SHOOK, William Beattie. THE MEASUREMENT OF IMPACT STRESSES IN BRITTLE MATERIALS.

The Ohio State University, Ph.D., 1961
Engineering, chemical

University Microfilms, Inc., Ann Arbor, Michigan
THE MEASUREMENT OF IMPACT STRESSES IN BRITTLE MATERIALS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By


* * * * * * *

The Ohio State University
1961

Approved by

[Signature]
Adviser
Department of Ceramic Engineering
ACKNOWLEDGMENTS

The author is grateful for the opportunity to utilize the mathematical reasoning of Dr. Francis Niedenfuhr. He is also indebted to Dr. Theodore Hildebrandt, whose interest and perserverence resulted in the simplification of the Niedenfuhr equation. Many others have played a major role in the development of the impact machine and associated instrumentation. Special thanks are directed to Richard Wuske, William W. Brown III, and Stephen K. Bennett.

The continuing interest and guidance of Dr. J. O. Everhart has been both pleasurable and instructive. The financial support of the Engineering Experiment Station in the early work, and of the Edward Orton Jr. Ceramic Foundation for its continuing support, are gratefully acknowledged.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>INTRODUCTION 1</td>
</tr>
<tr>
<td></td>
<td>REVIEW OF THE LITERATURE 3</td>
</tr>
<tr>
<td>II</td>
<td>THEORY</td>
</tr>
<tr>
<td></td>
<td>Development of Stress Equations 13</td>
</tr>
<tr>
<td></td>
<td>Simplification of Stress Equations 26</td>
</tr>
<tr>
<td></td>
<td>Solution of the Stress Equation 31</td>
</tr>
<tr>
<td>III</td>
<td>EQUIPMENT</td>
</tr>
<tr>
<td></td>
<td>Hammer Transducer 55</td>
</tr>
<tr>
<td></td>
<td>Calibration 58</td>
</tr>
<tr>
<td></td>
<td>Associated Instrumentation 62</td>
</tr>
<tr>
<td></td>
<td>Oscilloscope 62</td>
</tr>
<tr>
<td></td>
<td>Photographic Recording 65</td>
</tr>
<tr>
<td></td>
<td>Impact Machine 65</td>
</tr>
<tr>
<td></td>
<td>Static Testing Machine 67</td>
</tr>
<tr>
<td></td>
<td>E-Scope 67</td>
</tr>
<tr>
<td>IV</td>
<td>EXPERIMENTAL PROCEDURE</td>
</tr>
<tr>
<td></td>
<td>Measurement of Physical Properties 69</td>
</tr>
<tr>
<td></td>
<td>Modulus of Elasticity 69</td>
</tr>
<tr>
<td></td>
<td>Impact Strength 70</td>
</tr>
<tr>
<td></td>
<td>Modulus of Rupture 71</td>
</tr>
<tr>
<td></td>
<td>Absorption, Apparent Porosity and Specific Gravity 71</td>
</tr>
</tbody>
</table>
Specimen Preparation
Materials 71
Preparation of Materials 71
Forming Procedure 72
Firing Procedure 73

V RESULTS AND DISCUSSION

Stress Calculations 74
Extruded Bars 74
Commercial Specimens 81

The Fracture Process

VI CONCLUSIONS 50
VII REFERENCES 52

AUTOBIOGRAPHY 95
**LIST OF FIGURES**

<table>
<thead>
<tr>
<th>Figure</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Stress Ratio as Predicted by Bending Energy</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>An Element of a Deformed Beam Showing Forces and Moments</td>
<td>14</td>
</tr>
<tr>
<td>3</td>
<td>Load and Support of a Simple Beam</td>
<td>17</td>
</tr>
<tr>
<td>4</td>
<td>Pressure Pulse Due to Impact</td>
<td>19</td>
</tr>
<tr>
<td>5</td>
<td>The Effect of Poisson's Ratio on the Function</td>
<td>38</td>
</tr>
<tr>
<td>6</td>
<td>The Time Function for $R = 0.15$ and $L = 4.00$</td>
<td>43</td>
</tr>
<tr>
<td>7a</td>
<td>The Time Function Modelled in $R/L$</td>
<td>49</td>
</tr>
<tr>
<td>7b</td>
<td>Stress Factor as a Function of Bar Geometry and Time</td>
<td>50</td>
</tr>
<tr>
<td>8</td>
<td>The Influence of Diameter on the Rate of Stressing</td>
<td>51</td>
</tr>
<tr>
<td>9</td>
<td>The Influence of Span on the Rate of Stressing</td>
<td>52</td>
</tr>
<tr>
<td>10</td>
<td>The Influence of Elasticity on the Rate of Stressing</td>
<td>53</td>
</tr>
<tr>
<td>11</td>
<td>The Piezoelectric Hammer</td>
<td>56</td>
</tr>
<tr>
<td>12</td>
<td>Calibration of Hammer at Rapid Unloading Rates</td>
<td>59</td>
</tr>
<tr>
<td></td>
<td>Title</td>
<td>Page</td>
</tr>
<tr>
<td>---</td>
<td>----------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>13</td>
<td>Calibration of Hammer at Intermediate Unloading Rates</td>
<td>60</td>
</tr>
<tr>
<td>14</td>
<td>Calibration of Hammer at Slow Unloading Rates</td>
<td>61</td>
</tr>
<tr>
<td>15</td>
<td>Calibration Curve</td>
<td>64</td>
</tr>
<tr>
<td>16</td>
<td>The Impact Machine</td>
<td>66</td>
</tr>
<tr>
<td>17</td>
<td>Representative Oscillographs of Impact</td>
<td>75</td>
</tr>
<tr>
<td>18</td>
<td>Measured Strength of Body S</td>
<td>77</td>
</tr>
<tr>
<td>19</td>
<td>The Influence of Span on the Measured Strength</td>
<td>79</td>
</tr>
<tr>
<td>20</td>
<td>Oscillograph Showing Z-axis Energized at Failure</td>
<td>84</td>
</tr>
<tr>
<td>21</td>
<td>Progress of Failure</td>
<td>85</td>
</tr>
<tr>
<td>22</td>
<td>Impact at Various Levels of Piezoelectric Signal</td>
<td>87</td>
</tr>
<tr>
<td>23</td>
<td>Impact at Various Times After Collision</td>
<td>88</td>
</tr>
<tr>
<td>24</td>
<td>Impact Failure Originating at a Flaw on Tension Side</td>
<td>89</td>
</tr>
</tbody>
</table>
### LIST OF TABLES

<table>
<thead>
<tr>
<th>Number</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1a</td>
<td>Fortran Program Notation</td>
<td>32</td>
</tr>
<tr>
<td>1b</td>
<td>Fortran Program</td>
<td>33</td>
</tr>
<tr>
<td>2</td>
<td>Computations for Variations in Poisson's Ratio</td>
<td>37</td>
</tr>
<tr>
<td>3</td>
<td>Variations Included in the Computer Program</td>
<td>39</td>
</tr>
<tr>
<td>4</td>
<td>Computations for the Case $R = 0.15$ and $L = 4.00$</td>
<td>42</td>
</tr>
<tr>
<td>5</td>
<td>Computations for Various Values of $R$ and $L$</td>
<td>45</td>
</tr>
<tr>
<td>6</td>
<td>Results of Calibration</td>
<td>63</td>
</tr>
<tr>
<td>7</td>
<td>Failure Stress Data</td>
<td>76</td>
</tr>
<tr>
<td>8</td>
<td>Commercial Specimens</td>
<td>82</td>
</tr>
</tbody>
</table>
CHAPTER I

INTRODUCTION

Mechanical failures of ceramic materials in service are generally due to impact conditions. The impact resistance of a ceramic is therefore a very important measure of strength. Unfortunately, the profound influences of the elastic properties, geometry of test specimen, and velocity of impact have not been clearly defined in determinations of impact strength.

The conventional testing machine is of the energy-measuring type. The specimen is either subjected to repeated blows of gradually increasing intensity until failure occurs or it is struck by a single blow which causes it to break. In the former case, the energy which just causes failure is recorded, while in the latter case the energy extracted from the hammer is taken as the impact strength.

It is not possible to calculate the stress level at the time of failure from energy measurements. There is no theory of strength of materials which can make use of such data, since the distribution of energy in the test specimen is not known.

The purpose of this work is to develop a means of impact strength measurement which allows the calculation of outer fiber stress at the instant of failure. The ability to calculate stress necessitates a mathematical statement describing the influence of the various physical parameters of the test
conditions. Chapter II is devoted to the development of the stress equation. The simplification and solution of this equation to allow its use in routine testing are also described.

The experimental data which must be obtained during the test are found to be the force applied and the duration of this application. The piezoelectric hammer and associated equipment used for this purpose are described in Chapter III, along with the description of the impact machine constructed for this study.
REVIEW OF THE LITERATURE

One treatment of impact problems utilizes the principles of impulse and momentum. The methods are adequate in cases where there is no destructive action; the colliding bodies are assumed to rebound after the impact.

In cases of beams loaded at high rates, the resulting stresses are calculated on the assumption that the material is not stressed beyond the yield point. This makes it possible to write an energy conservation equation involving the work done and the strain energy of bending of the beam. However, it is necessary to assume the shape of the deflection curve of the beam during the loading, and the assumption is made that the dynamic curvature is the same as the static curvature.

The deflection of a simply supported beam under concentrated loading may be defined in terms of the elasticity and geometry of the test specimen, and the applied load as

\[ \delta = \frac{PL^3}{48EI} \]  \hspace{1cm} 1.1

The force causing this deflection is therefore

\[ P = \delta \left( \frac{48EI}{L^3} \right) \]  \hspace{1cm} 1.2

The energy stored in the beam is equal to the work done by the force. Since the force was initially zero at zero deflection, and attained \( P \) at maximum deflection, the average force is \( P/2 \) applied through a distance \( \delta \). From
equation 1.2, the energy may be written

\[ \frac{P}{2} \delta = \delta^2 \frac{24EI}{L^3} \]  \hspace{1cm} 1.3

The weight causing the deflection in an impact test travels through a total height \((h + \delta)\) in the case of a drop test. Equating the external energy to the bending energy,

\[ W(h + \delta) = \delta^2 \frac{24EI}{L^3} \]  \hspace{1cm} 1.4

This may be written in quadratic form as

\[ \delta^2 - 2\delta_{ST} \cdot \delta - \delta_{ST} \cdot \frac{2gh}{g} = 0 \]  \hspace{1cm} 1.4

where \(\delta_{ST} = \frac{WL^3}{48EI}\). Since \(\sqrt{2gh}\), the solution is:

\[ \delta = \delta_{ST} + \sqrt{\delta_{ST}^2 + \delta_{ST} \cdot \frac{2}{g}} \]  \hspace{1cm} 1.5

As Timoshenko (1) points out, equation 1.5 represents the upper limit which the maximum dynamic deflection may attain if there are no energy losses. The assumption that all kinetic energy available in, or extracted from, the hammer is used in overcoming the inertia and in bending the specimen is inherent in impact stress calculations based on this equation.

The effect of the mass of the beam on the deflection was investigated by Cox (2). He shows that a reduced mass of the beam supported at the ends must be \((17/35)ql/g\), so that the more complete expression for impact deflection
becomes

\[ \sigma = \delta_{ST} + \sqrt{\delta_{ST}^2 + \frac{\delta_{ST}^2}{g}} \cdot \frac{1}{1 + \frac{17}{35} \frac{ql}{W}} \]  

For a horizontally struck beam, or where h is large compared to \( \delta_{ST} \), the equation may be reduced to

\[ \sigma = \sqrt{\frac{\delta_{ST}^2}{g}} \cdot \frac{1}{(1 + \frac{17}{35} \frac{ql}{W})} \]  

Since stress \( \sigma = \frac{Mc}{I} \), and M for a simply supported beam is \( PL/4 \), equation 1.7 may be expressed as

\[ \sigma_{\text{max}} = \sqrt{\frac{3W}{L^2}} \cdot \sqrt{\frac{Ec^2}{L+I}} \cdot \frac{1}{1 + \frac{17}{35} \frac{ql}{W}} \]  

For a beam of circular cross section, this becomes

\[ \sigma_{\text{max}} = \frac{\sqrt{W}}{2g} \cdot \frac{24E}{AL} \cdot \frac{1}{1 + \frac{17}{35} \frac{ql}{W}} \]  

and for a rectangular cross section,

\[ \sigma_{\text{max}} = \frac{\sqrt{W}}{2g} \cdot \frac{18E}{AL} \cdot \frac{1}{1 + \frac{17}{35} \frac{ql}{W}} \]  

Squaring both sides and collecting terms, we may write for the circular and rectangular sections respectively:

\[ \frac{\sigma_{\text{max}}^2}{24E} \cdot AL = \frac{Wh}{1 + \frac{17}{35} \frac{ql}{W}} \]
and
\[
\sigma_{\text{max}}^2 \cdot AL = \frac{Wh}{1 + \frac{17}{35} \frac{ql}{W}}
\]
\[1.10\, b\]

The same equations without the corrections for inertia of the bar have been developed by Preston (3). Referring to equations 1.5 and 1.6, an interesting result may be noted for the limiting case \( v = 0 \). It may be seen that the dynamic deflection in this case is
\[
\sigma = 2 \sigma_{\text{ST}}
\]
\[1.11\, a\]
or
\[
\sigma_{\text{dynamic}} = 2 \sigma_{\text{static}}
\]
\[1.11\, b\]

For the case of impact failures at impacting velocities much greater than zero, the dynamic deflection, and therefore the stress, should be proportional to the velocity as shown by equation 1.7. Figure 1 is constructed from the relationship described by equation 1.5. This clearly illustrates the magnitude of stress to be expected in an impact in terms of the static stress caused by the same weight. The weight employed in the impact test is used to calculate the deflection caused by its static application. The experimentally determined height of drop just causing failure, appropriately reduced by the Cox equation, may then be used to calculate the stress at failure according to this theory.

It should be emphasized that several important assumptions are
Figure 1. Stress Ratio as Predicted by Bending Energy Considerations.
inherent in the use of a bending energy equation. These may be listed as follows:

1. The shape of the flexure curve in the dynamic situation is the same as the static curvature, and the energy distribution is thereby defined.
2. The deflections are at all times proportional to the applied load.
3. The impacting weight and the deflected beam attain a state of rest at the instant of maximum deflection.
4. The effects of shear may be neglected.
5. There are no energy losses in the system, the energy stored in the beam being equal to the work done by the force.
6. The inertia effects are adequately accounted for by the Cox correction factor.
7. The yield point of the material is not exceeded.

The brittle nature of ceramics postulates that the failure will occur before yielding begins (4). The final assumption may therefore be considered valid. However, the remaining assumptions are not clearly justified. It may be concluded that the validity of this analysis is open to serious criticism.

Dr. Shand (5) has recently presented a form of equation 1.10 a and concludes with the comment:

It is doubtful that the standard impact test serves any useful purpose in the evaluation of glass and ceramics, at least, until the behavior of brittle bodies under these conditions has been investigated more thoroughly.
Among the earliest investigators to attempt measurement of the impact stresses in ceramic materials was A. E. Williams (6). As Preston later pointed out (3), the equation was not correctly written in this article. This led researchers for many years to report impact stresses based on an erroneous equation. Using the corrected energy expression, Preston (7) was then able to utilize the data of Guyer (8) and Sherwood (9) for comparative purposes. He concluded that the impact strength was greater than the static flexural strength by an amount of approximately forty percent, dependent upon the length of span used in testing. He reasoned that this apparent increase was due to the influence of the rate of loading. Later work by Preston (10) has shown the effect of loading rate on glass strength.

Illyn (11) has used an equation describing the impact energy in terms of the static flexural strength, as follows:

\[
 \frac{(K \times S_S^2)}{24 \, E} \times Al = \eta Ph
\]

1.12

where

\[ K = \text{measure of the stress-rate sensitivity of a material for the specific condition of load application.} \]

\[ S_S = \text{static transverse strength.} \]

\[ A = \text{cross sectional area of specimen.} \]
\[ E = \text{Young's modulus of elasticity} \]

\[ l = \text{length of span}. \]

\[ P = \text{effective weight of the pendulum}. \]

\[ h = \text{fall height just necessary to break the specimen}. \]

\[ \eta = \text{inertia coefficient, a measure of the effective work available in the impacting pendulum to deform and stress the sample}. \]

He reports that the K factor was found to be \( 1.8 \pm 0.2 \) for a variety of rods, both vitrified and at various porosities.

Equation 1.12 is seen to be equivalent to the bending energy statement for a circular cross section, equation 1.10 a, where \( \sigma_{\text{max}} \) is the maximum impact stress. Then Illyn has effectively discovered that \( \sigma_{\text{impact}} \) equals \( \sigma_{\text{static}} \) times a stress rate factor K. With \( K = 1.8 \pm 0.2 \),

\[ \sigma_{\text{impact}} = \sigma_{\text{static}} (1.8 \pm 0.2) \tag{1.13} \]

with a Cox correction for effective mass. This is equivalent to the previous statement regarding "sudden loading", in which case the stresses are exactly twice those resulting from static loading by the same weight.

A new impact machine has recently been described in private reports of a British research group. The machine is constructed so as to release the
specimen an instant before impact takes place. The operation is best described as a drop-test with incremental loading and accurate positioning. The assumption is made that all the kinetic energy of the specimen is transformed into bending energy, and an equation is written to this effect:

\[ \rho gh = K \frac{P^2}{E} \]  

1.14

where

- \( \rho \) = density of the material
- \( g \) = acceleration due to gravity
- \( h \) = impact height
- \( P \) = Modulus of Rupture
- \( E \) = Young's modulus of elasticity

The factor \( K \) is a numerical constant depending on the shape of the specimen and form of loading. Since the bar is not supported at the moment of impact, it is considered to be a centrally supported beam with uniform load, and \( K = 1/40 \).

It should be pointed out that this equation is again a modification of the expression 1.10 a. Since \( \sqrt{2} = 2gh \), we may substitute \( \frac{\sqrt{2}}{2} \) for \( gh \), and \( \rho = \text{mass per Unit Volume, W/gAl} \). This yields the statement

\[ \frac{W \sqrt{2}}{2 \text{gAl}} = \frac{KP^2}{E} \]  

1.15
With simple beam support and concentrated loading, the factor $K$ is $1/18$ for a rectangular bar and $1/24$ for a round bar. This is the equation 1.10 given previously, and also the Illyn statement, equation 1.12 with the exception of a "stress rate factor".

The British workers discovered by experiment that the plot of $\rho gh$ against $P^2/E$ gave a straight line with slope $1/20$. Since they expected slope $1/40$, they conclude that the static modulus of rupture should be increased by a factor 1.4. This corresponds to Illyn's factor $1.8 \pm 0.2$, and Preston's factors 1.2 to 1.4.
CHAPTER II

THEORY

Development of Stress Equation

In view of the significant assumptions inherent in the use of the bending energy equation, that theory has questionable application for calculation of impact stresses. It was reasoned that a solution of the wave equation might better describe the actual stress condition in the dynamic situation. Accordingly, a Fourier series solution is sought (12) for the specific case of a simply-supported beam loaded at the center of the span.

The general wave equation describing flexural vibrations of rods may be derived from considerations of the forces and moments acting upon an element of a deformed beam. Figure 2 pictures the element. The dotted lines indicate its position due to bending moments acting, while the curved solid outline of the element shows the additional deflection imposed by the transverse shear stresses. The dotted arrows indicate the inertia force and moment.

The symbols used are as follows:

\[ V = \text{shear force} \]
\[ M = \text{bending moment} \]
\[ \gamma = \text{shear strain at neutral axis} \]
\[ \phi = \text{rotation of the element} \]
\[ \rho = \text{mass density} \]
Figure 2. An Element of a Deformed Beam Showing Forces and Moments.
I = moment of inertia of the cross section of the beam
A = area of cross section
y = deflection
x = coordinate in the direction of beam length
p = distributed loading per unit length of the beam (Force/Length)

In the following analysis, a prime indicates differentiation with respect to x and a dot indicates differentiation with respect to time.

From elementary beam theory:

\[ \gamma = \frac{3 V}{2 AG} \]  

2.1

\[ EI \phi' = -M \]  

2.2

where

G = the shear modulus
E = Young's modulus of elasticity

From the geometry of the deformed element:

\[ y' = \phi - \gamma \]  

2.3

Summing forces and moments shown in Figure 2:

\[ \sum F_y = 0: \quad V' \, dx + \rho A \phi' \, dx = p \, dx \]  

2.4

\[ \sum M = 0: \quad M' \, dx = V \, dx + \rho I \, dx \]  

2.5
Combining equations 2.1 through 2.5 the following set of partial differential equations is obtained:

\[ \phi' + ay - y'' = bp \]

\[ \phi'' - c\phi - d\phi'' + cy' = 0 \] 

where

\[ a = \frac{3\phi}{2G}; \quad b = \frac{3}{2AG} \]

\[ c = \frac{2AG}{3EI}; \quad d = \frac{\phi}{E} \]

The wave equation for lateral vibration of beams may be obtained by elimination of \( \phi \) from equations 2.6, including the corrections for rotary inertia applied by Rayleigh (1894) and for shear of the bar elements as shown by Timoshenko (1921).

The maximum bending stress in the beam is

\[ \sigma = \frac{Mh}{2I} \]

According to equation 2.2, this may be written

\[ \sigma = -\frac{Eh}{2} \cdot \phi' \]

To find the stresses in the bent beam equations 2.6 must be solved for \( \phi' \).

In an impact test of a brittle beam the loading and support may be expected to appear in Figure 3, with the corresponding \( y(x) \) and \( \phi(x) \) as shown. A
Figure 3. Load and Support of a Simple Beam Showing Deflection and Slope.
solution is therefore attempted by separation of variables in the form:

\[ \phi = \sum T_n(t) \cos \alpha x \]
\[ y = \sum \tau_n(t) \sin \alpha x \]  \hspace{1cm} (2.8)

where \( T_n \) and \( \tau_n \) are functions of time, \( t \), alone and \( \alpha = \frac{n\pi}{L} \).

The load per unit length of beam, \( p(x,t) \), can also be expressed in the form of a Fourier series:

\[ p(x,t) = \sum a_n(t) \sin \alpha x \]  \hspace{1cm} (2.9)

Assuming that the force \( F \) exerted by the hammer on the beam is approximately linear with time to the fracture,

\[ F = kt \]  \hspace{1cm} (2.10)

where

\[ F(t) = \int_0^L p(x,t)dx \]  \hspace{1cm} (2.11)

It is reasonable to expect that \( p(x,t) \) is a narrow pulse of approximately uniform pressure of intensity \( p_o \) and width \( \delta \), as shown in Figure 4. Then, from equations 2.10 and 2.11:

\[ kt = \int_0^L p(x,t)dx = p_o(t) \delta \]

so that

\[ p_o(t) = \frac{k}{\delta} t \]  \hspace{1cm} (2.12)
Figure 4. Pressure Pulse Due to Impact.
It is thus possible to write

\[ p(x, t) = p_{0}(t)f(x) = kt \frac{f(x)}{\delta} \]

where \( f(x) \) is zero in the range \( 0 \leq x \leq L \) except for \( L/2 - \delta/2 \leq x \leq L/2 + \delta/2 \) where \( f(x) = 1 \). If \( \delta \) is so small as to be negligible with respect to \( L \), the Fourier expansion of \( f(x) \) is given by

\[ f(x) = \sum_{n=1}^{\infty} \left( \frac{2\delta}{L} \sin \frac{n\pi x}{2} \right) \sin \alpha x \]

Therefore, equation 2.9 reduces to

\[ p(x, t) = \frac{2kt}{L} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \sin \alpha x \]

Equations 2.9 and 2.15 may now be substituted into equations 2.6 to obtain the following pair of ordinary differential equations:

\[ -\alpha T_n + a \ddot{T}_n + \alpha^2 T_n = \frac{2kb \sin \alpha L/2}{L} t \]

\[ d\ddot{T}_n + (\alpha^2 + c) T_n - c \alpha \dot{T}_n = 0 \]

A particular solution to 2.16 is

\[ T_n = \frac{2kbc \sin \alpha L/2}{L \alpha^2} t \]

\[ \ddot{T}_n = \frac{2kb \sin \alpha L/2}{L \alpha^4} (\alpha^2 + c) t \]

where the constants \( l \) and \( m \) are introduced for convenience.
A complementary solution of equations 2.16 is found by making \( T_n \) and \( C_n \) both proportional to \( e^{pt} \). The secular equation is thus

\[
p^4 + 2p^2 \left[ \frac{\alpha^2}{2a} + \frac{\beta^2 + c}{2d} \right] + \frac{\alpha^4}{ad} = 0 \quad 2.18
\]

or, with an obvious change in notation

\[
p^4 + 2p^2 \xi^2 + \beta^2 = 0 \quad 2.19
\]

The solutions of 2.19 are

\[
p_{1,2}^2 = -\xi \pm \sqrt{\xi^2 - \beta^2} \quad 2.20
\]

Since both values of \( p^2 \) are real, negative, and distinct from each other, there are four distinct pure imaginary roots of 2.18. Denote these roots by \( \pm iw_{n1} \) and \( \pm iw_{n2} \),

where

\[
w_{n1}^2 = +\xi - \sqrt{\xi^2 - \beta^2} \quad 2.21
\]

\[
w_{n2}^2 = +\xi + \sqrt{\xi^2 - \beta^2}
\]

The subscripts "n" are attached to emphasize the dependence of these roots on the summation.

Making use of these notations the complementary solutions of equation 2.16 may be written as

\[
T_n = A_{n1} \cos w_{n2} t + A_{n2} \sin w_{n1} t + A_{n3} \cos w_{n2} t + A_{n4} \sin w_{n2} t
\]

\[
C_n = B_{n1} \cos w_{n1} t + B_{n2} \sin w_{n1} t + B_{n3} \cos w_{n2} t + B_{n4} \sin w_{n2} t
\]

\[2.22\]
The eight constants of integration, \( A_1 \cdots B_4 \), are not all independent, but are connected by the fact that expressions 2.22 are solutions of 2.16, so that:

\[
B_{n_1} = A_{n_1} \frac{\alpha}{\alpha^2 - \omega^2_{n_1}}; \quad B_{n_2} = A_{n_2} \frac{\alpha}{\alpha^2 - \omega^2_{n_2}}
\]

\[
B_{n_3} = A_{n_3} \frac{\alpha}{\alpha^2 - \omega^2_{n_3}}; \quad B_{n_4} = A_{n_4} \frac{\alpha}{\alpha^2 - \omega^2_{n_4}}
\]

Equations 2.23 may be written more concisely by introducing the obvious notations \( e_1 \) and \( e_2 \):

\[
B_{n_1} = e_1 A_{n_1}; \quad B_{n_2} = e_1 A_{n_2}
\]

\[
B_{n_3} = e_2 A_{n_3}; \quad B_{n_4} = e_2 A_{n_4}
\]

By combining equations 2.17, 2.22, and 2.24, the complete solution of equation 2.16 is obtained as

\[
T_n = A_{n_1} \cos \omega_{n_1} t + A_{n_2} \sin \omega_{n_1} t + A_{n_3} \cos \omega_{n_2} t + A_{n_4} \sin \omega_{n_2} t + n_{1} t\]

\[
\overline{T}_n = e_1 (A_{n_1} \cos \omega_{n_1} t + A_{n_2} \sin \omega_{n_1} t)
\]

\[
+ e_2 (A_{n_3} \cos \omega_{n_2} t + A_{n_4} \sin \omega_{n_2} t) + m_{1} t
\]

In the impact test equations 2.25 are subject to the initial conditions that the beam be at rest at \( t = 0 \), that is,

\[ T_n(0) = \dot{T}_n(0) = \overline{T}_n(0) = \dot{T}_n(0) = 0 \]
Hence:

\[
\begin{align*}
A_n^1 + A_n^3 &= 0 \\
e_1 A_n^1 + e_2 A_n^3 &= 0
\end{align*}
\]

so \(A_n^1 = A_n^3 = 0\) \hspace{1cm} 2.26

and

\[
\begin{align*}
w_n A_n^2 + w_n A_n^4 &= -l_n \\
e_1 w_n A_n^2 + e_2 w_n A_n^4 &= -m_n
\end{align*}
\]

therefore

\[
A_n^2 = \frac{m_n - e_2 l_n}{w_n (e_2 - e_1)}
\]

\[
A_n^4 = \frac{e_1 l_n - m_n}{w_n (e_2 - e_1)}
\] \hspace{1cm} 2.28

The formal solution of the problem is thus completed. The above expressions must be combined into the final equation for stress. From equation 2.7:

\[
\sigma = -\frac{E h}{2} \phi'
\]

From equation 2.8:

\[
\phi' = \sum T_n \cos \alpha \times x
\]

thus,

\[
\phi' = -\sum T_n \sin \alpha \times x
\] \hspace{1cm} 2.29
$$
\sigma = \frac{Eh}{2} \sum \alpha \ T_n \sin \alpha x
$$

By the physical argument that the bending moment will be maximum at \( x = L/2 \), the stress will be maximum for

$$
\sum T_n \sin \frac{\alpha L}{2} \tag{2.30}
$$

From equations 2.25 and 2.26:

$$
T_n = \frac{m_n - e_2 l_n}{w_n (e_2 - e_1)} \sin w_n t - \frac{m_n - e_1 l_n}{w_n (e_2 - e_1)} \sin w_{n+1} t \tag{2.31}
$$

Explicit forms for each coefficient are as follows:

From equation 2.17:

$$
m_n = \frac{3 k}{AGL} \cdot \frac{2 AG}{(\alpha^2 + 3 EI) \sin \frac{\alpha L}{2}} \tag{2.32}
$$

and

$$
l_n = \frac{2 k}{EIL\alpha^3} \sin \frac{\alpha L}{2} \tag{2.32}
$$

From equation 2.17 and 2.23:

$$
e_2 l_n = \frac{2 k}{EIL} \cdot \frac{1}{\alpha^2 (\alpha^2 - \frac{3 \rho}{2 G} w_n^2)} \sin \frac{\alpha L}{2} \tag{2.33}
$$

and

$$
e_1 l_n = \frac{2 k}{EIL} \cdot \frac{1}{\alpha^2 (\alpha^2 - \frac{3 \rho}{2 G} w_n^2)} \sin \frac{\alpha L}{2} \tag{2.34}
$$
From equation 2.23:

\[
e_{2} - e_{1} = \frac{3 \epsilon}{2G} \frac{\alpha (w_{n} - w_{n}^{2})}{\left(\alpha^{2} - \frac{3 \epsilon}{2G} w_{n}^{2}\right)\left(\alpha^{2} - \frac{3 \epsilon}{2G} w_{n}^{2}\right)} \tag{2.35}
\]

From equations 2.18, 2.19 and 2.21:

\[
w_{n2}^{2} - w_{n1}^{2} = 2 \sqrt{\epsilon^{2} - \beta^{2}} \tag{2.36}
\]

and

\[
w_{n1}^{2} = \epsilon - \sqrt{\epsilon^{2} - \beta^{2}} \tag{2.37}
\]

\[
w_{n2}^{2} = \epsilon + \sqrt{\epsilon^{2} - \beta^{2}} \tag{2.38}
\]

These may be calculated from \( \epsilon \) and \( \beta^{2} \), since

\[
\epsilon = \frac{\alpha^{2} G}{3 \rho} + \frac{\alpha^{2}}{2 \rho/E} + \frac{2AG}{3EI} \tag{2.39}
\]

and

\[
\beta^{2} = \frac{\alpha^{4}}{ad} = \frac{\alpha^{4}}{2GE} \tag{2.40}
\]

From equation 2.29 and the explicit expressions above, the equation for maximum stress is then written as

\[
\sigma = \frac{Eh}{2} \sum \alpha \sin \alpha x
\]
Substitution of the various explicit forms yields the complete expression:

\[
\delta_{\text{max}} = \frac{hk}{2IL} \sum_{n=1,3} \left\{ \frac{3\text{EI}}{\text{AG}} \left( \frac{2+2\frac{AG}{3\text{EI}}}{\alpha^2} \frac{2}{\frac{\alpha^2}{2G} - \frac{3\rho}{2G} w_n^2} \right) \right. \\
\left. \frac{w_n}{\left[ \frac{3\frac{\alpha^2}{2G} (w_n^2 - w_{n1}^2)}{\left(\alpha^2 - \frac{3\rho}{2G} w_n^2\right) \left(\alpha^2 - \frac{3\rho}{2G} w_{n1}^2\right)} \right]} \sin w_{n1} + \frac{2}{\alpha^2} \right\} \]

Simplification of Stress Equation

The several characteristics of equation 2.41 which aid in its simplification are as follows:

1. Wherever \( w_i^2 \), \( i = 1, 2 \) appears in the coefficients of \( \sin w_i t \), it is multiplied by the constant

\[
a = \frac{3\rho}{2G};
\]

therefore define

\[
\gamma_1^2 = aw_1^2.
\]

2. Every term in the expression under the summation sign contains the factor \( 1/\alpha^2 \).
3. The denominators of the coefficients of \( \frac{\sin w_1 t}{w_1} \) are identical.

4. The first term in the numerator of the coefficients of \( \frac{\sin w_1 t}{w_1} \)
can be written
\[
\frac{3 \, EI}{AG} \frac{\alpha^2 + 2}{\alpha^4} = \frac{2}{\alpha^4} \left( \frac{\alpha^2}{c} + 1 \right); \quad c = \frac{2 \, AG}{3 \, EI}
\]

By utilizing these relations, equation 2.41 may be written
\[
\frac{2 \, IL}{hk} \sigma_{\text{max}} = 2 \sum_{n=1,3,5} \frac{1}{\alpha^2} \left\{ \left( \alpha^2 - \gamma_2^2 \right) \left( \alpha^2 - \gamma_1^2 \right) \right\} \left( \frac{1}{c} + \frac{1}{\alpha^2} - \frac{1}{\gamma_2^2} \right) \frac{\sin w_1 t}{w_1}
\]
\[
+ 2 \, t \sum_{n=1,3,5} \frac{1}{\alpha^2}
\]

The last term of equation 2.42 may be written in closed form, since
\[
\sum_{n=1,3,5} \frac{1}{\alpha^2} = \sum_{n=1,3,5} \left( \frac{L}{n \pi} \right)^2 = \frac{L^2}{8}
\]

Since \( G = \frac{E}{2 \left( \nu + 1 \right)} \), then \( c = 3 \left( \nu + 1 \right) \frac{L}{A} = 3 \left( \nu + 1 \right) \frac{R^2}{A} \)

where \( R \) is the radius of gyration of the cross section about the neutral plane.
Because of the repeated occurrence of the combination, a term \( \Upsilon \) is defined:

\[
\Upsilon = 3 (\sqrt{\gamma} + 1)
\]

It is observed that \( \gamma_i^2 = a w_i^2 \) satisfies

\[
\gamma^4 - \left[ \alpha^2 + \frac{a}{d} (\alpha^2 + c) \right] \gamma^2 + \frac{a}{d} \alpha^4 = 0 \quad 2.44
\]

From the above, the equation for \( \gamma \) becomes

\[
\gamma^4 - \left[ (1 + \Upsilon) \alpha^2 + \eta^2 \right] \gamma^2 + \gamma \alpha^4 = 0 \quad 2.45
\]

where

\[
\eta^2 = \frac{1}{R^2}
\]

Considering the coefficients of \( \sin w_1 t \), it is noted that the \( \gamma_i^2 \) terms always occur in the combination \( \frac{1}{\alpha^2 - \gamma_i^2} \). The quadratic equation for \( \gamma^2 \) is therefore transformed into a quadratic equation for \( \mu^2 \), where

\[
\mu^2 = \frac{1}{\alpha^2 - \gamma^2}
\]

This results in the following:

\[
\mu^4 - \left[ (\gamma - 1) R^2 + \frac{1}{\alpha^2} \right] \mu^2 - \frac{R^2}{\alpha^2} = 0 \quad 2.46
\]

The solutions are

\[
\mu^2 = \frac{1}{2} \left[ (\gamma - 1) R^2 + \frac{1}{\alpha^2} \pm \sqrt{\left[(\gamma - 1) R^2 + \frac{1}{\alpha^2}\right]^2 + 4 \frac{R^2}{\alpha^2}} \right]
\]

\[
= \frac{1}{2} \left[ (\gamma - 1) R^2 + \frac{1}{\alpha^2} \pm \sqrt{(\gamma - 1)^2 R^4 + 2 (\gamma + 1) \frac{R^2}{\alpha^2} + \frac{1}{\alpha^4}} \right]
\]

2.47
Let $\mu_1^2 = B_1$ take the $+$ sign, and $\mu_2^2 = B_2$ take the $-$ sign.

and define

\[
T = \frac{1}{\alpha^2 - \gamma^2_1} - \frac{1}{\alpha^2 - \gamma^2_2} = B_1 - B_2
\]

\[
= \sqrt{\left(\gamma - 1\right)^2 R^4 + 2 \left(\gamma + 1\right) \frac{R^2}{\alpha^2} + \frac{1}{\alpha^4}}
\]

Then

\[
B_1 = \frac{1}{2} \left[ \left(\gamma - 1\right) \frac{R^2}{\alpha^2} + \frac{1}{\alpha^2} + T \right]
\]

\[
B_2 = \frac{1}{2} \left[ \left(\gamma - 1\right) \frac{R^2}{\alpha^2} + \frac{1}{\alpha^2} - T \right]
\]

The coefficient $\frac{1}{c} + \frac{1}{\alpha^2} - \frac{1}{\alpha^2 - \gamma^2_1}$ may be written

\[
\gamma \frac{R^2}{\alpha^2} + \frac{1}{\alpha^2} - B_1 = \Delta - B_1
\]

\[
\Delta - B_1 = \gamma \frac{R^2}{\alpha^2} + \frac{1}{\alpha^2} - \frac{1}{2} \left[ \left(\gamma - 1\right) \frac{R^2}{\alpha^2} + \frac{1}{\alpha^2} + T \right]
\]

\[
= \frac{1}{2} \left[ \left(\gamma + 1\right) \frac{R^2}{\alpha^2} + \frac{1}{\alpha^2} - T \right] = B_2 + R^2
\]

and

\[
\Delta - B_2 = \frac{1}{2} \left[ \left(\gamma + 1\right) \frac{R^2}{\alpha^2} + \frac{1}{\alpha^2} + T \right] = B_1 + R^2
\]
Combining these definitions the stress equation may be written

\[
\sigma_{\text{max}} = \frac{IL}{hk} \sum_{n = 1, 3, 5} \frac{1}{\alpha^2} \left\{ - \frac{1}{T} \left[ \left( \Delta - B_2 \right) \frac{\sin \omega_1 t}{\omega_1} \right] - \left( \Delta - B_1 \right) \frac{\sin \omega_2 t}{\omega_2} \right\} + \frac{L^2}{8} t
\]

from which

\[
\sigma_{\text{max}} = \frac{hklL}{I} \left\{ \frac{1}{\pi^2} \sum_{n = 1, 3, 5} \left\{ - \frac{1}{T} \left[ \left( \Delta - B_2 \right) \frac{\sin \omega_{n_1} t}{\omega_{n_1}} \right] - \left( \Delta - B_1 \right) \frac{\sin \omega_{n_2} t}{\omega_{n_2}} \right\} + \frac{L^2}{8} t \right\}
\]

The material constants of the specimen enter into the calculation of the coefficients \( \frac{\Delta - B_2}{T} \) and \( \frac{\Delta - B_1}{T} \) only in terms of Poisson's ratio in the form \( \gamma = 3 (\gamma + 1) \). These coefficients also contain information concerning the geometry of the specimen, namely the length of span \( L \) and the radius of gyration \( R \).

The numbers \( \gamma_i \) also depend only upon \( \gamma \), \( R \) and \( L \). Specifically,

\[
\gamma_i^2 = \alpha^2 - \frac{1}{B_1}
\]

Thus, the only way in which the material constants, other than \( \gamma \), enter is through the factor \( \sqrt{\frac{2 G}{3 \rho}} \) in the statement:

\[
w_i = \gamma_i \sqrt{\frac{2 G}{3 \rho}}.
\]
Since \( G = \frac{E}{2(\nu + 1)} \), this may be stated as

\[
\omega_i = \gamma_i \sqrt{\frac{E}{3(\nu + 1)\rho}} = \gamma_i \frac{\sqrt{E}}{\sqrt{\rho}}
\]

Defining \( \bar{c} = \sqrt{\frac{E}{\rho}} \) and \( \tau = \bar{c}t \) then the stress equation becomes

\[
\sigma_{\text{max}} = \frac{hkL}{I} \left\{ -\frac{1}{c} F(\tau; \nu, R, L) + \frac{t}{6} \right\}
\]

where

\[
F(\tau; \nu, R, L) = \sqrt{\frac{\nu}{\pi^2}} \sum_{n=1,3,5}^{\infty} \frac{1}{\tau} \left[ \frac{\Delta-B_2}{\gamma_1} \sin \left( \frac{\gamma_1}{\sqrt{\rho}} \tau \right) \right. \\
- \frac{\Delta-B_1}{\gamma_2^2} \sin \left( \frac{\gamma_2}{\sqrt{\rho}} \tau \right) \left. \right]
\]

Solution of the Stress Equation

The IBM 704 computer was programmed to yield \( F(\tau; \nu, R, L) \) for selected geometries and a range of \( \nu \). The notation in the Fortran program is shown in Table 1. A numerical change was introduced so that the specific gravity, \( \bar{\rho} \), may be used rather than mass density, \( \rho \). The computations are then in terms of \( e = \frac{\bar{\rho}}{E} \) and \( \bar{\tau} = \frac{t}{\sqrt{e}} \).

In order to evaluate the influence of Poisson's ratio a numerical experiment was performed varying \( \nu \) over the range, \( 0 \leq \nu \leq 0.25 \) for the bar geometry \( R = 0.15 \) and \( L = 4.0 \). For fixed \( t \), the function was constant to four significant digits. The results of this analysis are shown
**TABLE 1a**

**FORTRAN PROGRAM NOTATION**

<table>
<thead>
<tr>
<th>Formula</th>
<th>Fortran</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{V} + 1 )</td>
<td>( \mathcal{V} )</td>
</tr>
<tr>
<td>( R )</td>
<td>( R )</td>
</tr>
<tr>
<td>( L )</td>
<td>( S )</td>
</tr>
</tbody>
</table>

number of terms computed \( \text{I MAX} \)

\( \Delta \mathcal{T} \) \( \text{DT} \)

\( \mathcal{J} = 3 (\mathcal{V} + 1) \) \( \mathcal{Z} \)

\( t \) \( \mathcal{T} \)

\( R^2 \) \( \text{R2} \)

\( n \) \( 2^* I-1 \)

\( \alpha^2 \) \( \text{A2} \)

\( 1/\alpha^2 \) \( \text{A1} \)

\( \Gamma \) \( \text{GAM} \)

\( \gamma_j \) for \( \alpha = (2^i - 1) \frac{\pi}{L} \); \( j = 1, 2 \) \( \text{G (I, J)} \)

\[ \sum \frac{\Delta \cdot B_j}{T} \frac{\sin \gamma(3-j)}{\gamma(3-j)} \] \( \text{SIG (J)} \)

\( F (\mathcal{C}) \) \( \text{FT} \)

Print \( \text{KODE = 1} \)

Punch \( \text{KODE = 2} \)

Print and Punch \( \text{KODE = 3} \)
TABLE 1b

FORTRAN PROGRAM

DIMENSION D(25), G(25, 2), B(25, 2), SIG(2)

READ1, KODE, I MAX, DT, TMAX, R, S, V

FORMAT (212, 6F10.4)

Z = 3.*V

C1=SQRTF (0.93571929E-4*Z)

R2=R**2

PI2=3.1415927**2

AO=3.1415927/S

DO 2 I= 1, ImAX

A2=(AO*FLOATF (2*I-1))**2

A1=1./A2

GAM=SQRTF(((Z-1.)*R2)**2+2.*(Z+1.)*R2*A1+A1**2)

CO=1./(GAM*FLOATF((2*I-1)**2))

DO=(Z+1.)*R2+A1

DO2J=1,2

D(J)=(DO-((-1.)**J)*GAM)/2.

G(I, J)=SQRTF(A2-1./(D(J)-R2))

B(I, J)=CO*D(J)/G(I, J)
TABLE 1b (Contd.)

\[
\text{KMAX=TMAX/DT}
\]

\[
\text{GOTO (10, 11, 10), KODE}
\]

0010 PRINT13, IMAX, R, S, V

\[
\text{GOTO (12, 12, 11), KODE}
\]

0011 PUNCH13, IMAX, R, S, V

13 FORMAT (6H IMax=13, 3H R=F6.4, 3H L=F6.2, 6H NU+1=F6.2/26H OT)

0012 DO6K=1, KMAX

\[
T=DT*\text{FLOATF (K)}
\]

DO3J=1, 2

\[
\text{SIG(J)=0}
\]

DO3I=1, IMAX

3 SIG(J)=SIG(J)+B(I, J)*SINF(G(I, J)*T/C1)

\[
\text{FT}=(C1/\pi 2)*(\text{SIG(1)}-\text{SIG(2)})
\]

GOTO(4, 5, 4), KODE

4 PRINT7, T, FT

GOTO(6, 6, 5), KODE

5 PUNCH7, T, FT

6 CONTINUE
TABLE 1b (Contd.)

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>GOTO8</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>FORMAT(1HO4E16.8)</td>
</tr>
</tbody>
</table>
in Table 2. The data for the extremes of variation are used to plot Figure 5.

The foregoing results clearly indicate that further numerical consideration of Poisson's ratio is not necessary. The variables of interest then become $R$ and $L$.

Table 3 shows the range of variation considered in the computations. Case number 1 in this table is designed to explore the variation of $F(\tau)$ for the specific conditions $L = 4.0$ and $R = 0.1500$. In view of the previous analysis, $\psi$ was arbitrarily chosen as 0.20. The results of this calculation are shown in Table 4.

Since by definition, $\tau = \frac{t}{\sqrt{\psi}}$, it is possible to describe the variation of the function $F(\tau)$ in time for any assumed value $\sqrt{\psi}$.

Figure 6 is constructed from the data shown in Table 4, and pictures the $F(\tau)$ value as a function of time for the specific case

\begin{align*}
L &= 4.0 \\
R &= 0.1500 \\
e &= 2.4 \times 10^{-7}
\end{align*}

Because there is no energy-loss term in the stress equation, the function does not display a decaying amplitude with time. Since materials do exhibit internal friction and other accompanying energy dissipation (13),
<table>
<thead>
<tr>
<th>Poisson's Ratio</th>
<th>0.00</th>
<th>0.05</th>
<th>0.10</th>
<th>0.15</th>
<th>0.20</th>
<th>0.25</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 x 10^{-2}</td>
<td>0.11464 x 10^{-2}</td>
<td>0.11473 x 10^{-2}</td>
<td>0.11483 x 10^{-2}</td>
<td>0.11492 x 10^{-2}</td>
<td>0.11501 x 10^{-2}</td>
<td>0.11510 x 10^{-2}</td>
</tr>
<tr>
<td>2</td>
<td>0.21690</td>
<td>0.21704</td>
<td>0.21717</td>
<td>0.21730</td>
<td>0.21744</td>
<td>0.21756</td>
</tr>
<tr>
<td>3</td>
<td>0.31180</td>
<td>0.31198</td>
<td>0.31216</td>
<td>0.31234</td>
<td>0.31251</td>
<td>0.31269</td>
</tr>
<tr>
<td>4</td>
<td>0.40087</td>
<td>0.40108</td>
<td>0.40127</td>
<td>0.40146</td>
<td>0.40166</td>
<td>0.40186</td>
</tr>
<tr>
<td>5</td>
<td>0.48499</td>
<td>0.48525</td>
<td>0.48548</td>
<td>0.48569</td>
<td>0.48590</td>
<td>0.48610</td>
</tr>
<tr>
<td>6</td>
<td>0.56292</td>
<td>0.46358</td>
<td>0.56408</td>
<td>0.56453</td>
<td>0.56490</td>
<td>0.56526</td>
</tr>
<tr>
<td>7</td>
<td>0.64174</td>
<td>0.64118</td>
<td>0.64052</td>
<td>0.63972</td>
<td>0.63950</td>
<td>0.63986</td>
</tr>
<tr>
<td>8</td>
<td>0.73023</td>
<td>0.72968</td>
<td>0.72910</td>
<td>0.72851</td>
<td>0.72786</td>
<td>0.72719</td>
</tr>
<tr>
<td>9</td>
<td>0.81300</td>
<td>0.81274</td>
<td>0.81245</td>
<td>0.81217</td>
<td>0.81188</td>
<td>0.81159</td>
</tr>
<tr>
<td>10</td>
<td>0.88481</td>
<td>0.88485</td>
<td>0.88490</td>
<td>0.88501</td>
<td>0.88516</td>
<td>0.88532</td>
</tr>
<tr>
<td>11</td>
<td>0.94389</td>
<td>0.94431</td>
<td>0.94479</td>
<td>0.94531</td>
<td>0.94582</td>
<td>0.94628</td>
</tr>
<tr>
<td>12</td>
<td>0.99123</td>
<td>0.99170</td>
<td>0.99235</td>
<td>0.99303</td>
<td>0.99369</td>
<td>0.99433</td>
</tr>
</tbody>
</table>
Figure 5. The Effect of Poisson's Ratio on the Time Function.
### TABLE 3

**VARIATIONS INCLUDED IN COMPUTER PROGRAM**

<table>
<thead>
<tr>
<th>R/L</th>
<th>DT</th>
<th>T Max.</th>
<th>R</th>
<th>L</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.037500</td>
<td>.005</td>
<td>.500</td>
<td>0.150000</td>
<td>4.0</td>
</tr>
<tr>
<td>0.015625</td>
<td>.010</td>
<td>.130</td>
<td>0.093750</td>
<td>6.0</td>
</tr>
<tr>
<td>0.017045</td>
<td>.010</td>
<td>.130</td>
<td>0.093750</td>
<td>5.5</td>
</tr>
<tr>
<td>0.018750</td>
<td>.010</td>
<td>.130</td>
<td>0.093750</td>
<td>5.0</td>
</tr>
<tr>
<td>0.020833</td>
<td>.010</td>
<td>.130</td>
<td>0.093750</td>
<td>4.5</td>
</tr>
<tr>
<td>0.023438</td>
<td>.010</td>
<td>.130</td>
<td>0.093750</td>
<td>4.0</td>
</tr>
<tr>
<td>0.023438</td>
<td>.010</td>
<td>.130</td>
<td>0.140625</td>
<td>6.0</td>
</tr>
<tr>
<td>0.025568</td>
<td>.010</td>
<td>.130</td>
<td>0.140625</td>
<td>5.5</td>
</tr>
<tr>
<td>0.026786</td>
<td>.010</td>
<td>.130</td>
<td>0.093750</td>
<td>3.5</td>
</tr>
<tr>
<td>0.028125</td>
<td>.010</td>
<td>.130</td>
<td>0.140625</td>
<td>5.0</td>
</tr>
<tr>
<td>0.031250</td>
<td>.010</td>
<td>.130</td>
<td>0.093750</td>
<td>3.0</td>
</tr>
<tr>
<td>0.031250</td>
<td>.010</td>
<td>.130</td>
<td>0.140625</td>
<td>4.5</td>
</tr>
<tr>
<td>0.031250</td>
<td>.010</td>
<td>.130</td>
<td>0.187500</td>
<td>6.0</td>
</tr>
<tr>
<td>0.034091</td>
<td>.010</td>
<td>.130</td>
<td>0.187500</td>
<td>5.5</td>
</tr>
<tr>
<td>0.035156</td>
<td>.010</td>
<td>.130</td>
<td>0.140625</td>
<td>4.0</td>
</tr>
<tr>
<td>0.037500</td>
<td>.010</td>
<td>.130</td>
<td>0.187500</td>
<td>5.0</td>
</tr>
<tr>
<td>0.039063</td>
<td>.010</td>
<td>.130</td>
<td>0.234375</td>
<td>6.0</td>
</tr>
<tr>
<td>0.040179</td>
<td>.010</td>
<td>.130</td>
<td>0.140625</td>
<td>3.5</td>
</tr>
<tr>
<td>R/L</td>
<td>DT</td>
<td>T Max.</td>
<td>R.</td>
<td>L</td>
</tr>
<tr>
<td>------</td>
<td>-----</td>
<td>--------</td>
<td>--------</td>
<td>-----</td>
</tr>
<tr>
<td>0.041667</td>
<td>.010</td>
<td>.130</td>
<td>0.187500</td>
<td>4.5</td>
</tr>
<tr>
<td>0.042614</td>
<td>.010</td>
<td>.130</td>
<td>0.234375</td>
<td>5.5</td>
</tr>
<tr>
<td>0.046875</td>
<td>.010</td>
<td>.130</td>
<td>0.140625</td>
<td>3.0</td>
</tr>
<tr>
<td>0.046875</td>
<td>.010</td>
<td>.130</td>
<td>0.187500</td>
<td>4.0</td>
</tr>
<tr>
<td>0.046875</td>
<td>.010</td>
<td>.130</td>
<td>0.234375</td>
<td>5.0</td>
</tr>
<tr>
<td>0.046875</td>
<td>.010</td>
<td>.130</td>
<td>0.281250</td>
<td>6.0</td>
</tr>
<tr>
<td>0.051136</td>
<td>.010</td>
<td>.130</td>
<td>0.281250</td>
<td>5.5</td>
</tr>
<tr>
<td>0.052083</td>
<td>.010</td>
<td>.130</td>
<td>0.234375</td>
<td>4.5</td>
</tr>
<tr>
<td>0.053571</td>
<td>.010</td>
<td>.130</td>
<td>0.187500</td>
<td>3.5</td>
</tr>
<tr>
<td>0.054688</td>
<td>.010</td>
<td>.130</td>
<td>0.328125</td>
<td>6.0</td>
</tr>
<tr>
<td>0.056250</td>
<td>.010</td>
<td>.130</td>
<td>0.281250</td>
<td>5.0</td>
</tr>
<tr>
<td>0.058594</td>
<td>.010</td>
<td>.130</td>
<td>0.234375</td>
<td>4.0</td>
</tr>
<tr>
<td>0.059659</td>
<td>.010</td>
<td>.130</td>
<td>0.328125</td>
<td>5.5</td>
</tr>
<tr>
<td>0.062500</td>
<td>.010</td>
<td>.130</td>
<td>0.187500</td>
<td>3.0</td>
</tr>
<tr>
<td>0.062500</td>
<td>.010</td>
<td>.130</td>
<td>0.281250</td>
<td>4.5</td>
</tr>
<tr>
<td>0.062500</td>
<td>.010</td>
<td>.130</td>
<td>0.375000</td>
<td>6.0</td>
</tr>
<tr>
<td>0.065625</td>
<td>.010</td>
<td>.130</td>
<td>0.328125</td>
<td>5.0</td>
</tr>
<tr>
<td>0.066964</td>
<td>.010</td>
<td>.130</td>
<td>0.234375</td>
<td>3.5</td>
</tr>
<tr>
<td>0.068182</td>
<td>.010</td>
<td>.130</td>
<td>0.375000</td>
<td>5.5</td>
</tr>
</tbody>
</table>
### TABLE 3. (Contd.)

<table>
<thead>
<tr>
<th>R/L</th>
<th>DT</th>
<th>T Max.</th>
<th>R Max.</th>
<th>L Max.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.070313</td>
<td>.010</td>
<td>.130</td>
<td>0.281250</td>
<td>4.0</td>
</tr>
<tr>
<td>0.072917</td>
<td>.010</td>
<td>.130</td>
<td>0.328125</td>
<td>4.5</td>
</tr>
<tr>
<td>0.075000</td>
<td>.010</td>
<td>.130</td>
<td>0.375000</td>
<td>5.0</td>
</tr>
<tr>
<td>0.078125</td>
<td>.010</td>
<td>.130</td>
<td>0.234375</td>
<td>3.0</td>
</tr>
<tr>
<td>0.080321</td>
<td>.010</td>
<td>.130</td>
<td>0.281250</td>
<td>3.5</td>
</tr>
<tr>
<td>0.082031</td>
<td>.010</td>
<td>.130</td>
<td>0.328125</td>
<td>4.0</td>
</tr>
<tr>
<td>0.083333</td>
<td>.010</td>
<td>.130</td>
<td>0.375000</td>
<td>4.5</td>
</tr>
<tr>
<td>0.093750</td>
<td>.010</td>
<td>.130</td>
<td>0.281250</td>
<td>3.0</td>
</tr>
<tr>
<td>0.093750</td>
<td>.010</td>
<td>.130</td>
<td>0.328125</td>
<td>3.5</td>
</tr>
<tr>
<td>0.094250</td>
<td>.010</td>
<td>.130</td>
<td>0.375000</td>
<td>4.0</td>
</tr>
<tr>
<td>0.107143</td>
<td>.010</td>
<td>.130</td>
<td>0.375000</td>
<td>3.5</td>
</tr>
<tr>
<td>0.109375</td>
<td>.010</td>
<td>.130</td>
<td>0.328125</td>
<td>3.0</td>
</tr>
<tr>
<td>0.125000</td>
<td>.010</td>
<td>.130</td>
<td>0.375000</td>
<td>3.0</td>
</tr>
</tbody>
</table>
### TABLE 4

**COMPUTATION OF F( T ) FOR**

*R = 0.1500,  L = 4.00*

<table>
<thead>
<tr>
<th>T</th>
<th>F(T)</th>
<th>T</th>
<th>F(T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5 x 10^2</td>
<td>0.05980 x 10^{-2}</td>
<td>10.0 x 10^{-2}</td>
<td>0.88516 x 10^{-2}</td>
</tr>
<tr>
<td>1.0</td>
<td>0.11501</td>
<td>12.5</td>
<td>1.01317</td>
</tr>
<tr>
<td>1.5</td>
<td>0.16723</td>
<td>15.0</td>
<td>1.05858</td>
</tr>
<tr>
<td>2.0</td>
<td>0.21743</td>
<td>17.5</td>
<td>1.08676</td>
</tr>
<tr>
<td>2.5</td>
<td>0.26581</td>
<td>20.0</td>
<td>1.06185</td>
</tr>
<tr>
<td>3.0</td>
<td>0.31251</td>
<td>22.5</td>
<td>0.95084</td>
</tr>
<tr>
<td>3.5</td>
<td>0.35778</td>
<td>25.0</td>
<td>0.78558</td>
</tr>
<tr>
<td>4.0</td>
<td>0.40166</td>
<td>27.5</td>
<td>0.60808</td>
</tr>
<tr>
<td>4.5</td>
<td>0.44429</td>
<td>30.0</td>
<td>0.40047</td>
</tr>
<tr>
<td>5.0</td>
<td>0.48590</td>
<td>32.5</td>
<td>0.12781</td>
</tr>
<tr>
<td>5.5</td>
<td>0.52606</td>
<td>35.0</td>
<td>-0.12994</td>
</tr>
<tr>
<td>6.0</td>
<td>0.56490</td>
<td>37.5</td>
<td>-0.34623</td>
</tr>
<tr>
<td>6.5</td>
<td>0.60232</td>
<td>40.0</td>
<td>-0.58349</td>
</tr>
<tr>
<td>7.0</td>
<td>0.63950</td>
<td>42.5</td>
<td>-0.79963</td>
</tr>
<tr>
<td>7.5</td>
<td>0.68342</td>
<td>45.0</td>
<td>-0.94124</td>
</tr>
<tr>
<td>8.0</td>
<td>0.72786</td>
<td>47.5</td>
<td>-1.02693</td>
</tr>
<tr>
<td>8.5</td>
<td>0.77093</td>
<td>48.5</td>
<td>-1.05816</td>
</tr>
<tr>
<td>9.0</td>
<td>0.81188</td>
<td>49.0</td>
<td>-1.06798</td>
</tr>
<tr>
<td>9.5</td>
<td>0.85026</td>
<td>49.5</td>
<td>-1.07929</td>
</tr>
<tr>
<td>10.0</td>
<td>0.88516</td>
<td>50.0</td>
<td>-1.08929</td>
</tr>
</tbody>
</table>
Figure 6. The Time Function for $R = 0.15$, $L = 4.00$. 

\[ \tau = \frac{t}{\sqrt{\epsilon}} \]
the validity of the stress equation is obviously limited in time. It will be
shown later that the correction for stress calculation is the decaying
function \( F(\tau/\tau) \), and considerations of energy losses would contribute very
little change.

Table 5 shows the results of the variation of \( R \) and \( L \) through the
ranges \( 0.09375 \leq R \leq 0.37500 \) and \( 3.0 \leq L \leq 6.0 \).

The graphical presentation of these data may be accomplished by
plotting \( F(\tau,R,L) \) as a function of \( \tau/L \) for each value of \( R/L \). The re-
sulting family of curves then defines the function for all reasonable bar
sizes and spans. The curves are shown in Figure 7a.

On the basis of equation 2.41 and with the curves developed in
Figure 7, the stress-time curve for specific bars may be calculated.
Assuming variations in diameter, span and elasticity over the ranges

\[
0.375 \leq D \leq 0.750 \\
3.0 \leq L \leq 6.0 \\
2.5 \times 10^6 \leq E \leq 10^7
\]

equation 2.41 describes the stress as a function of time. These curves are
shown in Figures 8, 9, and 10. The hypothetical rate of load application is
such that all bars are stressed at the same rate according to simple beam
theory. It may be seen that the rate of stressing according to equation 2.41
is greatly different from that indicated by use of the flexure formula.
### TABLE 5

**COMPUTATION OF F(T) FOR VARIOUS VALUES OF R AND L**

$L = 3.00$

<table>
<thead>
<tr>
<th>R</th>
<th>0.093750</th>
<th>0.140625</th>
<th>0.187500</th>
<th>0.234375</th>
<th>0.281250</th>
<th>0.328125</th>
<th>0.375000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>0.11325x10^-2</td>
<td>0.11186x10^-2</td>
<td>0.11119x10^-2</td>
<td>0.11084x10^-2</td>
<td>0.11064x10^-2</td>
<td>0.11051x10^-2</td>
<td>0.11042x10^-2</td>
</tr>
<tr>
<td>1</td>
<td>0.21395</td>
<td>0.20763</td>
<td>0.20297</td>
<td>0.19568</td>
<td>0.19742</td>
<td>0.19586</td>
<td>0.19475</td>
</tr>
<tr>
<td>2</td>
<td>0.30697</td>
<td>0.29378</td>
<td>0.28373</td>
<td>0.27557</td>
<td>0.26893</td>
<td>0.26968</td>
<td>0.25928</td>
</tr>
<tr>
<td>3</td>
<td>0.39379</td>
<td>0.37269</td>
<td>0.35673</td>
<td>0.34292</td>
<td>0.32857</td>
<td>0.31445</td>
<td>0.30161</td>
</tr>
<tr>
<td>4</td>
<td>0.47393</td>
<td>0.44736</td>
<td>0.42800</td>
<td>0.40383</td>
<td>0.37828</td>
<td>0.35228</td>
<td>0.32659</td>
</tr>
<tr>
<td>5</td>
<td>0.55341</td>
<td>0.53076</td>
<td>0.48868</td>
<td>0.43692</td>
<td>0.38380</td>
<td>0.33529</td>
<td>0.29215</td>
</tr>
<tr>
<td>6</td>
<td>0.64210</td>
<td>0.59343</td>
<td>0.51371</td>
<td>0.43086</td>
<td>0.35305</td>
<td>0.28339</td>
<td>0.22478</td>
</tr>
<tr>
<td>7</td>
<td>0.72480</td>
<td>0.63328</td>
<td>0.51523</td>
<td>0.40288</td>
<td>0.30548</td>
<td>0.22157</td>
<td>0.14825</td>
</tr>
<tr>
<td>8</td>
<td>0.79531</td>
<td>0.65351</td>
<td>0.49907</td>
<td>0.36327</td>
<td>0.24349</td>
<td>0.14324</td>
<td>0.06122</td>
</tr>
<tr>
<td>9</td>
<td>0.85257</td>
<td>0.65525</td>
<td>0.47386</td>
<td>0.31236</td>
<td>0.16763</td>
<td>0.04708</td>
<td>-0.04367</td>
</tr>
<tr>
<td>10</td>
<td>0.89594</td>
<td>0.65469</td>
<td>0.44122</td>
<td>0.23043</td>
<td>0.05630</td>
<td>-0.07190</td>
<td>-0.16234</td>
</tr>
<tr>
<td>11</td>
<td>0.91782</td>
<td>0.64898</td>
<td>0.37342</td>
<td>0.12705</td>
<td>-0.05290</td>
<td>-0.16730</td>
<td>-0.23321</td>
</tr>
<tr>
<td>12</td>
<td>0.93720</td>
<td>0.62181</td>
<td>0.28568</td>
<td>0.02357</td>
<td>-0.14531</td>
<td>-0.24131</td>
<td>-0.28130</td>
</tr>
</tbody>
</table>

45
TABLE 5. (Contd.)

<table>
<thead>
<tr>
<th>L = 4.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
</tr>
<tr>
<td>T</td>
</tr>
<tr>
<td>1x10^{-2}</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>7</td>
</tr>
<tr>
<td>8</td>
</tr>
<tr>
<td>9</td>
</tr>
<tr>
<td>10</td>
</tr>
<tr>
<td>11</td>
</tr>
<tr>
<td>12</td>
</tr>
<tr>
<td>13</td>
</tr>
<tr>
<td>L = 6.00</td>
</tr>
<tr>
<td>---------</td>
</tr>
<tr>
<td>R</td>
</tr>
<tr>
<td>T 0.093750</td>
</tr>
<tr>
<td>1x10^-2</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>7</td>
</tr>
<tr>
<td>8</td>
</tr>
<tr>
<td>9</td>
</tr>
<tr>
<td>10</td>
</tr>
<tr>
<td>11</td>
</tr>
<tr>
<td>12</td>
</tr>
<tr>
<td>13</td>
</tr>
<tr>
<td>14</td>
</tr>
<tr>
<td>15</td>
</tr>
<tr>
<td>20</td>
</tr>
<tr>
<td>24</td>
</tr>
</tbody>
</table>

TABLE 5. (Contd.)
TABLE 5. (Contd.)

<table>
<thead>
<tr>
<th>R</th>
<th>T</th>
<th>0.093750</th>
<th>0.140625</th>
<th>0.187500</th>
<th>0.234375</th>
<th>0.281250</th>
<th>0.328125</th>
<th>0.375000</th>
</tr>
</thead>
<tbody>
<tr>
<td>27</td>
<td></td>
<td>x10⁻²</td>
<td>x10⁻²</td>
<td>x10⁻²</td>
<td>x10⁻²</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>28</td>
<td></td>
<td>2.50083</td>
<td>2.30642</td>
<td>1.94184</td>
<td>1.55103</td>
<td>1.14616</td>
<td></td>
<td></td>
</tr>
<tr>
<td>30</td>
<td></td>
<td></td>
<td></td>
<td>1.41090</td>
<td></td>
<td></td>
<td>0.55512</td>
<td>0.19146</td>
</tr>
<tr>
<td>32</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.83965</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>33</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.30491</td>
<td>-0.08027</td>
<td></td>
</tr>
<tr>
<td>35</td>
<td></td>
<td>2.49607</td>
<td>1.92048</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>36</td>
<td></td>
<td>3.17431</td>
<td></td>
<td>1.13823</td>
<td>0.53123</td>
<td></td>
<td>0.01248</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td></td>
<td></td>
<td></td>
<td>1.68335</td>
<td>0.85431</td>
<td>0.14967</td>
<td></td>
<td></td>
</tr>
<tr>
<td>42</td>
<td></td>
<td>2.60148</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>44</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.51216</td>
<td>-0.29586</td>
<td></td>
<td></td>
</tr>
<tr>
<td>45</td>
<td></td>
<td>3.52760</td>
<td>1.38109</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>48</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.07078</td>
<td></td>
<td></td>
</tr>
<tr>
<td>49</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.03386</td>
<td></td>
<td></td>
</tr>
<tr>
<td>54</td>
<td></td>
<td>3.78584</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>55</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.53446</td>
<td></td>
<td></td>
</tr>
<tr>
<td>56</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>63</td>
<td></td>
<td>3.85995</td>
<td>1.61323</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>70</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.93965</td>
<td></td>
<td></td>
</tr>
<tr>
<td>72</td>
<td></td>
<td>3.58762</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 7a. The Function Modelled in R/L.
Figure 7b. Correction Factor as a Function of Bar Geometry and Time.
Figure 8. Influence of Diameter on Rate of Stressing.
Figure 9. Influence of Span on Rate of Stressing.
Figure 10. Influence of Elasticity on Rate of Stressing.
The stress equation 2.54 may be written in simpler form by abbreviating the time function to \( F(T) \) and rearranging. The expression then becomes

\[
\sigma_{\text{max}} = \frac{Me}{I} \left( 1 - 8 \frac{F(T)}{T} \right)
\]

The \( Me \) multiplier has been computed from the values shown in Table 5 for a wide range of bar geometries. The dimensionless coordinates of Figure 7b permit a single curve representation of all spans and bar sizes used in impact testing, for which \( \frac{R}{L} \leq 0.0375 \).

The decaying nature of the curve in Figure 7b is of special interest. The time limit of application is dependent entirely upon bar geometry, density, and elasticity. It should be noted that the parameter \( \frac{T}{L} \) is a ratio of times, such that

\[
\frac{T}{L} = \frac{\text{time to failure in impact}}{\text{time for sound to travel the bar length}}
\]
CHAPTER III
EQUIPMENT

Hammer Transducer

Examination of equation 2.41 shows that the required experimental data for calculation of impact stress are (1) the duration of impact, and (2) the slope of the force-time curve. The general method yielding advantages of accurate positioning and controlled velocities of impact was reasoned to be the pendulum-type machine. It was also desired to include in the machine design all the features which allow the most accurate measurements of energy absorption by the specimen.

Design

There are several studies of impact in the literature which make use of piezoelectric transducers. Goland et al. (14) used piezoelectric Barium Titanate in an investigation of stress waves in struck beams, and Watanabe (15) employed x-cut quartz in determining the general features of an impact test. The general application of piezoelectricity to measurements of mechanical stresses has also been described by Bloch (16).

The transducer constructed for this study is shown in Figure 11. The construction is such that the crystal is completely surrounded by Teflon, and is precompressed by the connecting bolts. The precompression was found to be desirable for two reasons: (1) The Teflon undergoes plastic
Figure 11. The Piezoelectric Hammer.
deformation under sustained loading, thereby relieving the mounting pressure required for assured stress transfer. (2) The undamped crystal influences the time trace of voltage during the impact.

According to piezoelectric theory (17) and literature on commercially available quartz transducers (18), the amount of precompression does not influence the response magnitude of the system to a stress wave. This is verified by experiment for stresses well below the compressive failure, but deviations are reported as crystal failure loads are approached (19).

If the piezoelectric element is kept very dry and clean, Wenden (20) has illustrated that the charges developed at the surfaces may be maintained for long periods of time. A commercial quartz transducer is available which will retain the signal for several hours (18). However, if there is any dirt at the surfaces or sufficient moisture, the surface resistance is lowered to allow charge neutralization. The electrical conductors must also be extremely well insulated for charge retention.

Because of the inherent problems in constructing a hammer element with the refinements indicated above, an alternate procedure was chosen which allows less critical design. If the transducer can be calibrated at various loading rates, the measured load can be corrected for the effects of charge loss with time.
Calibration

Calibration of the hammer by static loading is accomplished through the rapid removal of known loads. The rate of unloading is read directly from the oscillograph.

Three methods are employed in the calibration procedure to yield unloading rates over a wide range. These are described below:

Method (1)-rapid unloading rates. The hammer element is mounted in a testing machine so that a wire in tension will produce a compressive load on the crystal. The method is described by Figure 12. At the desired load, as indicated by the testing machine, the wire is cut or otherwise released, thus effecting a sudden relief of the load. The average time of unloading by this method is 368 micro-seconds.

Method (2)-intermediate unloading rates. The physical arrangement is similar to that used in Method 1, and is shown in Figure 13. A sheet-metal band is used in place of the wire. Enclosed within the loop of the band is a small wedge positioned so as to spread the loop. If the wedge is sufficiently unstable, a slight twist causes sudden load release. The average unloading time by this method is $15.6 \times 10^{-3}$ seconds.

Method (3)-slow unloading rates. Figure 14 describes the test equipment used for slow release of controlled loads. A hydraulic jack,
Figure 12. Calibration of Hammer at Rapid Unloading Rates.
Figure 13. Calibration of Hammer at Intermediate Unloading Rates.
Figure 14. Calibration of Hammer at Slow Unloading Rates.
equipped with a spring return mechanism, is used to compress the piezoelectric hammer. Since the jack is resting on the bed of the testing machine, the actual load applied to the crystal is registered on the dial of the machine. After the desired load is reached, the relief value on the hydraulic system is opened as quickly as possible. The spring return features provide a consistent rate of unloading. The average time of unloading by this method was $40.65 \times 10^{-3}$ seconds.

Table 6 is constructed from the data obtained by the above three methods of calibration. It may be seen that the range of data is very limited, and a high degree of confidence may be placed in the measurement of loads by this method. Figure 15 describes the calibration curve.

**Associated Instrumentation**

**High Impedance Amplifier**

The measurement of piezoelectric voltages requires a very high impedance voltmeter. A Keithley Model 610 Multi-purpose Electrometer was used as an impedance matching amplifier between the oscilloscope and the piezoelectric hammer.

**Oscilloscope**

The oscilloscope used for all time-base measurements of voltage is a Dumont Model 329-A. The measurements reported in this work are easily within the capabilities of this instrument.
<table>
<thead>
<tr>
<th>Time, seconds</th>
<th>Rapid Pounds per Volt</th>
<th>Intermediate Pounds per Volt</th>
<th>Slow Pounds per Volt</th>
</tr>
</thead>
<tbody>
<tr>
<td>$360 \times 10^{-6}$</td>
<td>23.53</td>
<td>$16.0 \times 10^{-3}$</td>
<td>25.77</td>
</tr>
<tr>
<td>370</td>
<td>24.82</td>
<td>16.0</td>
<td>25.64</td>
</tr>
<tr>
<td>380</td>
<td>25.64</td>
<td>15.0</td>
<td>25.64</td>
</tr>
<tr>
<td>390</td>
<td>26.32</td>
<td>16.0</td>
<td>26.18</td>
</tr>
<tr>
<td>400</td>
<td>26.18</td>
<td>16.0</td>
<td>26.18</td>
</tr>
<tr>
<td>410</td>
<td>27.55</td>
<td>15.0</td>
<td>25.16</td>
</tr>
<tr>
<td>420</td>
<td>28.13</td>
<td>14.0</td>
<td>25.43</td>
</tr>
<tr>
<td>430</td>
<td>28.17</td>
<td>14.5</td>
<td>25.17</td>
</tr>
<tr>
<td>440</td>
<td>28.78</td>
<td>17.0</td>
<td>24.14</td>
</tr>
<tr>
<td>450</td>
<td>28.43</td>
<td>15.0</td>
<td>25.36</td>
</tr>
<tr>
<td>460</td>
<td>27.55</td>
<td>15.6</td>
<td>25.36</td>
</tr>
<tr>
<td>470</td>
<td>27.39</td>
<td>16.0</td>
<td>24.19</td>
</tr>
<tr>
<td>480</td>
<td>25.64</td>
<td>16.0</td>
<td>24.19</td>
</tr>
<tr>
<td>490</td>
<td>28.84</td>
<td>16.0</td>
<td>24.19</td>
</tr>
<tr>
<td>500</td>
<td>28.30</td>
<td>16.0</td>
<td>24.19</td>
</tr>
</tbody>
</table>

**Average: 368.4**

**Standard deviation: 81**
Figure 15. Calibration of Piezoelectric Hammer.
Photographic Recording

The oscillographs were obtained by photographing the sweep triggered by the incoming signal. For this purpose a Dumont Model 271-A 35 m.m. Oscillograph camera was used.

In the early studies it was considered desirable to include energy measurements in the testing program. A camera was therefore used to record the peak swing of the pendulum after the impact and also the movement imparted to the piezoelectric hammer by the impact. An Argus Model C-3 35 m.m. camera was provided for this purpose.

Impact Machine

As shown earlier, the experimental data required for the calculation of impact stress according to equation 2.41 are the duration of impact and the force exerted against the specimen in this time.

The oscillograph resulting from the piezoelectric hammer signal provides these measurements. The machine is pictured in Figure 16. The design utilizes the modification introduced by Maxwell and Rahm (21), in that the specimens are placed on the moving hammer element. The free-swinging hammer as proposed by Southwell (22) is also incorporated to eliminate the grounding energy. The machine described herein thus incorporates all of the refinements developed for energy measuring machines, and at the same time is a force-measuring machine.
Figure 16. The Impact Machine.
The design of the pendulum positioning device allows the machine to be used as an incremental impact machine with the specimen swinging. The hammer element may also be lifted to strike the stationary specimen, as performed in most incremental loading machines.

The specimen holder weighs 24 pounds, and the piezoelectric hammer element weighs 1.81 pounds.

**Static Testing Machine**

The machine is of the lever type, the ratio between the loads on the specimen and at the end of the lever being 10:1.

For normal testing speeds lead shot is allowed to flow into a collecting bucket at the end of the lever. The rate of loading thus obtained is 2.275 pounds per second on the specimen.

The same machine is used to effect different loading rates. Water supplied at a constant head is passed through orifices to provide loads on the test specimen at the rates of 0.144 and 1.040 pounds per second.

**E-Scope**

The apparatus employed to measure the resonant vibration frequency is an E-scope Model E 30 A obtained from the Kinetic Instrument Company, Highland Park, Illinois. The components of this machine are:

1. Audio oscillator, range 60 to 30 K cycles/second.
2. University Model MA-25 loudspeaker converted to provide mechanical drive to the specimen.

3. Frequency standard circuitry based on a 100 Kc quartz crystal. The standard frequency for internal calibration is maintained at 4000 cycles/second.


5. D.C. Milliameter for comparative amplitude measurements.

6. Astatic rochelle salt phonograph pickup.
CHAPTER IV

EXPERIMENTAL PROCEDURE

Measurement of Physical Properties

Modulus of Elasticity

The physical dimensions and modulus of elasticity were determined for each specimen. The sonic testing method was followed, in which the resonant frequency of flexural vibration is determined. The calculation of modulus of elasticity is based on the simplified formula shown by Pickett (23):

\[ E = C_1 W f^2 \]  

where:

- \( E \) = Young's modulus of elasticity
- \( W \) = Weight of the specimen, pounds
- \( f \) = resonant frequency of flexural vibration

\[ C_1 = 4.1632 \frac{1}{d^4} T_1 x 10^{-3} \] for a circular cross section

or

\[ C_1 = \frac{2.4523}{b^3} \frac{1}{t} T_1 x 10^{-3} \] for a rectangular cross section

and

\( T_1 \) = a modification of Goen's Correction Factor.
Values of $T_1$ may be calculated for various ratios of radius of gyration to length and assumed values of Poisson's ratio. Pickett has provided a graphical means of determination of $C_1$ for Poisson's ratio = $1/6$.

Impact Strength

The bars were placed in the specimen holder and allowed to swing into contact with the piezoelectric hammer from an arbitrarily chosen height. The time trace of the signal from the hammer was displayed on the oscillograph and photographed at the instant of impact. Subsequent projection of the picture thus allows voltage-time measurements.

Where energy measurements were desired, a photograph of the upswing of the specimen holder was obtained. This same time exposure recorded the movement of the piezoelectric hammer. The holder and hammer are provided with an outstanding light spot at the uppermost leading edge. The swing is therefore recorded on the exposed film as a streaked arc ending at the top of the swing. At the greatest height achieved by the pendulums, the light spots are in sharp focus.

At the start of a series of impacts the empty holder is swung from the predetermined height and the swing camera is exposed. Both the hammer and holder thus record their respective positions for zero energy extraction. Subsequent photographs of post-impact movements may be compared directly with the calibration shot.
In order to eliminate any parallax effects in this measurement, and to aid in scaling projected slides, a picture of a 1/4-inch grid was taken at the identical distance from the film. It is thereby simple to perform direct measurements of swing height at magnifications of four or five.

Modulus of Rupture

All modulus of rupture determinations were made with the specimen supported as a simple beam with a 5-inch span and concentrated center-point loading. The rate of loading was varied to allow the influence of stress rate to be determined for the body used.

Absorption, Apparent Porosity and Specific Gravity

Pieces resulting from the strength tests were thoroughly dried and then weighed. The ASTM Test Procedure C 373-56 for moisture absorption was followed. The suspended and saturated weights were then used to calculate absorption, apparent porosity, and specific gravity.

Specimen Preparation

Materials

The study is directed toward the measurement of impact strength rather than investigating the variation of this property in numerous ceramics. For this purpose it was necessary to consider only one material for the program. Variations in the physical properties are obtained by testing specimens fired to various degrees of vitrification.
The body used has the following composition:

55% Fluid-energy ground nepheline syenite (commercial grade A-400, American Nepheline Corporation)

5% Potters flint

25% Georgia Pioneer kaolin

15% Bell's dark clay

This was selected in view of previous investigations based on the identical composition (24). The method of preparation, green and dry properties are standardized, and the influence of firing procedure on material properties has been studied in detail (25).

Preparation of Materials

The raw materials were blunged four hours at a specific gravity of 1.50 to 1.52. The resulting slip was filter pressed at 75 p.s.i. The filter cakes were hand-wedged intermittently during a 10 hour drying period on a steel-topped table.

Forming Procedure

Laboratory specimens—when the plasticity appeared proper for extrusion, the body was passed twice through a deairing Fate-Root-Heath Laboratory extrusion machine, followed by a final extrusion at 28 in. Hg. vacuum. The column was cut to form specimens approximately 6 inches long. The die sizes used were 3/4 inch and 1/2 inch diameter.
The above procedure was followed for each batch preparation in batch sizes ranging from 50 pounds to 80 pounds.

All specimens used in the initial study were formed by extrusion.

Commercial specimens— in order to demonstrate the magnitude of impact strength of commercial bodies, a series of cast specimens was obtained from one sanitary ware, two fine china, one electrical porcelain, two hotel china manufacturers. These specimens were formed by slip casting in a set of plaster molds which was transported from plant to plant. Each mold cavity measures 3/8 x 1 x 7 inches.

Firing Procedure

The extruded bars were dried for several days and fired in groups of 20 to various cones in a globar kiln. Whenever bars from a new batch were needed, at least one firing was performed with bars from the preceding batch.

The specimens of commercial bodies were left with the manufacturer and fired along with the ware according to plant practices. The fired specimens were then sent to the author.
CHAPTER V

RESULTS AND DISCUSSION

Stress Calculations

Extruded Bars

Representative oscillographs of the impact are shown in Figure 17. The two groups of photographs describe the profound influence of the hardness of the striking face on the shape of the force-time curve. It is apparent from these curves that the assumption of linear load application is reasonably precise for soft impact, but is not always valid for an impact between hard surfaces. The value of \( k \), the slope of the force-time curve, is taken as the slope of the straight line drawn from the origin to the peak voltage, even in the case of double peaks on the force-time curve.

Stress calculations based on equation 2.56 and Figure 7b for a variety of spans and radii of gyration are shown in Table 7. The impact failure stress is seen to be highly dependent upon the velocity of impact in the case of hard striking surfaces.

The results are also shown graphically in Figure 18. It may be seen that low velocity tests with hard striking surfaces, which exhibit double humps in the force-time curve, result in failure stress values very similar to the static modulus of rupture. Intermediate velocities, with only one peak in the force curve, exhibit lower levels which average approximately 60 per
Figure 17. Representative Oscillographs of the Impact.
TABLE 7  

FAILURE STRESS DATA

<table>
<thead>
<tr>
<th>Firing Cone</th>
<th>06</th>
<th>03</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modulus of Rupture</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.144 lbs./sec.</td>
<td>-----</td>
<td>-----</td>
<td>8600(330)*</td>
<td>7280(420)</td>
</tr>
<tr>
<td>1.040 lbs./sec.</td>
<td>-----</td>
<td>-----</td>
<td>9250(275)</td>
<td>7700(380)</td>
</tr>
<tr>
<td>2.275 lbs./sec.</td>
<td>3400(260)</td>
<td>6700(230)</td>
<td>9890(320)</td>
<td>8950(460)</td>
</tr>
</tbody>
</table>

Hard Striking Face

Impact Stress

<table>
<thead>
<tr>
<th></th>
<th>Low velocity</th>
<th>Intermediate velocity</th>
<th>High Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low velocity</td>
<td>3320(320)</td>
<td>6830(480)</td>
<td>10,600(420)</td>
</tr>
<tr>
<td>Intermediate velocity</td>
<td>990(470)</td>
<td>2550(485)</td>
<td>5100(420)</td>
</tr>
</tbody>
</table>

High Velocity

Soft Striking Face

<table>
<thead>
<tr>
<th></th>
<th>Low velocity</th>
<th>Intermediate velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low velocity</td>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
<td>Intermediate velocity</td>
<td>-----</td>
<td>-----</td>
</tr>
</tbody>
</table>

*All standard deviations are shown in parentheses following the average of ten specimens.
Figure 18. Measured Strength of Body S.
cent of the static modulus of rupture. At higher testing velocities the single peak load occurs in about the same time as in the case of intermediate velocities. However, the force at the peak is greater as the velocity increases.

Figure 19 illustrates the influence of the span used in the impact test. The consistency of the impact stress equation in paralleling the modulus of rupture is evident. The bending stress calculated from the applied load at intermediate velocities is also shown. It should be noted that the percentage change in span is directly reflected as a percentage change in Mc/I.

One interpretation of the oscillating force curve is that the piezoelectric hammer is excited to resonance by the characteristic loading rate of hard surfaces. Investigation of the frequency response characteristics of the hammer within the range 60 to 30 K c.p.s. shows no resonant tendencies.

With the assumption that the oscillations are a reflection of impact conditions, the oscillograph may be interpreted as a series of blows. The elastic rebound of the bar and hammer causes load release, and the forward motion of the impacting bodies applies the force again.

If the specimen is strong enough to sustain the initial loading, then it is subjected to the continuing impulse of the striking hammer until
Figure 19. The Influence of Span on the Measured Strength.
failure or rebound relieves the load. During this latter phase, the re-
bounding of the elastic surfaces promotes high frequency oscillations
superimposed on the general loading curve. The low velocity impacts of
hard surfaces shown in Figure 17 illustrate this type of curve.

As the velocity of the impact increases, the intensity of the
first peak load also increases. If failure is not initiated during the first
load application, but the stress level at the peak load is sufficiently high,
the bar may fail during rebound from the hammer. There is thus no
possibility of recording the actual failure stress under these conditions.
The impact stress calculated from the higher rates of loading approaches
the static modulus of rupture of the material.

It has been demonstrated that the Niederfuhr equation is capable
of reflecting the influence of span in the impact test. The consistent results
obtained with widely different bar sizes illustrate the equally effective
corrections for bar geometry.

The inability of the flexure formula to correct for span effects
and bar cross-section, as well as lack of correlation with modulus of rupture,
is evidence that the simple equation cannot be applied under these conditions.
This implies that all measurements based on energy absorption have no
validity beyond certain critical speeds of testing. The breaking time at which
this occurs is entirely dependent upon the bar geometry, elasticity and
density, as described by the curve of Figure 7b.

Commercial Specimens

The results of impact measurements on various commercial whiteware bodies are shown in Table 8. Since there are no data available regarding the slow-speed strength of these bodies, it is not possible to indicate the reduction of modulus of rupture. The values calculated for Mc/I from the peak load illustrate the magnitude generally discovered in the conventional energy-measuring machine.

The Fracture Process

Shand (26) has shown that brittle fracture may be considered to occur in several distinct steps. The crack propagation velocity in slow speed testing may be rising to a maximum in the first stage following initiation. At a critical velocity the formation of multiple fracture surfaces takes place. If the load application is slow enough to permit relaxation of the specimen, the velocity of crack propagation will reduce. It is reasonable to expect that high rates of load application will minimize relaxation tendencies, so that the initial crack development continues to propagate at maximum velocity.
<table>
<thead>
<tr>
<th>Body</th>
<th>Failure Stress, psi (Low Velocity Impact)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>12,300</td>
</tr>
<tr>
<td>P</td>
<td>25,100</td>
</tr>
<tr>
<td>Q</td>
<td>13,650</td>
</tr>
<tr>
<td>R</td>
<td>18,600</td>
</tr>
<tr>
<td>S</td>
<td>18,800</td>
</tr>
<tr>
<td>T</td>
<td>13,100</td>
</tr>
</tbody>
</table>
A series of experiments was designed to study the progress of fracture. The oscilloscope is equipped with a Z-axis which modulates the beam intensity according to the signal applied at the Z terminal. If a rapid voltage change can be made to occur at the Z-axis coincident with bar failure, then an intensity change indicates the instant of first failure. A strip of conducting paint was applied across the tension side of the specimen, and the circuitry was such that separation of the strip would apply a battery voltage across the Z-axis. Figure 20 pictures the resulting oscillograph. Failure is indicated after the first peak but before any further peak loads.

A General Radio stroboscope type 1530-A was used to obtain photographs of the specimen in various stages of the impact. A pencil line applied on the specimen provided electrical continuity until the instant at which a crack severed the line. The strobe light emitted a flash of approximately 2 micro-seconds duration at that time. Test bars were positioned so that the pencil line was located (1) just above the point of impact, (2) 90 degrees from the point of impact, and (3) 180 degrees from the point of impact. The successive photographs are shown in Figure 21. These pictures indicate that the failure starts on the tension side. The separation is large with the pencil line just above the impact point, while the other positions result in smaller crack development. The fracture is barely visible with the line located at 180 degrees from the point of impact.
Figure 20. Oscillograph Showing Z-axis Energized at Failure.
Figure 21. Progress of Failure.
The strobe light may also be triggered by the signal from the piezoelectric hammer. A sensitivity adjustment provided on the unit allows the flash to coincide with various signal intensities. It was thus possible to photograph test specimens at any desired amplitude of the piezoelectric signal. A series of bars was broken at successive stages of impact by progressively decreasing the sensitivity adjustment. These are pictured in Figure 22. The development of cracking appears to progress from the tension side to the impacted side. All pictures taken with high sensitivity settings show no evidence of cracks.

Further studies were made by utilizing a microphone to energize the light. The stage of impact which is photographed by this means depends on the distance from the point of impact to the microphone. The pictures in Figure 23 show that secondary cracking occurs after the bar is severed. These secondary cracks emanate from the impacted side of the specimen, thus yielding fracture patterns which are reversed with respect to the original fracture.

Final proof of the bending failure is shown in Figure 24. A flaw was positioned on the tension side well away from the point of impact. Complete failure took place with no evidence of damage to the bar at the point of impact.
Figure 22. Impact at Various Levels of Piezoelectric Signal.

a. High sensitivity (Low signal).

b. Medium sensitivity.

c. Low sensitivity (High signal).
a. Microphone close to impact.

b. Microphone 24 inches from impact.

c. Microphone 48 inches from impact.

Figure 23. Impact at Various Times After Collision.
Figure 24. Impact Failure Originating at a Flaw on Tension Side.
CHAPTER VI

CONCLUSIONS

1. The hardness of the striking face strongly influences the rate of load application and the shape of the force-time curve during impact. A soft striking face, such as Teflon, applies a nearly linear load to the ceramic specimen until failure occurs. A hard striking face subjects the bar to one or more blows during the impact. At low testing velocities the hammer and specimen are in contact at the time of failure. At higher testing velocities, the specimen may fail in rebounding from the hammer.

2. The Niedenfuhr equation allows consistent consideration of bar geometry. Widely different spans and cross-sections may be impacted with little change in the failure stress.

3. The effects of inertia and shear resistance to bending may be described by the family of curves in Figure 7b. All usual bar geometries in impact testing have an \( R/L \) less than 0.0375, and may be represented by a single curve in that figure.

4. The computer solutions of the Niedenfuhr equation, as presented in Figure 7b, indicate a critical rate of loading dependent upon \( R, L, E, \) and \( \bar{c} \). If the duration of linear load application is greater than
the critical time for a specific bar, the breaking stress may be calculated from the simple flexure formula and the measured load at failure. It then follows that the flexure curve under these conditions of loading is adequately described by the static curvature.

5. Bending energy considerations are not valid at breaking times shorter than the critical time for the specific bar under test. It is doubtful that any impact test with a hard striking face can lead to accurate stress calculations from energy measurements alone.

6. Failure in impact is initiated on the tension side of the specimen. Secondary rupture may occur from the impacted side as the broken specimen is accelerated. This phenomenon is responsible for fracture surfaces originating from the impacted surface.

7. For the whiteware composition used in the tests, failure stress in impact with uniformly increasing load application is essentially the same as the static failure stress.
CHAPTER VII

REFERENCES


I, William Beattie Shook, was born in Columbus, Ohio, on October 3, 1928. I received my secondary education from Rocky River High School, Rocky River, Ohio, and from the Castle Heights Military Academy, Lebanon, Tennessee. In June, 1953, I received the degree Bachelor of Ceramic Engineering from the Ohio State University, Columbus, Ohio. I was employed at the Engineering Experiment Station as a Research Associate from 1953 to 1956. I was appointed Supervisor of Building Research from 1956 to 1958, and became Supervisor of Ceramic Research in 1958.