial and tangential components of \( F \) are found to be given by

\[ F_r = (\gamma \Omega^2 / g)(r - k \cos \theta) , \]
\[ F_\theta = (\gamma \Omega^2 / g)(k \sin \theta) , \]

where \( k \) = eccentricity. It is noted that for the familiar concentrically-rotating disk, \( k = 0 \); then, \( F_r \) is independent of \( \theta \) and \( F_\theta = 0 \).

\(^{12}\) The main application of this type which comes to the author's mind is in the problem of determining the centrifugal stresses in a harmonic-motion cam, which is circular in planform, mounted on a very-small-diameter shaft rotating at high speed.
Figure 7. Geometrical relationships and directions of forces for a typical point $P$ in an eccentrically-rotating disk.
Using the above expressions, Equations 2-4 are integrated to give these expressions for the body-force functions:

\[ V_\theta = -\frac{\gamma \Omega h_o}{g} kr^{1-n} \cos \theta + X_r(r), \quad 2-48 \]

\[ V_r = \frac{\gamma \Omega h_o}{g} \left( \frac{r^{2-n}}{3-n} - \frac{2k}{2-n} \frac{r^{1-n} \cos \theta}{r} + \frac{1}{r} \int X_r(r) \, dr + \frac{1}{r} X_\theta(\theta) \right), \quad 2-49 \]

where \( X_\theta(\theta) \) and \( X_r(r) \) are functions of integration.

Since this problem has not been solved previously for the uniform-stiffness, isotropic case, it is of interest to do so now. In the uniform-thickness case, it is always possible to make the two body-force functions identical by proper selection of \( X_\theta \) and \( X_r \). Therefore, by equating the right-hand sides of Equations 2-48 and 2-49, it is found that in order to have identical body-force functions, \( X_\theta = 0 \), and

\[ X_r = \frac{\gamma \Omega h_o}{g} \frac{r^2}{3} + \frac{1}{r} \int X_r(r) \, dr. \]

Thus,

\[ X_r = \frac{\gamma \Omega h_o}{g} \frac{r^2}{2}. \]

Then

\[ V = \frac{\gamma \Omega h_o}{g} \left( \frac{1}{2} r^2 - kr \cos \theta \right). \]

Using Equations 2-33 through 2-36, the particular solution for this case is found to be

\[ \phi_p = \frac{2-v}{16} \frac{\gamma \Omega h_o}{g} r^4. \]
Application of Equations 2-25, 2-27, and 2-28 give the following roots of characteristic equation 2-21: For \( p = 0 \): 0, 0, 2, 2; for \( p = 1 \): -1, 1, 3, 1. Noting the double roots, the complementary solution can be written as

\[
\phi_c/Ch_o = A_{01} + A_{02} \ln r + A_{03} r^2 + A_{04} r^2 \ln r
\]

\[
+ (A_{11} r^{-1} + A_{12} r + A_{13} r^3 + A_{14} r \ln r) \cos \theta + C_1 r \theta \sin \theta ,
\]

where \( C \) is an arbitrary constant to be selected for convenience later.

Consideration of the stresses produced by the stress function in Equation 2-50 shows that the coefficients \( A_{01} \) and \( A_{12} \) do not affect the stresses, so that they may be taken to be zero. Furthermore, a study of the displacements shows that in order for the displacements to be single-valued, \( A_{04} = 0 \) and

\[
A_{14} = -\frac{1}{2} (1 - \nu) C_1.
\]

Combining the particular and complementary solutions, the complete stress function can now be written as

\[
\phi/Ch_o = A_{02} \ln r + A_{03} r^2 - \frac{2 - \nu}{16} r^4
\]

\[
+ (A_{11} r^{-1} + A_{13} r^3) \cos \theta + C_1 r (\theta \sin \theta - \frac{1 - \nu}{2} \ln r \cos \theta) ,
\]

where \( C = \gamma \Omega^2 / g. \)

In order to evaluate the constants of integration in Equation 2-51, the physical boundary conditions must be specified. Assuming the outer
periphery ($r = b$) to be a free edge\footnote{13 It is assumed that the disk does not have a rim, blades, or other masses attached to its outer periphery.}, both the radial stress and the shear stress along it are zero. As is customary in manufacture of turbine disks and compressor impellers, it is assumed here that the disk is fitted to the shaft by means of a press or shrink fit. Thus, under static conditions (i. e., no rotational velocity) the radial stress at the inner boundary is equal to the negative of the contact pressure at the fit, which is assumed to be uniform around the circumference. In operation with an eccentricity, the contact pressure is modified, partly because it must provide the centripetal force to react the centrifugal force and partly due to differences between the centrifugal expansions of the disk and of the shaft at their common boundary. The problem of the contact stresses between a thin disk and a long shaft is a three-dimensional one which is quite difficult even when it involves only axisymmetric loading (87). Thus, any assumptions made as to the deformations in two dimensions are understood to be very approximate in character. In view of this, it is assumed here that the radial-pressure distribution consists of a uniform pressure $p$, which can conservatively be assumed to be equal to the static contact pressure, with a superimposed pressure increment, due to centripetal action, proportional to the cosine of the angle $\theta$ in Figure 7. Then the radial-stress

\begin{equation}
\end{equation}
distribution at the inner boundary is given by the following expression:

\[(\sigma_r)_{r=a} = -p(1 + C_p \cos \theta),\]  

where \(C_p\) is a dimensionless constant.

As a second boundary condition at the inner boundary, it is assumed that the shear stress is zero, the friction along this boundary being taken as negligible. These last two boundary conditions were used by Bickley (88) in his analysis of the related problem of determining the stresses in a plate with a hole containing an oversized rivet or pin pulled in the plane of the plate.

To evaluate the pressure coefficient \(C_p\), the resultant centripetal force is equated to the resultant force at the inner boundary as follows:

\[
\frac{\gamma \Omega^2 kV_o}{g} = -\int_0^{2\pi} (-C_p \cos \theta) \cos \theta a_{h_a} d\theta
\]

where \(V_o\) = volume of the disk and \(a_{h_a}\) = disk thickness at the inner radius. Thus, in general,

\[
C_p = \frac{\gamma \Omega^2 kV_o}{\pi gpah_a}
\]  

and for the uniform-thickness disk,

\[
C_p = \frac{C_kb}{p} \frac{1 - \frac{a^2}{b^2}}{a},
\]  

where \(a \leq a/b\).
Now substitution of the unevaluated terms of the stress function into the boundary conditions results in two equations in the constants $A_{01}$ and $A_{03}$ and four equations in the constants $A_{11}$ and $A_{13}$. However, the four equations have two different pairs of identical left-hand sides. Obviously, the four equations can be compatible only if the terms appearing in their right-hand sides are related in such a way that the appropriate right-hand sides are identical. It can be shown that this is a consequence of static equilibrium (61). This gives a relationship for determining the constant $C_1$. An alternate, but equally acceptable, way of determining $C_1$ is to insert the horizontal force resultant $\gamma y^2 kV_o / g$ into the right-hand side of Equation 2-17 and $V_r$ as well as the stress-function terms headed by $C_1$ on the left-hand side.

The resulting expressions for the constants are as follows:

\[
\begin{align*}
A_{02} &= \left(\frac{\nu}{4} - \frac{1 - \nu}{1 - a^2} \frac{P}{Cb^2}\right) a^2 b^4, \\
A_{03} &= \left[\nu(1 + a^2) + \frac{4a^2}{1 - a^2} \frac{P}{Cb^2}\right] b^2 / 8, \\
A_{11} &= -\frac{1 - \nu}{8} \frac{kb^4}{1 + 1/a^2}, \\
A_{13} &= -\frac{1 - \nu}{8} \frac{k}{1 - a^2}, \\
C_1 &= \frac{1}{2} kb^2.
\end{align*}
\]

Insertion of these values into the stress function and thence into the appropriate equations for the stress components, Equations 2-3, gives the stress distributions for various values of the quantity $Cb^2$, eccentricity $k$, and contact pressure $P$. For instance, assuming that
the contact pressure is the minimum required in order to prevent the
disk from coming loose on the shaft under operation with an eccen-
tricity, \( C_p = 1 \) and thus

\[
\frac{p}{C b^2} = \frac{k}{b} \frac{1 - a^2}{a}.
\]

Under this assumption, the tangential-stress distribution is given by
the following expression:

\[
\frac{\sigma_\theta}{C b^2} = -\left(\frac{\nu}{4} a^2 - \frac{k}{b} a\right)\left(\frac{b}{r}\right)^2 + \frac{\nu}{4} \left(1 + a^2\right) + \frac{k}{b} a - \left(\frac{4 - 3\nu}{4}\right)\left(\frac{r}{b}\right)^2
\]

\[\quad - \left[\left(\frac{b}{r}\right)^3 + \frac{b}{r} + \frac{2 + a^2}{1 - a^2}\left(\frac{r}{b}\right)\right] \frac{1 - \nu}{4} \frac{k}{b} \cos \theta.\]

In particular, for \( \nu = 1/3 \), \( a = 0.1 \), and \( k/b = 0.001 \), the highest stress
at the inner radius, which is usually the most critical in design, is
0.00688 \( C b^2 \). For a steel disk (\( \gamma = 0.283 \) lb/cu. in.), 30 inches in dia-
meter (\( b = 15 \) inches) and running at 6000 rpm (\( \Omega = 628 \) radians/sec),
this amounts to a stress of only 1790 psi.

Before turning attention to varying-thickness disks, it is inter-
esting to note that for problems involving boundary values (stresses or
displacements) varying with \( \cos \theta \) (i.e., for \( p = 1 \)) in uniform-thickness
disks with any arbitrary values of the elastic constants \( e \), \( c \), and \( \nu \)
(i.e., any arbitrary polar orthotropy), the complementary solution \( \Phi_c \)
always contains terms of the form $r \cos \theta$ and $r \ln r \cos \theta$, since unity is a double root of the characteristic equation under these conditions.

As an example of a varying-thickness, polar-orthotropic disk, a homogeneous disk having a thickness varying with $r^{-1}$ (i.e., $n = 1$), an orthotropic ratio $e = \frac{1}{4}$, and Poisson's ratio $\nu = \frac{1}{2}$, is next considered. (Although a value of $\frac{1}{2}$ is rather high, in fact, the theoretical maximum, for Poisson's ratio for an isotropic material, for a polar-orthotropic material with $e < 1$, this is a reasonable value.) In order to obtain an estimate of a reasonable value of the constant $c$, Equation 1-23, as proposed by Lang, is used, giving a value of $c = 5/8$.

Then, taking the functions $X_\theta$ and $X_r$ each to be zero, the body-force functions are found to be, by Equations 2-48 and 2-49,

$$V_\theta = -\mathrm{Ch}_0 k \cos \theta, \quad V_r = \mathrm{Ch}_0 \left(\frac{1}{2} r - 2k \cos \theta\right),$$

where $C = \frac{\gamma \Omega^2}{g}$. Using the results of the preceding section, the particular solution is found to be

$$\phi_p = \mathrm{Ch}_0 \left(\frac{-5}{126} r^3 + \frac{1}{2} kr^2 \cos \theta\right).$$

The roots of the characteristic equation are, for the symmetrical part ($p = 0$), $-\frac{1}{2}$, 0, 3/2, 1, and for the nonsymmetrical part ($p = 1$), -0.618, 0, 1.618, 1. Then the complementary solution is of the form
\[
\frac{\phi_c}{C} = A_{01}r^{-1/2} + A_{02} + A_{03}r^{3/2} + A_{04}r
\]

\[
+ (A_{11}r^{-0.618} + A_{12} + A_{13}r^{1.618} + A_{14}r)\cos \theta
\]

\[
+ C_1 r(\theta \sin \theta + \beta \ln r \cos \theta),
\]

where the constant \( \beta \) is given by

\[
\beta = \frac{2(e - \nu + \nu n)}{(1 - \nu)n - (1 + e + 2c - 2\nu)} = -\frac{1}{2}.
\]

As in the case of uniform-thickness, isotropic disks, two of the coefficients, here \( A_{02} \) and \( A_{14} \), do not affect the stresses and thus may be taken to be zero. Also, an investigation of the displacements shows that for single-valued displacements, \( A_{04} = 0 \) (since \( e + \nu n = 3/4 \neq 0 \)) and \( A_{12} = 0 \) (since \( (1 - \nu)n - (1 + e) - 3/4 \neq 0 \)).

Then, using the same boundary conditions as for the uniform-thickness, isotropic case, the constants of integration are evaluated in the same manner, with the following results:

\[
A_{01} = \left[ \frac{p}{C_b} a^{3/2} - \frac{13}{21} a^2 \left( 1 - a^{3/2} \right) \right] \frac{2b^{7/2}}{1 - a^2},
\]

\[
A_{03} = -\left[ \frac{p}{C_b} a^{3/2} - \frac{13}{21} a^{7/2} \right] \frac{2b^{3/2}}{3(1 - a^2)},
\]

\[
A_{11} = \frac{kb^{2.618}}{1.618 a} \frac{1 - \frac{1}{2} a - (\frac{3}{2} - a)a^{0.618}}{1 - a - 3},
\]

\[
A_{13} = \frac{kb^{0.382}}{0.618 a} \frac{1 - \frac{1}{2} a - (\frac{3}{2} - a)a^{-1.618}}{1 - a^{-2.236}},
\]

\[
C_1 = kb.
\]
Using the same assumption as used previously for the magnitude of the shrink-fit pressure, the tangential-stress distribution can be expressed as follows:

\[ \frac{\sigma_\theta}{Cb^2} = \left[ 2 \frac{k}{b} (1 - a) a^{3/2} - \frac{13}{21} a^2 (1 - a^{3/2}) \right] \frac{3}{2} \frac{(b/r)^{3/2}}{1 - a^2} \]

- \left[ 2 \frac{k}{b} (1 - a) a^{3/2} - \frac{13}{21} (1 - a^{7/2}) \right] \frac{1}{2} \frac{(r/b)^{1/2}}{1 - a^2} + \frac{5}{21} \left( \frac{r}{b} \right)^2

+ \left[ \frac{0.990}{b^2} (A_{11}^{-1.4} 1.618 + A_{13} r 0.318) - \frac{1}{2} \frac{k}{b} \right] \cos \theta.

In particular, for a disk having the same \( a \) and \( k/b \) values as for the uniform-thickness case, the highest tangential stress at the inner radius is equal to 0.01414 \( Cb^2 \), or 3680 psi for a disk of the same density and outside diameter and running at the same speed as the uniform-thickness isotropic one. This is more than twice the highest tangential-stress value for the uniform-thickness, isotropic one. Thus, it is seen that increasing the relative radial stiffness is not beneficial under the conditions assumed in these examples. It is also interesting to note that in neither case does the disk basic thickness \( h_0 \) affect the stress value.

The similarity between the varying-thickness, polar-orthotropic case considered here and the uniform-thickness, isotropic case is remarkable. However, it should be pointed out that for many
combinations of varying thickness and polar orthotropy, only two terms, instead of four, disappear from the original complementary solution due to considerations of null effectiveness on stresses and/or single-valuedness of displacements. This type of problem can be most easily illustrated by the case of a disk mounted on a perfectly rigid shaft. However, the displacement of the disk-shaft boundary is not taken to be zero, since it must be remembered that under loading, the disk center line undergoes a rigid-body motion consisting of a translation \( \Delta \) in the direction of the resultant centrifugal force and no rotation. Then the radial, tangential, and rotational displacements of all points on the disk-shaft intersection are as follows:

\[
\begin{align*}
  u &= -\Delta \cos \theta, \\
  v &= \Delta \sin \theta, \\
  \omega &= 0.
\end{align*}
\]

From these and from the general expressions for the displacements due to the various types of stress functions, as given in the preceding section, it is seen that the displacement integration constants \( G \) and \( H \) appearing in the general expressions are zero, and three relations remain. These three relations provide a means for determining the displacement constant \( F \) as well as the required two additional relations for determining the stress-function coefficients.

Finally, it should be noted that the value of \( \Delta \) used in Equation 2-60 can be estimated from the amount of radial bearing clearance if this is the primary source of eccentricity or from the shaft deflection.
calculated as a beam if this is the major displacement contribution, or from both if desired. In the case of a uniform-diameter shaft, this deflection is easily calculated by means of one of the various methods of elementary beam theory. For a stepped-diameter shaft, such as often used for rotating machinery, various methods of deflection calculations are discussed in a recent article by the present author (89).
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CHAPTER 3

GENERALIZED PLANE STRESS DUE TO ARBITRARY PLANE TEMPERATURE DISTRIBUTION

1. Introduction and Historical Background

Apparently the first study of stresses due to nonuniform temperature distributions was made in 1838 by Duhammel (1), who considered thin shells. Probably the first treatment of thermal stresses in a flat disk subject to the condition now known as generalized plane stress was made in 1842 by Neumann (2).

In spite of the importance of thermal stresses in cooling fins and other elements of heat-transfer equipment, in turbine disks, and in bimetallic thermostatic disks, the literature on generalized plane thermal stress is relatively sparse. However, as early as 1927, uniform-thickness circular disks subject to any arbitrary smoothly-varying temperature distributions were treated by Yamaguti (3).

An analogy between stresses induced by plane temperature distributions and those produced by plane dislocations was noted by

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1 Chapter 2, Section 1, contains a discussion of the assumptions made in connection with generalized plane stress.

2 Dislocations are discussed in Chapter 2, Sections 1 and 3.
Muskhelishvili (4) in 1916 and later by Biot (5). Since 1938 (6), this analogy has been widely used in simulating plane thermoelastic problems.

Probably Stodola (7) in 1906 was the first to treat thermal stresses in disks of varying thickness. He considered a circular disk of general hyperbolic profile subject to a power-function radial temperature distribution. Essentially this same problem was later treated by Binnie (8) and by Sen Gupta (9). Sen Gupta, in the same paper, analyzed the general-exponential-profile disk with a logarithmic radial temperature distribution.

Holzer (10) treated the problem of synthesis of the profile of a turbine disk to make maximum use of the material when the disk is subjected to nonuniform axisymmetric temperature distribution and centrifugal body forces.

In all of the analyses described above, it was assumed that the thermal-expansion coefficient and the elastic coefficients are independent of temperature. However, in most actual materials, all of these properties exhibit temperature dependence. One of the first investigations to take into consideration the temperature dependence of these properties was that of Suhara (11) in 1930. Reference is also made to the work of Thomson (12) and of Kovalenko (13).
In 1947 Manson (14) introduced an approximate method, based on finite differences, for calculating axisymmetric thermal stresses in circular disks of any arbitrary profile. He later extended this method to the determination of the disk profile shape required to give a desired axisymmetric distribution of radial stress, provided that the axisymmetric temperature distribution and the rotational velocity are known (15). Other approximate methods for calculating axisymmetric thermal-stress distributions in disks of arbitrary thickness profiles have been presented by Leopold (16), Giovannozi (17), Malinin (18), Strub (19), and Singh (20).

Perhaps the first treatment of generalized plane thermal stresses in anisotropic elastic materials is due to Carrier (21) in 1944. It is interesting to note that he considered not only orthotropic elastic coefficients but also orthotropic thermal-expansion characteristics, such as exhibited by certain crystals. Recently Teodorescu (22) presented a general treatment of the uniform-thickness, plane elasticity problem for an orthotropic material under an arbitrary temperature distribution.

In a recent paper, Kovalenko (23) suggested that the nonuniform, nonaxisymmetric temperature distribution in a varying-thickness turbine disk due to the effect of heat input from many individual blades could be approximated by an expression of the form $T = Cr^a \cos b\theta$, 
where \( b \) is the number of blades and \( a \) and \( C \) are constants. However, he did not actually carry out the analysis.

It is believed that thermally-stressed anisotropic disks of varying thickness have not been treated in the literature. A general treatment of this class of problems is presented in this chapter.

In all of the previously described analyses, generalized-plane-stress conditions were assumed. Two analyses which take into account the stress component in the thickness direction are the approximate calculations undertaken by Iliffe (24) and by Hoyle (25), using relaxation techniques.

2. **Derivation of the General Equation**

In the absence of body forces, the equations of equilibrium, Equations 1-1, for the radial and tangential directions become

\[
\begin{align*}
\frac{\partial}{\partial r} \left( r h \sigma_r \right) - h \sigma_\theta \theta + h \frac{\partial \sigma_\theta}{\partial \theta} &= 0, \\
\frac{\partial}{\partial r} \left( r^2 h \sigma_\theta \right) + r h \frac{\partial \sigma_\theta}{\partial \theta} &= 0.
\end{align*}
\]

Then the equilibrium equations can be satisfied by a stress function \( \Phi \) identical to that defined by Equations 2-3 except that here the body-force functions \( V_r \) and \( V_\theta \) can be omitted. Thus

\[
h \sigma_r = \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2}, \quad h \sigma_\theta = \frac{\partial^2 \Phi}{\partial r^2}, \quad h \sigma_r \theta = -\frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \right).
\]
Compatibility of strain requires that Equation 1-6, restated here, must hold.

\[ \frac{\partial}{\partial r} \left( r \frac{\partial \varepsilon_r}{\partial \theta} - r^2 \frac{\partial \varepsilon_\theta}{\partial r} \right) + r \frac{\partial \varepsilon_r}{\partial r} - \frac{\partial^2 \varepsilon_r}{\partial \theta^2} = 0. \]  

3-3

In order to permit more generality than usually encountered in thermoelastic analyses, it will be assumed that the material exhibits polar-orthotropic thermal-expansion characteristics as well as polar-orthotropic elasticity. Then the following thermoelastic stress-strain relations are obtained by adding the appropriate thermal strains to the stress-strain relations given by Equations 1-14:

\[ \varepsilon_r = \frac{e \sigma_r - \nu \varepsilon_\theta}{eE} + \alpha T_0, \quad \varepsilon_\theta = \frac{\sigma_\theta - \nu \varepsilon_r}{eE} + f \alpha T_0, \quad \varepsilon_{r\theta} = \frac{2c \sigma_r \theta}{eE}, \]  

3-4

where \( \alpha \) = coefficient of thermal expansion in the radial direction,

\( f \) = ratio of the coefficient of thermal expansion in the tangential direction to \( \alpha \), and \( T_0 \) = mean temperature through the thickness.

---

3 As is usual in thermoelasticity, it is assumed here that there is no mechanical coupling between the temperatures and the stresses. For most problems of engineering interest, this assumption is reasonable (26).
Combining Equations 2-7, 3-2, 3-3, and 3-4 gives the following result:

\[
S \left[ \frac{\partial^4 \Phi}{\partial r^4} + \frac{2}{r} \frac{\partial^3 \Phi}{\partial r^3} - \frac{e}{r^2} \frac{\partial^2 \Phi}{\partial r^2} + \frac{e}{3} \frac{\partial \Phi}{\partial r} + \frac{2 (e - v)}{r} \frac{\partial^4 \Phi}{\partial r^2 \partial \theta^2} \right] - \frac{2 (c - v)}{r^3} \frac{\partial^3 \Phi}{\partial r \partial \theta^2} + \frac{2 (c - v + e)}{r^4} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{e}{4} \frac{\partial^4 \Phi}{\partial \theta^4} \\
+ \frac{dS}{dr} \left[ 2 \frac{\partial^3 \Phi}{\partial r^3} + \frac{2 - v}{r} \frac{\partial^2 \Phi}{\partial r^2} - \frac{e}{2} \frac{\partial \Phi}{\partial r} + \frac{2 (c - v)}{r^2} \frac{\partial^3 \Phi}{\partial r \partial \theta^2} \right] + \frac{2 (c - v + e)}{r} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{e}{4} \frac{\partial^4 \Phi}{\partial \theta^4} \\
= -e \left[ f \frac{\partial^2 \Phi}{\partial r^2} + (2f - 1) \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{2} \frac{\partial^2 \Phi}{\partial \theta^2} \right] \frac{a T_o}{\sigma}.
\]

As might be expected, the left-hand side of Equation 3-5 is identical to the left-hand side of Equation 2-8. Furthermore, for the case of isotropic thermal expansion, the right-hand side of Equation 3-5 reduces to simply \(-e \sqrt{2} \frac{a T_o}{\sigma}\), regardless of whether the thickness is uniform or varying. However, this does not necessarily mean that the right-hand side reduces to a simple expression, as in the uniform-thickness case, since the heat-transfer equations are more complicated, as shown in the Appendix, Section 2.

In solving thermoelastic problems, use is often made of an analogy between the thermoelastic problem and a body-force problem involving the same elastic body at constant temperature (27). However, a comparison of Equations 2-8 and 3-5 shows that an exact analogy
exists only when \( V_r = V_\theta \) and there is isotropy with regard to both
elasticity and thermal expansion. Then \((1 - \nu)\) is analogous to unity
and \(SV\) corresponds to \(aT_o\).

3. General Solution for a Disk with a Power-Function
Stiffness Distribution

Inserting the expression for the compliance distribution given
in Equation 2-10 into the differential equation 3-5 results in a differen-
tial equation with exactly the same left-hand side as Equation 2-11.
Therefore, the complementary solution of the resulting equation is the
same as that for Equation 2-11, which is treated in detail in Chapter 2,
Section 3.

Assuming the following Fourier power series for the product
\(\alpha T_o\):
\[
\alpha T_o = \sum \sum C_{ij} r^i \cos j \theta, \tag{3-6}
\]
rather than for \(T_o\) alone, allows provision for variation of the thermal
coefficient of expansion \(\alpha\) with temperature. Substitution of the above
expression into Equation 3-5 gives the following expression for the
right-hand side of Equation 3-5:
\[
- e \sum \sum \left[(fi^2 + \phi - i - j^2) \cos j \theta \right] C_{ij} r^{-2}. \tag{3-7}
\]
The particular solution is now assumed to be of the form:
\[
\phi_p = \sum \sum C_{ij} r^{2+\xi} B_{ij} \cos j \theta. \tag{3-8}
\]
Finally, substitution of this expression into differential equation 2-11 with expression 3-7 for its right-hand side gives the following expressions for the particular constant $B_{ij}$:

$$B_{ij} K_{ij} = - e(f_i^2 + f_i - i - j^2),$$

where $K_{ij}$ has the value given by Equation 2-36.
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CHAPTER 4

BENDING OF PLATES WITH ARBITRARY LATERAL PRESSURE DISTRIBUTION

1. Introduction and Historical Background

The classes of problems treated here are limited by these assumptions of classical elastic-plate theory: homogeneity, small deflections, relatively small thickness, negligible normal stresses in the thickness direction, lines initially straight and normal to the neutral surface remain so during bending, flat initial neutral surface which is not stretched. Although classical theory is often limited to isotropy, polar orthotropy \(^1\), which includes isotropy as a special case, is assumed here. A circular annular planform, with thickness varying only with the radius, is also assumed.

Any arbitrary distribution of pressure \(q\) can be expanded in this Fourier-power series, provided enough terms are used:

\[
K \sum \sum a_{ij} r^i \cos j \theta
\]

\[4-1\]

\(^1\) Polar orthotropy is defined and discussed in Chapter 1, Section 3.
plus a similar sine series. Here $a_{ij}$, $i$, $j$, and $K$ are constants. For brevity, only the cosine series will be written hereafter; however, it is obvious that a sine series can readily be added when required.

The class of thickness distributions which will be discussed first can be characterized in terms of the flexural rigidity $D$ as follows:

$$D = D_0 r^{-n},$$  \hspace{1cm} 4-2

where $D_0$ and $n$ are constants.

The relationship of the general problem to various classes of special cases which have been previously treated in the literature can now be discussed.

**Isotropic plate theory.** -- The most simple class is that of an isotropic, uniform-thickness plate subjected to an axisymmetric system of line or point forces or couples, but no normal pressure ($K=0$).

This class was treated first by Poisson (1) in 1829 for solid circular plates, and, much later, by Wahl and Lobo (2) for annular plates. The same class, but with nonsymmetric forces and couples, is believed to have been considered first by H. Reissner (3) in 1929.

The next case is the familiar one of an isotropic, uniform-thickness plate subjected to uniform pressure ($i = j = 0$ only). This is a special case of the class having $j = 0$ only and arbitrary $i$, i.e., pressure varying only with radius.
For \( j > 0 \), the pressure distribution is nonsymmetric. Probably the first study of a case in this class was that of W. Flugge (4) for an isotropic, uniform-thickness plate subjected to a linearly-varying, nonsymmetric pressure distribution (\( i = j = 1 \) only). The general case of such plates subjected to concentrated lateral loads arbitrarily located was probably considered first by Clebsch (5) in 1862, while a closed-form solution for the clamped case was obtained by Michell (6) in 1902. Loadings consisting of arbitrary, analytic pressure distributions (\( i, j \) arbitrary) are believed to have been first treated by Jen (7) in 1948.

The first study of varying-thickness isotropic plates is probably due to Stodola (8), who in 1906 considered a horizontal plate with flexural rigidity given by Equation 4-2 and loaded by its own weight (\( i = \frac{n}{3}, \ j = 0 \)). Other early work on axisymmetrically-loaded, varying-thickness plates was conducted by Birkhoff (9) and Garabedian (10).

Some special kinds of thickness profiles which have been treated in the literature for the axisymmetric, isotropic case are summarized in Table 3.

Varying-thickness isotropic plates with nonsymmetric loadings, but no normal pressure, were perhaps considered first by Gran Olsson (11) in 1939. This class was later treated by Itenberg (12), Gradwell (13), Kovalenko (14), Umanski (15), and Conway (16).
### TABLE 3. Additional Plate Thickness Profiles Which Have Been Treated in the Literature for the Axisymmetric, Isotropic Case

<table>
<thead>
<tr>
<th>Thickness Function</th>
<th>Investigator</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h = Cr$</td>
<td>Conway (19)</td>
<td>1948</td>
</tr>
<tr>
<td>$h = 1 - (r/a)$</td>
<td>Bisshopp (20)</td>
<td>1944</td>
</tr>
<tr>
<td>$h = h_o; r &lt; a$</td>
<td>Conway (21)</td>
<td>1951</td>
</tr>
<tr>
<td>$h = h_o r/a; r = a$</td>
<td>Favre (22)</td>
<td>1949</td>
</tr>
<tr>
<td>$h = h_e e^{-ar^2}$</td>
<td>Pichler (25)</td>
<td>1928</td>
</tr>
<tr>
<td>$h = h_e e^{-ar^b}$</td>
<td>Gran Olsson (26)</td>
<td>1937</td>
</tr>
<tr>
<td>$h = [1 - (r/a)^b]c$</td>
<td>Conway (27)</td>
<td>1953</td>
</tr>
<tr>
<td>$h = h_o r^n [1 - (r/a)^m]^p$</td>
<td>Kovalenko (29)</td>
<td>1959</td>
</tr>
<tr>
<td>$h = h_o r^n e^{-r}$</td>
<td>Kovalenko (29)</td>
<td>1959</td>
</tr>
</tbody>
</table>
In addition to the particular type of axisymmetric normal-pressure distribution Stodola considered, Faedo (17) treated uniform pressure and Kito (18) considered a pressure distribution varying with the radius.

So far as is known, Gran Olsson (30) was the first to treat a problem of the class comprised of varying-thickness isotropic plates subjected to a nonsymmetric pressure distribution. He treated the same pressure distribution as did Flugge for uniform-thickness plates, namely \( i = j = 1 \) only. The general case of an arbitrary pressure distribution \( (i, j \) arbitrary) has recently been treated by Grigorenko (31) and by Kovalenko (32).

**Anisotropic plate theory.** --Lekhnitski (33) attributed the general theory of bending of anisotropic plates of uniform thickness to Gehring and to Boussinesq. The class consisting of uniform-thickness circular plates with polar orthotropy has been treated by Lekhnitski (33) for the axisymmetric case and by Carrier (34) for the nonsymmetric case characterized by \( i = j = 1 \) only. Apparently polar-orthotropic plates subject to an arbitrary pressure distribution have not been treated in the literature. This class of problem is treated in general fashion in the present chapter.

The first treatment of anisotropic plates of varying thickness is probably due to Shulezhko (35). Perhaps the only treatment of the class consisting of nonsymmetrically-loaded, polar-orthotropic plates of
varying thickness is due to Baclig and Conway (36), who treated the case of arbitrary n and nonsymmetric bending but no lateral pressure. It is believed that the class consisting of polar-orthotropic, varying-thickness plates subjected to nonsymmetric pressure distributions has not been treated previously. This class is considered later in this chapter.

**Analogies and approximate methods.** -- Wieghardt (37) in 1908 is believed to have been first to discuss the exact analogy between the deflection of a uniform-thickness isotropic plate and the stress function for a geometrically similar isotropic disk subject to a two-dimensional stress system. Holzer (38), followed by L. Foppl (39) and by Prager (40), extended this analogy qualitatively to plates and disks of radially-varying thickness and subjected to axisymmetric loading. Later Fox and Southwell (41) noted an analogy between the traction in a plate and the radial extension of the disk. Still later Vainberg (42) noted the applicability of the qualitative analogy to nonsymmetric, as well as axisymmetric, loadings. Thus, solutions which have been found for problems of generalized plane stress, as considered in Chapter 2, can be easily adapted to problems of the present type, since they are both of the same general form.

---

2 Strictly speaking, an analogy implies exact correspondence between the governing equations. However, here, in the case of varying thickness, some of the corresponding terms do not have the same coefficients.
It is believed that heretofore the qualitative analogy described above has not been extended to the polar-orthotropic case. This is accomplished in the next section of this chapter.

An approximate method for analyzing bending of varying-thickness plates by breaking them into a number of uniform-thickness annular elements, directly analogous to the Donath approximation for varying thickness rotating disks, was introduced, apparently independently, by Nilsson (43), Gawain and Curry (44), Gittleman (45), and Boston (46). Some improvements on this technique were later accomplished by Che-nea and Naghdi (47) and by Quinlan (48). The slope-deflection method of calculation was apparently first applied to varying-thickness plates by Narucka (49).

Experimental investigations and miscellaneous topics. -- There have been very few experimental investigations carried out to determine the validity of the theory of varying-thickness plates. Perhaps some reasons for this are the difficulties in conducting such experiments and the fact that the theory is probably more accurate than the experiments.

Herrmann and Ades (50) in 1956 made strain measurements, by means of electric-resistance wire strain gages, on a linearly-tapered annular plate subjected to a ring load at the inner edge and confirmed Conway's analysis (21) within reasonable experimental
accuracy. Another strain-gage investigation was carried out by Weese (51), who studied a linearly-tapered, solid circular plate simply supported at the periphery and subjected to uniform normal pressure. He obtained exceptionally close agreement with an extension of Conway's analysis (21) to cover this kind of loading.

Large deflections are considered to be beyond the scope of the present investigation. However, it is interesting to note that large deflections of uniform-thickness plates were considered as early as 1915 by Timoshenko (52), while perhaps the first treatment of large deflections of varying-thickness plates is due to Federhofer (53) in 1947. Recently, Grigerenko (54) has also taken into account the tensile forces developed in the large deflections of varying-thickness plates.

The effect of the deflection due to transverse shear, which becomes appreciable in moderately thick plates, was introduced in 1944 by E. Reissner (55) in a theory which is more accurate than the classical thin-plate theory. However, the effect of transverse shear deformation has only recently been included in the analysis of varying-thickness plates, by Essenburg (56).

Bending of circular plates of nonhomogeneous material which have a modulus of elasticity which varies through the thickness and along the radius has been treated by Kovalenko (57).
Engineering applications. --In addition to the usual applications involving isotropic plates of uniform or varying thickness subjected to point loadings or uniform loadings distributed over lines or areas, the following special or unique applications have arisen.

The case of a normal pressure varying as the square of the radius is of interest in connection with plates retaining fluids subjected to centrifugal force fields (58). Another important application in which a radially-varying normal-pressure distribution is encountered is in rotating disks having nonsymmetric cross section, i. e., ones in which the middle surfaces are not quite flat initially (Figure 8). Thus, in general, the centrifugal force acting on a typical annular element acts at a moment arm with respect to the location of the resultant of all such elemental centrifugal forces. This produces the same effect as a radially-varying normal-pressure distribution. This problem has been treated for the isotropic case by Lur'e (59), Matsuko and Nakajima (60), and Hodge and Papa (61), and for the polar-orthotropic case by Mikeladze (62).

The case of a linearly-varying nonsymmetric normal-pressure distribution, previously mentioned, is directly applicable, when superimposed upon a uniform pressure, to vertical plates (pressure-vessel

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3 It is assumed that the deflections are small so that the interaction between the centrifugal stretching and the bending is negligible, so that superposition of the two loadings is valid. This, of course, is always a conservative assumption.
Figure 8. A rotating disk with nonsymmetric profile.

Figure 9. Distribution of effective normal pressure due to centrifugal action is a slightly-tilted rotating plate.
heads) retaining hydrostatic pressure. Another application of a non-symmetrically-distributed normal pressure is in connection with the bending stresses in slightly-tilted rotating disks (Figure 9), treated by the author (63) for the uniform-thickness case. Here the tilt produces a moment arm through which the centrifugal forces act. This moment arm varies with both the radial and angular positions and the centrifugal forces vary with the radius in a fashion which depends upon the way in which the thickness varies. Thus, the net effect is the same as a nonsymmetric normal-pressure distribution which is dependent upon the disk profile. A somewhat similar loading occurs in rotating disks subject to gyroscopic loadings, a case which has been treated by Kovalenko (64), Corley (65), and Hirschberg and Mendelson (66).

In addition to the obvious application of orthotropic plate theory to orthotropic materials, such as wood, it is also applicable to certain kinds of nonhomogeneous plate structures which can be considered to be constructionally orthotropic. Examples of this latter category include plates of plywood, reinforced concrete, and reinforced plastics. In addition, corrugated plates and plates with stiffeners can be treated in this manner. However, in such instances, most of the available data (67) is for rectilinear orthotropy, rather than polar orthotropy. Nevertheless, Andreeva (68) has treated isotropic-material corrugated diaphragms, such as used in pressure capsules, by considering...
them as flat plates of polar-orthotropic material. Also, the equivalent polar-orthotropic elastic coefficients for isotropic circular plates stiffened by radial ribs have been recently determined by Rubach and Agranovich (69) and by Oster (70), who also treated circular plates stiffened by concentric rings.

2. Derivation of the General Equation

A typical polar-coordinate element of the plate is shown in Figure 10. The dimensions of the element in its midplane are 2 dr and 2 r dθ. Vertical shear force per unit length along the plate is denoted by Q; radial and tangential bending moments per unit length are denoted by Mr and Mθ; twisting moment per unit length is symbolized by Mrθ; and q represents the pressure normal to the plate middle surface. All of these quantities are taken to be positive when they act in the directions indicated in Figure 10.

---

The sign convention used here is the same as that used by Stoker (71), E. Reissner (55), and also Lekhnitski (33) and other Russian scholars. The convention used by Conway (16) and by Timoshenko and Woinowsky-Krieger (72) differs only in regard to the directions of positive Mrθ₁ and Mrθ₂.
Figure 10. Forces and moments acting on a typical polar-coordinate element of a plate.
Summing moments about tangential and radial axes passing
through the center of the element and summing forces in the axial di-
rection give the following equations of equilibrium:

\[
\begin{align*}
M_\theta r^2 & (r + dr)^2 d\theta - M_\theta r^1 (r - dr)^2 d\theta + (M_{r\theta 4} - M_{r\theta 3})^2 dr \cos d\theta \\
- (M_{\theta 3} + M_{\theta 4})^2 dr \sin d\theta - Q_r (r + dr)^2 d\theta dr - Q_r (r - dr)^2 d\theta dr = 0 \\
(M_{\theta 4} - M_{\theta 3})^2 dr \cos d\theta + M_{r\theta 2} (r + dr)^2 d\theta - M_{r\theta 1} (r - dr)^2 d\theta \\
+ (M_{r\theta 4} + M_{r\theta 3})^2 dr \sin d\theta - (Q_{\theta 4} + Q_{\theta 3})^2 dr r d\theta = 0 \\
Q_r (r + dr)^2 d\theta - Q_r (r - dr)^2 d\theta + (Q_{\theta 4} - Q_{\theta 3})^2 dr + q4 r dr d\theta = 0.
\end{align*}
\]

Assuming that the elemental angle \(d\theta\) is very small so that
\(\sin d\theta \approx d\theta\) and \(\cos d\theta \approx 1\), dividing by \(4 dr d\theta\), and neglecting differen-
tial quantities of second order give the following result:

\[
\begin{align*}
\frac{\partial}{\partial r} (rM_r) & + \frac{\partial M_r}{\partial \theta} r \theta - M_{r\theta} = 0, \\
\frac{\partial}{\partial r} (rM_{r\theta}) & + \frac{\partial M_{r\theta}}{\partial \theta} + M_{r\theta} - rQ_{\theta} = 0, \\
\frac{\partial}{\partial r} (rQ_r) & + \frac{\partial Q_r}{\partial \theta} + rq = 0.
\end{align*}
\]
Substituting the first two of Equations 4-3 into the third to eliminate \( Q_r \) and \( Q_\theta \) gives the following equation:

\[
\begin{align*}
\frac{\partial}{\partial r} \left( rM_r \right) + \frac{\partial}{\partial \theta} \left( \frac{\partial M_\theta}{\partial r} \right) - \frac{\partial M_\theta}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{\partial M_\theta}{\partial \theta} \right) + \frac{1}{r} \frac{\partial^2 M_\theta}{\partial \theta^2} + \frac{1}{r} \frac{\partial M_r \theta}{\partial \theta} + rq &= 0.
\end{align*}
\]

4-4

Since it is assumed that lines originally straight and normal to the neutral surface remain so during bending, the two-dimensional displacements \( u \) and \( v \), as shown in Figure 11, are given by

\[
u = -z \frac{\partial w}{\partial r}, \quad \text{and} \quad v = -z \frac{\partial w}{\partial \theta},
\]

4-5

where \( z = \) distance from the plate neutral surface (positive in the direction shown in Figure 11) and \( w = \) axial displacement of the neutral surface (positive downward).

Then, substituting Equations 4-5 into the strain-displacement relations in polar coordinates, Equations 1-2, 1-3, and 1-4, the strains are

\[
\begin{align*}
\varepsilon_r &= -z \frac{\partial^2 w}{\partial r^2}, \quad \varepsilon_\theta = -z \left( \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r} \frac{\partial^2 w}{\partial \theta^2} \right), \quad \varepsilon_{r\theta} = -2z \frac{\partial^2 w}{\partial r \partial \theta} \left( \frac{w}{r} \right).
\end{align*}
\]

4-6
Figure 11. Displacement of an arbitrary point 0 located at polar-coordinate position $r$, $\theta$ and a distance $z$ below the neutral surface.
Using the above expressions for the strains and the stress-strain relations for a polar-orthotropic material, Equations 1-14, and then integrating the various stress components through the thickness $h$ gives the following moment-deflection relations:

\[
\begin{align*}
M_r &= -D \left[ \frac{\partial^2 w}{\partial r^2} + \nu \left( \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) \right], \\
M_\theta &= -D_e \left( \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \nu \frac{\partial^2 w}{\partial r^2} \right), \\
M_{r\theta} &= -D_d \frac{\partial^2 w}{\partial r \partial \theta},
\end{align*}
\]

where

\[
D = \frac{Eh^3}{12(1 - \nu^2/e)} \quad \text{and} \quad d = \frac{Ch^3}{6D}.
\]

The quantities $D$ and $D_e$ are the flexural rigidities corresponding respectively to the radial and tangential directions; $D_d$ is the torsional rigidity corresponding to $\sigma_{r\theta}$.
Finally, substituting the moment-deflection relations 4-7 into
the combined equilibrium equation 4-4 and remembering that D varies
with r only gives the following result:

\[ D \left[ \frac{\partial^4 w}{\partial r^4} + \frac{2}{r} \frac{\partial^3 w}{\partial r^3} + \frac{e}{r^2} \frac{\partial^2 w}{\partial r^2} + \frac{3}{r \partial r} \frac{\partial w}{\partial r} + \frac{2(d + \nu)}{r^2} \frac{\partial^4 w}{\partial r^2 \partial \theta^2} \right] \\
\left[ - \frac{2(d + \nu)}{r^3} \frac{\partial^3 w}{\partial r \partial \theta^2} + \frac{2(d + \nu + \epsilon)}{r^4} \frac{\partial^2 w}{\partial \theta^2} + \frac{\epsilon}{r^4} \frac{\partial^4 w}{\partial \theta^4} \right] \\
+ \frac{dD}{dr} \left[ 2 \frac{\partial^3 w}{\partial r^3} + \frac{2 + \nu}{r} \frac{\partial^2 w}{\partial r^2} - \frac{e}{r^2} \frac{\partial w}{\partial r} + \frac{2(d + \nu)}{r^2} \frac{\partial^3 w}{\partial r \partial \theta^2} \right] \\
\left[ - \frac{2(d + \nu + \epsilon)}{r^3} \frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 D}{\partial r^2} \left[ \frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r} \frac{\partial w}{\partial r} + \frac{\nu}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right] \right] \\
= q. \]

Comparison of Equation 4-9 with Equation 2-8 shows that the
left-hand sides of the two equations are of the same form, except that
the sign in front of Poisson's ratio is reversed at every place in which
it appears. Thus, the qualitative analogy can be extended to the classes
consisting of nonsymmetrically-loaded, varying-thickness, polar-ortho-
tropic, simply-connected\(^5\) plates and disks, provided that the plate is
not subjected to normal pressure and that the disk is not subjected to
body-force functions \(V_r\) and \(V_\theta\). To extend the analogy to multiply-
connected\(^5\) plates and disks, it would be necessary to make an analysis

\(^5\)Simply-connected plates are solid ones; multiply-connected
plates are ones containing one or more holes.
taking into account the single-valued nature of the disk rotations and
the displacements, such as was done by Mindlin (73) for the uniform-
thickness, isotropic case.

In subsequent work, it will be necessary to have expressions
for the shear forces \( Q_r \) and \( Q_\theta \) in terms of the plate deflection. These
can readily be obtained by substituting the moment-displacement rela-
tions, Equations 4-7, into the first two of Equations 4-3 and solving for
the respective quantities with the following results:

\[
Q_r = -D \left[ \frac{\partial^3 w}{\partial r^3} + \frac{1}{r} \frac{\partial^2 w}{\partial r^2} - \frac{e}{r} \frac{\partial w}{\partial r} + \frac{(d+v)}{r^2} \frac{\partial^3 w}{\partial r \partial \theta^2} - \frac{(e+d+v)}{r^3} \frac{\partial^2 w}{\partial \theta^2} \right]
\]

\[
- \frac{dD}{dr} \left[ \frac{\partial^2 w}{\partial r^2} + \nu \frac{\partial w}{\partial r} + \frac{v}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right]
\]

\[
4-10
\]

and

\[
Q_\theta = -D \left[ \frac{d+v}{r} \frac{\partial^3 w}{\partial r^2 \partial \theta} + \frac{e}{r^2} \frac{\partial^2 w}{\partial r \partial \theta} + \frac{e}{r^3} \frac{\partial^3 w}{\partial \theta^3} \right]
\]

\[
- \frac{dD}{dr} \left[ \frac{d}{r} \frac{\partial^2 w}{\partial r \partial \theta} - \frac{d}{r^2} \frac{\partial w}{\partial \theta} \right].
\]

\[
4-11
\]

In connection with one of the boundary conditions for a free edge
of a circular plate, it is useful to have an expression for the effective
transverse shear force, which is the quantity \( Q_r \) minus the quantity
\((1/r) \partial M_r \partial \theta\), thus:

\[
N_r = -D \left[ \frac{\partial^3 w}{\partial r^3} + \frac{1}{r} \frac{\partial^2 w}{\partial r^2} - \frac{e}{r} \frac{\partial w}{\partial r} + \nu \frac{\partial^3 w}{\partial r \partial \theta^2} - \frac{(e+v) \partial^2 w}{\partial r \partial \theta^2} \right]
\]

\[
- \frac{dD}{dr} \left[ \frac{\partial^2 w}{\partial r^2} + \frac{v}{r^2} \frac{\partial w}{\partial r} + \frac{\nu \partial^2 w}{r^2} \right].
\]

\[
4-12
\]
3. General Solution for a Plate with a Power-Function Rigidity Distribution

In the preceding section, a plate with arbitrary radial variation of flexural rigidity was considered. Now the analysis is applied to the power-function rigidity distribution, Equation 4-2. Substituting this rigidity function into the differential equation of equilibrium, Equation 4-9, gives the following result:

\[
\begin{align*}
\frac{\partial^4 w}{\partial r^4} + \frac{2(1+n)}{r} \frac{\partial^3 w}{\partial r^3} + \frac{n^2 + vn + n - e}{r^2} \frac{\partial^2 w}{\partial r^2} + (e - vn) \frac{1-n}{r^3} \frac{\partial w}{\partial r} \\
+ \frac{2(d+v)}{r^2} \frac{\partial^4 w}{\partial r^2 \partial \theta^2} - 2(d+v) \frac{1-n}{r^3} \frac{\partial^3 w}{\partial r \partial \theta^2} \\
+ \frac{2(d+v)(1-n) - e(n-2) + vn(n-1)}{r^4} \frac{\partial^2 w}{\partial \theta^4} + \frac{e}{r^4} \frac{\partial^4 w}{\partial \theta^4} = \frac{q r^n}{D_o}.
\end{align*}
\]

It was shown that in the generalized-plane-stress problem (Chapter 2, Section 3) the stress function need not be periodic, provided that certain combinations of its derivatives (the stresses and displacements) are periodic. In plate bending, the corresponding primary function is the deflection. Here, the plate deflection itself must be periodic, assuming that there are no lateral-type dislocations present. This is a more severe restriction on the primary function than in the generalized-plane-stress problem; thus, the most general form of the deflection function, excluding dislocations, is merely the sum of Fourier cosine and sine series.
As in generalized plane stress, the differential equation is linear; thus, the general solution is equal to the sum of the complementary and particular solutions.

**Complementary solution.** -- Using the same method as in the solution of the generalized-plane-stress equation, Chapter 2, Section 3, the complementary solution is assumed to be of the following form:

\[ w = \sum_{p=0}^{\infty} R_p(r) \cos p\theta \]

plus a similar sine series, omitted here for brevity.

Substitution of the assumed solution into the differential equation 4-13 with the right-hand side set equal to zero gives the following Euler-type ordinary differential equation:

\[
r^4 R'''' + 2(1+n)r^3 R'''' + \left[ n^2 + \nu n + n - e - 2(d+\nu)p^2 \right] r^2 R'''' \]

\[ + \left[ e - \nu n + 2(d+\nu)p^2 \right] (1-n)rR' + \] \[ -\left[ 2(d+\nu)(1-n) - e(n-2) + \nu n(n-1) - ep^2 \right] p^2 R_p = 0 . \]

This equation is identical to the corresponding equation in the generalized-plane-stress solution, Equation 2-20, except that in every place Poisson's ratio \( \nu \) and \( c \) appear in the latter, they are replaced by \( -(\nu) \) and \( d \) in the former. Therefore, the rest of the present derivation is the same as in Chapter 2, Section 3, except that here \( w \) replaces \( \phi \), \( \nu \) replaces \( -(\nu) \), and \( d \) replaces \( c \).
Particular solution. --Provided that a sufficient number of terms is used, any arbitrary normal-pressure distribution \( q \) can be approximated by the Fourier-power series of expression 4-1.

Dimensional analysis of Equation 4-13 with expression 4-1 as its right-hand side shows that the particular solution must be of the form

\[
wp = \frac{K}{D_0} \sum_i \sum_j a_{ij} \frac{r^{i+n+4}}{B_{ij}} \cos j \theta ,
\]

where \( B_{ij} \) is determined, by substitution into the equation, to be given by:

\[
B_{ij} = \mu_i (\mu_i - 1)(\mu_i - 2)(\mu_i - 3) + 2(n + 1)\mu_i (\mu_i - 1)(\mu_i - 2)
\]

\[
+ (n^2 + vn - n - e)\mu_i (\mu_i - 1) + (e - vn)(1 - n)\mu_i - 2(d + v)\mu_i (\mu_i - 1)j^2
\]

\[
+ [2(d + v)(1 - n)(\mu_i - 1) + e(n - 2) - vn(n - 1) + ej^2]j^2 ,
\]

where \( \mu_i \equiv i + n + 4. \)

It is to be pointed out that for all pressure distributions which are expressible in the closed form of expression 4-1, the particular solution is also expressible in closed form. Values of the constant \( B_{ij} \) for some particular closed-form cases of interest are listed in Table 4.

It should be noted that when \( i = 0 \) only, the pressure must be distributed over an annular area, rather than over a solid circular one, in order to avoid a discontinuity in the pressure distribution at the origin, except when \( i \) and \( j \) are zero simultaneously.
TABLE 4. Values of the Constant $B_{ij}$ for Isotropic Plates with Some Particular Cases of Thickness Distribution and Pressure Distribution

<table>
<thead>
<tr>
<th>Pressure as a Function of $\theta$</th>
<th>Pressure as a Function of $r$</th>
<th>Thickness as a Function of $r$</th>
<th>Constant $B_{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Independent</td>
<td>Independent</td>
<td>Independent</td>
<td>64</td>
</tr>
<tr>
<td>Independent</td>
<td>Linear</td>
<td>Independent</td>
<td>225</td>
</tr>
<tr>
<td>Independent</td>
<td>Quadratic</td>
<td>Independent</td>
<td>576</td>
</tr>
<tr>
<td>Varying Thickness:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Independent</td>
<td>Independent</td>
<td>Inversely prop.</td>
<td>2968+168v</td>
</tr>
<tr>
<td>Independent</td>
<td>Linear</td>
<td>Inversely prop.</td>
<td>4968+216v</td>
</tr>
<tr>
<td>Independent</td>
<td>Quadratic</td>
<td>Inversely prop.</td>
<td>7830+2702v</td>
</tr>
</tbody>
</table>

Axisymmetric Cases

Uniform Thickness:

| Proportional to $\cos \theta$ | Independent | 0 | Independent | 0 | 45 |
| Proportional to $\cos \theta$ | Linear      | 1 | Independent | 0 | 192 |
| Proportional to $\cos \theta$ | Quadratic   | 2 | Independent | 0 | 525 |

Varying Thickness:

| Proportional to $\cos \theta$ | Independent | 0 | Inversely prop. | 3 | 2862+162v |
| Proportional to $\cos \theta$ | Linear      | 1 | Inversely prop. | 3 | 4830+210v |
| Proportional to $\cos \theta$ | Quadratic   | 2 | Inversely prop. | 3 | 7664+264v |

Nonsymmetric Cases

Uniform Thickness:

| Proportional to $\cos \theta$ | Independent | 0 | Independent | 0 | 45 |
| Proportional to $\cos \theta$ | Linear      | 1 | Independent | 0 | 192 |
| Proportional to $\cos \theta$ | Quadratic   | 2 | Independent | 0 | 525 |

Varying Thickness:

| Proportional to $\cos \theta$ | Independent | 0 | Inversely prop. | 3 | 2862+162v |
| Proportional to $\cos \theta$ | Linear      | 1 | Inversely prop. | 3 | 4830+210v |
| Proportional to $\cos \theta$ | Quadratic   | 2 | Inversely prop. | 3 | 7664+264v |
It is interesting to note that the constant $B_{ij}$ is independent of Poisson's ratio only in the constant-thickness case ($n = 0$). The value of $B_{ij}$ given in the table for the uniform-pressure, uniform-thickness, isotropic case ($i = j = 0$, $e = d + v = 1$) coincides with the familiar value of 64, as given by Timoshenko and Woinowsky-Kreiger (74), for instance. The case of $i = j = 1$, $n = 0$, and $e = d + v = 1$ is the linear-pressure-distribution, uniform-thickness, isotropic case considered by Flügge (4) and the value of 192 listed in the table for this case agrees with his result.

4. Application: Example 4 - Bending of a Slightly-Tilted Rotating Plate

Sometimes it is desired to determine the stresses in a rotating plate in which the plane of the plate is not quite perpendicular to the axis of rotation. An example is a compressor or turbine rotor which is not properly aligned during installation on the shaft.

It is assumed here that the angle of tilt $\delta$ is sufficiently small so that its cosine may be taken to be equal to unity. Then the tilt has a negligible effect on the generalized plane stresses in the plane of the plate and they may be assumed to be the same as those calculated for an untilted rotating disk of the same geometrical configuration. Furthermore, the deflections due to the tilt effect will be sufficiently small that interaction between lateral loading due to the tilt and the in-plane centrifugal stretching can be neglected. Then the total stress acting
at any given point by superposition is equal to the sum of the bending stress and the centrifugal generalized plane stress.

It is the purpose here to calculate only the bending stress, since centrifugal stresses in rotating disks of varying thickness have been quite adequately covered in the literature.

Figure 12 shows a typical small element with a surface area $A$ and located at a distance $r$ from the center of the plate. The centrifugal force acting on this element is given by

$$F = \left( \frac{\gamma \Omega^2 h A r \cos \delta}{g} \right), \quad 4-18$$

where $\gamma = \text{specific weight of the disk material}$, $g = \text{gravitational acceleration}$, $\Omega = \text{rotational velocity}$, $h = \text{thickness of the element}$, and $\delta = \text{angle between the radial line passing through the center of the element and the plane normal to the rotational axis}$. The angle $\delta$ is always smaller than the tilt angle $\delta_o$, since

$$\sin \delta = \sin \delta_o \cos \theta. \quad 4-19$$

The force $F$ can be resolved into a component in the plane of the plate and another component $Q$ normal to the plane of the plate, as shown in Figure 12. The normal component $Q$ is equal to the force $F$ times the sine of the angle $\delta$. Then the equivalent normal pressure $q$, which is the normal force per unit area or $Q/A$, is given by

$$q = \left( \frac{\gamma \Omega^2 h r \sin \delta \cos \delta}{g} \right). \quad 4-20$$
Projected view looking at the plane of the plate

Side view looking down the tilt axis

Section A-A through a typical element

Figure 12. Configuration of a slightly-tilted rotating plate and the force components acting on a typical element.
Now, since it has been assumed that angle \( \delta \) is small, the above exact expression for the normal pressure can be approximated by the following expression:

\[
q = \frac{(\gamma U^2 h r \sin \delta \cos \theta)}{g}.
\]

Since Equation 4-2 expresses the flexural rigidity, the thickness \( h \) at any radius \( r \) can be expressed as

\[
h = h_0 r^{-n/3}.
\]

Substituting this expression for the thickness into Equation 4-21 gives the following result:

\[
q = Kr^{1-n/3} \cos \theta,
\]

where \( K = \frac{\gamma U^2 h_0 \sin \delta}{g} \). It should be noted that the equivalent pressure distribution depends upon the thickness distribution in a fashion analogous to the dependence of centrifugal force upon the thickness distribution.

The results of the preceding section are now applied to a particular plate as an example. It is assumed that the material is isotropic (i.e., \( e = d + \nu = 1 \)), with a Poisson's ratio of \( 1/3 \). The plate thickness is taken to vary inversely as the radius, so that the constant \( n \) in Equation 4-2 for the flexural-rigidity distribution is 3. Then using \( p = 1 \) and substituting \( d \) and \( (-\nu) \) for \( c \) and \( \nu \) in Equations 2-25 and 2-27, it
is found that $\beta_1 = \beta_2 = 3/2$. Then the roots $\lambda_k$ of the characteristic equation are -2, -2, 1, 1. In view of the double roots, the solution takes the form

$$w = (C_1 \rho^{-2} + C_2 \rho^{-2} \ln \rho + C_3 \rho + C_4 \rho \ln \rho + \rho^7) \frac{Kb^4 \cos \theta}{BD_b},$$

where $B = 2916$, from Equation 4-17 with $\mu_i = 7; \rho \equiv r/b$; and $D_b \equiv D$ at $r = b$.

The four constants of integration $C_k$ are evaluated by considering the four boundary conditions: deflection and slope are zero at the inner edge, which is clamped; the effective shear force and radial bending moment are zero at the outer edge, which is free. Mathematically, these boundary conditions are expressed by:

$$\begin{align*}
\left( w \right)_{\rho=a} &= 0; \\
\left( \frac{\partial w}{\partial \rho} \right)_{\rho=a} &= 0; \\
(N)_{\rho=1} &= 0; \\
(M)_{\rho=1} &= 0,
\end{align*}$$

where $a \equiv a/b$.

The boundary conditions give rise to the following set of simultaneous linear algebraic equations in the coefficients:

$$\begin{align*}
\alpha^{-2} C_1 + \alpha^{-2} \ln \alpha C_2 + \alpha C_3 + \alpha \ln \alpha C_4 &= -\alpha^7, \\
-2\alpha^{-2} C_1 + (1 - 2 \ln \alpha) \alpha^{-2} C_2 + \alpha C_3 + (1 + \ln \alpha) \alpha C_4 &= -7\alpha^7, \\
87 C_1 - 59 C_2 + 13 C_4 &= 732, \\
15 C_1 - 14 C_2 + 4 C_4 &= 132.
\end{align*}$$
Solving Equations 4-26 for a value of \( a \) of 0.1 and substituting the resulting coefficients into the expression for the displacement, Equation 4-24, and the displacement into the first of Equations 4-7 gives an expression for the radial moment. A similar procedure is followed for the tangential moment. It is found that the largest moment, and thus the largest stress, is in the radial direction at the inner edge. For the assumed geometry, the maximum stress can be expressed as \( \sigma_r / C = 1.633 \), where \( C = (\gamma \mu^2 b^3 \sin \delta_o) / \rho b \). This is a considerable reduction in stress when compared to the value of \( \sigma_r / C = 7.36 \) obtained previously by the author (63) for the case of uniform thickness and the same ratio \( a \). The explanation for the considerable reduction in stress is due in part to the additional material in the varying thickness plate, which is ten times as thick at its inner radius as the thickness of the uniform-thickness plate. However, since the volume of the varying-thickness disk is only 1.82 times that of the uniform-thickness one, this is not the complete explanation. Obviously, some of the improvement is due to a more efficient thickness distribution. For equal outer radius, thickness at the outer radius, density, tilt angle, angular velocity, and inner radius, the product of maximum stress and volume for the varying-thickness plate is still only 40.3 percent of that of the uniform-thickness one. Although no attempt has been made here to obtain an optimum thickness distribution, it is obvious that the
particular thickness distribution chosen in this example is considerably more efficient for this type of loading than a uniform-thickness plate.

In order to assess the potential error in using the uniform-thickness solution to predict the stresses in the varying-thickness case, it is of interest to determine the "effective" outer radius \( b \), which, if used in the uniform-thickness solution, gives the correct result, as determined in the present analysis. It turns out that the effective radius in this case is 60.5 percent of the actual radius. Since this value is quite small, the large error inherent in using an empirical method such as this, instead of a more accurate analysis such as the present one, is obvious.
References


15. Umanski, E. S., "Bending of a Disk of Hyperbolic Profile with a Cyclically Symmetric Load at the Edge" (in Ukrainian), Sbornik Trudy Institut Stroiteln'oi Mekhaniki, Akademii Nauk Ukrainskaya SSR, no. 15 (1950); "Bending of a Disk of Varying Stiffness under Arbitrary Unsymmetrical Loads" (in Ukrainian), Izvestia Kievskikh Politekhicheskogo Instituta, vol. 12 (1953), according to Korenev, loc. cit., Reference 12.


CHAPTER 5

BENDING OF PLATES WITH ARBITRARY DISTRIBUTIONS OF LINEAR THERMAL GRADIENTS

1. Introduction and Historical Background

The first analysis of plate bending due to thermal gradients through the thickness was probably made by Nadai (1) in 1925, followed by Yamaguti (2) in 1927. However, general formulation of the thermoelastic plate equations is credited to Marguerre (3) in 1935; he assumed that generalized-plane-stress conditions prevail on the plate surfaces. Later the Sokolnikoffs (4) derived similar equations without resorting to this assumption.

Applications of thermoelastic circular-plate theory have been mostly limited to axisymmetric cases, e. g., the work of Goldberg (5), Forray and Newman (6), and Zaid and Forray (7). An analogy to unheated plates subject to normal pressure has been used by Boley and Weiner (8).

Thermal bending of anisotropic plates was treated by Pell (9) in 1946 and later by Durgar'ian (10). Williams (11) considered isotropic plates subject to large thermal deflections; later Langhaar, Miller, and Boresi (12) analyzed orthotropic plates under these
conditions. Rowley (13), and later, Sorenson (14), extended the E.
Reissner thick-plate theory to thermal bending.

All of the foregoing analyses have been limited to homogeneous plates of uniform thickness. The first analysis of thermal bending of varying-thickness plates is believed to be due to Gran Olsson (15), who in 1932 treated rectangular plates of varying thickness. The first investigation of thermal bending of varying-thickness circular plates is probably that of Herrmann (16) in 1955. He analyzed annular plates having hyperbolic thickness profiles and subjected to axisymmetric distributions of temperature gradient expressed as a power function of the radius. He applied his analysis to a turbine disk subjected to thermal gradients measured experimentally by Manson (17) during the startup of a turbojet engine and compared his results favorably with the strain values measured by Manson. Varying-thickness circular plates subjected to axisymmetric thermal-gradient distributions were also considered by Kovalenko (18).

The case of the thermal bending of a varying-thickness circular plate with a modulus of elasticity which varies through the thickness of the plate has been recently treated by Kovalenko (19).

From the foregoing review of the literature on the present topic, it is readily apparent that the following classes of thermal-bending problems have not been treated in the literature:

1. Plates subjected to nonsymmetric distributions of thermal gradient.
2. Polar-orthotropic plates of varying thickness.

Both of these classes are included in the general analysis given in the succeeding sections of this chapter.

2. Derivation of the General Equation

In the absence of normal pressure\(^1\), the combined equilibrium equation, Equation 4-4, becomes

\[
\frac{\partial^2}{\partial r^2} (rM_r) + \frac{\partial^2 M_{r\theta}}{\partial r \partial \theta} - \frac{\partial M_{\theta}}{\partial r} + \frac{1}{r} \frac{\partial^2}{\partial \theta^2} (rM_{r\theta})
\]

\[
+ \frac{1}{r} \frac{\partial^2 M_{\theta}}{\partial \theta^2} + \frac{\partial^2 M_{\theta}}{\partial \theta^2} + \frac{1}{r} \frac{\partial M_{r\theta}}{\partial \theta} = 0.
\]

The stress-strain relations for a material possessing polar orthotropy with regard to both elasticity and thermal expansion\(^2\), taking into account a linear temperature gradient through the thickness, are

\[
\begin{align*}
\varepsilon_r &= \varepsilon_r^r - \nu \varepsilon_{\theta} + \frac{2\alpha T \cdot z}{h}, \\
\varepsilon_{\theta} &= \varepsilon_{\theta}^r - \nu \varepsilon_r + \frac{2\alpha T \cdot z}{h}, \\
\varepsilon_r \theta &= \frac{2 \sigma_{r\theta}}{E},
\end{align*}
\]

\(^1\)For cases involving bending due to both thermal gradients through the thickness and normal pressure, provided that the deflections are sufficiently small, the two solutions can be superimposed.

\(^2\)As in the case of generalized plane thermal stresses (Chapter 3, Section 2), it is assumed here that there is no coupling between the temperatures and the stresses.
where the elastic coefficients $E$, $v$, $e$, and $c$ are as defined in Chapter 1, Section 3; the thermal coefficients $a$ and $f$ are as defined in Chapter 3, Section 2; $z$ = distance from the middle surface of the plate; and $T_1$ = temperature difference between the hottest surface and the middle surface.

Using the stress-strain relations and the strain-displacement relations 4-6, integration of the various stress components through the thickness $h$ gives the following thermoelastic moment-deflection relations:

$$
M_r = -D \left[ \frac{\partial^2 w}{\partial r^2} + v \left( \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) + 2(1 + \nu e) \frac{a T_1}{h} \right],
$$

$$
M_\theta = -eD \left[ \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{\nu}{e} \frac{\partial^2 w}{\partial r^2} + 2(1 + \nu) \frac{a T_1}{h} \right],
$$

$$
M_{r\theta} = -D d \frac{\partial^2}{\partial r \partial \theta} \left( \frac{w}{r} \right),
$$

where $D$ and $d$ are as defined in Chapter 4, Section 2.
Finally, substitution of Equations 5-3 into Equation 5-1 and remembering that $D$ is a function of $r$ only, gives the following result:

\[
D\left[\frac{\partial^4 w}{\partial r^4} + \frac{2}{r} \frac{\partial^3 w}{\partial r^3} - \frac{e}{r^2} \frac{\partial^2 w}{\partial r^2} + \frac{e}{r^3} \frac{\partial w}{\partial r} + \frac{2(d+\nu)}{r^2} \frac{\partial^4 w}{\partial r^2 \partial \theta^2}\right]
\]

\[
- \frac{2(d+\nu)}{r^3} \frac{\partial^3 w}{\partial r \partial \theta^2} + \frac{2(d+\nu+e)}{r^4} \frac{\partial^2 w}{\partial \theta^2} + \frac{e}{r^4} \frac{\partial^4 w}{\partial \theta^4} + \frac{dD}{dr} \left[\frac{2}{r^3} \frac{\partial^3 w}{\partial \theta^2}\right]
\]

\[
+ \frac{2 + \nu}{r} \frac{\partial^2 w}{\partial r^2} - \frac{e}{r^2} \frac{\partial w}{\partial r} - \frac{2(d+\nu)}{r^2} \frac{\partial^3 w}{\partial r \partial \theta^2} - \frac{2(d+\nu)+e}{r^3} \frac{\partial^2 w}{\partial \theta^2}
\]

\[
+ \frac{d^2 D}{dr^2} \left[\frac{\partial^2 w}{\partial r^2} + \frac{v}{r} \frac{\partial w}{\partial r} + \frac{v}{r^2} \frac{\partial^2 w}{\partial \theta^2}\right] = 2D[(1+\nu e f) \frac{\partial^2 T}{\partial r^2} + (2 - \nu - ef
\]

\[
+ 2\nu ef) \frac{1}{r} \frac{\partial}{\partial r} + \frac{ef + \nu}{r^2} \frac{\partial^2 T}{\partial \theta^2} \frac{dT_1}{h} (\frac{aT_1}{r})
\]

\[
+ 2 \frac{dD}{dr} [2(1+\nu e f) \frac{\partial}{\partial r} + (2 - \nu - ef + 2\nu ef) \frac{1}{r} \frac{aT_1}{h}]
\]

\[
+ 2 \frac{d^2 D}{dr^2} (1+\nu e f)(\frac{aT_1}{h}).
\]

As is to be expected, the left-hand side of Equation 5-4 is identical to the left-hand side of Equation 4-9. Furthermore, for the case of elasticity and thermal expansion which are both isotropic, i.e.,

$e = d - \nu = f = 1$, the right-hand side of Equation 5-4 reduces to

\[
2D(1+\nu) \frac{\partial^2 T_1}{2} (\frac{aT_1}{h}) + 2\frac{dD}{dr} (1+\nu)(2 \frac{\partial}{\partial r} + \frac{1}{r})(\frac{aT_1}{h}) + 2 \frac{d^2 D}{dr^2} (1+\nu)(\frac{aT_1}{h}).
\]
For thermal and elastic isotropy and an axisymmetric thermal-gradient distribution, Equation 5-4 is equivalent to an equation given by Herrmann (16).

3. General Solution for a Plate with a Power-Function Rigidity Distribution

Inserting the expression for the rigidity distribution given in Equation 4-2 into the differential equation 5-4 results in a differential equation with exactly the same left-hand side as Equation 4-13. Thus, the complementary solution of the resulting equation is the same as that for Equation 4-13, which is given in Chapter 4, Section 2.

In order to determine an expression for the particular solution, it is first expedient to assume the following Fourier power series for the quantity \( aT_1/h \):

\[
\frac{aT_1}{h} = \sum_{s} \sum_{t} a_{st} r^s \cos t \theta ,
\]

where here \( a \) is taken to be the coefficient of thermal expansion corresponding to the mean of the temperatures on the two surfaces of the plate. Thus, by permitting \( a \) to vary with position \((r, \theta)\), the variation of the thermal-expansion coefficient with temperature can be taken into account.
Substitution of Equation 5-5 into Equation 5-4 along with the rigidity distribution 4-2 gives the following expression for the right-hand side of Equation 5-4:

\[ 2D \frac{1}{n-s} \left\{ \sum_{st} \left[ (1 + vef)(s^2 + s - 2ns) + (1 - vef)(n^2 - n - 2) - s(v + ef) \right] \right\} \]

\[ \cdot \sum_{st} ^{1} r^{s-2} \cos \theta \} \]. \hspace{1cm} 5-5

Now the particular solution is assumed to be of the form

\[ w = \sum_{st} \sum B_{st} \frac{1}{r^{2+s}} \cos \theta \] \hspace{1cm} 5-6

Substitution of the above expression into differential equation 5-4 with expression 5-5 for its right-hand side gives the following expressions for the particular constant \( B_{st} \):

\[ B_{ij} \sum_{st} = 2(1 + vef)(s^2 + s - 2ns) + 2(1 - vef)(n^2 - n - 2) - 2s(v + ef) \], \hspace{1cm} 5-7

where the constant \( B_{ij} \) is as given by Equation 4-17 with \( i = s - n - 2 \) and \( j = t \).
References


CHAPTER 6
CONCLUDING REMARKS

1. Conclusion

In conclusion, it can be stated that all basic types of stress problems involving conditions of generalized plane stress (GPS) in circular disks with polar-orthotropic elasticity and with radial variations in thickness and in modulus of elasticity can be adequately treated, within the usual limitations of the assumptions of GPS, by using an Airy-type stress function and at most two body-force functions. Furthermore, this same approach is valid, without limitation, to problems involving conditions of plane strain (PS) in relatively long circular members, solid or hollow, with polar-orthotropic elasticity and with a radially-varying modulus of elasticity. Also, the classical theory of thin elastic plates can be extended to include all basic types of problems involving plate bending (PB) in circular plates with polar-orthotropic elasticity and with radial variations in thickness and in modulus of elasticity.

It is important to note that the elastic behavior of circular elements under the three conditions GPS, PS, and PB, as governed by Equations 2-8, Equation 2-9, and Equation 4-9, respectively, are
governed by the same differential equation, provided that for GPS and PS, the only body forces acting are derivable from body-force functions which make the right-hand side of Equation 2-8 or 2-9 equal to zero, and for PB no normal pressure is acting. The equation can be written as follows:

\[
\psi \left[ \frac{\partial^4}{\partial r^4} + \frac{2}{r} \frac{\partial^3}{\partial r^3} - \frac{e}{r^2} \frac{\partial^2}{\partial r^2} + \frac{e}{r^3} \frac{\partial}{\partial r} + \frac{2(c^*-\mu)}{r} \frac{\partial^4}{\partial r^2 \partial \theta^2} \right.
\]

\[
- \frac{2(c^*-\mu)}{r^3} \frac{\partial^3}{\partial r \partial \theta^2} + \frac{2(c^*-\mu+e)}{r^4} \frac{\partial^2}{\partial \theta^2} + \frac{e}{r^4} \frac{\partial^4}{\partial \theta^4} \Phi
\]

\[
+ \frac{d\psi}{dr} \left[ \frac{2}{r \partial r} - \frac{2}{r} \frac{\partial^2}{\partial r^2} - \frac{e}{r^2} \frac{\partial}{\partial r} + \frac{2(c^*-\mu)}{r} \frac{\partial^3}{\partial r \partial \theta^2} \right] \Phi
\]

\[
- \frac{2(c^*-\mu)}{r^3} \frac{\partial^2}{\partial \theta^2} \right] \Phi + \frac{2}{r \partial r} \left[ \frac{\partial^2}{\partial r^2} - \frac{\mu}{r} \frac{\partial}{\partial r} - \frac{\mu}{r} \frac{\partial^2}{\partial \theta^2} \right] \Phi = 0.
\]

where the functions \( \Phi \) and \( \psi \) and the constants \( \mu \) and \( c^* \) have the following definitions for the various stress conditions:

<table>
<thead>
<tr>
<th>Stress Condition</th>
<th>( \Phi )</th>
<th>( \psi )</th>
<th>( \mu )</th>
<th>( c^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>GPS</td>
<td>( \phi )</td>
<td>( S )</td>
<td>( \nu )</td>
<td>( c )</td>
</tr>
<tr>
<td>PS</td>
<td>( \phi )</td>
<td>( E^{-1} \nu/(1-\nu) )</td>
<td>( c )</td>
<td></td>
</tr>
<tr>
<td>PB</td>
<td>( w )</td>
<td>( D )</td>
<td>( -\nu )</td>
<td>( d )</td>
</tr>
</tbody>
</table>

where \( \phi \) is the stress function, \( w \) is lateral deflection, \( \nu \) is Poisson's ratio, \( S \) is compliance \( [1/(Eh)] \), \( E \) is the modulus of elasticity which may vary with radius, \( D \) is the flexural rigidity \( [(Eh^3/12)/(1-\nu^2)] \), \( h \) is the thickness, and \( c \) and \( d \) are shear coefficients.
2. **Suggestions for Future Research**

The scope of the present investigation has been limited to analysis of circular disks. Other than this one limitation on the planform and configuration, the analysis has been fairly general in that it has considered nonsymmetric distributions, varying thickness, and non-homogeneous and polar-orthotropic material properties. The main classes of problems considered include "disks" subjected to generalized plane stresses (GPS), "plates" undergoing plate bending (PB), and heat transfer in disks with plane temperature distributions but not with thermal gradients through the thickness.

Some of the related topics of technological importance which have been omitted in this investigation are discussed briefly in the remainder of this section.

**Generalized plane stresses** -- The only types of dislocations considered in this study were the classical Volterra-type dislocations, in which the displacements but not the stresses were discontinuous across the dislocational barrier. However, as described in detail in Chapter 2, Section 1, more complex dislocations, having discontinuities in stress as well as in displacement, have recently been treated in the literature for the uniform-thickness case. It appears to be worthwhile to study the behavior of such dislocations in varying-thickness disks.
The only type of nonhomogeneity considered in this study was the familiar kind, first treated by Hruban (Chapter 2, Section 1), for the isotropic case, in which only the moduli of elasticity were varied. The moduli of elasticity in the radial and tangential directions were varied in such a manner that their ratio remained constant (i.e., the orthotropic ratio was held constant). However, in order to apply polar-orthotropic disk theory to a compressor impeller, for example, with radial vanes having stiffness decreasing at increasing radii, it is necessary to consider the equivalent polar-orthotropic disk as having both varying thickness (unless the impeller disk itself has uniform thickness) and varying orthotropic ratio (to account for the varying stiffness of the blade). The theory presented in Chapter 2 could be modified to account for such a variation by treating the orthotropic ratio \( \varepsilon \), as well as the compliance \( S \), as a function of the radius.

Apparently the only exact analyses involving GPS in noncircular disks of varying thickness are those of Shepherd (1) and of Conway (2), who both treated a rectangular planform. Recently, Armstrong (3) gave an approximate solution for a varying-thickness circular disk with a single noncentral circular hole.

In view of the paucity of analyses of GPS in varying-thickness disks with noncircular or nonsymmetric planforms, it appears that such an analysis in general curvilinear coordinates, such as those carried out for the general three-dimensional elastic case by Deuker (4)
and later, independently, by Freiberger (5), would be worthwhile. An important application of this analysis would be a solution of a varying-thickness circular disk with a noncentral circular hole, such as treated by Jeffrey (6) for the uniform-thickness case using bipolar coordinates.

Recently there has been considerable success in the solution of plane elasticity problems involving uniform thickness by application of the complex-variable methods of Kolosoff and Muskhelishvili (7). Thus, it would seem to be of interest to develop these techniques so that varying-thickness problems can be treated.

**Plate bending.** -- The same remarks concerning the need for an analysis considering varying orthotropic ratio in GPS apply here. Here the application would be to plates reinforced with radial or circumferential-ring stiffeners.

Although the governing equations are analogous, there appears to have been considerably more work on the analysis of PB of noncircular planforms of varying thickness than on GPS involving the same geometry. Perhaps the first analysis in the PB category is that of Gran Olsson (8), who treated the rectangular planform in 1932. Gran Olsson was also first to treat varying-thickness plates with an elliptic planform (9). For the case of varying-thickness PB, the triangular planform was probably first analyzed by Göttlicher (10). Apparently, for varying-thickness PB, the skew (or parallelogram) planform was treated first by Fung (11) and the circular sector planform by Aronson (12).
It is interesting to note that since exact solutions are now available for the varying-thickness circular plate, it would seem that the time is ripe for the application of the point-matching method, an approximate method of solution introduced by Barta (13) in 1937 and recently extended in application by Conway and his associates (14), to problems of noncircular-planform plates that vary in rigidity with radial location only. Examples include a slightly-tilted turbine rotor with internal splines, bending of a varying-thickness circular plate with a small crack or notch at either the internal or external boundary, and bending of a varying-thickness semicircular plate.

Literature on bending of varying-thickness elastic plates resting on Winklerian-elastic foundations is sparse. The only such analysis known to the author is that of Conway (15), who in 1955 treated circular annular plates and noted an analogy between bending of a plate on an elastic foundation and lateral vibration of the same plate.

**Vibration of plates and disks.** --Analysis of lateral bending vibration of circular plates goes back to 1850, when Kirchhoff (16) published what is attributed to be the first successful analysis of the problem, for the uniform-thickness case. Perhaps the first treatment of such vibrations in varying-thickness plates is an approximate analysis by Stodola (17) in 1914, using the Rayleigh-Ritz energy method. The first exact solution of this problem is credited to Dubois (18), who in 1927 considered plates with general hyperbolic and quadratic
thickness profiles. An interesting analogy between lateral vibration of a linearly-varying-thickness plate and those of a right-circular cone was noted by Conway (19).

The effect of rotational speed on lateral vibration of plates, of interest in turbine rotors, was considered by Lamb and Southwell (20) in 1921 and by Southwell (21) alone the next year.

Torsional oscillation of disks of arbitrary thickness profile was treated by Grammel (22), who also analyzed in-plane vibration of such disks (23).

**Elastic stability.** Perhaps the first analysis of elastic stability in varying-thickness plates is that of Gran Olsson (24), who in 1938 treated the buckling of an annular plate with quadratically-varying thickness and subjected to uniform radial compression. Probably the first treatment of elastic stability in anisotropic varying-thickness plates is due to Shulezhko (25) in 1942. The effect of varying body forces on buckling of varying-thickness plates was treated by Federhofer (26) in 1940 for circular plates subjected to radially-varying body forces. Buckling of varying-thickness plates subjected to in-plane shearing stresses was considered by Federhofer and Egger (27).

Under sufficiently nonuniform temperature distributions or with sufficient mechanical restraint, changes in temperature can produce a buckling phenomenon known as thermal buckling. Apparently this was first treated, for uniform-thickness plates, in 1950 by Grigolyuk (28).
and also by Lindholm (29). In 1958 Bogdanoff, Goldberg, and Helms (30) considered thermally-induced buckling of varying-thickness circular plates and were able to predict correctly this type of failure in gas-turbine wheels.

Thermal buckling of anisotropic, nonhomogeneous plates, which, as shown in Chapter 4 are equivalent to isotropic plates of varying thickness, were treated in 1958 by Langhaar, Miller, and Boresi (31).

As is apparent from this brief review of the state of technology in this field, it appears to be a fruitful area for future research, particularly in view of the trend toward more efficient design, i. e., varying thickness, in architectural and space-vehicle applications.

Elastic shells: stresses, elastic stability, and vibration. --

Apparently the first analysis of stresses in an elastic shell with varying wall thickness is due to H. Reissner (32), who in 1908 treated a circular cylindrical tank with linearly-varying wall thickness. In 1925 Steuermann (33) considered a tank with quadratically-varying wall thickness, and later Sonntag (34) analyzed a tank having a periodic variation in wall thickness. A very practical application of varying-thickness cylindrical-shell theory was recently made by Rodabaugh (35) in connection with a linearly-tapered-wall transition joint for connecting circular pipes having two different wall thicknesses but the same outside diameter.
Spherical shells of varying wall thickness are believed to have been considered first by Meissner (36) in 1915. Honegger (37) was perhaps the first to treat conical shells with varying wall thickness.

General analyses of varying-wall-thickness shells of revolution have been contributed by E. Reissner (38) in 1947 for shallow shells and recently by Herzog (39) for general shells of revolution subject to arbitrary loadings. Apparently no general analyses for varying-thickness shells which are not shells of revolution have yet been carried out. However, it is the belief of the author that the Airy-stress-function approach applied in this study to varying-thickness disks and applied to membrane shells of revolution by Langhaar (40) should prove to be quite successful.

Many general classes of shells have not yet been analyzed for the varying-thickness case. Some of these are of practical importance in either engineering or architecture. These include toroidal shells, which have applications as pipeline expansion joints and in the housings of fluid-power machinery, such as centrifugal pumps and hydraulic turbines. Ellipsoidal shells of revolution can be made in such a manner that the wall thickness varies with the radius and such shells are widely used as heads for cylindrical pressure vessels, yet there appears to be no analysis available for them. Paraboloidal and hyperboloidal shells have both seen architectural application in recent years, yet they have not been treated for the varying-thickness case in any analyses.
known to the author. Helicoidal shells, as exemplified by hollow-steel propeller blades, have not yet been treated for the case of varying thickness.

The only elastic-stability analysis of a varying-thickness shell known to the author is that of Federhofer (41) in 1952, in which he treated a circular cylindrical shell. In view of the paucity of work in this aspect of shell analysis and its importance in missile and space-vehicle applications, this should prove to be a fertile, although difficult, field for further study.

The literature on vibration of shells with varying wall thickness is almost nonexistent, with two noteworthy exceptions: an analysis of a cylindrical shell by Falkewicz (42) in 1951 and an investigation of a bell-shaped shell by Aoki (43) in 1954. Obviously, this aspect of shell analysis is still wide open for further study.

Inelastic stresses in disks and plates. -- In many engineering applications, the determining factor in the design of a structural element is its inelastic behavior instead of its elastic performance. Here inelastic behavior is understood to include behavior both in the plastic range and under creep conditions (viscoelastic or viscoplastic). The plastic range is of importance in connection with all room- or low-temperature applications in which deflection is critical or in which prevention of fracture is required, with the possible exception of so-called brittle fracture and fatigue failure after a large number of
loading cycles. Creep is important in most high-temperature applications, including steam, gas-turbine, and nuclear powerplants, and aircraft, missile, and space-vehicle structures.

One of the earliest GPS analyses of disks subject to plastic-range stresses was the study of Nadai and Donnell (44) on rotating disks in 1929. Varying-thickness disks of arbitrary profile were treated by Manson (45) in 1952. The optimum profile shape for a rotating disk in the plastic range was determined by Rabotnov (46) in 1948. Recently a more general approach for the optimum design of thickness profiles for disks subject to plastic-range GPS due to various types of loading was contributed by Hu and Shield (47).

Plastic-range bending of varying-thickness plates was considered by Grigor'ev (48) in 1954 and the optimum thickness variation for such plates was later discussed by Onat, Schumann, and Shield (49).

Perhaps the first analysis of creep stresses in varying-thickness disks under GPS conditions was that of Odqvist (50) in 1934. Creep stresses in disks with arbitrary thickness variation were treated by Popov (51) in 1947, followed by Millenson and Manson (52) a year later. So far as known to the author, the optimum thickness variation for a rotating disk under creep conditions has not yet been determined.

The author is not aware of any studies of creep stresses in varying-thickness plates subject to bending, although this problem was treated in the uniform-thickness case as early as 1956 by Lin (53).
Heat transfer. -- There appears to have been very few analyses of heat transfer in noncircular-planform disks that vary in thickness, with the exception of those of rectangular planform. As in the problem of stresses in noncircular disks, this problem could perhaps best be attacked in approximate fashion by use of the point-matching method, provided that the thickness varies with radial location only. To the best of the author's knowledge, this method has not yet been applied to any heat-transfer problems, yet its general concept does not limit it to problems of the types it has been applied to so far, namely, plane elasticity, plates, buckling, torsion, and vibration.

Finally, a reliable, yet easily applicable, approximate theory for treating thermal gradients through the thickness of a varying-thickness plate still remains to be developed.
References


39. Herzog, M., "The Fundamental Equations of the Surface of Thin Elastic Shells with Varying Wall Thickness and Finite Deformations Subject to Arbitrary Loading and Temperature Variation" (in German), Bautechnik, vol. 37, pp. 27-29 (1960). (AMR 14, Rev. 147, 1961)


APPENDIX

HEAT TRANSFER IN DISKS

1. Introduction and Historical Background

Heat-transfer analysis of disks first became important in connection with cooling fins of air-cooled reciprocating engines. Figure 13 shows this and other examples of "extended surfaces".

Apparently the first analysis of cooling fins was that of Parsons and Harper (1) for uniform-thickness rectangular-planform fins in 1922. In the same year, Harper and Brown (2) analyzed uniform-thickness annular fins, as well as tapered-thickness rectangular fins. They also introduced a figure of merit, called the fin effectiveness, for extended surfaces. It is defined as the ratio of the rate at which heat is actually dissipated by both fin surfaces to that at which it would be dissipated if the surfaces were maintained at the temperature of the base of the fin.

Tapered-thickness annular fins apparently were first considered by Schmidt (3) in 1926 and independently by Binnie (4) in the same year. Schmidt also enunciated a criterion for selecting the optimum profile
Figure 13. Various types of extended surfaces for heat transfer. Heated surfaces are marked H; primary cooling surfaces are denoted by C.
shape for a fin. He stated that a fin which has a uniform heat flux\(^1\) along its length will have a minimum volume of material. This criterion was later placed on a sound mathematical basis, with the use of variational calculus, by Duffin (5). Schmidt determined the optimum profile for a rectangular fin; the optimum profile for an annular fin was reported by Jakob (6).

Solutions for a whole class of varying-thickness fins, either rectangular or annular, were introduced by Gardner (7) in terms of Bessel functions. Mathematical aspects of various assumed profile shapes have been further discussed by Tate and Cartinhour (8).

In all of the analyses previously described, the surface heat loss was assumed to be directly proportional to the difference between the fin temperature and the temperature of the surrounding medium and the latter temperature was taken to be either uniform or zero. Now this heat-loss relation, often called Newton's law of cooling, is reasonably applicable to heat transfer by forced convection or even approximately so to heat transfer by a combination of forced and free convection. The presence of forced convection in many technical applications, such as in cooling fins for both air- and liquid-cooled reciprocating engines, as well as to rotating disks, and also the mathematical simplicity of Newton's law of cooling, has led to the widespread usefulness

\(^{1}\) For a material with constant thermal conductivity, this requires a uniform temperature gradient.
of the linear heat loss assumption. An analysis for a fin having a non-linear heat loss proportional to the 5/4 power of the temperature, i.e., subject to free convection entirely, has been given by Hutcheon and Spalding (9). Extended surfaces in which radiation is the primary mode of heat transfer, as in gas turbines operating at very high altitudes, have recently been treated by Bartas (10), Chambers and Somers (11), Bartas and Sellers (12), Liu (13), and Wilkins (14).

In all of the early analyses of extended surfaces subject to forced convection, it was assumed that the surface heat-transfer coefficient was uniform over the entire surface. However, an experimental study conducted by Chai (15) has shown that there is a large variation in the heat-transfer coefficient over the surface, the coefficient being significantly greater at the tip than at the root of the fin. Two recent analyses which have taken this variation into consideration for rectangular fins are those of Melese (16), who assumed a linearly-varying coefficient, and Han and Lefkowitz (17), who treated a coefficient varying with any power of the distance along the fin.

In the analyses mentioned above, it was assumed that there was no internal heat generation within the fin. However, in connection with the cooling of nuclear reactors, internal heat generation should be considered. Two recent analyses taking this into account are those of Saadeh (18) and Minkler and Rouleau (19).
In most engineering applications, the extended surface is sufficiently thin in comparison to its other dimensions to enable temperature gradients through the thickness to be neglected. It is interesting to note here the analogy to the assumption of generalized plane stress in elasticity. Thick uniform-thickness fins, in which the axial temperature gradients cannot be neglected\(^2\), have been analyzed for the case of rectangular planform by Avrami and Little (20) and for annular planform by Keller and Somers (21). Such analyses are said to be two-dimensional in nature, since they consider temperature gradients in the radial and axial directions.

All of the researches described above which involved disks assumed that the heat flow and the temperature distribution were axisymmetric. Thus, with the exception of the analyses which considered axial heat flow, heat flow was assumed to take place in only one direction: radial. It appears that the only treatment of fins in which non-symmetric heat flow is considered is that of Murray (22) for annular fins of uniform thickness. However, his analysis was still restricted to one radial plane of symmetry. It is readily apparent that the temperature distribution in an annular fin on a cylinder which has its axis

\(^2\)However, the boundary conditions are such that the temperature distribution is symmetrical about the disk midplane. The problem of heat transfer for boundary conditions which give rise to temperature distributions which are not symmetrical about the midplane is discussed later.
perpendicular to the cooling-fluid flow direction would certainly not be axisymmetric.

All of the extended-surface heat-transfer analyses known to the author assume steady-state flow, i. e., that heat flow and temperature are independent of time. For purposes of economic evaluation this assumption is justified, since the time required to reach steady-state conditions and to shut down is usually very short compared to the time spent operating at steady-state conditions. However, in many cases, the maximum thermal-stress conditions would be expected to be reached under transient conditions rather than in the steady state.

Another usual assumption in extended-surface heat-transfer analysis is that the material is homogeneous and isotropic with respect to its thermal conductivity. This assumption certainly appears to be justifiable at least on a macroscopic scale for the large majority of engineering applications.

The usual assumption that the thermal conductivity is independent of the temperature is not strictly true for most materials over very wide ranges of temperature. However, in many cases either the unit change is small or the temperature range is sufficiently narrow, so that good results can be obtained by using a mean value of the conductivity over the temperature range involved.
In the analysis of heat transfer in fins, the following boundary conditions are usually used:

1. The temperature at the base of the fin is a known constant.
2. There is no radial heat flow at the tip of the fin, i.e., the temperature gradient is zero at the tip.

The latter assumption implies that the peripheral surface at the tip is perfectly insulated. This, of course, is not strictly correct. However, in their 1922 paper Harper and Brown (2) suggested using a mathematical or corrected length greater than the actual length so that the heat which physically leaves at the tip of the actual fin would leave at the added length section and none would leave at the mathematical tip. For a rectangular fin, Harper and Brown suggested an addition to the length of one-half the actual tip thickness. The validity of this assumption was confirmed by Harper and Brown and later by Dusinberre (23).

The principal difference between heat transfer in a heated rotating disk and that in an annular cooling fin is in the nature of the dependency of the surface heat-transfer coefficient with radius. There are two basically different regimes for fluid flow, and thus heat loss, from rotating disks operating in an open space (not in a housing).
Laminar flow (24) occurs near rotating disks at Reynolds numbers below approximately 500,000 (25) and thus is usually encountered over only a small percentage of the surface of present-day high-speed disks, such as turbine disks. Turbulent flow occurs at Reynolds numbers above approximately 500,000 and thus is of considerable technical importance. Experimental studies conducted independently by Cobb and Saunders (26) and by Zaloudek (27) have indicated that the local surface heat-transfer coefficient for turbulent flow is proportional to the 0.8 power of the Reynolds number, and thus proportional to the 1.6 power of the radius. Recently, Rotem (28) made an analysis which indicated that the same relationship is applicable to disks rotating in a casing or housing with small clearance, except that with the housing, the constant of proportionality is somewhat smaller. However, there is some evidence to the contrary.

Reynolds number $N_{Re}$ is a dimensionless parameter associated with frictional effects in fluid flow. It can be defined as flow velocity times the characteristic dimension divided by the kinematic viscosity $\nu$ of the fluid. For flow to a rotating disk,

$$N_{Re} = 4\pi \Omega r^2 / \nu,$$

where $\Omega =$ rotational velocity and $r =$ radius.
In the case of a turbine disk, the heat input is at the outer periphery, rather than at the inside radius as it is in the case of a conventional external cooling fin. Also, since a turbine disk is thickest at the inside radius, the assumption of perfect insulation at an effective distance from the actual boundary, in order to simplify the second boundary condition is not as realistic as in the case of a cooling fin. In particular, the effective-distance concept breaks down completely for disks having a thickness \( h_i \) at the inside radius \( r_i \) which is larger than \( r_i \). \(^4\)

In certain applications, the temperatures of the environmental media adjacent to the upper and lower surfaces of the plate are quite different. Thus, the axial thermal gradients (i.e., thermal gradients through the thickness) are much greater than the thermal gradients in the plane and thus certainly cannot be neglected. Furthermore, the temperature distribution is no longer symmetric about the disk midplane. Some examples in which such conditions occur are turbine disks, flat heads of high-temperature boilers, and possibly cooling fins with surfaces perpendicular to the coolant flow.

\(^4\)This statement is proved as follows: The actual heat-transfer area at the root is \( 2\pi r_i h_i \), while the substitute area used in the approximation is \( 2(\pi r_i^2) \). Thus, when \( h_i > r_i \), the substitute area is less than the actual root area.
For plates of any profile shape and any planform, the exact
differential equation which governs, assuming uniform thermal prop-
erties, is

\[
\frac{\nabla^2 T}{3} + \rho c \frac{\partial T}{\partial t} = 0, \tag{A-1}
\]

where \( T \) = temperature at a point within the plate; \( \rho \) = mass density of
the material; \( c \) = specific heat of the material at constant strain;
\( t \) = time; \( \nabla^2 \) is the three-dimensional Laplacian operator, which is

\[
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}
\]
in cartesian coordinates and

\[
\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}
\]
in cylindrical coordinates. Even for a circular planform, exact solu-
tions of Equation A-1 have been obtained for only the uniform-thickness
case. For a rectangular plate, a series solution can be obtained; for a
solid or annular circular plate, a solution in terms of Bessel functions
is indicated.

In view of the technical importance of the varying-thickness
case, particularly in turbine rotors, it is only natural that approximate
methods would be used. To the best of the author's knowledge, the
first attempt at an approximate solution considering the axial tempera-
ture drop was made by Marguerre (30) in 1935. He considered
uniform-thickness rectangular plates with different coefficients of convective heat transfer at the upper and lower surfaces of the plate. However, he did not illustrate the application of his analysis to any specific problems.

Another approximate method was given by Kostiuk (31). He outlined a method and stated its applicability to the varying-thickness case; however, he applied it to only the uniform-thickness axisymmetric case.

2. Derivation of the General Equation for Generalized Plane Thermal Conditions

In this derivation, the following assumptions are made:

1. The disk itself contains no heat sources.

2. The thickness of the disk varies smoothly and as a function of the radius only.

3. The thickness of the disk is sufficiently small compared to its other dimensions that temperature gradients through the thickness are negligible. However, this does not preclude heat loss from the disk surfaces provided that the heat loss at the top and bottom surfaces are equal, i.e., symmetric about the midplane of the disk. In other words, the disk may be said to be subject to generalized plane thermal conditions, roughly analogous to generalized plane stress conditions in the theory of elasticity.
4. Newton's linear law of cooling is applicable to heat flow from the disk surfaces, i.e., surface heat transfer takes place by forced convection.

Figure 14 shows a typical disk element for which the heat-flow rates across the various surfaces are to be considered.

The heat-flow rate into the surface at the inner radius of the element is given by

\[ q_{r_1} = 2(f_r - \frac{\partial f_r}{\partial r} dr)(h - dh)(r - dr)d\theta, \]

where \( f_r \) denotes the mean heat flux in the radial direction.

Neglecting higher products of differentials, the above equation reduces to

\[ q_{r_1} = 2 [rhf_r - \frac{\partial}{\partial r} (rhf_r)dr]d\theta. \]

In similar fashion, the following expression is obtained for the heat-flow rate out of the surface cut by the outer radius of the element:

\[ q_{r_2} = 2 [rhf_r + \frac{\partial}{\partial r} (rhf_r)dr]d\theta. \]

The heat-flow rate into the right-hand surface of the element is given by

\[ q_{\theta_1} = 2(f_{\theta} - \frac{\partial f_{\theta}}{\partial \theta} d\theta)hdr, \]

where \( f_{\theta} \) denotes the mean heat flux in the tangential direction.
Figure 14. The disk element considered in the derivation.
Similarly, the rate of heat flow out of the left-hand surface is expressed as

$$q_\theta = 2(f_\theta + \frac{\partial f_\theta}{\partial \theta} d\theta) h dr .$$

The total rate of heat transferred by forced convection out of the upper and lower surfaces of the plate is given by

$$q_{z1} + q_{z2} = 8C(T_0 - T') r dr d\theta ,$$

where $C$ = convection heat-transfer coefficient for each surface, $T_0$ = disk temperature, $T'$ = temperature of the surrounding environment, which may be a function of $r$ and $\theta$.

Adding the input heat-flow rates and subtracting the output rates gives the following expression for the total rate of heat gain:

$$4 \frac{\partial}{\partial r} (rhf_{r}) dr d\theta + 4 h \frac{\partial f_\theta}{\partial \theta} dr d\theta$$

$$+ 8C(T_0 - T') r dr d\theta .$$

Now the total rate of heat gain is also given by

$$4 \rho c \frac{\partial T_0}{\partial t} hr dr d\theta ,$$

where $\rho$, $c$, and $t$ are as defined in the preceding section.

The heat fluxes are proportional to their respective temperature gradients, that is,

$$f_r = -k \frac{\partial T_0}{r \partial r} , \quad f_\theta = -\frac{k_\theta}{r} \frac{\partial T_0}{\partial \theta} ,$$

$$A-2$$
where $k_r$ and $k_\theta$ are the thermal conductivities corresponding to heat flow in the radial and tangential directions, respectively.

Using Equations A-2, equating the two expressions of the total heat rate, and simplifying give the following result:

$$
\frac{1}{r} \frac{\partial}{\partial r} \left( r k \frac{\partial T}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( k_\theta h \frac{\partial T}{\partial \theta} \right) - 2C (T_o' - T) + \rho c h \frac{\partial T}{\partial t} = 0. \quad A-3
$$

For steady-state heat transfer and constant thermal conductivities, Equation A-3 reduces to the following expression:

$$
k_r \frac{\partial}{\partial r} \left( r h \frac{\partial T}{\partial r} \right) + k_\theta \frac{\partial}{\partial \theta} \left( h \frac{\partial T}{\partial \theta} \right) - 2C (T_o' - T) = 0. \quad A-4
$$

For the case of isotropic thermal conductivity, this equation further reduces to:

$$
\frac{1}{r} \frac{\partial}{\partial r} \left( r h \frac{\partial T}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( h \frac{\partial T}{\partial \theta} \right) - \frac{2C}{k} (T_o' - T) = 0, \quad A-5
$$

where $k = \text{common value of } k_r \text{ and } k_\theta$.

Finally, for uniform thickness ($h = H$), the above equation simplifies to the following expression:

$$
\left[ \frac{\nabla^2}{2} - \frac{2C}{kH} \right] (T_o') = - \frac{2C T'}{kH}, \quad A-6
$$

where $\nabla^2$ represents the two-dimensional Laplacian operator.
3. **General Solution for a Disk with Power-Function Distributions of Thickness and Local Heat-Transfer Coefficient**

In addition to the four basic assumptions listed at the beginning of the preceding section, the following assumptions are now made:

2a. The thickness of the disk is given by \( h = h_0 r^{-n} \), where \( h_0 \) and \( n \) are constants.

5. The local heat-transfer coefficient on each surface is given by \( C = C_0 r^s \), where \( C_0 \) and \( s \) are constants.

6. The thermal conductivities in the radial and tangential directions are independent of temperature and of position on the disk.

7. Heat flow and temperature distribution are independent of time.

8. One boundary \( (r = r_1) \) is insulated \( (\partial T/\partial r = 0) \) and the other boundary \( (r = r_2) \) has a temperature distribution given by the following Fourier series:

\[
T = \sum_{m} a_m \cos m\theta
\]

plus a similar sine series omitted here for brevity.
Under the conditions stated above, the governing differential equation \( A - 4 \) reduces to the following expression:

\[
\frac{\partial^2 T}{\partial r^2} + \frac{1 - n}{r} \frac{\partial T}{\partial r} + \frac{b}{r^2} \frac{\partial^2 T}{\partial \theta^2} - C_1^2 r^{n+s} T = - C_1^2 r^{n+s} T' \quad A - 8
\]

where \( b \equiv \frac{k_\theta}{k_r} \) and \( C_1^2 \equiv \frac{2C_o}{k_{ho}} \).

Now attention will be concentrated upon the solution of Equation \( A - 8 \). A complementary solution of the following form is assumed:

\[
T_o = f(r) \cos m \theta \quad A - 9
\]

where \( m \) is an integer constant, and the form of \( f(r) \) is yet to be determined. Substituting the solution form into the differential equation and multiplying by \( r^2 \) results in the following ordinary differential equation

\[
r^2 f'' + (1 - n) rf' - (C_1^2 r^{2+n+s} + bm^2) f = 0 \quad A - 10
\]

where a prime denotes an ordinary derivative with respect to \( r \).

Now the so-called generalized Bessel equation, as given by Sherwood and Reed (32) following a suggestion by R. D. Douglass, is

\[
r^2 f'' + [(1 - 2M) r - 2ar^2] f' + \left[ \frac{p}{a_1} \frac{2}{r^p} + a^2 \frac{r^2}{r^2} + a(2M - 1)r + M^2 - p^2 N^2 \right] f = 0 \quad A - 11
\]

and has the following general solution:

\[
f = r^M e^{ar} [A J_N(a_1 r^p) + BY_N(a_1 r^p)] \quad A - 12
\]

where \( A \) and \( B \) are arbitrary constants of integration and \( J_N \) and \( Y_N \) are Bessel functions of the first and second kind of order \( N \).
Comparison of Equation A-12 with Equation A-10 gives these relationships:
\[ a = 0, \quad M = \frac{n}{2}, \quad a_1 = \frac{iC_1}{p}, \quad N = \frac{1}{p} \sqrt{4bm^2 + n^2}, \quad p = \frac{1}{2} (2 + n + s). \]

Thus, the complementary solution of Equation A-8 can be expressed as
\[ T_o = [A J_N \left( \frac{iC r^p}{p} \right) + B Y_N \left( \frac{iC r^p}{p} \right)] r^2 \cos \theta, \quad A-13 \]

where
\[ N = \frac{1}{p} \sqrt{4bm^2 + n^2}, \quad p = \frac{1}{2} (2 + n + s), \quad \text{and} \quad C_1 = \sqrt{\frac{2C_o}{kh}}. \]

4. Application: Example 3 - Nonsymmetric Steady-State Heat Transfer in a Cooling Fin

As an example, a circular annular cooling fin, with varying thickness and varying local convection coefficient and subjected to a nonaxissymmetric temperature distribution on a boundary, is treated here.

The thickness is given by \( h = 0.25 \, r^{-1} \), where \( h \) and \( r \) are both measured in inches; thus, \( h_0 = 0.25 \) and \( n = 1 \). The local surface heat-transfer coefficient is expressed by \( C = 1.25 \, r^{1.47} \), where \( C \) is in units of Btu hr\(^{-1}\) ft\(^{-2}\) F\(^{-1}\) and \( r \) is in inches (i.e., \( C_0 = 1.25 \) and \( s = 1.47 \)).

The thermal conductivity is independent of direction (i.e., \( b = 1 \)) and is constant at a value of \( k = 24 \) Btu hr\(^{-1}\) ft\(^{-1}\) F\(^{-1}\).

The environmental temperature is assumed to be uniform and constant. Therefore, all temperatures referred to hereafter are measured above the environmental temperature and \( T' \) can be taken to
be equal to zero. The fin is assumed to be insulated at an effective outside radius \( r_1 = 2 \) inches; at its inside radius \( r_2 = 1 \) inch, it is subjected to a steady-state temperature distribution given by \( 100 \cos \theta \), i.e., \( a_m = 100 \) and \( m = 1 \).

The problem is to determine an expression for the temperature distribution in the fin.

The first step in the solution is to compute the values of the three basic parameters appearing in Equation A-13. These are \( N = 1/2 \), \( p = 2.235 \), and \( C_1 = 2.235 \). Then, according to Equation A-13, the solution is expressed by

\[
T_o = [A J_{1/2}(ix) + B Y_{1/2}(ix)] r^{1/2} \cos \theta ,
\]

where \( x = c_1 r^p/p = C_1 r^{2.235}/2.235 \).

Now it so happens that Bessel functions of the first and second kinds of order \( 1/2 \) for imaginary arguments can be expressed in terms of hyperbolic functions as follows (33):

\[
J_{1/2}(ix) = \frac{1 + i}{\sqrt{\pi x}} \sinh x , \quad Y_{1/2}(ix) = -\frac{1 + i}{\sqrt{\pi x}} \cosh x .
\]

The boundary conditions can best be expressed in terms of the function \( f(r) \) appearing in Equations A-9 through A-12. In view of the above relations, \( f \) can be expressed as:

\[
f = (A_1 \sinh x + B_1 \cosh x) x^{-0.2765} .
\]
The boundary conditions, expressed in terms of \( f \) and \( x \) are:

\[
\left. \frac{df}{dx} \right|_{x=x_1} = 0 , \quad (f)_{x=x_2} = T_m = 100 ,
\]

where \( x_1 = C_1 r_1^p/p = 4.7 \) and \( x_2 = C_1 r_2^p/p = 1 \). These lead to two simultaneous equations in \( A_1 \) and \( B_1 \), which, when solved, give \( A_1 = -272 \) and \( B_1 = 272 \).

Thus, the final expression for the temperature distribution in the fin is:

\[
T_o = 272 (\cosh x - \sinh x) x^{-0.2765} \cos \theta ,
\]

where \( x = r^2.235 \).
References


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From 1951 to 1959, I was successively employed as a junior design engineer and aeronautical design engineer in industry and as principal mechanical engineer at Battelle Memorial Institute, and served on active duty as an Air Force project officer. During this period, I pursued graduate extension courses from the University of Florida and the University of Maryland, attended a special summer seminar in creative engineering at the Massachusetts Institute of Technology, and began studies toward the doctorate at The Ohio State University.

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