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CATEGORIES OF CONCEPT FORMATION WITH AN EMPHASIS
ON GEOMETRICAL CONCEPTS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

ALVERN WALTER KAUFMANN, B.A., M.A.

* * * * * * *

The Ohio State University
1960

Approved by:

[Signature]
Adviser
Department of Education
ACKNOWLEDGMENTS

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Background of the Problem

"To know what we think, to be masters of our own meaning, will make a solid foundation for great and weighty thought." This statement emphasizes very well the topic that is being treated in this study. The teacher of mathematics certainly recognizes that the student, as well as teacher, must be master of his own meaning if he is to be prepared for the study of principles of "great and weighty thought." How many teachers have asked for definitions of mathematical terms and received a group of words which were not coherent, let alone acceptable? As examples, consider the following definitions of "angle" given by college junior and senior mathematics majors:

1. An angle is the figure represented by two rays emanating from a single point, the rays not being coincident.

2. An angle is the intersection of any two lines in space.

3. An angle is determined by two lines drawn from a given vertex point.

Notice that these say nothing about rotation; do not allow

\[^{1}\text{C. S. Peirce, Chance, Love and Logic (ed. M. R. Cohen), "How to Make Our Ideas Clear," p. 36.}\]
for angles such as 0°, 180°, or 360°; and are not definitive regarding the angle in the figure formed by two intersecting lines, or even by two rays.

For the purpose of illustrating deficiencies that appear in the ability to define mathematical terms, a group of college mathematics majors (juniors and seniors) was asked to define "circle," "ellipse," "angle" and function. The results are categorized below.

<table>
<thead>
<tr>
<th>Definition given</th>
<th>Number giving definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>A circle is:</td>
<td></td>
</tr>
<tr>
<td>1. The locus of points equidistant from a given point</td>
<td>5</td>
</tr>
<tr>
<td>(with no mention of the points' being limited to a plane).</td>
<td></td>
</tr>
<tr>
<td>2. The locus of points equidistant from a given point</td>
<td>3</td>
</tr>
<tr>
<td>in a plane.</td>
<td></td>
</tr>
<tr>
<td>3. The intersection of a plane and a sphere.</td>
<td>2</td>
</tr>
<tr>
<td>4. The line in a plane, through three noncollinear</td>
<td>1</td>
</tr>
<tr>
<td>points, with constant curvature.</td>
<td></td>
</tr>
<tr>
<td>An ellipse is:</td>
<td></td>
</tr>
<tr>
<td>1. An oval or oblong.</td>
<td>1</td>
</tr>
<tr>
<td>2. The locus of points in a plane the sum of whose</td>
<td>6</td>
</tr>
<tr>
<td>distances from two fixed points is a constant or which satisfy the equation ( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 ).</td>
<td></td>
</tr>
<tr>
<td>3. A special intersection of a plane and an ellipsoid</td>
<td>1</td>
</tr>
<tr>
<td>of revolution.</td>
<td></td>
</tr>
<tr>
<td>An angle is:</td>
<td></td>
</tr>
<tr>
<td>1. The inclination of one line with another.</td>
<td>6</td>
</tr>
<tr>
<td>2. The amount of rotation of one line with respect to another.</td>
<td>2</td>
</tr>
</tbody>
</table>
A function is:

1. A correspondence between two variables.
2. A determinate correspondence between two variables.
3. A relationship between two variables such that a change in one causes a change in the other.

Interpretation of the Chart

The definitions of circle show a lack of precision in that almost half the group neglected to make the stipulation that the locus must be confined to a plane. There is little doubt, however, that any of these students could recognize, on sight, a circle and reject a non-circle in a test of identification.

The ellipse was defined practically unanimously in terms of analytic geometry whether by the algebraic equation, \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \), the locus of points in a plane the sum of whose distances from two fixed points is a constant (which they learned in analytic geometry), or both.

The definitions given for angle are interesting in that only two definitions contain the concept of rotation. Although the majority of the students had thus ruled out some important angles by their definition, they readily identified them as such when confronted with a set of examples and counterexamples.

The defining of function in terms of one variable's changing so as to bring about a change in a related variable was characteristic of the thinking of almost half the group. These same students however demonstrated that the graph of \( x = 2 \) is a straight line just as
much as the graph of \( x + y = 4 \), although only the latter of the two involves related change in two variables.

This procedure of testing for the degree of understanding of mathematical concepts by requiring a definition of the term applied to the concept seems a rather common practice. Interestingly enough, of the four mathematical terms presented above for definition the results on three indicate that a sizeable proportion of students answering give evidence of an operational, or recognitional, type of understanding of the word, although they are not able to provide an acceptable definition. Such results pose the question of the wisdom of depending upon the provision of a definition as the sole evidence for the attainment of a given concept.

Such de-emphasis of definition as a test of understanding is not new, for much has been written regarding the advisability of providing meaning before definition.

For example Breslich says:

The contention has been made in the foregoing pages that understanding of a mathematical concept should be the outgrowth of experience, and that the definition, if given at all, should be taught when the pupil knows the meaning. The test of understanding is not the correct reproduction of a memorized statement but the ability of the pupil to use it correctly in a variety of situations.\(^2\)

Here Breslich not only advocates relegating the process of defining

terms to a secondary position, but also condemns using the giving of a definition as a "test of understanding."

Speaking of the same problem, Godfrey and Siddons say:

In the early stages a boy should not be bothered with formal definitions; he wants to get a working knowledge of the language of geometry; he must understand the meaning of words and expressions rather than be able to define them in set terms.3

In this quotation and in the one above from Breslich, both of which give a fair representation of what methodology books have to say on the subject of meaning and definition, it is noticeable that although they state quite clearly that definition should follow meaning, they do not describe any procedure for the development of meaning.

Statement of the Problem of This Study

The general problem of this study is that of determining the nature of the development of meaning and definition in mathematics. This involves analyzing the method by which meaning is developed, and determining just what part definition plays in such development. It is the opinion of the author that the meaning of mathematical terms can be developed, in most cases, entirely separately from and prior to the giving of a definition.

Procedure of This Study

It is the plan of this study to trace the development of certain mathematical concepts from their simplest mode of apprehension by the learner up to the level of the definition. In tracing the development of the meaning of geometrical and other mathematical terms and symbols such use will be made of studies in concept formation where these have implications for the development of the meaning of such terms and symbols. In order to include all pertinent aspects of this developmental approach, use will be made of studies of concept formation in animals and in young children to ascertain any similarity in such development at the non-verbal level. However, the studies included at this level will deal only with those that involve the development of concepts of geometric form rather than of words and symbols because the writer has found no evidence that animals can be taught to read or respond to verbal clues or commands except in a very limited manner.

In studying the nature of the acquisition of mathematical concepts there are two general principles which it is the purpose of this study to confirm. These are that (1) concepts may be developed prior to the meaning of words, and (2) a certain type of meaning of words, terms, or symbols which involves the proper selection of things, relationships, or groups of words to which the word refers, is possible of attainment prior to the cultivation of ability to give a definition of the word.
A simple example of each of the above principles is given. Birds have been shown to understand the basic concept of number in that they can tell when one of their offspring is missing upon returning to the nest, provided that the original number of fledglings is small. If the total number was four, for example, and the bird evidenced alarm upon returning and finding only three offspring, we would say this indicated a primitive understanding of the concept "four." This understanding is certainly non-verbal. As an example of the second principle we have merely to refer to the initial examples of this chapter, in which students showed that they could identify examples of the word to be defined without being able to give a satisfactory definition.

Scope and Limitations of This Study

The material for this study is taken from four major disciplines: psychology, logic, mathematics and education. The bulk of the material is drawn from the field of psychology in the form of reports of studies made on the formation of concepts in man and animal. Wherever it has been possible, use has been made of those studies which deal with the formation of concepts of geometrical form and of mathematical terms and symbols. Where others are used, their relevance will be pointed out. Besides these experimental reports, use will be made of studies in such areas as comparative and experimental psychology and such aspects of educational psychology as theories of learning. The discipline of logic is implicated since
Definition is a fundamental concept of that discipline. The relevant implications of logic to be examined will be limited to logic's contribution to the meaning and definition of terms and symbols. Mathematics is the central discipline about which the study is developed. The majority of experiments (in the literature of psychology) dealing with the formation of mathematical concepts treat the concepts of number and geometric form. Since the concept of number has received recent treatment by Lowry<sup>1</sup>, the major emphasis of this study will be upon geometric concepts. Some other mathematical concepts will be treated, however, and thus the adjective "mathematical" as well as "geometrical" will be used. For the most part the mathematics books included will be texts used at all levels — elementary school through graduate school. Scholarship in the area of education is drawn upon wherever findings point to ways and means of improving practices and procedures in the teaching of mathematics. Sources in this area will include educational psychology books, periodical articles and books on the methodology of teaching.

**Definition of Terms Used in This Study**

Most of the terms needing definition in this study will be familiar to readers conversant with the particular discipline in

context of which the word is used. However since the scope of this study includes several disciplines it will be assumed that important terms need to be clarified. Therefore such terms will be defined at the first point of significant use and subsequently in cases where it is important that a special meaning be given for a word or term. In some other cases terms will be introduced with meanings peculiar to this study. For these a stipulative\(^5\) definition will be given upon the first appearance of the term.

A Review of Studies Similar to the Present One

General Concept Formation

Russell (1956). Although making no particular connection between general concept formation and the formation of mathematical concepts, Russell\(^6\) does present a valuable resume' of studies of concept formation in humans in two chapters entitled "Children's Concepts" (5) and "Concept Formation" (8).

Leeper (1951). Leeper\(^7\) presented an extensive summary of research in concept formation in a chapter of the Handbook of Experimental Psychology entitled "Cognitive Processes." In this

\(^5\)Stipulative definition means a definition which the author suggests as the one to be used for a given term.

\(^6\)D. H. Russell, Children's Thinking, Chapters 5 and 8.

he included research on animals as well as human beings. The research includes that involving geometric forms and other geometric concepts, but no attempt is made to correlate this with the teaching of geometry, or the development of the definition of mathematical terms.

Munn (1955). Munn⁸ comprehensively summarizes research in concept formation in animals and humans from the level of conditioning in inframammals through the verbal level in adult humans.

Thompson (1952). In Child Psychology, Thompson⁹ presents a treatment of concept formation which is very similar to that of Munn.

Piaget (1953). Piaget has written several books presenting his findings in the realm of concept formation. His particular clinical method of obtaining data is controversial, but his findings are considered significant even by those who criticize his methods. Some of the criticisms are: (1) he tends to generalize on too few cases; (2) his procedures are not standardized; and (3) the very way in which he asks questions tends to suggest the answer to his subjects. An example of the latter might be the question, "Who put the clouds in the sky?" rather than, "How did the clouds get in the sky?"


⁹G. G. Thompson, Child Psychology.
Typical of his numerous books is the one entitled The Child's Conception of the World. In this, as in others, he records questions he asked his subjects pertaining to certain concepts under study, their answers and his interpretation of the significance of these answers. Other psychologists, such as Dennis and Russell, have adopted his general method with improvements and have obtained strikingly similar results (see p. 13).

Studies Suggesting Identifiable Stages in Concept Development

Piaget (1930). Piaget found that there were stages in the development of the meaning of a word. For example he ascribed four stages to the development of the meaning of "life" as shown by the change in referents which the child accepted as denoting life. These may be summarized as follows:

The first stage is that in which life is attributed to the referent of any type of activity, as the sun — "gives light"; tree "if it gives fruit"; an oven — "it cooks dinner." i.e., activity which doesn't necessarily involve motion.

The second stage is characterized by the attributing of life to anything that moves i.e., a stone is alive if it is rolling — table — no — it can't move.

The third stage comes when the child considers "alive" only those things which move spontaneously. A horse is alive because "he helps man" etc. — Streams


11 Ibid., pp. 196-204.
are alive because they are moving all the time — lake —
no it is all alone and can't move by itself.
The fourth stage is that in which the concept "life"
is restricted to plants and animals.

Such findings are important to the present study in the
sense that they suggest the development of concepts in recognizable
stages. The concept "life," is not a mathematical one, but the
general implications are nonetheless important.

Piaget (1953). In a later article concerning the way in which
children develop mathematical concepts, Piaget reports a very
interesting discovery concerning the growth of concepts. He says
that though, historically, geometry developed as (1) Euclidean
(figures, angles etc.), (2) projective (perspective), and (3)
topological (describing spatial relationships),

A child begins with the last.... At the age of three
he readily distinguishes between open and closed
figures...and he can also draw a small circle outside
or attached to the edge of the large one. All this
he can do before he can draw a rectangle or express
the Euclidean characteristics (number of sides, angles,
etc.) of a figure.13

Here again Piaget presents evidence that the development of con­
cepts is by stages.

12J. Piaget, "How Children Form Mathematical Concepts."
Scientific American, CLXXXIX (November, 1953), 74-79.

13Ibid., p. 75.
Dennis and Russell (1939-1943). Of the studies incorporating Piaget's methods, those by Dennis\(^1\) - 17 and Russell\(^1\) - 16 are perhaps the best known. Dennis working alone, and with Russell, applied Piaget's questions on the concept, "alive," to American children and to Zuni and Hopi Indian children. Although Dennis and Russell used Piaget's questions they made efforts toward standardizing the method of presenting these questions to the children. Even with these modifications they found results that verified Piaget's findings on the evolution of concepts.

Welch (1940). Welch,\(^1\)\(^8\),\(^1\)\(^9\) reporting on studies in concept formation, outlines these stages in the development of concepts in children (the listing represents a summary of his analysis):

\(^1\)\(^1\)R. W. Russell and W. Dennis, "Studies of Animism: I, A Standardized Procedure for the Investigation of Animism." Journal of Genetic Psychology, LV (1939), 389-400; II, LVI (1940), 353-68 (Russell only); III, LVII (1940), 57-63, Two authors plus F. E. Ash.


1. The pre-abstract stage: The child of 12 months can discriminate without benefit of words, as on those occasions when it refuses pabulum and chooses the bottle.

2. The first stage (about 26 months): The child is able to handle first hierarchy concepts of a single genus-species relationship. The child, for instance, understands that men and women are people or that apples and potatoes are foods.

3. The second stage (about 54 months): The child is able to handle second hierarchy concepts as, for example, food-vegetable-potato relationships.

4. From this point on the stages are demarcated by the child's ability to handle third, fourth and higher hierarchies.

In these studies, Welch provides corroboration for the position that concepts are developed in recognizable stages in children.

Heidbreder (1946). In reporting an experiment investigating the order in which certain concepts are grasped by adults, Heidbreder20 says:

The data indicate that the nine concepts used in the experiment were attained by the subjects in a regular order; concepts of concrete objects first, concepts of spatial forms next, and concepts of number last.21

The findings of Heidbreder's study, along with those of the several studies reported above, support a general thesis of this study that concepts, specifically geometrical concepts, are


21Ibid., II, p. 221
developed in recognizable stages. But the stages in this study, although identifiable, are not assumed to form an hierarchy. A fundamental purpose of the study is the establishment of the fact that the various stages do not necessarily occur in a set pattern or order.

Plan of the Remainder of This Study

Chapter II contains a discussion of concept, concept formation and the meaning of "meaning" to be used in the present study. The major portion of the chapter is devoted to the presentation of a list of categories, into which the methods of the acquisition of mathematical concepts may be placed, plus a short discussion of each category. This list then forms the outline for the remaining chapters of the thesis.

Chapter III incorporates numerous reports of experimental findings with animal and human subjects in the area of conditioning, or what is here called Pre-conceptual Knowledge. This forms the first category in the list presented.

In Chapter IV are collected those experimental findings, again from animal and human studies, which illuminate the analysis of Conceptual Knowledge.

Chapter V presents a treatment of the third category, that of Referential Meaning. Here collected are reports of studies on
concept formation in human beings at the verbal level. Several methods of developing what is called referential meaning are outlined.

Part I of Chapter VI presents the fourth category in the development of geometrical concepts, that of Pre-Analytical Meaning. In this part of the chapter an attempt is made to show how such meaning can be conveyed independently of the methods implicit in those stages both below and above it in the list. The second part of the chapter presents the category of Analytical Meaning as that of abstract definition. This represents the most complex method in the acquisition of mathematical concepts.

Chapter VII consists of a summary of the study and the implications of the study for the teaching of mathematics.
In speaking of the way in which children form mathematical concepts, Piaget says:

It is a great mistake to suppose that a child acquires the notion of number and other mathematical concepts just from teaching. On the contrary, to a remarkable degree he develops them himself, independently and spontaneously. When adults try to impose mathematical concepts on a child prematurely his learning is merely verbal; true understanding of them comes only with his mental growth.1

This is not to discount the value of teaching, but Piaget does suggest here that we must have an appreciation of the manner in which concepts are formed if teaching methods are to be coordinated with natural learning processes, and concepts learned most naturally and efficiently. Russell,2 too, points out that children develop concepts naturally without direct instruction. He says it is at those times when adults wish to select the most important topics or concepts for development that the issue of methodology is raised.

**The Meaning of "Concept"

Stebbing says: "By concepts we mean abstracta, or universal notions, such as causality, space, time, fatherhood, number."3

---


2D. Russell, Children's Thinking, p. 227.

3L. S. Stebbing, A Modern Introduction to Logic, p. 439
Curti, in the *Monroe Encyclopedia of Educational Research*, says, "a concept may be defined, tentatively, as a general idea of a class of objects or relationships." These definitions describe "concept" so as to avoid a pitfall which the present study aims to skirt. They are not so restrictive as to "tie" a concept to a verbal symbol. In other words, a dog might have a "general idea of a class of objects," such as food, or a "universal notion" of what constitutes a master. That concepts may be grasped apart from any verbalization, oral or written, is a fundamental assertion of this study and demands a definition of "concept" which properly distinguishes the particular 'idea' from the word or symbol which might be given as a name for it. Hence in this study "concept" will mean a universal notion of a class of objects or relationships which can be known but cannot be ascribed that kind of meaning which inheres in words. The mathematical term or symbol with which meaning is considered to be associated in this study is merely a name for the mathematical concept involved. Since the meaning of such a term or symbol is so intimately tied in with the concept itself it is reasonable to assume types of meaning attainment an integral part of a study on the development of mathematical concepts, although we may wish to show that the two exist independently.

---

In summary, then, concepts are not dependent upon words for their appreciation. When we study the meaning of the name given them, we need words. This interpretation allows for the attainment of concepts by animals, an aptitude the reality of which has been amply validated in the literature of psychology.

The Meaning of "Meaning"

The word "meaning" has been used above as an attribute of words only, but, as yet, its meaning for use in this study has not been clarified. Let us illustrate the problem of isolating a particular meaning of "meaning" with the following list:

<table>
<thead>
<tr>
<th>Statement</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>2. &quot;I mean to improve.&quot;</td>
<td>2. &quot;I intend to improve.&quot;</td>
</tr>
<tr>
<td>3. &quot;I had a dream, I wonder what its meaning is.&quot;</td>
<td>3. Multiple meaning:</td>
</tr>
<tr>
<td></td>
<td>a. Predictive - &quot;I wonder if it will really happen.&quot;</td>
</tr>
<tr>
<td></td>
<td>b. Physical - &quot;I wonder if it was because I overate.&quot;</td>
</tr>
<tr>
<td></td>
<td>c. Psychological - &quot;I wonder if I have been thinking too much about that subject.&quot;</td>
</tr>
<tr>
<td></td>
<td>d. Psychoanalytic - &quot;I wonder what hidden desires it indicates.&quot;</td>
</tr>
<tr>
<td>5. The word &quot;frown&quot; means a facial expression in which the eyebrows are lowered and the face takes on a sober look as of displeasure.</td>
<td>This is an analytic description of the word &quot;frown.&quot;</td>
</tr>
<tr>
<td>6. &quot;Chair&quot; means that (pointing to a chair).&quot;</td>
<td>6. Here the act of pointing establishes a correspondence between the word and the referent, or the thing referred to.</td>
</tr>
</tbody>
</table>
7. "'Chair' means something you sit on." This is a verbal description which falls short of an analytical definition.

It is obvious that these uses of the word "meaning" are not all apropos of the meaning of mathematical terms. As stated above, the type of meaning to be considered in this study is that which inheres in words, groups of words or mathematical symbols encountered in the study of mathematics. It is thus clear that the word "meaning" as it is applied to experienced phenomena is ruled out, as, for example, in #3 above. In this instance it is quite obvious that the various meanings all apply to "the dream" as an experience rather than to the word "dream." Likewise #1, and #2 do not apply to the meaning of words. The use of "means" in the translation of words (#1 above), although indisputably applied to words, is considered inapplicable to the meaning of mathematical terms and symbols. Examples five through seven, on the other hand, do illustrate meaning as applied to words.

The Three Meanings of "Meaning" for Geometrical Terms

I. Referential Meaning

Example #6 above illustrates a type of meaning, called, by some, "ostensive definition" or "ostensive method." It is here named referential meaning. It establishes a correspondence between a word and the thing referred to (referent) by a verbal or physical pointing. Physical pointing has been illustrated (#6 above) —
verbal pointing would consist of describing in words the location, in space-time, of the object or relationship referred to. For example, a person might say, "The thing you saw in the truck parked in our driveway is a sulky."

II. Pre-Analytical Meaning

The second type of meaning applicable to mathematical terms is called pre-analytical meaning. It is illustrated by #7 in the list above and involves a verbal description which is not a verbal pointing and yet is not accurate or complete enough to be classed as an analytical definition.

III. Analytical Meaning

Analytical meaning is the third and final type of meaning to be considered in this study. It is represented by example #5 and is that level or type of meaning which is called analytical definition.

The Limitations of Concept Formation Experiments

For this study much material is being drawn from studies on concept formation in animals and human beings. Vinacke says concept formation is poorly understood for several reasons, the main ones being that

...the data utilized in discussions of the subject are too narrow, since they are usually drawn from limited experimental situations, and usually emphasize simple,
readily quantifiable overt responses. The methods of investigation have usually been restricted in scope, and have lent themselves better to showing the results of concept formation than to revealing its nature.\textsuperscript{5}

Obviously, with no present method for checking the process of the mind while it is in action, we must be satisfied with testing for products. With limitations on types of subjects usable in a laboratory setting (this is especially true of animal experiments) we must base conclusions on the limited type of behavior there testable and upon the limited types of subjects adaptable to laboratory experiment. This difficulty is recognised by Hullfish who says there is no measure of a concept; "there is simply, a measurement of products, of reactions to... stimulating conditions."\textsuperscript{6}

Within the bounds of the techniques of current psychology there seems to be no choice other than to evaluate present experimental findings, at least until a better method of studying concept formation is devised. There is, of course, the possibility of studying concept formation by introspection but this procedure is of no use when the subjects are young children or animals.


\textsuperscript{6} H. G. Hullfish, Aspects of Thorndike's Psychology in Their Relation to Educational Theory and Practice. (The Ohio State University Press, Columbus, Ohio, 1926), p. 77
Concept Development Typed According to Categories

That animals and children can be taught to distinguish geometric figures, discriminate size differences, and distinguish between differing frequencies of vibration on two distinct levels, (1) that of conditioning and (2) that of concept formation, has been demonstrated abundantly, and reported in studies which will be reviewed in the next two chapters. Consider as an illustration of these two levels, the acquisition of the concept triangularity by two monkeys (A and B). Suppose monkey A is trained by a conditioning process to respond positively to a triangle and negatively to a circle, but is able to choose the triangle only if it is accompanied by a circle as the negative stimulus. Now consider monkey B which is so trained (through a process of variation of positive and negative stimuli) that it can choose the triangle regardless of the negative stimulus, or even in the absence of one. That monkeys can be trained in either way has been demonstrated and will be discussed more fully later. For the present let us examine the types of meaning attained. Is there a difference? That monkey B possesses a meaning superior to that possessed by A is taken for granted.

Katz calls these two levels of apprehension relative and absolute respectively. He says:

If an animal is trained to a definite impression, such as a color, without any essential connection of this impression with others playing its part in the training, then we say that we have an absolute apprehension of this impression. Comparatively recently
it was discovered that the absolute training of an animal takes place much more rarely than a superficial impression would lead one to believe, and that in reality under natural conditions there usually occurs something which is now termed a relative apprehension. 7

This first type of response (relative apprehension), is commonly called a conditioned response where the second type (absolute apprehension) is called concept formation, or the apprehension of form per se, albeit at a very primitive level.

That the level of understanding attained by either monkey is only rudimentary, is demonstrated by its inability to:

1. draw a triangle.
2. point to a triangle when it sees the word in a context.
3. spell the word triangle.
4. measure the degrees in a triangle.
5. find the area of a triangle.
6. prove that a given triangle is a right triangle.
7. give an analytical definition of a triangle.

This evidence of at least three different levels (conditioning, concept formation, and a more advanced type of understanding) in the attainment of concepts supports the thesis that there is a gradation in such learning. It is suggested that there are five levels in the development of a mathematical concept. A listing of the five levels and a description of each follows:

1. Conditioning -- Pre-conceptual Knowledge.
3. Interrelationship of Term and Referent -- Referential Meaning.

7D. Katz, Animals and Men, Studies in Comparative Psychology, p. 68.
First Level of Verbalization Involving Description of Referent — Pre-analytical Meaning.

Second Level of Verbalization Involving Analytical Definition — Analytical Meaning.

1. **Pre-conceptual Knowledge** is that which exists at the level of conditioning. In certain animals such as the inframammalia it is the highest level of concept development possible. At the infrahuman mammal and human levels, the extent of the use of the process will depend upon training methods used. It is illustrated by the behavior of monkey A described at the beginning of the present section.

2. Conceptual Knowledge as here used is that demonstrated by a basic response to form per se which Kats calls absolute apprehension. It is non-verbal and is demonstrated by the correct choice of a stimulus, representing the class of a concept, regardless of surrounding phenomena. It has been demonstrated in animal and human subjects as will be substantiated later and is the type of behavior shown by monkey B in the above-mentioned illustration.

3. Referential Meaning, by definition, exists at the verbal level and consists of relating a word or symbol to a referent of that sign, which may be a geometric form, an object, a relation, or a process. It is illustrated by questions 1-3 above in which there exists a relationship between a word and a form (a triangle would need to be presented in connection with question #3).

4. Pre-analytical Meaning is an intermediate stage between Referential Meaning and Analytical Meaning in that it represents
a level of verbalisation which involves a description or analysis of the referent which although unacceptable as an analytical definition is not a mere connection between referent and term. A very simple example of this type would be a so-called definition given by a pre-school child for a "hole," "A hole is to dig." 8

5. Analytical Meaning comprises the final category of concept development in the proposed list. Meaning at this level demands the ability to provide an analysis of the referent in terms of structure and implications. Possession of such meaning would be demonstrated by the ability to answer questions 4-7 above.

The Test of Meaning

If it is assumed that a satisfactory definition of understanding at the various levels can be agreed upon, there remains the problem of determining just when such understanding has been attained; posed as a question, this might be phrased, "what is the proper test of a certain type of understanding?" It is assumed that the test of understanding at the Pre-conceptual and Conceptual levels will be an operational test involving a set number of correct choices made by a subject, the number to be arbitrarily set by the experimenter. At the Referential level meaning will be said to be attained if the learner is able to recognize referents and use the word or

8 R. Krauss, A Hole is to Dig.
symbol properly. The requirement at the level of Pre-analytical and Analytical meaning will be that a verbal description or definition suitable to that level be given. That there are at least these two levels of verbal meaning (Referential as tested by proper use and Analytic as tested by use-plus-definition) seems to be substantiated by the findings of Smoke. He says:

It [his experiment] also indicated that individuals who have learned concepts, in the sense of being able to make consistently correct discriminations while going through a crucial test series, frequently fail to define verbally these concepts with accuracy.

In other words Smoke says that subjects who at the verbal level show evidence of referential meaning may not be able to give evidence of analytical meaning.

Such an operational test as outlined above is advocated in principle by Wilson, Buckingham, and Lowry who accepts such a test and defines it as follows:

...the meaning of a term or statement according to the operational point of view is in terms of a set of operations which one performs in applying that term or statement in a concrete situation.


10G. T. Wilson, Teaching the New Arithmetic.


12W.L. Lowry, op. cit., p. 144.
Bridgman says:

The fundamental idea back of an operational analysis is simple enough; namely that we do not know the meaning of a concept unless we can specify the operations which were used by us or our neighbors in applying the concept in any concrete situation.13

This latter is an operational test of meaning in a more sophisticated sense than that of merely being able to use a term properly. It is applicable more at the level of Analytical Meaning as stated above than at the Referential level. The two levels which are here apparent are (1) the level of operational meaning, at which level a person is able to demonstrate understanding of referential meaning by correct use, or by correct response in the case of pre-conceptual and conceptual knowledge, and (2) the level of operational definition in which case the person can specify the operations to be used "in applying the concept in a concrete situation," as quoted above.

As an example of the difference between the two levels of operational meaning, (1) by use and (2) by specification of operations, consider the term "circle." The statement, "A wheel is an illustration of a circle," shows correct use or application of the term in the referential sense, whereas the statement, "The locus of points obtained in a plane by tracing a point moving at a constant distance from a fixed point is a circle," is an operational definition in that the operations for obtaining a circle are set forth.

The first type of operational test is necessary to show possession of referential meaning whereas the second type is sufficient to show possession of analytical meaning but is not necessary since there are other types of definitions which might be given at this level. Therefore the operational test at the highest level, that of analytical meaning as it is described above, will be the giving of a definition suitable to that level. Such an operational test at the appropriate level will be considered the test of meaning in this study.

Summary of the Chapter

For this study concept is defined so as to distinguish it from words, as an element of learning experience. Words are used as names for concepts and as such are subject to definitions or explanations of meaning. Of the varied meanings of "meaning," three are selected for this study. They all apply to the meaning of words and are called (1) referential meaning, (2) pre-analytical meaning and (3) analytical meaning.

In keeping with the announced intention of setting forth a list of various types of concept development, the following set was given (using the terms introduced in this study):

1. Pre-conceptual Knowledge.
2. Conceptual Knowledge.
3. Referential Meaning.
4. Pre-analytical Meaning.
5. Analytical Meaning.
These are briefly elaborated on as representative of the development of concepts from the level of conditioning in animals and pre-verbal humans through the level of analytical definition, considered the most desirable level of concept development involving the meaning of words and symbols in mathematics. The five levels of concept development are to receive detailed treatment in the four succeeding chapters. The operational test with appropriate variations for the various categories is to be considered the test of meaning attainment or conceptual knowledge throughout.
CHAPTER III

CONDITIONING — PRE-CONCEPTUAL KNOWLEDGE

Implications of Animal Studies for Human Learning

In this and the following chapter the categories in the development of mathematical concepts involve analysis of animal as well as human studies in the areas of conditioning and concept formation. This is undertaken because it is felt there is a definite relationship between both kinds of learning at the lower levels. Hilgard says of this comparative method:

Only is a process demonstrable in human learning can also be demonstrated in lower animals is the comparative method useful in studying it[ all italicized as shown]....The chief aim of the comparative method...is ultimately to understand the human being.1

In studies reported, this similarity should be clearly apparent either in the parallel nature of the experiment, with animal and with human subjects, or in the obviously human aspects of a process studied in the animal subject.

There are certain difficulties encountered in studying animal behavior, however, which render a comparative treatment partially invalid. Munn2 sets forth a number of such difficulties which may

1E. R. Hilgard, Theories of Learning, p. 329.

be summarized as follows:

1. The size of the animal definitely limits the possibility of its being tested in a laboratory setting, i.e., it would be most difficult and the cost prohibitive, to set up the apparatus for conditioning a whale (note the present difficulties encountered by Dr. White in getting an electrocardiogram of a whale in connection with his heart research).

2. The disposition of the animal limits the usefulness of certain species for laboratory testing, as for example, a lion would be much more irascible than a monkey.

3. The change in the response of the animal because of being taken outside of its natural habitat is an unknown factor.

4. Testing the learning ability of animals beyond evaluating the products of learning is impossible.

5. Motivational differences from animal to animal, as between a monkey and an ant responding to hunger, are difficult to ascertain.

Fortunately for our purposes, these difficulties seem to suggest, for the most part, that the similarities between animal and human learning may be greater, not less, than superficial results would indicate. Thus we are led to further evaluate and compare the complete data.

Varieties of Conditioning

I. Classical Conditioning.

One of the basic ways of studying animal learning is that of conditioning. The procedure used in the typical conditioning experiment is as follows:
1. A certain natural response is selected for conditioning, say salivation, or withdrawal of the leg.

2. The usual natural stimulus for this response is determined, (a dish of food for salivation or a shock for withdrawal).

3. An arbitrary stimulus, which originally fails to elicit the response in question, is selected, such as the ringing of a bell.

4. The two stimuli, arbitrary and natural, are presented simultaneously or close together in point of time, and the observer checks to see if the arbitrary stimulus comes to elicit the response which was previously associated only with the natural stimulus.

Pavlov (1927). Pavlov is the originator, in his classical experiments with dogs, of the type of training called conditioning. The dog selected for experimentation was strapped by a harness into a stall-type stand. A tube was affixed from the salivary gland to a bottle with some device for counting the drops of salivary flow. Food placed before the hungry animal caused an attendant flow of saliva. In subsequent trials, a bell would ring when the food was placed before the dog, with the sequence changing from bell following food in the initial trials to bell preceding food in the final trials until finally salivation resulted from the mere ringing of the bell.

Pavlov described a particular aspect of his conditioning, in which he was able to show in the dogs, a discriminatory ability with geometrical figures. He describes this as follows:

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*I. P. Pavlov, Conditioned Reflexes.*
A projection of a luminous circle on to a screen in front of the animal was repeatedly accompanied by feeding. After the reflex had become well established a differentiation between the circle and an ellipse with a ratio of the semi-axes $2:1$, of the same luminosity and the same surface area, was obtained by the usual method of contrast [i.e., food for correct choice, none for incorrect choice]. A complete and constant differentiation was obtained comparatively quickly. The shape of the ellipse was now approximated by stages to that of the circle (ratios of the semi-axes of $3:2$, $4:3$ and so on) and the development of the differentiation continued through successive ellipses. The differentiation proceeded with some fluctuations, progressing at first more and more quickly, and then again slower until an ellipse with ratio of semi-axes $9:8$ was reached.¹

At this point the dog exhibited a pathological reaction, apparently brought on by the exacting discriminatory behavior, which grew more intense until the animal was unsuitable for experimental purposes. This experimental neurosis, as it has come to be called, was quite a surprise to Pavlov but has since become important as a means toward a comparative approach to the study of neurotic tendencies in human beings and animals. The complete implications are of course irrelevant to the present problem, but the major significance, in this study, of Pavlov's findings is that the dog was able to discriminate between two geometric forms so nearly identical. The break at a ratio of $9:8$ is important in that it indicates the upper

¹Ibid., pp. 290-91.
limit of the dog's discriminatory ability with the circle and ellipse.

Others such as Liddell, Bayne, Anderson and Gantt have since worked along similar lines with dogs, goats, sheep and pigs, but with emphasis upon producing experimental neuroses. Gantt\(^5\) reports an experiment in which the dog was trained to distinguish musical notes of different frequencies. The dog was tested for differentiation of tones as close as 1000 and 1012 vibrations per second. This ability to discriminate between two stimuli, called relative apprehension by Kats, represents a level of conceptualization which is classed as conditioning.

II. Transposition

In the typical transposition experiment, the subject is presented with two stimuli of different sizes or colors. The subject is then trained by reward to choose one, say the larger, "a," of two circles, "a" and "b." After a suitable training period, "b" is presented along with "c," which is smaller than "b." If the subject in such test situations consistently chooses the larger of the two stimuli, it is said to have "transposed" according to the relationship "larger than." This type of experimentation has been conducted with both animal and human subjects at the non-verbal level and with humans at the verbal level. The verbal level will

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be discussed in a later chapter in connection with concept formation, because transposition at the verbal level has been shown to give results indicative of learning of a different type than conditioning.

Early workers with transposition at the pre-verbal level found a transposition when the test stimulus was very close to one of the training stimuli, with respect to the relationship being tested, but failure when the test stimulus was far removed in the continuum. This failure they attempted to explain by various unsatisfactory methods, still maintaining that the animal or human subject had learned to recognise a relationship "larger than," "smaller than," etc. Spence 6-8 was the first to come forward with a reasonable explanation of this failure to transpose at a distance. He says:

> A differential response to two members of a stimulus-series, such as steps in brightness...has been interpreted by the Gestalt psychologists...as being based upon the perception of some relational property of the stimulus-


7 "The Differential Response in Animals to Stimuli Varying Within a Single Dimension." Psychological Review, XLIV (1937), 430-44.

configuration. In opposition to that interpretation, the writer has proposed an explanation that is based on the principles of conditioning... His theory... does not assume that the animal is responding to some abstract relational property.... He holds, on the contrary, that the positive member in the stimulus-series acquires excitatory properties as a result of training to the response of approaching it and, furthermore, that there is a gradient of generalization of this excitation to other members of the stimulus-series, including the negative stimulus-object. With failure of reinforcement of response to the latter, however, an inhibitory tendency develops which, in turn, is assumed to generalize to adjoining members of the stimulus-series.

His theory that animals do not acquire concepts involving certain relationships like "larger than," at least by the methods of transposition, has been further verified in experiments conducted by Kuenne,\textsuperscript{10} Kendler,\textsuperscript{11} and Alberts and Ehrenfreund\textsuperscript{12}. The findings of Kuenne and of Alberts and Ehrenfreund also verify the additional hypothesis that children at the pre-verbal stage transpose as animals do.

\textsuperscript{9}Ibid., p. 223.


Transposition of a size relationship at the non-verbal level is thus classed as a type of conditioning somewhat similar to the circle-ellipse discrimination by Pavlov's dog, in that it is a relative apprehension.

III. Matching from Sample

Weinstein (1941). Weinstein\textsuperscript{13} studied a type of learning behavior, classified as stimulus-response learning or simple conditioning, which has been called matching from sample. In this learning situation the subject, human or animal, is confronted with a series of stimuli such as geometrical solids of different shapes (sphere, cone, pyramid, etc.). On the same tray with these solids and set apart from the group is a stimulus object, such as a pyramid, which is identical with the pyramid in the group. Reward is given for choosing the object in the group which matches the reference object, and no reward is given for wrong choice. Time is allowed for only one selection in a given trial. Weinstein found that non-verbal experiments with children gave results comparable to those obtained with monkeys.

Nissen, Blum and Blum (1948). Nissen, Blum and Blum\textsuperscript{14} report on similar experiments with chimpanzees in which they obtained

\textsuperscript{13}B. Weinstein, "Matching from Sample by Rhesus Monkeys and by Children." Journal of Comparative Psychology, XXXI (1941), 195-213.

results indicative of the same type of learning as reported by Weinstein.

IV. Oddity Problem

Spaet, Harlow, and Meyer (1943,1949). Using a conditioning approach which is somewhat different from those preceding, Spaet and Harlow,15 and, Meyer and Harlow,16 report on experiments with monkeys in which the subjects were trained to choose the odd member (as determined by shapes or color) of a group of objects, such as a triangular prism from a field of circular cylinders of varying sizes and colors. In other experiments, the color of the object to be selected was purposely made different from the color of the rest of the group of differently shaped blocks. Although color, size and shape were varied from block to block and from trial to trial, reward was always given for recognizing the object which was singular in either shape or color, and the authors based their conclusions on demonstration of ability to make such correct choices.


Nissen and McCulloch (1937). Nissen and McCulloch performed similar experiments with chimpanzees and obtained evidence of rapidly developed ability to choose the odd element in a field of 12 objects, 11 of which were alike in shape or color.

Success in such experiments is again classed as evidence of relative apprehension, or conditioning.

Summary of the Chapter

Conditioning, as it applies to the development of mathematical concepts, has been discussed as consisting of four major types:

1. Classical Conditioning.
2. Transposition.
3. Matching From Sample.
4. The Oddity Problem.

These all have been shown to fit the classification of relative apprehension and as such do not satisfy the requirements of absolute apprehension, or concept formation. For this reason the type of learning called conditioning has been named pre-conceptual knowledge for purposes of classification in this study.

Implications for Teaching

I. Distinction between the non-verbal ability to differentiate stimuli and the next stage, in which concept formation is coupled

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with verbalization, becomes an important matter for the teacher. It means that a child can see the difference between, say, a triangle and a square, long before he can even count the sides and realize that the difference is related to the difference between the numbers 3 and 4, which of course is not the only difference, or perhaps even the most important difference. It means that he can tell that red and green are not the same color, long before he can even point to a green object when the word green is mentioned (a fact which every parent realizes). It means, in short, that there exists a level of conceptual development of this discriminatory sort, which precedes and is basic to all subsequent levels.

II. In a sense, the educational implications at this non-verbal level are restricted to the understanding of how a concept is developed, since formal teaching is not begun with children at a strictly non-verbal level. Evidence supports the thesis, however, that non-verbal methods combined with verbal clues, change the level of learning from pre-conceptual to conceptual and thus move into the realm of active pedagogy.

III. At the non-verbal level, learning in animals and humans is strikingly similar when similar methods of training are used. When strict conditioning methods are used, the type of learning achieved is conditioning or what has been called pre-conceptual knowledge in this study.
CHAPTER IV

THE ACQUISITION OF CONCEPTS AT
THE NON-VERBAL LEVEL — CONCEPTUAL KNOWLEDGE

Part I — Concept Formation in Animals

The last chapter indicated that human beings, on a non-verbal level, and animals can be trained to discriminate between various types of geometrical and auditory stimuli. The dog, for instance, can be trained to salivate when confronted with a circle and to stop salivation when the stimulus resembles an ellipse. The success of such experiments of a conditioning nature have caused psychologists to ponder certain questions such as: (1) Can a dog be trained in such a manner that its reaction to a geometrical form, such as a circle, reflects a level of understanding which might be classed as concept formation? and (2) Can a dog be trained so that it will show some positive response to a circle, regardless of size, and regardless of whether the circle is accompanied by some other geometrical form?

Not all psychologists are in agreement, however, about conceptualization in animals. For instance, Bierens deHaan says:

...animals do not form concepts.... If they did so... they would certainly have a word to denote it.... The animal has no words, and therefore has no concepts....

Bierens deHaan's assertion would undoubtedly be valid if the test of concept formation were the subject's possession of a verbal symbol for it. This is not necessary, however, for Smoke found evidence that human adults may possess a given concept without being able to represent it verbally. Interestingly enough, Bierens deHaan later describes a kind of behavior on the part of animals which we would class as concept formation. He says:

The animal...is not able to form abstractions. But on the other hand it must be admitted that sometimes animals have proved to be able to perform something which may be regarded as a precursor to such abstractions.... Animals that were trained to choose a box which was marked with a triangle, and to disregard another one marked with a circle, afterwards also chose this box if it was marked not with the triangle used up till then, but with some other kind of triangle, say a right-angled triangle instead of an equilateral.... In such a case we must admit that the animal has grasped the sensorial generalisation of "something with three sides and three angles."3

This "sensorial generalisation" describes behavior which we have previously called discrimination of form per se, "absolute apprehension," or "concept formation" in harmony with common usage in present-day experimental psychology. The fact that it exists on a non-verbal level does not forbid its being called concept formation, since, as Smoke has shown, verbalisation is

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3J. A. Bierens deHaan, op. cit., p. 115.
not necessary for concept formation. Moreover, concept formation as it is defined in this study and the test to be used for determining whether an animal has a concept indicate that words are not necessary.

With one important change, we would accept the following definition of concept formation which is ably set forth by Smoke in another article:

By "concept formation," "generalization," or "concept learning" we refer to the process whereby an organism develops a symbolic response (usually, but not necessarily, linguistic) which is made to the members of stimulus patterns but not to other stimuli. The change we would advocate is the substitution of the word "significant" for "symbolic," since the latter term has a verbal connotation, even though Smoke continues to say the response need not be linguistic. This change for the purpose of clarification would more definitely accommodate concept formation as observed in animals.

Studies on Concept Formation in Animals

This section will present reports of experiments which demonstrate that animals at certain levels are capable of a type of learning which we call concept formation, or the discrimination of form per se. The two terms are used synonymously and mean the

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ability to select a given positive stimulus or a conceptually related stimulus when it is entirely independent of the negative stimulus present, or even in the absence of a negative stimulus. Thus an animal trained to recognize forms of a triangular nature would react positively to any type of triangle and ignore any non-triangle, or it would choose the more triangular of two non-triangles.

Andrew and Harlow (1948). Andrew and Harlow, working with sixteen Macaque monkeys, found that when the monkeys were trained merely to discriminate between an equilateral triangle as positive stimulus and a circle as negative stimulus, choice of the more triangular of two stimuli occurred barely above the chance level. The monkeys were trained with only one special kind of triangle (equilateral) and apparently did not develop a general concept of triangularity. When the monkeys were given an additional period of training in "concept formation," in which a variety of different kinds of triangles and more or less triangular objects, varied as to position, were employed as the positive stimulus (Figure 1), the testing gave results which were 25 per cent above chance.

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6. In a training or testing situation, two or more stimuli are presented to the subject at once. The choice of only one of them (the positive stimulus) brings reward. The remaining non-rewarding stimuli are called negative stimuli.
Some examples of the type of positive — negative discrimination pairs used by Andrew and Harlow with monkeys in training and testing for the formation of the concept "triangularity."

The monkeys were credited with correct responses based on form per se, and also with forming the concept triangularity, because they were able to discriminate any triangle from any non-triangle, or the more nearly triangular of two non-triangles. For example, they demonstrated an ability to choose any of the forms marked with a + in figure 1 in preference to the accompanying negative form, recognising that the positive form had characteristics involving triangularity to a more marked degree than did the accompanying negative form.

7"Discriminate" is here used as it is found in the literature of psychology. It here means to demonstrate by some overt act the choice of a given stimulus to the exclusion of surrounding stimuli.
Meet (1933). Meet, also working with monkeys, was able to train them to discriminate between an equilateral triangle and a circle of equal area. Changing the background form or color after the training period did not affect the subject's choices. For instance, the training series may have consisted of a triangle and circle mounted on blue rectangles. In testing for change of background the positive and negative stimuli might be mounted on rhombuses of the same color or red, so that either form or color, or both, could be changed for one stimulus or two. Only one of the four monkeys demonstrated an ability to discriminate form per se, while, the author claims, the other three discriminated on the basis of "retinal distribution of light." It should be pointed out that Meet did not test his monkeys on a variety of types of triangles, but used only equilateral triangles with no differences except for a little rotation. After the findings of Andrew and Harlow (reported above) and their success with training for concept formation by using a variety of types of triangles as positive stimulus, we should not be surprised at the failure to develop the concept of form considering the type of training stimuli Meet used.

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Gellerman (1933). In an experiment involving two chimpanzees and two two-year-old children, Gellerman\(^9\) obtained evidence on form discrimination per se. Using an alternation-box apparatus with a form presentation frame and identical procedures, he found that the recognition of a triangle by all subjects was not affected by rotation, by changes in the type of triangle, by changes in relative size, by the introduction of new negative figures, or by reversal of black and white as respective colors of form and background. He found the children to be definitely superior to the chimpanzees in discriminating between figures outlined in dots rather than solid lines. The children were also superior when the negative figures closely approached the positive. This is not surprising when one views some of the positive and negative stimulus-figures used. For example, in one case, the positive stimulus was a triangle while the negative stimulus was almost identical, except that the very tip of one vertex was cut off.

A second part of the experiment involved training for discrimination between a cross and a square on a diamond background. All subjects were able to choose the cross as positive figure regardless of background variation, negative figure, or rotation. They were

also able to react positively to the cross presented alone, and to react negatively to other figures presented alone. The conclusion was drawn that both chimpanzees and children were able to discriminate form per se, against any kind of background.

**Buylendyck (1936).** Buylendyck, speaking with reference to his own experiments with dogs as well as reporting the work of others, says: "...Like all higher animals, the dog directs its attention to the shapes of objects and it can recognize similarity of form."10

He proceeds to caution, however, that the dog is above all a smelling animal, and that great care must be taken in experimentation to control this factor and make certain that results are based on vision alone. Another word of caution is given by the same author:

Thus, after a hundred tests, another dog learned to distinguish a square from a circle, and in this experiment the clearness of the square was four times as great as that of the circle. When this difference in light was done away with, no habit was formed.11

Buylendyck thus attributes to dogs a sense of form discrimination, but he does so with reservations.

**Karn and Munn (1932).** Karn and Munn,12 working with dogs in a Munn-type apparatus,13 found evidence that dogs do possess an

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10F. J. J. Buylendyck, *The Mind of a Dog*, p. 38

11Buylendyck, *op.cit.*, p. 127


13This is a cage as shown diagramatically at right. The animal upon entering stands upon the grill. If it pushes against the correct door (so chosen by the experimenter) it opens and he gets the food. If the wrong door is pushed, the animal gets a shock from the grill.
ability to discriminate between visual patterns, but they did not obtain discrimination at the level of form per se, i.e., the concept "triangularity" was not formed. As one reviews the report of their experiment, it is easy to see why they did not elicit this kind of learning. First, they seem to have used, as positive stimuli, various shades and degrees of an upright equilateral triangle, varying it by deleting certain parts but always preserving the upright equilateral triangle, while the negative figure was an inverted equilateral triangle with similar deletions! One must judge this an absurd setup for testing for the concept of triangularity. The second objection, obviously related to the first, is that the training period was carried through the initial stage of training merely for form discrimination, rather than testing for the concept of triangularity as Andrew and Harlow did. The positive evidence of concept formation obtained by Andrew and Harlow, who used many different triangles in the training period, would suggest that Karn and Munn too would have achieved the same results if they had used a similar training procedure.

Smith (1934). Smith, using a two-door apparatus with shock for the wrong choice and food as a reward for the right choice found

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that cats could be taught to discriminate between a triangle and a circle. Here again no more than discrimination is indicated, with the question still unanswered: How would the cats behave under further training with a variety of triangles?

F{}ields (1932). In an investigation, Fields\textsuperscript{15} found that white rats could discriminate form per se. He worked with eleven white rats in a modified Lashley jumping apparatus.\textsuperscript{16} The rat was trained to jump at one of two 9 X 11 cards on which was drawn a triangle of a given type in varied sizes and positions of rotation. The card would fall down and let the rat get at a dish of food. If the wrong choice was made, the card would not fall and the rat fell into a net below. The negative stimulus used in the training period was a circle, but in the actual testing, other non-triangular forms were used as negative stimuli. When the rats were trained with only an upright equilateral triangle versus a circle of equal area, and then tested on equilateral triangles in various other positions versus a rectangle or cross, their accuracy of choice was low.

\textsuperscript{15}P. E. Fields "Studies in Concept Formation I, The Development of the Concept of Triangularity by the White Rat." \textit{Comparative Psychology Monographs}, IX (1932), No. 2.

\textsuperscript{16}In this apparatus the rat runs up a ramp to a pedestal from which it views two doors upon which are placed the positive and negative stimuli respectively. To get to either door it must jump through space a distance of several inches. If it jumps at the proper door, the door falls back so that the rat lands on the other side and can get at food as a reward. If the wrong choice is made the door does not open and the rat falls into a net below.
When the training period included various other positions, shapes and sizes of triangles versus a circle, their accuracy was above 90 per cent. Thus, the variation of stimuli in the training period is established as essential for insuring the formation of a concept, or as Fields concludes:

When white rats are given a training period specifically designed to provide a large number of different "triangle experiences," the rats are able to perfect a type of behavior which is fully described by the implications inherent in our use of the term "concept."17

Fields (1936). In a further study, the ability of raccoons to form concepts of geometric form was compared with that of white rats. Fields18 trained both types of animals to distinguish an equilateral triangle from a circle of equal area. Raccoons made an immediate transfer to rotated positions of equilateral triangles and right triangles, in a manner far superior to that of the rats, whose behavior resulting from this limited training approach was correspondingly limited.

To date there is no evidence that concept formation is possible below the mammalian level. There have been, however, extensive

17 Fields, op.cit., p. 69

experiments which give indication of a type of discriminatory
behavior in the lower animals such as the fish, turtle, chicken
and even bees, but the evidence is not presented as proof of
concept formation, since such behavior was merely a type of con­
ditioning.19

Summary of Part I -- Chapter IV

In the preceding reports of experiments dealing with the
formation of concepts of geometric form, it has been shown that the
type of learning demonstrated by animals is dependent upon the type
of training to which they were subjected. When the training pro­
cedure uses stimuli representing special cases of geometric forms,
the type of learning is something other than concept formation; it
has here been classified as pre-conceptual knowledge. When the
training stimuli are changed to include more general cases of a
given geometric form, the level of learning is changed to conceptual
learning, or concept formation. Furthermore, experiments which did
not indicate learning at the level of concept formation have revealed
a lack of one important element, that of generalised training.
Specifically, animals have been shown to conceptualise, without any
verbal means, about geometric form.

19 See Norman L. Munn, The Evolution and Growth of Human Behavior,
pp. 119-123.
From the experiments cited in this section, we may conclude that: (1) the mammals tested have demonstrated an ability to differentiate simple geometrical figures of similar and dissimilar types; and (2) when the training stimuli include limited or no variation of a geometric form, the resulting form discrimination is a type of conditioning or pre-conceptual learning, whereas when the training stimuli are extended to include more generalised variations, the type of learning demonstrated is defined by the term "concept formation."

**Implications of Part I for the Teaching of Mathematics**

The inability of animals to discriminate form per se when they are trained only on one type of positive figure has an interesting parallel in the writer’s recent experience (albeit on the verbal level) with his eight-year-old son, Lonnie, who had recently completed the second grade. In playing a game of connect-the-dot in a rectangular array, the writer had occasion to sketch a right triangle, using the term "triangle" as he did so. Lonnie said, "But that isn't a triangle!" Another triangle was sketched which accidently happened to have sides almost equal and Lonnie was asked if this were a triangle. When his answer was in the affirmative, he was asked why this one was a triangle and the other was not. He said, "In that one the sides are not the same (equal)." Evidently his schooling and experience had included only equilateral triangles,
and the concept of triangularity was not yet developed on a formal
per se basis.

The fact that animals can be trained to give evidence, by non-verbal
means, of the possession of concepts of geometric form seems
to prove that human subjects can grasp concepts before they can
understand any terms applied as names to such concepts.

As was seen in the reports of animal studies, in true concept
formation, variation in the positive stimulus is much more important
than variation in, or even the presence of, a negative stimulus
(counter-example). This would indicate the greater importance of
positive examples and their variations than the use of negative
instances in teaching.

Part II — Concept Formation in Humans at the Pre-verbal Level

After considering the formation of concepts in animals, it is
logical to look to experiments which have tested for the non-verbal
formation of concepts in human beings. These experiments fall into
two categories: (1) those like Gellerman's 20 (see p. 48), in which
human and animal subjects were tested and compared, and (2) those in
which only human subjects were used. Studies which include both
animal and human subjects are relatively few in number; there are

20L. W. Gellerman, loc. cit.
therefore only two such studies included in this section.

Studies on Concept Formation in Children

Ling (1941). Ling\textsuperscript{21} studied form discrimination in 50 infants between the ages of six and fifteen months. In his experiment he used a horizontal display board (Figure 2) with five small holes drilled in a semicircle, so that when the infant was seated in front of the board, the concave part of the semicircle faced him and flat stimulus objects could be placed over the holes. The stimuli consisted of two sets of five half-inch-thick cutouts of a circle, an oval, a square, a triangle and a cross. The two sets were identical except for size. In any one testing series, from two to five forms might be used. If less than five were used, the unused holes were filled with inconspicuous plugs painted the same color as the board. The positive stimulus, which was not fastened down,

\begin{center}
\textbf{Figure 2}
\end{center}

\begin{center}
\includegraphics[width=0.5\textwidth]{figure2.png}
\end{center}

\textsuperscript{21}B. C. Ling, "Form Discrimination as a Learning Cue in Infants," \textit{Comparative Psychology Monographs}, XVII (1941), No. 2.
was sterilized and coated with a saccharine solution which made the block sweet to the taste; it could thus be picked up and sucked as a reward. The negative stimulus or stimuli were fastened through the holes so that the child could not lift them from the board. The stimuli were arranged in an order of apparently increasing complexity so that each member of the set was slightly similar to its neighbors on both sides: circle — oval — square — triangle — cross.

At the beginning of the test, the child was seated in front of the board, with a curtain dropped between him and the board. The form to be used as the positive stimulus (say the circle) was placed over one of the holes (the remaining holes being plugged). The curtain was raised, and the child either reached naturally for the circle and placed it in his mouth or was assisted to do so by the experimenter. After an interval, the block was returned and the curtain dropped. After several preliminary trials with just the positive stimulus, the testing proper was begun by placing one of the other forms (oval, square, triangle or cross) over another hole and fastening it to the board through the hole. This form then served as a negative stimulus which if chosen carried no reward, since it could not be picked up and was not saccharine-coated. Trials were then carried out, with the position of positive and negative stimulus being changed from time to time in a random fashion. The trials were carried on until the child chose the positive stimulus in eight out of ten consecutive trials. This was judged successful
form discrimination in a field of two forms (with only two forms on the board at one time, a positive stimulus and a negative stimulus). In a given series of tests the child was trained and tested for choosing one of the forms (the positive stimulus in that series) from a field of two, three, four and five forms. The criterion for acceptable form discrimination in each field was arbitrarily set at eight correct choices out of ten consecutive trials in a field of two forms, seven of ten for a field of three forms, six of ten for a field of four forms, and five of ten for a field of five forms.

In separate series of tests, the forms were arranged and combined in ways to elicit the distinguishing of each particular form in turn from every other form, separately as well as from groups of two, three, and four other forms. By this method, the relative difficulty of discriminating between all of the stimulus-forms could be determined. In still further testing, the positive and negative stimuli were switched in a two-form field. For example, if the circle had been used as positive stimulus with, say, the triangle as negative for a number of trials, there was a sudden change so that the subject found the circle fastened down, on the next trial, and the triangle saccharine coated. This changing was not done erratically, for the new positive stimulus was retained through a subsequent series of testings. Here the purpose was not to confuse, but rather to see how such a sudden reversal would affect the subject, and how long it would take him to adapt to the new positive stimulus.
In still other trials, the size of one or more of the forms might be changed, to note the effect on form discrimination.

Ling's findings are summarized as follows:

1. A child as young as six months is able to discriminate between simple geometric forms.
2. This discrimination is of the nature of concept formation, i.e., of form per se.
3. Change of relative position, spatial orientation (rotation) or size of either positive or negative stimuli has little effect on ability to discriminate.
4. It is more difficult to discriminate the positive stimulus as the field is increased in number.
5. In a field of three to five blocks it is harder to discriminate between a positive and a negative stimulus which are similar, than between dissimilar stimuli (i.e., it is harder to discriminate the circle from the oval than the circle from the cross).
6. Sudden reversal of positive to negative stimuli brings confusion, which soon subsides.

Munn and Steining (1931). Munn and Steining 22 trained a fifteen-month-old child to distinguish a triangle from various other simple geometric forms such as a square, a circle, a cross and an X. He used a two-door form discrimination apparatus in which the stimuli were hung on the doors. The child was first shown that the door with the triangle on it could be opened, and that he could then pick up a piece of chocolate inside. After this preliminary coaching, the child was confronted with two doors at a

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distance and allowed to approach and choose one, which he would attempt to open. If he chose the one with the triangle, he could open it and get a piece of chocolate; if he chose the other door on which was hung one of the other objects (square, circle, cross, or X) he found it locked and received no candy. When the child was judged to have achieved form discrimination with a specific combination of stimuli, other tests were conducted in which only the negative stimulus was changed from trial to trial and the relative position of positive and negative stimulus was changed at random throughout all the testing. In other tests, the shape of the background was repeatedly changed, to determine whether choices were being made on the basis of the stimulus object alone, or total configuration. Munn and Steinig found "unequivocal evidence of a response to triangularity, to form per se," even though they worked only with equilateral triangles and with 180° rotations. They found also that the ability to distinguish a positive stimulus was unaffected by change in the background or in negative stimulus. These findings would, of course, be acceptable evidence that the subjects had developed the concept of triangularity.

Terman and Merrill (1937), Skeels (1933). The form board used in intelligence testing by Terman and Merrill and Gesell has also

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23 Ibid., p. 88
24 L. M. Terman and M. A. Merrill, Measuring Intelligence.
25 A. Gesell, The Mental Growth of the Pre-School Child, pp. 120-22.
been used in form discrimination studies. An extensive study has been reported by Skeels in which he trained and tested 11 children between the ages of 15 and 46 months. He used a form board with variously shaped flat wooden objects that fit into corresponding openings in four separate rectangular wooden blocks called units, which in turn fit into the master board (Figure 3). This device is used in two ways. In the first method, the cutouts are all left out on the board so that a set of holes and a corresponding set of flat cutouts comprise the type of matching test generally used in intelligence testing. The second method consists of starting with the cutouts in their respective holes and training the subject to distinguish a specified form, the positive stimulus, from the

![Four Form Four Unit Board](image)

Figure 3

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remaining forms, the negative stimuli. It is this latter procedure which involves conceptual knowledge of the type tested in the previously reported experiments in this chapter.

In his experiment on form discrimination in very young children, Sokols used a four-form, four-unit board in which the forms were a square, a circle, a lozenge and a Maltese cross, all of approximately equal area. Under each form and concealed when the form was in place, was a metal cup for holding the reward when that form was the positive stimulus. Before each test, the cutouts were placed in their respective holes, with a cookie in the cup beneath the form which was chosen as positive stimulus and no cookie, of course, in the other cups. The training and testing was performed at a time of day when the cookie would be desirable (not immediately following a meal when the child might not care for anything more to eat). In preliminary training, the child was allowed to experiment with lifting the various cutouts until he found the cookie. If, after a short period of time, the child still had not noticed the cookie, the experimenter pointed it out.

In a given test series, the children were trained and tested to choose one of the forms, say the circle, and ignore the others. This was done by rewarding the circle choice, the positive stimulus, and not rewarding the other choices, the negative stimuli. The position of the unit containing the circle was changed at random from trial to trial. Success was arbitrarily judged on the basis
of eight correct choices out of ten successive trials. Of the 41 children tested, all but one met this criterion, and that one achieved seven out of ten. Skeels thus concluded that these children were able to distinguish geometric form at this non-verbal level.

In a separate but related experiment, Skeels tested some of the same subjects in order to correlate ability to recognize form, as in the test just mentioned, and ability to match each form with its corresponding opening when the subject is confronted with all four cutouts and holes, a procedure which involves some generalization. Accordingly, he worked with a subgroup of eight younger children, aged 15 to 23 months, taken from the larger group of 41 children. He trained and tested them in the matching-type use of the four-form, four-unit board in order to see if there was any correlation between success in simple form discrimination and success in placing each form in its proper opening. He found that the younger children could recognize form but could not fit the forms into their proper place. He therefore concluded that successful performance in the form-board matching test did not demonstrate true ability to generalize about form. Skeels found form discrimination, as established by the earlier part of his experiment, in children of all ages, with no indication of greater success for the older children.

Long (1940). Long,27 using 13 children, aged three to six years,

trained and tested by strictly non-verbal means, for the concept "roundness." The roundness he tested for was of three types, each developed in a separate series: roundness as in (a) a sphere, (b) a circular cylinder and (c) a circle. His method was one of paired choices in which an object of the positive stimulus-type, linked with reward, was paired with an object of a negative stimulus-type. In one series, for example, he tested for the concept of spherical roundness, by pairing various sizes and samples of spheres as positive stimuli with such non-spherical solids as cubes, prisms and rectangular solids. In another series, he paired circular disks of various sizes and colors with other geometric forms. In a third series, he paired circular cylinders with prisms, etc. The objects were placed in two glass-front, lighted boxes which were constructed so that pressure on the glass in front of the correct choice would release a piece of candy into the tray before the box. From his results, Long concluded that the concept of roundness was established by an 85 per cent record of success.

Studies on Concept Formation

Using Both Children and Animals as Subjects

Gellerman (1933), Weinstein (1941). The study by Gellerman in which two chimpanzees and two two-year-old children were taught to

28Gellerman, loc. cit.
distinguish the more triangular of two stimuli has been referred to in the section on Concept Formation in Animals. There is a somewhat similar experiment with monkeys and children reported by Weinstein who used Rhesus monkeys and two children, aged 34 and 37 months, in a setting which developed the concept of congruence of geometric figures. The subjects were trained with a given set of paired objects, being given one member of a pair and then required to select the identical mate from a field of several objects, only one of which was congruent in the geometrical sense of equality of parts. After the training period, they were tested on a number of pairs of entirely new objects. Weinstein found the children much faster and more adaptable than the monkeys, even when they were not able to formulate verbally the principle involved. The fact that one of the children was finally able to express something verbally that indicated a knowledge of likeness led Weinstein to make the following observations:

The final explicit verbal formulation of the relationship by Sadel [one of the children] strongly suggests that in the human subjects at least [italics mine], an implicit cue in the nature of a symbol or concept had been effective even when they were unable to respond verbally to questions concerning the reasons for their correct choices.


30Ibid., p. 211.
This shows that verbalisation has a pronounced effect upon the speed with which a concept is formed.31

Summary of Part II — Chapter IV

Experiments dealing with concept formation in the higher infrahuman mammals and very young children point to a striking similarity between both groups on the non-verbal level. This similarity is of importance in this study mainly in establishing that the attainment of concepts at the non-verbal level is quite the same in higher animals and human beings — a similarity which has already been noted at the conditioning or pre-conceptual level. There is, however, a disparity in speed and ability even at this level, with the human attainment surpassing that of the animals, as demonstrated by Gellerman (see p. 48) and Weinstein above. Once the child is able to use words, the ability of animals and of human beings can no longer be compared since as Weinstein points out, even when verbal clues are not furnished, a subject who is at the verbal level tends to promote the development of concepts by verbalisation of his own.

The evidence of the experiments reported in this part of the chapter establishes several facts about non-verbal learning in children and animals:

1. Concepts of geometric form may occur in children at the pre-verbal level at as early an age as six months (see findings of Ling).

31See reference to findings of Kuenne, Kendler, and Alberts and Ehrenfreund, p. 37.
2. Children form concepts at the non-verbal level in the same manner as animals, but much more rapidly (Gellerman and Weinstein).

3. Children tend to promote concept development through their own verbalisation (as indicated in the experiments of Gellerman and Weinstein).

**Implications of Part II for Teaching**

Since, as Piaget points out (see Chapter II), concept formation in the child is mainly an informal process even at the verbal level, the findings reported in this and the preceding chapters are more important for understanding the acquisition of geometrical concepts than for changing any teaching methods at this level. We would say, then, that at least in the pre-verbal stage concepts develop through natural processes of growth and maturation. The teacher ordinarily does not concern himself with deliberately developing such concepts. It is at the verbal level that the teacher is concerned with changing the speed or direction of concept development. It is at this level that active pedagogy begins in the development of concepts.

These findings imply that children as well as animals can form concepts without, or prior to, a verbal definition or, in fact, any verbal means. This is affirmative evidence that neither definition nor verbal reference need take precedence in the development of a new concept. A further implication is that since many important concepts are formed before school age and since it has been shown that concepts are developed at the pre-verbal level, there may be more that the parent and pre-school teacher can do in properly developing important concepts.
Definition of the "Referential Meaning" of Words

Concept formation at the verbal level develops from an awareness and appreciation of a relationship between a word or other sign (the symbol) and an object, form or thing (the referent). Consider, for example, the child who picks up, or points to, a triangle when the word "triangle" is spoken or written; the child who points to a circle and says, "That is a circle;" or the student who says, "The figure in the upper right-hand corner of page 67 of the text is a rectangle." Each of these persons is exhibiting evidence of an awareness of such interrelationship. This type of meaning attainment, as applied to words, is here called referential meaning. It is produced in the learner by several methods of establishing a connection between a word and its referent. Ostensive definition, for example, is really such a method, though the term is misleading. In this method, the meaning of a term or symbol is established by pointing to the referent. Referential meaning may also be established by a written statement which points to the referent, locating it rather than describing it. The person who isolates the referent as being
in the upper right-hand corner of page 67," as above, is employing this latter approach. But the child who says, "Rain is so you can make mud pies," is using a type of meaning other than referential, since this statement contains a crude type of analysis of the referent. The other methods of establishing referential meaning are by the listing of examples (denotation) and the listing of examples plus counterexamples.

Hendrix points out that what is here called referential meaning is, by definition, impossible at the strictly pre-verbal level, because there can be no relationship between word and referent in this case. In reporting a class-room teaching experiment involving the presentation of the concept prime number, Hendrix says that all of the pupils except one grasped a subverbal awareness of prime numbers before these numbers were named. This, she says,

...could not have been behavioral evidence of meaning, because the entity called meaning is a triadic relation—a relation involving a person, a referent, and a symbol. It seems indisputable to me that for all but one member of that group ability to select prime numbers was behavioral evidence of awareness of the existence of such a class of numbers, not behavioral evidence of meaning. It became behavioral evidence of meaning only after a name had been attached to the class.1

Concepts may be formed at the pre-verbal level but the term referential meaning is not applicable until a word, group of words

or a symbol is attached to it. As Stebbing says: "A word then, has meaning for us when we know what it is to which the word refers."

Referential Meaning as a Category of Concept Development

By the time a child begins school, he has acquired quite a list of words which he can use more or less correctly. The speaking vocabulary is of course smaller than the vocabulary of recognition. Russell says "He [the child] may know about the table legs, and that they are hard when you bump them with your head, long before he is able to say the word table."

He further says, "In some experiments, children show clear understanding of a concept but inability to verbalize it." In other words, the child does not know the term or phrase for the referent. This is in agreement with the findings of Smoke, with the exception that Smoke does not limit the situation to children but includes adults. Hendrix says that "mathematical generalizations (and others) may be held on a dynamic subverbal level as in the John R. Clark report of army illiterates who possessed surprising arithmetical power." Further, the list

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3David Russell, *Children's Thinking*, p. 117.

4Ibid., p. 118.


of words for which the child can give even an elementary definition is much smaller than either his speaking or recognition vocabularies. A child's ability to give even an elementary definition indicates his advanced level, compared with that of a child who can only point to a referent.

**Experiments With Concept Formation at the Verbal Level**

Certain experiments with concept formation at the verbal level establish the possibility of linking a symbol with its class of referents, without any definition or explanation of the symbol. A number of such experiments are reported here.

**Welch and Long (1940).** Over the past forty years, there has been extensive experimentation with young children, feebleminded adults and college students, in the area of concept formation at the verbal level. Welch and Long,\(^7\-^9\) working separately and together, conducted and reported several experiments dealing with concept formation in children. The experiment reported by Welch and Long was

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\[8\text{L. Welch, "The Genetic Development of the Associational Structures of Abstract Thinking." Journal of Genetic Psychology, LVI (1940), 175-206.}

\[9\text{The Development of Discrimination of Form and Area." Journal of Psychology, VII (1939), 37-54.}
with 103 children, 42 to 83 months of age, who were trained in
the recognition of form. Through verbal symbols the subjects were
tested for the acquisition of two types of referential meanings:
(1) "object level," in which the child was expected to attach a
given nonsense syllable-name to a specific class of referents and
(2) "hierarchical categories," involving relationships between
geometric forms. In this second part, nonsense syllable-names were
again given to the referents (geometric forms), and the problem was
that of recognizing relationships between these referents.

In the object level test, three one-inch wooden beads of different
shapes were used: a rectangular prism, a hexagonal prism and a sphere.
The training consisted of riveting the child's attention to one of
the beads and saying a nonsense syllable such as (a) vim, (b) jid
or (c) quap. The child was then called upon to identify the objects
indicated at random, and he was rewarded with praise for the correct
response. The criterion for success was set at 15 successive correct
responses, which marked the attainment of referential meaning at this
level.

In the second part of this experiment, blocks of various geo-
metric shapes were used as pictured in figure 4. The subjects were
first trained to give the experimenter an object referred to as
"MRF" or "TOV" upon request. (The objects and their corresponding
names are shown in figure 4). Such behavior showed the experimenter
that the subject had achieved referential meaning of the "object
level" type. After success with this test, the subjects were taught
that either a MEF or a TOV was a VIC (a higher category including all triangles). In the same manner, the referents of "TOP" and "ZIL" were first recognized at the object level and then grouped in the category of

![Tree Diagram]

"DAX" (as the lines in the diagram indicate). This was called the first hierarchy test, and the criterion for success was set at 15 perfect successive choices. After satisfactory completion of this step, the subject was taught that all VIC and DAX, i.e., triangles and rectangles respectively, belonged to a higher category, the XIP (certain polygons). Testing for this type of response was then conducted, with successful attainment of this second hierarchy concept set at 35 successive perfect choices.

A sample test follows: From a pile of blocks the subject was asked to select all XIPs (referents of "XIP"). He selected all
the blocks of the types pictured above, i.e., all the MEFs, TOVs, YOPs and ZILs. He was then asked to put back all the DAX (rectangles), leaving only the VICs (triangles). Of 45 children tested in this experiment, 42 were able to pass the first hierarchy test, or were able to give the experimenter either a MEF or a TOV if he asked for a VIC, or select a YOP or a ZIL if he asked for a DAX. Only 22 of the 45 were able to pass the second hierarchy test, however. Welch and Long found no correlation between chronological age and ability to solve the above problems.

A further test series consisted of training for discrimination between two triangles or two rectangles which were progressively more difficult to distinguish, as for example, two isosceles triangles with equal altitudes but slightly different base lengths (Figure 5).

![Diagram](image-url)
The authors found that their subjects could more easily identify three objects, connecting the referent with its symbol in the object level test, than learn to identify a hierarchical structure of two objects and their class (all MEF and TOV belong to VIC). They found that the same subjects who could identify objects could also discriminate for the first hierarchy, but the second operation took twice as many learning trials. The second hierarchy was very difficult to teach. As might be expected, the experimenters found, in the discrimination between two members of a given set, that the smaller the difference between MEF and TOV, the more difficult was the distinction. And finally, they found that those with high scores on the hierarchy tests came up with high scores in the memory and discrimination tests.

Welch (1939). Welch, using 39 children, 12 months to five years of age, tested for ability to recognize form and size differences in squares and rectangles of plywood. In the initial training period, the subjects were taught to pick up plate 1, an 8" x 8" square of plywood, in response to the symbol "ate." A piece of candy, used as a reward, was found under the plate. Following this training period they were tested for discriminating between this 8" x 8" square and a rectangle. The rectangles were ordered in a series which ranged

10 Ibid.
from a figure very dissimilar to the square to one almost identical with it. Two fourteen-month-old children were able to discriminate an 8" x 8" square from a 12 3/4" x 3 1/4" rectangle. Welch found that fine discrimination of form and the distinction of large from small appeared at approximately the same chronological age in his subjects.

Stevenson and Iscoe (1954). As the section on Transposition in Chapter III explains, evidence supports the belief that subjects performing in transposition experiments at the verbal level demonstrate a type of behavior which is describable as concept formation. In an experiment, conducted at the verbal level, involving size relationships, Stevenson and Iscoe\(^\text{11}\) worked with 140 children, in grades four through seven, testing for transposition of the relations "smaller than" and "larger than." In a transposition experiment the subject is trained to respond discriminately to two stimuli, for example, to the larger of two similar objects, and is then tested with other pairs of stimuli, representing the same relationship.

If the relationship was one of size, then the stimuli would be of the same form but have varying degrees of size difference between any two of a given pair of stimulus objects. In this experiment the authors departed from the typical transposition experiment in that

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they worked for differentiation from a field of three stimuli rather than the customary two, so that there were three forms of different size in any given trial and choice was to be based on the largest, smallest or middle size depending upon the experiment. They used hollow cubes for the stimulus forms with the bottom open so that an eraser could be hidden under the proper box to serve as a game-type reward for correct choice. They divided the children into two equal groups and trained and tested one group to choose the largest of three stimuli and the other group to choose the smallest of three. Transposition in two subgroups was accomplished following training periods consisting of four and ten correct consecutive responses respectively before transposing. Those subjects required to achieve ten correct consecutive responses before transposing were called overtrained and demonstrated better results in transposing.

Stevenson and Iscoe (1955). The same authors report another transposition experiment in which the subjects were mildly feebleminded adults with MA above seven years. In this experiment two stimulus squares were used and the subject was to choose the smaller one. Transposition was achieved with no attendant decrease upon increasing the amount of separation of training and test stimuli with respect to

the relationship being tested. The findings of these two studies give evidence of the formation of the concept "larger than" or "smaller than."

All of the experiments reported thus far in this chapter tend to verify the theory that referential meaning is possible of attainment prior to the development of a more difficult level of meaning in which a definition or explanation of the term or symbol can be stated.

Sullivan (1927). A study of the ability of adults to acquire the concept or mathematical relation, "contains" or, "is contained in," was undertaken by Sullivan. In this experiment the three geometric figures, triangle, circle and square were given nonsense syllable-names, (Figure 6), which were learned by the subjects.

![Figure 6](image)

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These geometric forms were then arranged in various patterns involving one, two or all three of the set. For example, the three geometric shapes might be combined and associated with the combination of syllables as shown in figure 7. From a series of similar configurations the subject was to generalize that the first syllable in the name of the configuration, which he associated with its proper geometric form, signified that that form was the innermost, with the second syllable signifying the enclosing form. This was, in turn, enclosed by the third form whose nonsense syllable-name would be the last syllable of the name. The third part of figure 7 indicates a corresponding configuration involving only two of the forms and syllables.

The experiment was divided into three parts depending upon the amount and timing of the instruction given the subject about the required generalization, or relationship. Group I was not told in advance that a generalization would be requested. Group II

Figure 7

turopa
ropatu
patu
received in their verbal instructions the directive to look for a generalization as they were viewing the various configurations. Group III was told in advance that the instructor would make a generalization after the series was ended.

Some of Sullivan's findings were as follows:

1. The chance of getting a suitable generalisation was less, and the time taken longer with method I, — that in which there was no advance information given — than with the other two methods.
2. Method I insures greater recall, over a longer duration, than methods II or III.
3. Method II gave better results on recall tests than method III after one week.

Additional Studies

Other studies having to do with concept formation as it is related to mental set and verbal instructions given have been reported by Rees and Israel,\(^ \text{14} \) Reed\(^ \text{15} \) and Ewert and Lambert.\(^ \text{16} \) Some of the general findings of these studies, including Sullivan's are


listed below:

1. The time taken to learn a memory series is increased by knowledge of failure and decreased by knowledge of success.
2. Women are more affected by such knowledge than men.
3. Subjects with high I.Q.'s are more affected by knowledge of failure than success, while those with lower I.Q.'s are affected conversely.
4. The value of such knowledge for later recall is greater in success reports, but less in those of failure.
5. In a manipulatory problem more complete verbal instructions were accompanied by fewer manipulatory movements necessary for problem solution.
6. There is a high correlation between performance and intelligence.
7. Learning curves are made smoother and more gradual by increasing the amount of instruction.
8. Those of higher intelligence rely more on language symbols than those with lower I.Q.'s.

Implications of Findings

Regarding Procedure in Concept Formation

Such findings as these are important to this study because they emphasize the fact that the formation of concepts is definitely dependent upon teaching methods. These experiments have of course been conducted at the verbal level and the implications are therefore primarily for teaching methods at this level. This is in keeping with the previous statement that the major interest of the teacher is directed toward concept formation and the development of meaning at the verbal level.
Methods for Developing Referential Meaning

The preceding chapters, and sections of this chapter, have dealt with various types of experiments having to do with learning at the several levels of concept development up to and including that involving the referential meaning of words. These experiments, conducted for the most part at the non-verbal level, presented findings pertaining to the acquisition of mathematical concepts by animals and human beings without the help of a teacher. The methods of the experiments conducted at the verbal level have, of course, the complexion of a teacher-learner setting. The inseparability of teaching methods and the development of concepts will become more and more evident in the succeeding sections of this chapter, and of the chapters to follow. Such methods have dual implications: first, they represent means of concept development, and second, they present by their very nature methods of instruction for such concept development.

I. The Ostensive Method

A. Ostensive Definition and the Ostensive Method

Johnson\(^{17}\) originated, at least in name, a method of developing referential meaning which he called "ostensive definition." This, \(^{17}\)W. E. Johnson, *Logic*, Part I, Vol. I, pp. 94-96.
he applied to the giving of proper names, and adjectival names such as those of colors, to referents which cannot be initially defined except by pointing to examples. He uses the word "cochineal," for instance, to show that no verbal definition is sufficient to establish the referential meaning of "cochineal" without pointing to a sample of that color. He says that an ostensive type definition can be used only when the object (sample) can be directly pointed to.

This "ostensive definition" as Johnson calls it is included in this chapter because it is not generally accepted as a type of definition, since it does not consist of two equal verbal expressions. Stabbing agrees that there are non-definitional methods of developing meaning but resists calling these definitions in the following:

We must be careful not to use "definition" so widely that it comes to stand for any process enabling us to learn the application of words.... The process of pointing (whether metaphorically or otherwise) to the referent is not a process of defining since the referent is not another expression equivalent to the defined expression;... Thus we cannot agree with Mr. Johnson that proper names can be defined. He recognizes a form of definition which consists in 'the act of indicating, presenting, or introducing the object to which the name is to apply', to which he gives the name ostensive definition. This is open even more strongly to the objections raised against extensive definition.18

18L. S. Stebbing, A Modern Introduction to Logic, pp. 422-23.
In this study, "ostensive definition" will be considered as a method of developing referential meaning, whether by the physical or the verbal isolation of referents, rather than as a type of definition. Therefore with the exception of treatment in quotations it will be referred to as ostensive method, or simply, ostension. It can be seen that it includes the two types of "so-called definitions" discussed by Stebbing, i.e., the "ostensive definition" of Johnson and "extensive definition or definition by examples."

B. The Need for Ostension.

In justifying the need for ostension Ayer says:

In the first place, it is not true that we are able to use or understand a language when we are acquainted only with its formation and transformation rules. These rules are indeed sufficient for the characterization of a purely abstract system of logic or mathematics, so long as no attempt is made to give the system a material interpretation; but they are not sufficient for the characterization of any language that serves to communicate propositions about matters of fact.... For it to become a language it is necessary that some at least of the expressions that it contains should be given a meaning. And this is affected by the method of ostensive definition, that is, by correlating these expressions not with other expressions, but with what is actually observed.19

Feigl says that unlimited regress in definition is impossible. He points to the need of finally using what is called "ostensive definition" thus:

As to whether this last step in the definition of an empirical concept is to be considered itself a kind of definition is a mere question of terminology. It is rather fashionable nowadays to speak of "ostensive definition."  

This need for an experiential base for meaning is further emphasized by Werkmeister in the following quotation:

Whenever doubts assail us at the connotative level, whenever we are suspicious or uncertain of the meaning of words, we have at our disposal only one procedure by which we can re-establish certainty: we must return to the demonstrative level of definition and lay anew the denotative foundation for the connotative meanings. If the words we employ have lost all connection with the world of experience, they cease to be terms, they have no meanings and are but "sounds" or "marks on paper" which signify nothing. It is therefore imperative for clear thinking that, regardless of how far the process of abstraction and connotative definition carries us away from the immediacy of demonstrable experience, it must at all times be possible to retrace our steps, to leave the realm of abstractions and to return to the realm of demonstrable contents of experience.  


21 W. H. Werkmeister, An Introduction to Clear Thinking, p. 169.
C. The Use of the Ostensive Method in Developing
the Referential Meaning of Words.

Russell says: "Our empirical vocabulary is based upon words having ostensive definitions, and an ostensive definition consists of a series of percepts which generate a habit." He illustrates this by the following:

Let us begin with "meaning" and let us take the word "hot" for purposes of illustration. . . . there was an open fire in my nursery, and every time I went near it someone said "hot!" . . . they used the same word when I perspired on a hot summer's day, and when, accidentally, I spilled scalding tea over myself. The result was that I uttered the word "hot" whenever I noticed sensations of a certain kind.

Russell thus equates one way of learning the meaning of a word with the establishing of referential meaning by the methods shown above. That this is considered a method of building concepts is further shown by the "identical-elements" approach, through the use of the ostensive method, considered by Hull in the formation of the concept "dog." Hull says:

A young child finds himself in a certain situation, reacts to it by approach say, and hears it called "dog." After an indeterminate intervening period he finds himself in a somewhat different situation and hears that called "dog." Later he finds himself in a somewhat different situation still, and hears that called "dog" also . . . . the intervals between the "dog" experiences are filled with all sorts of other absorbing experiences which are

23 Ibid.
contributing to the formation of other concepts. At length the time arrives when the child has a "meaning" for the word dog. Upon examination this "meaning" is found to be actually a characteristic more or less common to all dogs and not common to cats, dolls and teddy bears. But to the child the process of arriving at this meaning or concept has been largely unconscious....Such in brief is our "standard" or normal type of concept evaluation.\footnote{C. L. Hull, "Quantitative Aspects of the Evolution of Concepts," Psychological Monographs, XXVIII (1920), No.1, p. 5.}

This meaning that Hull refers to is what is here classed as referential meaning and is seen to come from a repeated use of the ostensive method. The development of this referential meaning is probably accomplished before formal schooling begins. It need not be so, however, as the same result may be accomplished in a formal setting, i.e., that involving a teacher and learner.

A very early stage, then, in meaning attainment, is what Johnson has called "ostensive definition" and consists of two methods of connecting the term used with its referent. One is by actually pointing out one or more of the referents and saying, for example, "That is called a circle." The second method is by describing in unambiguous words the location of the referent, as for example: "The figure in the upper right hand corner is a circle"; or the animal we saw on T.V. last night is a kangaroo." Neither of these is an analytical definition as defined in this study since there is not the equating of two verbal phrases, one of which may be logically substituted for the other.
II. The Denotative Method

The second method of establishing referential meaning is that of listing examples or, what is called by some, the denotative method. This method would be used when the concept to be developed involves a relationship or characteristic which is not evident in one example. Consider the referential meaning of "mammal." Clearly, one example would not be sufficient to establish such meaning. The fact that in such a case the listing of many examples may still not accomplish the purpose renders this method of little practical use, unless the listing of examples is accompanied by some descriptive phrases. The inclusive of such description involves a type of verbalization which is not a part of referential meaning. For this reason the material related to this method is included in the discussion of the next category, that of verbalization involving description or pre-analytical meaning (see Chapter VI -- Part I).

III. The Method of Example, Counterexample and Variation

A method of developing referential meaning which involves the two preceding methods is that of giving real and verbal illustrations accompanied by negative instances. For example, we may point to a circle and say, "That is a circle," and then point to an ellipse and say, "That is not a circle." Indeed the only difference between this and the ostensive and denotative methods is that this method purports to facilitate the connection between term or symbol and referent, by the use of counterexamples.
Smoke (1933), Dickinson and Tyler (1944). Smoke reports an experiment in concept formation using 30 undergraduates and incorporating negative instances. He found that there was no appreciable difference between the length of time taken to learn a concept by the use of positive and negative instances and the time taken when positive instances alone were provided. He does admit that most of the subjects (23 or 30) preferred the use of both positive and negative instances. Later, Dickinson and Tyler conducted a similar experiment with practically the same results.

Animal experiments on concept formation which have already been reported throw some light on the subject of the use of negative instances. Munn mentions the use of variation in the negative stimulus in a concept formation experiment as a test for determining whether or not the supposed learning is mere negative conditioning to the negative stimulus. This means that if throughout the training period the subject were making its choice, and being rewarded,


on the basis of choosing the form which was not a circle it would continue to choose any second object which was not a circle even though that object might be different from the original positive stimulus (triangle) of the training period; if this were true, the experiment could not be interpreted as demonstrating recognition of form per se. The fact that the animal did change its pattern of choice when the positive stimulus was changed showed two things: first, that it was discriminating on the basis of the positive stimulus, and second, that the negative stimulus was relatively unimportant. The relative unimportance of the negative stimulus in the recognition of form, per se, has been pointed out in connection with several experiments reported in Chapter IV in which variation in the negative stimulus had little effect on the recognition of the positive stimulus.

There are some other implications of the findings of experiments in concept formation for the topic of this section. The findings of Fields and others mentioned in Chapter IV show that a variation in the positive stimulus is not only helpful but is essential in eliciting concept formation. It may be recalled that Fields' rats, when trained with only an equilateral triangle versus a circle, were not able to distinguish form per se, and did not evidence any understanding of the concept of triangularity, but when the training period included more cases and positions of the positive stimulus the concept was formed. This means that they were able
to discriminate any type of triangle from the circle or any other negative stimulus which was used in testing. It would seem just as likely that such variation of positive stimulus or variation in examples of the referent would enhance the chances for the development of the referential meaning of words. Such a method might be called, "Developing Meaning Through Variation," and could include the use of negative instances. This method can be explained best by an illustration. Suppose that the concept of perpendicular lines is to be taught. Instead of showing merely figure 8 (part a), the other examples in the figure would be shown as well, in order that the students not get the idea that lines are perpendicular only if one is horizontal and the other vertical.

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Figure 8
Smith (1940). Smith points out this need for variation in developing a concept as follows:

It is the tendency in textbooks to define a term, perform a construction, or prove a theorem with the use of as simple a figure as possible and to expect the pupil to apply what he has learned (without help) to more complex figures. 28

To show that this carry-over is not always made, Smith devised some tests which he gave to 114 students with I.Q.'s ranging from 90 to 146 (mean 117.9). One test involving perpendicularity of lines was given after students were shown what perpendicular lines were by means of a simple example involving a horizontal line with a perpendicular drawn to it, at approximately its midpoint. A part 29 of this test is given below in a condensed form:

Test 6. Draw a perpendicular:

1. to AB at 0

A ___________ B

0


29Ibid., p. 107
2. to MN from P

3. to RS from T

4. to AB from C

5. to AB at A and B

6. to AB from D

7. to AB from C
He found that errors increased as the figure became more complex, i.e., farther removed from the illustrative example given as an introduction. With further training and testing, in which varied examples were given in the training period, the test results showed significant improvement, thus suggesting that the use of variation in the positive stimulus has possibilities as a teaching method.

**Summary of the Chapter**

The referential meaning of a word or symbol was defined as that meaning involving the interrelationship of a word or sign, a referent and an interpreter. Various experiments, dealing with concept formation at the verbal level and applying to this kind of meaning, were reported. Findings of these experiments indicate that the referential meaning of words can be established prior to the analytical meaning.
of a word but that the referential meaning of words standing for concepts of relationship are more difficult to establish than the referential meaning of words dealing with things.

Some of the findings of experiments involving methods used in the formation of concepts of relationship indicate that the method of presentation is very important.

Three general methods of developing the referential meaning of words were presented as (a) the ostensive method, (b) the denotative method and (c) the method of example, counterexample and variation. In connection with the first of these methods the individual's ability to single out, hypothetically at least, the referent for a word was presented as a criterion for meaning attainment. The second method was deferred for inclusion in the next category because the method of denotation is so rarely found without accompanying descriptions of the examples (which descriptions change the method to one involving verbalization). The method of example, counterexample and variation was illustrated by studies conducted along these lines. The negative results obtained in experiments on the use of examples and counterexamples alone, in developing a concept, were pointed out as being in agreement with the previously mentioned implications of concept formation experiments with animals and humans in Chapter IV. The method of variation gives much greater promise and is recommended for further study.
Implications for Teaching

The results of experiments on concept formation at the verbal level show definitely that referential meaning is possible of attainment without being based on the prior analytical definition or explanation of a word or term. This is an important observation from the standpoint of the teaching of mathematics, in its implications for the presentation of new concepts.

The findings of Smoke and of Dickinson and Tyler tend to indicate that the use of negative instances is not a definite asset in developing referential meaning. It has not been established, however, that it is a liability, and study needs to be given such matters as retention of learning when negative instances are used, and the results compared with those obtained in cases where positive instances alone are used. Experiments in concept formation with animals and children do point to the relative unimportance of the negative stimulus (negative instance) and thus support the negative findings reported above.

Other findings in concept formation experiments do point to another closely related method of developing referential meaning, namely, the use of variation of the positive stimulus. The limited experimentation along this line has yielded positive results as to its value and it deserves further study as a method of developing the referential meaning of a mathematical term.
CHAPTER VI

METHODS OF DEVELOPING THE
MEANING OF WORDS THROUGH VERBALIZATION

Part I -- Meaning as Developed by
Explanation or Description -- Pre-analytical Meaning

We have seen that referential meaning involves the identification of referents by the physical or verbal pointing-out of such, or by the listing of examples of the referent. Now, consider the following statement: A paraboloid is a solid which looks like half of a football. This is not an example of the referential meaning of the word, paraboloid, for there is neither oral nor verbal isolation of a referent. It is a verbalization which contains a description of the referent. This is an example of what we choose to call pre-analytical meaning. One purpose of such a verbalization is the establishment of a means whereby referential meaning can be acquired, but the method of doing this is not that of verbal or physical pointing.

Krauss (1952). Krauss has compiled a group of so-called "first definitions" of common terms as given by four- and five-year-olds such as:

(1) "Cats are so you can have kittens."
(2) "A floor is so you don't fall in the hole your house is in."^1

^1R. Krauss, A Hole is to Dig, (pages not numbered).
These show that the child at this level has acquired a certain type of understanding, so that he can recognize the referent of the word and, perhaps, use the word properly in conversation even though he is unable to give an explanation, or a definition of the word which would be satisfactory to an expert in the field. It is evidence of something other than referential meaning, however, for these statements do more than establish a connection between a referent and a word; they are crude attempts at an analysis.

The difference between the use of the ostensive method for establishing referential meaning and this method of verbalization which is pre-analytical meaning, may be further clarified by the following: Suppose the question, "What is an ellipsoid?" is asked.

1. One person points to an ellipsoid. This shows referential meaning.

2. Another person points to one and says, "That is an ellipsoid." This shows referential meaning.

3. Another says, "An ellipsoid is an egg-shaped solid." This contains a description and demonstrates pre-analytical meaning.

4. One says, "It is the solid that we discussed in calculus class yesterday." This contains a description but the verbal pointing is pronounced so it is considered to demonstrate pre-analytical meaning. It is actually a borderline case.

5. Still another says, "That's what we were looking at in the display case last night." This shows referential meaning because there is no analysis. It is an example of verbal pointing.
Ostension and Denotation Accompanied by Description

The establishment of referential meaning, as pointed out in the last chapter, includes the methods of physical or verbal pointing (ostension) and the listing of examples (denotation). In either case the eventual establishment of referential meaning follows much the same pattern as the identical-element approach in concept formation in which the elements common to many examples, when abstracted by the mind, lead to the firm establishment of referential meaning. This is, at best, a slow and uncertain process if the example-giving is not accompanied by some description, as can be seen from the following illustration. Suppose we were to try to develop the meaning of "bobo" as an adjective applying to a class of musical instruments, by merely giving examples of bobo instruments:

1. Kettle drum
2. Bass Viol
3. Violin
4. Viola
5. Cello
6. Guitar
7. Banjo
8. Ukelele
9. Trombone
10. Autoharp

Now it is clear that "bobo" does not mean "stringed" as shown by the presence of numbers 1 and 9. What it does mean is probably
not yet apparent to the reader. Suppose, therefore, that we list
a group of "non-bobo" instruments, i.e., counterexamples:

2. Marimba 10. Saxophone
3. French horn 11. Sousaphone
4. Baritone 12. Trumpet
5. Cymbals 13. Organ
6. Tambourine 14. Flute
7. Triangle 15. Bagpipes
9. Piano 17. Oboe

If many, many examples and counterexamples were listed or
pointed to it is still not likely that the meaning of "bobo" would
become apparent to any but a small percentage of those well acquainted
with musical instruments. This points to the need for an accompanying
description.

Now, if the denotation, provided in a list of examples, is
accompanied by descriptive phrases, the prospects of establishing
some type of meaning improve. Add to the above information a few
definitive statements, such as, (1) "bobo" is a characteristic of
a particular type of instrument that is not as apparent in its
physical construction as in its use; (2) this characteristic has
to do with the manner in which the instrument is tuned; (3) the
characteristic, more precisely, involves the possibility of an
instrument's being tuned to any desired interval; and finally (4)
it describes an instrument which can be so tuned while the musician
is playing it (the piano and harp are thereby ruled out). With these
statements and the previous listings a rather definite meaning is
established, but it is not strictly referential since the accompanying
descriptions make it either pre-analytical or analytical depending upon the precision of the analysis.

That there are words whose referential meaning can be developed by an ostensive or denotative approach alone seems evident from the fact that children develop their understanding of initial concepts in such a manner. But even here, we must be careful, for, while it may at first appear that the informal type of learning which goes on outside the classroom, and in the early years of childhood, is denotation or ostension without description, it is difficult to say for sure that some type of explanation or description does not accompany the pointing out of examples. For instance, consider the word "hot." A child hears the word in many situations, but explanatory statements may very well accompany such example-giving, as: "It's hot, it burns"; "It's hot, see how red it is"; It's hot, the flames are leaping"; It's hot, it will hurt." All of these are accompanying descriptions which suggest that referential meaning is not usually attained independently of description. In fact, the burning sensation that a child receives when touching a hot object is a type of accompanying explanation, though it is non-verbal.

There may be some concepts which can be developed satisfactorily by ostensive means alone but these are limited for the most part to proper names, or limited classes such as simple geometric forms (e.g. circle), and even here any accompanying explanation would be an analysis changing the meaning to pre-analytical. Suppose, for an
example of a limited class in which the listing of examples is insufficient, that we consider the meaning of the word president as applied to president of the United States. Now obviously we can develop the referential meaning of this class by ostensive means so that the learner can call up any member, namely, by listing all members of the class to date. The question that remains is, "Does this mean that the learner can designate henceforth any president of the United States?" The answer is clearly, "No!" Without some listing of properties or attributes, the mere learning of the total class does not fit one for the task of recognizing a president of the United States. Such can be assured only if along with the denotation (list of examples) there is provided an accompanying description such as, "this class includes all such people who shall in the future be elected to the highest office of the United States," assuming, of course, there is no change in our present form of government. This type of meaning is pre-analytical.

Denotation alone is satisfactory in the case of proper nouns where there is an unambiguous set of elements of the class. An example would be, "John L. Lewis was president of the United Mine Workers in 1958," provided there was not another John L. Lewis who was president of another organization by that name at that time. The practical uses of such an approach are limited.
The Identical-Elements Approach to Concept Formation as an Example of the Denotative Method With and Without Description

Hull (1920). The use of denotation for the establishment of referential or pre-analytical meaning has been compared to the procedure of finding the common element in a list of examples. Hull reported the results of an experiment along this line in 1920. In this experiment the adult subjects were presented with Chinese characters, one at a time, shown through the window of a memory apparatus. There were several series of characters, each member of a given series possessing some marking common to the other members of that series. For example one series consisted of the characters with the common element being . The characters in any one with the common were presented in a random order consisting of characters from various series accompanied by a nonsense syllable which was the same for all members of a given series but naturally not the same for all members of a given test set. The subject was required to learn that a given term applied to all members of a given series, having a given common element. For example, in a given test series, the character containing the element would be accompanied by the syllable with corresponding syllables for all other characters.

Learning was established when the subject could give the sound / when any character with the \m{\bar{u}} was shown, or the sound / when a character containing another common element was shown. The procedure was so continued through the other characters of a series.

The major drawback of this method, in the opinion of some, is that it applies only to externally apprehended common elements and not to such properties as relationships, as in the "bobo" illustration above. For instance, how could one use such an approach to develop the referential meaning of the word "uncle" or "father?" Certainly pointing out examples would not establish such meaning if unaccompanied by explanatory statements. Smoke\(^3\) has somewhat revolted against Hull's common element approach, substituting for it a process of searching for relationships. He says:

> It seems doubtful to us whether one can any longer hold that the discovery of one or more "common elements" either constitutes, or results in, the formation of a concept unless there is at the same time a response to the relationships existing between these "elements" in the stimulus pattern.\(^4\)

This response to relationships seems to involve what is here regarded as a type of analysis which changes the level of meaning from referential to pre-analytical.

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\(^3\)K. L. Smoke, \textit{loc. cit.}

\(^4\)\textit{Ibid.}, p. 7.
He revolts further against the common element approach in saying, "Concept formation is not such a process [purely analytical, like the common element approach], for it is both analytic and synthetic."

Smoke's dynamic whole method of testing for concept formation was quite similar to Hull's, in that series of stimuli accompanied by nonsense syllable names were presented in much the same manner. The difference was that Smoke used geometric forms, with the relationship between parts as the key. For example a circle with two dots, one inside and one outside, was always accompanied by the syllable 'dax'. Close discrimination was required in the case of such variations as a circle with two dots both inside or both outside. The subject was considered to have learned the particular concept when he could give the name when presented any member of the family. Smoke's findings tend to confirm the importance of an example's being accompanied by explanation.

Further evidence of the necessity of explanatory instructions for establishing referential meaning was presented by Ewert and Lambert in reporting an experiment in concept formation with adults. The problem consisted of transferring discs from one place to another

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5 Ibid., p. 8.

in a set order. The 140 subjects were divided into four groups.
These groups were given different amounts of instruction, ranging
very little to complete instructions plus a demonstration. The
findings confirm the importance of verbal instructions in the formation
of a concept, in this case one of relationships. It was found, for
example, that the more complete the verbal instructions the fewer
the manipulatory moves necessary for the solution of the problem.
They also found that the more complete the instructions, the higher
the correlation between performance and intelligence. Those given
a generalization with regard to the procedure during the course of
the instruction could in most cases repeat it at the end while those
who were not given such a generalization in no case were able to
make one at the end.

Along this same line Wickens says, "Experimental evidence shows
that when subjects were forewarned to look for an underlying principle
they much more readily discovered it."7

Pre-analytical Meaning Attained From a Verbal Context

The teaching or learning of the pre-analytical meaning of a
word is accomplished or evidenced by the giving of descriptive
statements regarding the referents of the word. In a sense this

7D. D. Wickens and D. R. Meyers, Psychology, p. 255.
is developing meaning by verbal context as the following illustrates:

A circle might be described as

1. that which is perfectly round.
2. something like a ball.
3. a form that rolls without bumping.
4. the outline of a slice through an orange.
5. a thing rounder than an ellipse.

Now the sum total of these descriptive or contextual statements produces a rather clear notion as to what a circle is, so that a referent might be isolated. On the other hand, the ability, on the part of the learner to give these descriptions is analytical evidence of more than referential meaning. It is what is here called pre-analytical meaning.

Werner and Kaplan (1950). In a study directed at developing the meaning of a word through verbal context, Werner and Kaplan worked with 125 children, aged 8 1/2 to 13 1/2 years, with I.Q.'s ranging from 101 to 111. They divided the children into five groups of 25 according to age. The study dealt with the acquisition of word meanings through verbal contexts with no outside clues. The child's job was to find the meaning of a particular coined word by studying it as it was encountered in six different verbal settings. There was

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a total of twelve sets of six sentences each. The twelve words to be learned denoted either an object or an action.

For example, the artificial word in the first set of six sentences was corplum, for which the correct translation was "stick" or "piece of wood." The contexts for corplum were as follows:

1. A corplum may be used for support.
2. Corplums may be used to close off an open place.
3. A corplum may be long or short, thick or thin, strong or weak.
4. A wet corplum does not burn.
5. You can make a corplum smooth with sandpaper.
6. The painter used a corplum to mix his paints.  

In the procedure, the experimenter presented a card with sentence "1" on it and the child was asked what he would say the word meant, and why. While this card was still in view he was presented with the second card and asked the same question plus the question as to whether the second meaning given would fit that on the first card. This was continued through card #6 and the responses were recorded. The responses were then analyzed by three judges within five main categories. Three of the given categories are of interest in this connection. First was correctness and conventionalization, in which they found a steady growth through the age levels. Second was signification or interdependence of word and sentence meaning. In this category it was noted that the younger children tended to confuse the entire sentence with the

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9Ibid., p. 251.
meaning of the word whereas older children were able to separate
the two. Symbolization -- their fourth category -- referred to
the connection between symbol and referent. In this category the
younger children tended to identify one with the other; i.e., there
was no separation of symbol from referent. The other two categories
-- the third (grammatization) and the fifth (rigidity) -- have impli-
cations mainly for the field of semantics.

The general procedure of this experiment illustrates the method
of developing the pre-analytical meaning of the coined word since
the contextual statements were, in effect, descriptive of the
referent. The authors found a direct correspondence between ability
and age.

Summary of Part I -- Chapter VI

In Part I pre-analytical meaning was presented as a type of
meaning involving a verbalization which is descriptive of the
referent but is not satisfactory as a definition. It is a category
of concept development intermediate between the category which
involves referential meaning and the one which involves analytical
meaning. Its major purpose was given as that of promoting the
recognition of referents but it has another purpose, implicit in
the fact that the possession of such meaning evidences an under-
standing of basic qualities of the referent which involve more
than the ability merely to recognize it.
A major portion of this part of the chapter was devoted to a discussion of the importance of combining description with the provision of examples in establishing referential meaning by ostensive and denotative methods, but it was pointed out that such a procedure changes the type of meaning to what is here called pre-analytical meaning. The difficulty or impossibility of establishing referential meaning by example alone was made apparent by the "bobo" instrument illustration.

The similarity between this method of establishing referential meaning (denotation or ostension with accompanying explanations) and the common- or identical-elements approach in concept formation experiments was pointed out and certain relevant experiments were reported. Smoke's dynamic whole approach to such experimentation was shown to compare favorably with this method in that he employed the technique of looking for relationships rather than for the common element alone.

**Implications of Part I for Teaching**

The evidence of experiments on concept formation using the identical-elements approach, coupled with an evaluation of the actual use of a pure ostensive method of establishing referential meaning, suggest that while both are possible they are not nearly so likely or productive of learning as what has been called "denotation or ostension-plus-description," which is a method of
establishing pre-analytical meaning. A further implication is that this level of meaning is of much greater usefulness in the development of mathematical concepts than any level thus far discussed because of the inherent difficulty of using either the denotative or the ostensive method without some accompanying explanation, intended or otherwise.

Part II — The Level of Definition — Analytical Meaning

The Purpose of Analytical Definition

The stages of concept development thus far treated have all involved situations in which the referents were readily recognizable. For such referents the earlier categories of concept development have been sufficient. It is when we encounter referents about which there is disagreement as to their inclusion in a certain class that we see the need for an analytical definition which makes possible positive identification of referents by a classificatory means which not only categorizes existing referents but can be used to include or exclude new referents.

Robinson, in Definition, makes an observation on the function of definition at the level of analysis which is in substantial agreement with the function outlined above:

Such mathematical improvements or analyses of concepts often seem absurd to persons whose only conception of definition is teaching someone the meaning of a word he does not yet know, because they are often obscure, and when they are improvements instead of analyses they alter the meaning of the word. But they
are not absurd; it is merely that the critic has mistaken their purpose; their aim is not to give people more vocabulary, but to create systems of concepts.

The functions of analytical definition, then, are three, (1) identification of referents, (2) classification of referents for the purpose of developing a logical system, and (3) analysis for determining the constituents of the referents.

**Definition of Analytical Meaning**

Analytical meaning is considered to be that meaning which is evidenced by the ability to give a definition of a word or symbol which adequately describes the composition or generation of the referent of that word or symbol, or accurately isolates the referent within its relevant class (as per genus et species).

**Examples of Analytical Definitions**

I. By Composition

An example of a definition which describes the composition of a referent for the word "circle" is the following: A circle is the set of points or number pairs \((x,y)\) in a cartesian plane such that \((x-a)^2 + (y-b)^2 = r^2\), where \((a,b)\) is a fixed number pair.

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1R. Robinson, *Definition*, p. 197.
II. By Generation

Consider the following definition: A circle is the locus of a point which moves in a plane so that its distance from a given fixed point is always constant. Clearly this describes an operation for generating a circle rather than the composition of the circle.

III. By Isolation

Definition by isolation denotes the singling out of a particular member of a family of referents by naming the connotation which sets it apart from other members of the family. This is the familiar genus et species definitional approach of Aristotle. The circle may be defined by this method as follows: A circle is a simple closed plane curve with constant curvature. Here the circle is singled out from the family of simple closed plane curves by the property, "with constant curvature."

Robinson's List of Types of Mathematical Definition

Robinson (1954). Robinson lists six types of mathematical definitions, three of which seem to comprise what we call analytical meaning. His list is as follows:

1. Abbreviations....
2. The analysis of concepts.
3. The analysis of concepts into specified concepts of the system.
He places these in two groups: the second, third and fourth comprising what he calls "real definition," and the first, fifth and sixth comprising "nominal definitions." The types in the latter group (nominal definitions) belong either to the kind of meaning studied in the first part of this chapter, that is, pre-analytical or to referential. This can be seen if the three types in this group are examined. His first type, abbreviation, is a stipulative-type definition which establishes a relationship between a word or symbol and its referent which may well be another symbol as in the case of \( i \) for \( \sqrt{-1} \). It is a type of verbal pointing and thus belongs in the category of referential meaning. In an important sense any definition is an abbreviation in that the term being defined may be substituted for the entire connotation such as "acute angle" may be substituted for "an angle between zero and ninety degrees." This is not what is here meant by "abbreviation" since Robinson's meaning is a "stipulative-type" abbreviation.

The fifth type refers to the informal type of definition which involves a description of undefined terms in a system so as to remove ambiguity, but the example he cites sounds very much like

his first type or a stipulative-type abbreviation, for he quotes the following from *Principia Mathematica*, p. 93, as his example: "If \( p \) and \( q \) are any propositions, the proposition '\( p \) or \( q \)', i.e., 'either \( p \) is true or \( q \) is true,' where the alternatives are to be not mutually exclusive, will be represented by '\( p \lor q \)'."³ He says that Whitehead and Russell call this an explanation of a primitive idea rather than a definition. This example establishes a close relation between his first and fifth types of definition if they are distinct. He then explains his sixth type as one that refers to the citing of examples to which the system applies i.e., a set of postulates is presented as part of a definition followed by an example which illustrates the definition. This latter is what Robinson calls a coordinating definition, and he adds that he is not convinced that it is really different from his type five, or "nominal definition of the symbols of a system." However these are classified, it seems clear that numbers one, five, and six all involve a category of meaning which is not analytical.

The group that Robinson calls "real definition" (numbers two, three and four) represents the type of mathematical definition which is considered in this chapter, namely, analytical definition. Robinson says that his second type, (the analysis of a concept) differs from

³A. N. Whitehead and B. Russell, *Principia Mathematica*, p. 93
the first type (abbreviation) in that it involves the verbalization of a discovery that a certain familiar symbol may be substituted for a certain set of familiar symbols, whereas abbreviation involves a declaration that a new symbol is to stand for a certain combination of old symbols. For example, "sphere" may be defined as, "a solid enclosing a maximum volume with a given surface area." In this case all the words are familiar but the analysis is the new thought.

His third type is merely the analysis of a concept, with the restriction that the symbols used in this analysis must be parts of the mathematical system. This type is illustrated by the example given above in which the words, "sphere," "volume" and "surface area" were chosen for a mathematical definition rather than, "ball," "substance" and "skin" respectively. Robinson says that the improvement of a concept often accompanies an analysis of it. This refinement, although more a process than a type, characterizes his fourth type listed and the third of the group of "real definitions" here included under those dealing with analytical meaning.

Analytical Definition and Referents

It is necessary of course for the words used in an analytical definition to have been already established in referential meaning with the possible exception of the word or symbol being defined (definiendum), which may possibly have no referent. Thus the analytical definition of "water" as, "a chemical compound composed of two parts hydrogen and one part oxygen," is given in terms of words already
caressing referential meaning, but in the definition of a "unicorn" as, "a mythical animal having one horn," all words have referential meaning except the word "unicorn" itself. The lack of such referential meaning in the case of the word "unicorn" does not detract from the meaning presented by the definition, for the referential meaning associated with each of the other words in the definition enable the mind to "picture" a referent or "model," even though such is nonexistent in the world of touch and sight. In the same way, "untriangle," defined as, "a triangle having three right angles," although nonexistent in the Euclidean sense, has a certain meaning for the reader because the other words in the definition have referential meaning.

**Analytical Definition Typed According to the Existential Nature of the Referents**

As examples given in the previous section suggested, analytical definition may vary depending upon the existence or non-existence of referents of its definiendum. If the nature of the referents is used as the criterion for typing analytical definitions, we obtain the following interesting outline which will be further elaborated in the course of the present chapter, by discussion and example:

I. Definition in which the referents are assumed to exist.

A. The number of referents is unchanged although the definition may change.

B. The number of referents is varied through an historically changing definition.
C. The number of referents is varied through a pedagogical procedure which presents a different or extended definition at various levels.

II. Definition in which the existence of the referents is either unknown or beyond perception but whose properties are assumed.

A. The referents are ethical ideals or religious concepts.
B. The referents are created in the mind to satisfy a scientific hypothesis.

III. Definition in which the non-existence of the referents is known or assumed but for which analytical meaning is still possible.

I. Definition in Which the Referents are Assumed to Exist

A. The Number of Referents is Unchanged Although the Definition may Change

Examples of this type are rather limited, but any of the conic sections will serve as an example. If the circle, for example, is defined as in plane geometry, in terms of a locus, viz., "the locus of points in a plane at a given distance from a given fixed point"; or if it is defined later in solid geometric terms as, "the intersection between a non-tangent plane and a sphere"; or, still later, in analytic geometry as, "the set of points or number pairs, \((x,y)\), such that \(x^2 + y^2 = r^2\)," we have the same set of referents of the class called "circle." There has been no change in the set of referents through the steps of such a definitional process.
B. The Number of Referents is Varied Through An Historically Changing Definition

That definitions do change as new progress is made in a science is certain. If the set of referents is in any way tied to the definition it seems reasonable to suppose that this set may change in number.

An excellent example of the historical change that takes place in the definition of a term is that of "acid," given by Mills.

...the definitions in the sciences are also constantly varying. A striking instance is afforded by the words acid and alkali, especially the former. As experimental discovery advanced, the substances classed with acids have been constantly multiplying, and by a natural consequence the attributes connoted by the word have receded and become fewer. At first it connoted the attributes of combining with an alkali to form a neutral substance (called a salt), being compounded of a base and oxygen, causticity to the taste and touch, fluidity, etc...the formation of neutral bodies by combination with alkalis, together with such electro-chemical peculiarities as this is supposed to imply, are now the only differentiae which form the fixed connotation of the word acid as a term of chemical science. [4]

In the history of the definition of "acid there is seen to be a constant interplay between the referents and their connotation, but in the present consideration the important change is that which takes place in the set of referents of "acid," through the historical change in the definition. For a contemporary definition of "acid" the following is taken from the Handbook of Chemistry and

Physics:

For many purposes it is sufficient to say that an acid is a hydrogen-containing substance which dissociates on solution in water to produce one or more hydrogen ions. More generally, however, acids are defined according to other concepts. The Brønsted concept states that an acid is any compound which can furnish a proton. Thus NH₄⁺ is an acid since it can give up a proton: \( \text{NH}_4^+ \rightarrow \text{NH}_3^+ + \text{H}^+ \), and NH₃ is a base since it accepts a proton.⁵

These two definitions, plus a third one which employs the idea of sharing electrons, show that the process of definitional change is still going on, and also, that more than one definition may be in use simultaneously, depending upon the use being made of it. Speaking of changing definitions, Leonard says:

> With advancing biological knowledge, the physiological characteristics of "fish" and other kinds of animal came to be more exactly known and understood...these and other biological discoveries eventually led to a redefinition of the word "fish"...In much more recent years, still a different basis of definition has been introduced in biology; having to do with the closeness of "family connection" in the evolutionary pattern.⁶

In each succeeding redefinition the set of referents is seen to change, deleting some, such as "whale," and adding others.

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Van Evera speaks of the historical changes in the definition and denotation of the word "element." In referring to his early student days, he says:

I [just] defined an element as being a substance all of whose atoms have the same atomic number. When I had my first introduction to chemistry in 1919, we defined an element as being a kind of matter, all of whose atoms were exactly alike. Isotopes had not been discovered at that time....Many students and some adults complain because definitions must be changed....Changes of this nature must take place in any living, growing science.7

Still another concept whose definition has undergone repeated change through the centuries is that of "light," as given by the Pythagoreans, Newton, Huygens, Young-Fresnel, Maxwell and present day quantum theorists.

Speaking of perpetually changing definitions, Robinson says:

Even the scientist, consciously aiming at fixity of reference, often finds his words changing in his hands, or deliberately changes their meanings to meet a new situation or avoid a greater inconvenience....The mathematician...too, finds them doing so.8

Robinson thus admits that definitions in mathematics as well as in the other sciences are subject to constant change.

7B. Van Evera, "Definition, Didactics and Deliberations, Science Teacher, XIV (1958), 156.

8R. Robinson, Definition, p. 53.
1. **Number**

Consider, as an example of such historical change, the referential meaning or set of referents for "number." The early historical use of "number" included in its extension, denotation, or set of referents only the positive integers or counting numbers. Through historical development, this denotation grew to include positive fractions or rationals, introduced to make division always possible. Later negative integers and rationals were added in order to make subtraction always possible, and so on, through the years, adding irrationals and imaginary numbers in order to make certain mathematical operations possible. This concept surely "changed or grew," in the sense that its class of referents expanded, in the historical development of mathematics.

2. **Angle**

Again, consider the definition of the geometrical term, "angle." Euclid's definition was, "A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line." The extension of the term in this definition was indeed very limited, including only angles between zero

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and $180^\circ$. With changes, coming some 1800 years later, which gave a definition based upon the idea of rotation instead of inclination, the extension was increased to include the straight angle, reflex angles, negative angles and other angles equal to or greater than $360^\circ$.\textsuperscript{10} This historical development in the definition of angle has had a marked effect upon the pedagogical treatment of the concept. This fact in itself is not strange, as the historical development of a subject is often reflected in the pedagogical treatment, but the interesting fact in the case of "angle" is that Euclid's original limitation of the set of referents to angles between zero and $180^\circ$ is still detectable in geometry texts of today. For example, the writer was unable to find a single present-day text in plane geometry (published in the United States) that contained any but a casual reference to reflex angles.\textsuperscript{11}

\textsuperscript{10}This definition in terms of rotation is credited by Kokomoor to a German, W. Schmid, in 1539. See F. W. Kokomoor, Mathematics in Human Affairs, p. 282.

\textsuperscript{11}For a recent attempt in this direction, see a report by W. R. Tata, "The Role of Reflex and Negative Angles in Plane Geometry." Unpublished Master of Education field service report, Columbus, Ohio: The Ohio State University, 1960.
3. **Function**

For a more modern example, consider the change in the definition of "function," as used in mathematics over the past three-hundred years. In giving typical definitions of function fifty years ago (1910), Young writes as follows:

...the popular definition of a function, viz., that a quantity \( y \) is said to be a function of another quantity \( x \), if \( y \) depends upon \( x \)....

and later:

A variable \( y \) is said to be a one-valued function of a second variable \( x \), provided there exists a correspondence between the class of the variable \( x \) and the class of the variable \( y \) such that to every value of \( x \) there corresponds a unique value of \( y \).

This definition although given fifty years ago is very much like those found today.

Young then continues with a short section on the "History of the Concept of a Function," in which he traces the development of the definition of function from that used by Descartes in 1637.

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(the referents were powers of the variable $x$) through that given by Dirichlet which Young says is very similar to the one quoted above. He says:

This conception of a function is a very general one, and was reached only after many years of gradual extension. Indeed, the history of the term function forms an interesting example of the tendency of mathematics to generalize its concepts.

The definition of "function," given by Young as the accepted one, with the addition of some terminology involving number pairs, is seen to be much the same as the present-day definition of "function," typical of which is the following by Denbow and Goedcke: "A function is a set of ordered pairs of numbers $(x, y)$, such that for each $x$ in the set there is exactly one $y$ in the set."\(^{15}\)

In the transition of the definition of function over the years, the class of referents is seen to have grown from the powers of $x$, as used by Descartes, through the type in which there was a determinate correspondence between two variables, to the present set of referents which only need be a set of number pairs, with any correspondence whatever existing between them. The only requirement in this latter definition is that the set of first members be paired with

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\(^{14}\) Ibid., p. 194.

\(^{15}\) C. H. Denbow and V. Goedcke, Foundations of Mathematics, p. 103.
the set of second members such that each first member appears with only one second member. Thus a table of values relating time of birth with number of babies born in a certain city would be an example of a function, provided no two babies were born at the same time.

C. The Number of Referents is Varied Through Pedagogical Procedure

While there are cases in which there may be a pedagogical change\(^{16}\) in definition without change in referents (e.g., the circle and other conic sections), it seems more likely that a development in definition will involve a change of number in the set of referents, as in the concept, angle. As the last section demonstrated, changes in the pedagogical handling of definition, as well as other changes in the analysis of a concept, very often stem from the historical changes that have taken place in the development of the concept. This is not always the case, however, for as Piaget points out the historical development of a subject (geometry), which includes the development of a number of definitions, is not necessarily the way in which a child learns the subject most easily. It will be remembered from Chapter I that he shows that, while historically geometry developed as (1) Euclidean, (2) projective and (3) topological, the child in learning geometry does not follow this line of development (see p. 12

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\(^{16}\)By "pedagogical change" is meant a developmental process in the definition of a term brought about by the teacher.
above). In addition there is the well-known example of the order in which the calculus is taught, proceeding from differential to integral, although the historical development was just the reverse. Thus, the psychological pattern followed in the development of a mathematical concept, while often being identical with the historical pattern, need not be so.

II. Definition in Which the Existence of the Referents is Either Unknown or Beyond Perception.

A. The Referents are Ethical Ideals or Religious Concepts

Definitions dealing with ethical ideals or religious concepts have little effect upon the teaching of mathematics, except in some of the logical considerations where a concept such as "truth" might be involved. The meaning of such concepts as "good," "virtue," "god," "love," etc. must be developed by analysis in terms of other words which do have referential meaning inasmuch as the actual referents of the above words are beyond perception.

B. The Referents are Created in the Mind to Satisfy a Scientific Hypothesis

In definitions of this type the term to be defined, while having no referent, must be defined in terms of properties or connotations which do have referents. Thus it is seen that such definition represents the most complex form, of an analytical definition, in that the term defined is at the moment devoid of referential meaning; but even
here the other words in the definition must carry referential meaning, or they must be defined in terms that have such meaning, so that eventually we get back to referential meaning. Examples of this type would be such definitions as are found in the atomic hypothesis, the molecular hypothesis or in the quantum theory of light. A mathematical example would be the definition of an n-dimensional space. Apropos of this point, Stebbing says, "It is possible to understand a given word when we are provided with a correct description of its referend, or with a definition of it in words, the referends of which we know." It is the latter statement which allows for the definition of a word whose referend is beyond the reach of our ordinary senses.

As an illustration of the difficulty of visualization of referents in connection with some definitions, consider the concept of dimensionality and particularly the functions: \( x + y = 4 \); \( x + y + z = 4 \); \( x + y + z + w = 4 \); and \( x + y + z + w + u = 4 \). Now, the first can be shown to be a straight line in a plane, the second, a plane in three dimensional space; but where do we go for an example of the third? Likewise \( x^2 = 4 \) yields two points on a line or two lines in a plane; \( x^2 + y^2 = 4 \) gives a circle in a plane or a cylinder in 3-space; and \( x^2 + y^2 + z^2 = 4 \)

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gives a sphere in 3-space but what does it give in 4-space or what
does \( x^2 + y^2 + z^2 + w^2 = 4 \) give in 4-space or n-space?

As an example of concepts whose referents are beyond present
perception consider the following:

A wave electron, a photon, a wave of probability,
cannot be visualised; they are simply symbols useful
in expressing the mathematical relationships of the
microcosm...because the equations of quantum physics
define more accurately than any mechanical model the
fundamental phenomena beyond the range of vision.
In short; they work, as the calculations which hatched
the atomic bomb spectacularly proved.\(^\text{18}\)

In mathematics analytical definition exists more for the purpose
of the analysis, classification and improvement of concepts, or the
establishment or extension of a mathematical system, than for the
purpose of teaching someone the referential meaning of a word,
although such meaning may be involved in the process.

III. Definition in Which the Non-existence of the
    Referents is Known or Assumed but for Which
    Analytical Meaning is Still Possible

This is the last type listed in the outline involving analytical
definitions typed according to referents. It is important because it
illustrates the fact that the meaning of an analytical definition is

\(^\text{18}\) L. Barnett, The Universe and Dr. Einstein, pp. 28, 29.
independent of the existence of its referent. Consider, for example the definitions of "utopia" and the various mythical beings such as "satyrs," "mermaids," "unicorns" etc. Clearly in each case the referent is known not to exist but the analytical definition which could be stated does transmit meaning, because it is in terms of words which do have referential meaning. A number of analytical definitions of mathematical and pseudo-mathematical terms will be analyzed to show that such are, possible, and productive of learning, despite the nonexistence of the referents of the term defined. Consider the follow ing:

1. A trioid is an equilateral triangle whose interior angles are unequal.

Clearly there is no such geometrical figure and thus no referent, and yet, the definition has meaning in the sense that the reader understands what is being said and what this figure would consist of if it did exist.

2. A hexod is an equiangular nonequilateral hexagon which can be inscribed in a circle.

The importance of the analytical definition as a means of classification of referents which at times results in new knowledge is illustrated by this definition. The definition has meaning but does it have referents? Clearly, it poses a geometrical problem which calls for a demonstration. If it is proven that there are no referents a proposition has been established as false, but the mean ing set forth by the definition is not destroyed. It happens in this case that referents exist.
3. A pentod is an equiangular nonequilateral pentagon which can be inscribed in a circle.

The same statements apply here as were applied to example #2 above, but in this case referents do not exist. Again, the definition retains its meaning regardless of the nonexistence of referents.

4. A crypto-n-gon is an equiangular nonequilateral polygon which can be inscribed in a circle.

Here again the definition suggests a proposition which upon investigation may yield new information, but regardless of the outcome (the existence or nonexistence of referents), meaning obtains in the definition.

Several valid generalizations should be clear from the preceding examples and discussion:

1. The place of analytical definition in a mathematical system is not affected by the nonexistence of referents.

2. The meaning of such definitions is not dependent upon the existence of referents.

3. The statement of a definition which may appear to have no referents may later be shown to have them -- in fact, the very possibility may cause a conjecture which could lead to the establishment of a proposition.

4. The existence of referents may be dependent upon the system involved, i.e., a definition which may have no referents under the postulates of one system may be found to have them in another system. For example, we may define parallelism, of a line through a point not on a given line in a plane, in terms of nonintersection. Now if we use the postulates of Euclidean geometry we get one referent in a given situation; if the postulates of Lobachevsky, many referents; and if the postulates of Riemann, no referents.
Summary of Part II — Chapter VI

In this part of the chapter, the analytical meaning of a word or term was defined as that meaning, the possession of which is evidenced by an ability to give a verbal description of a word or symbol which describes the composition or generation of the referent, or accurately isolates it within its relevant class. Examples of each were given. Robinson's list of six types of mathematical definition was analyzed and found to contain three types that properly belong to the category of pre-analytic meaning and three which belong to the category of analytical meaning.

It was pointed out that the process of definition comprises at least three functions: (1) identifying referents of a word or symbol, (2) classifying referents in a logical system, and (3) analyzing the referent for its constituent parts. The first function is best served by the methods of concept development implicit in the categories of referential and pre-analytical meaning. The second and third functions are realized on the level of analytical meaning.

It has been shown that, although the existence of referents is not essential to a definition, the way in which a class of referents for a given definition change in number is connected with a change in the definition of a term or symbol whether brought about by historical or pedagogical change. Both types of changing definition have effects upon the teaching of subject matter, since they may involve alterations in anything from a single concept to the logical structure of an entire discipline.
Implications of Part II for Teaching

The distinction made in this part of the chapter between that procedure in which the aim of definition is to develop the referential meaning of a word or symbol, and that in which the aim of definition is the systematization and analysis of concepts is very important in that it helps answer the question of the proper time for introducing definitions in teaching mathematics. The problem is met by tailoring the definition to the age and abilities of the learner. Such a dichotomy as Robinson has suggested (p. 111) fits in very well with the categorization of the ways of acquiring concepts established in this study. By classifying the first purpose of definition listed above as that satisfied by the development of referential meaning through the use of any definitional method, the problem of the premature introduction of explanations, descriptions or analytical definitions is averted. Ordinarily then, analytical definitions should not be introduced until the need is felt for a systematization or analysis of concepts as in the first introduction to a deductive system by way of demonstrative geometry.

Definitions presented in the teaching of mathematics have been shown to undergo, at times, a development in which there is a corresponding change in the set of referents. Such definitions may be of two types: (1) the definition given at one level is consistent with the final definition but is incomplete in the sense that it does not include all the referents to which the later definition applies; and
(2) the definition given at one time must be changed at a higher level in favor of a more accurate definition which includes referents previously excluded. A discussion of the relative merits of these two types follows.

Partial Definition

The early presentation of a partial or incomplete definition is psychologically acceptable and desirable, as Leonard insists:

Suppose that a child should ask his father, "Daddy, what's an ostrich?" His father might reply, "An ostrich is a great big bird."... The child's concern in the example above was probably such as would be satisfied by such an incomplete definition. In fact, if the father were to enter into enough details to make the definition complete, it would threaten to become for the child, with his limited knowledge, too complicated, and thus fail to serve the child's immediate purpose.19

The reader who is a parent has probably had the experience of being asked for an explanation of a word only to see the child's interest waning after the first sentence of explanation. The child's present interest and need has apparently been met by a partial definition.

The early definition of number as integers, whether made explicitly or implicitly, is a partial definition. The initial definition of angle in terms of inclination is partial. The early definition of common fractions, say 1/2 or 2/3, is seen to be only a partial definition when it is compared with their later meaning and definition. Van Engen says:

The child in his early work in elementary school.

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has many experiences with the actions associated with taking $2/3$ of some physical object such as a sheet of paper. Later this same symbol may bring forth a different response — that of taking $2/3$ of a number of objects.  

Later Reversal of Early Definition

At times the practice in developing a mathematical concept involves beginning with a definition which in a sense is partial but is better described as being quite different from the final definition which the student is expected to carry with him. An example of this is the practice of initially teaching that subtraction can only be applied to taking a smaller number from a larger. This must later be reversed when the concept of negative numbers is presented. The definition of the trigonometric functions, first in terms of a right angle and later in terms of the general angle, is another example of such a reversal in definition. When such an early definition is taught, it may or may not be recognized by the teacher as inconsistent with the final form. It is taught as being temporarily sufficient until such time as it is to be corrected or replaced by the accepted definition. There is actually a fine line between this type of changing definition and the partial definition approach previously described. Both involve

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$^{20}$H. VanEngen, Twenty-first Yearbook of the National Council of Teachers of Mathematics, p. 80.
a seeming revision of an earlier definition but the second type involves cases where the change is more radical, involving apparent or actual contradiction.

A prominent example of this is that definition of function which suggests that the concept necessarily involves change. In the past few years there has been notable improvement in the formulation of definitions in general and in the formulation of the definition of function in particular. This change is especially pronounced at the lower secondary level where definitions were formerly not presented in so many words and the meaning of function which was given implied "the change of one variable as produced by a change in a related variable." Although there has been improvement the writer found the following treatment in a second year algebra text bearing a copyright date of 1957:

When the value of one variable depends on the value of another, the first is said to be a function of the second.\textsuperscript{21}

This definition is all right, but it is expanded a few statements later:

A table shows that one variable is a function of another when, for every change in the value of the independent variable, there is a corresponding change in the value of the dependent variable.\textsuperscript{22}

\textsuperscript{21} J. Freilich, S. L. Berman and E. P. Johnson, Algebra For Problem Solving, p. 55.

\textsuperscript{22} Ibid.
How would such a function as \( x = 4 \) be handled, employing the idea of change inherent in the meaning of function as it is here developed? That such ambiguity or inaccuracy does prevail is attested to by the writer’s experience with college freshmen over a period of ten years of teaching. It seems never to fail that the presentation of such functions as \( x = k \) meets with surprise. The reason seems quite apparent — the idea is entirely foreign in the light of previous experience with functions and graphs where change was always present. The problem here could be relieved and the idea of change retained if such functions as the above were treated as functions involving zero change.

Another example of a concept which seems to be learned improperly because of early misdefinition is that of the value of \( \pi \). Many students at the college level come up with 22/7 as the value of \( \pi \) when asked, and seem to have difficulty in accepting the fact that this is but a grade school approximation for \( \pi \). The difficulty is undoubtedly brought about by the early definition (with referential meaning) of \( \pi \) as being equal to 22/7, perhaps not because the teacher explicitly defines it that way but because the proper distinction between the two numbers \( \pi \) and 22/7 is not presented or taught.

These are both unfortunate occasions for later rather drastic revision, caused by the early introduction of a definition which was faulty or misleading. There may be examples which are not undesirable but none seem apparent to this writer. Neither should the impression be left that the examples given imply that the idea of change in
definition or the giving of an approximate value of $\pi$ is wrong

($3 \frac{1}{7}$ is the value of $\pi$ to the nearest unit fraction). It seems

a valid assumption, however, that with minor changes in approach, in-

volving mainly a recognition of the problem by the teacher, these

topics might be so presented that the earlier definition and meaning

could be considered as a partial definition with later modifications

being in the form of additions rather than contradictions.

Van Evera supports this position with the following:

Once we have given these fundamental definitions we

must be true to them in our subsequent teaching. Moreover, if...

we find that our concepts so change that one of our

original definitions is no longer acceptable, then we

must very carefully, with an eye to the student, change

that definition...use this change as a teaching tool to
demonstrate the growth of science and how concepts change

as our total knowledge increases.23

Thus he says that where possible, we should use definitions which

will remain steady through subsequent teaching but that if changes

are necessary we as teachers should make a learning situation out of

it.

CHAPTER VII

SUMMARY IMPLICATIONS AND RECOMMENDATIONS

Summary

The ability of college students of mathematics to recognize referents and properly use a given term despite an inability to give an acceptable definition of it suggests that such learning should be tested by some means other than that of requiring the giving of a definition, unless it is a term without referents.

The fact that educators have suggested that meaning should be developed separately from definition without suggesting any procedure for such development nor even clarifying what "meaning" they refer to provided an impetus for this study. The general problem is that of determining just how meaning and the definition of geometrical terms are related to each other and to the development of mathematical concepts in general.

Having stated, in Chapter II, that meaning and definition are applied to terms whereas concepts are subject to analysis the study was centered around a proposed set of categories of concept development. The five categories were listed as:

1. Conditioning -- Pre-conceptual Knowledge.
3. Interrelationship of Term and Referent -- Referential Meaning.
4. First Level of Verbalization Involving Description of Referent — Pre-analytical Meaning.
5. Second Level of Verbalization Involving Analytical Definition — Analytical Meaning.

A detailed treatment of these five categories comprises the major part of the study. This treatment covers the development of concepts by conditioning in animals and humans, non-verbal concept formation in animals, pre-verbal and verbal concept formation in human beings, and proceeds through the stages of concept development commonly thought of as types of definition to the level of analytical definition. At all levels the emphasis is upon the development of geometrical concepts or the meaning of mathematical words, terms or symbols. The test of meaning employs as its criterion the ability to give operational evidence of the understanding of concepts.

In Chapter III on conditioning, it was found that there is a fine line between such meaning and that of the second level, conceptual knowledge, and that distinguishing between the two categories along morphological lines was not possible inasmuch as both animals and human beings are capable of both, as tests have shown. Use was made of a dichotomy suggested by Katz in which two terms, relative and absolute apprehension, are defined. All concept development at the level of conditioning may be described as relative apprehension while that at the level of conceptual knowledge may be described as absolute apprehension. It was further pointed out that experimental evidence
indicates that meaning which may exist at the level of conditioning is changed to the level of conceptual knowledge by verbalization, a transition possible only in humans.

Conceptual knowledge is possible at two levels, non-verbal and verbal. Because of the great difference between these, as to the degree of understanding that each represents, the lower form, non-verbal, has been classed as conceptual knowledge while the verbal type is considered a part of referential meaning.

Chapter IV treats conceptual knowledge in terms of various experiments on the formation of concepts at the non-verbal level in animals and human beings. A very important experimental procedure was pointed out in connection with concept formation experiments. This had to do with the type and variation in stimuli used in training and testing. It was found that when the training period consisted of special cases and positioning of the geometrical figures the type of learning attained was of a more primitive level (conditioning) than when greater variation took place in the stimuli used in training. When the latter prevailed, conceptual knowledge was attained.

It was further established that the attainment of concepts at the non-verbal level by human beings is remarkably similar to that of animals though human beings attained meaning, in this way, much more quickly. This similarity immediately disappears as soon as the human shows the slightest propensity for taking advantage of verbal clues.
In Chapter V, referential meaning was defined as being that kind of meaning in which there is developed, by either physical or verbal pointing, an appreciation and awareness of a relationship between a given symbol, usually verbal, and an object, form or thing called a referent. Such a level includes a number of methods of developing meaning which have been loosely called definition by others. The acceptance for this study of a definition of "definition" as being the equality of two verbal statements, one "the defining statement" and the other "the defined" thus excludes these methods and types from the kind of meaning acquired through verbalization (pre-analytical and analytical meaning).

Part I of Chapter VI was devoted to the study of what has been called pre-analytical meaning. This category of concept development subsumes examples of that type of meaning developed by verbalization involving description and explanation which is not categorized as an analytical definition. Part II of this chapter treats analytical meaning, the verbalization of which is called analytical definition. The purpose of this type of definition was described as being three fold: (1) identification of referents of a word or symbol, (2) classification of referents for the purpose of developing a logical system, and (3) analysis for determining the constituents of the referents. 

Analytical meaning attained through the means of analytical definition was analyzed from the standpoint of the existence or non-existence of the referent of the term defined. It was pointed out that the analytical definition itself, although imparting meaning, does not guarantee the existence of referents.
Findings and Implications

1. Experimental results show that both animals and human beings are able to differentiate between two stimuli which differ in size, shape, tone or color without being able to conceptualize in the sense of being able to perform any meaningful operation with a single stimulus. This means that a child is able to distinguish the difference between say a triangle and a square long before he can even count the number of sides and realize that the difference is related to the difference between the numbers three and four. It means that he can tell that red and green are not the same color (if he is not color blind) long before he can ever point to a green object when the word "green" is mentioned.

2. Certain experiments related to conditioning and concept formation, notably those on transposition, indicate that the ability to verbalize, even though verbal clues are not given, changes the method of concept development from pre-conceptual to conceptual. This finding has implications favoring the use of the comparative method. It highlights, too, the manner in which analytical meaning differs from referential meaning, namely in the ability on the part of the learner to verbalize.

3. It has been shown in the reports of experiments dealing with the formation of concepts of geometric form that the type of learning demonstrated by animals is dependent upon the type of training to which they are subjected. When the training period includes stimuli repre-
senting special cases only, the learning is that classed as pre-conceptual. When the training period is changed or extended to include more variations in stimuli, the method of concept development is changed to that of conceptual knowledge. From the similarity of the experimental approach (with animal and with human subjects), it seems evident that such a finding has a definite bearing on the methods of developing referential meaning in a teaching situation. Referential meaning, it will be remembered, involves conceptual knowledge at the verbal level where active pedagogy begins.

4. Experiments have shown that in actual concept formation involving geometric forms the negative stimulus and/or background is relatively unimportant, the positive stimulus playing the predominant role. This ties in with negative results reported on specific experiments conducted to test the relative importance of the use of negative instances in developing concepts. This finding coupled with the previous one suggests a much more fruitful method of developing concepts, namely the use of variation in the positive stimulus.

5. The results of comparative experiments with animals and human beings point out the similarity between the ways both acquire concepts. This similarity of results warrants the further use of animal subjects in studying aspects of learning for which animals are adapted.

6. Some general findings on concept formation in children are:

a. Concepts of geometric form may be formed in children at the pre-verbal level at as young an age as six months.

b. Children form such concepts at the pre-verbal level in much the same manner as do animals but much more rapidly.
c. Children tend to promote their development of concepts by verbalization on their own.

The fact that children and animals form concepts without verbalization supports the view that neither analytical definition nor referential meaning have priority in the development of a mathematical concept.

7. Experiments reported on concept formation at the verbal level tend to verify the theory that referential meaning is possible of attainment prior to the attainment of a level of meaning on which a description or a definition of the term or symbol can be stated. This is essentially the theory espoused by many educators, that the giving of a definition should follow the giving of examples (establishing referential meaning).

9. The literature also seems to indicate that the mere pointing out of examples is not sufficient in itself to establish referential meaning, but when coupled with analysis in the form of connotative statements it constitutes a quite satisfactory method. The covert presence of these accompanying connotative statements in the exercise of the "ostensive method" was pointed out.

9. Referential meaning as it is developed by any of its various constituent methods represents a very basic level of meaning without verbalization. It should be observed, however, that the ability to recognize referents may be a by-product of an analytical definition although this is not its primary purpose.

10. The definition of mathematical terms was shown to be constantly changing, both historically and pedagogically. The first — historical change — is somewhat beyond the control of the mathematics
educator but the second is certainly within his province. More important than the change in the definition, however, is the relation of such change to the set of referents of the defined term. The change in the number of referents of a given term as it undergoes change has a profound influence upon the referential meaning accompanying an analytical definition. It was demonstrated that meaning was not dependent upon the existence of a referent, for examples were given in which meaning was conveyed despite the nonexistence of referents.

Some Recommended Principles for Use in the Development of Meaning

Two ideas of special importance have been documented in the course of this study with respect to the development of concepts. The first is that the child and adult develop concepts by various methods. The second is that the meaning of a word, term, or symbol itself undergoes a process of change or may be thought of as passing through stages. Such change may be historical or pedagogical. In either case the stages through which the meaning is developed should support the pattern of concept development of the learner. Some principles for use in developing concepts so as to fashion what we might call a "constructive order for concept attainment" are as follows:

1. No concept should be taught which must later be unlearned unless something is gained which cannot be by any other method, (unless there is no other way).

   Example: In early mathematics division by zero is forbidden and ruled out as being meaningless because it involves aspects of the limit concept which are beyond the student. Later
this 'meaningless' idea must be replaced by an idea consistent with the possibility of a quantity's approaching infinity. This is a case of there being no other way than to reverse a meaning given earlier.

2. No concept should be taught by one method if a more accepted method is available, unless there is virtue in both methods.

Example: The meaning of "one-half" is quite often so exclusively taught as meaning one of two equal physical segments that the meaning one-half of a group of objects is neglected. Here neither should be taught to the exclusion of the other.

3. No concept should be taught which is false.

Example: Giving the meaning of \( \pi \) as being 22/7, without stating whenever it is used that it is but an approximation, is teaching a falsehood. It may be true that when \( \pi \) is first used the students may not be able to grasp what an irrational number is but this does not prevent the teaching of the meaning of "approximate value."

4. The meaning of a word, term, or symbol, for which referents are available, should not be established by beginning with the final analytical definition, in the learning of elementary mathematics. Such analytical definition if given should follow the establishment of the referential meaning by one of the methods outlined in Chapters V and VI.

Example: The introduction of a new concept, say the slope of a tangent to a curve, should begin with the establishment of referential meaning, by citing or pointing to examples, rather than giving an analytical definition.

5. The meaning of a word, term, or symbol is not apt to be attained by one method of presentation. Ordinarily, the teacher should begin with the level of referential meaning, as in the example above, but should expect to make use of other methods as appropriate.
All of these principles, if adhered to in the development of concepts, should result in a pedagogy which contributes to the learning of mathematics along natural lines.

Recommendations for Further Study

1. In the light of findings on concept formation in animals and humans and on the use of negative instances with humans, it is recommended that use be made of the method of variation in teaching a new concept in mathematics rather than the exclusive use of special cases or such cases coupled with negative instances. However, due to the somewhat inconclusive results obtained in the experiments on the use of negative instances in developing concepts with children at the verbal level, it is recommended that additional study be made of the use of negative instances coupled with the method of variation. Besides coupling this technique with the method of variation, special emphasis should be laid on testing for the retention of such meaning with and without the use of negative instances for various lengths of time.

2. It has been the expressed purpose of this study to set forth a theory as to the method of development of mathematical concepts and the meaning of mathematical terms. The list of categories has, in the main, been based upon accepted experimental findings. There is one method for the development of pre-analytical meaning, however, which deserves further study. It is the denotative method with
description. That description must accompany denotation in order to provide an efficient method of developing referential meaning. It seems obvious but it should be interesting and profitable to study the development of referential meaning by denotation with and without analysis. Great care should be taken to insure that an analysis is not given inadvertently in the "without-analysis part" for it was shown that in any process involving the ostensive method, analyzing statements may unconsciously be given.
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AUTObIOGRAPHY

I, Alvern Walter Kaufmann, was born in Cleveland, Ohio, February 21, 1924. I received my secondary school education in the public schools of Maple Heights, Ohio, and graduated from Greenville College, Greenville, Illinois, in 1947 with a Bachelor of Arts degree. While serving in the United States Army Air Force (1943-46), I received undergraduate training at Vanderbilt and Harvard Universities which was applied toward my degree from Greenville College. In 1948, I received the Master of Arts degree from the Ohio State University. From 1948 to 1950 I taught mathematics at Aurora College, Aurora, Illinois. I then taught at Central College, McPherson, Kansas, for two years before returning to Columbus, Ohio, in 1952. While in Columbus, I taught in the Columbus public schools for two years (1952-54) and in the Department of Mathematics of The Ohio State University for three years (1954-57). After completing residence requirements in 1957, I went to Roberts Wesleyan College, North Chili, New York, where I taught while completing the dissertation for the degree Doctor of Philosophy.