This dissertation has been microfilmed exactly as received

McLARNAN, Charles Walter. KINEMATIC SYNTHESIS OF COMPLEX LINKAGES.

The Ohio State University, Ph.D., 1960
Engineering, mechanical

University Microfilms, Inc., Ann Arbor, Michigan
KINEMATIC SYNTHESIS OF COMPLEX LINKAGES

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By


* * * * * *

The Ohio State University
1960

Approved by

[Signature]
Adviser
Department of Mechanical Engineering
ACKNOWLEDGMENTS

The author wishes to acknowledge his indebtedness to his adviser, Professor Walter L. Starkey, and to Dr. Roy F. Reeves and Miss F. Mery Gong of the Numerical Computation Laboratory of The Ohio State University, whose generous assistance have appreciably aided this research. The use of the computational facilities was made possible by financial support from the Numerical Computation Laboratory.

A special debt is owed to his wife, whose patience and complete cooperation made this work possible.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><strong>Statement of the Problem</strong></td>
<td>viii</td>
</tr>
<tr>
<td></td>
<td><strong>Scope of the Problem</strong></td>
<td>ix</td>
</tr>
<tr>
<td></td>
<td><strong>Brief Summary of Results</strong></td>
<td>x</td>
</tr>
<tr>
<td>I</td>
<td><strong>Introduction</strong></td>
<td>1</td>
</tr>
<tr>
<td>II</td>
<td><strong>Description of the Four Complex-Linkage Mechanisms to be Studied</strong></td>
<td>9</td>
</tr>
<tr>
<td>III</td>
<td><strong>The Equations of Position for the Selected Mechanisms</strong></td>
<td>14</td>
</tr>
<tr>
<td>IV</td>
<td><strong>Transformation of the Position Equations by the Elimination of Variables</strong></td>
<td>22</td>
</tr>
<tr>
<td>V</td>
<td><strong>Development of a Method of Solution of the Position Equation Subject to the Precision Point Constraints</strong></td>
<td>31</td>
</tr>
<tr>
<td>VI</td>
<td><strong>Description of a Digital-Computer Program for Solving the Constrained Position Equations</strong></td>
<td>44</td>
</tr>
<tr>
<td>VII</td>
<td><strong>Example Problem</strong></td>
<td>63</td>
</tr>
<tr>
<td>VIII</td>
<td><strong>Summary of Results and Conclusions</strong></td>
<td>77</td>
</tr>
<tr>
<td></td>
<td><strong>Bibliography</strong></td>
<td>83</td>
</tr>
<tr>
<td>Appendix</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td><strong>Calculation of Angular Position of Output Link for Mechanism A</strong></td>
<td>85</td>
</tr>
<tr>
<td>B</td>
<td><strong>Computer Program for Designing Four-Bar Linkages with Five Precision Points</strong></td>
<td>87</td>
</tr>
<tr>
<td>Appendix</td>
<td>FORTRAN Statements for Computer Program to Design Mechanism A with Six Precision Points</td>
<td>Page</td>
</tr>
<tr>
<td>----------</td>
<td>-------------------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>C</td>
<td></td>
<td>108</td>
</tr>
<tr>
<td>D</td>
<td>FORTRAN Statements for Computer Program to Design Mechanism A with Seven Precision Points</td>
<td>111</td>
</tr>
<tr>
<td>E</td>
<td>FORTRAN Statements for Computer Program to Design Mechanism A with Eight Precision Points</td>
<td>114</td>
</tr>
<tr>
<td>F</td>
<td>FORTRAN Statements for Computer Program to Design Mechanism A with Nine Precision Points</td>
<td>118</td>
</tr>
<tr>
<td>G</td>
<td>FORTRAN Statements for Computer Program to Compute Angular Positions of Output Link for Mechanism A</td>
<td>121</td>
</tr>
<tr>
<td></td>
<td>Autobiography</td>
<td>125</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>1</td>
<td>The Four-Bar Linkage as a Function Generator</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>Comparison of Linkage Output $\phi = g(\theta)$ with Given Function $y = f(x)$ for an Example Having Four Precision Points</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>Mechanism A</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>Mechanism B</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td>Mechanism C</td>
<td>10</td>
</tr>
<tr>
<td>6</td>
<td>Mechanism D</td>
<td>10</td>
</tr>
<tr>
<td>7</td>
<td>Representation of Links of Mechanism A by Complex Numbers</td>
<td>15</td>
</tr>
<tr>
<td>8</td>
<td>Complex Representation of Mechanism B</td>
<td>16</td>
</tr>
<tr>
<td>9</td>
<td>Complex Representation of Mechanism C</td>
<td>17</td>
</tr>
<tr>
<td>10</td>
<td>Complex Representation of Mechanism D</td>
<td>18</td>
</tr>
<tr>
<td>11</td>
<td>Sketches of the Six Most Accurate Solutions to the Example Problem</td>
<td>81</td>
</tr>
<tr>
<td>A-1</td>
<td>Mechanism A</td>
<td>85</td>
</tr>
<tr>
<td>B-1</td>
<td>Four-Bar Linkage</td>
<td>88</td>
</tr>
</tbody>
</table>
### LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Partial Derivatives</td>
<td>40</td>
</tr>
<tr>
<td>2</td>
<td>Solutions for Left-Hand Loop of Mechanism A</td>
<td>66</td>
</tr>
<tr>
<td>3</td>
<td>Solutions for Right-Hand Loop of Mechanism A</td>
<td>67</td>
</tr>
<tr>
<td>4</td>
<td>Maximum Values of Error in the Nine Approximations to a Solution</td>
<td>69</td>
</tr>
<tr>
<td>5</td>
<td>Results of Nine Trials with the Six Precision Point Program</td>
<td>71</td>
</tr>
<tr>
<td>6</td>
<td>Dimensions of the Solutions to the Six-Point Program</td>
<td>72</td>
</tr>
<tr>
<td>7</td>
<td>Results of Five Trials with the Seven Precision Point Program</td>
<td>73</td>
</tr>
<tr>
<td>8</td>
<td>Dimensions of the Solutions to the Seven-Point Program</td>
<td>74</td>
</tr>
<tr>
<td>9</td>
<td>Results of Four Trials with the Eight Precision Point Program</td>
<td>75</td>
</tr>
<tr>
<td>10</td>
<td>Dimensions of the Solutions to the Eight-Point Program</td>
<td>76</td>
</tr>
<tr>
<td>11</td>
<td>Summary of Most Accurate Solutions to the Generation of the Function $y = x^2$ for $0 \leq x \leq 1$ by Mechanism A with 90 Degree Rotations of Input and Output Cranks</td>
<td>79</td>
</tr>
</tbody>
</table>
**NOMENCLATURE**

- **a** the length of a link
- **b** auxiliary variable which is a combination of the linkage parameters
- **c** coefficient of the n linear equations in the Newton-Raphson iteration
- **e** base of the natural logarithm system
- **G** dimensionless function of the linkage parameters
- **i, k, j** subscript
- **j** imaginary unit ($\sqrt{-1}$)
- **k_\theta, k_\phi** scale factor relating the range of an angle ($\theta, \phi$) to the range of a given variable ($x, y$)
- **n** the number of precision points used in a given problem
- **N** number of iterations performed
- **r** dimensionless ratio of the linkage dimensions
- **t, v** angles (variable) used to designate the positions of the driver and follower links with respect to their initial positions, radians.
- **x, y, w** independent and dependent variables of the function to be mechanized
- **z** a complex number in the plane of motion of the mechanism
- **\alpha, \lambda** angles (constant) used as linkage dimensions
- **\beta, \gamma, \delta** angles (variable) used to designate the positions of links other than the driver and follower
- **\theta** input angle of the mechanism used for function generation
- **\phi** output angle of the mechanism used for function generation
- **\psi** second input angle for a two-degree-of-freedom linkage
- **z** "is defined to be"
The objectives of this dissertation were to investigate the feasibility of using complex, rigid-body, plane linkages for function generation whereby such linkages are synthesized to produce required functional relationships among the rotations of individual links, and to develop methods which would make it possible for engineers having an interest in accurate function generation to design these mechanisms in a systematic manner.
SCOPE OF THE PROBLEM

The use of the four-bar linkage as a function generator has been studied with successful results in recent years. Analytical as well as graphical methods have been presented which make it possible to mechanize functions of one variable. Freudenstein (11)\(^1\) has shown how a mechanism may be designed which will approximate a given function with minimum error throughout the range of the motion. Allen (3) has studied the use of a two degree of freedom linkage to generate functions of two variables. In order to obtain an accuracy not possible with four-bar linkages, one might naturally expect that the use of more complex mechanisms could result in a better approximation of a given function.

The scope of this dissertation was limited to a study of three six-link mechanisms, each having one degree of freedom, and one seven-link mechanism having two degrees of freedom. This study involved an investigation of the basic equations governing the positions of the links in each of the mechanisms, and the development of a method of design which could permit the synthesis of any of the four mechanisms for function generation.

\(^1\)Numbers in parentheses refer to articles listed in the Bibliography.
BRIEF SUMMARY OF RESULTS

The basic equations relating the input and output motions of four complex linkages were developed. The basic similarity of these equations to each other was noted. A design procedure was developed which was applicable to each of the four mechanisms. The method consisted of choosing the dimensions of a linkage which gave approximately the correct output and then varying those dimensions so as to more nearly approach the proper solution. Four computer programs were developed by which this method was applied to the simplest of the four linkages. This linkage consisted of two four-bar linkages in series. These programs used six, seven, eight, and nine precision points, respectively. A numerical example was solved using these programs on an IBM 704 computer at the Numerical Computation Laboratory of The Ohio State University.

It was found that an appreciable increase in accuracy was possible when a four-bar linkage was replaced by a six-bar linkage. The method of calculation was iterative in nature and did not converge in all cases. Nine trials with the six-point program yielded five solutions. When these five trials were used as the original choices for the seven-point program, four solutions were obtained. These four solutions, when used in the eight-point program, yielded two solutions. No solutions were obtained from the nine-point program.
For reasons discussed in Chapter VIII, not all of the mathematical solutions resulted in useful linkages. Thus the useful results were five mechanisms with six precision points and one mechanism with seven precision points.
CHAPTER I

INTRODUCTION

1.1 USE OF THE FOUR-BAR LINKAGE

The simplicity and versatility of the four-bar linkage have made it a widely used configuration for the mechanical transmission of motion and/or force. Consisting of only three moving parts (two cranks and a connecting link or coupler) with simple pinned connections, it is easy to manufacture. Its motion characteristics can be easily determined if the parts can safely be assumed rigid and if the bearing clearances are negligible.

In spite of the variety of its applications, however, the design of four-bar linkages to accomplish a given objective has often been prohibitively difficult. Methods for the synthesis of linkages have not been developed at the same rate as methods for their analysis. All too frequently the available methods for linkage synthesis have required such a high degree of training and ingenuity that only a few specialists have ever mastered them.

In recent years, however, a number of engineers in this country as well as abroad have focused their attention on problems of this kind. The methods for kinematic synthesis of the four-bar linkage have been improved so that it is now possible to solve a number of problems by systematic methods which require a minimum of inventiveness on the part of the designer.
1.2 KINEMATIC SYNTHESIS

Problems in kinematic synthesis can conveniently be divided into three classes. The first, called path generation, is concerned with finding a mechanism which will cause a point on a moving body to trace a given path. The second, which could be called coupler positioning, is concerned with designing a mechanism such that one of its links moves to a number of specified positions. The third class is concerned with designing a mechanism such that one body, usually a crank or a sliding block, moves in a given relation to the motion of another crank or slider. If both the input link and output link are cranks, the problem is to find a linkage with a given functional relationship between the crank rotations. For this reason this class of problems has been given the designation of "function generation." Figure 1 shows how a four-bar linkage can be used as a function generator, and illustrates some of the terminology described below. This paper discusses the use of complex mechanisms as function generators. The term "complex mechanism" will be used to refer to a mechanism having more than three movable links.

Figure 1. The Four-Bar Linkage as a Function Generator.
1.3 THE PROBLEM OF SYNTHESIS FOR FUNCTION GENERATION AND TERMINOLOGY

In using one of the mechanisms discussed here for function generation we wish to have the relationship between the angular rotation $\phi$ of one crank and the angular rotation of $\Theta$ of another be as much as possible like a given functional relationship between two variables $x$ and $y$ over some range ($x_0$ through $x_{n+1}$) of the independent variable $x$. Usually only a portion of the total range of motion of the cranks is used. The angular position $\Theta$ of the input crank is chosen to represent the variable $x$, and the angular position $\phi$ of the output crank represents the dependent variable $y$. The symbols $x_0$ and $x_{n+1}$ are used to denote the limits of the range of $x$. Corresponding values of the dependent variable are given the symbols $y_0$ and $y_{n+1}$.

It is customary to specify the scale factors $k\Theta$ and $k\phi$ which relate the range of $\Theta$ to the range of $x$ and the range of $\phi$ to the range of $y$. These are generally chosen so that the ranges of $\Theta$ and $\phi$ are between $45^\circ$ and $135^\circ$, greater than $45^\circ$ in order to obtain appreciable link rotations, but not too large because mechanisms with smaller ranges are often easier to synthesize.

The nature of linkage synthesis makes it impossible to get an exact correspondence between a given function $y = f(x)$ and a linkage output $\phi = g(\Theta)$, unless $f(x)$ would happen to be of the same mathematical form as $g(\Theta)$, which would be improbable. The best that can be done is to have the two functions exhibit the desired correspondence at a finite number of points. The number $n$ is used to signify the number of points at which the correspondence is exact. These points are called the "precision points" of the linkage for obvious reasons.

Figure 2 illustrates the relationship of the given function with a
linkage output in a case where four precision points have been obtained.

The functional relationship exhibited by the linkage will generally be in error at all positions other than the \( n \) precision positions, although for a continuous function with little curvature

\[
\begin{align*}
\phi &= G(\Theta) \\
\gamma &= F(x) \\
\end{align*}
\]

\[\begin{array}{ccccccc}
\theta_0 & \theta_1 & \theta_2 & \theta_3 & \theta_4 & \theta_5 & \Theta \\
\end{array}\]

**Figure 2.** Comparison of Linkage Output \( \phi = g(\Theta) \) with Given Function \( \gamma = f(x) \) for an Example Having Four Precision Points.

this error may be so small as to be scarcely detectable. There are a number of reasons why a linkage output might be in error. Manufacturing errors, bearing clearances, and deflection of the members under load are three which are of great practical significance, but which will not be considered here. The errors referred to in this paper are the errors obtained when considering the mechanism as consisting of rigid links of known dimensions being connected with bearings having no clearance.
The values of the input and output angles for the precision point positions are given the subscript $i, i=1,2, \ldots, n$. Thus $\theta_1$ is the lowest value of $\theta$ for which the correspondence is exact, $\theta_2$ is the next, etc. The initial values of these angles, $\theta_1$ and $\phi_1$ are not specified but are determined as a part of the calculations. The values of the independent variable $x_1$ which are to be used for precision points are selected as a preliminary part of the problem.

When the scale factors $k_\theta$ and $k_\phi$ and the precision point locations have been chosen, it is possible to determine the angular positions of both cranks in terms of the initial positions $\theta_1$ and $\phi_1$ by the formulas:

$$\theta_i = \theta_1 + k_\theta (x_i - x_1),$$

and

$$\phi_i = \phi_1 + k_\phi (y_i - y_1).$$

Thus the problem of designing a linkage whose cranks occupy corresponding positions is the same as the problem of designing the linkage so that the cranks undergo $n-1$ successive rotations from some yet-to-be-determined starting position.

1.4 RECENT PROGRESS IN KINEMATIC SYNTHESIS

The use of analytical methods in kinematic synthesis was stimulated greatly by the 1940 publication in Russian of an article by Bloch. This article introduced the applications of complex numbers to plane kinematic mechanisms, and suggested the matrix methods which have been used with success by later investigators.

One of the principal American proponents and developers of
analytical methods has been Professor Ferdinand Freudenstein of Columbia University. Some of his contributions are summarized as follows: In 1955 (8) he presented methods for the synthesis of four-bar linkages as function generators having three, four, or five precision points; in 1958 (11) he presented a discussion of the theoretical errors inherent in linkages used as function generators and a procedure whereby the precision points could be chosen so as to minimize the error; in 1959 (13) he (with G. N. Sandor) presented a method for the synthesis of path generating mechanisms having as many as five precision points on the path to be generated. Another important result of this latest work is that the mathematical techniques presented there could be applied to the synthesis of four-bar linkages as function generators with a great reduction over previous methods in the amount of calculations required for a solution. These calculations were suggested to the author by Professor Freudenstein and are summarized in Appendix B of this dissertation.

Professors Hartenberg and Denavit of Northwestern University have also contributed significantly to the literature on analytical kinematic synthesis, two of their recent papers being a discussion of a matrix-type method of classifying kinematic constraints (5) and a presentation of methods of synthesis of space mechanisms (6).

At the present time, the analytical methods of synthesis of plane, four-bar linkages for function generation permit a maximum of five precision points. It sometimes happens that a sixth or even a seventh precision point occurs between those specified but this is unusual. This number (five) of precision points is a natural
consequence of the number of links of the mechanism as discussed in Chapter II. Mechanisms having more links or more degrees of freedom permit more precision points. Mechanisms which do not move in a plane also permit more precision points because of the additional constraints which must be supplied. Thus Hartenberg and Denavit (6) have obtained as many as six precision points in the synthesis of three-dimensional mechanisms.

Graphical methods for kinematic synthesis are more numerous than analytical and are generally older historically. Since this dissertation presents a method which is predominantly analytical, no attempt is made to completely summarise graphical methods, but a few examples of recent interest should be mentioned. Examples of the fine German publications are Professor Beyer's *Kinematische Getriebesynthese* (4) and Kurt Hain's *Angewandte Getriebeslehre* (14). An American book which gives many concepts and methods of kinematic synthesis is Svoboda's *Computing Mechanisms and Linkages* (16). This book is especially valuable because of the introduction which the author gives to the subject of the approximate mechanization of functions with linkages. The methods are predominantly graphical in nature and are dependent upon the inventiveness and perseverance of the designer. Emphasis is placed on finding the mechanism which gives the "best fit" with respect to the desired input-output relationship and little or no attention is paid to the number or location of "precision points."

A recent paper by Adams (1) utilizes one of the techniques of Svoboda and shows how the linkage dimensions can be varied in a systematic manner to obtain a solution with a minimum (though still a great
deal) of effort.

Recent papers by Allen (2,3) are particularly appropriate to the subject of this dissertation. The first gives graphical methods described by Hain (14) for designing a four-bar linkage with three, four, or five precision points. The second gives graphical techniques for designing linkages with two degrees of freedom to synthesize functions of two variables. The latter problem is treated analytically in this dissertation.

1.5 OBJECTIVES

The objectives of this dissertation were to show how complex, rigid-body, plane linkages might be used to obtain function-generation approximations with more than five precision points. Three six-bar linkages were studied, one of which permitted nine precision points; the others allowing eleven. In addition, a seven-bar linkage having two degrees of freedom was considered. It is hoped that the methods developed here may eventually be applied to this latter linkage so that functions of two variables may be generated with thirteen precision points.

In order to illustrate the application of these procedures, a number of examples were solved using the first-mentioned linkage above. This method of calculation is iterative. Due to the lengthy calculations required, it has been prepared for computation on an IBM 704 computer. At this time only the program for the first linkage is available, but it is hoped that programs for the others will be developed in the near future.
CHAPTER II

DESCRIPTION OF THE FOUR COMPLEX-LINKAGE MECHANISMS TO BE STUDIED

2.1 CONFIGURATION OF THE MECHANISMS

The mechanisms to be studied are shown in Figures 3, 4, 5, and 6 on the following page. Mechanisms A, B, and C are the three most important six-link, pin-connected mechanisms which can be used for function generation. The left-hand portion of each of mechanisms A and B is seen to consist of a four-bar linkage. Linkage A is completed by attaching one end of the remaining two links to a general point on the follower of the first four-bar linkage and the other end to the fixed link. To complete linkage B, the moving end of the last two links is attached to a general point on the coupler.

Mechanism A is basically the simpler of the two since it consists of two four-bar linkages in series. This is demonstrated by the motion of the movable end of the latter two links mentioned above. In mechanism A, this motion is circular, and in mechanism B it is on a coupler curve. Since the coupler curve is of higher order than a circle, this permits more variety of motion of the output crank.

Mechanism C is a kinematic inversion of mechanism B. It is the only one of these mechanisms with only two cranks. While it is not evident at a casual glance, this mechanism has as much flexibility of output motion as mechanism B, but is more difficult to analyze.
Figure 3. Mechanism A.

Figure 4. Mechanism B.

Figure 5. Mechanism C.

Figure 6. Mechanism D.
Mechanisms A, B, and C possess one degree of freedom. By this it is meant that specifying the angular position of just one of the cranks is sufficient to permit the determination of the positions of all other links in the mechanism. Mechanism D, however, has two degrees of freedom, so that the positions of two of the cranks must be specified, whereupon the angular position of the third crank can be found. This makes it possible to use mechanism D to represent functions of two variables, if it should be desired to mechanise a given function \( w = f(x, y) \). Its use as a "three dimensional" function generator makes this linkage potentially the most important of the group considered here.

2.2 NUMBER OF PRECISION POINTS POSSIBLE

2.2.1 Linkage A

The synthesis problem for mechanism A may now be stated as follows: Given a set of corresponding values for \( \theta \) and \( \phi \) (input and output rotations) what linkage parameters \( a_1, a_2, \ldots, a_8 \) and \( a \) should be used, and from what initial positions should we measure the angles \( \theta \) and \( \phi \). At first glance it might seem that there are eleven constants to be determined: eight lengths \( a_1, \ldots, a_8 \); the phase angle \( \alpha \) between the left- and right-hand loops; and the two starting values for the input and output rotations \( \theta_1 \) and \( \phi_1 \). A closer look, however, shows that nine are sufficient to completely describe the mechanism. Since the right and left loops have no element in common, it can be seen that exactly the same motion may be transferred by this mechanism if every link in the left-hand loop is doubled in size, or, for that matter, if any scale model of the left-hand loop is used in place of the one shown. We can
choose the scale factor to be used for the left loop by arbitrarily assigning a length to any one of its links.

A similar discussion could be made for the right-hand loop. Therefore one of its links may also be arbitrarily chosen. In the following work it will be convenient to take the fixed lengths $a_1$ and $a_2$ equal to unity.

It might also seem that we could introduce a new parameter $\lambda$ by placing the center of rotation of the output link somewhere off the line of centers of the other two cranks. This is not true however, because we could then rotate the entire right-hand loop through the angle $-\lambda$, increasing the phase angle $\alpha$ by $\lambda$ and decreasing the initial output angle $\beta_1$ by $\lambda$. The mechanism would then have the output crank center on the original line of centers and would generate exactly the same function of $\beta$ vs. $\theta$.

Thus we have a linkage for which nine parameters must be determined in order to define the mechanism completely. We can therefore expect that a solution to this problem exists if we take nine sets of values of $\theta$ and $\beta$ and write the position equations nine times using a new set of values each time. This will give nine equations with the nine linkage parameters as the unknowns. Thus this linkage can be used for function generation with nine precision points. It can also be used for any lesser number $n$ precision points by assigning arbitrary numerical values to $9-n$ of the linkage parameters.
2.2.2 Linkages B and C

For linkages B and C the parameters which must be determined to specify completely the linkage are the eight lengths $a_1, a_2, \ldots, a_8$, the two angles $\alpha$ and $\lambda$ and the initial values of the input and output angles $\theta_1$ and $\phi_1$.

It is obvious that any scale model of these linkages will also generate the same function. This reduces the number of linkage parameters to eleven. No further reduction in the number of parameters is possible. Therefore eleven precision points are possible with these linkages.

2.2.3 Linkage D

Linkage D has two degrees of freedom. Therefore there must be two inputs $\theta_1$ and $\psi_1$ given for each value of the output angle $\phi_1$.

There are fourteen linkage parameters to be determined for this mechanism. These are the nine lengths $a_1, a_2, \ldots, a_9$ and the angles $\alpha, \lambda, \theta_1, \psi_1$, and $\phi_1$. Since, as for A, B, and C any scale model will do the same job, we can arbitrarily assume a value for one of the lengths. There can be no further reduction in the number of unknowns. Therefore it should be possible to design this linkage to have thirteen precision points.
CHAPTER III

THE EQUATIONS OF POSITION FOR THE SELECTED MECHANISMS

3.1 GENERAL DISCUSSION

The basic equations relating the positions of each of the links in any of the mechanisms above can most conveniently be written in terms of complex numbers. The equations merely state that for any position of the mechanism, the links join to form a closed loop or loops. Each of the above mechanisms contains three loops, two of which are independent. The position equation for a given loop is written by letting each link of that loop be represented in magnitude and direction by a complex number, and noting the relationship that exists between the complex numbers for that loop.

3.2 MECHANISM A

For mechanism A, the complex numbers are defined as shown in Figure 7. Thus the complex numbers of the left-hand loop can be seen to form a quadrilateral whose equation is

\[ z_2 + z_3 = z_1 + z_4, \]  

and the right-hand loop has its equation given by

\[ z_5 + z_6 = z_8 + z_7. \]
Figure 7. Representation of Links of Mechanism A by Complex Numbers.

written in the complex polar notation using the magnitudes and directions defined in Figure 3, equations (1a) and (2a) can be written as

$$a_2 e^{j\theta} + a_3 e^{j\gamma} = a_1 + a_4 e^{j\beta} \quad (3a)$$

and

$$a_5 e^{j(\beta - \alpha)} + a_6 e^{j\delta} = a_8 + a_7 e^{j\phi}, \quad (4a)$$

where $j$ is the imaginary unit $\sqrt{-1}$. 
3.3 MECHANISM B

Figures 8, 9, and 10 show the definition of the complex representations for mechanisms B, C, and D.

For mechanism B, the position equations are

\[ z_2 + z_3 = z_1 + z_4, \]  
\[ z_2 + z_5 + z_6 = z_8 + z_7, \]

or in terms of the parameters from Figure 4,

\[ a_2 e^{j \theta} + a_3 e^{j \phi} = a_1 + a_4 e^{j \psi}, \]  
\[ a_2 e^{j \theta} + a_5 e^{j (\beta + \alpha)} + a_6 e^{j \delta} = a_8 e^{j \lambda} + a_7 e^{j \delta}. \]
3.4 MECHANISM C

For mechanism C the position equations are

\[ z_2 + z_3 = z_1 + z_5 + z_4, \]  
(1c)

and

\[ z_2 + z_6 = z_1 + z_8 + z_7, \]  
(2c)

or,

\[ a_2 e^{i\theta} + a_3 e^{i\beta} = a_1 + a_5 e^{i(\phi + \lambda)} + a_4 e^{i\gamma} \]  
(3c)

and

\[ a_2 e^{i\theta} + a_6 e^{i(\beta + \alpha)} = a_1 + a_8 e^{i\phi} + a_7 e^{i\delta}. \]  
(4c)
3.5 MECHANISM D

For mechanism D, the position equations are

\[ z_2 + z_3 = z_1 + z_5 + z_4, \]  \hspace{1cm} (1d)

and

\[ z_1 + z_5 + z_6 + z_7 = z_9 + z_8, \]  \hspace{1cm} (2d)

or

\[ a_2 e^{j\theta} + a_3 e^{j\gamma} = a_1 + a_5 e^{j\gamma} + a_4 e^{j\beta} \]  \hspace{1cm} (3d)

and

\[ a_1 + a_5 e^{j\gamma} + a_6 e^{j(\beta - \alpha)} + a_7 e^{j\lambda} = a_9 e^{j\lambda} + a_8 e^{j\theta}. \]  \hspace{1cm} (4d)
3.6 **ANALYSIS OF POSITIONS**

To develop a method for synthesizing mechanisms it is necessary first to clarify certain aspects of mechanism analysis. A typical problem involving kinematic analysis of these mechanisms might well be: Given the dimensions of the linkage and the angular position(s) of the input link(s), what is the angular position of the output link. In terms of the symbols previously defined, this problem might read:

For mechanism A of Figure 3: Given \( \alpha_1 \) through \( \alpha_8, \alpha, \) and \( \theta \), find \( \phi \).

For mechanisms B and C of Figures 4 and 5: Given \( \alpha_1 \) through \( \alpha_8, \alpha, \lambda \), and \( \theta \), find \( \phi \).

For mechanism D of Figure 6: Given \( \alpha_1 \) through \( \alpha_9, \alpha, \lambda, \theta \) and \( \psi \), find \( \phi \).

Upon examination of the position equations we find that in addition to the given information we have only four quantities which are not known. These are the angles \( \beta, \gamma, \delta \) and \( \phi \). The output angle \( \phi \) is to be found. The angles \( \beta, \gamma \) and \( \delta \) are variable angles which do not interest the problem solver, and could be eliminated from the equation if desired.

Each of the complex equations is equivalent to two scalar equations. Thus each pair of position equations is equivalent to four scalar equations. In the problem stated above, the angles \( \beta, \gamma, \delta \) and \( \phi \) can be thought of as the four unknowns for which the pair of position equations are to be solved.
To determine the output angle $\phi$ of one of these mechanisms it is usually not convenient to eliminate $\beta$, $\gamma$, and $\delta$ and attempt to solve directly for $\phi$. The positions can be evaluated much more rapidly by dividing the linkage into a series of triangles which are solved by applications of the laws of cosines and sines. The method of calculation used to evaluate the angle $\phi$ for linkage A when the linkage parameters and $\theta$ are given is illustrated in Appendix A. Similar methods would be used for linkages B, C, and D.

A given example of mechanism A could be assembled in any of four different ways with either of the left-hand or right-hand loops open or crossed. Each of these assemblies would give a different value of the output angle $\phi$ for a given input angle $\theta$. Therefore any one of these four values of $\phi$ will satisfy the position equations. This fact that $\phi$ is multiple-valued will be of importance in a discussion of the solutions to the synthesis problems to follow. It will be discussed further in Chapter VIII.

3.7 THE SYNTHESIS PROBLEM

For mechanism A it has been stated that nine precision points are possible. That is, we can design the mechanism to provide eight given rotations $\phi_1$ of the follower for eight given rotations $\theta_1$ of the driver from initial positions $\theta_1$ and $\phi_1$. In general terms, this means that each of the nine rotations ($\theta_1$, $\phi_1$) can be substituted into the position equations. Each substitution results in a new pair of complex equations relating the linkage parameters, the given angles ($\theta_1$, $\phi_1$) and the variable position angles $\beta$, $\gamma$ and $\delta$. Since each pair of complex
position equations is equivalent to four scalar equations, this is equivalent to writing 36 equations with 36 unknowns. The unknowns would be the nine linkage parameters and nine values each of the angles \( \beta \), \( \gamma \), and \( \delta \).

The above paragraph illustrates the general complexity of synthesis problems. The usual approach to this multitude of unknowns is to eliminate some or all of the unwanted angles from the position equations in order to reduce the unknowns to a manageable few.

Freudenstein (13) has shown how these angles may be kept in the basic position equation for a four-bar linkage and evaluated as a part of the computation. The methods discussed here for mechanism A, as well as that proposed for B, C, and D will require that all three angles \( \beta \), \( \gamma \), and \( \delta \) be eliminated from the position equations, with the result that a single position equation remains which relates the linkage parameters and the input and output angles. The method by which the two complex (or four scalar) equations (3a) and (4a) are reduced to a single scalar position equation is described in the next section.

Each of the nine sets of input-output rotations is then substituted into this remaining position equation. For mechanism A this results in nine equations in the nine unknowns \( a_2/a_1 \), \( a_3/a_1 \), \( a_4/a_1 \), \( a_5/a_8 \), \( a_6/a_8 \), \( a_7/a_8 \), \( a \), \( \theta_1 \), and \( \theta_1 \). These equations are non-linear and will be solved by an iterative method.

Mechanisms B, C, and D present precisely the same problems. The greatest difference being in the total number of equations involved. It is proposed that the same general method of solution will apply to all four of these linkages.
CHAPTER IV

TRANSFORMATION OF THE POSITION EQUATIONS

BY THE ELIMINATION OF VARIABLES

4.1 GENERAL METHOD

The procedure for eliminating the angles $\beta$, $\gamma$ and $\delta$ from the original equations is the same for each of the mechanisms A, B, C, and D. The steps will be carried out in detail here for mechanism A and the important results will be indicated for the other mechanisms.

The method can briefly be described as follows: The angle $\gamma$ is eliminated from equation (3) and the angle $\delta$ is eliminated from equation (4). The resulting two equations are then combined to eliminate the angle $\beta$.

4.2 MECHANISM A

To eliminate $\gamma$ from equation (3) we solve equation (3) for the term involving $\gamma$ as follows.

$$a_3 e^{j\gamma} = a_1 - a_2 e^{j\theta} + a_4 e^{j\beta}.$$  

Multiplying each side of the equation above by its complex conjugate gives

$$a_3^2 = a_1^2 + a_2^2 + a_4^2 - a_1 a_2 (e^{j\theta} + e^{-j\theta}) + a_1 a_4 (e^{j\beta} + e^{-j\beta})$$

$$- a_2 a_4 \left[ e^{j(\theta - \beta)} + e^{-j(\theta - \beta)} \right],$$
and since $2 \cos \sigma = e^{i\sigma} + e^{-i\sigma}$, we can write

$$a_3^2 = a_1^2 + a_2^2 + a_4^2 - 2a_1 a_2 \cos \theta + 2a_1 a_4 \cos \beta - 2a_2 a_4 \cos (\theta - \beta). \quad (5a)$$

Dividing by $2a_2 a_4$ and rearranging gives

$$\cos (\theta - \beta) - \frac{a_1}{a_2} \cos \beta = \frac{a_2^2 - a_3^2 + a_4^2}{2a_2 a_4} - \frac{a_1}{a_4} \cos \theta. \quad (6a)$$

Introducing new variables,

$$r_1 = a_1/a_2,$$
$$r_2 = a_1/a_4,$$ and

$$r_3 = (a_2^2 - a_3^2 + a_4^2)/2a_2 a_4,$$

where the symbol $\equiv$ is read as "is defined to be," we have

$$\cos (\theta - \beta) - r_1 \cos \beta = r_3 - r_2 \cos \theta. \quad (7a)$$

Eliminating $\theta$ from equation (4a) in similar fashion gives:

$$a_6^2 = a_8^2 + a_7^2 + a_5^2 - 2a_5 a_8 \cos (\beta - \alpha) + 2a_7 a_8 \cos \beta - 2a_5 a_7 \cos (\beta - \beta + \alpha). \quad (8a)$$

Introducing the new variables

$$r_4 = a_8/a_7,$$
$$r_5 = a_8/a_5,$$
$$r_6 = (a_5^2 - a_6^2 + a_7^2 + a_8^2)/2a_5 a_7$$

(9a)
and rearranging gives
\[ \cos (\theta - \beta + \alpha) + r_4 \cos (\beta - \alpha) = r_6 + r_5 \cos \phi. \] (10a)

Equations (7a) and (10a) can be written in terms of \( \sin \beta \) and \( \cos \beta \) by the use of trigonometric identities in the following manner:

\[
\begin{align*}
\cos \theta \cos \beta + \sin \theta \sin \beta = r_1 \cos \beta - r_2 \\
\cos (\beta + \alpha) \cos \beta + \sin (\beta + \alpha) \sin \beta + r_4 \cos \beta \cos \alpha \\
+ r_4 \sin \beta \sin \alpha = r_6 + r_5 \cos \phi,
\end{align*}
\]

or

\[
\begin{align*}
\cos \beta (\cos \theta - r_1) + \sin \beta (\sin \theta) = r_3 - r_2 \cos \theta \\
\cos \beta [\cos(\beta + \alpha) + r_4 \cos \alpha] + \sin \beta [\sin(\beta + \alpha) + r_4 \sin \alpha] \\
= r_6 + r_5 \cos \phi.
\end{align*}
\]

These equations are of the form:

\[ b_1 \cos \beta + b_2 \sin \beta = b_3 \] (11a)

and

\[ b_4 \cos \beta + b_5 \sin \beta = b_6, \] (12a)

where

\[ b_1 = \cos \theta - r_1, \]
\[ b_2 = \sin \theta, \]
\[ b_3 = r_3 - r_2 \cos \theta, \]
\[ b_4 = \cos (\beta + \alpha) + r_4 \cos \alpha, \] (13a)
\[ b_5 = \sin (\beta + \alpha) + r_4 \sin \alpha, \]

and

\[ b_6 = r_6 + r_5 \cos \phi. \]
Equations (11a) and (12a) can be solved simultaneously for \( \cos \beta \) and \( \sin \beta \) in terms of \( b_1, b_2, ..., b_6 \), to give

\[
\cos \beta = \frac{(b_3b_5 - b_2b_6)}{(b_1b_5 - b_2b_6)}
\]

and

\[
\sin \beta = \frac{(b_1b_6 - b_3b_6)}{(b_1b_5 - b_2b_6)}.
\]

Since \( \cos^2 \beta + \sin^2 \beta = 1 \), we can write

\[
\frac{(b_3b_5 - b_2b_6)^2}{(b_1b_5 - b_2b_6)^2} + \frac{(b_1b_6 - b_3b_6)^2}{(b_1b_5 - b_2b_6)^2} = 1
\]

or

\[
(b_3b_5 - b_2b_6)^2 + (b_1b_6 - b_3b_6)^2 = (b_1b_5 - b_2b_6)^2. \quad (11a)
\]

This is the basic position equation of the mechanism. Since the parameters \( b_1, b_2, ..., b_6 \) are ultimately functions of the nine linkage dimensions the equation is in the desired form.

4.3 MECHANISM B

The corresponding equations for mechanism B are:

\[
a_4^2 = a_1^2 + a_2^2 + a_3^2 - 2a_1a_2 \cos \theta - 2a_1a_3 \cos \beta + 2a_2a_3 \cos (\theta - \beta). \quad (5b)
\]
Introducing the variables,
\[ r_1 = \frac{a_1}{a_2}, \]
\[ r_2 = \frac{a_1}{a_3}, \]
and
\[ r_3 = (a_1^2 + a_2^2 + a_3^2 - a_4^2)/2a_2a_3, \]
we have
\[ -\cos(\theta - \beta) + r_1 \cos \beta = r_3 - r_2 \cos \theta. \] (7b)

Equation (6b) becomes
\[ a_6^2 = a_2^2 + a_5^2 + a_7^2 + a_8^2 + 2a_2a_5 \cos(\theta - \beta - \alpha) \]
\[ -2a_2a_8 \cos(\theta - \lambda) - 2a_2a_7 \cos(\theta - \phi) \]
\[ -2a_5a_8 \cos(\beta + \alpha - \lambda) - 2a_5a_7 \cos(\phi - \beta - \alpha) \]
\[ + 2a_7a_8 \cos(\phi - \lambda) \] (8b)

Defining:
\[ r_4 = \frac{a_5}{a_7}, \]
\[ r_5 = \frac{a_8}{a_7}, \]
\[ r_6 = \frac{a_7}{a_2}, \]
and
\[ r_7 = (a_2^2 + a_5^2 - a_6^2 + a_7^2 + a_8^2)/2a_2a_7 \]
and rearranging gives
\[ -r_4 \cos(\theta - \beta - \alpha) + r_4r_5r_6 \cos(\beta + \alpha - \lambda) + r_4r_5 \cos(\phi - \beta - \alpha) = \]
\[ r_7 - r_5 \cos(\theta - \lambda) - \cos(\theta - \phi) + r_5r_6 \cos(\phi - \lambda) \] (10b)

Taking
\[ b_1 = r_1 - \cos \theta, \]
\[ b_2 = -\sin \theta, \]
\[ b_3 = r_3 - r_2 \cos \theta, \]
\[ b_4 = r_4 \{r_6[\cos(\phi - \alpha) + r_5 \cos(\lambda - \alpha)] - \cos(\theta - \alpha)}, \] (13b)
\[ b_5 = r_4 \left\{ r_6 \left[ \sin(\theta - \phi) + r_5 \sin(\lambda - \phi) \right] - \sin(\theta - \phi) \right\} \]

and

\[ b_6 = r_7 - r_5 \cos(\theta - \lambda) - \cos(\theta - \beta) + r_5 r_6 \cos(\phi - \lambda) \]

and, as before

\[ (b_3 b_5 - b_2 b_6)^2 + (b_1 b_6 - b_3 b_4)^2 = (b_1 b_5 - b_2 b_4)^2 \quad (14b) \]

4.4 MECHANISM C

For mechanism C, the equations are

\[ a_4^2 = a_1^2 + a_2^2 + a_3^2 + a_5^2 - 2a_1 a_2 \cos \theta - 2a_1 a_3 \cos \beta + 2a_1 a_5 \cos(\phi + \lambda) + 2a_2 a_3 \cos(\phi - \beta) - 2a_2 a_5 \cos(\phi - \beta - \lambda) - 2a_3 a_5 \cos(\phi - \beta + \lambda) \quad (5c) \]

Defining:

\[ r_1 = a_1/a_2 , \]
\[ r_2 = a_3/a_2 , \]
\[ r_3 = a_2/a_5 , \]

and

\[ r_4 = (a_1^2 + a_2^2 + a_3^2 - a_4^2 + a_5^2)/2a_2 a_5 , \]

and rearranging gives

\[ r_1 r_2 r_3 \cos \beta - r_2 r_3 \cos(\theta - \beta) + r_2 \cos(\phi - \beta + \lambda) = \]
\[ r_4 - r_1 r_3 \cos \theta + r_1 \cos(\phi + \lambda) - \cos(\phi - \beta - \lambda). \quad (7c) \]

Similar operation on equation (4c) gives

\[ a_7^2 = a_1^2 + a_2^2 + a_6^2 + a_8^2 - 2a_1 a_2 \cos \theta - 2a_1 a_6 \cos(\beta + \alpha) + 2a_1 a_8 \cos \phi + 2a_2 a_6 \cos(\theta - \beta - \alpha) - 2a_2 a_8 \cos(\theta - \phi) - 2a_6 a_8 \cos(\phi - \beta - \alpha). \quad (8c) \]
Defining

\[ r_5 = \frac{a_6}{a_2}, \]
\[ r_6 = \frac{a_2}{a_8}, \]

and

\[ r_7 = \frac{(a_1^2 + a_2^2 + a_6^2 - a_7^2 + a_8^2)/2a_2a_8} \]

and rearranging gives

\[ r_1r_6 \cos(\beta + \alpha) - r_5r_6\cos(\theta - \beta - \alpha) + r_5 \cos(\phi - \beta - \alpha) = r_7 - r_1r_6 \cos \theta + r_1 \cos \phi - \cos(\theta - \phi). \] (10c)

Then taking

\[ b_1 = r_2 \left\{ \cos(\phi + \lambda) + r_3 \left[ r_1 - \cos \theta \right] \right\}, \]
\[ b_2 = r_2 \left\{ \sin(\phi + \lambda) - r_3 \sin \theta \right\}, \]
\[ b_3 = r_4 - r_1r_3 \cos \theta + r_1 \cos(\phi + \lambda) - \cos(\theta - \phi - \lambda), \]
\[ b_4 = r_5 \left\{ \cos(\phi - \alpha) - r_3 \left[ \cos(\theta - a) - r_1 \cos \alpha \right] \right\}, \] (13c)
\[ b_5 = r_5 \left\{ \sin(\phi - \alpha) - r_6 \left[ \sin(\theta - a) + r_1 \sin \alpha \right] \right\}, \]

and

\[ b_6 = r_7 - r_1r_6 \cos \theta + r_1 \cos \phi - \cos(\theta - \phi) \]
gives, as before,

\[ (b_3b_5 - b_2b_6)^2 + (b_1b_6 - b_3b_4)^2 = (b_1b_5 - b_2b_4)^2. \] (14c)

4.5 MECHANISM D

For mechanism D, the equations are

\[ a_3^2 = a_1^2 + a_2^2 + a_4^2 + a_5^2 - 2a_1a_2 \cos \theta + 2a_1a_5 \cos \Psi + 2a_4a_5 \cos \phi \]
\[ - 2a_2a_5 \cos(\theta - \Psi) - 2a_2a_4 \cos(\theta - \beta) + 2a_1a_5 \cos(\Psi - \beta). \] (5d)
Defining:
\[ \begin{align*}
  r_1 & = a_1/a_5, \\
  r_2 & = a_4/a_5, \\
  r_3 & = a_9/a_2, \\
  r_4 & = (a_1^2 + a_2^2 - a_3^2 + a_4^2 + a_5^2)/2a_2a_5,
\end{align*} \]
and rearranging gives
\[ -r_1r_2r_3 \cos \beta + r_2 \cos(\theta - \beta) - r_2r_3 \cos(\psi - \beta) = r_4 - r_1 \cos \theta + r_1r_3 \cos \psi - \cos(\theta - \psi). \quad (7d) \]

Operating similarly on equation (4d) we get
\[ a_7^2 = a_1^2 + a_2^2 + a_6^2 + a_8^2 + a_9^2 + 2a_1a_5 \cos \psi + 2a_1a_6 \cos(\beta - \alpha) - 2a_1a_9 \cos \lambda - 2a_1a_8 \cos \beta + 2a_5a_6 \cos(\psi - \beta + \alpha) - 2a_5a_9 \cos(\psi - \lambda) - 2a_6a_9 \cos(\psi - \alpha - \lambda) - 2a_6a_8 \cos(\psi - \beta + \alpha) + 2a_8a_9 \cos(\psi - \lambda) \quad (8d) \]

Defining:
\[ \begin{align*}
  r_5 & = a_6/a_5, \\
  r_6 & = a_9/a_5, \\
  r_7 & = a_9/a_8, \\
  r_8 & = (a_1^2 + a_2^2 + a_6^2 - a_7^2 + a_8^2 + a_9^2)/2a_5a_8,
\end{align*} \]
and rearranging gives
\[ -r_1r_5r_7 \cos(\beta - \alpha) - r_5r_7 \cos(\psi - \beta + \alpha) + r_5r_6r_7 \cos(\beta - \alpha - \lambda) + r_5 \cos(\beta - \beta + \alpha) = r_8 + r_1r_7 \cos \psi - r_1r_5r_7 \cos \lambda - r_1 \cos \theta - r_6 \cos(\psi - \lambda). \quad (10d) \]
Then taking

\[ b_1 = r_2 \left[ \cos \theta - r_3 \left( \cos \phi + r_1 \right) \right], \]

\[ b_2 = r_2 \left( \sin \theta - r_3 \sin \psi \right), \]

\[ b_3 = r_4 - r_1 \cos \theta + r_1 r_3 \cos \phi - \cos(\theta - \psi), \]

\[ b_4 = r_5 \{ \cos(\phi + a) + r_7 [r_6 \cos(a + \lambda) - r_1 \cos a - \cos(\psi + a)] \}, \]

\[ b_5 = r_5 \{ \sin(\phi + a) + r_7 [r_6 \sin(a + \lambda) - r_1 \sin a - \sin(\psi + a)] \}, \]

and

\[ b_6 = r_8 + r_1 r_7 \cos \psi - r_1 r_8 \cos \lambda - r_1 \cos \phi \]

\[ - r_6 r_7 \cos(\psi - \lambda) - \cos(\psi - \phi) + r_6 \cos(\phi - \lambda) \]

gives, as before,

\[ (b_3 b_5 - b_2 b_6)^2 + (b_1 b_6 - b_3 b_4)^2 = (b_1 b_5 - b_2 b_4)^2. \]
CHAPTER V

DEVELOPMENT OF A METHOD OF SOLUTION OF THE POSITION EQUATION
SUBJECT TO THE PRECISION POINT CONSTRAINTS

5.1 THE BASIC METHOD

The following discussion applies only to the equations for mechanism A. It is applicable, however, with obvious modifications to mechanisms B and C. Mechanism D will require special modifications discussed in Section 5.3.

If the position equation (14a) for mechanism A is written in terms of the linkage constants \( r_1, r_2, \ldots, r_6 \) and \( \alpha \), and the variable angles \( \Theta \) and \( \Phi \), it becomes excessively unwieldy. It can, however, be thought of as a nonlinear algebraic equation relating those nine parameters.

For convenience in writing the equations which follow, we will define new symbols for the linkage parameters \( \alpha, \Theta_1, \) and \( \Phi_1 \). Since the symbol \( r \) has been chosen to represent the ratios of the linkage lengths it will be desirable to use the same designation for the remainder of the linkage parameters. Thus in the work which follows we will use

\[
\begin{align*}
  r_7 &= \alpha, \text{ radians}, \\
  r_8 &= \Theta_1, \text{ radians}, \\
  r_9 &= \Phi_1, \text{ radians}. 
\end{align*}
\]
Mechanism A permits motion synthesis with as many as nine precision points. It will also permit synthesis with any lesser number. Since it is possible to design four-bar linkages with five precision points, interest in mechanism A centers around its use with six or more points.

To design mechanism A with \( n \) precision points, values are assigned to \( 9 - n \) of the linkage parameters \( r_1, \ldots, r_9 \). Thus if we wish to design for seven precision points, \( r_8 \) and \( r_9 \) would be given numerical values, and the position equation would then contain only seven (or in the general case \( n \)) unknown values.

Next the \( n \) precision point values of the independent variable \( x \) can be chosen. These are the values of the independent variable at which the correspondence between the given function and the generated function is to be exact. These values are labeled \( x_i \) for \( i = 1, 2, \ldots, n \). From these values of \( x \), corresponding values of the dependent variable \( y \) are calculated from the given function so that \( y_i = f(x_i) \).

At these \( n \) precision points the relationship between \( \theta \) and \( x \) and \( \phi \) and \( y \) is to be

\[
\theta_1 = r_8 + k_\theta (x_1 - x_1)
\]

and

\[
\phi_1 = r_9 + k_\phi (y_1 - y_1).
\]

The angles \( k_\theta (x_1 - x_1) \) and \( k_\phi (y_1 - y_1) \) will be given the symbols \( t_1 \) and \( v_1 \) respectively. Hence,

\[
\theta_1 = r_8 + t_1
\]

and

\[
\phi_1 = r_9 + v_1
\]
Since the scale factors \( k_0 \) and \( k_f \) were chosen earlier, the \( n \) values of \( t_1 \) and \( v_1 \) are known. If a set of values of \( \theta_1 \) and \( \phi_1 \) is substituted into equation (14a), the result is a nonlinear algebraic equation which relates the \( n \) unknowns \( r_1, r_2, ..., r_n \). Substituting each of the sets of values of \( \theta_1 \) and \( \phi_1 \) into equation (14a) gives \( n \) equations in the \( n \) unknowns \( r_1, r_2, ..., r_n \). Mathematically then the problem is one of solving the set of \( n \) nonlinear algebraic equations in \( n \) unknowns.

The author has studied these equations in their expanded form with the hope that some rearrangement of the variables or other transformation might put them into more convenient form for solution. Such efforts have not been successful.

A number of iterative methods are available, however, which will permit the evaluation of a single solution to such a set of equations if a suitable approximation of the solution is available. The Newton-Raphson method of iteration (15) has been selected as the method to be used for the refinement of a solution. The work of Freudenstein (10), Freudenstein and Sandor (12,13) and Svoboda (16) has been used in order to find a first approximation to a solution.

5.2 THE FIRST APPROXIMATION

The initial estimate of a solution of a problem was obtained with the aid of a five precision point approximation of the desired function with a four-bar linkage. Two uses could be made of this five-point approximation. The simpler method could be used whenever the ranges of \( \theta \) and \( \phi \) were equal. It consisted of using identical linkages
for the left- and right-hand loops of the mechanism. This meant that the function generated by either the left-hand linkage or the right-hand linkage was not the function ultimately desired, but rather the "square root" of that function as described by Svoboda (16).

The dimensions of the "square-root generator" could be obtained by either of the methods mentioned below for determining four-bar linkage dimensions. When the dimensions $a_1$, $a_2$, $a_3$, $a_4$, $\theta_1$, and $\phi_1$ for the basic four-bar were obtained, the right-hand loop would then be formed by taking $a_5 = a_2$, $a_6 = a_3$, $a_7 = a_4$, $a_8 = a_1$, and $a = \phi_1 - \theta_1$. The nine initial values $r_1$, $r_2$, ..., $r_9$ could then be determined and the iteration begun.

The second method consisted of using different four-bar linkages for the two loops of the mechanism. There are a number of ways in which this could be done. If the ranges of $\theta$ and $\phi$ were equal, and if more than one solution resulted from the synthesis of the four-bar linkage for the "square-root" of the desired function, two of these different four-bar linkages could be used for the left- and right-hand loops of the six-bar linkage. It might be that these four-bar linkages could be chosen so that their errors between precision points tended to cancel each other, thus giving as a first approximation a linkage which generated the desired function with more accuracy than possible with two identical four-bar linkages.

Another method which could be used would be to "factor" the

---

1 First suggested in a private communication by Dr. T.P. Goodman, General Electric Company.
desired function in some other manner than by finding its "square root." For example, the function of \( y = x^6 \) could be first approximated by taking for the left-hand loop a linkage which approximated \( y = x^2 \) and for the right-hand loop a linkage which approximated \( y = x^3 \).

If the ranges of \( \theta \) and \( \phi \) are not equal it is difficult to use identical linkages for both sides of the mechanism. In such a case it would be simpler to design the left four-bar linkage such that its input crank moved with the same range as \( \theta \) and its output crank moved with a range intermediate between the ranges of \( \theta \) and \( \phi \). The function to be generated by this left-hand loop could be some intermediate function between \( y = x \) and \( y = f(x) \). The right-hand linkage could subsequently be designed so that its input crank has the same motion as the output crank of the left-hand linkage and its output crank moves to the desired angular position \( \phi_1 \).

The attempt in each of the methods described above has been to produce a six-bar linkage having five precision points with respect to the desired function. One other method of doing this has not proved to be successful. If the left-hand linkage generates the desired function with five precision points and the right-hand loop is a parallelogram, the above-mentioned result is attained. If this method is used, however, the system of linear equations to be solved in the Newton-Raphson iteration contains one column which is the negative of another. This results from the fact that the links labeled \( a_5 \) and \( a_7 \) are parallel throughout the entire range of the motion. This causes the determinant of the coefficients of the set of equations to be zero and makes the equations unsolvable.
The four-bar linkages for the five-point approximations were obtained with the aid of an IBM 704 Computer at The Ohio State University Numerical Computation Laboratory. The program used in the calculation of their dimensions is based on the method developed by Sandor and Freudenstein (12, 13). The development of the equations is summarized and the program is written in full in Appendix B.

The precision point values chosen for the four-bar linkages in the example problem are described in Chapter VII.

The determination of the initial approximation is, in the opinion of the author, the weakest part of this entire method. The question of when the iteration will converge to a solution from a given approximation is as yet unanswered. In a troublesome problem, when the methods described above do not lead to a solution, the ingenuity of the designer in choosing additional approximations will be the determining factor as to whether or not a solution can be found.

5.3 THE FIRST APPROXIMATION FOR MECHANISM D

Since linkage D has two degrees of freedom, a four-bar linkage can no longer be used to approximate the function to be generated. A four-bar linkage can, however, be used to represent the relation between the output and one of the input variables if the other input variable is held fixed. Therefore, if we fix the input link a5 (hold \( V = V_c \)), then we could design the remainder of the linkage as described in the preceding section. This would result in a linkage having five precision points with respect to the function \( z = f(x, y_c) \).

We could then fix the input crank a2 at some convenient
position \( \theta_c \), and determine the location of the center of rotation of crank \( a_5 \) in order to approximate the proper relationship between the linkage output and the function \( z = f(x_c, y) \). This basically is the procedure described by Allen (3) and will not be discussed further at this time.

An alternative procedure would be to complete the design of a mechanism of the type of mechanism A in order to obtain more than five precision points with the function \( z = f(x, y_c) \). This might be practical if the relationship between \( z \) and \( x \) is of paramount importance and the different values of \( y \) to be used are merely to provide small corrections in the output motion. Allen's method could then be used to obtain an original estimate for the remainder of the linkage.

5.4 THE APPLICATION OF THE NEWTON-RAPHSON ITERATION

In using the Newton-Raphson iteration in the solution of the \( n \) position equations (14), we define

\[
G(r_1, r_2, \ldots, r_9, t, v) = (b_3b_5 - b_2b_6)^2 + (b_1b_6 - b_3b_4)^2
- (b_1b_5 - b_2b_4)^2
\]

Since the parameters \( b_1 \) through \( b_6 \) are functions of the angles \( \theta \) and \( \phi \) as well as the linkage parameters \( r_1 \) through \( r_6 \), for the precision point calculations, the system of \( n \) equations can be written

\[
G_i(r_1, r_2, \ldots, r_9, t_i, v_1) = 0 \quad i = 1, 2, \ldots, n
\]
This method requires evaluation of $G_1$ and its partial derivatives with respect to the linkage parameters $r_1$ through $r_n$.

The function and its derivatives are computed using the values of $r_1$, $r_2$, ..., $r_9$ which were chosen as the first assumption of the solution.

The system of equations

$$
\frac{\delta G_1}{\delta r_1} \Delta r_1 + \frac{\delta G_1}{\delta r_2} \Delta r_2 + \cdots + \frac{\delta G_1}{\delta r_n} \Delta r_n = -G_1, \quad i = 1, 2, \ldots, n \quad (15)
$$

is then solved for the "correction factors" $\Delta r_1, \Delta r_2, \ldots, \Delta r_n$. The original values of $r_1$, $r_2$, etc. are amended by taking

$$
\begin{align*}
    r'_1 &= r_1 + \Delta r_1 \\
    r'_2 &= r_2 + \Delta r_2 \\
    \vdots & \quad \vdots \\
    r'_n &= r_n + \Delta r_n
\end{align*}
$$

and the calculation is repeated using the new values $r'_1, r'_2, \ldots, r'_n$ until the "correction factors" become sufficiently small or until it is obvious that the "correction factors" are not going to become small. If the original choices of the parameters $r$ are wisely made, the iteration should converge to a solution after only a few trials. If the original choice is unwise, a solution might never be obtained.

5.5 EVALUATION OF $G$ AND ITS DERIVATIVES

As an aid to the writing of the equations, let us define

$$
\begin{align*}
    b_7 &= b_3b_5 - b_2b_6, \\
    b_8 &= b_1b_6 - b_3b_4, \\
    b_9 &= b_1b_5 - b_2b_4
\end{align*}
$$

and

$$
\begin{align*}
\end{align*}
$$
Since the parameters $b_1, b_2, \ldots, b_9$ are functions of $\theta$ and $\phi$, they should also carry the subscript $i$ when used in the precision point calculations. Thus $b_6$, for example, shall later be written as $b_{i6}$ where the $i$ distinguishes the values of $\theta_i$ and $\phi_i$ used in its evaluation.

As a result of the above definition we have

$$G = b_7^2 + b_8^2 + b_9^2,$$

and therefore

$$\frac{\partial G}{\partial r_k} = 2b_7 \frac{\partial b_7}{\partial r_k} + 2b_8 \frac{\partial b_8}{\partial r_k} - 2b_9 \frac{\partial b_9}{\partial r_k} \quad (18)$$

Now

$$\frac{\partial b_7}{\partial r_k} = b_3 \frac{\partial b_5}{\partial r_k} + b_5 \frac{\partial b_3}{\partial r_k} - b_2 \frac{\partial b_6}{\partial r_k} - b_6 \frac{\partial b_2}{\partial r_k},$$

$$\frac{\partial b_8}{\partial r_k} = b_1 \frac{\partial b_6}{\partial r_k} + b_6 \frac{\partial b_1}{\partial r_k} - b_3 \frac{\partial b_4}{\partial r_k} - b_4 \frac{\partial b_3}{\partial r_k}, \quad (19)$$

and

$$\frac{\partial b_9}{\partial r_k} = b_1 \frac{\partial b_5}{\partial r_k} + b_5 \frac{\partial b_1}{\partial r_k} - b_2 \frac{\partial b_4}{\partial r_k} - b_4 \frac{\partial b_2}{\partial r_k}.$$

The partial derivatives $\frac{\partial b_j}{\partial r_k} (j = 1, \ldots, 6; k = 1, \ldots, 9)$ can best be expressed with the aid of Table 1 below. This table was prepared from the definitions of $b_1, b_2, \ldots, b_6$ in Equations (13a), remembering that

$$\psi_1 = r_8 + t(i),$$

and

$$\phi_1 = r_9 + v(i),$$

and

$$a = r_7.$$
### TABLE 1

**PARTIAL DERIVATIVES**

<table>
<thead>
<tr>
<th></th>
<th>( b_1 )</th>
<th>( b_2 )</th>
<th>( b_3 )</th>
<th>( b_4 )</th>
<th>( b_5 )</th>
<th>( b_6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\partial}{\partial r_1} )</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{\partial}{\partial r_2} )</td>
<td>0</td>
<td>0</td>
<td>-( \cos \theta )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{\partial}{\partial r_3} )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{\partial}{\partial r_4} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \cos \alpha )</td>
<td>( \sin \alpha )</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{\partial}{\partial r_5} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \cos \phi )</td>
</tr>
<tr>
<td>( \frac{\partial}{\partial r_6} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( \frac{\partial}{\partial r_7} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-( b_5 )</td>
<td>( b_6 )</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{\partial}{\partial r_8} )</td>
<td>-( \sin \theta )</td>
<td>( \cos \theta )</td>
<td>( r_2 \sin \theta )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{\partial}{\partial r_9} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-( \sin(\phi+a) )</td>
<td>( \cos(\phi+a) )</td>
<td>-( r_5 \sin \phi )</td>
</tr>
</tbody>
</table>
Therefore,

\[(1/2)(dG/d\theta_1) = -b_6b_6 + b_9b_9,\]

\[(1/2)(dG/d\theta_2) = -b_7b_5 \cos \theta + b_8b_4 \cos \theta,\]

\[(1/2)(dG/d\theta_3) = b_7b_5 - b_8b_4,\]

\[(1/2)(dG/d\theta_4) = b_7b_3 \sin \alpha - b_8b_3 \cos \alpha - b_9b_1 \sin \alpha + b_9b_2 \cos \alpha,\]

\[(1/2)(dG/d\theta_5) = -b_7b_2 \cos \beta + b_8b_1 \cos \beta,\]

\[(1/2)(dG/d\theta_6) = -b_7b_2 + b_8b_1,\]

\[(1/2)(dG/d\theta_7) = b_7b_4b_4 + b_8b_3b_5 - b_9b_1b_4 - b_9b_2b_5,\]

\[(1/2)(dG/d\theta_8) = b_7b_5r_2 \sin \theta - b_7b_6 \cos \theta - b_8b_6 \sin \theta - b_8b_4r_2 \sin \theta + b_9b_5 \sin \theta + b_9b_4 \cos \theta,\]

and

\[(1/2)(dG/d\theta_9) = b_7b_3 \cos(\beta + \alpha) + b_7b_2r_5 \sin \beta - b_8b_1r_5 \sin \beta + b_8b_3 \sin(\beta + \alpha) - b_9b_1 \cos(\beta + \alpha) - b_9b_2 \sin(\beta + \alpha).\]

5.6 **ARRANGEMENT FOR COMPUTATION**

For the precision point calculations, the above derivatives must be evaluated for each pair of values \(\theta_1, \phi_1\). In order to avoid repetitious writing of 2's, equation (15) was divided by 2. The coefficients of the resulting equation were designated as \(C_{ij}\), where

\[C_{ij} = (1/2)(dG_i/d\theta_j) \quad i = 1, \ldots, n\]

\[j = 1, \ldots, n\]

and

\[C_{i,n+1} = -C_i/2 \quad i = 1, \ldots, n\]
The set of \( n \) linear equations to be solved for the correction factors \( \Delta r_1, \Delta r_2, \ldots, \Delta r_n \) then can be written

\[
\sum_{j=1}^{n} C_{ij} \Delta r_j = C_{i,n+1} \quad i = 1, 2, \ldots, n \tag{21}
\]

The equations by which these coefficients \( C_{ij} \) can be evaluated for \( n = 9 \) are given below, where the substitutions

\[
a = \tau \gamma, \\
\theta = \tau + t, \\
\phi = \tau + v,
\]

and

\[
\theta = \tau + v, \\
\phi = \tau + v,
\]

have been made. As a convenience the equations (13a) and (17) have been similarly transformed and are repeated.

\[
t_1 = k_9(x_1 - x), \\
v_1 = k_9(y_1 - y),
\]

\[
b_{11} = \cos(\tau + t) - r, \\
b_{12} = \sin(\tau + t), \\
b_{13} = r_3 - r_2 \cos(\tau + t), \\
b_{14} = \cos(\tau + r + v) + r_4 \cos \tau r, \\
b_{15} = \sin(\tau + r + v) + r_4 \sin \tau r, \\
b_{16} = r_6 + r_5 \cos(r + v), \\
b_{17} = b_{13}b_{15} - b_{12}b_{16}, \\
b_{18} = b_{11}b_{16} - b_{13}b_{14},
\]

\[
(22)
\]

\[
(23)
\]
\[
\begin{align*}
    b_{19} &= b_{11}b_{15} - b_{12}b_{14}, \\
    c_{11} &= b_{15}b_{19} - b_{16}b_{18}, \\
    c_{12} &= (b_{14}b_{18} - b_{15}b_{17}) \cos(r_8 + t_1), \\
    c_{13} &= b_{15}b_{17} - b_{14}b_{18}, \\
    c_{14} &= (b_{13}b_{17} - b_{11}b_{19}) \sin r_7 + (b_{12}b_{19} - b_{13}b_{18}) \cos r_7, \\
    c_{15} &= (b_{11}b_{18} - b_{12}b_{17}) \cos(r_9 + v_1), \\
    c_{16} &= b_{11}b_{18} - b_{12}b_{17}, \\
    c_{17} &= (b_{13}b_{17} - b_{11}b_{19})b_{14} + (b_{13}b_{18} - b_{12}b_{19})b_{15}, \\
    c_{18} &= \left[ (b_{15}b_{17} - b_{14}b_{18})r_2 + (b_{15}b_{19} - b_{16}b_{18}) \right] \sin(r_8 + t_1) \\
    &\quad + (b_{14}b_{19} - b_{16}b_{17}) \cos(r_8 + t_1), \\
    c_{19} &= (b_{13}b_{17} - b_{11}b_{19}) \cos(r_7 + r_9 + v_1) \\
    &\quad + (b_{13}b_{18} - b_{12}b_{19}) \sin(r_7 + r_9 + v_1) \\
    &\quad + (b_{12}b_{17} - b_{11}b_{18})r_5 \sin(r_9 + v_1), \\
    c_{1,10} &= (1/2)(b_{19}^2 - b_{18}^2 - b_{17}^2).
\end{align*}
\]

For \( n < 9 \), all coefficients through \( c_{1,n} \) (including \( b_{17}, b_{18}, \) and \( b_{19} \)) will be calculated as given. The coefficient \( c_{1,n+1} \) will be replaced by \( c_{1,10} \) above, and all other \( C \)'s will be disregarded.

When the \( n(n+1) \) coefficients \( c_{ij} \) have been calculated the system of equations (21) can be solved. If the solution is to be made with aid of a desk calculator, equations (24) could be slightly modified to provide some reduction in effort. The author has prepared this problem for solution by an IBM Computer. The computer program in FORTRAN language is presented in Appendix C. The calculations performed in the operation of this program are summarized in the following chapter.
6.1 GENERAL DISCUSSION

As mentioned previously, four computer programs were developed, one each for designing mechanism A to generate a function with six, seven, eight or nine precision points. Each of these programs performs the same general operations, which are described below. Input to and output from each of these programs is slightly different and will be described in Sections 6-2 through 6-5. A fifth program was also developed by means of which the output of an example of mechanism A can be calculated. This fifth program is described in Section 6-6.

The program instructions for all of these programs are given in Appendices C through G. The equivalent machine instructions have not been included because of their length and because the FORTRAN statements are much easier for a reader to follow. Persons not familiar with the FORTRAN language are referred to International Business Machines Corporation publications: "Programmers Primer for FORTRAN" (Form 32-0306-1) and "Programmer's Reference Manual for FORTRAN" (Form 32-7026).

The essential difference in the four programs is that each program is set up to solve a different number of equations. In the six-point program, the numerical values of the linkage parameters
r7, r8, and r9 are held constant and the program is set up to solve six simultaneous equations with the six unknowns r1, r2, ..., r6. In the seven-point program, r8 and r9 are held constant, and in the eight-point program r9 is held constant.

The following statements are applicable to all four major programs. The input to each of the first four programs must give all information required to perform the calculations described in Chapter V. All input is by punched IBM cards. The first few (four or six) cards give the precision point values and the ranges of the given variables x and y. The next two cards give the nine dimensions of the linkage which is the initial approximation of the solution to the problem. The next card, which is the last, contains five numbers. The first is the range of the mechanism input angle θ. The second is the range of the output angle φ. The third is a small number ε1 which is used in an instruction to stop the iteration if the correction factors Δr become sufficiently small. The fourth is the number of times the iteration will be repeated (unless stopped by the aforementioned instruction) before the machine shuts down. The fifth number is a small number ε3 which is used in the comparison of two angles which would be equal but for small errors in the calculation.

All angles used in the input to the first four programs are to be in degrees. All numbers are written with six significant figures.

When the information above has been read into the machine, the linkage parameters r1, r2, ..., r9 are calculated from the information
on the cards giving the linkage dimensions. Constants $k_\theta$ and $k_\phi$ are then calculated from the given ranges of $x$, $y$, $\theta$, and $\phi$. Using the precision point data on the first cards and $k_\theta$ and $k_\phi$ the machine then calculates $t_1$ and $v_1$.

With $r_1$, ..., $r_9$, $t_1$, and $v_1$ known, calculations then continue to determine $b_1$, ..., $b_9$ and $c_1$, ..., $c_{1,n+1}$ for each value of $i$ from 1 to $n$. The set of linear equations (21) is then solved for the correction factors $\Delta r_1$, $\Delta r_2$, ..., $\Delta r_n$. These correction factors are added to the original estimates of $r_1$, ..., $r_n$. The new values of $r_1$, ..., $r_9$ and the sum of the absolute values of the $\Delta r$'s are then either punched or printed for the later examination of the operator.

The machine then compares the sum of the absolute values of the correction factors to the sum of the absolute values of the $r$'s. If the ratio of $\sum |\Delta r|$ to $\sum |r|$ is greater than $\varepsilon_1$ the machine repeats the iteration as many times as called for in the input, or until $\sum |\Delta r| / \sum |r|$ becomes less than $\varepsilon_1$, whichever occurs sooner.

When the end of the iterations is reached, the machine punches on a card identified by the numbers 2 $N$ on the left side (where $N$ is the number of times the iteration has been performed) the five current values of $r_1$, $r_2$, $r_3$, $r_4$ and $r_5$. Then on a card identified by the numbers 3 $N$ it punches the values of $r_6$, $r_7$, $r_8$, $r_9$, and $\sum |\Delta r|$. A third output card identified by 4 $N$ receives the five numbers $\varepsilon_3$, $k_\theta$, the range of $\theta$, $x_1$, and $x_0$. These three cards may be used as the input to the fifth program if the operator wishes to determine the input-output characteristics of this final linkage. As a general rule he will
if the iteration has converged to a solution, but will not otherwise.
The operator can tell whether convergence has occurred by observing
the number $\sum |\Delta r|$. If this number is small (usually less than $10^{-4}$)
then a solution has been obtained.

As a final part of the output the machine uses the current
values of $r_1$, ..., $r_9$ to calculate $(a_2/a_1)$, $(a_3/a_1)$, $(a_4/a_1)$, $\theta_1$, $\phi_1$
$(a_5/a_8)$, $(a_6/a_8)$, $(a_7/a_8)$, and $\alpha$. These numbers are then punched on
output cards with the identifying numbers $5N$ and $5(N + 1)$ in the
same order (five numbers on card $5N$ and four numbers on card $5(N + 1)$).
The machine then stops. These last two cards may be used to replace
the original cards giving the linkage dimensions if the operator
wishes to continue the iterations from this point.

The operator can determine from the number of iterations
performed and from the printed value of $\sum |\Delta r|$ whether the computations
have converged to a solution or not. If a solution has been obtained,
output cards 2, 3 and 4 can be used as input cards to the fifth program.
The result of this program is a calculation and printing of the angular
position of each of the three cranks of mechanism A for one degree
increments of the input angle $\Theta$ throughout the entire range of interest.
A plot of $\phi$ vs. $\Theta$ can then be made and compared to the desired function
$y = f(x)$ to check the precision points and to determine the error be-
tween precision points. The operator can then decide whether respacing
of the precision points should be done in order to reduce the maximum
deviation of the generated function from the given function.

If a solution has not been reached, the operator will want to
examine the cards giving the values of $r_1$ through $r_9$ and $\sum |\Delta r|$ to see
if a solution is being approached. If the values of \( r \) seem to be converging toward limits, the computation can be continued by replacing the cards giving the original linkage dimensions by the output cards \( 5N \) and \( 5(N+1) \). Restarting the program will then repeat the computations described above, with the latest computed set of linkage dimensions as the assumption of the solution.

If the values of \( r_1 \) through \( r_9 \) are not converging toward a solution or if the convergence is so slow that one cannot be sure, it may be desirable to restart the computations with a different set of linkage parameters. It is in this event that the judgment and ingenuity of the designer will be needed.

6.2 INPUT AND OUTPUT DATA FOR SIX PRECISION POINT PROGRAM

6.2.1 Input Format

Before discussing the input data itself a few words of explanation about the format according to which these data are placed on the cards are in order. The program has been written so that each card is supposed to carry two identifying numbers plus five items of input data with each of these items consisting of a number having six significant figures. The first seventy columns of the card carry this information and the remainder of the card is left blank. The identifying numbers occupy the first five columns, the first of these numbers using the first two columns and the second using columns 3, 4, and 5. These identifying numbers are read into the machine, but are not used for any part of the computation, therefore they may be any numbers which help the operator to identify the cards. The author
found it convenient to use the first number to signify the order in which the cards are to be read and the second number to designate the problem number. Thus the first and second input cards for problem 1, would have their first five columns occupied by the numbers b1b14 and b2b14, respectively, where the symbol b designates a blank column. The next 65 columns are reserved for the five items of input data on that card, with each item occupying 13 columns. These numbers are written in the form ±0.JXXXXX × 10YY, where the I's represent the six significant figures of the number and the Y's give the location of the decimal point. The numbers must be written in the manner indicated with the first significant digit immediately following the decimal point, then the letter E followed by an exponent of 10 such that (0.JXXXXX) × (10^Y) is the number. Thus the number 38-1/2 would be written +0.38500E + 02. It can quickly be noted that each number written in this manner requires 13 columns. In the discussions which follow, no mention will be made of the identifying numbers in the first five columns. If the "second number" is referred to this is to be understood to mean the second item of input data, which will occupy columns 19 through 31.

6.2.2 Input Data to the Six-Point Program

The input to this program is contained on seven cards. The first two contain ten values of the independent variable x. The first number is the initial value of that variable, not necessarily the first precision point but rather the starting value of the range of x. The next six numbers (the last four on card one and the first two on card two) give the six precision point values of x. The eighth number is
the final value in the range of $x$. The ninth and tenth numbers are the lower and upper limits of the range of $x$, respectively. In most cases these last two numbers are the same as the first and eighth values.

Input cards 3 and 4 carry ten corresponding values of the dependent variable $y$. The first eight values of $y$ are calculated from the given function $y = f(x)$ and the first eight values of $x$. The ninth and tenth are the lower and upper limits of the range of $y$. As above these last two numbers may be the same as the first and eighth but they will not be if the function is not monotonic.

The next two cards, 5 and 6, give the nine significant dimensions of the linkage which is to be the initial approximation to the solution. The numbers on card 5 are $(a_2/a_1), (a_3/a_1), (a_4/a_1), \theta_1,$ and $\phi_1$, respectively. The numbers on card 6 are $(a_5/a_8), (a_6/a_8), (a_7/a_8)$ and $\alpha$, respectively. The fifth number on card 6 is left blank.

Card 7 contains five numbers as follows: (1) the range $\Delta \theta$ of the input angle $\theta$ in degrees; (2) the range $\Delta \phi$ of the output angle $\phi$ in degrees; (3) the small number $\epsilon_1$ which will end the iteration if $(\sum |\Delta r| / \sum |r|) < \epsilon_1$; (4) the number $\epsilon_2$ which is the number of times the iteration is to be performed before stopping regardless of the value of $\sum |\Delta r| / \sum |r|$; and (5) the small number $\epsilon_3$ which is not used in this program, but is carried along and punched on an output card for use in the fifth program for the calculation of the input–output characteristics of the linkage which is the solution. The actual use of $\epsilon_3$ is described in section 6-6. In the example problems described in Chapter VII the author took $\epsilon_1 = 10^{-5}, \epsilon_2 = 50, \text{ and } \epsilon_3 = 10^{-3}$.
6.2.3 Output Data from the Six-Point Program

Output from this program is by means of a line printer as well as punched cards. Two lines of output information are printed after each iteration is performed. When the calculations are finished, two additional lines are printed and five cards are punched. The purpose of the printed data is to permit the operator to observe the results of each iteration and to determine whether the calculations are proceeding toward a solution or not. The punched cards are made so as to be available for use as input cards to this or other programs if desired.

The format for the printed output is the same as the format for the punched input and output cards. Thus each line of output data begins with two identifying numbers. The two output lines which are printed after each iteration start with the identifying numbers 2 and 3. The second number on each of these lines is the number of times the iteration has been performed. These identifying numbers are followed by five six-digit numbers. The numbers on line 2 are the new values of \( r_1, r_2, r_3, r_4, \) and \( r_5 \) which have resulted from that iteration. The significant numbers on line 3 are the new values of \( r_6, r_7, r_8, r_9, \) and \( \sum |\Delta r| \). The most important of these ten numbers is the last one on line 3, \( \sum |\Delta r| \). By observing this number the operator can tell whether the calculations are approaching a limit (\( \sum |\Delta r| \) small) or whether the changes in the linkage parameters are large as the machine hunts for a solution without success.

When all the iterations are finished, the machine uses the current values of \( r_1, r_2, \ldots, r_9 \) to compute new values of the linkage
dimensions \((a_2/a_1), (a_3/a_1), (a_4/a_1), \theta_1, \beta_1, (a_5/a_8), (a_6/a_8), (a_7/a_8), \) and \(a\). The first five of these numbers are printed on a line labeled \(S \, N\) where \(N\) is the number of iterations and the last four are printed on a line labeled \(S \, (N+1)\). They are printed in the order in which they are listed above. There is no more printed output from this program.

The punched output from this program consists of five cards. The information punched on these cards gives no new information to the operator. Their only function is that they can be used as input cards to other programs. The first three output cards may be used as input cards to the fifth program if the operator wishes to determine the output angle \(\theta\) as a function of \(\theta\) over the entire range of the motion. The format of these output cards is necessarily the same as the input format mentioned in Section 6.2.1. These three cards carry identifying numbers \(2 \, N\), \(3 \, N\), and \(4 \, N\). Card number 2 carries the five latest values of the linkage parameters \(r_1, r_2, r_3, r_4, \) and \(r_5\). Card 3 carries the latest values of \(r_6, r_7, r_8, r_9, \) and \(\sum |\Delta r|\). Card 4 contains the numbers \(c_3, k_9, \Delta \theta, x_1, \) and \(x_0\).

The last two output cards carry the identifying numbers \(5 \, N\) and \(5 \, (N+1)\). They carry the nine linkage dimensions \((a_2/a_1), (a_3/a_1), (a_4/a_1), \theta_1, \beta_1\) on card \(5 \, N\) and \((a_5/a_8), (a_6/a_8), (a_7/a_8), \) and \(a\) on card \(5 \, (N+1)\). They can be used to replace cards 5 and 6 in the input cards to this program if the iteration of this problem is to be continued or can be used as the equivalent input cards to any of the other programs of this set.
It might be well to note that the information printed on the last four lines of output is exactly the same as the information punched on the output cards labeled 2 N, 3 N, 5 N, and 5 (N+1). There is no output card labeled 1.

The instructions for this program in FORTRAN language are given in Appendix C.

6.3 INPUT AND OUTPUT DATA FOR SEVEN PRECISION POINT PROGRAM

6.3.1 Input Data

The input to this program is contained on seven cards. The format according to which the data is punched on these cards is the same as that discussed in Section 6.2.1 for the previous program. The first two cards contain ten values of the independent variable x. The first number is the initial value of that variable, not necessarily the first precision point, but rather the starting value of the range of x. The next seven numbers (the last four on card one and the first three on card two) give the seven precision point values of x. The remaining two numbers on card two are the final value in the range of x and the total range of the variable x, respectively. In most cases this last number will be equal to the difference between the ninth and the first numbers.

Input cards 3 and 4 carry ten corresponding values of the dependent variable y. The first nine values of y are calculated from the first nine values of x according to the given function \( y = f(x) \). The tenth number is the total range of the variable y. As mentioned above for the variable x, the range of y may be the difference.
between the final and initial values of that variable, but it will
not be unless the function is continuously increasing or decreasing.

The last three cards, 5, 6, and 7 contain the same in-
formation that is on cards 5, 6, and 7 for the preceding program,
i.e., card 5 carries the values of \((a_2/a_1), (a_3/a_1), (a_4/a_1), \theta_1\),
number left blank; and card 7 carries \(\Delta \theta, \Delta \phi, \epsilon_1, \epsilon_2 \text{ and } \epsilon_3\); fifth
number left blank; and card 7 carries \(\Delta \theta, \Delta \phi, \epsilon_1, \epsilon_2 \text{ and } \epsilon_3\).
The linkage dimensions on cards 5 and 6 may come from a four-bar
linkage analysis as discussed in Section 5.2 or they may come from
a solution to a six-precision point problem from the preceding
program.

6.3.2 Output Data from the Seven-Point Program

The output from this program is by means of a line printer
as well as punched cards. The printer types approximately the same
information that was printed as output from the previous program, but
it has been arranged so as to be more conveniently displayed.

The format by which the output is printed is to have one
identifying number at the left of each line, followed by nine sig-
nificant items of output data. The identifying number gives the
number of iterations which have been completed.

After the input data have been read into the machine, the
linkage dimensions are printed on a line in the order: \((a_2/a_1),
(a_3/a_1), (a_4/a_1), a, (a_5/a_8), (a_6/a_8), (a_7/a_8), \theta_1, \phi_1\). Then before
any iterations have been performed the values of the linkage parame-
ters \(r_1, r_2, \ldots, r_8\) are printed on another line. Both of these
lines carry the identifying number zero.
After each iteration one additional line of output data is printed. This line contains the number of the iteration and the new values of \( r_1, r_2, \ldots, r_9 \) and \( \sum |\Delta r| \) which have resulted from that iteration. This makes it easier for the operator to follow the progress of \( \sum |\Delta r| \), since he can check the last number in each line with much less difficulty than the last number in every other line as was the case with the preceding program.

After the last iteration the current values of \( r_1, r_2, \ldots, r_9 \) are used to calculate the nine linkage dimensions. They are then printed on the last line in the same order in which they were printed on the first line.

The machine then punches five output cards with the same format as the input card format and giving the same information as the five output cards for the preceding program. That is, the first card is labeled 2 N and carries \( r_1, r_2, r_3, r_4, \) and \( r_5 \). The next card is labeled 3 N and carries \( r_6, r_7, r_8, r_9, \) and \( \sum |\Delta r| \). The next card is labeled 4 N and carries \( e_3, k_0, \Delta \theta, x_1 \) and \( x_0 \). Next is output card 5 N which carries \( (a_2/a_1), (a_3/a_1), (a_4/a_1), \phi_1, \) and \( \phi_1 \). The last output card is labeled 5 (N+1) and carries \( (a_5/a_8), (a_6/a_8), (a_7/a_8), \) and \( a \). When these five cards are punched the machine stops.

The instructions for this program in FORTRAN language are given in Appendix D.
6.4 INPUT AND OUTPUT DATA FOR EIGHT PRECISION POINT PROGRAM

6.4.1 Input Data

The input to this program is contained on seven cards. The format according to which the data is punched on these cards is the same as that discussed in Section 6.2.1 for the first program.

The first two cards contain ten values of the independent variable \(x\). The first number is the initial value of that variable, not necessarily the first precision point, but rather the starting value of the range of \(x\). The next eight numbers (the last four on card one and the first four on card two) give the eight precision point values of \(x\). The remaining number on card two is the range of the variable \(x\).

Input cards 3 and 4 carry ten corresponding values of the dependent variable \(y\). The first nine values of \(y\) are calculated from the first nine values of \(x\) according to the given function \(y = f(x)\). The tenth number is the total range of the variable \(y\).

The last three cards 5, 6, and 7, contain the same information that is on cards 5, 6, and 7 for the preceding programs, i.e., card 5 carries the values of \((a_2/a_1), (a_3/a_1), (a_4/a_1), \theta_1\), and \(\phi_1\); card 6 has \((a_5/a_9), (a_6/a_9), (a_7/a_9), \) and \(a\), with its fifth number left blank; and card 7 carries \(\Delta \theta, \Delta \phi, \epsilon_1, \epsilon_2,\) and \(\epsilon_3\). The linkage dimensions on cards 5 and 6 may come from a four-bar linkage synthesis as discussed in Section 5.2, or they may come from a solution to a six- or seven-precision point problem from one of the preceding programs.
6.4.2 Output Data from the Eight-Point Program

The output from this program is by means of the line printer and punched cards, and is exactly the same as the output from the seven-point program. The reader is directed to Section 6.3.2 for a discussion of the content and form of this output.

The instructions for this program in FORTRAN language are given in Appendix E.

6.5 INPUT AND OUTPUT DATA FOR NINE PRECISION POINT PROGRAM

6.5.1 Input Data

The input to this program is contained on nine cards. The format according to which the data is punched on these cards is the same as that discussed in Section 6.2.1 for the first program.

The first three cards contain thirteen values of the independent variable $x$. The first number is the initial value of that variable, not necessarily the first precision point, but rather the starting value of the range of $x$. The next nine numbers (the last four on card one and all five on card two) give the nine precision point values of $x$. The three numbers on card three are, respectively; the final value in the range of $x$, the lower limit of the range of $x$, and the upper limit of the range of $x$. The remaining space for two numbers on card three is left blank.

Input cards 4, 5, and 6 carry thirteen corresponding values of the dependent variable $y$. The first eleven values of $y$ are calculated from the first eleven values of $x$ according to the given function $y = f(x)$. The twelfth and thirteenth are the lower and
upper limits of the range of $y$, respectively. The remaining space for two numbers on card 6 is left blank.

The last three cards, 7, 8, and 9, contain the same information that is on cards 5, 6, and 7 for the preceding programs, i.e., card 7 has the values of $(a_2/a_1)$, $(a_3/a_1)$, $(a_4/a_1)$, $\theta_1$, and $\phi_1$; card 8 carries $(a_5/a_8)$, $(a_6/a_8)$, $(a_7/a_8)$, and $\sigma$ with its fifth number left blank; and card 9 contains $\Delta\theta$, $\Delta\phi$, $\varepsilon_1$, $\varepsilon_2$, and $\varepsilon_3$.

The linkage dimensions on cards 7 and 8 may come from a four-bar linkage synthesis as discussed in Section 5.2, or they may be the dimensions of a solution to a six, seven or eight precision point problem from one of the preceding programs.

6.5.2 Output Data from the Nine-Point Program

The nine-point program was the first of the programs to be developed, and its output is not in as convenient a form for observation as is the output for the preceding programs. There is no printed output from this program, all the output being punched on cards. The format for the data on the output cards is the same as the format for the input cards which is described in Section 6.2.1.

Two output cards are punched at the end of each iteration. These cards carry the identifying numbers $2N$ and $3N$, respectively, where $N$ is the number of times the iteration has been performed.

The five six-digit numbers on the card labeled $2N$ are the new values of $r_1$, $r_2$, $r_3$, $r_4$, and $r_5$ which have resulted from that iteration. The five additional numbers on card $3N$ are the new values of $r_6$, $r_7$, $r_8$, $r_9$, and $\sum |\Delta r|$.
After the last iteration has been performed, the machine punches three additional output cards. These are the cards identified by the numbers 4 N, 5 N, and 5 (N+1) which have been described for the preceding programs and which contain, respectively, the current values of $\varepsilon_3$, $k_\theta$, $\Delta \theta$, $x_1$ and $x_0$ in card 4 N; $(a_2/a_1)$, $(a_3/a_1)$, $(a_4/a_1)$, $\Theta_1$ and $A_1$ on card 5 N; and $(a_5/a_8)$, $(a_6/a_8)$, $(a_7/a_8)$, and $a$ on card 5 (N+1) with the last space for a number left blank.

In order to observe the progress of the iterations it is desirable to have the information on the cards printed for easier reading. When the output cards have been printed this printed output is almost the same as the printed output of the six-point program, the only difference being that the information on output card 4 N is not printed for the six-point program, but would be for this one. When these cards have been printed the operator can tell how well the calculations are approaching a limit by observing the behavior of the last number ($\sum |\Delta r|$ ) on each of the lines bearing the identifying numbers 3 N.

The instructions for this program in FORTRAN language are given in Appendix F.

6.6 A PROGRAM FOR COMPUTING THE LINKAGE OUTPUT OF MECHANISM A

6.6.1 Input Data

The input to this program is on three IBM cards. The format for these cards is the same as the input format of the preceding programs which is described in Section 6.2.1.
The five input numbers on the first card are the values of \( r_1, r_2, r_3, r_4, \) and \( r_5 \) for the linkage whose output is to be determined. The values on the second card are \( r_6, r_7, r_8, \) and \( r_9 \). The fifth space for a number on this card may be left blank. Whatever number is in this fifth space on the second card will be ignored by the machine. The third input card contains values of \( E_3, k_9, \Delta \theta, x_1 \) and \( x_0 \), in that order.

It might be advisable here to remind the reader that \( r_1, r_2, r_3, r_4, r_5, \) and \( r_6 \) are dimensionless ratios of the lengths of the links in the mechanism. The parameters \( r_7, r_8, \) and \( r_9 \) are angles which must be expressed in radians. The number \( E_3 \) is a small number which the author has successfully taken as \( 10^{-3} \). The factor \( k_9 \) is the ratio of the range of \( \theta \) in radians to the range of \( x \). The number \( \Delta \theta \) is the range of \( \theta \) in degrees. The value of \( x_1 \) is the first precision point value of \( x \), and \( x_0 \) is the initial value of the range of \( x \).

The three input cards may be the three output cards labeled \( 2 N, 3 N \) and \( 4 N \) from one of the four preceding programs, or if it is desired to compute the output for any other linkage, input cards may be punched which give the above information. Formulas by which \( r_1, \ldots, r_6 \) may be calculated are given in Section 4.2, equations (6a) and (9a).

6.6.2 Calculations and Output Data

To begin the computations for this program the values of \( r_1, \ldots, r_9 \) are used to calculate the linkage dimensions \( (a_2/a_1) \),
Using these dimensions and the initial input angle $\Theta_1$, calculations are made to find the angular position $\phi_1$ of the output link using the method described in Appendix A. Four answers are obtained from this computation because both the left- and right-hand loops of the mechanism may be either open or crossed. These four answers are then checked against the input value of $\phi_1$ to see which is correct. It is in this comparison that the input number $\epsilon_3$ is used. If $\phi_1$ and one of the four answers differ by less than $\epsilon_3$, that answer is identified as correct and by its identification it is known whether the right- and left-hand loops are open or crossed. If none of the four answers are sufficiently close to $\phi_1$, the machine multiplies $\epsilon_3$ by ten and repeats the comparison, after printing on the output sheet the statement: INITIAL VALUES OF THETA AND PHI ARE NOT COMPATIBLE, REPEAT WITH $E3 \times 10$. The comparison usually identifies the proper solution the first time if the input cards come from a problem solution, but if the operator merely wishes to determine the output characteristics of a given linkage and calculates $\phi_1$ with the aid of a slide rule or a set of five-place trigonometric tables, the chances are that the value of $r_9$ ($\phi_1$ in radians) read into the machine will not be as accurate as the machine-calculated answers. In such cases it is not unusual for the above statement to be printed once or twice before the machine proceeds on to the next step.

After the correct phase of the mechanism is identified,
the machine prints on two output lines the values of the linkage dimensions. These lines are printed using the same format as for the input cards and carry the identifying numbers 5 1 and 5 2. Line 5 1 has the values of the dimensions \(a_2/a_1\), \(a_3/a_1\), \(a_4/a_1\), \(\theta_1\) and \(\phi_1\). Line 5 2 has the values of the dimensions \(a_5/a_5\), \(a_6/a_5\), \(a_7/a_5\), \(\alpha\) and \(\beta_1\), where \(\beta_1\) is the angular position of the middle crank at the first precision position.

The computer then returns to the value of \(\theta\) which corresponds to the initial value \(x_0\) of the independent variables, and calculates \(\phi\) for each value of \(\theta\) by one degree increments throughout the range of \(\theta\). The machine prints one line for each of these calculations. The format used for these lines is the same as for the input cards. The identifying numbers are 6 M, where M designates the difference \(\theta - \theta_0\). Each line of this output carries three numbers aside from the identifiers. These are the values of \(\theta\), \(\beta\) and \(\phi\), printed in that order. After proceeding through the range of \(\theta\), the machine stops.

This program is tabulated in FORTRAN language in Appendix G.
CHAPTER VII

EXAMPLE PROBLEM

7.1 THE PROBLEM

The problem which was selected for use in these programs was to generate the function \( y = x^2 \), for \( 0 \leq x \leq 1 \). This function was to be mechanized by an example of mechanism A, with its input and output cranks moving through ranges of 90°. Both cranks were to rotate counterclockwise in the directions of increasing \( x \) and \( y \).

This problem was selected because of the mathematical simplicity of the function, the fact that the function is smooth and continuously increasing in the range of interest, and the fact that this same problem has been solved by Freudenstein using a four-bar linkage and the results published (10,11). A similar problem was solved by Denavit and Hartenberg (6) when they synthesized the function \( y = x^2 \) for \( -1 \leq x \leq 1 \) with nine precision points using a space mechanism. The best accuracy which was obtainable with the four-bar linkage resulted in a maximum error of only 0.067 degrees anywhere within the range of the motion. Using the space mechanism designed by Denavit and Hartenberg, the maximum error is 0.13 degrees, although this figure cannot be compared with the above because the function generated here was much more difficult to mechanize accurately.
In order to obtain a reasonably accurate representation of the function on the initial attempts, the precision points were located according to the method suggested in reference (11) for Chebichev spacing. This Chebichev spacing is the same as the spacing of the zeroes of a Chebichev polynomial over the interval in $x$. These points are more closely spaced near the ends of the interval and therefore constrain the generated function more at the ends, where otherwise it would be likely to diverge most from the given function. This spacing of the precision points can be taken as the projection on the interval of $2n$ points evenly distributed around the circumference of a circle of which the interval in $x$ is a diameter. These $2n$ points should be symmetrically spaced with respect to the bisector of the interval.

Since each of the programs used for this analysis requires specification of the corresponding values of $\theta$ and $\phi$ at the first precision point, the first precision point was taken as $x = 0.0075961$ for all four of the programs. This value of $x_1$ is proper for Chebichev spacing of the precision points for the nine-point program. The remaining precision points were chosen to be approximately the same as Chebichev spacing, but were usually taken to be rounded numbers in the appropriate neighborhood of the Chebichev points. As mentioned earlier, the values of $\epsilon_1$, $\epsilon_2$ and $\epsilon_3$ on the last input card were taken to be $10^{-5}$, $50$, and $10^{-3}$, respectively.

One interesting point which plagues the user of these
programs is caused by the multiplicity of values of the output angle \( \phi \). Since \( \phi \) is a quadruple-valued function of \( \theta \) it is entirely possible that in a solution to one of the example problems some of the precision points will occur for one phase of the mechanism such as with the left loop open and the right loop crossed and the rest of the precision points will occur for some other phase of the same mechanism. In such a case the position equation would be satisfied at all of the precision points and the only way to observe that this had happened would be to calculate \( \phi \) as a function of \( \theta \) for one of the phases of the mechanism and compare it with the desired function. Since it is not usually practical to go from one phase of a mechanism to another, these solutions are not as valuable as those in which all the precision points occur in one phase. In the examples which follow, the number of precision points in one phase will be noted as a part of the summary information.

7.2 THE INITIAL APPROXIMATIONS TO A SOLUTION OF THE PROBLEM

The initial approximations to a solution of the problem were obtained from a four-bar linkage synthesis. The method of synthesis and a program based upon it are described in Appendix B.

The left-hand loop of the mechanism was made of a four-bar linkage which generated the function \( y' = x\sqrt{2} \). The five precision points for this analysis were taken as the odd-numbered precision points for the nine-point program.

Thus the precision points for the four-bar linkage synthesis of the left-hand loop for mechanism A were:
Three solutions were obtained to this problem. The pertinent linkage dimensions are tabulated as solutions one, two, and three in Table 2 below.

### TABLE 2

SOLUTIONS FOR LEFT-HAND LOOP OF MECHANISM A

<table>
<thead>
<tr>
<th>Dimension</th>
<th>One</th>
<th>Two</th>
<th>Three</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_2/a_1)</td>
<td>4.34842</td>
<td>1.06619</td>
<td>6.22179</td>
</tr>
<tr>
<td>(a_3/a_1)</td>
<td>0.754340</td>
<td>1.49892</td>
<td>3.11169</td>
</tr>
<tr>
<td>(a_4/a_1)</td>
<td>3.97079</td>
<td>0.673763</td>
<td>2.76572</td>
</tr>
<tr>
<td>(\theta_1)</td>
<td>254.261°</td>
<td>171.802°</td>
<td>-58.7709°</td>
</tr>
<tr>
<td>(\beta_1)</td>
<td>241.209°</td>
<td>203.661°</td>
<td>-79.1008°</td>
</tr>
</tbody>
</table>

The right-hand loop of the mechanism was formed by a four-bar linkage which took as input the angular positions of the output link of the left-hand loop. The output from the right-hand loop was, of course, taken to be the square of the input to the left-hand loop. The dimensions of the mechanisms for the right-hand loop were
obtained from the same four-bar linkage program using the precision points:

\[ x_1 = 0.0010061 \quad y_1 = 0.0000577007 \]
\[ x_2 = 0.087504 \quad y_2 = 0.0319002 \]
\[ x_3 = 0.37522 \quad y_3 = 0.250000 \]
\[ x_4 = 0.75712 \quad y_4 = 0.674688 \]
\[ x_5 = 0.98926 \quad y_5 = 0.984866 \]

As for the previous problem, three solutions were obtained from the four-bar linkage program. They are summarized as solutions four, five, and six in Table 3.

**TABLE 3**

SOLUTIONS FOR RIGHT-HAND LOOP OF MECHANISM A

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Four</td>
</tr>
<tr>
<td>(a_5/a_8)</td>
<td>3.88650</td>
</tr>
<tr>
<td>(a_6/a_9)</td>
<td>0.875366</td>
</tr>
<tr>
<td>(a_7/a_8)</td>
<td>3.31425</td>
</tr>
<tr>
<td>(\beta_1 - \alpha)</td>
<td>261.267°</td>
</tr>
<tr>
<td>(\phi_1)</td>
<td>243.889°</td>
</tr>
</tbody>
</table>

Thus the first approximation to a solution to this problem may be taken by choosing any of the three linkages of Table 2 for the left-hand loop and any of the three linkages of Table 3 for the
right-hand loop. Each of these nine possibilities was used as the initial approximation to a solution in the six-point program. These approximations were labeled one through nine with approximations one, two, and three using solution one for the left loop and solutions four, five, and six for the right loop, respectively. Approximations four, five, and six used solution two for the left loop and solutions four, five and six for the right loop, respectively. Approximations seven, eight, and nine used solution three for the left loop and solutions four, five, and six for the right loop, respectively.

Under normal conditions, the characteristics of the initial linkages would not be calculated, but in order to get some experience in the kind of accuracy that could be obtained with a five-point approximation and to see if the six-point calculations had more of a tendency to converge if the initial linkage generated the function quite accurately, the input-output angles of each of these nine mechanisms were computed using the program described in Section 6-6. The results of those calculations are summarized in Table 4 below. The maximum error in degrees occurring at any point throughout the range of the motion is given in the third column. In the fourth column is listed the location of that point of maximum error with respect to the five precision points. If the location is given as 2-3 this means that the maximum error occurred between the second and third precision points, i.e., between $x = 0.178606$ and $x = 0.500000$. If the location is given as 0-1 (or 5-6) the maximum error occurred
between the first (or fifth) precision point and the beginning (or end) of the range of the motion.

TABLE 4
MAXIMUM VALUES OF ERROR IN THE NINE APPROXIMATIONS TO A SOLUTION

<table>
<thead>
<tr>
<th>Approximation Number</th>
<th>Left Loop</th>
<th>Right Loop</th>
<th>Maximum Value, deg.</th>
<th>Location</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>4</td>
<td>-0.343</td>
<td>3-4</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>5</td>
<td>-0.095</td>
<td>0-1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>6</td>
<td>+0.161</td>
<td>2-3</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>4</td>
<td>-0.102</td>
<td>3-4</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>5</td>
<td>-0.364</td>
<td>2-3</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>6</td>
<td>-0.115</td>
<td>2-3</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>4</td>
<td>-0.206</td>
<td>3-4</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>5</td>
<td>-0.223</td>
<td>2-3</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>6</td>
<td>+0.027</td>
<td>2-3</td>
</tr>
</tbody>
</table>

7.3 USE OF THE SIX-POINT PROGRAM

Each of the nine approximations listed above was used as an input to the six-point program described in Section 6.2. The precision points for this program were taken to be the following:
\[ x_1 = 0.0075961 \quad \quad \quad \quad y_1 = 0.0000577007 \]
\[ x_2 = 0.150000 \quad \quad \quad \quad y_2 = 0.0225000 \]
\[ x_3 = 0.350000 \quad \quad \quad \quad y_3 = 0.122500 \]
\[ x_4 = 0.650000 \quad \quad \quad \quad y_4 = 0.422500 \]
\[ x_5 = 0.850000 \quad \quad \quad \quad y_5 = 0.722500 \]
\[ x_6 = 0.992404 \quad \quad \quad \quad y_6 = 0.984866 \]

The results of these calculations are summarized in Table 5 below. It can be seen that five of the nine attempts actually converged to solutions.

A comparison of Table 5 with Table 4 does not indicate that the more accurate approximations are any more likely to converge than the others. The number of iterations shown in the third column of Table 5 shows that all but one of the solutions was obtained after less than ten iterations. The numbers 6/50 in the fourth row signify that the calculations had essentially reached a limit after six iterations, but since \( C_1 \) was so small, the calculations were not stopped until 50 iterations had occurred. Each of these solutions had the proper number of precision points all in one phase of the mechanism. The last solution even had an extra one between \( x_2 \) and \( x_3 \). The most accurate of the five solutions is the last one, which has its maximum error of 0.022° at the end of the interval of interest. The last column gives the number which will be used to identify these solutions in the future. Their linkage dimensions are given in Table 6. A negative length, such as those in Solution 11 is to be regarded in the vector sense, i.e., -0.394237 at 171.802° is the same as +0.394237 at -8.198°.
<table>
<thead>
<tr>
<th>Approximation Number</th>
<th>Converged?</th>
<th>Number of Iterations</th>
<th>No. of Prec.Pts. in one Phase</th>
<th>Maximum Error, deg.</th>
<th>Location Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>No</td>
<td>50</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>No</td>
<td>50</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>No</td>
<td>50</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>Yes</td>
<td>6/50</td>
<td>6</td>
<td>+0.059</td>
<td>1-2</td>
</tr>
<tr>
<td>5</td>
<td>Yes</td>
<td>27</td>
<td>6</td>
<td>+0.064</td>
<td>3-4</td>
</tr>
<tr>
<td>6</td>
<td>Yes</td>
<td>9</td>
<td>6</td>
<td>-0.030</td>
<td>1-2</td>
</tr>
<tr>
<td>7</td>
<td>No</td>
<td>50</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>8</td>
<td>Yes</td>
<td>7</td>
<td>6</td>
<td>-0.179</td>
<td>1-2</td>
</tr>
<tr>
<td>9</td>
<td>Yes</td>
<td>7/26</td>
<td>7</td>
<td>-0.022</td>
<td>5-6</td>
</tr>
</tbody>
</table>
TABLE 6

DIMENSIONS OF THE SOLUTIONS TO THE SIX-POINT PROGRAM

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Solution Number</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10</td>
</tr>
<tr>
<td>$a_2/a_1$</td>
<td>0.919425</td>
</tr>
<tr>
<td>$\theta_1, \text{deg.}$</td>
<td>171.802</td>
</tr>
<tr>
<td>$a_3/a_1$</td>
<td>1.44622</td>
</tr>
<tr>
<td>$a_4/a_1$</td>
<td>0.553549</td>
</tr>
<tr>
<td>$\alpha, \text{deg.}$</td>
<td>-57.606</td>
</tr>
<tr>
<td>$a_5/a_8$</td>
<td>4.25172</td>
</tr>
<tr>
<td>$a_6/a_8$</td>
<td>1.07991</td>
</tr>
<tr>
<td>$a_7/a_8$</td>
<td>3.48357</td>
</tr>
<tr>
<td>$\phi_1, \text{deg.}$</td>
<td>243.889</td>
</tr>
</tbody>
</table>

7.4 USE OF THE SEVEN-POINT PROGRAM

Each of the five solutions to the six-point problem were used as an initial approximation to a solution in the seven-point program. The precision points for the seven-point program were taken to be the following:

$$x_1 = 0.0075961 \quad y_1 = 0.00000577007$$
$$x_2 = 0.100000 \quad y_2 = 0.010000$$
$$x_3 = 0.250000 \quad y_3 = 0.062500$$
$$x_4 = 0.500000 \quad y_4 = 0.250000$$
$$x_5 = 0.750000 \quad y_5 = 0.562500$$

72
\[ x_6 = 0.900000 \quad y_6 = 0.810000 \]
\[ x_7 = 0.992404 \quad y_7 = 0.984866 \]

The results of these calculations are shown in Table 7 below, and the dimensions of the four solutions obtained are listed in Table 8.

**TABLE 7**

RESULTS OF FIVE TRIALS WITH THE SEVEN PRECISION POINT PROGRAM

<table>
<thead>
<tr>
<th>Initial Solution Number</th>
<th>Converged?</th>
<th>Number of Iterations</th>
<th>No. of Prec. Pts. in One Phase</th>
<th>Location of Missing Prec. Pt(s)</th>
<th>Max. Error, Location</th>
<th>Solution Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>Yes</td>
<td>17</td>
<td>5</td>
<td>1 and 7</td>
<td>0.263</td>
<td>0-1</td>
</tr>
<tr>
<td>11</td>
<td>Yes</td>
<td>6</td>
<td>6</td>
<td>3</td>
<td>-0.180</td>
<td>3-4</td>
</tr>
<tr>
<td>12</td>
<td>Yes</td>
<td>25/50</td>
<td>6</td>
<td>2</td>
<td>-1.659</td>
<td>1-2</td>
</tr>
<tr>
<td>13</td>
<td>Yes</td>
<td>14</td>
<td>3</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>14</td>
<td>No</td>
<td>50</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

A solution with seven precision points should be more accurate than the preceding solutions, but since only five or six of the precision points for the above solutions are in the same phase of the mechanism, the precision points which remain are not as well spaced as they were for the previous solutions, and the maximum errors are larger.

Examination of Table 7 shows that the maximum error occurs at a location where a precision point is missing. It is somewhat surprising that solution 14, which originally had seven precision points, and was
the most accurate of the six-point mechanisms, did not cause convergence when used in the seven-point program.

TABLE 8

DIMENSIONS OF THE SOLUTIONS TO THE SEVEN-POINT PROGRAM

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Solution Number</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_2/\alpha_1$</td>
<td></td>
<td>1.82417</td>
<td>-0.408519</td>
<td>1.65844</td>
<td>0.95944</td>
</tr>
<tr>
<td>$\Theta_1$, deg.</td>
<td></td>
<td>171.802</td>
<td>171.802</td>
<td>171.802</td>
<td>-58.771</td>
</tr>
<tr>
<td>$\alpha_3/\alpha_1$</td>
<td></td>
<td>1.50935</td>
<td>0.967454</td>
<td>0.340044</td>
<td>1.04994</td>
</tr>
<tr>
<td>$\alpha_4/\alpha_1$</td>
<td></td>
<td>1.33160</td>
<td>-0.439820</td>
<td>2.32102</td>
<td>1.07794</td>
</tr>
<tr>
<td>$\alpha$, deg.</td>
<td></td>
<td>-109.373</td>
<td>26.816</td>
<td>214.441</td>
<td>52.163</td>
</tr>
<tr>
<td>$\alpha_5/\alpha_8$</td>
<td></td>
<td>4.44317</td>
<td>0.566524</td>
<td>1.74562</td>
<td>1.21401</td>
</tr>
<tr>
<td>$\alpha_6/\alpha_8$</td>
<td></td>
<td>3.08539</td>
<td>1.07232</td>
<td>0.289659</td>
<td>1.80876</td>
</tr>
<tr>
<td>$\alpha_7/\alpha_8$</td>
<td></td>
<td>1.48617</td>
<td>0.624530</td>
<td>0.919577</td>
<td>-0.904862</td>
</tr>
<tr>
<td>$\Theta_1$, deg.</td>
<td></td>
<td>243.389</td>
<td>208.977</td>
<td>-79.997</td>
<td>208.977</td>
</tr>
</tbody>
</table>

Solution 18 is the most striking example of a solution for which the precision points occur in different phases of the mechanism. The first two precision points occur when the left loop is crossed and the right loop is open. The third, fourth, and sixth points occur when both loops are open. The other points occur one each on the other two phases of the mechanism. The accuracy of any one phase of this mechanism is so poor that the mechanism has no practical use in the mechanization of this function.
7.5 USE OF THE EIGHT- AND NINE-POINT PROGRAMS

Each of the four solutions to the seven-point problem were used as an initial approximation to a solution in the eight-point program. The following precision points for this program were chosen:

\[
\begin{align*}
    x_1 &= 0.0075961 & y_1 &= 0.0000577007 \\
    x_2 &= 0.080000 & y_2 &= 0.006400 \\
    x_3 &= 0.220000 & y_3 &= 0.048400 \\
    x_4 &= 0.400000 & y_4 &= 0.160000 \\
    x_5 &= 0.600000 & y_5 &= 0.360000 \\
    x_6 &= 0.780000 & y_6 &= 0.608400 \\
    x_7 &= 0.920000 & y_7 &= 0.846400 \\
    x_8 &= 0.992404 & y_8 &= 0.984866 
\end{align*}
\]

The results of these calculations are summarized in Table 9 below and the dimensions of the two solutions obtained are listed in Table 10.

**TABLE 9**

RESULTS OF FOUR TRIALS WITH THE EIGHT PRECISION POINT PROGRAM

<table>
<thead>
<tr>
<th>Initial Solution Number</th>
<th>Converged?</th>
<th>Number of Converged?</th>
<th>Number of Prec. Points</th>
<th>Location Max. Error, Prec. deg.</th>
<th>Location Number</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>of Iterations in One Phase</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>Yes</td>
<td>9</td>
<td>6</td>
<td>1 and 8 0.534</td>
<td>0-1</td>
</tr>
<tr>
<td>16</td>
<td>Yes</td>
<td>10</td>
<td>7</td>
<td>4 0.082</td>
<td>3-4</td>
</tr>
<tr>
<td>17</td>
<td>No</td>
<td>50</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>18</td>
<td>No</td>
<td>50</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>
TABLE 10
DIMENSIONS OF THE SOLUTIONS TO THE EIGHT-POINT PROGRAM

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Solution Number</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>19</td>
</tr>
<tr>
<td></td>
<td>20</td>
</tr>
</tbody>
</table>

| $a_2/a_1$  | 2.58803        | -0.455427 |
| $\theta_1$, deg. | 165.764       | 185.134   |
| $a_3/a_1$  | 2.21553        | 1.10230   |
| $a_4/a_1$  | 1.40744        | -0.58281  |
| $\alpha$, deg. | -106.361      | 33.320    |
| $a_5/a_8$  | 6.21956        | 0.707055  |
| $a_6/a_8$  | 4.62181        | 1.19109   |
| $a_7/a_8$  | 1.83186        | 0.859816  |
| $\phi_1$, deg. | 213.889      | 208.977   |

The results of the calculations with the eight-point program are quite similar to those obtained with the seven-point program. In neither case were all eight of the precision points in one phase of the mechanism. The accuracy, therefore, is poorest near the location of one of the missing precision points and is not as good as could be obtained with six properly spaced points.

The two solutions to the eight-point problem were used as input linkages to the nine-point program. Chebichev spacing was used for the nine precision points. Neither of these attempts, however, converged to a solution after fifty iterations.
CHAPTER VIII

SUMMARY OF RESULTS AND CONCLUSIONS

8.1 SUMMARY OF RESULTS

The principal objective of this dissertation has been the development of a method for designing mechanisms having turning joints only in order to produce a given relationship between the rotations of the input and output cranks. This method is applicable to plane mechanisms with rigid links for which two independent vector position equations can be written.

The calculations are iterative in nature and act successively to improve, by the method of Newton, the dimensions of a linkage which was assumed to be of approximately the proper proportions. The calculations may or may not converge to a solution, depending upon the original assumption of linkage dimensions.

Six computer programs were developed by which an IBM 704 Computer was instructed to perform the necessary computations. The first of these programs had as its purpose the design of four-bar linkages to have five precision points with respect to any desired functions. The second, third, fourth, and fifth programs were developed to permit the design of a six-link mechanism to have respectively, six, seven, eight, or nine precision points. The
sixth program was designed to calculate the output rotations of the six-link mechanism as a function of the input rotations throughout the range of the motion to be used for function generation.

In the example problem chosen for solution, approximately half (11 of 20) of the attempts actually converged. However, since not all of the precision points in a given solution occurred in the same phase of the mechanism, only six of the eleven solutions resulted in a mechanism which generated the desired function more accurately than did the original assumption which led to that solution.

There were six solutions to the example problem which generated the desired function within $0.09^\circ$ over the entire range of interest. This maximum error is less than one-tenth of one percent of the total follower rotation. The dimensions of these mechanisms, their maximum error and the program by means of which they were obtained are listed in Table 11 below in order of decreasing accuracy. The mechanisms are drawn schematically in Figure 11. The most accurate solution is No. 14, with a maximum error of $0.022^\circ$. Somewhat surprisingly, the next most accurate solution is number 9, which resulted from the five precision point, four-bar linkage analysis. Four of these six most accurate mechanisms were the result of the six-point program. This occurred because the precision points for the seven- and eight-point programs did not all lie in the same phase of the mechanism, and therefore the precision points within one phase were poorly spaced over the interval.
TABLE 11

SUMMARY OF MOST ACCURATE SOLUTIONS TO THE GENERATION OF THE FUNCTION \( y = x^2 \) for \( 0 \leq x \leq 1 \) BY MECHANISM A WITH 90 DEGREES ROTATIONS OF INPUT AND OUTPUT CRANKS

<table>
<thead>
<tr>
<th>Dimension</th>
<th>14</th>
<th>9</th>
<th>12</th>
<th>10</th>
<th>11</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_1, \text{deg.} )</td>
<td>-58.771</td>
<td>-58.771</td>
<td>171.802</td>
<td>171.802</td>
<td>171.802</td>
<td>185.134</td>
</tr>
<tr>
<td>( \Phi_1, \text{deg.} )</td>
<td>-79.997</td>
<td>-79.997</td>
<td>-79.997</td>
<td>243.889</td>
<td>208.977</td>
<td>208.977</td>
</tr>
<tr>
<td>Source</td>
<td>Program 6-pt.</td>
<td>5-pt.</td>
<td>6-pt.</td>
<td>6-pt.</td>
<td>6-pt.</td>
<td>8-pt.</td>
</tr>
</tbody>
</table>

8.2 CONCLUSIONS AND SUGGESTIONS FOR FURTHER WORK

The six solutions tabulated in Table 11 indicate that the methods described herein can make possible the accurate function
generation of a rather well behaved function. The fact that no solutions were obtained which had eight or nine precision points in one phase of the mechanism, however, indicates that the accuracy obtained here can be increased still further. Two undertakings which might be worthy of additional study are (1) improving the methods of choosing an initial approximation to a solution, and (2) devising some procedure which would account for the phase of the mechanism under consideration. This latter suggestion would seem to be extremely difficult to accomplish, but would be equally valuable.

One use to which these programs might be put and which has not been explored is the generation of more complicated functions. One primary advantage of six-link over four-bar mechanisms is that they should permit the more accurate mechanization of functions which are complex in nature.

The three mechanisms B, C, and D for which the basic equations are presented in the earlier chapters should permit even more accuracy and generality due to the larger number of precision points possible. Computer programs should be developed particularly for mechanisms B and D in order to exploit these potentialities. Mechanism D, with its two degrees of freedom is a particular challenge, and the development of methods which permit the synthesis of this mechanism should be very rewarding.

The question of when and whether the iterative method of calculation presented here will converge has been a troublesome one.
FIGURE II. SKETCHES OF THE SIX MOST ACCURATE SOLUTIONS TO THE EXAMPLE PROBLEM.
If the (Jacobian) determinant of the coefficients of the n equations vanishes at or near a solution, the method may converge only very slowly or not at all. Conditions which are necessary for convergence can be stated, but have little significance here, since it has been easier to attempt a solution on the computer to see if convergence results than to examine the nature of the Jacobian.

There should in general be a number of solutions to any of the synthesis problems attempted in the preceding chapter. One basic difficulty of the method presented here is that only one solution is found at a time and no indication is given as to the total number of solutions possible. Also, there is at present no systematic way in which the additional solutions may be found after one or more are determined. A procedure which would permit the evaluation of multiple solutions after one is found would be extremely valuable.


APPENDIX A

CALCULATION OF ANGULAR POSITION OF OUTPUT LINK FOR MECHANISM A

Figure A-1. Mechanism A

To evaluate $\phi$ when the linkage dimensions ($a_1$ through $a_8$ and $a$) and the angular position $\theta$ of the driver are given, the following calculations are necessary:

$$h_1 = + \sqrt{a_2^2 + a_1^2 - 2a_1a_2 \cos \theta}$$

$$h_3 = \arcsin \left( \frac{a_2}{h_1} \sin \theta \right)$$

Note: If $a_2 \cos \theta \leq a_1$, $-\pi/2 \leq h_3 \leq \pi/2$.

If $a_2 \cos \theta > a_1$, $\pi/2 < h_3 < 3\pi/2$.

85
\[ h_4 = \arccos \left[ \frac{(h_2^2 + a_7^2 - a_8^2)}{2a_4h_1} \right] , \quad 0 \leq h_4 \leq \pi. \]

For the left-hand linkage open, \( \beta = \pi - h_3 - h_4 \).

For the left-hand linkage crossed (shown dotted), \( \beta = \pi - h_3 + h_4 \)

\[ h_2 = + \sqrt{\frac{2}{a_5^2 + a_8^2 - 2a_5a_8 \cos (\beta - \alpha)}}, \]

and

\[ h_5 = \arcsin \left[ \frac{a_5}{h_2} \sin (\beta - \alpha) \right]. \]

Note: If \( a_5 \cos (\beta - \alpha) \leq a_8, \ - \pi/2 \leq h_5 \leq \pi/2 \),

and if \( a_5 \cos (\beta - \alpha) > a_8, \ \pi/2 < h_5 < 3\pi/2 \).

\[ h_6 = \arccos \left[ \frac{(h_2^2 + a_7^2 - a_8^2)}{2a_7h_2} \right] , \quad 0 \leq h_4 \leq \pi. \]

For the right-hand linkage open,

\[ \phi = \pi - h_5 - h_6 \]

For the right-hand linkage crossed (not shown),

\[ \phi = \pi - h_5 + h_6 \]
B-I. Mathematical Development

The mathematical development of the four-bar linkage equations is essentially the same as that presented by Freudenstein and Sandor (12, 13). The problem is to find a four-bar linkage such that the input and output crank rotations assume five corresponding sets of values, \((\Theta_1, \Phi_1)\). As in the work described in the body of this dissertation, the initial values of \(\Theta_1\) and \(\Phi_1\) are to be determined, and we shall define \(t_1\) and \(v_1\) to represent input and output rotations respectively from those positions. Therefore

\[
\Theta_1 = \Theta_1 + t_1
\]

and

\[
\Phi_1 = \Phi_1 + v_1
\]

The rotation of the coupler will be represented by the corresponding angles \(\Psi_1\) and \(\Upsilon_1\).

The initial positions of the links of the mechanism will be represented by the complex numbers \(z_1, z_2, z_3\) and \(z_4\), as shown in Figure B-1. The position equation for the first precision position is therefore

\[
z_2 + z_3 = z_1 + z_4,
\]
and for the $i^{th}$ position, it is

$$z_2 e^{jt_1} + z_3 e^{j\mu_1} = z_1 + z_4 e^{j\nu_1} \quad i = 2, 3, 4, 5$$

where $j = \sqrt{-1}$.

Figure B-1. Four-Bar Linkage

If we define $\lambda_1 = e^{jt_1}$, $\mu_1 = e^{j\mu_1}$, and $\nu_1 = e^{j\nu_1}$, we can write

$$z_2 \lambda_1 + z_3 \mu_1 = z_1 + z_4 \nu_1 .$$

We can eliminate $z_1$ from the above equations by subtracting the first equation from each of the remaining four.

If at the same time we move all terms to the left of the equality,
we obtain

\[ z_2 \left( 1 - \lambda_2 \right) + z_3 \left( 1 - \lambda_3 \right) + z_4 \left( \nu_2 - 1 \right) = 0 \]
\[ z_2 \left( 1 - \lambda_3 \right) + z_3 \left( 1 - \lambda_4 \right) + z_4 \left( \nu_3 - 1 \right) = 0 \]
\[ z_2 \left( 1 - \lambda_4 \right) + z_3 \left( 1 - \lambda_5 \right) + z_4 \left( \nu_4 - 1 \right) = 0 \]
\[ z_2 \left( 1 - \lambda_5 \right) + z_3 \left( 1 - \lambda_6 \right) + z_4 \left( \nu_5 - 1 \right) = 0 \]

The above set of equations can be thought of as a set of four homogeneous linear complex equations relating the unknowns \( z_2, z_3, \) and \( z_4 \). The coefficients \( 1 - \lambda_1 \) and \( \nu_1 - 1 \) are known and the coefficients \( 1 - \lambda_1 \) are not. In order for a nontrivial solution for \( z_2, z_3 \) and \( z_4 \) to exist, the rank of the coefficient matrix above must be at most two. This can be guaranteed by the singularity of two independent third order submatrices, such as

\[
\begin{vmatrix}
1 - \lambda_2 & 1 - \lambda_3 & 1 - \lambda_4 \\
1 - \lambda_3 & 1 - \lambda_4 & 1 - \lambda_5 \\
1 - \lambda_4 & 1 - \lambda_5 & 1 - \lambda_6
\end{vmatrix} = 0 \quad (B-1)
\]

and

\[
\begin{vmatrix}
1 - \lambda_2 & 1 - \lambda_3 & 1 - \lambda_4 \\
1 - \lambda_3 & 1 - \lambda_4 & 1 - \lambda_5 \\
1 - \lambda_4 & 1 - \lambda_5 & 1 - \lambda_6
\end{vmatrix} = 0 \quad (B-2)
\]

In these two (complex) equations, the quantities \( \lambda_2, \lambda_3, \lambda_4 \), and \( \lambda_5 \) are unknown. Since \( \lambda_1 = e^{iu_1} \), we should be able to solve these equations for the values of \( u_1 \) which will lead to nontrivial solutions for \( z_2, z_3 \) and \( z_4 \).
The method of solution is essentially that given in references (12) and (13). It can be summarized as follows:

Expanding equations (B-1) and (B-2) gives the equations:

\[ A_1 + A_2 \mu_2 + A_3 \mu_3 + A_4 \mu_4 = 0 \]  (B-1)

and

\[ A_5 + A_6 \mu_2 + A_7 \mu_3 + A_8 \mu_5 = 0 \]  (B-2)

where

\[ A_2 = \begin{vmatrix} 1 - \lambda_3 & 1 - \nu_3 \\ 1 - \lambda_4 & 1 - \nu_4 \end{vmatrix}, \]

\[ A_3 = \begin{vmatrix} 1 - \lambda_4 & 1 - \nu_4 \\ 1 - \lambda_2 & 1 - \nu_2 \end{vmatrix}, \]

\[ A_4 = \begin{vmatrix} 1 - \lambda_2 & 1 - \nu_2 \\ 1 - \lambda_3 & 1 - \nu_3 \end{vmatrix}, \]

\[ A_6 = \begin{vmatrix} 1 - \lambda_3 & 1 - \nu_3 \\ 1 - \lambda_5 & 1 - \nu_5 \end{vmatrix}, \]

\[ A_7 = \begin{vmatrix} 1 - \lambda_5 & 1 - \nu_5 \\ 1 - \lambda_2 & 1 - \nu_2 \end{vmatrix}, \]

and

\[ A_5 = -A_6 - A_7 - A_8 \]

Eliminating \( \mu_4 \) from equation (B-1') and \( \mu_5 \) from equation (B-2') gives

\[ \mu_3 (B_1 + B_2 \mu_2) + B_5 \mu_2 + B_7 + B_5 \mu_2 + \mu_3 (B_1 + B_2 \mu_2) = 0 \]  (B-3)
and

\[ \mu_3(\beta_3 + \beta_4 \overline{m}_2) + \beta_6 \overline{m}_2 + B_8 \beta_6 \mu_2 + \overline{\mu}_3(\beta_3 + \beta_4 \mu_2) = 0 \quad (B-4) \]

where

- \( B_1 = \lambda_1 \overline{\lambda}_3 \)
- \( B_2 = \lambda_2 \overline{\lambda}_3 \)
- \( B_3 = \lambda_5 \overline{\lambda}_7 \)
- \( B_4 = \lambda_6 \overline{\lambda}_7 \)
- \( B_5 = \lambda_1 \lambda_2 \)
- \( B_6 = \lambda_5 \overline{\lambda}_6 \)
- \( B_7 = |A_1|^2 + |A_2|^2 + |A_3|^2 - |A_4|^2 \) (real)
- \( B_8 = |A_5|^2 + |A_6|^2 + |A_7|^2 - |A_4|^2 \), (real)

and \( \overline{\beta}_1 \) represents the complex conjugate of \( \beta_1 \).

Eliminating \( \mu_3 \) from equations (B-3) and (B-4) gives the equations:

\[
\begin{align*}
\c_1 \overline{c}_4 \overline{\mu}_2^3 + \overline{\mu}_2^2 (c_1 \overline{c}_3 + c_2 \overline{c}_4 + \overline{c}_5^2) + \overline{m}_2 (c_1 \overline{c}_2 + c_2 \overline{c}_3 + c_3 \overline{c}_4 + 2 \overline{c}_5 \overline{c}_6) \\
+ (|c_1|^2 + |c_2|^2 + |c_3|^2 + |c_4|^2 - 2 |c_5|^2 + c_6^2) \\
+ \mu_2 (\overline{c}_1 c_2 + \overline{c}_2 c_3 + \overline{c}_3 c_4 - 2 c_5 c_6) \\
+ \mu_2^2 (\overline{c}_1 c_3 + \overline{c}_2 c_4 + c_5^2) + \overline{c}_1 c_4 \mu_2^3 = 0, \quad (B-5)
\end{align*}
\]

where

- \( c_1 = \overline{B}_2 B_6 - \overline{B}_4 B_5 \),
- \( c_2 = \overline{B}_1 B_6 + B_2 B_8 - \overline{B}_3 B_5 - B_4 B_7 \),
- \( c_3 = \overline{B}_1 B_8 + B_2 B_6 - \overline{B}_3 B_7 - B_4 B_5 \).
\[ C_4 = \overline{B_1 B_6} - \overline{B_2 B_5}, \]
\[ C_5 = B_2 \overline{B_3} - B_1 \overline{B_4}, \]
and
\[ C_6 = \overline{B_1 B_3} + \overline{B_2 B_4} - B_1 \overline{B_3} - B_2 \overline{B_4} \] (Imaginary).

Now if we define
\[ D_1 = \overline{C_1 C_4}, \]
\[ D_2 = \overline{C_1 C_3} + \overline{C_2 C_4} + C_5, \]
\[ D_3 = \overline{C_1 C_2} + \overline{C_2 C_3} + \overline{C_3 C_4} - 2 C_5 C_6, \]
and
\[ D_4 = (1/2)(1 C_1^2 + 1 C_2^2 + 1 C_3^2 + 1 C_4^2 - 2 |C_5|^2 + C_6^2)(\text{real}), \]
equation (B-5) can be written
\[ \Re(D_1 \mu_2^3 + D_2 \mu_2^2 + D_3 \mu_2) + D_4 = 0, \] (B-6)
where \( \Re(z) \) indicates the real part of \( z \).

Since \( \mu_2 = e^{j \omega_2} = \cos \omega_2 + j \sin \omega_2, \)
\( \mu_2^2 = e^{j 2 \omega_2} = \cos 2 \omega_2 + j \sin 2 \omega_2, \)
and
\( \mu_2^3 = e^{j 3 \omega_2} = \cos 3 \omega_2 + j \sin 3 \omega_2, \)
we can express equation (B-6) in terms of these trigonometric functions. Then making use of the trigonometric identities:
\[ \cos 3\omega_2 = 4 \cos^3 \omega_2 - 3 \cos \omega_2, \]
\[ \sin 3\omega_2 = 3 \sin \omega_2 - 4 \sin^3 \omega_2, \]
\[ \cos 2\omega_2 = 2 \cos^2 \omega_2 - 1, \]
and
\[ \sin 2\omega_2 = 2 \cos \omega_2 \sin \omega_2, \]
the equation can be written in terms of powers of \( \sin \omega_2 \) and \( \cos \omega_2 \).
Then defining $\tau = \tan (u_2/2)$ and using the identities
\[
\cos u_2 = (1 - \tau^2)/(1 + \tau^2)
\]
and
\[
\sin u_2 = 2\tau/(1 + \tau^2)
\]
the following polynomial in $\tau$ is obtained:
\[
E_6\tau^6 + E_5\tau^5 + E_4\tau^4 + E_3\tau^3 + E_2\tau^2 + E_1\tau + E_0 = 0
\] (B-7)
where
\[
E_6 = -D_1x + D_2x - D_3x + D_4y,
E_5 = -6D_1y + 4D_2y - 2D_3y,
E_4 = 15D_1x - 5D_2x - D_3x + 3D_4y,
E_3 = 20D_1y - 4D_2y,
E_2 = -15D_1x - 5D_2x + D_3x + 3D_4y,
E_1 = -6D_1y - 4D_2y - 2D_3y,
\]
and
\[
E_0 = D_1x + D_2x + D_3x + D_4y = 0.
\]
$D_1x$ signifies the real part of $D_1$ and $D_1y$ signifies the imaginary part.

Looking back to equations (B-1) and (B-2) we can see that there are three values of $\mu_1$ which will lead to trivial solutions to the equations. These values are
\[
\mu_1 = 1,
\]
\[
\mu_1 = \lambda_1,
\]
and
\[
\mu_1 = \nu_1.
\]
The first of these requires that $\tau = 0$ and is indicated by the fact that $E_0 = 0$. Therefore we can divide the equation (B-7) by $\tau$ to obtain a fifth-order polynomial.
The other trivial solutions are
\[ \tau_4 = \tan \left( \frac{t_2}{2} \right) \]
and
\[ \tau_5 = \tan \left( \frac{v_2}{2} \right). \]
These can be divided out of the fifth-power equation, and the result will be a cubic equation in \( \tau \). In the program used, the quadratic expression \( \tau = p \tau + q \) was taken as an approximation to a factor of the fifth-order polynomial and was improved by means of a Bairstow iteration (15), where
\[ p = \tan \left( \frac{t_2}{2} \right) + \tan \left( \frac{v_2}{2} \right), \]
and
\[ q = \tan \left( \frac{t_2}{2} \tan \left( \frac{v_2}{2} \right) \right). \]

The resulting cubic equation was solved by an iterative method described by Sarafyan and used by Freudenstein (11).

The cubic equation in \( \tau \) will have either one or three real roots, indicating one or three solutions to the problem. If only one real root is obtained, the following calculations are carried out only once, but if multiple real roots are obtained, they are carried out for each of the roots in succession.

Once a root \( \tau \) has been found, the complex number \( \mu_2 \) can be evaluated from the definition of \( \tau \). Substituting this value of \( \mu_2 \) into equation (B-3) gives an equation from which \( \mu_3 \) can be determined. \( \mu_3 \), however, is double valued so that two answers are obtained, only one of which is correct. In order to determine the correct value of \( \mu_3 \), the given value of \( \mu_2 \) is also substituted into equation (B-4), and is computed again. The four values of \( \mu_3 \) are then compared and the proper solution is identified. The
quantities \( \mu_4 \) and \( \mu_5 \) are then determined from equations obtained by eliminating \( \mu_3 \) from equations (B-1') and (B-2'). As a check on the accuracy of the solution, the values of \( \mu_2, \mu_3, \mu_4 \) and \( \mu_5 \) are substituted into equations (B-1') and (B-2'). If the values are correct, both equations should give a zero remainder. The sum of the squares of the absolute values of the remainders is used as an indication of the errors in the solution.

With \( \mu_2 \) and \( \mu_3 \) known, the first three of the position equations can be used to evaluate \( z_2, z_3 \) and \( z_4 \) in terms of \( z_1 \). For convenience in interpreting the results, let us take \( z_1 = + 1 \). Then the three linear equations to be solved for \( z_2, z_3 \) and \( z_4 \) are

\[
\begin{align*}
z_2 + z_3 - z_4 &= 1, \\
z_2(1 - \lambda_2) + z_3(1 - \mu_2) + z_4(\psi_2 - 1) &= 0, \\
z_2(1 - \lambda_3) + z_3(1 - \mu_3) + z_4(\psi_3 - 1) &= 0
\end{align*}
\]

Upon solution of these equations, the linkage is determined and the results may be taken for use in the six-link program.

B-II. Input to Computer Program

The input to the computer is by means of five punched cards with five significant words per card. The first two columns are used for an identifying number, 01, 02, 03, 04, and 05 on successive cards. The first significant number on card 01 is the lower limit of the range of the independent variable \( x \). The remaining four numbers are the first four precision point values of \( x \). Each of these numbers
can be expressed to six significant figures. The format to be used is given by the first statement in the accompanying program. Only four numbers are given on card 02. These are the fifth precision point value of $x$, the upper limit of the range of $x$, the lower limit of the range of $x$ (repeated from card 01) and the upper limit of the range of $x$ (also repeated). The fifth word on card 02 is left blank.

Card 03 gives the values of the dependent variable $y$ corresponding to the values of $x$ given on card 01. Card 04 has as its first two words the values of $y$ corresponding to the first two values of $x$ given on card 02. The next two words on card 04 give the lower and upper limits, respectively, of the range of $y$. The fifth word on card 04 is left blank. There are five significant words on card 05. The first is the range of the input angle $\theta$. The second is the range of the output angle $\phi$. These angles are given in degrees. The third is a small number $\varepsilon_1$ which halts the Bairstow iteration in the synthetic division of the quadratic factor into the fifth-order equation when the remainder is sufficiently small. The fourth number is another small number $\varepsilon_2$ which stops the iteration in the solution of the cubic equation for $\tau$ when the result is sufficiently close to a solution. The fifth is a small number $\varepsilon_3$ used in the comparison of the four values obtained for $\mu_3$. If no two of the values differ by less than this number, the operator is notified by means of a specially punched card, the number $\varepsilon_3$ is multiplied by ten, and the comparison is repeated. These five cards complete the input to the
computer. In the example problems described in Chapter VII, each of the three numbers $C_1$, $C_2$, and $C_3$ were taken to be $10^{-6}$.

B-III. The Computer Output

Due to the length of the computations, the computer program had to be divided into two parts. The first part carries the work through the solution of the cubic equation for $t$, and the second part continues from that point to the end. Numbers which are needed in the second part of the computation are then printed on magnetic tape. There are 26 of these numbers, and they need not be listed here since they are of no particular interest to the user of this program. There is no external output from the first part.

The second half of the program completes the calculations. No external input is required. The machine need only be reset and started. The input to the second part is read from the magnetic tape. The calculations then continue using the first value of $t$ obtained in part one.

The output from the second part of the calculations is typed by a line printer. This output from part two consists of two lines for each of the real roots of the cubic equation for $t$. If there is only one real root, there will be only two lines, the first identified by the number 1 at the left, the second by 2. The first of these lines will contain four numbers other than the identification 1. These numbers are: (1) the value of the first root $t_1$; (2) the error obtained when this first root was substituted back into equations (B-1') and (B-2'); (3) the length of the driving
crank $a_2$ and (4) the value of the input crank angle $\Theta_1$ at the first precision point. The second output line, labeled 2, also has four numbers, which are: (1) the length of the coupler $a_3$; (2) the initial position of the coupler link $\psi_1$; (3) the length of the output crank $a_4$; and (4) the angular position of the output crank $\phi_1$ at the first precision point. It should be pointed out here that the length of the fixed link $a_1$ was taken as unity. If there are other real solutions to the cubic equation, output lines 3 and 4 will give corresponding information about the second root and lines 5 and 6 will give the same information using the third root. After printing this last output line the machine stops.

B-IV. The Computer Program

The programs which follow are written in FORTRAN language. The equivalent machine instructions have not been included because of their length and because the FORTRAN statements are much easier for a reader to follow. Persons not familiar with the FORTRAN language are referred to International Business Machines Corporation publications: "Programmers Primer for FORTRAN" (Form 32-0306-1) and "Programmer's Reference Manual for FORTRAN" (Form 32-7026).
COMPUTER PROGRAM FOR DESIGNING FOUR-BAR LINKAGES
WITH FIVE PRECISION POINTS

PART ONE

1 FORMAT(I2,5E13.6)
DIMENSION X(9),Y(9),T(9),V(9)
READ1,I*(X(K),K=1,9)
READ1,I*(X(K),K=6,9)
READ1,I*(Y(K),K=1,9)
READ1,I*(Y(K),K=6,9)
READ1,I*DT,DV,E1,E2,F3
2 FORMAT(41H COEFFICIENT OF FIFTH POWER OF TAU EQUALS
2 1 5H ZERO)
3 FORMAT(39H A68 = 0, PROCEEDING WITH CALCULATIONS,)
3 1 12H CHECK ERROR
RT=DT*0.017453293/(X(9)-X(8))
RV=DV*0.017453293/(Y(9)-Y(8))
6 DO91=1,5
7 T(I)=RT*(X(I+1)-X(I))
8 V(I)=RV*(Y(I+1)-Y(I))
9 CONTINUE
CT1=COSF(T(1))
CT2=COSF(T(2))
CT3=COSF(T(3))
CT4=COSF(T(4))
CT5=COSF(T(5))
ST1=SINF(T(1))
ST2=SINF(T(2))
ST3=SINF(T(3))
ST4=SINF(T(4))
ST5=SINF(T(5))
CV1=COSF(V(1))
CV2=COSF(V(2))
CV3=COSF(V(3))
CV4=COSF(V(4))
CV5=COSF(V(5))
SV1=SINF(V(1))
SV2=SINF(V(2))
SV3=SINF(V(3))
SV4=SINF(V(4))
SV5=SINF(V(5))
A02 = (1. - CT3) * (1. - CV4) - ST3*SV4
1 = (1. - CT4) * (1. - CV3) + ST4*SV3
A12 = ST4*(1. - CV3) + SV3*(1. - CT4)
1 = ST3*(1. - CV4) - SV4*(1. - CT3)
A03 = (1. - CT4) * (1. - CV2) - ST4*SV2
1 = (1. - CT2) * (1. - CV4) + ST2*SV4
A13 = ST2*(1. - CV4) + SV4*(1. - CT2)
1 = ST4*(1. - CV2) - SV2*(1. - CT4)
A04 = (1. - CT2) * (1. - CV3) - ST2*SV3
1 = (1. - CT3) * (1. - CV2) + ST3*SV2
A14 = ST3*(1. - CV2) + SV2*(1. - CT3)
1 = ST2*(1. - CV3) - SV3*(1. - CT2)
A01 = -A02 - A03 - A04
A11 = -A12 - A13 - A14
A06 = (1. - CT3) * (1. - CV5) - ST3*SV5
1 = (1. - CT5) * (1. - CV3) + ST5*SV3
A16 = ST5*(1. - CV3) + SV3*(1. - CT5)
1 = ST3*(1. - CV5) - SV5*(1. - CT3)
A07 = (1. - CT5) * (1. - CV2) - ST5*SV2
1 = (1. - CT2) * (1. - CV5) + ST2*SV5
A17 = ST2*(1. - CV5) + SV5*(1. - CT2)
1 = ST5*(1. - CV2) - SV2*(1. - CT5)
A05 = -A06 - A07 - A04
A15 = -A16 - A17 - A14
A41 = A01*A03 + A11*A13
A51 = A11*A03 - A01*A13
A42 = A02*A03 + A12*A13
A52 = A12*A03 - A02*A13
A43 = A05*A07 + A15*A17
A53 = A15*A07 - A05*A17
A44 = A06*A07 + A16*A17
A54 = A16*A07 - A06*A17
A45 = A01*A02 + A11*A17
A55 = A11*A12 - A11*A02
A46 = A05*A06 + A15*A16
A56 = A05*A16 - A15*A06
A47 = A01*A01 + A11*A11 + A02*A02 + A12*A12
+ A03*A03 + A13*A13 - A04*A04 - A14*A14
A48 = A05*A05 + A15*A15 + A06*A06 + A16*A16
+ A07*A07 + A17*A17 - A04*A04 - A14*A14
A61 = A42*A46 - A52*A56 - A44*A45 - A54*A55
A71 = A44*A55 + A54*A45 - A42*A56 - A52*A46
A62 = A41*A46 - A51*A56 + A42*A48 - A43*A45 + A53*A55 - A44*A47
A72 = A43*A55 + A45*A53 + A54*A47 - A51*A46 - A41*A56 - A52*A48
A63 = A41*A48 + A42*A46 + A52*A56 - A43*A47 - A44*A45 - A54*A55
A73 = A42*A56 - A51*A48 + A53*A47 - A52*A46 - A44*A55 - A54*A45
A64 = A41*A46 + A51*A56 - A43*A45 - A53*A55
A74 = A41*A56 - A51*A46 - A43*A55 + A53*A45
A65 = A42*A46+A52*A53-A41*A44-A51*A54
A75 = A51*A44-A41*A54+A52*A43-A42*A53
A76 = 2*A41*A53-A42*A54-A43*A54
A41 = A61*A64+A71*A74
A51 = A61*A74-A71*A64
A42 = A61*A63+A71*A73+A62*A72*A74+A65*A65-A75*A75
A52 = A61*A73-A71*A63+A62*A74-A72*A74+2*A65*A75
A43 = A61*A62+A71*A72+A62*A63+A72*A73
1 + A63*A64+A73*A74+2*A75*A76
1 + A63*A74-A73*A64-2*A65*A76
A44 = (A61*A61+A71*A71+A62*A72*A72*A72+A63*A63+A73*A73
1 + A64*A64+A74*A74-2*A65*A65-2*A75*A75-A76*A76/2
A61 = -6*A51+4*A52-2*A53
A62 = 15*A41-5*A42-A43+3*A44
A63 = 20*A51-4*A53
A64 = -15*A41-5*A42+A43+3*A44
A65 = -6*A51-4*A52-2*A53
A66 = A41+A42-A43+A44
IF(A66) 10*122*10

10 CONTINUE
A41 = A61/A66
A42 = A62/A66
A43 = A63/A66
A44 = A64/A66
A45 = A65/A66
A48 = (SINF(T(2)/2))/COSF(T(2)/2))
1 + (SINF(V(2)/2))/COSF(V(2)/2))
A49 = (SINF(T(2)/2))*SINF(V(2)/2))
1 / (COSF(T(2)/2)*COSF(V(2)/2))

11 A53 = A41+A48
A54 = A42+A48*A53-A49
A55 = A43+A48*A54-A49*A54
A56 = A44+A48*A55-A49*A55
A57 = A45+A48*A56-A49*A56
IF(A56) 107*106*107
106 IF.A57/107*14*107
107 A63 = A53+A48
A64 = A54+A48*A63-A49
A65 = A55+A48*A64-A49*A63
A67 = A48*A65-A49*A64
A68 = A65*A65-A67*A64
IF(A68) 108*123*108
108 A69 = A56*A65-A57*A64
109 A70 = A57*A65-A56*A67
IF(A48) 110*111*110
110 IF(A49) 112*113*112
111 IF(A49) 114*115*114
112 A71=-(A69/A68)/A48
113 A72=-(A70/A68)/A49
114 IF(ABS(A71-A71-E1))12,13,13
115 IF(ABS(A72-A72-E1))14,13,13
116 A48=A49*A71
117 A49=A48*A72
118 GOTO111
119 A48=-A69/A68
120 A72=-(A70/A68)/A49
121 IF(ABS(A48-E1))116,117,117
122 IF(ABS(A72-A72-E1))14,117,117
123 A49=A49*A72
124 GOTO111
125 A71=-(A69/A68)/A48
126 A49=A70/A68
127 IF(ABS(A71-A71-E1))118,119,119
128 IF(ABS(A49-E1))14,119,119
129 A48=A48*A71
130 GOTO111
131 A48=-A69/A68
132 A49=A70/A68
133 IF(ABS(A48-E1))120,121,121
134 IF(ABS(A49-E1))14,121,121
135 GOTO111
136 A58=A54-A53*A53/3.
137 A59=A53*A53*A53/13.5-A53*A54/3.+A55
138 IF(A58>15.15,15,24
139 A21=(ABS(A59)**(1./3.))*(ABS(A59)/(-A59))
140 A22=-A59-A58*A21
141 A20=(ABS(A22)**(1./3.))*(ABS(A22)/A22)
142 A23=-A59-A58*A20
143 A21=(ABS(A23)**(1./3.))*(ABS(A23)/A23)
144 IF(ABS(A21-A20)-E2*ABS(A21))17,16,16
145 ROOT1=A53/3.
146 IF((A59*A59/4.+(A58*A58*A58/27.))18,23,29
147 A21=A59^SQRTF(-A58/3.)/ABS(A59)
148 A22=-A59-A58*A21
149 A20=(ABS(A22)**(1./3.))*(ABS(A22)/A22)
150 A23=-A59-A58*A20
151 A21=(ABS(A23)**(1./3.))*(ABS(A23)/A23)
152 IF(ABS(A21-A2C)-E2*ABS(A21))12C,19,19
153 ROOT2=A21-A53/3.
154 A21=A59/(-A58)
155 A20=(A21*A21+A21+A59)/(-A58)
156 A21=(A20*A20*A20+A59)/(-A58)
157 IF(ABS(A21-A20)-E2*ABS(A21))22,21,21
158 ROOT3=A21-A53/3.
159 GOTO127
23  \text{ROOT2} = (A59/\text{ABS}(A59)) \times \text{SQRT}(1-A58/3) - A53/3.
\text{ROOT3} = 0.
\text{GOTO127}

24  \text{IF}(A58\times A58\times A58/27-A59\times A59/16) > 25.26
\text{GOTO29}

25  \text{ROOT1} = ((-A59)/\text{ABS}(A59)) \times \text{SQRT}(A58/3) - A53/3.
\text{GOTO29}

26  A21 = A59/(-A58)

27  A20 = (A21\times A21\times A21 + A59)/(-A58)
A21 = (A20\times A20\times A20 + A59)/(-A58)
\text{IF}(\text{ABS}(A21 - A20) - E2 \times \text{ABS}(A21) > 28.27.27

28  \text{ROOT1} = A21 - A53/3.

29  \text{ROOT2} = 0.

30  \text{ROOT3} = 0.

127  \text{WRITE} TAPE 1, \text{ROOT1}, \text{ROOT2}, \text{ROOT3}, E3, A01, A02, A12,
127  1 A03, A13, A04, A14, A05, A15, A06, A16, A07,
127  2 A17, CT2, ST2, CT3, ST3, CV2, SV2, CV3, SV3

\text{BACKSPACE1}

1^5 \text{STOP}

122 \text{PRINT} 2
\text{GOTO1C5}

123 \text{PRINT} 3
\text{GOTO14}
COMPUTER PROGRAM FOR DESIGNING FOUR-BAR LINKAGES

WITH FIVE PRECISION POINTS

PART TWO

1 FORMAT(12,5E13.6)
35 FORMAT(12,9H ROOT = *E13.6,9H ERROR = *E10.3,
36 \ 7H Z2 = *E14.7,11H THETA1 = *E14.7)
36 FORMAT(12,7H Z3 = *E14.7,9H PSI1 = *E14.7,
36 7H Z4 = *E14.7,9H PHI1 = *E14.7)
5 FORMAT(42H PSI3 WILL NOT CHECK CALCULATIONS CONTINUE
5 26H WITH E3 MULTIPLIED BY TEN)

READ TAPE1*ROOT1*ROOT2*ROOT3*E3*A01*A11*A02*A12
1 A03*A13*A04*A14*A05*A15*A06*A16*A07,
2 A17,CT2,ST2,CT3,ST3,CV2,SV2,CV3,SV3

L=1
32 CU2=(1.ROOT1*ROOT1)/(1. +ROOT1*ROOT1)
33 SU2=2.*ROOT1/(1. +ROOT1*ROOT1)
30 IF(CU2)33,34,35

33 U2=3.14159265+ ATANF(SU2/CU2)
GOTO36
34 U2=1.57079633*(ABSF(SU2)/SU2)
GOTO36
35 U2= ATANF(SU2/CU2)
36 A08=A01+A02*CU2-A12*SU2
A18=A11+A02*SU2+A12*CU2
A09=A05+A06*CU2-A16*SU2
A19=A15+A06*SU2+A16*CU2
A30=A04*A04+A14*A14-A03*A03-A13*A13-A08*A08-A18*A18
A31=2.*SQRTF((A08*A08+A18*A18)*(A03*A03+A13*A13))
CP1=A30/A31
SP1=SQRTF(1. -CP1*CP1)
IF(CP1)37,38,39

37 P1=3.14159265+ ATANF(SP1/CP1)
GOTO40
38 P1=1.57079633
GOTO40
39 P1= ATANF(SP1/CP1)
40 IF(A08)41,42,43

41 P2=3.14159265+ ATANF(A18/A08)
GOTO44
42 P2=(ABSF(A18)/A18)*1.57079633
GOTO44
43 P2= ATANF(A18/A08)
44 IF(A03)45,46,47

45 P3=3.14159265+ ATANF(A13/A03)
GOTO48
46  \( P_3 = (\text{ABSF}(A_{13})/A_{13}) \times 1.57079633 \)  
   GOTO 048
47  \( P_3 = \text{ATANF}(A_{13}/A_{03}) \)
48  \( P_7 = P_2 + P_1 - P_3 \)
49  \( P_8 = P_2 - P_1 - P_3 \)  
   IF \((A_{07} \times 130, 131, 132)\)  
130 \( P_4 = 3.14159265 \times \text{ATANF}(A_{17}/A_{07}) \)  
   GOTO 0133
131 \( P_4 = (\text{ABSF}(A_{17})/A_{17}) \times 1.57079633 \)  
   GOTO 0133
132 \( P_4 = \text{ATANF}(A_{17}/A_{07}) \)
133  \( A_{32} = A_{04} \times A_{04} + A_{14} \times A_{14} - A_{07} \times A_{07} - A_{17} \times A_{17} - A_{09} \times A_{09} - A_{19} \times A_{19} \)  
   \( A_{33} = 2 \times \text{SQRTF}\left((A_{07} \times A_{09} + A_{19} \times A_{19}) \times (A_{07} \times A_{07} + A_{17} \times A_{17})\right) \)  
   CP_5 = A_{32} / A_{33}  
   SP_5 = \text{SQRTF}\left(1 - \text{CP}_5^2 \times \text{CP}_5\right) \)  
   IF \((\text{CP}_5) \times 50, 51, 52\)
50  \( P_5 = 3.14159265 \times \text{ATANF}(\text{SP}_5 / \text{CP}_5) \)  
   GOTO 053
51  \( P_5 = 1.57079633 \)  
   GOTO 053
52  \( P_5 = \text{ATANF}(\text{SP}_5 / \text{CP}_5) \)
53  \( \text{IF}\left(\text{A}_09\right) \times 54, 55, 56\)  
54  \( P_6 = 3.14159265 \times \text{ATANF}(A_{19}/A_{09}) \)  
   GOTO 057
55  \( P_6 = 1.57079633 \times \text{ABSF}(A_{19}) / A_{19} \)  
   GOTO 057
56  \( P_6 = \text{ATANF}(A_{19}/A_{09}) \)
57  \( P_9 = P_6 + P_5 - P_4 \)
58  \( P_{10} = P_6 - P_5 - P_4 \)
59  \( P_7 = P_7 + 18.8495559 \)
60  \( \text{IF}\left(P_7 - 6.28318531 \times 62, 61, 61\right) \)  
61  \( P_7 = P_7 - 6.28318531 \)  
   GOTO 060
62  \( P_8 = P_8 + 18.8495559 \)
63  \( \text{IF}\left(P_8 - 6.28318531 \times 65, 64, 64\right) \)  
64  \( P_8 = P_8 - 6.28318531 \)  
   GOTO 063
65  \( P_9 = P_9 + 18.8495559 \)
66  \( \text{IF}\left(P_9 - 6.28318531 \times 69, 67, 67\right) \)  
67  \( P_9 = P_9 - 6.28318531 \)  
   GOTO 066
68  \( P_{10} = P_{10} + 18.8495559 \)
69  \( \text{IF}\left(P_{10} - 6.28318531 \times 71, 70, 70\right) \)  
70  \( P_{10} = P_{10} - 6.28318531 \)  
   GOTO 069
71  \( \text{IF}\left(\text{ABSF}(P_7 - P_9) - E_3\right) \times 75, 75, 75\)  
72  \( \text{IF}\left(\text{ABSF}(P_7 - P_{10}) - E_3\right) \times 76, 76, 76\)  
73  \( \text{IF}\left(\text{ABSF}(P_8 - P_9) - E_3\right) \times 77, 77, 77\)  
74  \( \text{IF}\left(\text{ABSF}(P_8 - P_{10}) - E_3\right) \times 78, 78, 125\)
75  \( U_3 = P_7 \)
    \( A_{38} = +1 \)
    \( A_{39} = +1 \).
GOTO 079
76 U3 = P7
    A38 = +1
    A39 = -1
GOTO 079
77 U3 = P8
    A38 = -1
    A39 = +1
GOTO 079
78 U3 = P8
    A38 = -1
    A39 = -1
79 A34 = A03 * A03 + A13 * A13 - A04 * A04 - A14 * A14 - A08 * A08 - A18 * A18
    A35 = 2 * SQRT( (A08 * A08 + A18 * A18) * (A04 * A04 + A14 * A14) )
    CP11 = A34 / A35
    SP11 = A38 * SQRT(1 - CP11 * CP11)
    IF(CP11) 80, 81, 82
80 P11 = 3.14159265 + ATAN(SP11 / CP11)
GOTO 083
81 P11 = 1.57079633 * (ABS(S11) / SP11)
GOTO 083
82 P11 = ATAN(SP11 / CP11)
83 IF(A04) 84, 85, 86
84 P12 = 3.14159265 + ATAN(A14 / A04)
GOTO 087
85 P12 = 1.57079633 * (ABS(A14) / A14)
GOTO 087
86 P12 = ATAN(A14 / A04)
87 U4 = P2 - P11 - P12
    A36 = A07 * A07 + A17 * A17 - A04 * A04 - A14 * A14 - A09 * A09 - A19 * A19
    A37 = 2 * SQRT( (A04 * A04 + A14 * A14) * (A09 * A09 + A19 * A19) )
    CP13 = A36 / A37
    SP13 = A39 * SQRT(1 - CP13 * CP13)
    IF(CP13) 88, 89, 90
88 P13 = 3.14159265 + ATAN(SP13 / CP13)
GOTO 091
89 P13 = 1.57079633 * (ABS(SP13) / SP13)
GOTO 091
90 P13 = ATAN(SP13 / CP13)
91 U5 = P6 - P13 - P12
    CU3 = COSF(U3)
    CU4 = COSF(U4)
    CU5 = COSF(U5)
    SU3 = SINF(U3)
    SU4 = SINF(U4)
    SU5 = SINF(U5)
    A73 = A01 + A02 * CU2 - A12 * SU2 + A03 * CU3
    - A13 * SU3 + A04 * CU4 - A14 * SU4
    A75 = A11 + A02 * SU2 + A12 * CU2 + A03 * SU3
    + A13 * CU3 + A04 * SU4 + A14 * CU4
    A74 = A05 + A06 * CU2 - A16 * SU2 + A07 * CU3
    - A17 * SU3 + A04 * CU5 - A14 * SU5
A76 = A15 + A06 * SU2 + A16 * CU2 + A07 * SU3
1 + A17 * CU3 + A04 * SU5 + A14 * CU5
ERROR = A73 * A73 + A75 * A75 + A74 * A74 + A76 * A76
A77 = (1 * -CT2) * (1 * -CU3) - ST2 * SU3
1 - (1 * -CT3) * (1 * -CU2) + ST3 * SU2
A87 = ST3 * (1 * -CU2) + SU2 * (1 * -CT3)
1 - ST2 * (1 * -CU3) - SU3 * (1 * -CT2)
A78 = (1 * -CV2) * (1 * -CU3) - SV2 * SU3
1 - (1 * -CV3) * (1 * -CU2) + SV3 * SU2
A88 = SV3 * (1 * -CU2) + SU2 * (1 * -CV3)
1 - SV2 * (1 * -CU3) - SU3 * (1 * -CV2)
A79 = - A77 + A78 + A04
A89 = - A87 + A88 + A14
Z02 = (A78 * A79 + A88 * A89) / (A79 * A79 + A89 * A89)
Z12 = (A88 * A79 - A78 * A89) / (A79 * A79 + A89 * A89)
Z03 = (A04 * A79 + A14 * A89) / (A79 * A79 + A89 * A89)
Z13 = (A14 * A79 - A04 * A89) / (A79 * A79 + A89 * A89)
Z04 = (A77 * A79 + A87 * A89) / (A79 * A79 + A89 * A89)
Z14 = (A79 * A87 - A77 * A89) / (A79 * A79 + A89 * A89)
Z2 = SQRTF(Z02 * Z02 + Z12 * Z12)
Z3 = SQRTF(Z03 * Z03 + Z13 * Z13)
Z4 = SQRTF(Z04 * Z04 + Z14 * Z14)
IF(Z02)92,93,94
THEETA = 180 * ATANF(Z12 / Z02) * 57.2957795
GOTO95
THEETA = (ARSF(Z12) / Z12) * 90.
GOTO95
THEETA = 57.2957795 * ATANF(Z12 / Z02)
IF(Z03)95,96,98
PSI = 180 * 57.2957795 * ATANF(Z13 / Z03)
GOTO99
PSI = 90 * ABSF(Z13) / Z13
GOTO99
PSI = 57.2957795 * ATANF(Z13 / Z03)
IF(Z04)100,101,102
PHI = 180 * 57.2957795 * ATANF(Z14 / Z04)
GOTO103
PHI = 90 * (ARSF(Z14) / Z14)
GOTO103
PHI = 57.2957795 * ATANF(Z14 / Z04)
PRINT135 * L * ROOT1 * ERROR * Z2 * THEETA
L = L + 1
PRINT136 * L * Z3 * PSI * Z4 * PHI
L = L + 1
ROOT1 = ROOT2
ROOT2 = ROOT3
ROOT3 = 0.
IF(ROOT1)32,105,32
STOP
125 PRINT5
E3 = 10. * E3
GOTO71
APPENDIX C

FORTRAN STATEMENTS FOR COMPUTER PROGRAM TO
DESIGN MECHANISM A WITH SIX PRECISION POINTS

DIMENSION X(10), Y(10), T(6), V(6), B(6,9), C(7,7)

FORMAT(I2, I3, 5E13.6)

READ1, I1, N, (X(I), I = 1, 10)
READ1, I1, N, (Y(I), I = 1, 10)
READ1, I1, N, (Z(I), I = 1, 10)
READ1, I1, N, (Z2, Z3, Z4, THETA, PHI)

READ1, I1, N, (Z5, Z6, Z7, ALPHA)
READ1, I1, N, (DT, DV, E1, E2, E3)

R1 = 1.0 / Z2
R2 = 1.0 / Z4
R3 = (Z2*Z2 + Z3*Z3 + Z4*Z4) / (2.0*Z2*Z4)
R4 = 1.0 / Z7
R5 = 1.0 / Z5
R6 = (Z1*Z5 + Z6*Z6 + Z7*Z7) / (2.0*Z5*Z7)
R7 = ALPHA*0.017453293
R8 = THETA*0.017453293
R9 = PHI*0.017453293

RT = (DT/(X(10) - X(9)))*0.017453293
RV = (DV/Y(10) - Y(9)))*0.017453293

DO 31 = 1, 6
T(I) = RT*(X(I + 1) - X(I))
V(I) = RV*(Y(I + 1) - Y(I))
CONTINUE

N = 1

DO 51 = 1, 6
B(I,1) = COSF(R8 + T(I)) - R1
B(I,2) = SINF(R8 + T(I))
B(I,3) = R3 - R2*COSF(R8 + T(I))
B(I,4) = COSF(R7 + R9 + V(I)) + R4*COSF(R7)
B(I,5) = SINF(R7 + R9 + V(I)) + R4*SINF(R7)
B(I,6) = R6 + R5*COSF(R9 + V(I))
B(I,7) = B(I,3)*B(I,5) - B(I,2)*B(I,6)
B(I,8) = B(I,1)*B(I,6) - B(I,3)*B(I,4)
B(I,9) = B(I,1)*B(I,5) - B(I,2)*B(I,4)
C(I,1) = B(I,9)*B(I,5) - B(I,4)*B(I,6)
C(I,2) = (B(I,8)*B(I,4) - B(I,7)*B(I,5)) * COSF(R8 + T(I))
C(I,3) = (B(I,7)*R(I,5) - B(I,8)*R(I,4)
C(I,4) = (B(I,7)*B(I,3) - B(I,9)*B(I,1)) * SINF(R7)

1 + (B(I,9)*B(I,2) - B(I,8)*B(I,3)) * COSF(R7)
C(I,5) = (B(I,8)*B(I,1) - B(I,7)*B(I,2)) * COSF(R9 + V(I))
C(I,6) = B(I,8)*B(I,1) - B(I,7)*B(I,2)
\( C(I, 7) = 0.5 \times (B(I, 9) \times B(I, 9) - B(I, 8) \times B(I, 8) - B(I, 7) \times B(I, 7)) \)

5 CONTINUE
L=1
11 J=2
M=1
12 IF(ABSF(C(M,1)) - ABSF(C(J,1)))13.14.14
13 M=J
14 IF(J+L-7)15,16,16
15 J=J+1
GOTO12
16 IF(M-1)26,26,17
17 K=1
18 C(7,K)=C(1,K)
IF(K+L-8)19,20,20
19 K=K+1
GOTO18
20 K=1
21 C(J,K)=C(M,K)
IF(K+L-8)22,23,23
22 K=K+1
GOTO21
23 K=1
24 C(M,K)=C(7,K)
IF(K+L-8)25,26,26
25 K=K+1
GOTO24
26 K=1
27 C(7,K)=C(1,K+1)/C(1,1)
IF(K+L-7)28,29,29
28 K=K+1
GOTO27
29 J=1
30 K=1
31 C(J,K)=C(J+1,K+1)-C(7,K)*C(J+1,1)
IF(K+L-7)32,33,33
32 K=K+1
GOTO31
33 IF(J-5)34,35,35
34 J=J+1
GOTO30
35 K=1
36 C(6,K)=C(7,K)
IF(K+L-7)37,38,38
37 K=K+1
GOTO36
38 IF(L-6)39,40,40
39 L=L+1
IF(L-6)11,26,26
40 DELR=ABSF(C(1,1))+ABSF(C(2,1))+ABSF(C(3,1)) + ABSF(C(4,1))+ABSF(C(5,1))+ABSF(C(6,1))
40 1 R1=R1+C(1,1)
R2 = R2 + C(2,1)
R3 = R3 + C(3,1)
R4 = R4 + C(4,1)
R5 = R5 + C(5,1)
R6 = R6 + C(6,1)
SUMR = ABSF(R1) + ABSF(R2) + ABSF(R3) + ABSF(R4) + ABSF(R5) + ABSF(R6)

I1 = 2
PRINT1, I1, N, R1, R2, R3, R4, R5
I1 = 3
PRINT1, I1, N, R6, R7, R8, R9, DELR
IF (DELR - E1 * SUMR) 43, 41, 41
IF (FLOATF(N) - E2) 42, 43, 43
N = N + 1
GOTO 4

I1 = 2
PUNCH1, I1, N, R1, R2, R3, R4, R5
I1 = 3
PUNCH1, I1, N, R6, R7, R8, R9, DELR
I1 = 4
PUNCH1, I1, N, E3, RT, DT, X(2), X(1)
Z2 = 1./R1
Z3 = SQRTF(1.+1./(R1*R1) + 1./(R2*R2) - 2.*R3/(R1*R2))
Z4 = 1./R2
Z5 = 1./R5
Z6 = SQRTF(1.+1./(R4*R4) + 1./(R5*R5) - 2.*R6/(R4*R5))
Z7 = 1./R4
THETA = R8 * 57.295730
ALPHA = R7 * 57.295780
PHI = R9 * 57.295780
I1 = 5
PUNCH1, I1, N, Z2, Z3, Z4, THETA, PHI
PRINT1, I1, N, Z2, Z3, Z4, THETA, PHI
N = N + 1
PUNCH1, I1, N, Z5, Z6, Z7, ALPHA
PRINT1, I1, N, Z5, Z6, Z7, ALPHA
STOP
FORTRAN STATEMENTS FOR COMPUTER PROGRAM TO
DESIGN MECHANISM A WITH SEVEN PRECISION POINTS

DIMENSION X(10), Y(10), T(7), V(7), B(7,9), C(8,8)
1 FORMAT(12,13,5E13.6)
2 FORMAT(13,9E13.6)
3 FORMAT(36H N Z2 Z3 Z4)
4 FORMAT(36H N R1 R2 R3)
5 FORMAT(36H N R4 R5 R6 R7)
6 FORMAT(36H N R8 DELR1)
7 READ1, I1, N, (X(I), I=1,5)
8 READ1, I1, N, (X(I), I=6,10)
9 READ1, I1, N, (Y(I), I=1,5)
10 READ1, I1, N, (Y(I), I=6,10)
11 READ1, I1, N, Z2, Z3, Z4, THETA, PHI
12 READ1, I1, N, Z5, Z6, Z7, ALPHA
13 READ1, I1, N, DT, DV, E1, F1, E3
14 R1 = 1.0/Z2
15 R2 = 1.0/Z4
16 R3 = (1.0+Z2*Z2-Z3*Z3+Z4*Z4)/(2.0+Z2+Z4)
17 R4 = 1.0/Z7
18 R5 = 1.0/Z5
19 R6 = (1.0+Z5*Z5-Z6*Z6+Z7*Z7)/(2.0+Z5+Z7)
20 R7 = ALPHA*0.017453293
21 R8 = THETA*0.017453293
22 R9 = PHI*0.017453293
23 N = 0
24 PRINT3
25 PRINT2, N, Z2, Z3, Z4, ALPHA, Z5, Z6, Z7, THETA, PHI
26 PRINT4
27 PRINT2, N, R1, R2, R3, R4, R5, R6, R7, R8
28 RT = (DT/(X(10)))*0.017453293
29 RV = (DV/(Y(10)))*0.017453293
30 DO81 = 1, 7
31 T(I) = RT*(X(I+1)-X(I))
32 V(I) = RV*(Y(I+1)-Y(I))
33 8 CONTINUE
34 N = 1
35 DO101 = 1, 7
36 B(I,1) = COSF(R8+T(I))-R1
37 B(I,2) = SINF(R8+T(I))
38 B(I,3) = R3-R2*COSF(R8+T(I))
39 B(I,4) = COSF(R7+R9+V(I))+R4*COSF(R7)
\[ B(I,5) = \sin(R7 + R9 + V(I)) + R4 \sin(R7) \]
\[ B(I,6) = R6 + R5 \cos(R9 + V(I)) \]
\[ B(I,7) = B(I,3) \cdot B(I,5) - B(I,2) \cdot B(I,6) \]
\[ B(I,8) = B(I,1) \cdot B(I,6) - B(I,3) \cdot B(I,4) \]
\[ B(I,9) = B(I,1) \cdot B(I,5) - B(I,2) \cdot B(I,4) \]
\[ C(I,1) = B(I,9) \cdot B(I,5) - B(I,8) \cdot B(I,6) \]
\[ C(I,2) = (B(I,8) \cdot B(I,4) - B(I,7) \cdot B(I,5)) \cdot \cos(R9 + V(I)) \]
\[ C(I,3) = B(I,7) \cdot B(I,5) - B(I,8) \cdot B(I,4) \]
\[ C(I,4) = B(I,7) \cdot B(I,3) - B(I,9) \cdot B(I,1) \cdot \sin(R7) \]
\[ 1 + (B(I,9) \cdot B(I,2) - B(I,8) \cdot B(I,3)) \cdot \cos(R7) \]
\[ C(I,5) = (B(I,8) \cdot B(I,1) - B(I,7) \cdot B(I,2)) \cdot \cos(R9 + V(I)) \]
\[ C(I,6) = B(I,8) \cdot B(I,1) - B(I,7) \cdot B(I,2) \]
\[ C(I,7) = B(I,4) \cdot B(I,3) \cdot B(I,7) - B(I,1) \cdot B(I,9) \]
\[ 1 + B(I,5) \cdot (B(I,3) \cdot B(I,8) - B(I,2) \cdot B(I,9)) \]
\[ C(I,8) = 0.5 \cdot (B(I,9) \cdot B(I,9) - B(I,8) \cdot B(I,8) - B(I,7) \cdot B(I,7)) \]

10 CONTINUE
L = 1
11 J = 2
M = 1
12 IF(ABSFC(M*1) - ABSFC(J*1)) 13, 14, 14
13 M = J
14 IF(J + L - 8) 15, 16, 16
15 J = J + 1
GOTO 12
16 IF(M - 1) 26, 26, 17
17 K = 1
18 C(8, K) = C(1, K)
19 IF(K + L - 9) 19, 20, 20
20 K = K + 1
GOTO 18
21 K = 1
22 C(1, K) = C(M, K)
23 IF(K + L - 9) 22, 23, 23
24 K = K + 1
GOTO 21
25 K = 1
26 C(M, K) = C(8, K)
27 IF(K + L - 9) 27, 28, 28
28 K = K + 1
GOTO 24
29 K = 1
30 C(8, K) = C(1, K + 1) / C(1, 1)
31 IF(K + L - 8) 28, 29, 29
32 K = K + 1
GOTO 27
33 J = 1
34 K = 1
35 C(J, K) = C(J + 1, K + 1) - C(8, K) \cdot C(J + 1, 1)
36 IF(K + L - 8) 32, 33, 33
37 K = K + 1
GOTO 31
33 IF(J-6)34,35,35
34 J=J+1
35 GOTO30
36 K=1
37 K=K+1
38 IF(L-7)39,40,40
39 L=L+1
40 1 DFLR=ABS(C(1+1)+ABS(C(2+1)+ABS(C(3+1)
40 2 +ABS(C(4+1)+ABS(C(5+1)+ABS(C(6+1)
40 3 +ABS(C(7+1))
40 4 R1=R1+C(1+1)
40 5 R2=R2+C(2+1)
40 6 R3=R3+C(3+1)
40 7 R4=R4+C(4+1)
40 8 R5=R5+C(5+1)
40 9 R6=R6+C(6+1)
40 10 R7=R7+C(7+1)
40 11 SUMR=ABS(R1)+ABS(R2)+ABS(R3)+ABS(R4)+ABS(R5)+ABS(R6)+ABS(R7)
41 IF(DFLR-E1*SUMR)43,41,41
42 N=N+1
43 GOTO9
44 I1=2
45 PUNCH1,1+1,N,R1,R2,R3,R4,R5
46 I1=3
47 PUNCH1,1+1,N,R6,R7,R8,R9,DELR
48 I1=4
49 PUNCH1,1+1,N,E3,RT,DT,X(2),X(1)
50 Z2=1/R1
51 Z3=SQR(TF(1+1./(R1*R1)+1./(R2*R2)-2.*R3/(R1*R2))
52 Z4=1/R2
53 Z5=1/R5
54 Z6=SQR(TF(1+1./(R4*R4)+1./(R5*R5)-2.*R6/(R4*R5))
55 Z7=1/R4
56 THETA=R8*57.295780
57 ALPHA=R7*57.295780
58 PHI=R9*57.295780
59 PRINT3
60 PRINT2,N,Z2,Z3,Z4,ALPHA,Z5,Z6,Z7,THETA,PHI
61 I1=5
62 PUNCH1,1+1,N,Z2,Z3,Z4,THETA,PHI
63 N=N+1
64 PUNCH1,1+1,N,Z5,Z6,Z7,ALPHA
65 STOP
APPENDIX E

FORTRAN STATEMENTS FOR COMPUTER PROGRAM TO
DESIGN MECHANISM A WITH EIGHT PRECISION POINTS

DIMENSION X(10),Y(10),T(87),V(8),B(8,9),C(9,9)
1 FORMAT(12,13,5E13.6)
2 FORMAT(I3,9E13.6)
3 FORMAT(36H N Z2 Z3 Z4)
3 1 39H ALPHA Z5 Z6
3 2 39H Z7 THEETA PHI
4 FORMAT(36H N R1 R2 R3)
4 1 39H R4 R5 R6
4 2 40H R7 R8 DELR
READ1,11,N,(X(I),I=1,5)
READ1,11,N,(Y(I),I=6,10)
READ1,11,N,(Z(I),I=1,5)
READ1,11,N,(Z(I),I=6,10)
READ1,11,N,Z2,Z3,Z4,Z7,Z5,Z6,ZA,PHA
READ1,11,N,DT,DV,E1,E2,E3
R1=1./Z2
R2=1./Z4
R3=1.+(Z2*Z2-Z3*Z3-Z4*Z4)/(2.*Z2*Z4)
R4=1./Z7
R5=1./Z5
R6=1.+(Z5*Z5-Z6*Z6+Z7*Z7)/(2.*Z5*Z7)
R7=ALPHA*0.017453293
R8=THETA*0.017453293
R9=PHI*0.017453293
N=0
PRINT3
PRINT2,N,Z2,Z3,Z4,ALPHA,Z5,Z6,Z7,THETA,PHA
PRINT4
PRINT2,N,R1,R2,R3,R4,R5,R6,R7,R8
RT=(DT/(X(10)))**0.017453293
RV=(DV/(Y(10)))**0.017453293
DO81=1.8
T(I)=RT*(X(I+1)-X(2))
V(I)=RV*(Y(I+1)-Y(2))
8 CONTINUE
N=1
9 DO101=1.8
B(I,1)=COSF(R8+T(I))-R1
B(I, 2) = \text{SINF}(R8 + T(I))
B(I, 3) = R3 - R2 * \text{COSF}(R8 + T(I))
B(I, 4) = \text{COSF}(R7 + R9 + V(I)) + R4 * \text{COSF}(R7)
B(I, 5) = \text{SINF}(R7 + R9 + V(I)) + R4 * \text{SINF}(R7)
B(I, 6) = R6 + R5 * \text{COSF}(R9 + V(I))
B(I, 7) = B(I, 3) * B(I, 5) - B(I, 2) * B(I, 6)
B(I, 8) = B(I, 1) * B(I, 6) - B(I, 3) * B(I, 4)
B(I, 9) = B(I, 1) * B(I, 5) - B(I, 2) * B(I, 4)
C(I, 1) = B(I, 9) * B(I, 5) - B(I, 8) * B(I, 6)
C(I, 2) = (B(I, 8) * B(I, 4) - B(I, 7) * B(I, 5)) * \text{COSF}(R8 + T(I))
C(I, 3) = B(I, 7) * B(I, 5) - B(I, 8) * B(I, 4)
C(I, 4) = (B(I, 7) * B(I, 3) - B(I, 9) * B(I, 1)) * \text{SINF}(R7)
1 + (B(I, 9) * B(I, 2) - B(I, 8) * B(I, 3)) * \text{COSF}(R7)
C(I, 5) = (B(I, 8) * B(I, 1) - B(I, 7) * B(I, 2)) * \text{COSF}(R9 + V(I))
C(I, 6) = B(I, 8) * B(I, 1) - B(I, 7) * B(I, 2)
C(I, 7) = B(I, 4) * (B(I, 3) * B(I, 7) - B(I, 1) * B(I, 9))
1 + B(I, 5) * (B(I, 3) * B(I, 8) - B(I, 2) * B(I, 9))
C(I, 8) = (B(I, 7) * B(I, 5) - B(I, 8) * B(I, 4)) * \text{R2 * SINF}(R8 + T(I))
1 + (B(I, 9) * B(I, 4) - B(I, 7) * B(I, 6)) * \text{COSF}(R8 + T(I))
2 + (B(I, 9) * B(I, 5) - B(I, 8) * B(I, 6)) * \text{SINF}(R8 + T(I))
C(I, 9) = C_5 * (B(I, 9) * B(I, 9) - B(I, 8) * B(I, 8)
1 - B(I, 7) * B(I, 7))
10 \text{CONTINUE}
L = 1
11 J = 2
M = 1
12 \text{IF} (\text{ABSF}(C(M, 1)) - \text{ABSF}(C(J, 1))) \geq 13 * 14 * 14
13 M = J
14 \text{IF} (J + L - 9) \geq 15, 16, 16
15 J = J + 1
\text{GO1012}
16 \text{IF} (M - 1) \geq 17 * 26, 26, 17
17 K = 1
18 C(9, K) = C(I, K)
19 \text{IF} (K + L - 10) \geq 19 * 20, 20
20 K = K + 1
\text{GO1018}
21 K = 1
22 C(1, K) = C(M, K)
23 \text{IF} (K + L - 10) \geq 22 * 23, 23
24 K = K + 1
\text{GO1021}
25 K = 1
26 C(M, K) = C(9, K)
27 \text{IF} (K + L - 10) \geq 25, 26, 26
28 K = K + 1
\text{GO1024}
29 K = 1
30 C(9, K) = C(I, K + 1) / C(I, 1)
IF(K+L-9) > 29 * 29
K=K+1
GOTO 27
J=1
K=1
C(J,K) = C(J+1,K+1) - C(9,K) * C(J+1,1)
IF(K+L-9) > 32, 33
K=K+1
GOTO 31
J=J-7
J=J+1
GOTO 30
K=1
C(8,K) = C(9,K)
IF(K+L-9) > 37, 38
K=K+1
GOTO 36
IF(L-8) > 39, 40
L=L+1
IF(L-8) > 11, 26
DLR = ABSF(C(1,1)) + ABSF(C(2,1)) + ABSF(C(3,1))
+ ABSF(C(4,1)) + ABSF(C(5,1)) + ABSF(C(6,1))
+ ABSF(C(7,1)) + ABSF(C(8,1))
R1=R1+C(1,1)
R2=R2+C(2,1)
R3=R3+C(3,1)
R4=R4+C(4,1)
R5=R5+C(5,1)
R6=R6+C(6,1)
R7=R7+C(7,1)
R8=R8+C(8,1)
SUMR=ABSF(R1) + ABSF(R2) + ABSF(R3) + ABSF(R4) + ABSF(R5)
+ ABSF(R6) + ABSF(R7) + ABSF(R8)
PRINT 2,N,R1,R2,R3,R4,R5,R6,R7,R8,DELR
IF(DLR-E1*SUMR) > 43, 41
N=N+1
GOTO 9
I1=2
PUNCH 1, I1, N, R1, R2, R3, R4, R5
I1=3
PUNCH 1, I1, N, R6, R7, R8, R9, DELR
I1=4
PUNCH 1, I1, N, E3, RT, DT, X(2), X(1)
Z2=1./R1
Z3=SQRTF(1.+1./(R1*R1)+1./(R2*R2)-2.*R3/(R1*R2))
Z4=1./R2
Z5=1./R5
Z6=SQRTF(1.+1./(R4*R4)+1./(R5*R5)-2.*R6/(R4*R5))
Z7 = 1 / R4
THETA = R8 * 57.295780
ALPHA = R7 * 57.295780
PHI = R9 * 57.295780
PRINT 3
PRINT 2, N, Z2, Z3, Z4, ALPHA, Z5, Z6, Z7, THETA, PHI
I1 = 5
PUNCH1, I1, N, Z2, Z3, Z4, THETA, PHI
N = N + 1
PUNCH1, I1, N, Z5, Z6, Z7, ALPHA
STOP
APPENDIX F

FORTRAN STATEMENTS FOR COMPUTER PROGRAM TO

DESIGN MECHANISM A WITH NINE PRECISION POINTS

DIMENSION X(13), Y(13), T(11), V(11), B(9,9), C(10,10)
FORMAT (12, I3, 5E13.6)
READ1 I1, N, (X(I), I=1,5)
READ1 I1, N, (X(I), I=6,10)
READ1 I1, N, (X(I), I=11,13), DR
READ1 I1, N, (Y(I), I=1,5)
READ1 I1, N, (Y(I), I=6,10)
READ1 I1, N, (Y(I), I=11,13)
READ1 I1, N, Z2, Z3, Z4, THETA, PHI
READ1 I1, N, Z5, Z6, Z7, ALPHA
READ1 I1, N, DT, DV, E1, F2, E3
R1 = 1.0/Z2
R2 = 1.0/Z4
R3 = (1.0+Z2*Z2-Z3*Z3+Z4*Z4)/(2.0*Z2*Z4)
R4 = 1.0/Z7
R5 = 1.0/Z5
R6 = (1.0+Z5*Z5-Z6*Z6+Z7*Z7)/(2.0*Z5*Z7)
R7 = ALPHA*0.017453293
R8 = THETA*0.017453293
R9 = PHI*0.017453293
RT = (DT/(X(13)-X(12)))*0.017453293
RV = (DV/(Y(13)-Y(12)))*0.017453293
DO31 = 1, 9
T(I) = RT*(X(I+1)-X(2))
V(I) = RV*(Y(I+1)-Y(2))
CONTINUE
N=1
DO51 = 1, 9
B(I, 1) = COSF(R8+T(I))-R1
B(I, 2) = SINF(R8+T(I))
B(I, 3) = R3-R2*COSF(R8+T(I))
B(I, 4) = COSF(R7+R9+V(I))+R4*COSF(R7)
B(I, 5) = SINF(R7+R9+V(I))+R4*SINF(R7)
B(I, 6) = R6+R5*COSF(R9+V(I))
B(I, 7) = B(I, 3)*B(I, 5)-B(I, 2)*B(I, 6)
B(I, 8) = B(I, 1)*B(I, 6)-B(I, 3)*B(I, 4)
B(I, 9) = B(I, 1)*B(I, 5)-B(I, 2)*B(I, 4)
C(I, 1) = B(I, 9)*B(I, 5)-B(I, 2)*B(I, 8)
C(I, 2) = (B(I, 8)*R(I, 4)-B(I, 7)*B(I, 5))*COSF(R8+T(I))
C(I, 3) = 6(B(I, 7)*B(I, 5)-R(I, 8)*B(I, 4)
C(I, 4) = (B(I, 7)*B(I, 3)-B(I, 9)*B(I, 1))*SINF(R7)
1 += (B(I, 9)*B(I, 2)-B(I, 8)*B(I, 3))*COSF(R7)
CONTINUE
L=1
J=2
M=1
IF(ABS(C(M,1))=ABS(C(J,1)))13,14,14
M=J
IF(J+L-10)15,16,16
J=J+1
GOTO12
IF(M-1)26,26,17
K=1
C(10,K)=C(1,K)
IF(K+L-11)19,20,20
K=K+1
GOTO18
K=1
C(1,K)=C(M,K)
IF(K+L-11)22,23,23
K=K+1
GOTO21
K=1
C(M,K)=C(10,K)
IF(K+L-11)25,26,26
K=K+1
GOTO24
K=1
PRINT1,N,L,C(1,1)
C(10,K)=C(1,K+1)/C(1,1)
IF(K+L-10)28,29,29
K=K+1
GOTO27
J=1
K=1
C(J,K)=C(J+1,K+1)-C(10,K)*C(J+1,1)
IF(K+L-10)32,33,33
K=K+1
GOTO31
IF(J-8)34,35,35
J=J+1
GOTO30
35 K=1
36 C(9,K)=C(10,K)
37 IF(K+L-10)<37,38,38
38 K=K+1
39 GOTO36
40 IF(L-9)<39,40,40
41 DFLR=ABS*(C(1,1))+ABS*(C(2,1))+ABS*(C(3,1))
42 IF(K+L-10)<41,42,41
43 R1=R1+C(1,1)*DR
44 R2=R2+C(2,1)*DR
45 R3=R3+C(3,1)*DR
46 R4=R4+C(4,1)*DR
47 R5=R5+C(5,1)*DR
48 R6=R6+C(6,1)*DR
49 R7=R7+C(7,1)*DR
50 R8=R8+C(8,1)*DR
51 R9=R9+C(9,1)*DR
52 SUMR=ABS*(R1)+ABS*(R2)+ABS*(R3)+ABS*(R4)+ABS*(R5)
53 IF(K+L-10)<53,54,54
55 N=N+1
56 GOTO4
57 IF(FLOATF(N)-E2)<57,58,58
59 N=N+1
60 PUNCH1,I1,N,R1,R2,R3,R4,R5
61 PUNCH1,I1,N,R6,R7,R8,R9,DELR
62 IF(FLTR-1*SUMR)<62,63,63
63 I1=2
64 PRINT1,I1,N,R1,R2,R3,R4,R5
65 I1=3
66 PRINT1,I1,N,R6,R7,R8,R9,DELR
67 IF(FLTR-1*SUMR)<67,68,68
68 N=N+1
69 GOTO4
70 I1=2
71 PUNCH1,I1,N,R1,R2,R3,R4,R5
72 PUNCH1,I1,N,R6,R7,R8,R9,DELR
73 Z2=1./R1
74 Z3=SQRTr(1.+1./R1+1./(R2*R2)-2.*(R3/(R1*R2)))
75 Z4=1./R2
76 Z5=1./R5
77 Z6=SQRTr(1.+1./(R4*R4)+1./(R5*R5)-2.*(R6/(R4*R5)))
78 Z7=1./R4
79 THEETA=R8*57.295780
80 ALPHA=R7*57.295780
81 PHI=R9*57.295780
82 N=N+1
83 PUNCH1,I1,N,Z2,Z3,Z4,THEETA,PHI
84 STOP
APPENDIX G

FORTRAN STATEMENTS FOR COMPUTER PROGRAM TO COMPUTE
ANGULAR POSITIONS OF OUTPUT LINK FOR MECHANISM A

1 FORMAT(12, I3, 5E13.6)
2 FORMAT(40H INITIAL VALUES OF THETA AND PHI ARE NOT
2 32H COMPATIBLE, REPEAT WITH E3 X 10)
6 FORMAT(38H THETA BETA PHI)
DIMENSION X(2)
READ1, I1, N, R1, R2, R3, R4, R5
READ1, I1, N, R6, R7, R8, R9
READ1, I1, N, E3, RT, DT, X(2), X(1)
Z2=1./R1
Z3=SQRRTF(1.+1./(R1*R1)+1./(R2*R2)-2.*R3/(R1*R2))
Z4=1./R2
Z5=1./R5
Z6=SQRRTF(1.+1./(R4*R4)+1./(R5*R5)-2.*R6/(R4*R5))
Z7=1./R4
44 IF(R7)45, 46, 46
45 R7=R7+6.2831853
GOTO44
46 IF(R7-6.2831853)48, 47, 47
47 R7=R7-6.2831853
GOTO46
48 ALPHA=R7*57.295780
49 IF(R8)50, 51, 51
50 R8=R8+6.2831853
GOTO49
51 IF(R8-6.2831853)53, 52, 52
52 R8=R8-6.2831853
GOTO51
53 THETA=R8*57.295780
54 IF(R9)55, 56, 56
55 R9=R9+6.2831853
GOTO54
56 IF(R9-6.2831853)58, 57, 57
57 R9=R9-6.2831853
GOTO56
58 PHI=R9*57.295780
H1=SQRRTF(1.+Z2*Z2-2.*Z2*COSF(R8))
SH3=Z2*SINF(R8)/H1
CH3=SQRRTF(1.-SH3*SH3)
IF(Z2*COSF(R8)-1.)61, 62, 63
61 H3=ATANF(SH3/CH3)
GOTO64
62 H3=(ABSF(SH3)/SH3)*1.5707963
GOT064
63 H3=3.1415927-ATANF(SH3/CH3)
64 CH4=(H1*H1+Z4*Z4-Z3*Z3)/(2.*H1*Z4)
   SH4=SQRTF(1.-CH4*CH4)
   IF(CH4)65,66,67
65 H4=3.1415927+ATANF(SH4/CH4)
   GOT068
66 H4=1.5707963
   GOT068
67 H4=ATANF(SH4/CH4)
68 H7=3.1415927-H3-H4
   H8=3.1415927-H3+H4
   H21=SQRTF(1.+Z5*Z5-2.*Z5*COSF(H7-R7))
   H22=SQRTF(1.+Z5*Z5-2.*Z5*COSF(H8-R7))
   SH51=Z5*SINF(H7-R7)/H21
   SH52=Z5*SINF(H8-R7)/H22
   CH51=SQRTF(1.-SH51*SH51)
   CH52=SQRTF(1.-SH52*SH52)
   IF(Z5*COSF(H7-R7)-1.)69,70,71
69 H51=ATANF(SH51/CH51)
   GOT072
70 H51=(ABSF(SH51)/SH51)*1.5707963
   GOT072
71 H51=3.1415927-ATANF(SH51/CH51)
72 IF(Z5*COSF(H8-R7)-1.)73,74,75
73 H52=ATANF(SH52/CH52)
   GOT076
74 H52=(ABSF(SH52)/SH52)*1.5707963
   GOT076
75 H52=3.1415927-ATANF(SH52/CH52)
76 CH61=(H21*H21+Z7*Z7-Z6*Z6)/(2.*H21*Z7)
   CH62=(H22*H22+Z7*Z7-Z6*Z6)/(2.*H22*Z7)
   SH61=SQRTF(1.-CH61*CH61)
   SH62=SQRTF(1.-CH62*CH62)
   IF(CH61)77,78,79
77 H61=3.1415927+ATANF(SH61/CH61)
   GOT080
78 H61=1.5707963
   GOT080
79 H61=ATANF(SH61/CH61)
80 IF(CH62)81,82,83
81 H62=3.1415927+ATANF(SH62/CH62)
   GOT084
82 H62=1.5707963
   GOT084
83 H62=ATANF(SH62/CH62)
84 H91=3.1415927-H51-H61
   H92=3.1415927-H51+H61
   H93=3.1415927-H52-H62
   H94=3.1415927-H52+H62
85 IF(H91)86,87,87
86 H91=H91+6.2831853
GOTO 85
87 IF(H91 - 6.2831853) 89, 88
88 H91 = H91 - 6.2831853
GOTO 87
89 IF(H92) 90, 91
90 H92 = H92 + 6.2831853
GOTO 89
91 IF(H92 - 6.2831853) 93, 92
92 H92 = H92 - 6.2831853
GOTO 91
93 IF(H93) 94, 95
94 H93 = H93 + 6.2831853
GOTO 93
95 IF(H93 - 6.2831853) 97, 96
96 H93 = H93 - 6.2831853
GOTO 95
97 IF(H94) 98, 99
98 H94 = H94 + 6.2831853
GOTO 97
99 IF(H94 - 6.2831853) 101, 100
100 H94 = H94 - 6.2831853
GOTO 99
101 IF(ABS(F(H91 - R9) - F3)) 106, 105
102 IF(ABS(F(H92 - R9) - F3)) 107, 106
103 IF(ABS(F(H93 - R9) - F3)) 108, 107
104 IF(ABS(F(H94 - R9) - F3)) 109, 108
105 PRINTZ
E3 = E3 * 10.
GOTO 101
106 H7 = H7 * 57.295780
Q1 = +1.
Q2 = +1.
GOTO 110
107 H7 = H7 * 57.295780
Q1 = +1.
Q2 = -1.
GOTO 110
108 H7 = H8 * 57.295780
Q1 = -1.
Q2 = +1.
GOTO 110
109 H7 = H8 * 57.295780
Q1 = -1.
Q2 = -1.
110 I1 = 5
N = 1
PRINT1, I1, N, Z2, Z3, Z4, THETA, PHI
N = 2
PRINT1, I1, N, Z5, Z6, Z7, ALPHA, H7
PRINT6
111 I1 = 6
112 N = 0
113 H9 = R8 - RT*(X(Z) - X(1))
114 \[ H_8 = H_9 + \text{FLOAT}(N) \times 0.017453293 \]
\[ H_1 = \text{SQRT}(1 + Z_2^2 Z_2 - 2 \times Z_2 \times \text{COS}(H_8)) \]
\[ SH_3 = Z_2 \times \text{SIN}(H_8) / H_1 \]
\[ CH_3 = \text{SQRT}(1 - SH_3^2 / SH_3) \]
\[ IF(Z_2 \times \text{COS}(H_8) - 1) \times 115 \times 116 \times 117 \]
115 \[ H_3 = \text{ATAN}(SH_3 / CH_3) \]
\[ \text{GOTO} 118 \]
116 \[ H_3 = (\text{ABSF}(SH_3) / SH_3) \times 1 \times 5707963 \]
\[ \text{GOTO} 118 \]
117 \[ H_3 = 3 \times 1415927 - \text{ATAN}(SH_3 / CH_3) \]
118 \[ CH_4 = (H_1^2 H_1 + Z_4^2 Z_4 - 2 \times Z_3 \times Z_4) / (2 \times H_1 \times Z_4) \]
\[ SH_4 = \text{SQRT}(1 - CH_4^2 / CH_4) \]
\[ IF(CH_4) \times 119 \times 120 \times 121 \]
119 \[ H_4 = 3 \times 1415927 + \text{ATAN}(SH_4 / CH_4) \]
\[ \text{GOTO} 122 \]
120 \[ H_4 = 1 \times 5707963 \]
\[ \text{GOTO} 122 \]
121 \[ H_4 = \text{ATAN}(SH_4 / CH_4) \]
122 \[ H_7 = 3 \times 1415927 - H_3 - Q_1 \times H_4 \]
123 \[ H_2 = \text{SQRT}(1 + Z_5^2 Z_5 - 2 \times Z_5 \times \text{COS}(H_7-R_7)) \]
\[ SH_5 = Z_5 \times \text{SIN}(H_7-R_7) / H_2 \]
\[ CH_5 = \text{SQRT}(1 - SH_5^2 / SH_5) \]
\[ IF(Z_5 \times \text{COS}(H_7-R_7) - 1) \times 124 \times 125 \times 126 \]
124 \[ H_5 = \text{ATAN}(SH_5 / CH_5) \]
\[ \text{GOTO} 127 \]
125 \[ H_5 = (\text{ABSF}(SH_5) / SH_5) \times 1 \times 5707963 \]
\[ \text{GOTO} 127 \]
126 \[ H_5 = 3 \times 1415927 - \text{ATAN}(SH_5 / CH_5) \]
127 \[ CH_6 = (H_2^2 H_2 + Z_7^2 Z_7 - 2 \times Z_6 \times Z_7) / (2 \times H_2 \times Z_7) \]
\[ SH_6 = \text{SQRT}(1 - CH_6^2 / CH_6) \]
\[ IF(CH_6) \times 128 \times 129 \times 130 \]
128 \[ H_6 = 3 \times 1415927 + \text{ATAN}(SH_6 / CH_6) \]
\[ \text{GOTO} 131 \]
129 \[ H_6 = 1 \times 5707963 \]
\[ \text{GOTO} 131 \]
130 \[ H_6 = \text{ATAN}(SH_6 / CH_6) \]
131 \[ H_10 = 3 \times 1415927 - H_5 - Q_2 \times H_6 \]
\[ \text{THETA} = H_8 \times 57.29578 \]
\[ \text{BETA} = H_7 \times 57.29578 \]
\[ \text{PHI} = H_10 \times 57.29578 \]
\[ \text{PRINT} 1 \times 11 \times N, \text{THETA, BETA, PHI} \]
\[ IF(\text{FLOAT}(N) - D_T) \times 132 \times 133 \times 134 \]
132 \[ N = N + 1 \]
\[ \text{GOTO} 114 \]
133 \[ \text{STOP} \]
I, Charles Walter McLarnan, was born in Mt. Vernon, Ohio, March 13, 1928. I received my secondary-school education in Mt. Vernon (Ohio) High School, and my undergraduate training at Ohio Wesleyan University and The Ohio State University, receiving the Bachelor of Arts degree from the former institution in June, 1951, and the Bachelor of Mechanical Engineering degree from the latter in June, 1954. While an undergraduate at The Ohio State University, I was employed as part-time instructor in the Department of Physics at Capital University, Columbus, Ohio. In December, 1955, I was granted the degree Master of Science from The Ohio State University. I have been employed as instructor in the Department of Mechanical Engineering of The Ohio State University since October, 1954.

I was married in June, 1952, to Marjorie Roseboom, and three children, Timothy, Linda and Margaret, have resulted.