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AN ANALYSIS OF A MOTOR FREIGHT

SCHEDULING PROBLEM

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Robert Frederick Miller, B.I.E., M.Sc.

* * * * * *

The Ohio State University

1961

Approved By

Paul N. Lchozky
Adviser
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ACKNOWLEDGMENTS

The scheduling problem which provides the referent for this investigation was initially studied as a part of a research project for which the writer served as Principal Investigator. This project was undertaken on a contract between the Systems Research Group, Engineering Experiment Station, The Ohio State University, and Suburban Motor Freight, Inc., 1100 King Avenue, Columbus, Ohio. The problem that was formulated during the early stages of this project presented a number of interesting theoretical questions. An independent study of certain of these questions led to the method of analysis and the results that are reported here.

To Dr. Daniel Howland, Director, Systems Research Group, and to Professor Robert S. Green, Associate Dean, College of Engineering and Executive Director, Engineering Experiment Station, I express my thanks for the support and encouragement that was generously provided. In addition, appreciation is due for the expert secretarial and report processing services that were made available. Mrs. Donna Entsminger's work in reducing a difficult manuscript to type is especially appreciated. Mr. J. R. Riley, President, Suburban Motor Freight, Inc. provided, to a large extent, my introduction to the problems of the motor freight industry, and constantly
encouraged the development and trial of new management procedures. This source and testing-ground of research ideas is gratefully acknowledged.

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CHAPTER 1

INTRODUCTION

Industrial Engineering Background

Since F. W. Taylor's initial development of stop watch time study in 1881, the number and variety of the techniques and areas of interest of the industrial engineer have multiplied to an impressive extent. The techniques that followed include micromotion and cyclegraph analysis, operations charting, predetermined time systems, a variety of incentive pay plans, job evaluation, cost accounting, budgeting, Gantt charting, statistical quality control, and organization analysis. During and following World War II came rapid developments in mathematical programming and applications of probability theory. More recent developments include the construction of normative decision models and their programming on computers to obtain a mechanized decision process for routine operations such as inventory control. The simulation of business operations on computers as an aid to planning future operations is another recent innovation of considerable significance.

These newer techniques have not replaced the more traditional approaches but have supplemented them and extended the scope of the problem.
areas in which the industrial engineer might contribute. The basic nature of
the problems which he faces has, however, significantly changed since 1881.
An outline of industrial engineering activity during three time periods will
serve to indicate a significant feature of this change and provide a motivation
for the method of analysis adopted here.

Taylor's method of "developing elementary scientific knowledge" of work
methods and performance times provides a convenient point of departure. In
The Principles of Scientific Management he outlines the stop watch time study
technique that was employed to find a "simple law" of work [23, pp. 117-118]¹.
For our purposes, it is significant to note that Taylor began his work by study­
ing "especially skillful" tradesmen to discover and standardize the best way of
performing a task. Outlined is a systematic way of transferring trade
knowledge of work methods and output capabilities from tradesmen to manage­
ment and the public domain.

If best methods were not thought to be currently practiced, Taylor
resorted to experimentation to develop them. His shoveling experiments are
a classic example. The "one best way" for a man to handle granular materials
was to use a shovel which held about twenty-one pounds. No other materials­
handling method was evidently considered if any alternatives in fact existed.
Shoveling was the materials-handling method.

¹ Numbers in brackets refer to the References which are collected at the end of this paper.
From these beginnings Taylor and his associates went on to develop the management systems which were published during the interval from 1900 to 1920. Observation, experimentation, and quantitative analysis were employed to discover and codify best work methods, management techniques, and principles of organization. The methods of physical science were to provide a model for building a science of management.

Villers has characterized the years 1921-1945 a "period of consolidation" [25, p. 14]. Although new techniques were devised during this period, perhaps the most significant development was the general adoption by industry of what was essentially the Taylor system of management. An interesting change in the character of management literature did, however, evolve during this period. Writers referred less frequently to the notion of "the one best way," and a large "methods improvement" literature developed. Operation Analysis, "the over-all study of the process," was proposed to improve materials, produce design, tools, materials-handling methods, operator motion path, and other characteristics of a production process [2]. The approach is characterized by a search for alternatives to present methods and the advocacy of a change if, with respect to some criterion, an alternative is an "improvement" of current practice. The problem of conflicting criteria seems to have been left to professional expertise as an element of the art of engineering. A rapidly developing technology provided competing methods of manufacture and alternative mechanisms of management control. A generalized interpretation
of the Gilbreth "search for the one best way" was required.

During World War II scientists and applied mathematicians undertook analytic studies of military operations, and the name "Operations Research" was coined to identify this kind of activity. After the war, a serious attempt was made to apply the approach and methods of analysis of "O.R." to the operations of civilian enterprises. In *Introduction to Operations Research*, Churchman, Ackoff, and Arnoff clearly state the aims and objectives of this "new science" [7, Ch. 1].

O. R. tries to find the best decisions relative to as large a portion of a total organization as is possible [p. 6].

O. R. is here defined in terms of its important goal: an over-all understanding of optimal solutions to executive-type problems in organizations [p. 7].

The concern of O. R. with finding an optimum decision, policy, or design is one of its essential characteristics. It does not seek merely to find a better solution to a problem than the one in use; it seeks the best solution [p. 8].

The general approach that is advocated can be characterized as follows: The primitive notion is the "operation." A particular operation is represented by a system of one or more mathematical expressions which constitute a "model" of the operation. A decision problem is obtained by identifying the "decision variables" of the model and specifying an appropriate "criterion function." The model is solved to find values of the decision variables for which the criterion function takes the desired extreme value.
A characteristic example of such a model is the "linear programming problem" [7, Ch. 11]. In this case, the model of an operation is constituted by an interpreted system of linear inequalities. A solution of the system for which a linear function of the indeterminates takes a maximum (or perhaps minimum) value is desired.

More generally, given any set of alternative courses of action, the problem considered is that of selecting the best alternative. Clearly, to select the "best" alternative, the characteristics of given alternatives which are relevant to a decision must be identified, and a principle of choice must be adopted. A problem of "value measurement" is then encountered. Titles such as "Measurement - The Unsolved Problem" [20, Ch. 23], Prediction and Optimal Decision [6], and Measurement: Definitions and Theories [8] are evidence of the concern. To an executive faced with responsibility for decision or to an engineer responsible for recommending a course of action or an engineering design, the criterion problem is not to be denied. The question is: What, if anything, can the methods of science contribute? The aspects of this general problem which are of particular interest here are outlined below.

Industrial Engineering is viewed as being generally concerned with the design of integrated systems of men, materials, and machines and system control mechanisms. For our immediate purposes design is the key word. One of the essential characteristics of a significant design problem is the
existence of alternatives. Another is that there be major aspects of the problem which are unique to a particular situation. If either element is missing we have what might be termed an "installation" problem.

A modern industrial engineer concerned with efficient handling of iron ore, ashes, or other granular materials has available, in contrast to Taylor's time, an almost endlessly diverse array of materials-handling methods to do the job. Almost certainly the use of a shovel will not even to considered; indeed, a benevolent technology will provide an especially designed mechanism to do the job if desired. Similarly, to control the flow path, quantity, and timing of in-process items through a plant, various production control methods are available. They range from a "let the foreman do it" plan to the use of competely mechanized systems controlled by a computer.

To assert that production is to be controlled by routinely solving a linear programming problem is to say that the design phase of the problem is complete. The basic mechanism of control has been selected, the problem has been formulated, and the basis for decision has been specified. All that remains is the routine solution of a particular type of problem. This, of course, may present difficulties which are far from trivial. Here, however, we are particularly interested in the design phase of the problem. Alternative problem formulations (designs) are of more interest than the identification of the attributes of and the restrictions imposed by a particular "operation"; an understanding of a particular situation is initially of more interest than an
understanding of the mechanism of a particular analytic representation; and, given an analytic representation, the implications of choice are of more interest than the derivation of an optimum decision rule. These, then, are the considerations that have provided the motivation for the approach used in the analysis to follow.

The Problem

The "highway operation" which provides the referent for this investigation is considered to be characteristic of that of a general class of motor freight companies. In brief, we treat a problem of scheduling the movements of personnel and equipment between the terminals of a network of a particular type. An analytic representation of this operation and a definition of a decision problem are given. It is initially assumed that a "mechanized" decision process is to be employed; that is, the schedule to be used is to be obtained by solving a given system of equations. We adopt the position that the basic representation of the operation, the given decision problem, and the assumed decision process are subject to change. In particular, it is not initially supposed that the utility function of a decision-maker (a criterion function) is known. We attempt, rather, to present an analysis that, at every stage, could be reviewed and modified as desired by the responsible operating manager. The problem formulated is a programming problem in integers. It is not, however, of the standard linear programming form.
Some Related Literature

Although approximately thirty research papers in the field of transportation are listed in *A Comprehensive Bibliography of Operations Research*, the particular problem type considered here does not appear to have been extensively studied [1]. The paper, "Scheduling Motor Freight Operations," presents background information concerning the industry, and outlines many of the problems involved in scheduling the operations of such an enterprise [21]. The paper, "The Hub Operation Scheduling Problem," presents a solution for the problem of distributing empty trailers among the terminals of such a system [19]. A problem with many of the characteristics of that considered here has been investigated by A. Charnes and M. H. Miller and reported as "A Model for the Optimal Programming of Railway Freight Train Movements" [5]. The authors of this paper suggest that the problem formulated may characterize an important new type of linear programming problem.

Since the publication of the Hitchcock "transportation problem" in 1941, a large literature which treats scheduling problems over directed networks has developed. "The Truck Dispatching Problem" of G. B. Dantzig and J. H. Ramser is an example [11]. This and other related linear programming applications can be viewed as problems of scheduling flows in one direction between the "nodes" of a network. Flow in two directions is one of the significant characteristics of the problem treated here.
The operating problem considered is defined by an interpreted system of linear equations with integer coefficients. A solution in integers is desired for any given integral values of the constants. We have, then, a system of linear Diophantine equations [13, Ch. 2]. A mathematical theory relating to the solutions of such systems in the rational integers, $0, \pm 1, \pm 2, \ldots$, was developed over one hundred years ago [22]. The results of this theory, however, are not in the desired form. Further, since only nonnegative integer solutions are feasible for applications of the type considered here, an added complication is present. A modification of the methods of modern algebra is used. Additional references and related research papers are cited throughout the text which follows.
CHAPTER 2

A MOTOR FREIGHT SCHEDULING PROBLEM

Introduction

Consider the following motor freight network which consists of terminals one through four and the indicated routes between them.

![An Illustrative Network](Fig. 1)

This illustrative network, as well as other members of its general class, is "connected"; that is, there is at least one route between every pair of terminals. A specified route between any two terminals which does not pass through an intermediate terminal is called a "division." It is identified by noting, in either order, the terminals at the ends of the route. A transit in a particular direction over a division will be referred to as a "division trip." The transit time for any division trip over the networks of the class considered here does not exceed a specified maximum driving time for an uninterrupted
trip by a single driver; further, in the case of some divisions or combinations
of divisions, more than one division trip can be made during this specified
time interval. It is assumed that each terminal is both a freight origin and
destination point and that a unique routing is used for all freight shipments
between each terminal pair.

Freight shipments are commonly transported over one or more divi-
sions of such networks by using tractor-trailer combinations. Movements
are scheduled by assigning a sequence of one or more division trips to a
single driver who may move different freight shipments and transport units
during each division trip. Such an assignment or "run" will be specified
by listing the terminals in the order visited. The sequence 3, 1, 4, for
example, denotes a run that consists of division trips 3, 1 and 1, 4. Such
a sequence is a "feasible run" if it traces an unbroken path over the divisions
of a network and if the run time, assumed to be constant, is no greater than
the specified maximum. After each run a minimum rest period is required
before a driver may be reassigned. Since the demand for loaded movement
between any pair of terminals commonly varies from period to period, not
all runs are scheduled on a periodic, fixed-departure-time basis but are,
at least in part, dispatched on demand.

This method of transporting freight shipments over a network with the
specified characteristics identifies the "highway operation," for which a
management problem is to be defined. A current "state" of the operation
is specified, in part, by noting (a) the number of drivers at each terminal who are available for a run; (b) the number of loads of freight available to move in each direction over every division; and (c) the number of runs of each type in progress. A forecast of the number, location, and future availability times of loads is assumed given. Control of future states of the operation is to be accomplished by prescheduling runs to service current and anticipated demand for loaded movement. Some aspects of this control problem are outlined below.

Suppose, for purposes of the immediate discussion, that every division trip over a network constitutes a run of the maximum duration. Then every outbound movement at a particular terminal requires a run which originates at that terminal. If during some period of time the number of available outbound loads and drivers is not the same, a lack of "balance" will be said to exist. In the case of an excess of drivers over loads, either a driver delay or an empty run will be required and if an empty run is to be scheduled, the question of a proper run destination may arise. If there are more loads than drivers, a load delay results since a driver from another terminal can be made available only after a period of time equal to the transit time between terminals plus the required rest period. If load delay is to be avoided, the provision of additional drivers must be initiated with this lead time. Given unbalance at a terminal, the available courses of action include load delay, driver delay, or empty movements. We suppose that certain penalties may be associated with
each of these actions and that load or driver delay is an acceptable alternative only if the unbalance is corrected within a relatively short period of time. A problem of selecting the least undesirable course of action may, then, arise.

If more than one division trip can be made during a single run, the identification of a condition of unbalance, in the sense of a requirement for penalty-incurring actions, cannot be independently made for at least some terminals. Suppose, for example, that runs 3, 1, 4 and 3, 1, 3 are feasible. With respect to terminal 1 these are "through" runs, and the balance between loads and drivers involves the available drivers at terminals 3 and 1, and the loads to be moved from 3 to 1, 1 to 3 and 1 to 4. If there were no through run which included the division trip 1, 4, then drivers must be available at terminal 1 to make all runs which include the division trip 1, 4. If through runs such as 3, 1, 3 and 3, 1, 4 are available, it may be possible to schedule all movements from 1 to 3 and 1 to 4 during these runs. The number of drivers required at terminal 1 to make runs which include these trips may vary from zero to the total number of movements from 1 to 3 and 1 to 4. The number required depends on the runs which can be made "through" terminal 1 and on the availability of drivers and loads to make such runs. For this general case, a means of identifying a penalty-free situation is a first requirement. A further elaboration of some problems of a run dispatcher will be presented in a more formal context.
The following statement will introduce the problem to be formulated. Let the number of drivers at each terminal who are available for run assignments determine the number of runs that are to originate at each terminal during some time period. Further let the desired number of division trips of each type that are to be made during the period be determined by the number of loads that are available to move from each origin terminal to each destination terminal. (Since there is a unique route for every load, the number of division trips of each type is established. If a given load movement cannot be completed during the time period, an intermediate destination is to be assigned.) The problem: Find a "run schedule" (the number of runs of each type to be initiated and completed during the period) that satisfies the above "requirements." Since it may happen that no such run schedule exists, a feasibility criterion is needed. If, for a particular scheduling period, no penalty-free schedule is feasible, then a requirements change is necessary. But alternative requirements changes may be possible, so a question of what change to make is relevant. The problem is complicated by the fact that, for alternative feasible requirements changes, the resulting sets of feasible run schedules are not identical. These considerations can be made explicit by a mathematical statement and analysis of the problem.

Since a set of feasible runs over the divisions of a network is required, a means of generating the set of all such runs precedes an analytic formulation.
Feasible Runs over the Divisions of a Motor Freight Network

Suppose that the driving times over the network of Figure 1 are as noted in the following division trip time matrix, hereafter for brevity to be called a transit matrix or a DTT matrix.

**TABLE 1**

**DIVISION TRANSIT TIMES**

<table>
<thead>
<tr>
<th>Terminal</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-</td>
<td>8</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>-</td>
<td>5</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>5</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>-</td>
<td>-</td>
<td>*</td>
</tr>
</tbody>
</table>

*Blanks denote "no division."

A numerical entry in the i\textsuperscript{th} row and j\textsuperscript{th} column of any DTT matrix indicates that a single division "connects" (is incident on) terminals i and j and that the driving time from i to j is \( d_{ij} \). It is assumed that all division trip times \( d_{ij} \) are less than or equal to the maximum permissible driving time for a single driver. In the following general procedure for generating the set of all feasible runs over the divisions of a network, it is not necessary that \( d_{ij} = d_{ji} \).

All runs that originate at a given terminal \( \delta \) and end after a single division trip can be noted by listing the number pairs \((\delta, j)\), where j takes the values for which there is an entry \( d_{\delta j} \) in the DTT matrix. The number pairs
(i,j) for which \( d_{ij} \) is defined constitute the set of all such single-division-trip
runs. A particular run \( \delta, \rho \) can be extended to form the additional runs \((\delta, \rho, j)\) for which there is an entry \( d_{\rho j} \) such that \( d_{\delta \rho} + d_{\rho j} \leq d^* \), the maximum permissible run time. All runs that consist of two division trips over the
divisions of the network are found by noting, in this way, all feasible single
division trip extensions of each of the runs \((i, j)\). In general, any sequence of
terminal numbers, \( t_1, t_2, \ldots, t_{\eta} \), represents a feasible run if every adjacent
number pair in the sequence identifies an existing division and if the sum of
the division trip times is no greater than the specified maximum. The set of
all feasible runs is generated by adjoining all sequences of \( \eta + 1 \) terminal
numbers (which represent feasible runs) to the list of sequences of \( \eta = 2, 3, \ldots \)
terminal numbers. The process is terminated when there is no feasible
extension of any of the sequences of \( \eta \) terminal numbers.

If there were a division connecting every terminal pair, then every
sequence of terminal numbers, \( t_1, t_2, \ldots, t, \ldots, \) with \( t \neq t_{\eta + 1} \), all \( \eta \),
represents a run. If, however, there is no division connecting some terminal
pair \( \delta, \rho \), a run from terminal \( \delta \) to terminal \( \rho \) must be made via one or more
intermediate terminals. Since the network is assumed to be connected, such
intermediate points exist and the transit can be made. If the transit time is
greater than the maximum for a feasible run, two or more runs are required.
A feasible multi-division-trip run has been termed a "through" run with
respect to the intermediate terminals. The number of such unbroken paths
over the divisions of a network can be readily determined.

The following results was given by Luce and Perry in another context [17]. Let every numerical entry of a DTT matrix be replaced by the entry "1" and let "0" fill all other cells so that an "incidence matrix" $V$ with entries $v_{ij} = 0, 1$ is obtained. An entry $v_{ij} = 1$ indicates that terminals $i$ and $j$ are the end points of a division. A transit from a terminal $\delta$ to a terminal $\rho$ via a single intermediate terminal $j$ can be made, over the divisions of the network, if and only if $v_{\delta j} = 1$ and $v_{j\rho} = 1$. The condition is that $v_{\delta j} v_{j\rho} = 1$. Then $\sum_{j} v_{\delta j} v_{j\rho}$ gives the number of ways of making a transit from $\delta$ to $\rho$ via any single intermediate point. The entries $v_{ij}$ of $V^2$ thus give the number of runs from any terminal $i$ to any terminal $j$ via one intermediate terminal. In general, if the entries of $V^n$ give the number of runs from $i$ to $j$ via $n-1$ intermediate terminals, then the entries of $V^n V^2V^{n+1}$ give the number of runs from $i$ to $j$ via $n$ intermediate terminals. This follows since there are $v_{n \delta j}$ runs from $\delta$ to $j$ via $n-1$ intermediate terminals; each of these can be extended to any terminal $\rho$ with $v_{j\rho} = 1$; and, $\sum_{j} v_{n j} v_{j\rho} = v_{n+1 \delta \rho}$. The proposition is trivially true for $n = 1$. Only some of the through runs would be expected to require feasible transit times.

If a maximum run time of ten hours is assumed, the set of all feasible runs over the network represented by the DTT matrix of Table 1 is as follows.
TABLE 2
THE SET OF FEASIBLE RUNS

<table>
<thead>
<tr>
<th>Run Number</th>
<th>Terminal Sequence</th>
<th>Run Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1, 2</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>1, 3</td>
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<tr>
<td>14</td>
<td>3, 2, 3</td>
<td>10</td>
</tr>
<tr>
<td>15</td>
<td>4, 1</td>
<td>6</td>
</tr>
<tr>
<td>16</td>
<td>4, 1, 3</td>
<td>-10-</td>
</tr>
</tbody>
</table>

Runs have been grouped and numbered by origin terminal to aid later identification.

The division trips that constitute each run, and run origin and destination terminals can be represented by the entries in the rows of a matrix which includes a column for each feasible run. In the case of the runs given in Table 2 the following 16 by 16 array is obtained.
### TABLE 3
ILLUSTRATIVE NETWORK RUN CHARACTERISTICS

<table>
<thead>
<tr>
<th>Run Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2, 1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1, 3</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3, 1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1, 4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4, 1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2, 3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3, 2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The entries in the first eight rows of the matrix give the number of division trips of each type that are made during the run identified by a column number.

The entries "1" in any one of the rows associated with an origin terminal identify the runs which originate at that terminal. Run destination terminals are similarly identified by the entries "1" in the last four rows of the matrix.

**An Illustrative Run Scheduling Problem**

In the case of the network of Figure 1 the scheduling problem considered can be formulated as follows. Let \( r_j \) denote the number of runs of type \( j = 1, 2, \ldots, 16 \) which are made during some fixed period of time, and let
the matrix $A_0$ be the 16 by 16 array of ones and zeros given in Table 3. Then
the column vector $r = (r_1', r_2', \ldots, r_{16}')^T$ represents a run schedule and the
linear transformation $A_0 r = c$ gives the number of division trips of each type
that are made, the number of runs that originated at each terminal, and the
number of runs that ended at each terminal. If these quantities are denoted
by $X_{tu}$, $O_t$, and $D_t$ respectively, then the vector $c$ can be represented by
$(X_{tu}, O_t, D_t)^T$ where the subscripts $tu$ and the subscripts $t$ correspond to the
row index numbers given in Table 3. The linear transformation can be written
as follows.  \(^1\)

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
r_1 \\
r_2 \\
r_3 \\
r_4 \\
r_5 \\
r_6 \\
r_7 \\
r_8 \\
r_9 \\
r_{10} \\
r_{11} \\
r_{12} \\
r_{13} \\
r_{14} \\
r_{15} \\
r_{16}
\end{bmatrix}
= \begin{bmatrix}
X_{12} \\
X_{21} \\
X_{13} \\
X_{31} \\
X_{14} \\
X_{41} \\
X_{23} \\
X_{32} \\
O_1 \\
O_2 \\
O_3 \\
O_4 \\
D_1 \\
D_2 \\
D_3 \\
D_4
\end{bmatrix}
\tag{2-1}
\]

\(^1\) Mathematical expressions are identified by noting, in parentheses,
the number of the chapter in which an expression first appears, followed by
a number that indicates its position in the sequence of expressions within
that chapter.
Since the $r_j$ take only nonnegative integral values, the coordinates of $c$ are also nonnegative integers.

If the $X_{tu}$, $O_t$, and $D_t$ are assigned nonnegative integral values which represent the schedule characteristics desired during some fixed time period, the scheduling problem considered here is obtained. The requirements specified by the coordinates of $c$ are the constants of a system of linear equations in the indeterminates $r_j$, and a particular set of nonnegative integers $(r'_1, \ldots, r'_n)$ that satisfy the system is a feasible run schedule. A method of obtaining a general solution of any problem of this type is desired.

General Formulation

We assume a given set of terminals and divisions that constitute a connected motor freight network. Two terminals which are end points of the same division will be called adjacent. Let the terminals be numbered $1, 2, \ldots, t, \ldots, q$, and let the number pair $t, u$ represent a single-division transit from any terminal $t$ to any adjacent terminal $u$. Since the network is connected, there is, for every terminal $t$, a nonvoid set, $U_t$, of such terminals $u$. Then any sequence of terminal numbers $t_1, t_2, \ldots, t_\eta, \ldots, t_\eta$, with $t_\eta + 1 \in U_{t_\eta}$, denotes a run over the divisions of the network. To each division there is to be assigned a positive real number called the transit time, and the sum of the transit times of the division trips that are made during any particular run will be called the run time. Any run with a run time no greater
than some specified maximum is a feasible run. A set of feasible runs with
the following properties is assumed:

1. For every division \( t, u \), there is a feasible run \( t, u \) and a
   feasible run \( u, t \); that is, there is a feasible run in each
direction between every pair of adjacent terminals.

2. There are some feasible runs which include more than
   one division trip.

3. If any particular run, \( t^1, t^2, \ldots, t^\eta \), with \( \eta > 2 \) is
   feasible, then the run \( t^{\eta-1}, t^\eta \) is also a
   member of the set of feasible runs.

The run characteristics of interest are the origin terminal, the destina-
tion terminal, and the number of division trips of each type, \( t, u \), which are
included in the run. These characteristics can be represented by the entries
of a matrix which includes a column for each run type, \( j = 1, \ldots, n \), in the set,
and a row for each division-trip type, each origin terminal, and each destina-
tion terminal. The rows and columns of any such matrix can be arranged to
obtain an \( m \) by \( n \) matrix \( A = \begin{bmatrix} a_{ij} \end{bmatrix} \) which can be partitioned as follows.

\[
\begin{bmatrix}
B_{pp} & B_{p, n-p} \\
G_{qp} & G_{q, n-p} \\
H_{qp} & H_{q, n-p}
\end{bmatrix}
\]

The subscripts of the partitioned submatrices \( B, G, \) and \( H \) indicate the
number of rows and columns of each submatrix.
Assume that the single-division-trip runs are numbered 1, \ldots, p, so that the characteristics of these runs are given by the entries in the first p columns of A. Then the characteristics of all other runs are given by the entries in the columns \( p + 1, \ldots, n-p \) of A. A unique number pair, \( t, u \), with \( t = 1, \ldots, q \) and \( u \in U_t \), will identify each row of B. These row index numbers are assigned so that the nonnegative integer in column \( j \), row \( t, u \) of B gives the number of trips, from terminal \( t \) to an adjacent terminal \( u \), that are included in run \( j \). Since there are exactly as many single-division-trip runs as there are division trips, the rows of B can be arranged so that \( B \) is an identity matrix; that is, \( a_{ii} = 1 \) and all other entries of \( B \) are zero. The entries of \( B \) are nonnegative integers which may be greater than one.

The terminal numbers 1, \ldots, t, \ldots, q are the row index numbers of \( G \) and \( H \). The entry "1" in row \( t \), column \( j \) of \( G \) identifies the origin terminal of run \( j \); similarly, the entry "1" in row \( t \), column \( j \) of \( H \) identifies the destination terminal of run \( j \). All other entries in these submatrices are zero.

Let the subscripts of the quantities \( (X^*_t) \), \( (O)_t \), and \( (D^*_t) \) correspond to the row index numbers of the submatrices \( B \), \( G \), and \( H \) respectively. We assume that each \( X_{tu} \) takes the nonnegative integral value given by the number of loads that are available for movement from a terminal \( t \) to an adjacent terminal \( u \) during some fixed time period. The value of each \( O_t \) is given by the number of available drivers at terminal \( t \), and the value of each \( D_t \) is given by the desired number of drivers who are to end runs at terminal \( t \).
The specifications \((D_t')\) are desired to control the location of drivers who will become available for reassignment to runs during subsequent time periods. When interpretation is not necessary, the run schedule specifications or "requirements" \((X_{tu}'), (O_t'),\) and \((D_t')\) will be represented by the quantities \(c_1, \ldots, c_m\). For any particular scheduling period, these quantities take given nonnegative integral values. It is assumed that a penalty is associated with a unit change in the given value of each particular requirement.

Given a particular requirements specification, the number, \(r_j\), of runs of type \(j = 1, \ldots, n\) to be scheduled is to be determined. It is assumed that all scheduled runs are initiated and completed during the scheduling period.

The conditions of the problem specify a system of \(m\) linear equations in the indeterminates \(r_1, \ldots, r_j, \ldots, r_n\). The coefficients, \(a_{ij}\), and constants, \(c_1, \ldots, c_i, \ldots, c_m\), of the system are nonnegative integers. For given values of the constants, a set of nonnegative integers, \(r_1', \ldots, r_n'\), which satisfy the system will be called a feasible run schedule. In matrix notation the system can be written:

\[
\begin{bmatrix}
B \\
G \\
H
\end{bmatrix}
\begin{bmatrix}
r_1 \\
r_j \\
r_n
\end{bmatrix} =
\begin{bmatrix}
X_{tu} \\
O_t \\
D_t
\end{bmatrix}
\]

\[(2-2)\]
When a simplified notation seems appropriate, the system \((2-2)\) will be symbolically written \(A_r = c\).

In applied problems of this kind, a given matrix, \(A_0\), would be expected to remain unchanged for extended periods of time. The desired values of the \(c_i\), however, would be expected to change from scheduling period to scheduling period. A general solution, then, provides a simple means of finding a feasible run schedule for any scheduling period for which a nonnegative integral solution of the system exists. If more than one feasible solution exists, a problem of selecting the "best" schedule is obtained. A representation of the set of all feasible solutions will provide a basis for selecting the best schedule. If, however, it happens that, for particular values \(c'_1, \ldots, c'_m\), there is no feasible solution, then these desired run schedule requirements must be modified. The problem of obtaining the "best" possible requirements specification must then be considered.

In general, the results desired are the following:

1. A set of necessary and sufficient conditions, on the constants \(c_i\), for the existence of a nonnegative integral solution of any particular system of linear equations with nonnegative integral coefficients.

2. A procedure for modifying a given set of values, \(c'_1, \ldots, c'_m\), for which there is no feasible solution, to obtain the "best" possible set of values for which a feasible solution exists.

3. A representation of the set of all feasible solutions.

4. A method of selecting the "best" solution of any particular system for which more than one feasible solution exists.
The initial investigation is to be restricted to systems of the type (2-2). The approach outlined provides a basis for an exploratory study of some fundamental questions of decision theory.
Let the symbolic relation \( Ar = c \) represent a system of nontrivial linear equations and let \( \alpha_i \) denote the \( i \)th row of any \( m \) by \( n \) coefficient matrix \( A \) considered as a (row) vector. If whenever \( \sum_{i=1}^{m} h_i \alpha_i = (0) \) it follows that all scalars \( h_i = 0 \), the rows of the matrix (or alternatively, the corresponding linear equations) are said to be "linearly independent." Here \((0)\) denotes the vector with all coordinates zero. Vectors which are not linearly independent are called "linearly dependent" and it follows that nonzero vectors \( \alpha_1, \ldots, \alpha_m \) are linearly dependent if and only if some one of the vectors \( \alpha_k \) is a linear combination of the preceding ones [4, p. 167]. The maximum number, \( w \), of linearly independent rows of a matrix is called the "row rank" of the matrix. Since every set of \( n + 1 \) vectors with \( n \) coordinates is linearly dependent, \( w \) is no greater than \( n \) [4, p. 169].

Assume that \( w < m \) and let \( \alpha_1, \ldots, \alpha_w \) be any linearly independent subset of the rows of \( A \). Then to each further row \( \alpha_{w+d} \) there corresponds
a unique set of real numbers, \((d_i)\), not all of which are zero, such that

\[
\alpha_i = \sum_{i=1}^{w+d} h_i \alpha_i, \quad d = 1, \ldots, m - w. \tag{3-1}
\]

That scalars exist which satisfy the vector equations (3-1) follows from the above proposition concerning sets of linearly dependent vectors.

Given independent \(\alpha_1, \ldots, \alpha_w\) and a particular further vector \(\alpha_{w+\delta}\), the scalars \(h_i\) which satisfy \(\alpha = \sum_{i=1}^{w+\delta} h_i \alpha_i\) must be unique since otherwise a contradiction of the premise that \(\alpha_1, \ldots, \alpha_w\) are linearly independent can be obtained [4, p. 182]. Because (3-1) implies that \(\alpha_{w+d}\) is the uniquely determined linear combination \(\sum_{i=1}^{w+d} h_i \alpha_i, d = 1, \ldots, m - w,\) in order to have \(\alpha_{w+d} = c\) consistent with \(\alpha_i = c, i = 1, \ldots, w,\) it is necessary that

\[
c = \sum_{i=1}^{w+d} h_i c = 0, \quad d = 1, \ldots, m - w. \tag{3-2}
\]

Therefore, given any particular set of \(w < m\) linearly independent equations, there exists a unique set of relations (3-2) which are necessary conditions for the solvability of the system of \(m\) linear equations. These relations are also sufficient conditions for the existence of real solutions of any system \(A r = c\). Because we need a generalization to the solution in integers of systems of linear equations with integral coefficients, we present a proof which will be adaptable to that situation.
Two systems of linear equations are said to be "equivalent" if any solution of either system is a solution of the other. Let \( ||A, c|| \) denote the augmented coefficient matrix of a system of linear equations and let \( ||E, b|| \) be any matrix obtained from \( ||A, c|| \) by performing a sequence of "elementary row operations;" viz.:

1. the interchange of any two rows
2. the multiplication of any row by a real number \( h \neq 0 \)
3. the addition of any row to any other row.

Then the matrices \( ||A, c|| \) and \( ||E, b|| \) are said to be "row equivalent" and the corresponding systems of linear equations are equivalent [4, pp. 44-48].

By using the elementary row operations 2 and 3 above any matrix can be reduced to a canonical form such that

1. every leading entry of a nonzero row is 1 and
2. every column containing a leading entry 1 has all its other entries zero [4, p. 173].

A matrix in this form is said to be "row reduced." Let \( E \) be a row-reduced matrix which has been obtained from \( A \) by performing the elementary row operations 2 and 3 on \( ||A, c|| \) to obtain \( ||E, b|| \), and let \( \beta_i \) denote the \( i \)th row of \( E \). For the nonzero rows of \( E \), \( \sum_i h_i \beta_i = (0) \) if and only if all \( h_i = 0 \); hence these rows are, by definition, linearly independent.
Suppose that the rank of A is less than M so that at least one row, \( \beta_{\delta} \), of E contains only the entries zero. The coordinate \( b_{\delta} \) of the column vector b is that linear function, \( c_{\delta} - \sum_{i \in I} h_{i} c_{i} \), of the \( c_{i} \) which was applied to the rows of A to obtain the row \( \beta_{\delta} = (0, \ldots, 0) \). In this way, a set of values \( h_{i} \) can be found, and an explicit set of relations (3-2) can be obtained. Since it is necessary that \( b_{d} = 0 \) for all \( d \) with \( \beta_{d} = (0) \), only \( w \) of the \( c_{i} \) may be assigned arbitrary values. The values of the remaining \( m - w \) constants must be given by (3-2).

Assume that \( w < m < n \) and that the necessary row operations of type 1 were included in the sequence used to obtain \( \| E, b \| \) so that the rows \( \beta_{w+1}, \ldots, \beta_{m} \) of the new coefficient matrix E contain only the entries zero. The equivalent system of equations which has been obtained can be written in vector form as follows

\[
\begin{align*}
\beta_{i} r &= b_{i} \quad i = 1, \ldots, w \\
(0) &= b_{i} \quad i = w + 1, \ldots, m
\end{align*}
\]

(3-3)

The interchange of any two columns \( \delta \) and \( \rho \) of E gives the change of variable \( r^{*} = r \) and \( r^{*} = r \). By a sequence of such column interchanges and changes of variable a system \( E^{*} r^{*} = b \) can be obtained such that for \( i = j \) the entries \( e_{ij} \), \( i = 1, \ldots, w \), of \( E^{*} \) are "1" and all other entries of this \( w \) by \( w \)
submatrix are zero. The system can be written

\[ r^* + e_{1, w+1} r^* + \ldots + e_{1, n} r^* = b_1 \\
\vdots \quad \vdots \quad \quad \vdots \quad \vdots \\
\vdots \quad \vdots \quad \quad \vdots \quad \vdots \\
r^* + e_{i, w+1} r^* + \ldots + e_{i, n} r^* = b_i \\
\vdots \quad \vdots \quad \quad \vdots \quad \vdots \\
\vdots \quad \vdots \quad \quad \vdots \quad \vdots \\
r^* + e_{w, w+1} r^* + \ldots + e_{wn, n} r^* = b_w \\
0 = b_{w+1} \\
\vdots \\
0 = b_m \]

(3-4)

and explicit expressions for the \( r^*_j \), \( j = 1, \ldots, w \), can be obtained in terms of

\( b_i \) and the indeterminates \( r^*_1, \ldots, r^*_n \). If \( b = 0 \), \( i = w + 1, \ldots, m \),

the system is satisfied for arbitrary values of \( r^*_j \), \( j = w + 1, \ldots, n \), and an

infinity of real solutions exists. Otherwise, there is some \( b \neq 0 \),

\( \delta \leq w + 1 \leq \delta \leq m \), the contradiction \( 0 = r^*_i = b \neq 0 \) is obtained, and the system

(3-4) cannot be satisfied for any values of the \( r^*_j \).

If \( w = m = n \) a single entry '1' is found in every row and column of \( E \),

all other entries are zero, and a unique solution exists for every set of

values \( (c_i \)'). If \( w = n \leq m \) there is no solution if any \( b_i \neq 0 \), \( i = w + 1, \ldots, m \);

otherwise, there is a unique solution for any given values of \( w \) of the \( c_i \). Since

(3-4) is equivalent to \( Ar = c \) the above results apply to this original system.
If \( w < m \), set \( b_{w+1}, \ldots, b_m \) equal to zero, let \( r^* = \lambda \) for \( d = 1, \ldots, n - w \), and let \( ||e_{id}|| \) denote the \( w \) by \( n - w \) matrix formed by the first \( w \) rows and last \( n - w \) columns of \( E^* \). Then the set of all solutions can be written

\[
    r^* = b - \sum_{i=1}^{n-w} e_{id} \lambda, \quad i = 1, \ldots, w. \tag{3-5}
\]

In (3-5) the parameters \( \lambda_d \) take arbitrary values. If all \( c_i = 0 \) then all \( b_i \) are zero and these relations generate the set of all solutions of the resulting homogeneous equations. This result can be summarized by noting that the set of all real solutions of any system of linear equations is given by any particular solution and the set of all solutions of the corresponding homogeneous equations.

The systems of linear equations of the type defined in Chapter 2 have integral coefficients and constants (which are, in fact, nonnegative) and non-negative integral solutions are desired. In the sequel such solutions will be termed "positive integral" solutions. Although the relations (3-2) are necessary conditions for the existence of integral solutions they are not, in general, sufficient; for example, the equation

\[
    2x + 4y = 5
\]

has no solution in the "rational integers" \( 0, \pm 1, \pm 2, \ldots \). A necessary condition for the solvability, in rational integers, of any linear equation
\[ a_1 x + a_2 x + \ldots + a_n x = c \] is that the greatest common divisor of \( a_1, \ldots, a_n \) divides \( c \) [13, p. 21]. This criterion, however, is not sufficient for the simultaneous solution of a number of such equations. A necessary and sufficient condition for the solvability, in rational integers, of systems of linear equations was given by H. J. S. Smith in 1861; viz.: 

A linear system is or is not resolvable in integral numbers, according as the greatest common divisor of the determinants of the matrix of the system is, or is not, equal to the corresponding greatest common divisor of its augmented matrix [22, p. 407].

The determinants of a matrix are, of course, the determinants of the greatest square matrices contained in it...[22, p. 387].

For a special case a previous result, given by M. Ignaz Heger, applies.

If the unaugmented matrix of an indeterminate system be prime the system is always resolvable. For, every determinate system, of which the matrix is a unit-matrix, is resolvable in integral numbers; and we may suppose the given indeterminate system to form part of such a determinate system [22, p. 387].

A prime matrix is one of which the greatest divisor is unity; i.e., the determinants of which are relatively prime. A prime square matrix (i.e., a matrix of which the determinant is unity) we shall call a unit-matrix [22, p. 368].

The latter criterion can be shown to apply for any system of network equations of the type considered here.

For systems of linear equations with integral coefficients an integral solution of \( Ar = c \) always exists for some sets of integral values \( (c_1) \); since, for any particular vector \( r' \) with integer coordinates, the linear transformation
A \( r^i = c \) generates such a set of values. For such systems we may, then, inquire: What conditions, on the \( c_i \), are implied by the requirement that a particular system \( A^o r = c \) possess nonnegative integral solutions. In the case of the network equations considered here, integral solutions exist for any integral values of \( w \) of the \( c_i \). The requirement that the solution be positive, however, imposes additional restrictions. These results are given in the following sections of this chapter.

Some Properties of The Network Equations

The network equations (2-2) are

\[
\begin{bmatrix}
B & r_1 \\
G & \vdots \\
H & r_n
\end{bmatrix}
\begin{bmatrix}
\{ r_1 \\
\vdots \\
r_n
\end{bmatrix}
= \begin{bmatrix}
X_{tu} \\
O_t \\
D_t
\end{bmatrix}
\]

The entries in the submatrix \( B \) are nonnegative integers, and the entries in the submatrices \( G \) and \( H \) are either 0 or 1.

There are \( q \) terminals and \( p \) division-trip types (one in each direction over every division), so the number of equations \( m = p + 2q \). It was assumed that there are \( p \) single-division-trip runs and some multi-division-trip runs; thus \( n > p \). Since only connected networks are considered, the number of
divisions, \( \frac{p}{2} \), is no less than \( q - 1 \) and the maximum number of divisions is \( \frac{q(q-1)}{2} \). The maximum is obtained when, for each of the \( q \) terminals, there are \( q - 1 \) distinct, incident divisions; hence \( q(q-1) \) is twice the maximum number of divisions in a network. Thus

\[
(q - 1) \leq \frac{p}{2} \leq \frac{q(q-1)}{2}
\]

and, since \( m = p + 2q \), it follows that \( 4q - 2 \leq m \leq q^2 + q \). We have \( n > p \geq 2q - 2 \); thus in some cases \( m \leq n \); in others it may happen that \( m > n \).

For any such system of linear equations the rank, \( w \), of the coefficient matrix is less than \( m \). Since, for example, there is a unique origin terminal and destination terminal for each run,

\[
\sum_{j=1}^{n} x = \sum_{t=1}^{q} O = \sum_{t=1}^{q} D \quad (3-6)
\]

and no more than \( 2q - 1 \) of the \( O_t \), \( D_t \) may be assigned arbitrary values. If the \( O_t \), \( t = 1, \ldots, q - 1 \), and all \( D_t \) are assigned given values, then

\[
O = \sum_{t=1}^{q} D - \sum_{t=1}^{q-1} O \quad (3-2)
\]

and an expression of the type (3-2) is obtained. The corresponding row,
, of the coefficient matrix must be given by
\[ \alpha = \sum_{t=1}^{p+q} \alpha_t - \sum_{t=1}^{p+q+t} \alpha_t. \] (3-7)

It follows that \( w \leq m - 1 \).

The relations (3-6) define a network property that is true if all runs originate and end at a terminal in the network. In writing the network equations, this was assumed to be the case; therefore, for any given values, \( r_{1, \ldots, r_n} \), the linear transformation \( \begin{bmatrix} r_1 \cdots r_n \end{bmatrix}^T \) gives values (scalars) \( (O_i', \ldots, D_i') \) for which (3-6) holds. To obtain the result (3-7), the noted elementary row operations are viewed as rules of logic. Given:

\[ \begin{bmatrix} a_{ij} \end{bmatrix} \begin{bmatrix} r_1 \cdots r_n \end{bmatrix}^T = \begin{bmatrix} c_1 \cdots c_m \end{bmatrix}^T, \]

a set of propositions (equalities), each of which is true for arbitrary values of \( r_1, \ldots, r_n \); and an equality

\[ c_\delta = L_\delta (c_i), \] which can independently be shown to be true for any values, \( c_i \), given by \( \begin{bmatrix} a_{ij} \end{bmatrix} \begin{bmatrix} r_1 \cdots r_n \end{bmatrix}^T \); it follows that row \( \alpha_\delta \) is given by

\[ L_\delta (\alpha_i). \] Since \( c_\delta - L_\delta (c_i) = 0 \), we must have \( \alpha_\delta - L_\delta (\alpha_i) = (0, \ldots, 0) \);

otherwise, \( (\alpha_\delta - L_\delta (\alpha_i)) \begin{bmatrix} r_1 \cdots r_n \end{bmatrix}^T \neq 0 \) for some values of the \( r_j \). With respect to the rows \( \alpha_i, i \neq \delta \), the row \( \alpha_\delta \) is dependent, and if

\( (L_\delta (\alpha_i)) \begin{bmatrix} r_1 \cdots r_n \end{bmatrix}^T \neq c_\delta \) when \( c_\delta = L_\delta (c_i) \), a contradiction is obtained, since any linear combination of equalities is an equality.

In (2-2), the rows of the submatrix \( H \) are dependent with respect to the rows of \( B \) and \( G \). This follows since the number of runs that originate at a terminal, less the number of runs that end at that terminal must equal the
difference between the number of division-trips to and from that terminal.

That is, for any terminal \( t \), \( t = 1, \ldots, q \),

\[
O_t - D_t = \sum_{u \in U} X_t^{u} - \sum_{u \in U} X_t^{u}, \quad (3-8)
\]

As before, for a given \( t \), \( U_t \) denotes the set of adjacent terminals \( u \). The relations (3-8) can be written

\[
D_t = \sum_{u \in U_t} X_t^{u} - \sum_{u \in U_t} X_t^{u} + O_t, \quad t = 1, \ldots, q \quad (3-9)
\]

and the rows \( \alpha_{p+q+t} \), \( t = 1, \ldots, q \), of the coefficient matrix are given by the functions (3-9) in terms of the rows \( \alpha_1, \ldots, \alpha_{p+q} \). The relations (3-8) could, of course, be solved for any \( q \) distinct terms.

It can be assumed, without loss of generality, that the values of the \( D_t \) are to be given by (3-9) for particular values of the \( O_t \) and \( X_t \). Then

\[
D_t - \sum_{u \in U_t} X_t^{u} + \sum_{u \in U_t} X_t^{u} - O_t = 0
\]

for \( t = 1, \ldots, q \). This linear function of the corresponding rows of the coefficient matrix gives

\[
\alpha_{p+q+t} - \sum_{u \in U_t} \alpha_{u} + \sum_{u \in U_t} \alpha_{u} - \alpha_{p+t} = 0 \quad (3-10)
\]

for \( t = 1, \ldots, q \).\(^1\) (Here the subscripts \( u_t \) and \( u_u \) are the row index numbers

\(^1\) See Appendix I for an alternative proof.
of the first $p$ rows of the coefficient matrix $A$. The other subscripts give a row number of this matrix.) The relations (3-10) specify a sequence of elementary row operations such that all entries in the last $q$ rows of the resulting row equivalent matrix are zero; hence, $w \leq m - q$. The entries in the first $p + q$ rows are, of course, unchanged.

If the relations (3-9) are satisfied, the linear equations

$$\begin{bmatrix} B \\ G \end{bmatrix} \begin{pmatrix} r_1 \\ \vdots \\ r_j \\ \vdots \\ r_n \end{pmatrix} = \begin{pmatrix} X_t \\ u \end{pmatrix}$$

t = 1, \ldots, q \text{ and } \forall u \in U \tag{3-11}$$

are equivalent to the original system. The columns of the coefficient matrix can be partitioned as before and written

$$\begin{bmatrix} B_{pp} & B_{p, n-p} \\ G_{qp} & G_{q, n-p} \end{bmatrix}$$

Assume that the rows and columns of this matrix are arranged so that $B_{pp}$ is an identity matrix. The first $p$ rows of this matrix are then in the row-reduced echelon form. By performing a sequence of elementary row operations, with $h = \pm 1, \pm 2, \ldots$, and by interchanging columns, the entire matrix can be reduced to this canonical form.
Row $p + 1$ of the coefficient matrix is the first row of the submatrix $G$.

Thus an entry $a_{p+1, \delta} = 1, 1 \leq \delta \leq p$, indicates that the single-division-trip run $\delta$ originates at terminal one. The only nonzero entry in column $\delta$ of $B_{pp}$ is $a_{\delta,\delta} = 1$ and the index number of row $\delta$ is one of the class $1u, u \in U_1$, where $1u$ is interpreted as a division trip from 1 to some terminal $u$. Then the row operations

$$a_{p+1} - \sum_{u \in U} a_{1u}$$

reduce all of the first $p$ coordinates of the row (vector) $p + 1$ to zero and the operations

$$a_{p+t} - \sum_{u \in U} a_{tu}, t = 1, \ldots, q,$$

reduce all of the entries of $G$ to zero. Suppose that after performing the row operations (3-13), there is at least one entry "-1" in every nonzero row of $G$, and that a "-1" in each of these rows can be selected such that each falls in a different column. This is in fact the case; hence, the row-reduced echelon form can be obtained by performing additional row operations with $h = \pm 1, \pm 2, \ldots$ and by interchanging columns. The proof depends on the run properties given on page 22.
After the elementary row operations (3-13), all entries in any given row, \( \delta \), of \( G \) are zero if either of the following cases is obtained:

\[ q, \ n-p \]

Case 1. There is no run \( j, p + 1 \leq j \leq n \), which includes any of the division-trips of the class \( \delta u \), \( u \in U \)

Case 2. The only runs \( j, p + 1 \leq j \leq n \), which include any division-trip \( \delta u \) originate at terminal \( \delta \), include a single division-trip type of the class \( \delta u \), \( u \in U \), and make one trip of the given type

Otherwise we have:

Case 3. There is some run \( j, p + 1 \leq j \leq n \), which originates at terminal \( \delta \) and makes exactly two trips of any type or types in the class \( \delta u \), \( \delta \in U \)

Case 4. There is some run \( j, p + 1 \leq j \leq n \), which does not originate at terminal \( \delta \), which includes a single division-trip type of the class \( \delta u \), \( u \in U \), and which makes one trip of the given type

In Case 3 or 4 a "-1" is found in any row \( \delta \) after the row operations (3-13).

Suppose that some runs \( j = p + 1, \ldots, n \) include more than one trip of any type or types in the class \( \delta u \), \( u \in U \), so that neither Case 1 nor Case 2 is obtained. These runs can be classified accordingly as they do or do not originate at terminal \( \delta \). If any of these runs originate at \( \delta \) then there is a run such that Case 3 is obtained since the runs considered include two or more trips of a type or types in the class \( \delta u \), \( u \in U \), and, by postulate 3, if there are runs which make more than two such trips there is at least one run which makes precisely two. A similar argument holds for Case 4. If with respect to any given row (terminal) \( \delta \) of \( G \), \( q, n-p \) neither Case 1 nor Case 2 is
obtained, then either Case 3, Case 4 or both Case 3 and Case 4 is obtained.

In Case 1 all entries in rows $\alpha_{\delta u}$, $u \in U$, of $B_{\delta p, n-p}$ and row $\alpha_{\delta}$ of $G_{\delta q, n-p}$ are zero before and after the operations (3-13). In Case 2, for every entry $a_{\delta j} = 1$ in row $\alpha_{\delta}$ of $G_{\delta q, n-p}$, there is, in $j$th position of the rows $\alpha_{\delta u}$, $u \in U$, a single nonzero entry, a "1"; hence, after the row operations (3-13), all of the entries of row $\alpha_{\delta}$ are zero. For Case 3 there is in row $\delta$ of $G_{\delta q, n-p}$ an entry $a_{\delta j} = 1$ for some column $j$ such that in this position of rows $\alpha_{\delta u}$, $u \in U$, there are exactly two nonzero entries, the entries "1"; or, one nonzero entry, the entry "2." After the operations (3-13) $a_{\delta j} = -1$. The same result is obtained for Case 4. After the row operations (3-13) there is at least one entry "-1" in every nonzero row of $G_{\delta q, n-p}$. If there is a run which is in Class 3 or 4 with respect to more than one terminal there is, by postulate 3, another run which is in Class 3 and a run which is in Class 4 with respect to just one of these terminals. Then for every nonzero row of $G_{\delta q, n-p}$ there is at least one column of this submatrix which includes an entry "-1."

Since all entries of $G_{\delta q, n-p}$ are reduced to zero, a sequence of row and column interchanges can be made such that every leading entry of a nonzero row of $G$ is "-1" and, further, that $a_{i i} = -1$, $i = p + 1, \ldots, w$. Since all other entries in columns $p + 1, \ldots, n$ are either zero or positive integers, all entries, $a_{ij}$, $i \neq j$ and $j = p + 1, \ldots, w$, can be reduced to zero by elementary row operations with $h = \pm 1, \pm 2 \ldots$ Then $a_{ij} = 0$ for $i \neq j$ and $j = 1, \ldots, w$. By multiplying rows $\alpha_{p+1}, \ldots, \alpha_w$ by -1 we obtain $a_{i i} = 1$ for $i = 1, \ldots, w$. 

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In this canonical form the entries in the columns \( w + 1, \ldots, n \) are rational integers.

The foregoing results can be summarized by noting that the rank, \( w \), of the \( m \) by \( n \) coefficient matrix

\[
A = \begin{pmatrix}
B_{pp} & B_{p, n-p} \\
G_{qp} & G_{q, n-p} \\
H_{qp} & H_{q, n-p}
\end{pmatrix}
\]

of the network equations \( Ar = c \) is no greater than \( m - q \). By performing a sequence of the operations

1. the interchange of any two rows or columns
2. the multiplication of any row by a scalar \( h = 1, 2 \ldots \)
3. the addition of any row to any other row

any such matrix can be row-reduced.

If these operations are performed on the augmented matrix \( | A, c | \) to obtain the row-reduced matrix \( | E^*, b | \), the corresponding equations are in the form (3–4), where each \( b_i \) is a given linear function of the \( c_i \). Since this canonical form was obtained by adding or subtracting multiples of rows (and by row and column interchanges), the coefficients of these functions are rational integers. Then for integral values of the \( c_i \), the \( b_i \) are integers.
The equations (3-5) can be written

\[ r^*_i = b_i - \sum_{d=1}^{n-w} e_{id} \lambda_d, \quad i = 1, \ldots, w, \]  

(3-14)

where \[ r^*_w + d/d = \lambda_d, \quad d = 1, \ldots, n-w. \]

For any system of network equations of the type considered, the \( e_{id} \) are integers. Then for integral values of the \( c_i \), integral values of the \( r^*_i \), \( i = 1, \ldots, n \), are generated by integral values of the \( \lambda_d \) and only by such values; further, if nonnegative values of these indeterminates can be obtained, they are generated by (3-14) if and only if the values of all parameters are nonnegative. If there are any real solutions of the original system \( A r = c \), then the system (3-14) is equivalent to this system, and the values of the \( r^*_i \) which are generated constitute a solution.

**Implications of Positivity Requirements**

Given \( r^*_i \geq 0, \ i = 1, \ldots, n \), we have, for the right-hand members of (3-14)

\[ \sum_{d=1}^{n-w} e_{id} \lambda_d \leq b_i, \quad i = 1, \ldots, w, \]  

(3-15)

\[ -\lambda_d \leq 0, \quad d = 1, \ldots, n-w, \]
a system of linear inequalities in the parameters $\lambda_d$ such that any solution of (3-15) gives values $(\lambda')$ which generate nonnegative values of the $r^*_i$. Suppose that for some one of these inequalities, say

$$\sum_{e\in E} \lambda \leq b,$$

it happens that $e_{\rho d} \geq 0$ for all $d$. Then for $\lambda \geq 0$ the left-hand member is nonnegative and unless $b \geq 0$ a contradiction is obtained and the system (3-15) cannot be satisfied. Hence, $b \geq 0$ is a necessary condition for the existence of positive solutions of the original system of network equations $A r = c$. Since $b$ is a known linear function of the requirements, this condition gives

$$b = \sum_{\rho} L(X, O, D) \geq 0,$$

where the $X_{tu}$, $O_t$, and $D_t$ take nonnegative integral values. Then the sum of the terms of $L_{\rho}$ with negative coefficients must be no greater than the sum of those terms with positive coefficients. It may, however, be possible to derive more restrictive conditions.

If the system (3-15) is consistent, then each of the inequalities is a true statement and any nonnegative linear combination of the inequalities is a true
statement. In particular, suppose that

$$0 \leq \sum_{i} h_{i} b_{i}$$

(3-16)

is a nonnegative combination of any of the n inequalities such that the (vector) sum of the row vectors formed by the coefficients of each of the left-hand members gives the vector (0). In the paper "Solvability and Consistency for Linear Equations and Inequalities" [16], H. W. Kuhn has shown that, for any system of inequalities, a method of elimination leads either to a solution or to a contradiction of this form; that is,

$$(0, \ldots, 0) (\lambda_{1}, \ldots, \lambda_{n-w})^{T} \leq \sum_{i} h_{i} b_{i} > 0.$$ 

Suppose that (3-15) is a system of inequalities which are necessary conditions for the existence of positive solutions of any particular system of linear equations for which real solutions exist. Further suppose that we have the set of all conditions of the form (3-16) which have been obtained from (3-15) by using Kuhn's process of elimination to obtain a general solution. Finally, suppose that it can be independently shown that the system of linear equations has positive solutions for at least some values of the right-hand members. Then the set of relations of the form (3-16) can be satisfied for some values of the $b_{i}$. Otherwise, there are no values ($b'_{i}$) for which the inequalities (3-15) possess a solution. But, by hypothesis, the solvability of (3-15) is a
necessary condition for the existence of positive solutions of the linear equations, and since positive solutions exist, a contradiction is obtained.

We conclude, therefore, that the set of conditions of the form (3-16) can be satisfied from some values \( b' \). A consistent set of conditions, then, is obtained, and it follows from Kuhn's result that this set is necessary and sufficient for the existence of real values \( \lambda' \) which satisfy the linear inequalities (3-15).

The above result obviously holds when (3-15) is obtained from the representation (3-14). Further, there are integral solutions of (3-15) for some integral values \( b' \); otherwise, there are no positive integral solutions of \( A r = c \) for any integral values of the \( c_i \), and a contradiction can again be obtained. Some integral values of the \( b_i \), then, must satisfy the conditions of the form (3-16) which are obtained during the solution process given by Kuhn. Since these are necessary conditions for the existence of any (positive) solutions of (3-15) they are, of course, necessary for the existence of (positive) integral solutions. It is also true that integral solutions of (3-15) exist for any integral \( b_i \) which satisfy an exhaustive set of conditions of the form (3-16). This result will be demonstrated following an outline of the solution procedure and the essentials of Kuhn's proof.
CHAPTER 4

ILLUSTRATIVE PROBLEM SOLUTION

The Network Equations

The network equations for the illustrative problem given in Chapter 2 are as follows:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_{12} \\
X_{21} \\
X_{13} \\
X_{31} \\
X_{14} \\
X_{41} \\
X_{23} \\
X_{42} \\
O_{32} \\
O^1 \\
O^2 \\
O^3 \\
D^4 \\
D^1 \\
D^2 \\
D^3 \\
D^4 \\
\end{bmatrix} = \begin{bmatrix}
\{r_1\} \\
\{r_2\} \\
\{r_3\} \\
\{r_4\} \\
\{r_5\} \\
\{r_6\} \\
\{r_7\} \\
\{r_8\} \\
\{r_9\} \\
\{r_{10}\} \\
\{r_{11}\} \\
\{r_{12}\} \\
\{r_{13}\} \\
\{r_{14}\} \\
\{r_{15}\} \\
\{r_{16}\} \\
\end{bmatrix} \quad (4-1)
\]

1 A series of column interchanges (and a corresponding renumbering of the $r_j$) can be used to place this coefficient matrix in the form (2-2). Such operations, however, are not required to obtain the row-reduced canonical form.
A general solution and a criterion for the existence of positive integral solutions are to be derived.

The Set of All Solutions

The coefficient matrix of the system (4-1) can be row-reduced by performing a sequence of elementary row operations with $h = \pm 1$. The following equivalent system of equations can be obtained.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_1 = X_{12}$</td>
<td></td>
</tr>
<tr>
<td>$r_2 - r_8 - r_9 = X_{31} - X_{32} + O_1 - X_{12} - X_{14}$</td>
<td></td>
</tr>
<tr>
<td>$r_3 + r_9 - r_{13} = X_{31} - O_3 + X_{21} + X_{23} - O_2$</td>
<td></td>
</tr>
<tr>
<td>$r_4 + r_8 + r_{13} = O_2 - X_{21} - X_{23} + X_{32}$</td>
<td></td>
</tr>
<tr>
<td>$r_5 + r_9 = X_{14}$</td>
<td></td>
</tr>
<tr>
<td>$r_6 = X_{21}$</td>
<td></td>
</tr>
<tr>
<td>$r_7 + r_8 + r_9 = O_2 - X_{21}$</td>
<td></td>
</tr>
<tr>
<td>$r_{10} + r_{13} - r_{16} = O_3 + O_1 - X_{12} - X_{13} - X_{14} + O_2 - X_{21} - X_{23}$</td>
<td>(4-2)</td>
</tr>
<tr>
<td>$r_{11} + r_{12} + r_{16} = X_{13} + X_{14} - O_1$</td>
<td></td>
</tr>
<tr>
<td>$r_{14} = X_{21} + X_{23} - O_2$</td>
<td></td>
</tr>
<tr>
<td>$r_{15} + r_{16} = X_{41}$</td>
<td></td>
</tr>
<tr>
<td>$0 = O_4 - X_{41}$</td>
<td></td>
</tr>
<tr>
<td>$0 = D - (X_{12} + X_{31} + X_{41}) - O_1 + (X_{12} + X_{13} + X_{14})$</td>
<td></td>
</tr>
<tr>
<td>$0 = D - (X_{21} + X_{32}) - O_2 + (X_{21} + X_{23})$</td>
<td></td>
</tr>
<tr>
<td>$0 = D - (X_{13} + X_{23}) - O_3 + (X_{31} + X_{32})$</td>
<td></td>
</tr>
<tr>
<td>$0 = D - X_{14}$</td>
<td></td>
</tr>
</tbody>
</table>

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The operations required to obtain this general solution of the system (4-1) are indicated by the right-hand members of these equations. The second of the equations (4-2), for example, can be obtained by adding the equations (4-1) with constants \( O_1 \) and \( O_3 \) and subtracting the equations with constants \( X_{31}, X_{32}, X_{12}, \) and \( X_{14} \). An equivalent set of row operations was used to reduce the augmented coefficient matrix. A means of checking the general solution is thus available. Another check is provided by the observation that

\[
\sum r = \sum O = \sum D. \quad \text{By adding the first eleven equations and making the substitution} \quad X_{41} = O_4, \text{the relation} \quad \sum r = \sum O \text{is obtained. On adding the last five equations we have} \quad \sum O = \sum D \text{as required.}
\]

The rank of the coefficient matrix is eleven and there are no solutions of the system (4-1) unless the last five equations of (4-2) are satisfied. The right-hand members of the first eleven equations of (4-2) are linear functions of the \( X \) and \( O, O, \text{and} O; \text{thus integral solutions exist for arbitrary integral values of these requirements if} \)

\[
O_4 = X_{41}, \\
D_1 = (X_{21} + X_{31} + X_{41}) + O - (X_1 + X_2 + X_3) \\
D_2 = (X_{21} + X_{22}) + O - (X_1 + X_2) \\
D_3 = (X_{13} + X_{23}) + O - (X_{31} + X_{32}) \\
D_4 = X_{14}.
\]

(4-3)
The first eleven equations could be made to include the terms $O_4$, $D_1$, $D_2$, $D_3$, and $D_4$ by solving the equations (4-3) for five other distinct terms and making the appropriate substitutions. Any one or more of these substitutions could, of course, be made. In any case, the value of some one of the requirements is fixed by each of the equations (4-3) and the expressions for the $r_j$ are given in terms of the other eleven requirements.

If the relations (4-3) are satisfied, the set of all solutions of (4-1) is given by the first eleven equations of (4-2) for arbitrary values of, say, $r_8$, $r_9$, $r_{12}$, $r_{13}$, and $r_{16}$. Let the values of these parameters be denoted by $\lambda_1$, $\lambda_2$, $\lambda_3$, $\lambda_4$, and $\lambda_5$ respectively, and let $r_i = b_i$, $i = 1, \ldots, 16$, be any particular solution of (4-2) with $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$. Then the set of all solutions is represented by the following vector sum:
For integral values of the requirements the $b_i$ are integers; hence values of the $r_i$ are generated by integral values of the parameter by such values.
Implications of Positivity Requirements

Given the general solution (4-4), the following inequalities result from the positivity condition \( r_i \geq 0, \ i = 1, \ldots, 16. \)

\[
egin{align*}
0 & \leq b_1 \\
-\lambda_1 - \lambda_2 - \lambda_3 & \leq b_2 \\
\lambda_2 - \lambda_4 & \leq b_3 \\
\lambda_1 + \lambda_4 & \leq b_4 \\
\lambda_3 & \leq b_5 \\
0 & \leq b_6 \\
\lambda_1 + \lambda_2 & \leq b_7 \\
-\lambda_1 & \leq 0 \\
-\lambda_2 & \leq 0 \\
\lambda_4 - \lambda_5 & \leq b_{10} \\
\lambda_3 + \lambda_5 & \leq b_{11} \\
-\lambda_3 & \leq 0 \\
-\lambda_4 & \leq 0 \\
0 & \leq b_{14} \\
\lambda_5 & \leq b_{15} \\
-\lambda_5 & \leq b_{16}
\end{align*}
\]
Since $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$, $\lambda_1 + \lambda_2 \leq b_7$ implies that $b_7 \geq 0$. From (4-2) we have $b_7 = O_2 - X_{21}$; hence, $O_2 \geq X_{21}$ is a necessary condition for the existence of a solution of (4-5). Since the only run which includes the division trip 2, 1 originates at terminal 2, an interpretation of this condition is immediately obtained. Additional conditions of this type are found by applying the method of elimination given by Kuhn to obtain a general solution of (4-5) [16].

Let $S_1$ denote the system of inequalities (4-5) and suppose that the relations $0 \leq b_1$, $0 \leq b_6$, and $0 \leq b_{14}$ are satisfied. Operating first on the inequalities which include the term $\lambda_1$ we have

\begin{align*}
- \lambda_1 & \leq b_2 + \lambda_2 + \lambda_3 \\
\lambda_1 & \leq b - \lambda_4 \\
\lambda_1 & \leq b - \lambda_7 \\
- \lambda_1 & \leq 0
\end{align*}

These inequalities can be written as follows:

\begin{align*}
- b_2 - \lambda_2 - \lambda_3 & \leq \lambda_1 \leq b - \lambda_4 \\
- b_2 - \lambda_2 - \lambda_3 & \leq \lambda_1 \leq b - \lambda_7 \\
0 & \leq \lambda_1 \leq b - \lambda_4 \\
0 & \leq \lambda_1 \leq b_7 - \lambda_2
\end{align*}  

(4-6)
If there is a solution of the system of inequalities which is obtained from $S_1$ by deleting those which include the term $\lambda_1$, then $S_1$ is solvable if there is a value of $\lambda_1$ which satisfies the inequalities (4-6). There is at least one such value if these inequalities are consistent; that is, if

$$-b_2 - \lambda_3 \lambda_2 \leq b_4 - \lambda_4$$
$$-b_2 - \lambda_3 \lambda_2 \leq b_7 - \lambda_2$$
$$0 \leq b_4 - \lambda_4$$
$$0 \leq b_7 - \lambda_2.$$

Rearranging terms with the parameters on the left and the constants on the right we have

$$-\lambda_2 - \lambda_3 + \lambda_4 \leq b_4 + b_2$$
$$-\lambda_3 \leq b_7 + b_2$$
$$\lambda_4 \leq b_4$$
$$\lambda_2 \leq b_7.$$

The relations (4-7) are those which are obtained if each of the inequalities of $S_1$ which includes the term $-\lambda_1$ is added, in turn, to each of the inequalities of $S$ which includes the term $\lambda_1$ with coefficient $+1$.

Delete from $S_1$ the inequalities that include the term $\pm \lambda_1$, append the inequalities (4-7) and let $S_2$ denote the resulting system. Given any solution of $S_2$, a solution of $S_1$ is obtained by assigning any value to $\lambda_1$.
which satisfies (4-6). Since the relations (4-7) are included in $S_2$ at least one such value. Any linear relations of the form $0 \leq L \left(b_{i}\right)\in S_2$ included in $S_2$ are necessary conditions for the solvability of $S_2$ at any of $S_1$. If some $L \left(b_{i}\right)$ were greater than zero, a contradiction is obtained and the system can be said to possess no solution. In a similar manner, systems $S_3$, $S_4$, $S_5$ and $S_6$ can be constructed. At the last stage, inequalities of the form $0 \leq L \left(b_{i}\right)$ remain.

Since the elimination procedure is one of adding certain inequalities by appending the resulting inequalities, and deleting those added, the results can be obtained by operating on coefficient matrices. If members of (4-2) are substituted for the corresponding $b_{i}$, the system of inequalities (4-5) can be written in matrix notation as follows:

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{array}{c}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\lambda_4 \\
\lambda_5
\end{array}
\leq
\begin{bmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[2\] See [16] for an alternative interpretation and discussion.
Let this be the system $S_1$. $S_2$ is obtained from $S_1$ by appending the new rows to each matrix which are given for each matrix by the operations row $2 + \text{row } 4$, $2 + 7$, $8 + 4$, and $8 + 7$, and by then deleting rows 2, 4, 7 and 8 from both matrices. The entries in column one of the parameter coefficient matrix of $S_2$ are all zero. In general $S_n (n = 2, 3, \ldots)$ is obtained from $S_{n-1}$ by appending to $S_{n-1}$ all inequalities which can be formed by adding an inequality with $-1$ in column $n-1$ of the left-hand matrix and an inequality with $+1$ in that column, and by then deleting those inequalities with $+1$ in the same column. Only the entries "0" are found in the parameter coefficient matrix of $S_n$; thus a system of inequalities in terms of the requirements has been obtained. Some of these inequalities may be superfluous; that is, they are given by adding others. Such relations may be deleted to obtain an "independent" set under addition. By construction, a solution of (4-8) exists if and only if these conditions can be satisfied.

For values of the requirements which satisfy $S_6$, values of the parameters which satisfy the system (4-8) are given by this process. The inequalities of $S_n$ with nonzero entries in column $n$ of the parameter coefficient matrix establish the range of admissible values of $\lambda_n$ in terms of the requirements and the parameters which are eliminated at later stages. The following derivation will illustrate the procedure.

---

3 In the case of this example the entries of the parameter coefficient matrix are, at every stage, 0 or $\pm 1$. 56
The inequalities \(0 \leq X_{12}, 0 \leq X_{21}, \) and \(0 \leq X_{21} + X_{23} - O \) of (4-8) are given as "restrictions" at the bottom of page 58. The coefficient matrices of the remaining inequalities are given above the horizontal line with the parameter coefficient matrix to the left of the vertical line and the requirements coefficient matrix to the right. The row operations required to obtain \(S_2\) are noted and the system \(S_2\) is rewritten (above the horizontal line) on the following page. The process continues through six stages and for \(S_6\) only restrictions remain.
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### Restrictions

**$S_1$**

$0 \leq X_{12}$

$0 \leq X_{21}$

$0 \leq X_{21} + X_{23} - O_2$

**$S_2$**

None
\[ S_2 \]

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\[ S_3 \text{ Restriction: } 0 \leq -x_{21} + o_2 \]
$$S_3$$

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$$0 \leq x_{14}$$

$S_4$ Restrictions: $$0 \leq x_{12} - x_{21} - x_{31} - x_{32} + O_1 + O_2 + O_3$$

$$0 \leq x_{12} + O_1$$
\[ S_4 \]

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\[ 0 \leq -X_{23}X_{23} + X_{32} + O_2 \]

\[ S_5 \text{ Restrictions:} \]

\[ 0 \leq X_{31} + X_{32} - O_3 \]

\[ 0 \leq -X_{12} - 2X_{21} - X_{31} - X_{32} + O_1 + 2O_2 + O_3 \]

\[ 0 \leq -X_{12} - X_{21} + O_1 + O_2 \]
**Note:** The starred rows are identical.
On collecting the foregoing restrictions and rearranging terms, we have the system presented on the following page. If the $X_{tu}$ are understood to be nonnegative, the first six inequalities and the inequality $O_2 \geq X_{21} - X_{13}$ may be deleted. The remaining inequalities that are not identified by a letter can be obtained by adding certain of the inequalities identified by the letters a through o. Such inequalities are superfluous and may be deleted. By construction, for any nonnegative values of the $X_{tu}$ and $O_t$ which satisfy the inequalities a through o, there are (positive) solutions of (4-8).

The range of values of the parameters which satisfy (4-8) are given on page 65. Consider the parameter $\lambda_5$. For integral values of the $X_{tu}$ and $O_t$, the upper and lower bounds on $\lambda_5$ are integers and for given values of the requirements a least upper bound and a greatest lower bound can be found. If given values of the requirements are nonnegative integers which satisfy the foregoing restrictions, these bounds are nonnegative integers and there is at least one integer in this interval. Assign an integral value in this interval to $\lambda_5$. The upper and lower bounds of $\lambda_4$ are given in terms of the requirements and $\lambda_5$, and with this modification the above remarks apply to $\lambda_4$. Proceeding in this way a (positive) integral solution of (4-8) can be obtained.
\[
\begin{align*}
0 & \equiv X_{12} \\
0 & \equiv X_{21} \\
0 & \equiv X_{13} \\
0 & \equiv X_{31} \\
0 & \equiv X_{14} \\
0 & \equiv X_{41}
\end{align*}
\]

\[
\begin{align*}
a & \equiv O_1 \equiv X_{12} \\
b & \equiv O_1 \equiv X_{12} + X_{14} - X_{31} \\
c & \equiv O_1 \equiv X_{12} + X_{13} - X_{31} + X_{14} - X_{41} \\
d & \equiv O_1 \equiv X_{12} + X_{13} + X_{14} \\
e & \equiv O_2 \equiv X_{21} \\
f & \equiv O_2 \equiv X_{21} - X_{13} \\
g & \equiv O_2 \equiv X_{21} + X_{23} \\
h & \equiv O_3 \equiv X_{91} + X_{32} \\
i & \equiv O_1 + O_2 \equiv X_{12} + X_{21} \\
j & \equiv O_1 + O_2 \equiv X_{12} + X_{21} + X_{14} - X_{31} \\
k & \equiv O_2 + O_3 \equiv X_{21} + X_{23} \\
l & \equiv O_2 + O_3 \equiv X_{21} + X_{23} + X_{31} - X_{13} \\
m & \equiv O_2 + O_3 \equiv 2X_{21} + X_{23} + X_{31} - X_{13} \\
n & \equiv O_2 + O_3 \equiv 2X_{21} + X_{23} + X_{31} + X_{32} \\
o & \equiv O_1 + 2O_2 + O_3 \equiv X_{12} + 2X_{21} + X_{31} + X_{23} \\
op & \equiv O_1 + 2O_2 + O_3 \equiv X_{12} + 2X_{21} + X_{14} + X_{23} \\
q & \equiv O_1 + 3O_2 + 2O_3 \equiv X_{12} + 2X_{21} + X_{31} + X_{14} + X_{23} + X_{32} \\
r & \equiv O_1 + 3O_2 + 2O_3 \equiv X_{12} + 3X_{21} + X_{31} + X_{14} + 2X_{23}
\end{align*}
\]
-\lambda_1 \leq \lambda_3 - X_{12} - X_{31} - X_{14} - X_{32} + O_1 + O_3

-\lambda_1 \leq 0

\lambda_1 \leq -\lambda_4 - X_{21} - X_{23} + X_{32} + O_2

\lambda_1 \leq -\lambda_2 - X_{21} + O_2

-\lambda_2 \leq \lambda_3 - X_{12} - X_{21} - X_{31} - X_{14} - X_{23} + O_1 + O_2 + O_3

-\lambda_2 \leq 0

\lambda_2 \leq \lambda_4 + X_{21} + X_{31} + X_{23} - O_2 - O_3

\lambda_2 \leq -X_{21} + O_2

\lambda_3 \leq -X_{12} - X_{21} - X_{31} - X_{14} - X_{32} + O_1 + O_2 + O_3

-\lambda_3 \leq -X_{12} - X_{14} + O_1

-\lambda_3 \leq -\lambda_4 - X_{12} - 2 X_{21} - X_{31} - X_{14} - X_{23} + O_1 + 2O_2 + O_3

-\lambda_3 \leq 0

\lambda_3 \leq -\lambda_5 + X_{12} + X_{13} + X_{14} - O_1

\lambda_3 \leq X_{14}

-\lambda_4 \leq X_{21} + X_{31} + X_{23} - O_2 - O_3

-\lambda_4 \leq 0

\lambda_4 \leq \lambda_5 - X_{12} - X_{21} - X_{13} - X_{14} - X_{23} + O_1 + O_2 + O_3

\lambda_4 \leq -X_{21} - X_{23} + X_{32} + O_2

\lambda_4 \leq -X_{12} - 2 X_{21} - X_{31} - X_{23} + O_1 + 2O_2 + O_3

\lambda_4 \leq -\lambda_5 - 2 X_{21} + X_{13} - X_{31} - X_{23} + 2O_2 + O_3

-\lambda_5 \leq -X_{12} - X_{21} - X_{13} - X_{14} - X_{23} + O_1 + O_2 + O_3

-\lambda_5 \leq -X_{12} - X_{13} + X_{31} - X_{14} + O_1

-\lambda_5 \leq 0

\lambda_5 \leq X_{41}

\lambda_5 \leq X_{12} + X_{13} + X_{14} - O_1

\lambda_5 \leq -X_{21} + X_{13} - X_{31} - X_{32} + O_2 + O_3

\lambda_5 \leq X_{13}

\lambda_5 \leq -2 X_{21} + X_{13} - X_{31} - X_{23} + 2O_2 + O_3

\lambda_5 \leq -X_{21} + X_{13} + O_2
Since we require that all values of the requirements, including $O_4$ and $D_1, \ldots, D_4$, be nonnegative integers, the relations (4-3) imply an additional positivity restriction. The fourth of the equations (4-3) is

$$D_3 = (X_{13} + X_{23}) + O_3 - (X_{31} + X_{32})$$

and $D_3 \geq 0$ implies that

$$O_3 \geq X_{31} - X_{13} + X_{32} - X_{23}.$$  

Let this be the restriction "p". All other such conditions that can be obtained from (4-3) are implied by the restrictions a through o.

On appending the restriction p and deleting the restrictions that are not identified by a letter, the following inequalities (4-10) are obtained. If these inequalities and the equations (4-3) are satisfied by nonnegative integral values of the requirements, then (nonnegative) integral values of the parameters exist which satisfy (4-8), and any such values (in (4-4) ) generate positive integral solutions of (4-1). The necessity of all of the conditions (4-10) except $10_o$ can readily be verified by inspection of the network. The necessity of $10_o$ can be demonstrated by a simple numerical example.
\[ \begin{align*}
a & \quad O_1 = X_{12} \\
b & \quad O_1 = X_{12} + X_{14} - X_{31} \\
c & \quad O_1 = X_{12} + X_{14} - X_{31} + X_{13} - X_{41} \\
d & \quad O_1 = X_{12} + X_{13} + X_{14} \\
e & \quad O_2 = X_{21} \\
f & \quad O_2 = X_{21} + X_{23} - X_{32} \\
g & \quad O_2 = X_{21} + X_{23} \\
h & \quad O_3 = X_{31} + X_{32} \\
i & \quad O_2 + O_3 = X_{21} + X_{23} \\
j & \quad O_2 + O_3 = X_{21} + X_{31} + X_{32} - X_{13} \\
k & \quad O_1 + O_2 + O_3 = X_{12} + X_{21} + X_{31} + X_{32} \\
l & \quad O_1 + O_2 + O_3 = X_{12} + X_{21} + X_{14} + X_{32} \\
m & \quad O_1 + O_2 + O_3 = X_{12} + X_{21} + X_{14} + X_{23} \\
n & \quad O_1 + O_2 + O_3 = X_{12} + X_{21} + X_{14} + X_{23} + X_{13} - X_{41} \\
o & \quad O_1 + 2O_2 + 2O_3 = X_{12} + 2X_{21} + X_{14} + X_{23} + X_{31} + X_{32} \\
p & \quad O_3 = X_{31} - X_{13} + X_{32} - X_{23} \end{align*} \]
Numerical Example

Let all $X_{tu} = 1$. Then the inequalities (4-10) reduce to

\[
\begin{align*}
1 & \leq O_1 \leq 3 \\
1 & \leq O_2 \leq 2 \\
0 & \leq O_3 \leq 2 \\
0 & \leq O_4 \leq 2 \\
O_1 + O_2 + O_3 & \geq 4 \\
O_1 + 2O_2 + 2O_3 & \geq 7
\end{align*}
\]

From (4-3), $O = X = 1$. The values $O_1 = 2$, $O_2 = 1$, $O_3 = 1$ and $O_4 = 1$ satisfy all but the last inequality, which is the condition $10^O$. By inspection of the network it can be established that no run schedule exists which meets these requirements if runs are restricted to ten hours. $O_1 = 3$, $O_2 = 1$, $O_3 = 1$ and $O_4 = 1$ satisfy the above restrictions and a feasible schedule exists.

The relations (4-9) give $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$, and for these values the following schedule is obtained from (4-2).

\[
\begin{align*}
r_1 &= 1 & 1, 2 & r_6 &= 1 & 2, 1 \\
r_3 &= 1 & 1, 3, 1 & r_{14} &= 1 & 3, 2, 3 \\
r_5 &= 1 & 1, 4 & r_{15} &= 1 & 4, 1
\end{align*}
\]
All other \( r_j \) are zero. The terminal sequences of assigned runs are noted to verify that the schedule satisfies all requirements.

In the case of this example there is, for fixed values of the \( X_{tu} \), a range of values of, say, \( O_1, O_2, \) and \( O_3 \) for which a feasible schedule exists; further, for fixed, feasible values of the requirements, the solution is not in general unique. These are general characteristics of systems of network equations of the type considered here (where \( w < n \)). There is, in this sense, a measure of scheduling "flexibility" which may be exploited to realize various operating objectives.

**Summary**

A general solution of (4-1) and a criterion for the existence of positive integral solutions was to be obtained. Since all coefficients of this system of equations are nonnegative, it immediately follows that positive solutions exist only if the values of all requirements are nonnegative. There are no solutions of (4-1) unless the equations (4-3) are satisfied and, if any solutions exist, the set of all solutions is given by (4-4) for arbitrary values of the parameters. Positive integral solutions are generated by and only by non-negative integral values of these parameters.

The positivity conditions were expressed by (4-8), and a general solution of this system of inequalities was obtained by Kuhn's method of elimination. The result is a set of necessary and sufficient conditions on the requirements.
for the existence of a (positive) solution of (4-8), and a set of upper and lower
bounds on the parameters which, for given values of the requirements, establish
the range of values of the parameters for which (4-8) is satisfied. For integral
values of the requirements these bounds are integers. Then for any nonnegative
integral values of the requirements which satisfy the conditions 10 through
10 \( \alpha \), there are (positive) integral solutions of (4-8).

It was assumed that \( O_4, D_1, D_2, D_3, \) and \( D_4 \) were to be assigned the
values given by (4-3). The requirement that these values be nonnegative imposes
the additional positivity requirement 10 \( \alpha \).

We conclude that for any nonnegative integral values of the requirements,
the conditions (4-10) together with those of (4-3) are necessary and sufficient
for the existence of positive integral solutions of (4-1). Further, since upper
and lower bounds on the values of the parameters are established by (4-9), the
number of positive integral solutions of (4-1) is finite.

The General Case

For any system of network equations of the type defined in Chapter 2, the
coefficient matrix contains only integer entries and the matrix can be row-
reduced by adding integral multiples of the rows. A system of linear inequal-
ities of the type (4-8) expresses the condition that all solutions be positive. In
this system the parameter coefficient matrix will contain only integer entries;
hence, the elimination procedure can be carried out by adding positive multiples
of the inequalities. This follows since, at every stage \( n \), there will be at least one entry in column \( n \) of the parameter coefficient matrix with a positive coefficient and at least one with a negative coefficient. If this were not the case, \( \lambda_n \) would be unbounded either above or below or both; an infinity of positive integral values of \( \lambda_n \) would satisfy the system; and an infinity of positive integral solutions of the linear equations could be generated. But each \( \lambda_n \) denotes some indeterminate \( r^\delta \) which can be assigned any feasible nonnegative integral value; hence, the system of inequalities includes the restriction (a least lower bound) \( \lambda_n \geq 0, \) all \( n \). Each of the runs, \( r_j \), originates at some terminal \( t \), and the maximum number of runs which can originate at \( t \) is given by \( O_t \). For each \( \lambda_n \), then, a greatest upper bound is established. It follows that the number of nonnegative integral solutions of the system of inequalities is finite.

If the elimination procedure is carried out by adding multiples of the inequalities, then, for integral values of the requirements, the upper and lower bounds given by the relations of the type (4-9) are integers. The sufficiency of the conditions of the type (4-10) for the existence of (positive) integral solutions of the system of linear inequalities is thus established. This completes the demonstration that for any system of network equations of the type considered here, conditions of the type (4-3) and (4-10) can be obtained which are necessary and sufficient for the existence of positive integral solutions.

Note that if at any stage of the solution process a division were required, it could happen that for some integral values of the requirements, the upper
and lower bounds given by relations of the type (4–9) would not all be integers and, further, that no integer would be included in some such interval. The system of inequalities of the type (4–8), then, would possess no integral solution. In this case the solution space of the system of linear equations includes no integral point for the given integral values of the requirements. For such systems of linear equations Smith's criterion must be applied (see p. 33).
CHAPTER 5

RUN SCHEDULING

The Decision Problem

We have a means of obtaining a general solution of any system of network equations of the type specified in Chapter 2. It consists of a criterion for the existence of positive integral solutions and a representation of the set of all solutions. In general, if any such solutions exist for a particular set of values of the requirements, there is a large, but finite, number of such solutions or feasible schedules.

A basic assumption that was made in the initial statement of the problem is that for any given scheduling period, a set of values of the requirements is given which is a desired or "ideal" specification of schedule characteristics. If these values of the requirements do not satisfy the feasibility criterion, a change in the value of one or more of the requirements must be made. In every such case there is more than one way of changing these values so that at least one feasible schedule exists. The question of determining the requirements changes that should be made is considered in the following chapter. Here we assume that at least one feasible schedule exists for given ideal values

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of the requirements so that the decision problem is restricted to that of selecting the particular feasible schedule to be employed. A formal statement of the problems follows.

The general solution of any of the systems of network equations considered here can be represented in the form given by the relations (3-14); that is,

\[ r^* = b_1 + \sum_{i=1}^{n-w} \left( -e_{i1} \right) \lambda_i , \quad i = 1, \ldots, w \]
\[ r^* = \lambda_d , \quad d = 1, \ldots, n-w. \]

The \( b_i \) and the \( e_{id} \) are integers. Let \( r^* = b_i, i = 1, \ldots, n, \) be the particular solution that is obtained when \( \lambda_d = 0, d = 1, \ldots, n-w, \) and let \( \mathcal{E} = \left| e_{id} \right| \) denote the \( n \) by \( n-w \) matrix formed by the coefficients of the parameters \( \lambda_d. \) (Note the change of sign.) Further let \( \mathcal{E}_d \) be the (column) vector formed by the entries in column \( d \) of \( \mathcal{E} \). Then, in vector notation, the set of all solutions can be written

\[ r^* = b + \lambda_1 \mathcal{E}_1 + \ldots + \lambda_d \mathcal{E}_d + \ldots + \lambda_{n-w} \mathcal{E}_{n-w}. \quad (5-1) \]

Here the \( \lambda_d \) take arbitrary values.

By using the methods given in Chapters 3 and 4, a set of upper and lower bounds on the \( \lambda_d \) can be obtained which define a domain \( D \) such that positive solutions are generated if and only if \( \lambda_1, \ldots, \lambda_d, \ldots, \lambda_{n-w} \in D. \) For any
feasible values of the requirements, there are integral points

$(\lambda_1, \ldots, \lambda_{d'}, \ldots, \lambda_{n-w})$ in $D$ and these and only these values of the parameters, in (5-1), generate feasible schedules (positive integral solutions $r^*$). The decision problem, then, is that of selecting an integral point

$(\lambda_1, \ldots, \lambda_{d'}, \ldots, \lambda_{n-w}) \in D$. The problem is nontrivial if more than one such point exists and if the decision-maker is not indifferent to the choice of the particular feasible schedule to be used.

Some Solutions Methods—A Review

One means of obtaining a solution to a problem of this kind is simply to present the decision-maker with the set of feasible schedules for a particular period and request that one be selected. The analyst might then claim a contribution to the decision process since an exhaustive set of alternatives has been presented—at least the decision need not be made in ignorance of what is possible. Such a procedure, however, suffers from a number of obvious difficulties. First of all, although the number of alternative schedules is finite, the number may be so large that even if a listing could be made available, it would be of little practical value. Further, a decision of this kind is an operating decision which must be made on a routine basis is schedule requirements change from period to period. In this case the cost of such a decision process would probably be prohibitive. Perhaps most importantly, however, such a procedure lacks objectivity; that is, the basis for choice is not made explicit. Our understanding of the method of solution is incomplete
in that the rational basis for choice, if any, is not known. If responsibility for operating decisions is to be delegated or if decision-making is to be "mechanized," an explicit basis for choice is required to control that decision process.

In the terminology of popular organization theory, the basis for a routine operating decision is called an operating "policy." Needed, then, is a statement of policy. But any statement of policy which leads to a decision is a solution of the stated decision problem. We may object to an arbitrary statement of policy simply on the ground that the objectives of the policy are not given. The statement of a policy determines how a decision is to be made but not the reason for the decision, and our understanding of the process remains incomplete. Finally, the recognition of an "exception principle" implies that the formulation of an acceptable, explicit statement of policy may be a difficult, if not an impossible, task.

Given an appropriate mathematical formulation of a decision problem, a solution may be obtained by analytic methods. The major elements of a decision problem which are postulated by modern theories of decision are:

1. a set of alternative courses of action
2. a measure of the desirability of "utility" of the alternatives
3. a set of rules for selecting the "best" alternative [20]

If the decision-maker could obtain a general analytic solution of a decision problem, the result would be a set of rules that would be recognized as a statement of policy. If some solution algorithm is used in every particular
case the policy is implicit. In either case the remarks of the preceding paragraph apply. A major advantage of this approach is that it may be applied in cases for which a general solution (an explicit policy statement) is not feasible. The approach may provide a solution method for which the exception principle is not needed. The objectives to be realized by a "best" solution are explicit, so, if the method is understood, the policy (expressed or implied) cannot be said to be arbitrary. This does not, however, mean that a policy which is obtained analytically is necessarily a "good" one.

Here an analytic solution method is to be used. We are interested not only in obtaining a decision mechanism but also in examining the possible contribution of the analysis to the problem of formulating policy. Needed is a basis for making the necessary value judgments. The approach used will be of particular interest in the following chapter.

Although the question of the cost of the decision process is of considerable engineering interest, this aspect of the decision problem will not be extensively treated here.

An Illustrative Utility Function

The initial formulation of the problem given in Chapter 2 is a statement of the "elements of value" which are here assumed to be relevant to a decision. Implied is a set of alternative courses of action which for any particular scheduling period is given by $(\lambda_1, \ldots, \lambda_{n-w}) \in D$. The mathematical
formulation of the problem, then, can be viewed as a general statement of policy which determines, in part, the kinds of judgments required to obtain a "best" decision. Only those stated elements of value whose utility changes with changing values of the parameters $\lambda_1, \ldots, \lambda_{n-w}$ are relevant to this decision.

Let $U_d(\lambda'_d)$ denote the utility associated with a particular value $\lambda'_d$ of any parameter $\lambda_d$ and let $U(\lambda'_1, \ldots, \lambda'_d, \ldots, \lambda'_{n-w})$ denote the utility of a particular feasible schedule where

$$U(\lambda'_1, \ldots, \lambda'_{n-w}) = \sum_{d=1}^{n-w} U_d(\lambda'_d).$$

Given any particular utility function we suppose that a feasible set of values $(\lambda^*_d)$ is desired such that for any other set of feasible values $(\lambda'_d)$,

$$U(\lambda^*_1, \ldots, \lambda^*_{n-w}) \geq U(\lambda'_1, \ldots, \lambda'_{n-w}).$$

That is, utility is to be maximized. If "cost" is to be the basis for choice, then the negative of a cost function is the appropriate utility function.

The vector equation (5-1) provides a basis for obtaining an appropriate utility function. For a unit change in the value of any parameter $\lambda_d$, the coordinate $e_{id}$ in the $i$th position in the column vector $\mathbf{C}_d$ gives the resulting change in the $i$th coordinate of $\mathbf{r}^*$; that is, the change in the number of runs of type $i$ that are scheduled. Given a measure of value in terms of the

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indeterminates, this observation provides a basis for determining the utility of a particular value of one of the parameters.

Suppose that total run cost is to be the basis for choice and that \( y_i \) is the cost of making a run of type \( i \). Then

\[
\sum_{i=1}^{n} y_{id}
\]

is the net change in total run cost occasioned by a unit increase in any given parameter \( \lambda_d \), and

\[
U_d(1) = -\sum_{i=1}^{n} y_{id}
\]

is the utility associated with a choice of \( \lambda_d = 1 \). In general,

\[
U_d(\lambda_d) = -\lambda_d \sum_{i=1}^{n} y_{id}, \quad d = 1, \ldots, n-w. \tag{5-3}
\]

Note that if \( U_d(\lambda_d) \neq 0 \) the utility may be either greater or less than zero.

The utility of a schedule is assumed to be given by

\[
U(\lambda_1, \ldots, \lambda_{n-w}) = -\sum_{d=1}^{n-w} \sum_{i=1}^{n} y_{id} \lambda_d. \tag{5-4}
\]

In the case of this example, the \( U_d(\lambda_d) \) and \( U(\lambda_1, \ldots, \lambda_{n-w}) \) are linear functions.

A set of values \( \lambda^*_1, \ldots, \lambda^*_{n-w} \in D \) for which the utility function (5-4) is a maximum generates a feasible run schedule which minimizes the total
run cost $C_r$. We have

$$C = \sum_{i=1}^{n} y_i r^*_i$$

where

$$r^*_i = b + \sum_{d=1}^{n-w} e_{id} \lambda_i$$

for $i = 1, \ldots, n$.

Then

$$C = \sum_{i=1}^{n} \sum_{d=1}^{n-w} y_i r^*_i = \sum_{i=1}^{n} y_i b + \sum_{i=1}^{n} \sum_{d=1}^{n-w} y_i e_{id} \lambda_i.$$

But the last expression for $C$ differs from the right-hand member of (5-4) only by the constant term $\sum_{i=1}^{n} y_i b$ and the sign of the second term. Hence, a set of values $(\lambda^*_i)$ for which the utility function (5-4) takes the maximum value is a set which minimizes the total run cost $C_r$.

Suppose that for the case of the illustrative problem of Chapter 4 the costs, $(y_i)$, of run types $i = 1, \ldots, 16$ are as follows:

<table>
<thead>
<tr>
<th>$i$</th>
<th>$y_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>13</td>
<td>2</td>
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<tr>
<td>14</td>
<td>3</td>
</tr>
<tr>
<td>15</td>
<td>2</td>
</tr>
<tr>
<td>16</td>
<td>4</td>
</tr>
</tbody>
</table>
From (5-2) above we obtain

\[
\begin{align*}
U_1(1) &= 1 \\
U_2(1) &= -1 \\
U_3(1) &= -1 \\
U_4(1) &= 1 \\
U_5(1) &= -1
\end{align*}
\]

where the \( \epsilon \) are given by (4-4). Then the utility of a schedule is

\[
U(id) = X_1 - X_2 - X_3 + X_4 - X_5 (5-5)
\]

Given this simple utility function, a general solution of the illustrative decision problem can be readily obtained. For this case, then, an explicit statement of a run scheduling policy which minimizes total run cost can be made available. It can be applied in any case for which a feasible schedule exists. The derivation follows.

**A General Solution of the Illustrative Scheduling Problem**

We require expressions for a set of values \( \lambda^*, \ldots, \lambda^*_5 \in D \) which maximize the utility function (5-5) where the domain \( D \) is defined by the inequalities (4-9). These inequalities are of the following form.
\[
\begin{align*}
0 & \leq \lambda_1 \\
-\lambda_2 - \lambda_3 - k_1 & \leq \lambda_1 \\
\lambda_1 & \leq k_2 - \lambda_4 \\
\lambda_1 & \leq k_3 - \lambda_2 \\
0 & \leq \lambda_2 \\
-\lambda_3 + \lambda_4 - k_4 & \leq \lambda_2 \\
\lambda_2 & \leq \lambda_4 + k_5 \\
\lambda_2 & \leq k_6 \\
0 & \leq \lambda_3 \\
- k_7 & \leq \lambda_3 \\
- k_8 & \leq \lambda_3 \\
\lambda_4 - k_9 & \leq \lambda_3 \\
\lambda_3 & \leq k_{10} - \lambda_5 \\
\lambda_3 & \leq k_{11} \\
0 & \leq \lambda_4 \\
- k_{12} & \leq \lambda_4 \\
\lambda_4 & \leq \lambda_5 + k_{13} \\
\lambda_4 & \leq k_{14} \\
\lambda_4 & \leq k_{15} \\
\lambda_4 & \leq k_{16} - \lambda_5 \\
0 & \leq \lambda_5 \\
- k_{17} & \leq \lambda_5 \\
- k_{18} & \leq \lambda_5 \\
\lambda_5 & \leq k_{19} \\
\lambda_5 & \leq k_{20} \\
\lambda_5 & \leq k_{21} \\
\lambda_5 & \leq k_{22} \\
\lambda_5 & \leq k_{23} \\
\lambda_5 & \leq k_{24}
\end{align*}
\]
Here the $k_1, \ldots, k_{24}$ are given by particular linear functions of the requirements. If the relations (4-10) are satisfied for given integral values of the requirements, these are integers such that the above inequalities form a consistent system. The range of feasible values of $\lambda_5$ is thus determined, and given a particular feasible value $\lambda'_5$, the range of $\lambda_4$ is established. Proceeding in this way, a solution $\lambda'_5, \ldots, \lambda'_1$ can be obtained.

Suppose that a value which satisfies the above system has been assigned each of the parameters $\lambda_2, \lambda_3, \lambda_4$, and $\lambda_5$ so that the problem is reduced to that of selecting a best feasible value for $\lambda_1$. Since $U_1(\lambda_1) = \lambda_1$, the maximum feasible value of $\lambda_1$ is desired. This value is given by the least upper bound of $\lambda_1$; hence,

$$\lambda^* = \min_{1} \left\{ \frac{k_2 - \lambda_4}{1}, \frac{k_3 - \lambda_2}{1} \right\}$$

This is an application of Bellman's "Dynamic Programming" (functional equation) approach which is based on the following principle:

**Principle of Optimality.** An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision [3, p. 83].

We propose to find, in turn, an expression for the optimal value of each of the remaining parameters $\lambda_2, \ldots, \lambda_5$ by using the utility function (5-5) as the basis for choice.
Note that the value of $\lambda^*$ may depend on $\lambda$ and that $U (\lambda) \leq 0$ for $\lambda_2 \geq 0$. But for $\lambda_2 \geq 0$, $\lambda^*$ can only be decreased by increasing values of $\lambda_2$; hence, $U (\lambda^*, \lambda) = \lambda^* - \lambda$ takes its maximum feasible value when $\lambda_2$ takes its minimum feasible value. Thus

$$\lambda^* = \text{Max} \begin{cases} 0 \\ -\lambda + \lambda - k \\ 3 \\ 4 \\ 4 \\ \end{cases}$$

In the case of $\lambda_3$, the situation is a bit more complicated. Required is an expression for $\lambda_3$ for which $U (\lambda^*, \lambda^*, \lambda) = \lambda^* - \lambda^* - \lambda$ is a maximum. We have

$$\lambda^* = \text{Min} \begin{cases} k_2 - \lambda \\ 4 \\ k_3 - \lambda \\ 2 \\ \end{cases} = \text{Min} \begin{cases} k_2 - \lambda \\ 4 \\ 3 - \text{Max} \begin{cases} 0 \\ 3 \\ -\lambda + \lambda - k \\ 4 \\ 4 \\ \end{cases} \\ 3 \\ 4 \\ 4 \\ \end{cases} = \text{Min} \begin{cases} k_3 \\ 3 \\ k_3 - (\lambda + \lambda - k) \\ 3 \\ 4 \\ 4 \\ \end{cases}$$

and an expression for $\lambda^*$. Suppose that $-\lambda + \lambda - k \leq 0$. Then neither $\lambda^*$ nor $\lambda^*$ depends on $\lambda$, and since $U (\lambda) \leq 0$ for $\lambda \geq 0$, the minimum feasible value of $\lambda_3$ is desired. If, however, we have $-\lambda + \lambda - k > 0$, then (for $\lambda_2 \geq 0$) a unit increase in $\lambda_3$ gives $\Delta U (\lambda_3) = -1$; $\Delta U (\lambda^*_{2}) = 1$, and $\Delta U_1 (\lambda^*_{1}) = 1$. Hence $\Delta U (\lambda^*, \lambda^*, \lambda) = 1$ and $\lambda^* = \lambda - k$ if this is a feasible value. For $\lambda_3 > \lambda - k$ the previous case is obtained and again

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we have $\lambda^*_3 = \lambda_4 - k$ if feasible. Thus

$$\lambda^*_3 = \min \left\{ \begin{array}{l} \lambda^*_4 - k \lambda^*_4 - k \lambda^*_4 - k \\ \lambda^*_4 - k \lambda^*_4 - k \lambda^*_4 - k \end{array} \right\}$$

Given $U(\lambda^*_1, \lambda^*_2, \lambda^*_3, \lambda^*_4) = \lambda^*_1 - \lambda^*_2 - \lambda^*_3 + \lambda^*_4$ and the above expressions for $\lambda^*_1, \lambda^*_2, \lambda^*_3, \lambda^*_4$, we observe that any one or more of these values may depend on $\lambda^*_4$, but if so then $U_1(\lambda^*_1), U_2(\lambda^*_2)$ or $U_3(\lambda^*_3)$ is reduced for increasing values of $\lambda^*_4 \geq 0$. It is sufficient to show that for any value of $\lambda^*_4 > 0$ the utility of the schedule is, at best, unchanged. We have $U_4(1) = 1$; but if $-\lambda^*_3 + \lambda^*_4 - k > 0$ then $\lambda^*_4$ is increased; otherwise, $\lambda^*_4$ is reduced. In either case

$$\lambda^*_4 = \max \left\{ \begin{array}{l} 0 \\ -k \\ 12 \end{array} \right\}$$

is always an optimal value.

Given the above decisions the only value which may depend on $\lambda_5$ is $\lambda^*_3$.

But for $\lambda_5 \geq 0$ and $\Delta \lambda_5 = 1$, $U_5(\lambda_5) = -1$ and $\Delta U_5(\lambda_5) = 0$ or 1. Positive values
of $\lambda_5$ never increase the value of the utility function; thus

$$\lambda^* = \max \begin{cases} 0 \\ -k_{17} \\ -k_{18} \end{cases}$$

is an optimal value.

In summary, the following functions define a policy which is optimal in the sense that, for feasible values of the requirements, a feasible schedule which minimizes $\sum_{i=1}^{16} y_{ri}^*$ is always generated by the given values of the parameters.
\[ \lambda^*_1 = \text{Min} \begin{cases} k_2 - \lambda_4 \\ k_3 - \lambda_2 \end{cases} \]

\[ \lambda^*_2 = \text{Max} \begin{cases} 0 \\ -\lambda_3 + \lambda_4 - k_4 \end{cases} \]

\[ \lambda^*_3 = \text{Min} \begin{cases} 0 \\ -k_7 \\ -k_8 \\ \lambda_4 - k_9 \\ \lambda_4 - k_4 \end{cases} \]

\[ \lambda^*_4 = \text{Max} \begin{cases} 0 \\ -k_{12} \end{cases} \]

\[ \lambda^*_5 = \text{Max} \begin{cases} 0 \\ -k_{17} \\ -k_{18} \end{cases} \]
For given feasible values of the requirements more than one optimal schedule may exist. The above policy, however, specifies a particular optimal schedule in every case.

The General Case

Note that at any stage $d$ in the above solution process,

$U(\lambda^*, \ldots, \lambda^*_d, \lambda)$ can be written as a function of $\lambda, \ldots, \lambda^*_d$ and that the number of cases to be considered at each stage is finite. For each case the expression for $\lambda^*_d$ can be determined by establishing that $U_d(z\lambda^*_d)$, $z$ an integer, is less than, equal to, or greater than zero. The writer, however, has not determined that a unique expression for $\lambda^*_d$, $d = 1, \ldots, n-w$, can always be obtained for every problem of this type. But since

$U(\lambda^*, \ldots, \lambda^*_{n-w-1}, \lambda_{n-w})$ is a function only of $\lambda_{n-w}$, and since a set of constants establish the range of $\lambda_{n-w}$, the method will always lead to a general solution. For large problems, however, an expression for $\lambda^*_d$ may be so complex that a general solution does not provide an acceptable hand computation method.

In such cases the following approaches may be feasible:

1. Apply some solution algorithm to solve the problem for each particular scheduling period using a high-speed computer if desirable.

2. Derive a simplified policy statement which approximates the desired policy.
Clearly a "trade-off" between solution cost and possible operating savings may arise. No general solution to a problem of this kind appears to be possible. It is, rather, a decision-process design problem to be considered in each individual case.

The above solution method, however, may be of some assistance in formulating an acceptable statement of policy even in cases for which an explicit utility function cannot be made available. Such cases, for example, may arise when (as for the general programming problem to follow) certain "costs" are of the "intangible" variety. A utility function with the following properties will lead to a general solution.

P-1. For any $\lambda_d$, the utility of a particular value $\lambda_d$ does not depend on the value of any other parameter.

P-2. It is known whether $U_d(1)$ is less than, equal to or greater than zero.

P-3. $|U_d(\lambda_d)| \leq |U_d(\lambda_d + 1)|$ for $\lambda = 0, 1, 2, \ldots$

P-4. $U(\lambda_1, \ldots, \lambda_n) = \sum_{d=1}^{n-w} U_d(\lambda_d)$

Suppose that the judgments of a decision-maker could be said to possess these properties and that at each stage of the solution process the implications of a particular judgment were understood. Then, if no simplifications were made, the resulting policy could be said to be optimal for that decision-maker. At least in principle, a utility function can be replaced by a decision-maker and an optimal policy can be derived. Such an approach would seem to be
particularly appropriate if a simplified policy statement is desired or in cases for which intangible costs were considered to be important. Further, research along these lines is indicated.

Numerical Example

Let \( r' = (1, 0, 0, 0, 3, 1, 0, 0, 2, 1, 0, 0, 1, 2, 0, 2) \) denote a run schedule over the network of our example. Assuming the run costs previously used to obtain the illustrative utility function, a total cost of 36 is obtained for this schedule. The linear transformation \( A_0 r' \) generates the following schedule characteristics.

\[
\begin{align*}
O_1 &= 4 & X_1 &= 1 & X_3 &= 3 \\
      & 1 & 12 & 32 & \\
O_2 &= 3 & X_2 &= 1 & X_4 &= 4 \\
      & 2 & 21 & 23 & \\
O_3 &= 4 & X_3 &= 3 & X_3 &= 3 \\
      & 3 & 31 & 14 & \\
O_4 &= 2 & X_4 &= 2 & X_4 &= 2 \\
      & 4 & 13 & 41 & \\
\end{align*}
\]

We suppose that these are given schedule requirements and that a schedule which minimizes total run cost is to be found. Here the set of all optimal schedules is given.
The inequalities (4-9) reduce to the following set in the case of this
numerical example.

\[
0 \leq \lambda_1 \leq 1 - \lambda_4 \\
2 - \lambda_2 - \lambda_3 \leq \lambda_1 \leq 2 - \lambda_2 \\
0 \leq \lambda_2 \leq 2 \\
1 - \lambda_3 + \lambda_4 \leq \lambda_2 \leq \lambda_4 + 1 \\
0 \leq \lambda_3 \leq 2 - \lambda_5 \\
0 \leq \lambda_4 \leq \lambda_5 \\
0 \leq \lambda_4 \leq 1 \\
0 \leq \lambda_5 \leq 2
\]

The utility function \( U(\lambda_1, \ldots, \lambda_5) = \lambda_1 - \lambda_2 - \lambda_3 + \lambda_4 - \lambda_5 \) is to be maximized.
Proceeding as for the general case we obtain the following solution.

\[
\lambda^* = \min\begin{cases}
1 - \lambda_4 \\
0
\end{cases}
\]

\[
\lambda^*_2 = \max\begin{cases}
0 \\
1 - \lambda_3 + \lambda_4
\end{cases}
\]

\[0 \leq \lambda^* \leq \lambda + 1\]

\[\lambda^*_4 = 0\]

\[\lambda^*_5 = 0\]

There are two optimal solutions; viz.,

\[
(\lambda^*_5 , \lambda^*_4 , \lambda^*_3 , \lambda^*_2 , \lambda^*_1 ) = (0, 0, 0, 1, 1)
\]

\[= (0, 0, 1, 0, 1)\]
If the preceding policy is applied the latter solution is obtained. The resulting run schedules are given by the vector equations (4-4) where, in this case, 

\[(b_1, \ldots, b_{16}) = (1, -2, 1, 1, 3, 1, 2, 0, 0, 2, 0, 0, 2, 2, 0, 0).\]

The optimal run schedules are

\[(r_1, \ldots, r_{16}) = (1, 0, 0, 0, 3, 1, 0, 1, 1, 0, 2, 0, 0, 2, 2, 0)\] and

\[(r_1, \ldots, r_{16}) = (1, 0, 1, 0, 2, 1, 1, 1, 0, 1, 1, 0, 2, 2, 0).\]

The total run cost for either schedule of 13 runs is 33. A substitution of these vectors \(r\) in (4-1) confirms that all schedule requirements are satisfied. By inspection of the vector equation (4-4) it can be established that these and only these schedules give minimum total run cost.
CHAPTER 6

THE GENERAL PROGRAMMING PROBLEM

Formulation

The network equations previously considered are of the form

\[ \sum_{j=1}^{n} a_{ij} r_{ij} = c_i, \quad i = 1, \ldots, m, \quad (6-1) \]

where the \( a_{ij} \) are nonnegative integers. The \( c_i \), \( i = 1, \ldots, p \), give the number of division trips of each of the \( p \) particular types which are made; \( c_{p+1}, \ldots, c_{p+q} \) gives the number of runs which originate at terminals 1, \ldots, q respectively; and \( c_{p+q+1}, \ldots, c_{p+2q} \) gives the number of runs which end at terminals 1, \ldots, q. Where necessary for interpretation, these "requirements" are represented by \( X_t, O_t, \) and \( D_t \) respectively. For each scheduling period an "ideal" nonnegative integral value is assigned each of these quantities.

A solution of the "general programming problem" is required in cases for which no feasible solution of the system (6-1) exists. For given nonnegative integers \( c'_i, \ i = 1, \ldots, m \), there is no solution if the system (6-1) is inconsistent, and if the \( (c'_i) \) do not satisfy a set of necessary conditions for
the existence of positive solutions, there is no feasible solution. In the former case one or more of a set of conditions of the form \{ L (c_i) = 0 \} is not satisfied. In the latter case the set of conditions is of the form \{ L (c_i) < 0 \}. In either case one or more of the values \( c'_i \) must be changed, and there is more than one \( c_i \) which can be assigned a new value such that a particular condition is satisfied.

Again, the problem can be considered in three parts:

1. Find the set of all possible courses of action.
2. Establish an operating policy.
3. Design an efficient decision process.

The first two aspects of the problem will be considered here.

To obtain a formulation of the general programming problem we admit the possibility of a change in any one or more of the requirements \( c_i \) and rewrite (6-1) above as

\[
\sum_{j=1}^{n} a_{ij} x_{ij} - \gamma_i = c_i, \quad i = 1, \ldots, m, \quad (6-2)
\]

so that \( \gamma_i > 0 \) represents an increase in the requirement of type \( i \); \( \gamma_i < 0 \) represents a decrease; and \( \gamma_i = 0 \) means that the original requirement of type \( i \) is unchanged. Since the \( a_{ij} \) are nonnegative integers, feasible solutions of (6-1) exist for some sets of integral values \( (c'_1, \ldots, c'_m) \); hence, there is always a solution of (6-2) of the form \( (r'_1, \ldots, r'_n, r'_m, \gamma'_i, \ldots, \gamma'_m) \).
where the $r_{ij}$ take the values 0, 1, 2, ... and the $\gamma_i$ are rational integers.

Given the set of conditions and requirements (6-1), the system (6-2) is an implicit representation of all possible courses of action. A general solution is desired.

The Set of All Solutions

A general solution of (6-2) can be obtained by the method used in Chapter 3. The coefficient matrix (6-2) is of the form $\begin{vmatrix} A & -I \end{vmatrix}$ where

$A = \begin{vmatrix} a_{ij} \end{vmatrix}$ and $I$ is an $m$ by $m$ identity matrix. In matrix notation the system (6-2) can be written

$$\begin{vmatrix} A & -I \end{vmatrix} \begin{vmatrix} r_1, \ldots, r_n, \gamma_1, \ldots, \gamma_m \end{vmatrix}^T = I \begin{vmatrix} c_1, \ldots, c_m \end{vmatrix}^T.$$  (6-3)

By performing a sequence of row operations on $\begin{vmatrix} A & -I \end{vmatrix}$, which reduces $\begin{vmatrix} A & -I \end{vmatrix}$ to the row-reduced canonical form, we can obtain

$$\begin{vmatrix} E_1, E_2, E_3 \end{vmatrix}$$

such that an equivalent system

$$\begin{vmatrix} E_1, E_2 \end{vmatrix} \begin{vmatrix} r_{11}, \ldots, r_{n1}, \gamma_1, \ldots, \gamma_m \end{vmatrix}^T = E_3 \begin{vmatrix} c_1, \ldots, c_m \end{vmatrix}^T.$$  (6-4)

is obtained. From (6-4), explicit expressions for $w$ of the $r_{ij}$ and $m-w$ of the indeterminates $\gamma_1, \ldots, \gamma_m$ can be obtained in terms of the $c_i$ and the other indeterminates.
A solution for the case of the example of Chapter 2 will illustrate some properties which are obtained for any system of the type (6-2). The requirements, \( c_i \), are denoted by the appropriate symbols \( X_{tu}^1 \), \( O_t \), and \( D_t \) as before.

Since values \( \gamma^1, \ldots, \gamma^m \) give the change in the original requirements specification, these quantities are represented by symbols of the form \( \Delta X_{tu}^i \), \( \Delta O_t \), and \( \Delta D_t \). If the previously used sequence of row operations is applied the following solution can be obtained.
\[ r_1 = b_1 = X_{12} \]
\[ r_2 = b_2 = O_3 - X_{31} - X_{32} + O_1 - X_{12} - X_{14} \]
\[ r_3 = b_3 = X_{31} - O_3 + X_{21} + X_{23} - O_2 \]
\[ r_4 = b_4 = O_2 - X_{21} - X_{23} + X_{32} \]
\[ r_5 = b_5 = X_{14} \]
\[ r_6 = b_6 = X_{21} \]
\[ r_7 = b_7 = O_2 - X_{21} \]
\[ r_{10} = b_{10} = O_3 + O_1 - X_{12} - X_{13} - X_{14} + O_2 - X_{21} - X_{23} \]
\[ r_{11} = b_{11} = X_{12} + X_{13} + X_{14} - O_1 \]
\[ r_{14} = b_{14} = X_{21} + X_{23} - O_2 \]
\[ r_{15} = b_{15} = X_{41} \]
\[ \Delta O_4 = b_{28} = O_4 - X_{41} \]
\[ \Delta D_1 = b_{29} = D_1 - (X_{21} + X_{31} + X_{41}) - O_1 + (X_{12} + X_{13} + X_{14}) \]
\[ \Delta D_2 = b_{30} = D_2 - (X_{12} + X_{32}) - O_2 + (X_{21} + X_{23}) \]
\[ \Delta D_3 = b_{31} = D_3 - (X_{13} + X_{23}) - O_3 + (X_{31} + X_{32}) \]
\[ \Delta D_4 = b_{32} = D_4 - X_{14} \]

\[ r_8 = r_9 = r_{12} = r_{13} = r_{16} = 0 \]

All \( \Delta X_{tu} = 0 \), and \( \Delta O_1 = \Delta O_2 = \Delta O_3 = 0 \)
The relations (6-5) always give a solution since requirements changes are automatically made if necessary to obtain a consistent system. Again let

\[ r_8 = \lambda_1, \quad r_9 = \lambda_2, \quad r_{12} = \lambda_3, \quad r_{13} = \lambda_4, \quad r_{16} = \lambda_5 \]

and let

\[ \Delta X_{12}, \ldots, \Delta X_{32}, \Delta O_1, \ldots, \Delta O_3 \]

take the arbitrary values denoted by the parameters \( \lambda_6, \ldots, \lambda_{16} \) respectively. Then the set of all solutions is given by the following system (6-6) where \( (b_1, \ldots, b_{32})^T \) is the particular solution (6-5). Since the parameters \( \lambda_6, \ldots, \lambda_{16} \) give the change in value of the corresponding requirements, the entries of the last eleven columns of the parameter coefficient matrix are, for rows identified by \( b_i \neq 0 \), the coefficients of the linear forms \( b_i = L(X_{tu}, O_t, D_t) \) given by (6-5). The first five columns give the previously obtained representation of the set of all solutions of (6-1) for the homogeneous case.
\[
\begin{array}{cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc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Feasible Operating Policies

The representation (6-6) provides a basis for obtaining an optimum solution of the illustrative programming problem since the methods developed in Chapters 4 and 5 can again be applied. The general programming problem, however, is considerably more difficult than the run scheduling problem previously considered. Here only an outline of some of the characteristics of the general problem will be given. A determination of the feasibility of obtaining an optimum policy is of particular interest.

Note that the conditions (4-10) for the existence of a feasible run schedule are (with the exception of 10_p) a consequence of the requirement that the r_j (and hence \(\lambda_1, \ldots, \lambda_5\)) be greater than zero. If these conditions are not satisfied, a change in value of one or more of the specified requirements must be made. Such changes are made by assigning values \(\pm 1, \pm 2, \ldots\) to one or more of the parameters \(\lambda_6, \ldots, \lambda_{16}\). For this example, then, the general programming problem is that of determining the feasible value to be assigned each of the parameters \(\lambda_1, \ldots, \lambda_{16}\). The determination of an appropriate utility function is required.

From (6-6) we have, for example \(\lambda_6 = \Delta X_{12}\). In the case of the interpretation of Chapter 2, the desired value of \(X_{12}\) is given by the number of loads available to be moved from terminal 1 to terminal 2; hence, \(\lambda_6 = 1\) implies that a "no load" transit is to be made and \(\lambda_6 = -1\) means that an available load will not be moved during the current scheduling period.
Similarly, \( \lambda_{14} = \Delta O_{1} \) and if \( O_{1} \) is given by the number of available drivers at terminal 1 then the restriction \( \lambda_{14} \leq 0 \) is implied. Here, however, we suppose that the \( O_{1} \) are given by the number of "layover" drivers at terminal \( t \) (those domiciled at another terminal who ended a run at terminal \( t \) during a previous period) and that a driver pool is provided by available drivers who are domiciled at terminal \( t \). In this case \( \lambda_{14}, \lambda_{15} \), and \( \lambda_{16} \) less than zero is assumed to imply a penalty payment, and if the value of any of these parameters is greater than zero, additional drivers (runs) are scheduled. Further, we have

\[
\Delta O_{i} = b_{28} - \lambda_{12} \quad \text{where} \quad b_{28} = O_{4} - X_{41} \quad \text{and} \quad \lambda_{12} = \Delta X_{41} ; \quad \text{hence,}
\]

\[
\Delta O_{4} = O_{4} - X_{41} - \lambda_{12} . \quad \text{Unless} \quad O_{4} = X_{41} \quad \text{an additional "cost" of some kind is always incurred.}
\]

If \( \lambda_{j} \neq 0, j = 6, \ldots, 16 \), imply penalties for the above reasons, then the values of the \( \Delta D_{t} \) are also of interest. These quantities give the change in the desired number of drivers (and transport units) who end runs at terminal \( t \) during the scheduling period. Since the "cost" of the next period's operation may depend on the values of the \( \Delta D_{t} \), it seems reasonable to assume a penalty for \( \Delta D_{t} \neq 0 \).

For purposes of the immediate discussion we neglect all feasibility considerations, such as the existence of upper bounds on the \( O_{t} \), and assume a programming problem for which one or more of the conditions (4-10) is not satisfied. Then some one or more of the parameters \( \lambda_{6}, \ldots, \lambda_{16} \) of (6-6) must take a value other than zero. Suppose, for example, that the relation

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that is, \( O_2 \geq X_{21} + X_{23} - X_{32} \), is not satisfied. Any one or more of these values may be changed and a feasible solution can be obtained if some other inequality is not thereby made inconsistent. Assume that any one of the values

\[
\Delta O_2 = 1, \; \Delta X_{21} = -1, \; \Delta X_{23} = -1, \; \text{or} \; \Delta X_{32} = 1
\]

leads to a feasible solution and that the decision problem is that of selecting one of these changes. That is, only one of the parameters \( \lambda_6, \ldots, \lambda_{16} \) of (6-6) is to be assigned a value other than zero and the nonzero assignment is to be one of the following:

\[
\lambda_7 = -1, \; \lambda_{12} = -1, \; \lambda_{13} = 1, \; \text{or} \; \lambda_{15} = 1.
\]

A determination of the utility of these alternatives is required.

Again, the representation (6-6) of the set of all solutions provides a basis for making the necessary judgments, since the resulting run schedule and requirements changes made for a change from \( \lambda_\delta = 0 \) to \( \lambda_\delta = 1, \; \delta = 1, \ldots, 16, \) are given by the entries in column \( \delta \) of the parameter coefficient matrix. For the above example, note that a change from \( \lambda_7 = 0 \) to \( \lambda_7 = -1 \) increases \( \Delta D_1 \) by one unit and decreases \( \Delta D_2 \) by one unit. A change from \( \lambda_{12} = 0 \) to \( \lambda_{12} = -1 \) decreases \( \Delta D_2 \) by one unit and increases \( \Delta D_3 \) by one unit. Similar observations apply for \( \lambda_{13} \) and \( \lambda_{15} \). But for \( \lambda_6, \ldots, \lambda_{16} \) equal to zero, the \( \Delta D_\delta \) are rational integers and a reduction in the absolute value of \( \Delta D_1, \; \Delta D_2, \) or \( \Delta D_3 \) is assumed to be of positive utility; hence, unless particular values \( \Delta D_1', \; \Delta D_2' \) and \( \Delta D_3' \) are given, it cannot be determined whether the utility

---

1 The result, however, may not be the "best" feasible requirements specification.
of the noted changes (in values of $\Delta D_1$, $\Delta D_2$, and $\Delta D_3$) is of positive or negative utility. Then $U_7(-1)$, $U_{12}(-1)$, $U_{13}(1)$ and $U_{15}(1)$ cannot be determined unless particular values of the requirements are specified. Further, the utility of the noted values of these parameters may change from positive to negative with changing values of the requirements.

The above result may immediately be generalized. Unless the decision-maker is indifferent to the value of the quantities $\Delta O_3$, $\Delta D_1$, $\Delta D_2$, $\Delta D_3$ and $\Delta D_4$, no unique utility function $U(\lambda_1, \ldots, \lambda_{16})$ applies for arbitrary values of the requirements. It follows that no optimum policy exists in the sense of Chapter 5. That is, a set of expressions for $\lambda^*_1, \ldots, \lambda^*_{16}$ which specify a "minimum cost" program for every scheduling period cannot be found. A utility function can be obtained for a particular period, but if new requirements are specified for a subsequent period a new utility function may be implied.

Clearly, this conclusion holds for any system of the type (6-2) if any change in specified values $c'_1, \ldots, c'_m$ implies a penalty. This follows since any matrix $\left| \begin{array}{c|c} a_{ij} \end{array} \right|$ of the specified type is singular. For the special case for which the relations (3-2) hold; viz.,

$$\begin{align*}
c_{w+d} &= \sum_{i=1}^{w} h_i c_i, \quad d = 1, \ldots, w, \\
\end{align*}$$

an optimum policy can be derived. (Here $\sum_{i=1}^{w} h_i d, \quad d = 1, \ldots, w$, defines the row operations used to obtain a general solution.) An optimum policy for
each of a number of special cases could, of course, be obtained. Given any such policy, the exception principle must be applied if the operation is to be programmed so that utility is always maximized. An efficient decision mechanism which can be applied to obtain an optimum solution for any particular programming period is needed.

A solution of the general programming problem by analytic methods may be made difficult by the complex utility functions which may be considered appropriate. It seems unlikely, for example, that the "cost" associated with a requirements change $\Delta X_{tu}$ would equal that for a requirements change $-\Delta X_{tu}$. Given the previous interpretation, a value $\Delta X'_{tu} > 0$ gives the number of "no load" transits made and $\Delta X'_{tu} < 0$ gives the number available loads which are not moved. In the former case an operating expense is incurred; in the latter case a penalty for failure to provide service is assumed. Similar cost differences might be considered appropriate for positive and negative values of the $\Delta O_t$ and the $\Delta D_t$. If the sum of run costs and requirements-change costs is to be minimized, a set of expressions for each $U(\lambda_\delta)$, $\delta = 1, \ldots, m$, is then obtained. The values of each of the functions $U(\lambda_\delta)$ could, of course, be tabulated. But the values of the $U(\lambda_\delta)$ are not independent, since a change in value of some $\lambda_\delta$ changes some $\Delta D_t$ and such a change, in turn, may change the expression for the utility of some other parameter. A series of tables is required to give the values of each of the functions $U(\lambda_d)$! For most applications the use of a high-speed computer
would seem to be necessary to first obtain the appropriate utility function, and then proceed to obtain an optimum requirements specification and an optimum run schedule.

In this and the preceding chapter we have investigated the general problem of deriving policy from a specified basis for decision. We conclude this section with an example of a difficulty which can arise if policy is specified a priori. Suppose that a "move all available loads" objective were specified. Then it may happen that no feasible schedule exists. Again consider the relations (4-10) and observe that for given values of the \( X_{tu} \) both upper and lower bounds are established for feasible values of \( O_1, O_2, \) and \( O_3. \) Since a limited number of drivers at each terminal establishes an upper bound on each of the \( \Delta O_t, \) it could happen that one or more of the relations (4-10) would be inconsistent for any feasible \( \Delta O_t \) if all \( \Delta X_{tu} \) were no less than zero. If such a policy is to be invariably applied it is necessary to accurately forecast demand (the values of the \( X_{tu} \)) and distribute drivers as required. The design and control of operations to achieve specified general objectives is, however, a distinct possibility. The basis for programming operations is then derived from the statement of policy. The decision rules used to program operations are a consequence instead of an elaboration of policy.

Given any means of solving the general programming problem, the preceding analysis provides a means of evaluating the policy or procedure used in terms of the basis for decision which the writer has specified. We may,
for example, observe a decision process in operation and record the data necessary to obtain a solution by the methods given here. A comparison of the operating costs which are incurred might then be made.

The approach used here is considered to be that of modern management science. It seems clear, however, that this approach is not that of traditional business practice. In the classic paper, "Derivation of a Linear Decision Rule for Production and Employment," Halt, Modigliani, and Muth suggest that the result of a study such as that suggested above is the ultimate test of a particular application of the methods of the new science [15].
CHAPTER 7

CONCLUSION

Operations Programming

We conclude by summarizing the problem considered and the approach used here and by reviewing some alternative problem formulations and solution methods. A tentative appraisal of the approach adopted, the recognition of some basic problems, and the identification of some areas for further research are the objectives of this discussion.

The problem considered here was obtained by identifying a "highway operation" which is considered to be characteristic of a relatively large number of motor freight companies. For a typical operation of the type identified, a large number of feasible runs over the divisions of the network are available and, in general, more than one run schedule with particular characteristics exists. The evident characteristics include the number of division trips of each type made, the number of runs that originate at each terminal, and the number of runs that end at each terminal. Further, the total run costs of schedules with identical characteristics are not, in general, equal. The possibility of a "cost" reduction by using linear programming
methods to obtain a minimum run cost schedule thus seems attractive.

A standard linear programming problem can, for example, be formulated by adopting a "move all available loads at minimum total run cost" objective subject to the restrictions imposed by the number of drivers available at each terminal. Using the previous notation with changed subscripts, the problem can be stated in the following form. Find nonnegative, integral values \((r'_j), j = 1, \ldots, n\), for which \(\sum_j y_j r_j\) is a minimum subject to the restrictions

\[
\sum_{ij} a_{ij} r_i \geq X_j, \quad i = 1, \ldots, p, \tag{7-1}
\]

\[
\sum_{ij} a_{ij} r_i \leq 0, \quad i = p + 1, \ldots, q.
\]

By appending "slack variables" and making the appropriate changes of sign, the problem is formulated as required to apply the simplex solution procedure [7]. A modification of this procedure to obtain integral solutions is available [9, 12].

Even for this simple formulation, however, a difficulty may arise, since it may happen that the total number of outbound loads at one or more terminals exceeds the total number of drivers (runs) which can be made available to move these loads. In this case the system (7-1) is inconsistent and there is no solution with all slack variables zero. The problem of assigning a cost to nonzero values of the slack variables is then obtained. If the possibility of a
reduction in value of one or more of $X_i$ is admitted, a difficult problem of
value measurement is presented since the cost of not providing service is
"intangible." The approach used in Chapter 5 provides a means of making
judgments of the extent of such "costs" in terms of current dollar costs of
providing (in this case) an additional driver. If it is objected that this
situation should not have been permitted to obtain we may append to (7-1) the
additional restrictions

$$\sum_{j \in I} a_{ij} r_j > D_i, i = p + q + 1, \ldots, 2q \quad (7-2)$$

and assign values $D_i'$ such that all of the loads which are expected to become
available during the next period can be moved with available drivers. The
problem is then more highly restricted, and it is even more likely that non-
zero values of the slack variables (requirements changes) would be required.

If a change in value of one or more of the $D_i$ is to be admitted, we have the
problem of determining a "trade-off" between certain costs of current
operations and those of the next period. Again, however, the approach of
Chapter 5 may provide some assistance in making the necessary judgments.

If the requirement for drivers during the next period is only probabilistically
known, a new and difficult element of complexity is introduced. This aspect
of the problem has not been considered here.

The first point made in conclusion, then, is that the approach used may
be of some assistance in making the value judgments necessary for a decision.
The second is concerned with the "elements of value" that are included in the definition of a decision.

In the case of the motor freight operations which have provided the referent for this research, the formulation (7-1) is such a simplified definition of the scheduling problem that it is considered unlikely that any highway dispatcher would recognize this formulation as even an approximation of the decision problem which he faces. Indeed, "The General Programming Problem" of Chapter 5, if understood, would probably be considered only a rough approximation of the decision problem as viewed by our dispatcher.

One of the deficiencies is that certain "elements of value" have been omitted. At least in principle, however, this deficiency can be remedied. One of the missing elements of value is the driver lay-over cost which is incurred when a run is ended at a terminal other than his domicile point. One means of recognizing this cost is obtained by partitioning each of the requirements $O_t$ into a set of requirements $(O_{tu})$, and interpreting these quantities as the number of available drivers at terminal $t$ who are domiciled at terminal $u$.

If the number of runs that originate at $t$ and end at $u$ is not $O_{tu}$, a requirements change is implied for which an appropriate penalty can be assessed. Considerations of this kind simply enlarge the size of the problem.

A further simplification of the writer's view of our dispatcher's decision problem is that the timing of runs has not been considered. Since this aspect of the problem may critically influence the level of equipment
utilization and, hence, fleet size requirements, it cannot be dismissed as unimportant. It would, however, appear to be generally necessary to consider this problem apart from the programming problem treated here. If the length of the programming period is reduced to the extent necessary to effectively schedule run departure times, a number of difficulties arise. One of these is that the problem (in common with many transportation problems) may become prohibitively large.\footnote{For an indication of the problem size involved see Luther and Walsh, "A Difficulty in Linear Programming for Transportation Problems," [18].} For reasonably comprehensive problem formulations, the complexity and size is such that hand-computation methods are almost certain to be impracticable. If a computer is not available or if computer capacity is exceeded some kind of simplification is required. Any generalization concerning the nature of acceptable simplifications would appear to be extremely speculative. In the case of a particular application, however, a suitable programming model might be constructed.

**Operations Design**

Given difficulties of the above kind it seems reasonable, for a particular operation, to at least consider approaches which differ in principle from that adopted here. One such possibility is that of designing a simplified operation, an efficient information-handling mechanism, and a simplified decision process.
which includes a decision-maker and explicit but perhaps general decision rules and statements of objectives. An approach of this kind is considered to be characteristic of the engineering profession in general. For the Industrial Engineer, analyses of the type presented here may provide a theoretical basis for design of the same essential kind as that which mechanics provides the machine designer. The judgments involved in industrial engineering activity of this nature, however, may be more readily challenged by the operating management served.

In the case of the operation considered here, an engineer might obtain general statements of objectives and operating "policies" and consider these tentative specifications for the design or redesign of the operation and the management control system. A set of statements, such as "move all available loads," "schedule runs to minimize driver lay-over and run costs," "domicile the number of drivers required at each terminal," and "design an efficient control mechanism for dispatchers" might constitute the initial specifications which would be obtained. It seems unrealistic to expect that a charge of this kind would adequately specify the system desired or even that the stated objectives would be consistent. A precise definition of the problem and the resolution of conflicting objectives are typical aspects of the design process.

For a general discussion of such an approach see Taylor, "System Simplification," [24].
The contribution of this kind of activity to a science of management, however, would seem to be limited. If a design problem is considered, then, by definition, there are major aspects of the problem which are unique to the particular situation. The approach described is at least as old as the engineering profession. Any new theoretical techniques, approaches, or concepts developed during the design process, however, may represent a general contribution. A continued division of labor between academic investigation and engineering design seems more likely than basic contributions by practitioners or especially "practical" results from theoretical investigations.

More General Applications

The standard formulation of the linear programming problem can be written:

\[
\begin{align*}
\text{minimize} & \quad \sum_{j=1}^{n} b_j x_j \\
\text{subject to} & \quad \sum_{i=1}^{m} a_{ij} x_j \geq c_i, \quad i = 1, \ldots, m, \\
& \quad x_j \geq 0, \quad j = 1, \ldots, n.
\end{align*}
\]
A problem formulation of this kind implies that the decision-maker is indifferent to all values of \( \sum_{j=1}^{n} a_{ij} x_j \), except the lower bound \( c_i \). If this is not the case a situation of the type identified as "The General Programming Problem" in Chapter 6 is obtained. If the rank, \( w \), of the coefficient matrix \( \sum_{i} a_{ij} \) of (7-4) is less than \( m \), then the problem is not of the standard form since the utility function is, in general, neither linear nor of any other simple form.

An example of a general problem class for which \( w < m \) is provided by the "multistage" programming problem [10]. This follows since for this problem class the outputs of one period or stage are the inputs of the next stage. The motor freight problem considered here can be viewed as a simple two-stage programming problem. Any linear programming problem with \( w < m \) for which a "cost" is associated with nonzero values of any "slack variable" is a member of the general problem class which includes that considered in Chapter 6. No such problem is of the standard form for arbitrary values of the right-hand members of the "structural equations."

Any system of linear equations with real coefficients and constants can be solved by using the elementary row operations to reduce the coefficient matrix to the row-reduced canonical form. The restriction that solutions be nonnegative can be expressed by a system of linear inequalities and the modification of Kuhn's method given in Chapters 3 and 4 can be applied. A necessary and sufficient set of conditions for the existence of feasible solutions
of the linear equations can thus be obtained if such (nonnegative) solutions exist for some values of the constants. If, however, integral solutions are required, it may happen that no such solutions exist for systems that possess real solutions. With this exception, the method can be used to reduce any programming problem of this type to a form such that the functional equation approach can be applied to obtain an optimum solution for particular values of the constants. The method, however, may not constitute a particularly efficient solution algorithm. In the view of the writer, the principal merit of this approach is its contribution to the understanding of the structure of a particular problem.

Further Research

The investigation reported here is primarily exploratory in nature. It began with a particular problem and an idea for an approach which seemed likely to produce results of value for engineering design activity. We conclude by noting some of the areas for further work which are of particular interest to the writer.

The variety and complexity of problems in the field of transportation appears to be sufficient to occupy researchers for a long time to come. Although a relatively large number of such problems and a variety of approaches have been reported in recent literature it seems, perhaps not surprisingly, that the really significant work is yet to be done. In the case of motor freight operations of the type considered here, research interest is
evident, but only a few published reports are currently available. Since a number of abstract formulations (including probabilistic decision models) seem promising, research in the operations of a particular organization may be of value as a means of selecting the most appropriate approach.

A recasting of the work reported here for the case of the general problem class of the previous section is of particular interest to the writer. The "general programming problem" seems likely to be of interest not only for a variety of industrial engineering applications but also as an approach to the study of a number of theoretical questions, including that of "value measurement." In the view of the writer, values cannot be "measured." The structure of problems, however, can be displayed, bases for judgment can be provided, and the implications of choice can be made explicit. It is to these ends that this investigation has been addressed and to which the writer hopes to contribute in the future.
APPENDIX I

AN ALTERNATIVE PROOF OF THE PROPOSITIONS (3-10)

We have the following system of network equations:

\[ \sum_{j=1}^{n} a_{ij} r = c_i, \quad i = 1, \ldots, p + 2q. \quad (I-1) \]

Here the entries in the coefficient matrix \( ||a_{ij}|| \) are interpreted as follows:

For \( i = 1, \ldots, p \), the entries in any column \( j \) give the number of trips from a terminal \( t \) to an adjacent terminal \( u \), which are made during a run \( j \). In the positions \( i = p + 1, \ldots, p + t, \ t = 1, \ldots, q \), of any column \( j \), there is a single entry "1". This entry identifies the origin terminal, \( t \), of run \( j \), and all other entries in these positions of column \( j \) are zero. Destination terminals are similarly identified by the entries "1" in the rows \( p + q + 1, \ldots, p + 2q \).

The relations (3-10) assert that, with respect to the first \( p + q \) rows of \( ||a_{ij}|| \), the last \( q \) rows are dependent. This follows since

\[ \alpha_{p+q+t} = \left( \sum_{u \in U} \alpha_{ut} \right) - \left( \sum_{u \in U} \alpha_{tu} \right) + \alpha_{p+t} \quad (I-2) \]

for \( t = 1, \ldots, q \). (Here \( \alpha_{ut} \) and \( \alpha_{tu} \) denote certain of the first \( p \) rows of \( ||a_{ij}|| \).)
where the entry in the jth position of a row \( \alpha_{ut} \) gives the number of transits from a terminal u to an adjacent terminal t. Subscripts other than ut and tu identify a row number of \( ||a_{ij}|| \). In the text, a proof of (I-2) is given by an argument on the right-hand members of (I-1). The following proof was suggested by Dr. Paul M. Pepper.

We are to show that each entry in any row \( \alpha_{p+q+t} \), \( t = 1, \ldots, q \), is given by the right-member of (I-2). The entry "1" in position j of a row \( p+q+t \) indicates that t is the destination terminal of run j. All other entries in the positions \( p+q+1, \ldots, p+2q \) of column j are zero. Consider the terms of (I-2) that are in parentheses. The entry in the jth position of the row given by this linear combination is "0" if: (a) run j neither originates nor ends at terminal t; or, (b) run j both originates and ends at t. This follows, since in both cases (a) and case (b), the number of trips to t which are made during run j is equal to the number of trips made outbound from t. In case (a), the entry in the jth position of row \( \alpha_{p+t} \) is zero; hence, the entry in the jth position of the row given by the right-hand member of (I-2) is zero as required. In case (b), the entry in the jth position of row \( \alpha_{p+t} \) is "1"; hence, the entry in the jth position of the row given by the right-hand member of (I-2) is "1" as required. If run j originates at a terminal t but ends at some other terminal, then the entry in the jth position of the row given by the terms in parentheses is "-1"; the entry in this position of row \( \alpha_{p+t} \) is "1"; and, the vector sum of these two rows gives the entry "0" in the jth position as required. This follows
since, if a run originates but does not end at t, the number of outbound trips
is one more than the number of inbound trips. The form of the argument for
the case of a run j which ends but does not originate at t is identical and the
proposition is thus established.
REFERENCES


AUTOBIOGRAPHY

I, Robert Frederick Miller, was born in Putnam County, Ohio, August 16, 1929. I received my secondary-school education in the public schools of Continental, Ohio, and my undergraduate training at The Ohio State University, which granted me the Bachelor of Industrial Engineering degree in 1952. A three-year tour of duty as a U. S. Navy destroyer officer followed.

In 1956, I received an appointment as Research Assistant, Operations Research Group, Engineering Experiment Station, The Ohio State University, and undertook graduate study specialized in the Department of Industrial Engineering. I received the Master of Science Degree in 1957. While completing the requirements for the Doctor of Philosophy degree, I held the position of Instructor in the Department of Industrial Engineering.