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A TRANSFORMATION THEORY
FOR MULTIPLICITY FUNCTIONS

DISSERTATION
Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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The Ohio State University
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Let $D$ be a bounded domain in Euclidean $n$-space $\mathbb{R}^n$, and let $T$ be a bounded continuous transformation from $D$ into $\mathbb{R}^n$. A non-negative multiplicity function $M = M(x, T, D)$ is a function defined for each $x$ in $\mathbb{R}^n$ and each domain $D$ contained in $D$. The values of $M$ are non-negative integers or $+\infty$. Suppose in addition that $M$ satisfies the following conditions:

(i) If $(x, D)$ is any pair such that $x$ is in $\mathbb{R}^n$, $D$ is a domain in $D$, and $x$ is not in $TD$, then $M(x, T, D) = 0$.

(ii) $M$ is a subadditive function of domains in $D$. Explicitly, if $\{D_j\}$ is a sequence of pairwise disjoint domains in a domain $D$ in $D$, then for all $x$ in $\mathbb{R}^n$

$$\sum_j M(x, T, D_j) \leq M(x, T, D).$$

(iii) Let $\{D_j\}$ be a sequence of domains in a domain $D$ in $D$. If the closure of each domain $D_j$ is contained in $D_{j+1}$, all $j$, and if for every compact set $F$ contained in $D$ there exists an integer $j$, depending on $F$, such that $F$ is contained in $D_j$, then for all $x$ in $\mathbb{R}^n$

$$\lim_j M(x, T, D_j) = M(x, T, D).$$

(iv) For each domain $D$ in $D$, $M(x, T, D)$ is a Lebesgue measurable function of $x$ in $\mathbb{R}^n$.

(v) The function $b(u, M)$ (termed a base function) defined
by

\[ b(u, M) = \inf_{u \in D \in \mathcal{D}} m(Tu, T, D) \]

is a Borel measurable function of \( u \) in \( \mathcal{D} \).

In IV.3 of [2] (the number in [ ] refers to the bibliography) Rado and Reichelderfer refer to such a multiplicity function \( M \) as a non-negative admissible multiplicity function, and they show that if the Lebesgue integral \( \int_{\mathbb{R}^n} m(x, T, D) \, dL \) of \( m(x, T, D) \) over \( \mathbb{R}^n \) is finite, then \( \int_{\mathbb{R}^n} m(x, T, D) \, dL, D \in \mathcal{D} \), is a finite subadditive function of domains \( D \in \mathcal{D} \), and its derivative \( D(u, M) \) exists a.e. in \( \mathcal{D} \), and it is Lebesgue measurable and summable. One of the main theorems in IV.3 of [2] regarding a non-negative admissible multiplicity function \( M \) states that:

If \( H(x) \) is a real valued, Lebesgue measurable function defined on \( \mathbb{R}^n \), if

\[ \int_{\mathcal{D}} D(u, M) \, dL = \int_{\mathbb{R}^n} m(x, T, D) \, dL < +\infty, \]

and if either \( H(Tu) D(u, M) \) is Lebesgue summable over \( \mathcal{D} \) or \( H(x) M(x, T, D) \) is Lebesgue summable over \( \mathbb{R}^n \), then they are both Lebesgue summable and

\[ \int_{\mathcal{D}} H(Tu) D(u, M) \, dL = \int_{\mathbb{R}^n} H(x) M(x, T, D) \, dL, \quad (1) \]

where \( \mathcal{D} \) is any domain in \( \mathcal{D} \).

The transformation formula (1) is then applied by Rado and Reichelderfer to obtain further transformation formulas. In particular, if \( \mathcal{D} \) is a domain whose closure is in \( \mathcal{D} \) and whose frontier is trans-
formed by \( T \) into a set of Lebesgue measure zero, then

\[
\int_D H(Tu) J_e(u, T) dL = \int_{\mathbb{R}^n} H(x) \mu(x, T, D) dL
\]  

(2)

whenever the integral on the left exists and is finite. \( J_e(u, T) \) is the essential generalized Jacobian, and \( \mu(x, T, D) \) is the topological index defined in \([2]\).

Roughly, the connection between formulas (1) and (2) is as follows:

The topological index \( \mu(x, T, D) \) generates (among others) the non-negative multiplicity functions \( K^+(x, T, D) \) and \( K^-(x, T, D) \). If the Lebesgue integrals

\[
\int_{\mathbb{R}^n} K^+(x, T, D) dL \quad \text{and} \quad \int_{\mathbb{R}^n} K^-(x, T, D) dL
\]

are finite, then

\[
\int_{\mathbb{R}^n} K^+(x, T, D) dL \quad \text{and} \quad \int_{\mathbb{R}^n} K^-(x, T, D) dL
\]

are subadditive functions of domains \( D \) in \( D \), and their respective derivatives \( D(u, K^+) \) and \( D(u, K^-) \) exist a.e. in \( D \). It turns out that if \( D \) is a domain whose closure is in \( D \) and whose frontier is transformed by \( T \) into a set of Lebesgue measure zero, then

\[
\mu(x, T, D) = K^+(x, T, D) - K^-(x, T, D) \quad \text{a.e. in } \mathbb{R}^n.
\]

Furthermore

\[
J_e(u, T) = D(u, K^+) - D(u, K^-) \quad \text{a.e. in } D.
\]

Formula (1) is then applied to \( K^+(x, T, D) \) and \( K^-(x, T, D) \) to obtain
formula (2).

Let \( M \) be a non-negative, admissible multiplicity function. A summatory function \( W(x, T, S) \) is defined by

\[
W(x, T, S) = \sum_{u \in T^{-1}(x \cap S)} b(u, M)
\]

where \( S \) is any set in \( D \), and \( x \) is any point in \( \mathbb{R}^n \). If we restrict the sets \( S \) to the class of Borel sets \( B \) in \( D \), then condition (v), which requires that \( b(u, M) \) be Borel measurable, ensures that

\[
\nu(B) = \int_{\mathbb{R}^n} W(x, T, B) dL
\]

is a finite measure on the Borel sets \( B \) in \( D \), whenever

\[
\int_{\mathbb{R}^n} M(x, T, D) dL < +\infty.
\]

A lemma basic to the method used in [2] to obtain formulas (1) and (2) is:

If \( M(x, T, D) \) is a non-negative, admissible multiplicity function such that

\[
\int_{\mathbb{R}^n} M(x, T, D) dL < +\infty,
\]

then

\[
D(u, \nu) = D(u, M) \text{ a.e. in } D,
\]

where \( D(u, \nu) \) is the derivative of the measure \( \nu(B) \).

The original purpose of this work was to generalize the non-negative, admissible multiplicity function \( M \) and the topological index \( \mu \), and then to parallel the development in [2] to obtain formulas corresponding to (1) and (2). As a first step in this direction the condition that the values of \( M \) be non-negative integers or \( +\infty \) was relaxed to the condition that the values of \( M \) be non-negative, extended real numbers. The development with respect to these non-negative, ex-
tended real valued, admissible multiplicity functions \( M \) paralleled
[2] nicely until the lemma yielding formula (3) was reached. Whether
or not formula (3) holds for such multiplicity functions \( M \) remains an
open question. The fact that we were unable to prove this lemma led us
to discover that formulas (1) and (2) could be obtained more simply
without the help of (3), and without introducing the base function
\( b(u, M) \) or the summatory function \( W(x, T, S) \) derived from the base
function. Thus when we define in section II.1 a non-negative, extended
real valued multiplicity function to be admissible, condition (v)
is dropped. In section II.2 we develop a transformation formula for
non-negative, extended real valued, admissible multiplicity functions
\( M \). In the latter part of section II.1 we extend the concept of admissi-
bility to signed multiplicity functions \( N \). A transformation formula is
developed in section II.3 for these multiplicity functions \( N \). In
Chapter III we apply the results of Chapter II to extended real valued
multiplicity functions generated by index functions \( \nu(x, T, D) \),
which may be thought of as generalizations of the topological index
\( \mu(x, T, D) \), and we obtain formulas corresponding to (2).
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CHAPTER I

PRELIMINARIES

I.1. Notation and Terminology

Throughout this work we shall designate Euclidean n-space by \( \mathbb{R}^n \). If \( E \) is a set contained in \( \mathbb{R}^n \) (\( E \subset \mathbb{R}^n \)), then \( \overline{E} \), \( C_E \), and \( \text{fr}E \) will designate the closure, complement, and frontier respectively of \( E \) relative to \( \mathbb{R}^n \). The null set will be indicated by \( \emptyset \). If "Lebesgue" is used as an adjective we shall generally abbreviate it as L. For example: The L-measure of an L-measurable set \( E \) will be indicated by \( L\mu(E) \). If \( f(x) \) has an L-integral over an L-measurable set \( E \), we shall denote that integral by \( \int_{\overline{E}} f(x) \), or, if \( E = \mathbb{R}^n \), simply by \( \int f(x) \). We shall use "a. e." to mean "almost everywhere" with respect to L-measure. The word countable will mean finite or denumerable. We shall also say that the empty set \( \emptyset \) is countable.

In an expression such as \( [1; 2, 25] \) the first number refers to the correspondingly numbered item in the bibliography, and the following numbers refer to pages in that item.

I.1.1. Definition. Let \( \mathbb{R} \) be the set of all real numbers. Let \( \mathbb{R}_1 = \mathbb{R} \cup \{+\infty, -\infty\} \), and let \( \mathbb{R}_2 \) be the set consisting of all non-negative real numbers and \( +\infty \). We shall refer to the elements of \( \mathbb{R}_1 \) (\( \mathbb{R}_2 \)) as extended (non-negative extended) real numbers. We shall say that a function \( f(x) \) is real valued, extended real valued, or non-negative extended real valued according as its range is contained
1.1.2. Convention. If $a$ is any real number then $a \leq +\infty$ and $a > -\infty$.

1.1.3. Convention. When extended real valued functions defined on a subset of $\mathbb{R}^n$ are subjected to the operations of addition, subtraction, or multiplication, we shall follow the rules:

- $a + (\pm \infty) = (\pm \infty) + a = \pm \infty$, where $a$ is any real number;
- $(+\infty) + (-\infty) = (-\infty) + (+\infty) = (\pm \infty)$
- $-(\pm \infty) = 0$;
- $a \cdot (\pm \infty) = (\pm \infty) \cdot a = \pm \infty$ or $a \cdot (\pm \infty) = (\mp \infty) \cdot a = \mp \infty$ according as $a > 0$ or $a < 0$;
- $0 \cdot (\pm \infty) = (\pm \infty) \cdot 0 = 0$;
- $-(+\infty) = -\infty$, and $-(-\infty) = +\infty$.

In the sequel this convention will be applied only at points $x$ in a set of $L$-measure zero. For example, if $f_1 = f_1(x)$ and $f_2 = f_2(x)$ are extended real valued functions defined on a set $E \subset \mathbb{R}^n$, and if we put

$$g = f_1 \pm f_2, \quad \text{or} \quad g = f_1 \cdot f_2,$$

then it will be clear from the context, or it will be proved, that both $f_1$ and $f_2$ are finite a.e. in $E$.

1.1.4. Definition. Let $f = f(x)$ be an extended, real valued function defined on a set $E \subset \mathbb{R}^n$. Put

$$f^0(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) < 0 \end{cases}, \quad x \in E,$$

and

$$f_{\circ}(x) = \begin{cases} 0 & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}, \quad x \in E.$$
$f^o$ and $f_0$ are said to be the positive and negative parts respectively of $f$. Clearly $f^o - f_0 = f$.

**I.1.5. Remark.** If the extended real valued function $f$ is defined on an $L$-measurable set $E \subseteq \mathbb{R}^n$, then clearly $f$ is $L$-measurable in $E$ if and only if $f^o$ and $f_0$ are $L$-measurable in $E$.

**I.1.6. Definition.** Let $f = f(x)$ be a non-negative, extended real valued, $L$-measurable function defined a.e. on an $L$-measurable set $E \subseteq \mathbb{R}^n$. We shall follow Saks \[3\; 19, 20\] in saying that the $L$-integral of $f$ over $E$, $\int_E f(x)$, is the supremum (possibly $+\infty$) of the sums

$$\sum_{k=1}^m v_k \mathcal{L}E_k,$$

where $\left\{ E_k \right\}_{k=1}^m$ is an arbitrary finite sequence of pairwise disjoint $L$-measurable sets in $\mathbb{R}^n$ such that $E = \bigcup_k E_k$, and where $v_k$ is 0 if $f$ is not defined at any point in $E_k$, and $v_k$ is the infimum of $f$ on $E_k$ otherwise, $k = 1, \ldots, m$. If $f$ is an arbitrary, extended real valued, $L$-measurable function defined a.e. on an $L$-measurable set $E \subseteq \mathbb{R}^n$, we shall say that $f$ possesses an $L$-integral if the $L$-integral of either $f^o$ or $f_0$ is finite, and if this condition is satisfied, then the $L$-integral of $f$ over $E$ is given by

$$\int_E f(x) = \int_E f^o(x) - \int_E f_0(x).$$

In case $\int_E f(x)$ is finite we shall say that $f$ is $L$-summable over $E$. 
I.2. Results from the Theory of Measure and Integration

For ease of reference we list as lemmas the following results.

**I.2.1. Lemma.** Every linear combination of L-measurable functions with constant coefficients is an L-measurable function \([3; 15]\).

**I.2.2. Lemma.** Let \(f(x)\) be a real valued, L-measurable function defined on \(\mathbb{R}^n\). Then there exists a real valued, Borel measurable function \(g(x)\) defined on \(\mathbb{R}^n\) such that \(g(x) = f(x)\) a. e. on \(\mathbb{R}^n\) \([3; 75]\).

**I.2.3. Lemma.** (Lusin's Theorem). If \(f = f(x)\) is an extended real valued, L-measurable function defined a. e. and finite a. e. on an L-measurable set \(E \subseteq \mathbb{R}^n\), then for every real number \(h > 0\) there exists a closed set \(F\), depending on \(h\), such that \(F \subseteq E, L(E - F) < h\), and \(f\) is continuous on \(F\) relative to \(F \subseteq \mathbb{R}^n\) \([3; 72]\).

**I.2.4. Lemma.** Every linear combination with constant coefficients \(af + bg\) of two L-summable functions defined a. e. on an L-measurable set \(E \subseteq \mathbb{R}^n\) is L-summable, and

\[
\int_E (af + bg)(x) = a \int_E f(x) + b \int_E g(x)
\]

\([3; 24]\).

**I.2.5. Lemma.** Let \(f = f(x)\) be an extended real valued, L-measurable function defined a. e. on an L-measurable set \(E \subseteq \mathbb{R}^n\).

(i) \(f\) is L-summable over \(E\) if and only if the absolute value \(|f|\) of \(f\) is L-summable over \(E\).

(ii) If there exists a non-negative L-summable function \(g = g(x)\) defined a. e. on \(E\) such that \(|f| \leq g\) a. e. on \(E\), then \(f\) is L-summable over \(E\).

(iii) Hence \(f\) is L-summable over \(E\) if and only
If \( f^0 \) and \( f_0 \) are L-summable over \( E \) [3; 25].

1.2.6. Lemma. If an extended real valued, L-measurable function \( f(x) \) defined a. e. on an L-measurable set \( E \subseteq \mathbb{R}^n \) has an integral over \( E \) different from \(+\infty\), the set of points \( x \) in \( E \) at which \( f(x) = +\infty \) has L-measure zero [3; 23]. If in addition \( f(x) \geq 0 \) and

\[
\int_E f(x) = 0 ,
\]

then \( f(x) = 0 \) a. e. in \( E \) [1; 183].

1.2.7. Lemma. Let \( f(x) \) be an extended real valued, L-measurable function defined a. e. on an L-measurable set \( E \subseteq \mathbb{R}^n \). If \( f \) is L-summable over \( E \), then for every real number \( h > 0 \) there exists a real number \( k > 0 \) such that

\[
\int_S |f(x)| < h
\]

whenever \( S \) is an L-measurable set such that \( S \subseteq E \) and \( LS < k \) [1; 191].

1.2.8. Lemma. Let \( f(x) = \sum_j f_j(x) \) be a series of non-negative extended real valued, L-measurable functions defined a. e. on an L-measurable set \( E \subseteq \mathbb{R}^n \). Then

\[
\int_E f(x) = \sum_j \int_E f_j(x)
\]

[3; 27].

1.2.9. Lemma. Let \( \{E_j\} \) be a sequence of pairwise disjoint, L-measurable sets in \( \mathbb{R}^n \), and let \( f(x) \) be an extended real valued, L-measurable function defined a. e. on \( E = \bigcup_j E_j \). If \( f(x) \) is L-summable over \( E \), then
\[
\int_E f(x) = \sum_j \int_{E_j} f(x)
\]

[3; 28].

1.2.10. Lemma. (Lebesgue's Theorem on integration of monotone sequences of functions). If \( \{ f_j(x) \} \) is a non-decreasing sequence of non-negative extended real valued, L-measurable functions defined a.e. on an L-measurable set \( E \subset \mathbb{R}^n \), and if

\[
f(x) = \lim_j f_j(x) \text{ a.e. on } E,
\]

then

\[
\int_E f(x) = \lim_j \int_{E_j} f_j(x)
\]

[3; 28].

1.2.11. Lemma. (Lebesgue's Theorem on term by term integration). Let \( f(x) \) and \( f_j(x), j = 1, 2, \ldots \), be extended real valued, L-measurable functions defined a.e. on an L-measurable set \( E \subset \mathbb{R}^n \), such that \( \lim_j f_j(x) = f(x) \) a.e. on \( E \). If there exists a non-negative extended real valued, L-summable function \( g(x) \) defined a.e. on \( E \) such that

\[
|f_j(x)| \leq g(x) \text{ a.e. on } E, \ j = 1, 2, \ldots,
\]

then \( f(x) \) and \( f_j(x), j = 1, 2, \ldots \), are L-summable over \( E \), and

\[
\int_E f(x) = \lim_j \int_{E_j} f_j(x)
\]

[3; 29].

1.2.12. Lemma. Let \( f_1 = f_1(x) \) and \( f_2 = f_2(x) \) be non-negative extended real valued, L-measurable functions defined a.e. on an L-measurable set \( E \subset \mathbb{R}^n \), and let \( g = g(x) \) be an extended real valued, L-measurable
function defined a.e. on E. If \((f_1 + f_2)g\) is L-summable over E, so are \(f_1g\) and \(f_2g\).

Proof. Since

\[ |f_1 g| = |f_1| |g| \leq (f_1 + f_2) |g| = |(f_1 + f_2) g|, \quad i = 1, 2, \]

I.2.12 follows from I.2.5.

The proof of the following lemma is equally trivial.

I.2.13. Lemma. Let \(f = f(x)\) and \(g = g(x)\) be extended real valued, L-measurable functions defined a.e. on an L-measurable set \(E \subseteq \mathbb{R}^n\). If \(fg\) is L-summable over E so are \(f^0g\) and \(f^0g\).


The lemmas in this section are either well known results, or they follow readily from the definitions involved.

I.3.1. Definition. An open oriented n-cube \(Q\) in \(\mathbb{R}^n\) consists of all points \(x\) whose coordinates \(x^i\) satisfy a system of inequalities of the form

\[ a_1 < x^i < a_1 + a_0, \quad 1 = 1, 2, \ldots, n, \quad (1) \]

where the \(a_i, \ i = 1, 2, \ldots, n,\) are arbitrary real numbers, and \(a_0\) is an arbitrary positive real number. If the \(a_i, \ i = 0, 1, 2, \ldots, n,\) are rational numbers, the open oriented n-cube is said to have rational vertices. A closed oriented n-cube \(Q'\) is obtained by replacing the inequality signs \(<\) in (1) by \(\leq\), and \(Q' = \overline{Q}\). Unless otherwise indicated, we shall be concerned only with open oriented n-cubes, and the term n-cube alone will always mean open oriented n-cube.
1.3.2. Definition. We shall designate by $D^*$ the set of all domains $D$ such that $D$ is the interior of the union of the closures of a finite number of $n$-cubes with rational vertices.

1.3.3. Lemma. $D^*$ is countable.

1.3.4. Definition. Let $m$ be a positive integer. The open grid $\Delta_m$ on $\mathbb{R}^n$ consists of all (open oriented) $n$-cubes $Q_m$ determined by a system of inequalities of the form

$$\frac{k_i}{2^m} < x^i < \frac{k_i + 1}{2^m}, \quad i = 1, 2, \ldots, n,$$

where the $k_i$ are arbitrary integers. Since we shall be concerned only with open grids, we shall generally use the term grid in place of open grid.

1.3.5. Lemma. Let $Q_m^1$ and $Q_m^2$ be elements of a grid $\Delta_m$ for any positive integer. Then either $Q_m^1 = Q_m^2$ or $Q_m^1 \cap Q_m^2 = \emptyset$.

1.3.6. Lemma. If $m_1$, $m_2$, and $m_3$ are positive integers such that $m_1 < m_2 < m_3$, and if $Q_m^3$ is in $\Delta_{m_3}$, then

(i) there exists an $n$-cube $Q_m^1$ in $\Delta_{m_1}$ such that

$$Q_m^1 \subseteq Q_m^3.$$

(ii) If $Q_m^1$ is any $n$-cube in $\Delta_{m_1}$, then either

$$Q_m^2 \subseteq Q_m^1$$

or

$$Q_m^2 \cap Q_m^1 = \emptyset.$$

(iii) $Q_m^3$ is the union of the closures of (a finite number of) $n$-cubes $Q_m^1$ in $\Delta_{m_3}$.

In view of this lemma we shall say that the sequence $\{\Delta_m\}_{m=1}^{\infty}$ of grids is nested.
1.3.7. Definition. Let $D$ be a bounded domain in $\mathbb{R}^n$. A sequence $\{D_j\}$ of domains is said to fill up $D$ from the interior provided
\[ D_j \subset D_{j+1} \subset D, \quad \text{all } j, \]
and for every closed set $F$ contained in $D$, there exists an integer $j$, depending on $F$, such that $F \subset D_j$.

1.3.8. Lemma. Let $D$ be a bounded domain in $\mathbb{R}^n$. Then there exists a sequence $\{D_j\}$ of domains such that
(i) $D_j$ is the interior of the union of the closures of a finite number of $n$-cubes of the grid $\Delta_{m_j}$, where
\[ m_1 < m_2 < \cdots < m_j < \cdots. \]
(ii) The sequence $\{D_j\}$ fills up $D$ from the interior.

1.3.9. Remark. Observe that for each $D_j$ in the preceding lemma, we have $D_j$ in $\mathcal{G}^*$ and $\text{Lfr}D_j = 0$.

1.3.10. Definition. Let $G$ be an open set in $\mathbb{R}^n$, and let $w$ be a real valued set function defined at least on all $n$-cubes in $G$. Let $u$ be a point in $G$. The upper derivative $\overline{D}(u, w)$ of $w$ at $u$ is the supremum of all extended real numbers $\mathcal{L}$ for which there exists a sequence $\{Q_j\}$ of $n$-cubes in $G$ such that
\[ u \text{ is in } Q_j, \text{ all } j; \lim_j \frac{w(Q_j)}{LQ_j} = 0; \lim_j \frac{w(Q_j)}{LQ_j} = \mathcal{L}. \]
The lower derivative $\underline{D}(u, w)$ of $w$ at $u$ is the infimum of these same numbers $\mathcal{L}$. If $\overline{D}(u, w)$ and $\underline{D}(u, w)$ are finite and equal at a point $u$ in $G$, then their common value $\overline{D}(u, w)$ is termed the derivative of $w$ at $u$. 
Lemma 1.3.11. If $D(u, w)$ exists at a point $u$ in the open set $G$, and if \( \{ Q_j \} \) is any sequence of n-cubes in $G$ such that

\[ u \text{ is in } Q_j, \text{ all } j; \text{ and } \lim_j LQ_j = 0, \]

then (the finite)

\[ \lim_j \frac{w(Q_j)}{LQ_j} \]

exists and is equal to $D(u, w)$.

Lemma 1.3.12. Let $v$ and $w$ be real valued set functions which are defined at least on all n-cubes in an open set $G$. If both $D(u, v)$ and $D(u, w)$ exist at a point $u$ in $G$, and if $a$ and $b$ are real numbers, then $D(u, av + bw)$ exists at $u$ and

\[ aD(u, v) + bD(u, w) = D(u, av + bw). \]
CHAPTER II

MULTIPLICITY FUNCTIONS

II.1. Admissible Multiplicity Functions

Throughout Chapter II unless otherwise indicated D will be a fixed bounded domain in $\mathbb{R}^n$. We shall designate by $\mathcal{D}$ the set of all domains $D \subset D$, and we shall designate by $\mathcal{T}$ the set of all bounded, continuous transformations $T$ from $D$ into $\mathbb{R}^n$.

II.1.1. Definition. A multiplicity function $N = N(x, T, D)$ is a function from the Cartesian product $\mathbb{R}^n \times \mathcal{T} \times \mathcal{D}$ into the extended real numbers.

Throughout Chapter II, T will be thought of as fixed in $\mathcal{T}$, and if $N = N(x, T, D)$ is a multiplicity function, we shall denote $N(x, T, D)$ by $N(x, D)$.

If $N = N(x, D)$ is a non-negative extended real valued multiplicity function we shall say simply that $N$ is a non-negative multiplicity function.

II.1.2. Definition. Let $N$ be a multiplicity function, and let $x$ be a point in $\mathbb{R}^n$. If for each $D$ in $\mathcal{D}$ and every sequence $\{D_j\}$ of domains which fill up $D$ from the interior (see I.3.7), it follows that

$$\lim_{j} N(x, D_j) = N(x, D)$$

then $N$ is said to have the filling up property at $x$. If $N$ has the filling up property at each point $x$ in a set $E \subset \mathbb{R}^n$, then $N$ is said to have the filling up property on $E$. 

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II.1.3. Definition. Let $M$ be a non-negative multiplicity function and let $x$ be a point in $\mathbb{R}^n$. If for each domain $D$ in $\mathcal{D}$ and every finite collection \( \{ D_i \} \) of pairwise disjoint domains in $D$ it follows that

\[
\sum_{i} M(x, D_i) \leq M(x, D),
\]

then $M$ is said to be \textit{subadditive} at $x$. If $M$ is subadditive at each point $x$ in a set $E \subseteq \mathbb{R}^n$, then $M$ is said to be subadditive on $E$.

Clearly if $M$ is subadditive at a point $x$ in $\mathbb{R}^n$ then

\[
\sum_{j} M(x, D_j) \leq M(x, D)
\]

for every countable sequence \( \{ D_j \} \) of pairwise disjoint domains in $D$.

II.1.4. Definition. Let $M$ be a non-negative multiplicity function. If there exists a set $X \subseteq \mathbb{R}^n$ such that $LX = 0$ and

(i) $x$ in $CTD \cap CX$ implies $M(x, D) = 0$, all $D$ in $\mathcal{D}$,

(ii) $M$ is subadditive on $CX$,

(iii) $M$ has the filling up property on $CX$,

(iv) $x$ in $CX$ implies $M(x, D)$ is finite, all $D$ in $\mathcal{D}$,

and if

(v) for each $D$ in $\mathcal{D}$, $M = M(x, D)$ is an $L$-measurable function of $x$ in $\mathbb{R}^n$,

then $M$ is said to be \textit{admissible in the restricted sense}, and $X$ is said to be an \textit{exceptional set} for $M$.

We shall designate by $\mathcal{M}$ the set of all non-negative multiplicity functions which are admissible in the restricted sense.
II.1.5. Remarks. (i) Clearly if $X$ is an exceptional set for $M$ in $\mathcal{M}$, then any set $Y \subset \mathbb{R}^n$ such that $Y \not\supset X$ and $LY = 0$ is also an exceptional set for $M$. It follows that if $M_1, M_2, \ldots$ is a countable sequence of elements in $\mathcal{M}$, then there exists a set $Y \subset \mathbb{R}^n$ such that $Y$ is an exceptional set for $M_i$, $i = 1, 2, \ldots$. Indeed if $X_i$ is an exceptional set for $M_i$, then $Y = \bigcup_i X_i$ is an exceptional set for $M_i$, $i = 1, 2, \ldots$. If $Y$ is an exceptional set for $M_i$, $i = 1, 2, \ldots$, we shall say that $Y$ is an exceptional set for the sequence $(M_1, M_2, \ldots)$.

(ii) If $M$ is admissible in the restricted sense and if $X$ is an exceptional set for $M$, it follows from condition (ii) in II.1.4 that for fixed $x$ in $CX$ $M$ is a monotone function of domains in $\mathcal{G}$. Explicitly, if $x$ is in $CX$ and if $D_1 \subset D$ in $\mathcal{G}$, then $M(x, D_1) \leq M(x, D)$. Hence condition (iv) of II.1.4 could be replaced by the apparently weaker condition

(iv') $x$ in $CX$ implies $M(x, D)$ is finite.

II.1.6. Definition. We shall find it convenient to introduce a notation for some special multiplicity functions in $\mathcal{M}$. Let $X$ be a set in $\mathbb{R}^n$ such that $LX = 0$. We define a multiplicity function $M$ as follows: $M(x, D) = 0$ if $x$ is in $CX$ and $D$ is in $\mathcal{G}$. If $x$ is in $X$ and $D$ is in $\mathcal{G}$, $M(x, D)$ may be any non-negative real number or $+\infty$. It is clear that $M$ is in $\mathcal{M}$ and that $X$ is an exceptional set for $M$. Since for all $D$ in $\mathcal{G}$, $M(x, D) = 0$ a.e. in $\mathbb{R}^n$, we shall designate such a multiplicity function $M$ by 0.

II.1.7. Lemma. Let $M_1$ and $M_2$ be in $\mathcal{M}$, and let $a$ and $b$ be real numbers. If (see I.1.3)
\[ N = a M_1 + b M_2 \]

then

for each \( D \) in \( \mathcal{O} \), \( N(x, D) \) is an \( L \)-measurable function of \( x \) in \( \mathbb{R}^n \). \hspace{1cm} (1)

Furthermore, if \( Y \) is an exceptional set for \( (M_1, M_2) \), then

\[ x \text{ in } CTD \cap CY \text{ implies } N(x, D) = 0, \text{ all } D \text{ in } \mathcal{O}, \hspace{1cm} (2) \]

and

\[ x \text{ in } CY \text{ implies } N(x, D) \text{ is finite, all } D \text{ in } \mathcal{O}. \hspace{1cm} (3) \]

**Proof.** (1) follows from I.2.1 and (v) in II.1.4. (2) and (3) follow respectively from (i) and (iv) in II.1.4.

**II.1.8. Lemma.** Let \( M_1 \) and \( M_2 \) be in \( \mathcal{M} \), and let \( Y \) be an exceptional set for \( (M_1, M_2) \). Let \( a \) and \( b \) be non-negative real numbers. Then

\[ M = a M_1 + b M_2 \]

is subadditive on \( CY \).

**Proof.** Fix \( D \) in \( \mathcal{O} \) and let \( \{ D_k \} \) be any finite collection of pairwise disjoint domains in \( D \). It suffices to prove

\[ \sum_{i} (a M_1(x, D_k) + b M_2(x, D_k)) \leq a M_1(x, D) + b M_2(x, D), \hspace{1cm} x \text{ in } CY. \hspace{1cm} (1) \]

Since \( M_1 \) and \( M_2 \) are in \( \mathcal{M} \) and \( Y \) is an exceptional set for \( (M_1, M_2) \), it follows from II.1.4 that both \( M_1 \) and \( M_2 \) are subadditive on \( CY \).

Hence

\[ \sum_{i} M_k(x, D_k) \leq M_k(x, D), \hspace{1cm} k = 1, 2; \hspace{1cm} x \text{ in } CY. \hspace{1cm} (2) \]

(1) follows readily from (2).
Lemma. Let $M_1$ and $M_2$ be in $\mathcal{M}$, and let $Y$ be an exceptional set for $(M_1, M_2)$. Let $a$ and $b$ be real numbers. Then

$$N = aM_1 + bM_2$$

has the filling up property on $CY$.

Proof. Fix $D$ in $\mathcal{J}$, and let $\{D_j\}$ be a sequence of domains filling up $D$ from the interior. Since $M_1$ and $M_2$ are in $\mathcal{M}$ and $Y$ is an exceptional set for $(M_1, M_2)$, it follows from II.1.4 that

$$M_k(x, D) \leq +\infty, k = 1, 2; x \in CY, \quad (1)$$

and

$$\lim_j M_k(x, D_j) = M_k(x, D), k = 1, 2; x \in CY. \quad (2)$$

With the help of (1) and (2) we obtain

$$\lim_j (aM_1(x, D_j) + bM_2(x, D_j)) = aM_1(x, D) + bM_2(x, D),$$

or

$$\lim_j N(x, D_j) = N(x, D), \quad x \in CY. \quad (3)$$

Since $D$ was arbitrary in $\mathcal{J}$, and $\{D_j\}$ was an arbitrary sequence of domains filling up $D$ from the interior, the lemma follows from (3).

Lemmas II.1.7, II.1.8, and II.1.9 yield

Lemma. $\mathcal{M}$ is positively linear. Explicitly, if $M_1$ and $M_2$ are in $\mathcal{M}$, and if $a$ and $b$ are non-negative real numbers, then

$$M = aM_1 + bM_2$$

is in $\mathcal{M}$.

We wish now to extend $\mathcal{M}$ to a linear class. That is to say, we
wish to define a class \( \mathcal{M} \) of multiplicity functions such that
\[ m \in \mathcal{M}, \text{ and if } m_1 \text{ and } m_2 \text{ are in } \mathcal{M}, \text{ then } N = am_1 + bm_2 \text{ is in } \mathcal{M} \text{ for all real numbers } a \text{ and } b. \]

**II.1.11. Definition.** Let \( N, m_1, \) and \( m_2 \) be multiplicity functions and let \( Y \) be a set in \( \mathbb{R}^n \). We shall say that \((m_1, m_2; Y)\) is a representation of \( N \) if

(i) \( m_1 \) and \( m_2 \) are in \( \mathcal{M} \).

(ii) \( Y \) is an exceptional set for \((m_1, m_2)\).

(iii) For all \( D \) in \( \mathcal{D} \) and all \( x \) in \( CY \), \( N(x, D) = m_1(x, D) - m_2(x, D) \).

Observe that \( N(x, D) \) is finite for all \( D \) in \( \mathcal{D} \) whenever \( x \) is in \( CY \).

**II.1.12. Remark.** Clearly if \((m_1, m_2; Y)\) is a representation of \( N \), then \((m_2, m_1; Y)\) is a representation of \(-N\).

**II.1.13. Definition.** Let \( N \) be a multiplicity function. Then \( N \) is said to be admissible if there exists a representation of \( N \). We shall designate by \( \mathcal{N} \) the set of all admissible multiplicity functions.

**II.1.14. Lemma.** \( m \in \mathcal{N} \).

**Proof.** Let \( M \) be in \( \mathcal{M} \), and let \( X \) be an exceptional set for \( M \). Then \( X \) is an exceptional set for a multiplicity function \( 0 \) in \( \mathcal{M} \) (see II.1.6). It follows that \((M, 0; X)\) is a representation of \( M \) so that \( M \) is in \( \mathcal{N} \). Hence \( m \in \mathcal{N} \).

**II.1.15. Lemma.** Let \( m_1 \) and \( m_2 \) be in \( \mathcal{N} \). Then \( N = m_1 + m_2 \) is in \( \mathcal{N} \).

**Proof.** Let \((m_1, m_2; X_1)\) and \((m_3, m_4; X_2)\) be representations of \( m_1 \) and \( m_2 \) respectively. Then for all \( D \) in \( \mathcal{D} \),
\[ N_1(x, D) = M_1(x, D) - M_2(x, D), \text{ } x \text{ in } C Y_1, \]

and

\[ N_2(x, D) = M_3(x, D) - M_4(x, D), \text{ } x \text{ in } C Y_2, \]

so that for all \( D \) in \( \mathcal{D} \)

\[ N_1(x, D) + N_2(x, D) = M_1(x, D) + M_3(x, D) - M_2(x, D) - M_4(x, D), \]

\[ x \text{ in } C(Y_1 \cup Y_2) \]  

(1)

But \( LY_1 = 0 \) and \( LY_2 = 0 \). Hence \( L(Y_1 \cup Y_2) = 0 \). By II.1.10

\( M_1 + M_3 \) and \( M_2 + M_4 \) are in \( \mathcal{M} \). Thus in view of (1)

\( (M_1 + M_3, M_2 + M_4; Y_1 \cup Y_2) \) is a representation of \( N_1 + N_2 \).

Consequently \( N \) in \( \mathcal{N} \).

**II.1.16. Lemma.** Let \( N \) be in \( \mathcal{N} \), and let \( a \) be a non-negative real number. Then \( aN \) is in \( \mathcal{N} \).

**Proof.** Let \( (M_1, M_2; Y) \) be a representation of \( N \). Then for all \( D \) in \( \mathcal{D} \)

\[ N(x, D) = M_1(x, D) - M_2(x, D), \text{ } x \text{ in } C Y. \]

Hence for all \( D \) in \( \mathcal{D} \)

\[ aN(x, D) = aM_1(x, D) - aM_2(x, D), \text{ } x \text{ in } C Y. \]  

(1)

But by II.1.10 \( aM_1 \) and \( aM_2 \) are in \( \mathcal{M} \). Consequently it follows

from (1) that \( (aM_1, aM_2; Y) \) is a representation of \( aN \) and so \( aN \) is in \( \mathcal{N} \).

By remark II.1.12 \( N \) in \( \mathcal{N} \) implies \( -N \) in \( \mathcal{N} \). Hence we obtain

from lemmas II.1.15 and II.1.16

**II.1.17. Lemma.** \( \mathcal{N} \) is linear.

We are unable to exhibit a procedure for finding a representation of an arbitrary element in \( \mathcal{N} \). However we devote the remainder of this section to an examination of a procedure which is sometimes helpful.
II.1.18. Definition. For each $D$ in $\mathcal{D}$ let $s(D)$ be generic for a finite or empty class of pairwise disjoint domains $D'$ in $\mathcal{D}$ such that $\overline{D'} \subset D$. Let $N$ be a multiplicity function. For each $x$ in $\mathbb{R}^n$ and each $D$ in $\mathcal{D}$ let $s^+(x, D, N)$ ($s^-(x, D, N)$) be generic for a finite or empty class of pairwise disjoint domains $D'$ in $\mathcal{D}$ such that

(i) $\overline{D'} \subset D$

(ii) $N(x, D') > 0$ ($N(x, D') < 0$).

In case no confusion can arise we shall designate $s^+(x, D, N)$ by $s^+(x, D)$ and $s^-(x, D, N)$ by $s^-(x, D)$.

II.1.19. Definition. Let $N$ be a multiplicity function. For each $x$ in $\mathbb{R}^n$ and $D$ in $\mathcal{D}$ put

$$N^+(x, D) = \sup_{s(D)} \sum_{D' \text{ in } s(D)} N(x, D')$$

$$N^-(x, D) = \inf_{s(D)} \sum_{D' \text{ in } s(D)} N(x, D')$$

where a summation over an empty set is defined to be zero.

II.1.20. Remark. Clearly $N^+$ and $N^-$ are non-negative multiplicity functions. Furthermore, if $D$ is in $\mathcal{D}$ and $D'$ is a domain contained in $D$, then

$$N^+(x, D') \leq N^+(x, D), \ x \text{ in } \mathbb{R}^n.$$ (1)
**II.1.21. Remark.** It follows readily that

\[ N^*(x, D) = \sup_{D': D' \text{ in } s^*(x, D)} \sum_{D' \text{ in } s^*(x, D)} |N(x, D')|. \]

**II.1.22. Lemma.** If \( N \) is a multiplicity function, then \( N^* \) and \( N^- \) are subadditive on \( \mathbb{R}^n \).

**Proof.** Let \( x \in \mathbb{R}^n \) and \( D \) in \( \mathcal{D} \) be fixed. Let \( \{D_i\}_{i=1}^m \) be a finite class of pairwise disjoint domains in \( D \). To prove that \( N^+ \) is subadditive on \( \mathbb{R}^n \) it suffices to show that

\[ \sum_{i} N^+(x, D_i) \leq N^+(x, D). \] (1)

If \( N^+(x, D) = +\infty \), then (1) is trivial. Assume \( N^+(x, D) \leq +\infty \).

By remark II.1.20 it follows that \( N^+(x, D_i) \leq +\infty \), \( i = 1, \ldots, m \). Let \( h \) be an arbitrary positive real number. Then by II.1.21 there exists for each \( i \) a (possibly empty) class \( s^+(x, D_i) \), depending on \( h \), such that

\[ \sum_{D' \text{ in } s^+(x, D_i)} N(x, D') > N^+(x, D_i) - \frac{h}{m}. \]

Hence

\[ \sum_{D' \text{ in } s^+(x, D_i)} N(x, D') > \sum_{i} N^+(x, D_i) - h. \] (2)

Since the sets \( D_i \) are pairwise disjoint and contained in \( D \), it follows that \( \bigcup_{i} s^+(x, D_i) \) is a class \( s^+(x, D) \) and
\[
\sum_{D_1} \sum_{D' \in s^+(x, D_1)} N(x, D') = \sum_{D' \in s^+(x, D)} N(x, D').
\]  

By II.1.21

\[
N^+(x, D) \geq \sum_{D' \in s^+(x, D)} N(x, D').
\]

(2), (3), and (4) yield

\[
N^+(x, D) > \sum_{i} N^+(x, D_i) - h.
\]

Since \( h \) was an arbitrary positive real number, (1) follows from (5).

The proof with respect to \( N^- \) is similar.

**II.1.23, Lemma.** If \( N \) is a multiplicity function, then \( N^+ \) and \( N^- \) have the filling up property on \( \mathbb{R}^n \).

**Proof.** Since the proofs are similar we shall prove the lemma only for \( N^+ \). Fix \( x \) in \( \mathbb{R}^n \) and \( D \) in \( \mathcal{D} \). Let \( \{D_j\} \) be any sequence of domains filling up \( D \) from the interior. To prove that \( N^+ \) has the filling up property on \( \mathbb{R}^n \) it suffices to show that

\[
\lim_j N^+(x, D_j) = N^+(x, D).
\]

Let \( a \) be any real number such that

\[
a < N^+(x, D).
\]

Then by definition of \( N^+ \) there exists a (possibly empty) class \( s(D) \) such that

\[
\sum_{D' \in s(D)} N(x, D') > a.
\]

Put

\[
F = \bigcup_{D' \in s(D)} D'.
\]
Then $F$ is compact and $F \subseteq D$. Since the domains $D_j$ fill up $D$ from the interior, there exists an integer $j'$, depending on $F$, such that

$$F \subseteq D_{j'}.$$  \hspace{1cm} (5)

With the help of (4) and (5) we see that the class $s(D)$ is a class $s(D_{j'})$ so that

$$N^+(x, D_{j'}) \geq \sum_{D' \text{ in } s(D)} N(x, D')$$  \hspace{1cm} (6)

Since $D \supset D_j$, all $j$, and since $j > j'$ implies $D_j \supset D_{j'}$, it follows from the subadditivity of $N^+$ on $\mathbb{R}^n$ (II.1.22) that

$$N^+(x, D_j) \geq N^+(x, D_{j'}) \text{, } j > j',$$  \hspace{1cm} (7)

and

$$N^+(x, D) \geq N^+(x, D_j) \text{, all } j.$$  \hspace{1cm} (8)

(3), (6), and (7) yield

$$N^+(x, D_j) > a \text{, } j > j'.$$  \hspace{1cm} (9)

Since $N^+$ is subadditive on $\mathbb{R}^n$, the sequence $\{N^+(x, D_j)\}$ is non-descending. Hence it has a (possibly infinite) limit and by (9)

$$\lim_{j} N^+(x, D_j) \geq a.$$  \hspace{1cm} (10)

Since $a$ was an arbitrary real number satisfying (2), (10) yields

$$\lim_{j} N^+(x, D_j) \geq N^+(x, D).$$  \hspace{1cm} (11)

By (8)

$$\lim_{j} N^+(x, D_j) \leq N^+(x, D).$$  \hspace{1cm} (12)

(11) and (12) yield (1).

We recall that $\mathcal{D}^*$ consists of all those domains $D$ such that
D is the interior of a finite union of closed n-cubes with rational vertices (see 1.3.1 and 1.3.2).

II.1.24. Definition. For D in $\mathcal{D}$ let $s^*(D)$ be generic for a finite or empty class of pairwise disjoint domains $D'$ in $\mathcal{D}^*$ such that $\overline{D'} \subseteq D$.

II.1.25. Lemma. If $N$ is a multiplicity function with the filling up property at a point $x$ in $\mathbb{R}^n$, and if $D$ in $\mathcal{D}$, then

$$N^+(x, D) = \sup_{s^*(D), D' \text{ in } s^*(D)} \sum N(x, D')$$

(1)

$$N^-(x, D) = -\inf_{s^*(D), D' \text{ in } s^*(D)} \sum N(x, D')$$

(2)

Proof. Since the proofs of (1) and (2) are similar, we shall give only the proof of (1). Since each class $s^*(D)$ is a class $s(D)$,

$$N^+(x, D) \geq \sup_{s^*(D), D' \text{ in } s^*(D)} \sum N(x, D')$$

It remains to prove that

$$N^+(x, D) \leq \sup_{s^*(D), D' \text{ in } s^*(D)} \sum N(x, D')$$

(3)

If $N^+(x, D) = 0$, then (3) is trivial. Suppose then that $N^+(x, D) > 0$. Let $b$ be any real number such that

$$0 \leq b \leq N^+(x, D)$$

(4)

Then there exists a class $s(D) = \left\{ D_i \right\}_{i=1}^{m}$
such that
\[ \sum_i N(x, D_i) > b. \]  

(5)

Since by our convention \((+\infty) + (-\infty) = (-\infty) + (+\infty) = 0\), it follows readily that we may assume all terms in the summation in (5) to be positive or \(+\infty\). Let \(a_i, i = 1, \ldots, m\) be any real numbers such that
\[ 0 < a_i \leq N(x, D_i). \]

Furthermore, in view of (5), we may require that
\[ \sum_i a_i > b. \]  

(6)

By I.3.8 and I.3.9 there exists for each \(i\) a sequence \(\{D_i^j\}\) of domains in \(\mathcal{O}^*\) which fill up \(D_i\) from the interior. Since \(N\) has the filling up property at \(x\), there exists for each \(i\) an integer \(j_i\) such that
\[ N(x, D_i^{j_i}) > a_i. \]

Hence by (6)
\[ \sum_i N(x, D_i^{j_i}) > b. \]  

(7)

Since the domains \(D_i^{j_i}\) fill up \(D_i\) from the interior it follows that
\[ D_i^{j_i} \subseteq D_i, \]
and since the domains \(D_i\) are pairwise disjoint, the domains \(D_i^{j_i}, i = 1, \ldots, m\) form a class \(s^*(D)\). Consequently
\[ \sup_{D' \text{ in } s^*(D)} \sum N(x, D') \geq \sum_i N(x, D_i^{j_i}). \]  

(8)
Since b was an arbitrary real number satisfying (4), we obtain (3) from (7) and (8).

**II.1.26. Lemma.** Suppose D in $\mathcal{D}$. Then the collection $\mathcal{S}^*(D)$ of all classes $s^*(D)$ is countable.

**Proof.** The collection $\mathcal{D}^*$ is countable (see I.3.3). Hence the collection $\mathcal{S}^*$ of all finite subsets of $\mathcal{D}^*$ is countable. But $\mathcal{S}^*(D) \subseteq \mathcal{S}^*$ so that $\mathcal{S}^*(D)$ is countable.

**II.1.27. Lemma.** If $N = N(x, D)$ is a multiplicity function which has the filling up property on $\mathbb{R}^n$ and which for each $D$ in $\mathcal{D}$ is an $L$-measurable function of $x$ in $\mathbb{R}^n$, then for each $D$ in $\mathcal{D}$ $N^+(x, D)$ and $N^-(x, D)$ are $L$-measurable functions of $x$ in $\mathbb{R}^n$.

**Proof.** Fix $D$ in $\mathcal{D}$. Let $a$ be any real number, and put

$$H = \{x / x \text{ in } \mathbb{R}^n \text{ and } N^+(x, D) > a\}.$$  

To prove that $N^+$ is an $L$-measurable function of $x$ in $\mathbb{R}^n$ it suffices to show that $H$ is $L$-measurable. Let $x_0$ be in $H$. By II.1.25 there exists a class $s^*(D)$, depending on $x_0$, such that

$$\sum_{D' \text{ in } s^*(D)} N(x_0, D') > a. \quad (1)$$

For each $x_0$ in $H$, let $s^*(x_0)$ be a class $s^*(D)$ for which (1) is satisfied. Note that by II.1.26 the collection $\mathcal{S}^*$ of distinct classes $s^*(x_0)$ obtained by varying $x_0$ in $H$ is countable. For each $x_0$ in $H$ put

$$E(x_0) = \{x / x \text{ is in } \mathbb{R}^n \text{ and } \sum_{D' \text{ in } s^*(x_0)} N(x, D') > a\} \quad (2)$$

Observe that $x_0$ in $E(x_0)$. Since for each $D'$ in $s^*(x_0)$, $N(x, D')$ is an $L$-measurable function of $x$ in $\mathbb{R}^n$, it follows from I.2.1 that the finite sum
is an L-measurable function of x in $\mathbb{R}^n$. Hence $E(x_0)$ is L-measurable. Since $\mathcal{F}^*$ is countable, it follows that the collection of distinct sets $E(x_0)$ obtained by varying $x_0$ in $H$ is countable. Hence

$$E = \bigcup_{x_0 \in H} E(x_0)$$  \hspace{1cm} (3)

is L-measurable. We shall show that $H = E$. If $x'$ in $H$, then $x'$ in $E(x')$, and hence $x'$ in $E$. Conversely, if $x'$ in $E$, then by (3), $x'$ is in some set, say, $E(x_0)$. Hence by (2)

$$\sum_{D' \in \mathcal{S}^*(x_0)} N(x', D') > a$$  \hspace{1cm} (4)

Since $\mathcal{S}^*(x_0)$ is a set s(D), it follows from the definition of $N^+(x', D)$ and (4) that $N(x', D) \geq a$. Hence $x'$ in $H$. Thus $H = E$.

Since $E$ is L-measurable so is $H$. The proof for $N^-$ is similar.

**II.1.26. Lemma.** If $N$ in $\mathcal{H}$, then for each $D$ in $\mathcal{D}$, $N^+$ and $N^-$ are L-measurable functions of x in $\mathbb{R}^n$.

**Proof.** Let $(M_1, M_2; Y)$ be a representation of $N$. Then by II.1.9 with $a = 1$ and $b = -1$ $N$ has the filling up property on CY. Put

$$N_1(x, D) = \begin{cases} N(x, D) & \text{if } x \in CY \text{ and } D \in \mathcal{D} \\ +\infty & \text{if } x \in Y \text{ and } D \in \mathcal{D} \end{cases}$$

Clearly $N_1$ has the filling up property on $\mathbb{R}^n$. Since $LY = 0$ and $N$ is an L-measurable function of x on $\mathbb{R}^n$, $N_1$ is also an L-measurable function on $\mathbb{R}^n$ for each $D$ in $\mathcal{D}$. It follows from the preceding lemma
that \( N_1^+ \) and \( N_1^- \) are L-measurable functions of \( x \) in \( \mathbb{R}^n \) for each \( D \) in \( \mathcal{D} \). But if \( x \) in \( CY \), then for every \( D \) in \( \mathcal{D} \) and every class \( s(D) \)

\[
\sum_{D' \in s(D)} N(x, D') = \sum_{D' \in s(D)} N_1(x, D).
\]

Hence for each \( D \) in \( \mathcal{D} \)

\[
N^\pm(x, D) = N_1^\pm(x, D), \quad x \in CY.
\]

Since \( LY = 0 \) and \( N_1^+ \) and \( N_1^- \) are L-measurable functions of \( x \) in \( \mathbb{R}^n \), it follows from (1) that for each \( D \) in \( \mathcal{D} \), \( N^+(x, D) \) and \( N^-(x, D) \) are L-measurable functions of \( x \) in \( \mathbb{R}^n \).

**II.1.29. Lemma.** Let \( N \) be in \( \mathcal{N} \), and let \( (M_1, M_2; Y) \) be a representation of \( N \). Then

\[
\text{x in } CTD \cap CY \text{ implies } N^\pm(x, D) = 0, \text{ all } D \text{ in } \mathcal{D}. \quad (1)
\]

**Proof.** Fix \( D \) in \( \mathcal{D} \). Let \( s(D) \) be arbitrary. Since \( M_1 \) and \( M_2 \) are in \( \mathcal{M} \) and \( Y \) is an exceptional set for \( (M_1, M_2) \) (see II.1.1.1.(ii)) it follows from (i) of II.1.4 that for all \( D' \) in \( \mathcal{D} \)

\[
M_k(x, D') = 0, \quad k = 1, 2, \text{ whenever } x \text{ in } CTD' \cap CY. \quad \text{But } x \text{ in } CY \text{ implies } N(x, D') = 0, \quad M_1(x, D') - M_2(x, D'), \text{ all } D' \text{ in } \mathcal{D}. \quad \text{Hence } x \text{ in } CTD' \cap CY \text{ implies that } N(x, D') = 0, D' \text{ in } \mathcal{D}. \text{ But } D' \subset D \text{ and } x \text{ in } CTD \cap CY \text{ imply that } x \text{ is in } CTD' \cap CY. \text{ Hence}
\]

\[
\sum_{D' \in s(D)} N(x, D') = 0 \text{ when } x \text{ in } CTD \cap CY. \quad (2)
\]

Since \( D \) and \( s(D) \) were arbitrary (1) follows from (2) and the definitions of \( N^+ \) and \( N^- \).

**II.1.30. Lemma.** Let \( N \) be in \( \mathcal{N} \) and let \( (M_1, M_2; Y) \) be a representation of \( N \). Then if \( D \) in \( \mathcal{D} \)

\[
x \text{ in } CY \text{ implies } N^\pm(x, D) \text{ is finite}. \quad (1)
\]
Proof. We give the proof of the lemma for $N^+$. 

Since $M_1$ and $M_2$ are in $\mathcal{M}$, and $Y$ is an exceptional set for $(M_1, M_2)$, it follows from (ii) and (iv) of II.1.4 that if $x$ in $CY$, and if $\{D_i\}$ is any finite collection of pairwise disjoint sets in $\mathcal{D}$, then

$$0 \leq \sum_k M_k(x, D_i) \leq M_k(x, D) < + \infty, \quad k = 1, 2.\quad(2)$$

Hence if $x$ in $CY$ and $s^+(x, D)$ are arbitrary

then

$$M_1(x, D) \geq \sum_{D' \text{ in } s^+(x, D)} M_1(x, D')$$

$$\geq \sum_{D' \text{ in } s^+(x, D)} (M_1(x, D') - M_2(x, D'))$$

$$= \sum_{D' \text{ in } s^+(x, D)} N(x, D') \geq 0$$

Thus since $s^+(x, D)$ was arbitrary

$$M_1(x, D) \geq N^+(x, D).\quad(4)$$

(2), (3), and (4) yield (1) for $N^+$. 

II.1.31. Lemma. If $N$ in $\mathcal{M}$, then $N^+$ and $N^-$ are in $\mathcal{M}$.

Proof. Let $(M_1, M_2; Y)$ be a representation of $N$. Then by II.1.29

$$x \text{ in } CTD \cap CY \text{ implies } N^+(x, D) = 0, \quad D \text{ in } \mathcal{D}.\quad(1)$$

By II.1.22

$N^\pm$ is subadditive on $\mathbb{R}^n$. \quad(2)
By II.1.23

\[ N^+ \text{ has the filling up property on } \mathbb{R}^n \]  

(3)

By II.1.30

\[ N^+ (x, D) \text{ is finite for } x \text{ in } C Y \text{ and } D \text{ in } D \]  

(4)

By II.1.28

\[ N^+ \text{ is a } L\text{-measurable function of } x \text{ in } \mathbb{R}^n, \quad \text{all } D \text{ in } D \]  

(5)

(1) through (5) imply that \( N^+ \) and \( N^- \) are in \( M \).

II.1.32. Remark. Note that (1) through (5) in the preceding lemma also imply that \( Y \) may be taken as an exceptional set for \( (N^+, N^-) \). Hence if there exists a set \( Z \) such that \( L Z = 0 \) and for all \( D \) in \( D \)

\[ N(x, D) = N^+(x, D) - N^-(x, D), \quad x \text{ in } C Z \]

then \( (N^+, N^-; Y \cup Z) \) is a representation of \( N \). The following example shows that it is not always possible to obtain for \( N \) in \( M \) a representation of the form \( (N^+, N^-; Y') \) where \( Y' \subset \mathbb{R}^n \) and \( LY' = 0 \). On the other hand if \( N \) is in \( M \), then, as we shall see in lemma II.1.36, \( N \) does have a representation of the form \( (N^+, N^-; Y') \) where \( Y' \subset \mathbb{R}^n \) and \( LY' = 0 \). A similar result can be proved when \( -N \) is in \( M \).

II.1.33. Example. Let: \( \mathbb{R}^n = \mathbb{R}^1 \), \( D = (-1, 1) \), \( D_0 = (-\frac{1}{2}, 0) \), and \( T \) be the identity transformation from \( D \) onto \( D \). If \( D = (a, b) \subset D \) and \( x \) in \( \mathbb{R}^n \) define

\[ M_1(x, D) = \begin{cases} 1 & \text{if } x \text{ in } D_0 \text{ and } \overline{D_0} \subset D \\ 0 & \text{otherwise} \end{cases} \]

\[ M_2(x, D) = \begin{cases} b & \text{if } x \text{ in } D_0 \text{ and } \overline{D_0} \subset D \\ 0 & \text{otherwise} \end{cases} \]
One readily checks that $M_1$ and $M_2$ are admissible in the restricted sense, and that $\emptyset$ is an exceptional set for $(M_1, M_2)$. Let $N = M_1 - M_2$. Then $N$ is admissible with $(M_1, M_2; \emptyset)$ as a representation. Suppose now that $x$ in $D_0$ and $D \supset D = (a, b) \supset D_0$. Then
\[ l \geq b > 0 \]
\[ N(x, D) = 1 - b \]
\[ N^+(x, D) = 1 \]
\[ N^-(x, D) = 0 \]
Hence $N(x, D) \neq N^+(x, D) - N^-(x, D)$, $x$ in $D_0$, $D = (a, b)$. Since $L D_0 = \frac{1}{2}$, there is no representation of $N$ of the form $(N^+, N^-; Y)$ where $Y \subset R^1$ and $LY = 0$.

**II.1.34. Lemma.** Let $M$ be any non-negative multiplicity function which is subadditive and has the filling up property on a set $E \subset R^n$. Then
\[ M(x, D) = M^+(x, D), \quad x \text{ in } E \text{ and } D \text{ in } \mathcal{D}. \]

**Proof.** Fix $x$ in $E$ and $D$ in $\mathcal{D}$. Let $a$ be any real number such that
\[ a < M^+(x, D). \quad (1) \]
Then there exists a set $s(D)$ such that
\[ \sum_{D' \in s(D)} M(x, D') > a. \quad (2) \]
If $s(D)$ is not empty, then it follows by the subadditivity of $M$ that
\[ \sum_{D' \in s(D)} M(x, D') \leq M(x, D). \quad (3) \]
If $s(D)$ is empty, then $\sum_{D' \in s(D)} M(x, D') = 0$.

Since $M$ is non-negative (3) must again hold when $s(D)$ is empty. Hence (2) and (3) yield

$$M(x, D) > a.$$  

Since $a$ was an arbitrary real number satisfying (1), it follows that

$$M(x, D) \geq M^+(x, D).$$  \hspace{1cm} (4)

Let $b$ be any real number such that

$$b < M(x, D),$$  \hspace{1cm} (5)

and let $\{D_i\}$ be a sequence of domains filling up $D$ from the interior. Since $M$ has the filling up property on $E$, there exists an integer $i_0$, depending on $b$, such that

$$M(x, D_{i_0}) > b.$$  \hspace{1cm} (6)

Since the domains $D_i$ fill up $D$ from the interior, $\overline{D_{i_0}} \subseteq D$.

Hence $D_{i_0}$ is itself a set $s(D)$. Consequently

$$M^+(x, D) \geq M(x, D_{i_0}).$$  \hspace{1cm} (7)

(6) and (7) yield

$$M^+(x, D) > b.$$  \hspace{1cm} (8)

Since $b$ was an arbitrary real number satisfying (5), (8) implies

$$M^+(x, D) \geq M(x, D).$$  \hspace{1cm} (9)

(4) and (9) yield

$$M(x, D) = M^+(x, D)$$

Since $x$ was arbitrary in $E$ and $D$ was arbitrary in $\mathcal{D}$, the lemma follows.
II.1.35. Lemma. Let $M$ be a non-negative multiplicity function. Then

$$M^-(x, D) = 0, \ x \in \mathbb{R}^n, \ D \in \mathcal{D}. \quad (1)$$

Proof. Since $M$ is non-negative, any set $s^-(x, D)$ is empty, and (1) follows (see II.1.21).

II.1.36. Lemma. Let $M$ be in $\mathcal{M}$, and let $X$ be an exceptional set for $M$. Then $(M^+, M^-; X)$ is a representation of $M$.

Proof. This follows immediately from II.1.34 and II.1.35.

II.2. Bounded Variation and Absolute Continuity with Respect to Non-Negative Admissible Multiplicity Functions

Throughout this section $M$ will be a fixed (non-negative) multiplicity function in $\mathcal{M}$ (see II.1.4). Since $M \geq 0$ and $M$ is an $L$-measurable function of $x$ in $\mathbb{R}^n$ for each $D \in \mathcal{D}$, we may form the Lebesgue integral

$$\int M(x, D), \ D \in \mathcal{D},$$

over $\mathbb{R}^n$ where the value of the integral is a non-negative real number or $+\infty$ (see I.1.6).

Recall that $T$ is a fixed, bounded, continuous transformation from the fixed bounded domain $D$ into $\mathbb{R}^n$, and that $\mathcal{D}$ designates the collection of all domains in $D$.

II.2.1. Definition. If $D$ is a domain in $\mathcal{D}$ and if

$$\int M(x, D) < +\infty,$$

that is to say, if $M(x, D)$ is $L$-summable over $\mathbb{R}^n$, then $T$ is said to be of bounded variation in $D$ with respect to $M$ (briefly: $T$ - BVM in $D$).
Actually the concept of bounded variation with respect to a multiplicity function \( M \) could just as well be applied to multiplicity functions \( M \) which satisfy all conditions for restricted admissibility except the finiteness condition, that is, condition (iv) of II.1.4. Indeed suppose that \( M \) is a multiplicity function which satisfies all conditions except possibly (iv) in II.1.4, and suppose that \( M(x, D) \) is \( L \)-summable over \( \mathbb{R}^n \). Then by I.2.6 \( M(x, D) \) is finite except possibly in a set \( X \subseteq \mathbb{R}^n \) of \( L \)-measure zero. By the subadditivity of \( M \) it follows that if \( \mathcal{D} \) is the set of all domains \( D' \subseteq D \), then \( M(x, D') \) is finite except possibly in the set \( X \), all \( D' \) in \( \mathcal{D} \). Hence if \( T-BVM \) in \( D \), then condition (iv) with \( \mathcal{D} \) in place of \( \mathcal{D} \) is automatically satisfied.

**II.2.2. Lemma.** If \( T-BVM \) in a domain \( D_0 \) in \( \mathcal{D} \), then \( T-BVM \) in every domain \( D \subseteq D_0 \).

**Proof.** Since \( M \) is in \( \mathcal{M} \), it follows by II.1.4 that \( M \) is subadditive except possibly on an exceptional set \( X \) where \( L_X = 0 \). Hence if \( D \subseteq D_0 \) then \( M(x, D) \leq M(x, D_0) \), \( x \) in \( X \). Consequently
\[
\int M(x, D) \leq \int M(x, D_0).
\] T-BVM in \( D_0 \) implies \( \int M(x, D_0) < +\infty \). Therefore \( \int M(x, D) < +\infty \) so that \( T-BVM \) in \( D \).

**II.2.3. Lemma.** If \( T-BVM \) in a domain \( D_0 \) in \( \mathcal{D} \), then

(i) \( \int M(x, D) \) is a subadditive function of domains \( D \subseteq D_0 \).

(ii) The (finite, non-negative) derivative \( D(u, M) \) of \( \int M(x, D) \) (see I.3.10) exists a.e. in \( D_0 \), and \( D(u, M) \) is \( L \)-measurable and \( L \)-summable over \( D_0 \).

(iii) \( \int D(u, M) \leq \int M(x, D), \quad D \subseteq D_0 \).
(iv) If $D \subseteq D_0$ and $\{D_j\}$ is a sequence of domains filling up $D$ from the interior, then
\[ \int M(x, D) = \lim_j \int M(x, D_j). \]

**Proof.** Our concept of a non-negative admissible multiplicity function is slightly more general in some respects than that in [2; 232, 233]. Thus we have dropped the requirements that $M(x, D)$ be a non-negative integer whenever $M(x, D) \neq +\infty$, that the base function generated by $M$ be Borel measurable, and that conditions (i), (ii), and (iii) in II.1.4 hold for all $x$ in $\mathbb{R}^n$ instead of for $x$ in $\mathcal{D}$. Nevertheless the argument in lemma 2 [2; 239] may be applied with $D_0$ in place of $D$ to obtain (i), (ii), and (iii). (iv) is a consequence of (ii) and (iii) in II.1.4 and of I.2.10.

**II.2.4. Lemma.** If $T$-BVM in a domain $D_0$ in $\mathcal{D}$ and if $h$ is a positive real number, then there exists a domain $D$, depending on $h$, such that
\[ \bar{D} \subseteq D_0, \ LfrD = 0 \]
and
\[ \int M(x, D_0) < \int M(x, D) + h. \]

**Proof.** By I.3.8 and I.3.9 there exists a sequence $\{D_j\}$ of domains filling up $D_0$ from the interior and having the property that $LfrD_j = 0$, all $j$. By II.2.3.(iv)
\[ \int M(x, D_0) = \lim_j \int M(x, D_j). \]
Hence for $j$ sufficiently large
\[ \int M(x, D_0) < \int M(x, D_j) + h \]
and the lemma follows.
II.2.5. Lemma. Suppose T-BVM in a domain D in $\mathcal{O}$. Let $\{D_j\}$ be a countable collection of pairwise disjoint domains in D. Then

$$\int M(x, D) = \sum_j \int M(x, D_j) \text{ if and only if } M(x, D) = \sum_j M(x, D_j) \text{ a.e. in } \mathbb{R}^n.$$ 

Proof. If

$$M(x, D) = \sum_j M(x, D_j) \text{ a.e. in } \mathbb{R}^n \quad (1)$$

then by I.2.8

$$\int M(x, D) = \sum_j \int M(x, D_j) \quad (2)$$

Suppose now that (2) holds. Let X be an exceptional set for M. Then

$$M(x, D) \text{ is finite for } x \text{ in } CX, \text{ and by the observation in II.1.3}$$

$$M(x, D) = \sum_j M(x, D_j) \geq 0, \text{ } x \text{ in } CX. \quad (3)$$

T-BVM in D implies

$$\int M(x, D) < +\infty. \quad (4)$$

By I.2.8

$$\sum_j \int M(x, D_j) = \int \sum_j M(x, D_j) \quad (5)$$

(2), (4), and (5) yield

$$\int (M(x, D) - \sum_j M(x, D_j)) = 0. \quad (6)$$

(3), (6), and I.2.6 yield (1). This completes the proof.

II.2.6. Lemma. If T-BVM in a domain D in $\mathcal{O}$, and if E is a set in $\mathbb{R}^n$ such that $LE = 0$, then $D(u, M) = 0$ a.e. in $T^{-1}E \cap D$.

Proof. Since T-BVM in D, $M(x, D)$ is L-summable over $\mathbb{R}^n$. By I.2.7 it follows that for any real number $h > 0$ there exists a real number $k > 0$, depending on $h$, such that if $S$ is an L-measurable set
in \( \mathbb{R}^n \), then

\[
\text{LS} < k \text{ implies } \int_S M(x, D) < h. \tag{1}
\]

Since \( IE = 0 \), there exists an open set \( G \), depending on \( h \), such that \( E \subseteq G \) and \( IE < k \). By hypothesis \( T \) is continuous and \( D \) is open in \( \mathbb{R}^n \). Hence \( T^{-1}G \cap D \) is open in \( \mathbb{R}^n \). Let \( \{ D_j \} \) be the (pairwise disjoint, open) components of the open set \( T^{-1}G \cap D \).

Since \( E \subseteq G \), it follows that

\[
T^{-1}E \cap D \subseteq T^{-1}G \cap D = \bigcup_j D_j \tag{2}
\]

Since the sets \( D_j \) are pairwise disjoint it follows from (2) and I.2.9 that

\[
\int_{T^{-1}G \cap D} D(u, M) = \int \bigcup_j D_j D(u, M) = \sum_j \int D_j D(u, M). \tag{3}
\]

By lemma II.2.3.(iii)

\[
\sum_j \int D_j D(u, M) \leq \sum_j \int M(x, D_j). \tag{4}
\]

\( M \) in \( M \) implies \( M(x, D_j) = 0 \) a.e. in \( CTD_j \), all \( j \). Hence

\[
\sum_j \int M(x, D_j) = \sum_j \int_{TD_j} M(x, D_j). \tag{5}
\]

Since \( D_j \) is a component of \( T^{-1}G \cap D \), it follows that \( TD_j \subseteq G \), all \( j \). Hence

\[
\sum_j \int_{TD_j} M(x, D_j) \leq \sum_j \int G M(x, D_j). \tag{6}
\]
By I.2.8

\[ \sum_{j} \int_{G} \mathbf{M}(x, D_j) = \int_{G} \sum_{j} \mathbf{M}(x, D_j) . \]  

By the observation in II.1.3

\[ \sum_{j} \mathbf{M}(x, D_j) \leq \mathbf{M}(x, D) \text{ a. e. in } \mathbb{R}^n \]

so that

\[ \int_{G} \sum_{j} \mathbf{M}(x, D_j) \leq \int_{G} \mathbf{M}(x, D) . \]  

Since \( C \mathcal{G} < k \), (1) and (3) through (8) yield

\[ \int_{T^{-1} \mathcal{G} \cap D} D(u, M) \leq h . \]

Now let \( \{h_i\} \) be any sequence of positive real numbers tending to zero. Then for each \( i \) we obtain as above an open set \( G_i \) such that

\[ E \subset G_i \]  

and

\[ \int_{T^{-1} G_i \cap D} D(u, M) \leq h_i \]  

Put

\[ U = \bigcap_{i} (T^{-1} G_i \cap D) . \]

From (10) it then follows that

\[ \int_{U} D(u, M) = 0 . \]  

Since \( D(u, M) \geq 0 \) a. e. in \( \mathcal{D} \) and therefore in \( U \), (11) and I.2.6 yield

\[ D(u, M) = 0 \text{ a. e. in } U \]  

(12)
With the help of (9) we see that

\[ T^{-1}E \cap D \subseteq U \]

(12) and (13) imply \( D(u, M) = 0 \) a. e. in \( T^{-1}E \cap D \).

**II.2.7. Lemma.** Suppose T-BVM in a domain \( D \) in \( \mathcal{D} \). Let \( H'(x) \) and \( H''(x) \) be two real valued functions on \( R^n \) such that \( H'(x) = H''(x) \) a. e. in \( R^n \). Then

\[ H'(x) M(x, D) = H''(x) M(x, D) \text{ a. e. in } R^n, \]

and

\[ H'(Tu) D(u, M) = H''(Tu) D(u, M) \text{ a. e. in } D. \]

**Proof.** Since \( M \) is in \( \mathcal{M} \), \( M(x, D) \) is finite a. e. in \( R^n \), and (1) follows readily. To prove (2) put

\[ E = \left\{ x \mid x \text{ in } R^n \text{ and } H'(x) \neq H''(x) \right\} \]

Then \( LE = 0 \), and by II.2.6 \( D(u, M) = 0 \) a. e. in \( T^{-1}E \cap D \). Hence

\[ H'(Tu) D(u, M) = 0 = H''(Tu) D(u, M) \text{ a. e. in } T^{-1}E \cap D. \]

Since \( H'(Tu) = H''(Tu) \) for \( u \) in \( D - T^{-1}E \) (and \( D(u, M) \) exists a. e. in \( D \)),

\[ H'(Tu) D(u, M) = H''(Tu) D(u, M) \text{ a. e. in } D - T^{-1}E. \]

Since \( D = (D \cap T^{-1}E) \cup (D - T^{-1}E) \), (2) follows from (3) and (4).

**II.2.8. Lemma.** Suppose T-BVM in a domain \( D \) in \( \mathcal{D} \). Let \( H(x) \) and \( H_m(x), m = 1, 2, \ldots \), be real valued functions on \( R^n \). If

\[ \lim_m H_m(x) = H(x) \text{ a. e. in } TD, \]

then

\[ \lim_m H_m(x) M(x, D) = H(x) M(x, D) \text{ a. e. in } R^n, \]

and

\[ \lim_m H_m(Tu) D(u, M) = H(Tu) D(u, M) \text{ a. e. in } D. \]
Proof. Since $M$ is in $\mathcal{M}$, it follows by II.1.4.(1) that

\[ M(x, D) = 0 \text{ a.e. in } CTD. \]

Thus

\[ \lim_m H_m(x) M(x, D) = H(x) M(x, D) \text{ a.e. in } CTD. \tag{4} \]

Let $X$ be an exceptional set for $M$, and let $Y$ be those points $x$ in $TD$ at which $\lim_m H_m(x)$ fails to exist or $\lim_m H_m(x) \neq H(x)$. Let $Z = X \cup Y$. Evidently

\[ LZ = 0 \tag{5} \]

$M(x, D)$ is finite in $CZ$ \tag{6}

\[ \lim_m H_m(x) = H(x) \text{ in } TD - Z \tag{7} \]

(5), (6), and (7) yield

\[ \lim_m H_m(x) M(x, D) = H(x) M(x, D) \text{ a.e. in } TD. \tag{8} \]

(4) and (8) yield (2). Now let

\[ U_1 = \{ u \in D \text{ and } D(u, M) \text{ fails to exist} \} \tag{9} \]

since $T = \text{BVM in } D$

\[ IU_1 = 0. \tag{10} \]

Let $U_2 = T^{-1}Z \cap D$. Since $LZ = 0$, it follows by II.2.6 that

\[ D(u, M) = 0 \text{ a.e. in } U_2. \]

Hence

\[ \lim_m H_m(Tu) D(u, M) = H(Tu) D(u, M) \text{ a.e. in } U_2 \tag{11} \]

On the other hand if $u$ is in $D - U_2$ then $Tu$ is in $TD - Z$. Hence by (7), (9) and (10)

\[ \lim_m H_m(Tu) D(u, M) = H(Tu) D(u, M) \text{ a.e. in } D - U_2. \tag{12} \]

(11) and (12) yield (3).
II.2.9. Lemma. Let $H(x)$ be a real valued, Borel measurable function on $\mathbb{R}^n$. Then $H(Tu)$ is a Borel measurable function on $D$ in $D$.

Proof. With minor modifications the proof of theorem 2 in [2; 213] may be applied here.

II.2.10. Lemma. Suppose $T$-BVM in a domain $D$ in $\mathcal{D}$. Let $H(x)$ be a real valued $L$-measurable function on $\mathbb{R}^n$. Then

$$H(x) M(x, D) \text{ is } L\text{-measurable in } \mathbb{R}^n \quad (1)$$

and

$$H(Tu) D(u, M) \text{ is } L\text{-measurable in } D. \quad (2)$$

Proof. Since $M$ is in $\mathcal{M}$, $M(x, D)$ is $L$-measurable in $\mathbb{R}^n$. Since $H(x)$ is also $L$-measurable in $\mathbb{R}^n$, the product $H(x) M(x, D)$ is $L$-measurable in $\mathbb{R}^n$, and (1) holds. By I.2.2 there exists a real valued, Borel measurable function $G(x)$ such that $G(x) = H(x)$ a.e. in $\mathbb{R}^n$. By II.2.9 $G(Tu)$ is Borel measurable in $D$. Since $T$-BVM in $D$, $D(u, M)$ exists a.e. in $D$, and it is $L$-measurable in $D$ by II.2.3. Hence $G(Tu) D(u, M)$ is $L$-measurable in $D$. By II.2.7 $H(Tu) D(u, M) = G(Tu) D(u, M)$ a.e. in $D$. Hence $H(Tu) D(u, M)$ is $L$-measurable in $D$.

Remark. Suppose $T$-BVM in a domain $D$ in $\mathcal{D}$. Let $H(x)$ be a real valued, non-negative, $L$-measurable function on $\mathbb{R}^n$. By the preceding lemma $H(x) M(x, D)$ and $H(Tu) D(u, M)$ are $L$-measurable in $\mathbb{R}^n$ and $D$ respectively. Since $H(x) M(x, D)$ and $H(Tu) D(u, M)$ are also non-negative we may form the integrals (which may be infinite):

$$\int H(x) M(x, D) \text{ and } \int_D H(Tu) D(u, M).$$
In the future, when we wish to consider such integrals, we shall generally not mention that the measurability conditions necessary for their consideration are satisfied.

II.2.11. Definition. Let $D$ be in $\mathcal{D}$. $T$ is said to be absolutely continuous in $D$ with respect to the multiplicity function $M$ (briefly: $T$-ACM in $D$) if $T$-BVM in $D$ and
\[
\int_D D(u, M) = \int M(x, D).
\]
Thus for $T$ to be ACM in $D$ we require that $T$ be BVM in $D$ and that equality hold in (iii) of II.2.3.

II.2.12. Lemma. Let $D$ be in $\mathcal{D}$, and let $\{D_j\}$ be a countable sequence of domains in $\mathcal{D}$ such that
\[
\begin{align*}
D_j &\subset D, \text{ all } j, \\
D_j \cap D_k &\neq \emptyset, j \neq k, \\
D &\subset \bigcup D_j = E_1 \cup E_2 \text{ where } E_1 = 0 \\
&\text{and } LTE_2 = 0.
\end{align*}
\]
If $T$-ACM in $D$, then
\[
\int M(x, D) = \sum_j \int M(x, D_j)
\]
and
\[
M(x, D) = \sum_j M(x, D_j) \text{ a.e. in } \mathbb{R}^n.
\]
\[\text{Proof. Since } T \text{-ACM in } D,
\int M(x, D) = \int_D D(u, M).
\]
Since $LTE_2 = 0$, it follows from II.2.6 that
\[
D(u, M) = 0 \text{ a.e. in } E_2.
\]
Since $I E_1 = 0$,
\[ \int_{E_1} D(u, M) = 0. \quad (8) \]

(7) and (8) yield
\[ \int_{E_1 \cup E_2} D(u, M) = 0. \quad (9) \]

(1), (2), (3), (9), and II.2.9 yield
\[ \int_D D(u, M) = \int \cup_{D_j} D(u, M) = \sum_j \int_{D_j} D(u, M). \quad (10) \]

T-ACM in $D$ implies T-BVM in $D$. T-BVM in $D$, $D_j \subseteq D$, and II.2.2 imply T-BVM in $D_j$, all $j$. Hence by II.2.3(iii)
\[ \int_{D_j} D(u, M) \leq \int M(x, D_j), \text{ all } j. \quad (11) \]

(11) yields
\[ \sum_j \int_{D_j} D(u, M) \leq \sum_j \int M(x, D_j). \quad (12) \]

By (1), (2), and II.2.3(i)
\[ \sum_j \int M(x, D_j) \leq \int M(x, D). \quad (13) \]

(6), (10), (12), and (13) yield (4). (4) and II.2.5 yield (5).

II.2.13. Lemma. If T-ACM in a domain $D_0$ in $\emptyset$, then T-ACM in every domain $D \subseteq D_0$.

Proof. Since T-ACM in $D_0$ implies T-BVM in $D_0$ which in turn implies T-BVM in $D \subseteq D_0$ (see II.2.2), we need only prove
\[

\int_D D(u, M) = \int M(x, D), \quad D \subset D_0. 
\] \hspace{1cm} (1)

**Case 1:** \(\overline{D} \subset D_0\) and \(\text{LfrD} = 0\).

Since \(\overline{D} \subset D_0\), \(D_0 - \overline{D}\) is open in \(\mathbb{R}^n\) and

\[

D_0 - \overline{D} = \bigcup_j D_j,
\] \hspace{1cm} (2)

where the domains \(D_j\) are the pairwise disjoint components of \(D_0 - \overline{D}\).

Since \(\text{LfrD} = 0\),

\[

L(D_0 - (\bigcup_j D_j \cup D)) = 0.
\] \hspace{1cm} (3)

Hence we may apply II.2.12 with \(E_1 = D_0 - (\bigcup_j D_j \cup D)\) and \(E_2 = \emptyset\) to obtain

\[

\int M(x, D_0) = \sum_j \int M(x, D_j) + \int M(x, D)
\] \hspace{1cm} (4)

\(T\)-ACM in \(D_0\) implies

\[

\int M(x, D_0) = \int_{D_0} D(u, M)
\] \hspace{1cm} (5)

Since the domains \(D_j\) are pairwise disjoint, we obtain with the help of (2) and (3)

\[

\int_{D_0} D(u, M) = \sum_j \int_{D_j} D(u, M) + \int_D D(u, M).
\] \hspace{1cm} (6)

(4), (5), and (6) yield

\[

\sum_j \int M(x, D_j) + \int M(x, D) = \sum_j \int_{D_j} D(u, M)
\]

\[

+ \int_D D(u, M).
\] \hspace{1cm} (7)

Since \(T\)-BVM in \(D_0\), it follows from II.2.2 that \(T\)-BVM in \(D\) and in
\(D_j\), all \(j\). Hence it follows from II.2.3.(iii) that
\[
\int_{D_j} D(u, M) \leq \int M(x, D_j), \text{ all } j, \tag{8}
\]
and
\[
\int_{D} D(u, M) \leq \int M(x, D). \tag{9}
\]
(7), (8), and (9) yield (1).

**Case 2. (General case):** \(D \subset D_0\).

Let \(h\) be an arbitrary positive real number. By II.2.4 there exists a domain \(D'\), depending on \(h\), such that
\[
\overline{D'} \subset D, \quad \text{Lfr}D' = 0
\]
and
\[
\int M(x, D') > \int M(x, D) - h. \tag{10}
\]
By Case 1
\[
\int M(x, D') = \int_{D'} D(u, M). \tag{11}
\]
\(0 \leq D(u, M) \text{ a. e. in } D\) and \(D' \subset D\) imply
\[
\int_{D} D(u, M) \geq \int_{D'} D(u, M). \tag{12}
\]
(10), (11) and (12) yield
\[
\int_{D} D(u, M) > \int M(x, D) - h
\]
Since \(h\) was an arbitrary positive number,
\[
\int_{D} D(u, M) \geq \int M(x, D). \tag{13}
\]
But T-BVM in \(D\) implies by II.2.3
\[
\int_{D} D(u, M) \leq \int M(x, D). \tag{14}
\]
(13) and (14) yield (1).
Theorem. If T-BVM in a domain $D_0$ in $\mathcal{D}$, then a necessary and sufficient condition that $T$ be ACM in $D_0$ is

$$\int_{D} D(u, M) = \int_{M(x, D)}, \text{ all } D \subset D_0.$$ (1)

Proof. The sufficiency of (1) follows by putting $D = D_0$ in the equality in (1). Necessity follows from the preceding lemma.

Definition. Assume that T-BVM in a domain $D$ in $\mathcal{D}$.

We designate by $\mathcal{H}(D)$ the set of all real valued, L-measurable functions $H = H(x)$ on $\mathbb{R}^n$ such that $H(Tu) - D(u, M)$ and $H(x) - M(x, D)$ are L-summable over $D$ and $\mathbb{R}^n$ respectively and their respective integrals are equal. That is

$$\int_{D} H(Tu) - D(u, M) = \int_{H(x) - M(x, D)} \neq \pm \infty \quad (1)$$

Note that by II.2.10, $H(x) - M(x, D)$ and $H(Tu) - D(u, M)$ are L-measurable in $\mathbb{R}^n$ and $D$ respectively so that the measurability conditions necessary for the consideration of the existence of their integrals are satisfied. (With regard to the existence of an integral see I.1.6).

The main result of this section states that if $T$-ACM in $D$ in $\mathcal{D}$, and if $H = H(x)$ is a real valued, L-measurable function on $\mathbb{R}^n$ such that either $H(Tu) - D(u, M)$ is L-summable over $D$ or $H(x) - M(x, D)$ is L-summable over $\mathbb{R}^n$, then $H$ in $\mathcal{H}(D)$. To prove this we shall show that $\mathcal{H}(D)$ contains certain special functions. We then obtain $H$ as a limit of certain of these special functions and show that $\mathcal{H}(D)$ contains this limit.
II.2.16. Lemma. Let $T$ be BVM in a domain $D$ in $\Omega$. If $H_1, \ldots, H_m$ is a finite sequence of functions in $\mathcal{H}(D)$, and if $a_1, \ldots, a_m$ are real numbers, then

$$\sum_{i=1}^{m} a_i H_i \text{ is in } \mathcal{H}(D). \quad (1)$$

Proof. By (1) in II.2.15 it follows that

$$\sum_{i=1}^{m} a_i \int_D H_i(Tu) \, D(u, M) = \sum_{i=1}^{m} a_i \int H_i(x) \, M(x, D)$$

$$\neq \pm \infty. \quad (2)$$

Since the terms in the sums of (2) are finite, it follows by I.2.4 that

$$\sum_{i=1}^{m} a_i \int_D H_i(Tu) \, D(u, M) = \int_D \sum_{i=1}^{m} a_i H_i(Tu) \, D(u, M). \quad (3)$$

and

$$\sum_{i=1}^{m} a_i \int H_i(x) \, M(x, D) = \int \sum_{i=1}^{m} a_i H_i(x) \, M(x, D). \quad (4)$$

(2), (3) and (4) yield (1).

II.2.17. Definition. Let $E$ be a set in $\mathbb{R}^n$.

Put

$$H_E(x) = \begin{cases} 1 & \text{if } x \text{ in } E \\ 0 & \text{if } x \text{ in } C E \end{cases}.$$

Then $H_E = H_E(x)$ is said to be the characteristic function of $E$.

Observe that $H_E$ is Borel measurable (L-measurable) if and only if $E$ is Borel measurable (L-measurable).
II.2.13. **Lemma.** Let $T$ be ACM in a domain $D_0$ in $\mathcal{D}$, and suppose that $Q$ is an (open, oriented) $n$-cube in $\mathbb{R}^n$. Then $H_Q$ is in $\mathcal{H}(D_0)$.

**Proof.** Since $Q$ is open, $H_Q$ is $L$-measurable. Since $T$-ACM in $D_0$ and $0 \leq H_Q(x) \leq 1$, $x$ in $\mathbb{R}^n$, it follows readily that $H_Q(Tu) D(u, M)$ and $H_Q(x) M(x, D_0)$ are $L$-summable over $D_0$ and $\mathbb{R}^n$ respectively.

Since the sets $Q$, $CQ$, and $frQ$ are pairwise disjoint, so are their inverses under $T$. Let $\mathcal{Q}$ and $\mathcal{Q}''$ be the set of components of $T^{-1}Q \cap D_0$ and $T^{-1}CQ \cap D_0$ respectively. Then $\mathcal{Q} = \mathcal{Q}' \cup \mathcal{Q}''$ is a countable collection of pairwise disjoint domains in $D_0$ and

$$D_0 = \bigcup_{D \in \mathcal{Q}} D \cup (T^{-1}frQ \cap D_0).$$

Since $LfrQ = 0$,

$$LT(D_0) = \bigcup_{D \in \mathcal{Q}} D = 0,$$  \hspace{1cm} (1)

and hence by II.2.12

$$M(x, D_0) = \sum_{D \in \mathcal{Q}} M(x, D) \text{ a.e. in } \mathbb{R}^n.$$  \hspace{1cm} Thus

$$H_Q(x) M(x, D_0) = \sum_{D \in \mathcal{Q}} H_Q(x) M(x, D) \text{ a.e. in } \mathbb{R}^n. \hspace{1cm} (2)$$

Since $D''$ in $\mathcal{Q}''$ implies $TD'' \subset CQ$, it follows that $H_Q(x) = 0$ for $x$ in $TD''$. Since $M$ is admissible, $M(x, D'') = 0$ a.e. in $CTD''$ for $D''$ in $\mathcal{Q}''$. Hence

$$\sum_{D'' \in \mathcal{Q}''} H_Q(x) M(x, D'') = 0 \text{ a.e. in } \mathbb{R}^n. \hspace{1cm} (3)$$

(2) and (3) yield

$$\int H_Q(x) M(x, D_0) = \int \sum_{D' \in \mathcal{Q}'} H_Q(x) M(x, D'). \hspace{1cm} (4)$$
The functions \( H_q(x) M(x, D') \), \( D' \) in \( \mathbb{R}' \), are all \( \geq 0 \) so that by I.2.8
\[
\int \sum_{D' \in \mathbb{R}'} H_q(x) M(x, D') = \sum_{D' \in \mathbb{R}'} \int H_q(x) M(x, D'). \tag{5}
\]
Since \( M \) is in \( \mathcal{M} \),
\[
M(x, D') = 0 \text{ a.e. in } \mathcal{C}T D', D' \in \mathbb{R}'. \tag{6}
\]
Thus
\[
\sum_{D' \in \mathbb{R}'} \int H_q(x) M(x, D') = \sum_{D' \in \mathbb{R}'} \int H_q(x) M(x, D'). \tag{7}
\]
Since \( TD' \subset \mathcal{Q} \) for all \( D' \) in \( \mathbb{R}' \), \( H_q(x) = 1 \) for \( x \) in \( TD' \).
Hence
\[
\sum_{D' \in \mathbb{R}'} \int H_q(x) M(x, D') = \sum_{D' \in \mathbb{R}'} \int H_q(x) M(x, D'). \tag{8}
\]
Making use of (6) again, we obtain
\[
\sum_{D' \in \mathbb{R}'} \int M(x, D') = \sum_{D' \in \mathbb{R}'} \int M(x, D'). \tag{9}
\]
(4), (5), (7), (8), and (9) yield
\[
\int H_q(x) M(x, D_0) = \sum_{D' \in \mathbb{R}'} \int M(x, D'). \tag{10}
\]
Since T-ACM in \( D_0 \), we have T-ACM in \( D' \), \( D' \) in \( \mathbb{R}' \) (see II.2.13).
Hence
\[
\int M(x, D') = \int_{D'} D(u, M), D' \in \mathbb{R}',
\]
and so
\[
\int H_q(x) M(x, D_0) = \sum_{D' \in \mathbb{R}'} \int_{D'} D(u, M). \tag{11}
\]
$H_Q(T_u) = 1$, $u$ in $D'$ in $\mathbb{A}'$, implies

$$\sum_{D \in \mathbb{A}} \int_{D'} D(u, M) = \sum_{D \in \mathbb{A}} \int_{D'} H_Q(T_u) D(u, M).$$  \hspace{1cm} (12)

Since $\mathbb{A}$ is a countable collection of pairwise disjoint domains in $D_0$, it follows from (1), I.2.9, and II.2.6 that

$$\sum_{D \in \mathbb{A}} \int_{D'} H_Q(T_u) D(u, M) = \int_{D'} H_Q(T_u) D(u, M).$$  \hspace{1cm} (13)

$H_Q(T_u) = 0$, $u$ in $D''$ in $\mathbb{A}''$, and (13) imply

$$\sum_{D' \in \mathbb{A}'} \int_{D'} H_Q(T_u) D(u, M) = \int_{D'} H_Q(T_u) D(u, M).$$  \hspace{1cm} (14)

(11), (12), and (14) yield

$$\int_{H_Q(x) M(x, D_0)} = \int_{D_0} H(T_u) D(u, M),$$

and the lemma follows.

**II.2.19. Lemma.** Assume T-ACM in a domain $D$ in $\mathbb{D}$. Let $m$ be a positive integer, and $\Delta_m$ be a grid on $\mathbb{R}^n$ (see I.3.4). Let $\{Q_i\}_{i=1}^r$ be any finite subcollection of $n$-cubes in $\Delta_m$, and let $a_1, \ldots, a_r$ be real numbers. Then

$$\sum_{i=1}^r a_i H_{Q_i}$$

is in $\mathcal{H}(D)$.  \hspace{1cm} (1)

**Proof.** By II.2.18, $H_{Q_i}$ in $\mathcal{H}(D)$ for each $i$. Hence by II.2.16, (1) holds.

**II.2.20. Lemma.** Suppose T-ACM in a domain $D$ in $\mathbb{D}$, and $F \subset TD$ is closed relative to $\mathbb{R}^n$. Let $H = H(x)$ be a real valued, non-negative
function on \( \mathbb{R}^n \) such that \( H \) is continuous on \( F \) relative to \( F \) and identically zero on \( CF \). Then \( H \) is in \( \mathcal{N}(D) \).

**Proof.** Let \( \triangle_m, m = 1, 2, \ldots \) be the sequence of nested grids defined in I.3.4 (see also I.3.6).

**Case 1.** There exists an integer \( m \) such that

\[
F \subset \bigcup_{Q_m \text{ in } \triangle_m} \text{fr } Q_m
\]

(1)

Since \( L \bigcup_{Q_m \text{ in } \triangle_m} \text{fr } Q_m = 0 \), it follows that \( LF = 0 \). But \( H(x) = 0 \), \( x \) in \( CF \). Hence \( H(x) = 0 \) a.e. in \( \mathbb{R}^n \). Since \( M \) is admissible, \( M(x, D) \) is finite a.e. in \( \mathbb{R}^n \). Thus \( H(x) M(x, D) = 0 \) a.e. in \( \mathbb{R}^n \) and

\[
\int H(x) M(x, D) = 0
\]

(2)

Since \( LF = 0 \), it follows from II.2.6 that \( D(u, M) = 0 \) a.e. in \( D \cap T^{-1}F \). But \( u \) in \( D - T^{-1}F \) implies \( H(Tu) = 0 \). Hence \( H(Tu) D(u, M) = 0 \) a.e. in \( D \) and consequently

\[
\int_D H(Tu) D(u, M) = 0
\]

(3)

(2) and (3) yield \( H \) in \( \mathcal{N}(D) \).

**Case 2.** For all \( m \), (1) fails to hold.

Since (1) fails to hold for each \( m \), it follows that for each \( m \) there exists at least one (open) \( n \)-cube \( Q_m \) in \( \triangle_m \) such that \( Q_m \cap F \neq \emptyset \). For each \( m \) let \( \left\{ Q_{m_i} \right\}_{i=1}^n \) be those \( n \)-cubes in \( \triangle_m \) which have a non-vacuous intersection with \( F \). For each \( m \) and each
i, i = 1, \ldots, r_m, \quad Q_m^i \cap F \text{ is closed and bounded and hence compact so that the continuous function } H \text{ has a (non-negative) maximum } a_m^i \\
on Q_m^i \cap F. \text{ For each } m \text{ put}
\[ H_m(x) = \sum_{i=1}^{r_m} a_m^i H_{Q_m^i}(x), \quad x \in \mathbb{R}^n. \]

By II.2.19
\[ H_m \text{ in } \mathcal{H}(D), \text{ all } m. \quad (1) \]

Suppose that we have proved
\[ \lim_m H_m(x) = H(x) \text{ a. e. in } F. \quad (2) \]

Since T-ACM in D implies T-BVM in D, and since \( H(x) = 0 \) on \( CF \), it follows readily from II.2.8 that
\[ \lim_m H_m(x) M(x, D) = H(x) M(x, D) \text{ a. e. in } \mathbb{R}^n, \quad (3) \]

and
\[ \lim_m H_m(Tu) D(u, M) = H(Tu) D(u, M) \text{ a. e. in } D. \quad (4) \]

By I.3.6 the grids \( \Delta_m \) are nested so that if \( Q_m \) is in \( \Delta_m \), then \( m_0 > m \) implies that \( \overline{Q_m} \) is a (finite) union of closed n-cubes in \( \Delta_{m_0} \).

Fix \( m \) and \( i, 1 \leq i \leq r_m \). Suppose \( m_0 > m \). Let
\[ \{ Q_m^{i'} \} \] be those n-cubes in \( \Delta_{m_0} \) such that \( \overline{Q_m} = \bigcup_{i'} Q_m^{i'} i'. \)

In case \( Q_m^{i'} \cap F = \emptyset \) for some \( i' \), put \( a_m^{i'} = 0 \). Then since
\[ a_m^i = a_m^{i'} \text{ for all } i', \quad \text{ and since the } Q_m^{i'} \text{ are pairwise disjoint,} \]
\[ a_m^i H_{Q_m^i}(x) \geq \sum_{i'} a_m^{i'} H_{Q_m^{i'}}(x). \]
It now follows readily that

the functions $H_m$ are monotonically non-increasing. \hspace{1cm} (5)\]

From (1)

$H_1$ in $\mathcal{H}(D)$. \hspace{1cm} (6)

In view of (3), (4), (5), and (6) we may apply Lebesgue's theorem on term by term integration (1.2.11) to conclude that

$H(x) M(x,D)$ is $L$-summable over $\mathbb{R}^n$, \hspace{1cm} (7)

$H(Tu) D(u,M)$ is $L$-summable over $D$,

$$\lim_{m} \int H_m(x) M(x,D) = \int H(x) M(x,D)$$ \hspace{1cm} (8)

and

$$\lim_{m} \int_D H_m(Tu) D(u,M) = \int_D H(Tu) D(u,M)$$ \hspace{1cm} (9)

By (1)

$$\int_D H_m(Tu) D(u,M) = \int_D H_m(x) M(x,D), \text{ all } m.$$ \hspace{1cm} (10)

(7), (8), (9), and (10) imply $H$ in $\mathcal{H}(D)$.

It remains to prove (2). Put

$$X = \bigcup_m \bigcup_{\text{fr } Q_m} Q_m \text{ in } \Delta_m,$$

Clearly $LX = 0$. Hence it suffices to show that (2) holds for a fixed point $x$ in $F - X$. Suppose then that $x$ is fixed in $F - X$. For each integer $m$ there exists an integer $i'$, depending also on $x$, such that
x in \( Q_m^{i'} \). Since the \( Q_m^{i'} \) close down on \( x \) as \( m \) tends to \( +\infty \), and since \( H \) is continuous on \( F \), it follows that \( a_m^{i'} \) tends to \( H(x) \) as \( m \) tends to \( +\infty \). Hence

\[
a_m^{i'} \to H(x) \quad \text{as} \quad m \to +\infty .
\] (11)

But for each \( m \), if \( j \neq i' \), then \( H_m^{j}(x) = 0 \). Hence

\[
a_m^{i'} H_m^{i'}(x) = H_m(x).
\] (12)

(11) and (12) imply (2) holds for the fixed point \( x \) in \( F - X \).

**II.2.21. Lemma.** Suppose T-ACM in a domain \( D \) in \( \mathcal{O} \). Let 

\( H = H(x) \) be a real valued, non-negative, L-measurable function on \( \mathbb{R}^n \).

If \( H(T_u) D(u, M) \) is L-summable over \( D \), or if \( H(x) M(x, D) \) is L-summable over \( \mathbb{R}^n \), then \( H \) is in \( \mathcal{H}(D) \).

**Proof.** By Lusin's Theorem (I.2.3) there exists for each positive integer \( m \) a set \( F_m' \) such that \( F_m' \) is closed in \( \mathbb{R}^n \), \( F_m' \subset TD \),

\( L(TD - F_m') < \frac{1}{2^m} \), and \( H \) is continuous on \( F_m' \) relative to \( F_m' \).

For each \( m \) put

\[
F_m = \bigcap_{k > m} F_k' \]

and

\[
H_m(x) = \begin{cases} 
H(x) & \text{if } x \in F_m \\
0 & \text{if } x \in \partial F_m .
\end{cases}
\]

We assert that

(i) \( \{ F_m \} \) is a non-descending sequence.
(ii) $F_m \subseteq TD$ and $F_m$ closed in $\mathbb{R}^n$, all $m$.

(iii) $L(TD - F_m) \leq \frac{1}{2^m}$ and $L(TD - \bigcup F_m) = 0$.

(iv) $\left\{ H_m(x) \right\}$ is a non-descending sequence, $x \in \mathbb{R}^n$.

(v) $H_m(x) \leq H(x)$, all $m$, $x \in \mathbb{R}^n$.

(vi) $H_m(x)$ is continuous on $F_m$ relative to $F_m$, and $H_m(x)$ is identically zero on $CF_m$, all $m$.

We shall prove (iii). The proofs of the remainder follow immediately from the definitions involved. Observe that.

$$TD - F_m = \bigcup_{k > m} (TD - F_k').$$

Hence

$$L(TD - F_m) = L \left( \bigcup_{k > m} (TD - F_k') \right) \leq \sum_{k > m} L(TD - F_k') \leq \sum_{k > m} \frac{1}{2k} = \frac{1}{2^m}.$$

Thus

$$L(TD - F_m) \leq \frac{1}{2^m}. \quad (1)$$

With the help of (i) we see that

$$TD - \bigcup_{m = 1}^j F_m \subseteq TD - F_j.$$

Hence by (1)

$$L(TD - \bigcup_{m = 1}^j F_m) \leq \frac{1}{2j}$$

so that $L(TD - \bigcup_{m = 1}^j F_m) \rightarrow 0$ as $j \rightarrow +\infty$.

Consequently
\[ L(TD - \bigcup_{m} F_{m}) = 0. \]  \hspace{1cm} (2)

(1) and (2) yield (iii).

In view of (ii) and (vi) we may apply II.2.20 to obtain

\[ H_{m} \text{ in } \mathcal{H}(D), \text{ all } m. \]  \hspace{1cm} (3)

Since \( L(TD - \bigcup_{m} F_{m}) = 0 \), it follows with the help of (i) and the definition of \( H_{m}(x) \) that

\[ \lim_{m} H_{m}(x) = H(x) \text{ a.e. in TD.} \]

Hence by II.2.8

\[ \lim_{m} H_{m}(x) M(x, D) = H(x) M(x, D) \text{ a.e. in } \mathbb{R}^{n}. \]  \hspace{1cm} (4)

\[ \lim_{m} H_{m}(Tu) D(u, M) = H(Tu) D(u, M) \text{ a.e. in } D. \]  \hspace{1cm} (5)

Since \( M(x, D) \) is non-negative and \( D(u, M) \) is non-negative wherever it is defined, it follows from (iv) that

\[ \{ H_{m}(x) M(x, D) \} \text{ and } \{ H_{m}(Tu) D(u, M) \} \text{ are non-descending.} \]  \hspace{1cm} (6)

(4), (5), (6) and Lebesgue's theorem on term by term integration (I.2.11) yield

\[ \lim_{m} \int_{D} H_{m}(x) M(x, D) = \int_{D} H(x) M(x, D). \]  \hspace{1cm} (7)

\[ \lim_{m} \int_{D} H_{m}(Tu) D(u, M) = \int_{D} H(Tu) D(u, M). \]  \hspace{1cm} (8)

By (3)

\[ \int_{D} H_{m}(Tu) D(u, M) = \int_{D} H_{m}(x) M(x, D), \text{ all } m. \]  \hspace{1cm} (9)

(7), (8), and (9) yield
It follows that if either of the integrals in (10) is finite, then they are both finite and \( H \) is in \( \mathcal{H}(D) \).

II.2.22. Convention. Let \( H = H(x) \) be a real valued function on \( \mathbb{R}^n \), and let \( H^0(x) \) be the positive part of \( H(x) \) (see I.1.4). Then we may form the functions \( H(Tu) \) and \( H^0(Tu) \) defined for all \( u \) in \( D \). Now let \( G(u) = H(Tu), \) \( u \) in \( D \), and consider \( G^0(u) \). We think of \( G^0(u) \) as the positive part of \( G(u) \), whereas we think of \( H^0(Tu) \) as a function on \( D \) generated by the function \( H^0(x) \) on \( \mathbb{R}^n \). Nevertheless \( G^0(u) = H^0(Tu) \) for all \( u \) in \( D \), and consequently for our purposes it will not be important to distinguish between \( G^0(u) \) and \( H^0(Tu) \). To simplify the notation in the sequel we shall follow the convention of designating \( G^0(u) \) by \( H^0(Tu) \). Similarly we shall designate \( G^0(u) \) by \( H^0(Tu) \).

II.2.23. Theorem. Suppose \( T \)-ACM in a domain \( D \) in \( \mathcal{D} \). Let \( H = H(x) \) be a real valued, \( L \)-measurable function on \( \mathbb{R}^n \). If either \( H(x) M(x, D) \) is \( L \)-summable over \( \mathbb{R}^n \) or \( H(Tu) D(u, M) \) is \( L \)-summable over \( D \), then \( H \) is in \( \mathcal{H}(D) \).

Proof. Throughout the proof we shall follow the preceding convention. Since \( D(u, M) \) is non-negative, the positive and negative parts of \( H(Tu) D(u, M) \) are \( H^0(Tu) D(u, M) \) and \( H^0(Tu) D(u, M) \) respectively. Similarly the positive and negative parts of \( H(x) M(x, D) \) are \( H^0(x) M(x, D) \) and \( H^0(x) M(x, D) \) respectively. Suppose that \( H(x) M(x, D) \) is \( L \)-summable over \( \mathbb{R}^n \). Then by I.2.5 \( H^0(x) M(x, D) \)
and \( H_0(x) M(x, D) \) are \( L \)-summable over \( \mathbb{R}^n \), and by II.2.21

\[
\int_D H^0(Tu) D(u, M) = \int_H^0(x) M(x, D) < +\infty \tag{1}
\]

and

\[
\int_D H_0(Tu) D(u, M) = \int_H_0(x) M(x, D) < +\infty \tag{2}
\]

so that

\[
\int_D (H^0(Tu) - H_0(Tu)) D(u, M) = \int_H^0(x) - H_0(x) M(x, D). \tag{3}
\]

But

\[
H^0(Tu) - H_0(Tu) = H(Tu) \tag{4}
\]

and

\[
H^0(x) - H_0(x) = H(x). \tag{5}
\]

(1), (2), (3), (4), and (5) imply \( H \) in \( \mathcal{H}(D) \).

Suppose now that \( H(Tu) D(u, M) \) is \( L \)-summable over \( D \). Then by I.2.5 \( H^0(Tu) D(u, M) \) and \( H_0(Tu) D(u, M) \) are \( L \)-summable over \( D \), and by II.2.21, (1) and (2) again hold. Hence (3) holds, and as before \( H \) in \( \mathcal{H}(D) \).

**II.2.24 Corollary.** Suppose T-ACM in a domain \( D \) in \( \mathcal{O} \). Let \( H = H(x) \) be a real valued, bounded, \( L \)-measurable function on \( \mathbb{R}^n \). Then \( H \) in \( \mathcal{H}(D) \).

**Proof.** T-ACM in \( D \) implies \( M(x, D) \) is \( L \)-summable over \( \mathbb{R}^n \). Since \( H \) is bounded and \( L \)-measurable, it follows that \( H(x) M(x, D) \) is \( L \)-summable over \( \mathbb{R}^n \). Hence \( H \) in \( \mathcal{H}(D) \) by the preceding theorem.

**Remark.** Since \( M \) is in \( \mathcal{M} \), \( M(x, D) = 0 \) a. d. e. in CTD. Hence in II.2.24, we could have required that \( H \) be bounded only a. d. e. in TD.
II. 3. Bounded Variation and Absolute Continuity with Respect to Admissible Multiplicity Functions

In II.2.1 and II.2.11 we introduced the concepts of bounded variation and absolute continuity with respect to multiplicity functions $M$ in $\mathcal{M}$ (see II.1.4). In this section we extend the concepts of bounded variation and absolute continuity to multiplicity functions $N$ in $\mathcal{N}$ (see II.1.13).

II.3.1. Definition. Let $D$ be in $\mathcal{D}$, let $N$ be in $\mathcal{N}$, and let $(M_1, M_2; Y)$ be a representation of $N$ (see II.1.11). If $T-BVM_i$ in $D$, $i = 1, 2$, we shall say that the representation $(M_1, M_2; Y)$ is BV in $D$. If $T-ACM_i$ in $D$, $i = 1, 2$, we shall say that $(M_1, M_2; Y)$ is AC in $D$.

II.3.2. Lemma. Let $D$ be a domain in $\mathcal{D}$, and let $N$ be in $\mathcal{N}$. Suppose that $N$ has a representation $(M_1, M_2; Y)$ which is BV in $D$. Then $N(x, D)$ is L-summable over $\mathbb{R}^n$, and

$$
\int N(x, D) = \int (M_1(x, D) - M_2(x, D)) = \int M_1(x, D) - \int M_2(x, D). \tag{1}
$$

Proof. Since $(M_1, M_2; Y)$ is BV in $D$, $T-BVM_i$ in $D$, $i = 1, 2$.

Hence by II.2.1

$$M_1(x, D), i = 1, 2, \text{ is L-summable over } \mathbb{R}^n. \tag{2}$$

Since $(M_1, M_2, Y)$ is a representation of $N$,

$$N(x, D) = M_1(x, D) - M_2(x, D) \quad \text{a.e. in } \mathbb{R}^n. \tag{3}$$

(2) and (3) imply that $N(x, D)$ is L-summable over $\mathbb{R}^n$ and that (1) holds.
II.3.3. Definition. Let D be a domain in $\mathcal{D}$, and let N be in $\mathcal{N}$. If N has a representation $(M_1, M_2; Y)$ which is BV in D, then T is said to be of bounded variation in D with respect to N (briefly, T-BVN in D).

The following lemma shows that the definition of bounded variation in II.3.3 is compatible with the definition of bounded variation in II.2.1.

II.3.4. Lemma. Let D be in $\mathcal{D}$, and let M be in $\mathcal{M}$. Then T-BVM in D in the sense of definition II.2.1 if and only if T-BVM in D in the sense of definition II.3.3.

Proof. Suppose T-BVM in D in the sense of definition II.2.1. Since M is in $\mathcal{M}$, M is admissible in the restricted sense. Hence by II.3.1 there is an exceptional set $X \subseteq \mathbb{R}^n$ for M. It follows that $(M, 0; X)$ may be taken as a representation of M (see II.1.6 in regard to O). Since $O(x, D)$ is obviously L-summable over $\mathbb{R}^n$, it follows that T-BVO in D in the sense of definition II.2.1. By hypothesis T-BVM in D in the sense of definition II.2.1. Hence the representation $(M, 0; X)$ of M is BV in D, and consequently T-BVM in D in the sense of definition II.3.3. On the other hand if T-BVM in D in the sense of definition II.3.3, then $M(x, D)$ is L-summable over $\mathbb{R}^n$ by II.3.2, and consequently T-BVM in D in the sense of definition II.2.1.

II.3.5. Lemma. Let N be in $\mathcal{N}$. If T-BVN in a domain $D_0$ in $\mathcal{D}$, then T-BVN in every domain $D \subseteq D_0$.

Proof. Let $(M_1, M_2; Y)$ be a representation of N such that $(M_1, M_2; Y)$ is BV in $D_0$. Then by II.3.1, T-BVM in $D_0$, $i = 1, 2$, and hence by II.2.2 T-BVM in every domain $D \subseteq D_0$, $i = 1, 2$. Hence
if \( D \subset D_0 \) then \((M_1, M_2; Y)\) is a BV in \( D \) representation of \( N \), and therefore T-BVN in \( D \).

Let \( N \) be in \( \mathcal{N} \). Suppose T-BVN in a domain \( D_0 \) in \( \mathcal{D} \). By II.3.2 and II.3.5 it follows that \( N(x, D) \) is \( L \)-summable over \( \mathbb{R}^n \), all \( D \subset D_0 \). Hence \( \int N(x, D) \) is a function of domains \( D \subset D_0 \), and if \( u \) in \( D_0 \), we may consider the existence of the derivative \( D(u, N) \) of \( \int N(x, D) \) at the point \( u \) (see I.3.10).

**II.3.6. Lemma.** Let \( N \) be in \( \mathcal{N} \). If T-BVN in a domain \( D \) in \( \mathcal{D} \), then

1. \( D(u, N) \) exists a.e. in \( D \).
2. \( D(u, N) \) is \( L \)-measurable and \( L \)-summable over \( D \).

Furthermore if \((M_1, M_2; Y)\) is a BV in \( D \) representation of \( N \), then

3. \( D(u, N) = D(u, M_1) - D(u, M_2) \) a.e. in \( D \)
4. \( \int_D D(u, N) = \int_D D(u, M_1) - \int_D D(u, M_2) \).

**Proof.** Let \((M_1, M_2; Y)\) be a representation of \( N \) such that \((M_1, M_2; Y)\) is BV in \( D \). Then by II.2.3 \( D(u, M_i), i = 1, 2 \), exists a.e. in \( D \) and is \( L \)-summable over \( D \). In view of I.3.12, the lemma now follows.

**II.3.7. Lemma.** Let \( D \) be a domain in \( \mathcal{D} \), and let \( N \) be in \( \mathcal{N} \). Suppose that \((M_1, M_2; Y)\) is a representation of \( N \) and that \((M_1, M_2; Y)\) is AC in \( D \). Then \((M_1, M_2; Y)\) is BV in \( D \).

**Proof.** By II.2.11 T-ACM\(_i\) in \( D \) implies T-BVM\(_i\) in \( D \), \( i = 1, 2 \). In view of definition II.3.1 the lemma follows.

**II.3.8. Lemma.** Let \( D \) be a domain in \( \mathcal{D} \) and let \( N \) be in \( \mathcal{N} \). Suppose that \( N \) has a representation \((M_1, M_2; Y)\) which is AC in \( D \). Then \( D(u, N) \) is \( L \)-summable over \( D \), \( N(x, D) \) is \( L \)-summable over \( \mathbb{R}^n \),
and
\[ \int_D D(u, N) = \int N(x, D). \]

**Proof.** By the preceding lemma \((M_1, M_2; Y)\) is BV in \(D\). Hence by II.3.2
\[ N(x, D) \text{ is } L\text{-summable over } \mathbb{R}^n \quad (1) \]

and
\[ \int N(x, D) = \int M_1(x, D) - \int M_2(x, D) \quad (2) \]

By II.3.6
\[ D(u, N) \text{ is } L\text{-summable over } D, \quad (3) \]

and
\[ \int_D D(u, N) = \int_D D(u, M_1) - \int_D D(u, M_2) \quad (4) \]

T-ACM\(_i\) in \(D\) implies
\[ \int_D D(u, M_1) = \int M_1(x, D), \quad i = 1, 2. \quad (5) \]

(2), (4), and (5) yield
\[ \int_D D(u, N) = \int N(x, D). \quad (6) \]

(1), (3) and (6) yield the lemma.

**II.3.9. Definition.** Let \(D\) be a domain in \(\mathscr{D}\) and let \(N\) be in \(\mathcal{M}\). If \(N\) has a representation \((M_1, M_2; Y)\) which is AC in \(D\), then \(T\) is said to be absolutely continuous in \(D\) with respect to \(N\) (briefly, T-ACN in \(D\)).

To justify this definition of absolute continuity we must show that it is compatible with the definition of absolute continuity in II.2.11.
**II.3.10. Lemma.** Let $D$ be a domain in $\mathcal{D}$ and let $M$ be in $M$. Then $T$-ACM in $D$ in the sense of definition II.2.11 if and only if $T$-ACM in $D$ in the sense of definition II.3.9.

**Proof.** Suppose $T$-ACM in $D$ in the sense of II.2.11. Since $M$ is in $M$, $M$ is admissible in the restricted sense. Hence there exists an exceptional set $X \subseteq \mathbb{R}^n$ for $M$. It follows that $(M, 0; X)$ is a representation of $M$ (see II.1.6 with regard to 0). Clearly $T$-ACO in $D$, in the sense of II.2.11. By hypothesis $T$-ACM in $D$ in the sense of II.2.11. Hence $(M, 0; X)$ is AC in $D$, and consequently $T$-ACM in $D$ in the sense of II.3.9. Suppose on the other hand that $T$-ACM in $D$ in the sense of II.3.9. Then $M$ has a representation $(M_1, M_2; Y)$ which is AC in $D$, and thus by II.3.8 $D(u, M)$ is $L$-summable over $D$ and

$$\int_D D(u, M) = \int M(x, D).$$

Hence $T$-ACM in $D$ in the sense of II.2.11.

**Henceforth in section II.3, $N$ will be thought of as fixed in $M$.**

**II.3.11. Lemma.** If $T$-ACN in a domain $D$ in $\mathcal{D}$, then $T$-BVN in $D$.

**Proof.** This is a corollary of II.3.7 and II.3.9.

**II.3.12. Lemma.** If $T$-ACN in a domain $D$ in $\mathcal{D}$, then $D(u, N)$ is $L$-summable over $D$ and

$$\int_D D(u, N) = \int N(x, D).$$

**Proof.** This is a corollary of II.3.8 and II.3.9.
II.3.13. Lemma. If T-ACN in a domain $D_0$ in $\mathscr{D}$, then T-ACN
in every domain $D \subset D_0$.

Proof. Let $(M_1, M_2; Y)$ be a representation of $N$ such that
$(M_1, M_2; Y)$ is AC in $D_0$. Then T-ACM$_1$ in $D_0$, $i = 1, 2$. Since $M_i$ is
in $\mathcal{M}$, $i = 1, 2$, it follows from II.2.13 that T-ACM$_i$ in every
domain $D \subset D_0$, $i = 1, 2$. Hence if $D \subset D_0$, then $(M_1, M_2; Y)$ is
AC in $D$, and the lemma follows.

II.3.14. Lemma. If T-BVN in a domain $D$ in $\mathscr{D}$, and if $H(x)$
is a real valued $L$-measurable function on $\mathbb{R}^n$, then

$$H(x) \ N(x, D) \text{ is an } L\text{-measurable function on } \mathbb{R}^n, \quad (1)$$

and

$$H(Tu) \ D(u, N) \text{ is an } L\text{-measurable function on } D. \quad (2)$$

Proof. Let $(M_1, M_2; Y)$ be a representation of $N$ such that
$(M_1, M_2; Y)$ is BV in $D$. Then $M_1$ and $M_2$ are in $\mathcal{M}$ and T-BVM$_1$ in $D$.
Hence by II.2.10 $H(x) M_1(x, D)$ is L-measurable over $\mathbb{R}^n$ and
$H(Tu) D(u, M_1)$ is L-measurable over $D$, $i = 1, 2$. Hence

$$H(x) (M_1(x, D) - M_2(x, D)) \text{ is L-measurable over } \mathbb{R}^n, \quad (3)$$

and

$$H(Tu) (D(u, M_1) - D(u, M_2)) \text{ is L-measurable over } D. \quad (4)$$

Since $(M_1, M_2; Y)$ is a representation of $N$,

$$N(x, D) = M_1(x, D) - M_2(x, D) \text{ a.e. in } \mathbb{R}^n. \quad (5)$$

By II.3.6

$$D(u, N) = D(u, M_1) - D(u, M_2) \text{ a.e. in } D. \quad (6)$$

(3), (4), (5), and (6) yield (1) and (2).
In the future when T-ACN in a domain D in $\mathcal{D}$ (and hence T-BVN in D) we shall consider L-integrals of $H(x) N(x, D)$ and $H(Tu) D(u, N)$, and we shall generally not mention that the measurability conditions necessary for their consideration are satisfied.

II.3.15. Lemma. Let D be a domain in $\mathcal{D}$, and let $H = H(x)$ be a real valued, bounded, L-measurable function on $\mathbb{R}^n$. If T-ACN in D, then $H(x) N(x, D)$ is L-summable over $\mathbb{R}^n$, $H(Tu) D(u, N)$ is L-summable over D, and

$$\int_D H(Tu) D(u, N) = \int H(x) N(x, D).$$

(1)

Proof. Let $(M_1, M_2, Y)$ be a representation of N such that $(M_1, M_2; Y)$ is AC in D. Then

$$N(x, D) = M_1(x, D) - M_2(x, D) \quad a.e. \text{ in } \mathbb{R}^n,$$

(2)

and by II.3.6 and II.3.11

$$D(u, N) = D(u, M_1) - D(u, M_2) \quad a.e. \text{ in } D.$$

(3)

Since T-ACM in D, and since H is bounded, it follows with the help of II.2.11 that $H(x) M_i(x, D)$ is L-summable over $\mathbb{R}^n$ and $H(Tu) D(u, M_i)$ is L-summable over D, $i = 1, 2$. Hence

$$H(x) (M_1(x, D) - M_2(x, D)) \text{ is L-summable over } \mathbb{R}^n,$$

(4)

and

$$H(Tu) (D(u, M_1) - D(u, M_2)) \text{ is L-summable over } D.$$

(5)

From (2), (3), (4), and (5) it follows that $H(x) N(x, D)$ is L-summable over $\mathbb{R}^n$ and $H(Tu) D(u, N)$ is L-summable over D. It remains to prove (1). Since T-ACM in D, $i = 1, 2$, and since H is bounded it follows from II.2.24 that
\[ \int_D H(Tu) \, D(u, M_i) = \int H(x) \, M_i(x, D) = \pm \infty \quad i = 1, 2. \]

Hence on subtracting
\[ \int_D H(Tu) \, (D(u, M_1) - D(u, M_2)) = \int H(x) \, (M_1(x, D) - M_2(x, D)) \quad (6) \]

(2), (3), and (6) yield (1).

**II.3.16. Lemma.** Suppose T-BVN in a domain D in \( \mathcal{D} \). Let \( H(x) \) and \( H_m(x), m = 1, 2, \ldots, \) be real valued functions on \( \mathbb{R}^n \). If

\[ \lim_m H_m(x) = H(x) \quad \text{a.e. in TD,} \]

then

\[ \lim_m H_m(x) \, N(x, D) = H(x) \, N(x, D) \quad \text{a.e. in } \mathbb{R}^n, \quad (1) \]

and

\[ \lim_m H_m(Tu) \, D(u, N) = H(Tu) \, D(u, N) \quad \text{a.e. in } D. \quad (2) \]

**Proof.** Let \((M_1, M_2; Y)\) be a representation of \( N \) such that \((M_1, M_2; Y)\) is BV in \( D \). Then T-BV in \( D \), and

\[ D(u, M_i) \quad \text{exists a.e. in } D, \quad i = 1, 2. \quad (3) \]

By II.3.6

\[ D(u, N) = D(u, M_1) - D(u, M_2) \quad \text{a.e. in } D. \quad (4) \]

By II.2.8

\[ \lim_m H_m(x) \, M_i(x, D) = H(x) \, M_i(x, D) \quad \text{a.e. in } \mathbb{R}^n, \quad i = 1, 2. \quad (5) \]

and

\[ \lim_m H_m(Tu) \, D(u, M_i) = H(Tu) \, D(u, M_i) \quad \text{a.e. in } D, \quad i = 1, 2. \quad (6) \]

Since \((M_1, M_2; Y)\) is a representation of \( N \),
\( M_1(x, D) \) is finite a.e. in \( \mathbb{R}^n \), \( i = 1, 2 \). (7)

and

\[ N(x, D) = M_1(x, D) - M_2(x, D) \text{ a.e. in } \mathbb{R}^n. \] (8)

(3), (5), (6) and (7) yield

\[ \lim_m H_m(x) (M_1(x, D) - M_2(x, D)) = H(x) (M_1(x, D) - M_2(x, D)) \text{ a.e. in } \mathbb{R}^n \] (9)

and

\[ \lim_m H_m(Tu) (D(u, M_1) - D(u, M_2)) = H(Tu) (D(u, M_1) - D(u, M_2)) \text{ a.e. in } D. \] (10)

(4), (8), (9), and (10) yield (1) and (2).

**II.3.17 Lemma.** Let \( D \) be a domain in \( \mathcal{D} \), and let \( H = H(x) \) be a real valued, non-negative, L-measurable function on \( \mathbb{R}^n \). Suppose \( T \)-ACN in \( D \), and that \( H(Tu) D(u, N) \) is L-summable over \( D \), then

\[ H(x) N(x, D) \text{ is L-summable over } \mathbb{R}^n, \] (1)

and

\[ \int_D H(Tu) D(u, N) = \int H(x) N(x, D). \] (2)

**Proof.** Let \( (M_1, M_2; Y) \) be a representation of \( N \) such that \( (M_1, M_2; Y) \) is AC in \( D \). Let

\[ E_1 = \{ x / x \text{ in } \mathbb{R}^n \text{ and } M_1(x, D) \geq M_2(x, D) \} \] (3)

\[ E_2 = \{ x / x \text{ in } \mathbb{R}^n \text{ and } M_2(x, D) > M_1(x, D) \} \] (4)

Then \( E_1 \) and \( E_2 \) are L-measurable and

\[ E_1 \cap E_2 = \emptyset, \ E_1 \cup E_2 = \mathbb{R}^n \] (5)

\[ T^{-1}E_1 \cap T^{-1}E_2 \cap D = \emptyset, \ (T^{-1}E_1 \cup T^{-1}E_2) \cap D = D \] (6)
Observe also that
\[ N(x, D) \geq 0 \text{ a.e. on } E_1 \]  
\[ N(x, D) \leq 0 \text{ a.e. on } E_2 \]  

Put
\[ H^j(x) = \begin{cases} H(x) & \text{if } x \in E_j \\ 0 & \text{if } x \in C E_j \end{cases}, \quad j = 1, 2. \]

It follows readily with the help of (5) and (6) that
\[ H^1(x) + H^2(x) = H(x), \quad x \in \mathbb{R}^n \]  
\[ H^1(Tu) + H^2(Tu) = H(Tu), \quad u \in D. \]

Since \( E_j \) and \( H(x) \) are \( L \)-measurable, so is \( H^j(x) \), \( j = 1, 2. \)

Since \( H^j(Tu) \geq 0; j = 1, 2, \) and \( H(Tu) ) D(u, N) \) is \( L \)-summable over \( D, \) it follows by I.2.12 that
\[ H^j(Tu) D(u, N) \text{ is } L \text{-summable over } D, \quad j = 1, 2. \]  

(11) and I.2.5 yield
\[ \| H^j(Tu) D(u, N) \| \text{ is } L \text{-summable over } D, \quad j = 1, 2. \]  

Since \( H(x) \geq 0, \) it follows from (3), (4), (7), and (8) that
\[ H^1(x) N(x, D) \geq 0 \text{ and } H^2(x) N(x, D) \leq 0 \text{ a.e. in } \mathbb{R}^n. \]

For each positive integer \( m \) and each \( x \) in \( \mathbb{R}^n \) put
\[ H^j_m = \begin{cases} H^j(x) & \text{if } H^j(x) < m \\ m & \text{if } H^j(x) \geq m \end{cases}, \quad j = 1, 2. \]

We observe that
(i) $H^j_m(x)$ is $L$-measurable and bounded, $j = 1, 2; m = 1, 2, \ldots$.

(ii) $0 \leq H^j_m(x) \leq H^j(x), j = 1, 2; m = 1, 2, \ldots$.

(iii) $|H^j_m(Tu) D(u, N)| = |H^j_m(Tu) / D(u, N)| \\
\leq |H^j(Tu) / D(u, N)| = |H^j(Tu) D(u, N)| \text{ a.e. in } D, \\
j = 1, 2; m = 1, 2, \ldots$.

(iv) $\lim_{m} H^j_m(x) = H^j(x), j = 1, 2; x \in \mathbb{R}^n$.

(v) $\lim_{m} H^j_m(x) N(x, D) = H^j(x) N(x, D) \text{ a.e. in } \mathbb{R}^n, \\
j = 1, 2 \text{ (see II.3.16)}.$

(vi) $\lim_{m} H^j_m(Tu) D(u, N) = H^j(Tu) D(u, N) \text{ a.e. in } D, \\
j = 1, 2 \text{ (see II.3.16)}.$

(vii) $H^1_m(x) N(x, D) \geq 0$ and $H^2_m(x) N(x, D) \leq 0 \\
a.e. in \mathbb{R}^n, m = 1, 2, \ldots \text{ (see 13 and 14)}.$

(viii) $\left\{ H^1_m(x) N(x, D) \right\}_m$ is a non-descending sequence \\
a.e. in $\mathbb{R}^n$.

(ix) $\left\{ -H^2_m(x) N(x, D) \right\}_m$ is a non-descending sequence \\
a.e. in $\mathbb{R}^n$.

(i) and II.3.15 yield for all $m$ and $j = 1, 2$

\[
\int_D H^j_m(Tu) D(u, N) \text{ is } L\text{-summable over } D \\
\int_D H^j_m(x) N(x, D) \text{ is } L\text{-summable over } \mathbb{R}^n
\]

and

\[
\int_D H^j_m(Tu) D(u, N) = \int_D H^j_m(x) N(x, D). (16)
\]

It follows from (11), (12), (15), (iii), and (vi) that we may apply I.2.11 to obtain
(16) and (17) yield
\[ \int_D H^j(Tu) D(u, N) = \lim_m \int H^j_m(x) N(x, D), j = 1, 2. \] (18)

(v), (vii), (viii), (ix), and Lebesgue's theorem on integration of monotone sequences of functions (I.2.10) yield
\[ \lim_m \int H^j_m(x) N(x, D) = \int H^j(x) N(x, D), j = 1, 2. \] (19)

(18) and (19) yield
\[ \int_D H^j(Tu) D(u, N) = \int H^j(x) N(x, D), j = 1, 2. \] (20)

(11) and (20) imply that
\[ H^j(x) N(x, D) \text{ is } L\text{-summable over } R^n, j = 1, 2. \] (21)

(9) and (21) yield (1). (9), (10) and (20) yield (2).

II.3.18. Theorem. Let D be a domain in \( \mathcal{D} \), and let \( H = H(x) \) be a real valued, L-measurable function on \( R^n \). Suppose that T-ACN in D, and that \( H(Tu) D(u, N) \text{ is } L\text{-summable over } D \). Then
\[ H(x) N(x, D) \text{ is } L\text{-summable over } R^n, \] (1)
and
\[ \int_D H(Tu) D(u, N) = \int H(x) N(x, D). \] (2)

Proof. We shall follow the convention in II.2.22 throughout the proof. Let \( H^0(x) \) and \( H_0(x) \) be the positive and negative parts respectively of \( H(x) \). Since \( H(Tu) D(u, N) \text{ is } L\text{-summable over } D \), it follows from I.2.13 that \( H^0(Tu) D(u, N) \) and \( H_0(Tu) D(u, N) \) are L-summable over D. Since \( H^0(x) \geq 0 \) and \( H_0(x) \geq 0 \), II.3.17 yields
and $H^0(x) N(x, D)$ and $H_0(x) N(x, D)$ are L-summable over $\mathbb{R}^n$, \[ \int_D H^0(Tu) D(u, N) = \int H^0(x) N(x, D), \tag{4} \]

and \[ \int_D H_0(Tu) D(u, N) = \int H_0(x) N(x, D). \tag{5} \]

Since $H(x) = H^0(x) - H_0(x)$ and $H(Tu) = H^0(Tu) - H_0(Tu)$, (1) and (2) follow from (3), (4), and (5).
CHAPTER III

SOME SPECIAL MULTIPLICITY FUNCTIONS

III.1. Multiplicity Functions Generated by Index Functions

Throughout Chapter III, unless otherwise indicated, $D$ will be a fixed, bounded domain in Euclidean $n$-space $\mathbb{R}^n$, $n$ fixed, and $T$ will be a fixed, bounded continuous transformation from $D$ into $\mathbb{R}^n$. $x$ will be generic for a point in $\mathbb{R}^n$, and $u$ will be generic for a point in $D$. $D$ will designate the set of all domains $D \subset D$.

III.1.1. Definition. A pair $(x, D)$, $x$ in $\mathbb{R}^n$ and $D$ in $\mathcal{D}$, is said to be admissible with respect to $T$ provided $\overline{D} \subset D$ and $x$ is in $CTfrD$. Since $T$ is fixed, we shall generally omit the phrase "with respect to $T$.”

III.1.2. Definition. An index function, $v = v(x, D)$, with respect to $T$ is a real valued function defined on the set of all admissible pairs. The set of all index functions with respect to $T$ will be designated by $\mathcal{V}$.

Clearly $\mathcal{V}$ is a linear class. In particular if $v$ is in $\mathcal{V}$, then $-v$ is in $\mathcal{V}$.

III.1.3. Definition. Let $x$ be in $\mathbb{R}^n$ and $D$ in $\mathcal{D}$. Then $D'$ is said to be an indicator domain for $(x, D)$ with respect to an index function $v$ in $\mathcal{V}$ if

(i) $\overline{D'} \subset D$
(ii) \((x, D')\) is admissible 
(iii) \(v(x, D') \neq 0\).

If \(v(x, D') > 0\) \((< 0)\), \(D'\) is said to be a positive (negative) indicator domain for \((x, D)\) with respect to \(v\).

Note that the pair \((x, D)\) need not be admissible since we do not require that \(\overline{D}\) be contained in \(D\) or that \(x\) be in \(\text{CTfrD}\).

III.1.4. Definition. Let \(x\) be in \(\mathbb{R}^n\) and \(D\) in \(\mathcal{D}\). An indicator class for \((x, D)\) with respect to an index function \(v\) in \(\mathcal{V}\) is a finite or empty class of pairwise disjoint indicator domains \(D'\) for \((x, D)\) with respect to \(v\). The general notation for an indicator class for \((x, D)\) with respect to \(v\) will be \(\mathcal{O}(x, D, v)\), or in case no confusion can arise, simply \(\mathcal{O}(x, D)\). We designate by \(+ \mathcal{O}(x, D)\) \((- \mathcal{O}(x, D)\) ) the possibly empty class consisting of all positive (negative) indicator domains in \(\mathcal{O}(x, D)\). We designate by \(\mathcal{O}^+(x, D)\) \((\mathcal{O}^-(x, D)\) ) a possibly empty indicator class in which all the indicator domains are positive (negative).

Clearly \(+ \mathcal{O}(x, D)\) and \(- \mathcal{O}(x, D)\) are indicator classes.

III.1.5. Definition. Let \(x\) be in \(\mathbb{R}^n\) and \(D\) in \(\mathcal{D}\). An indicator system for \((x, D)\) with respect to \(v\) in \(\mathcal{V}\) is a collection \(S\) of indicator classes \(\mathcal{O}(x, D)\) with respect to \(v\) such that

(i) The null set \(\emptyset\) is in \(S\)
(ii) If \(\mathcal{O}(x, D)\) is in \(S\), then \(+ \mathcal{O}(x, D)\) and \(- \mathcal{O}(x, D)\) are in \(S\).

(iii) If \(\mathcal{O}_1(x, D)\) and \(\mathcal{O}_2(x, D)\) in \(S\), and if \(\mathcal{O}_1(x, D) \cup \mathcal{O}_2(x, D)\) is an indicator class \(\mathcal{O}(x, D)\).
with respect to \( v \), then \( \sigma(x, D) \) is in \( S \).

The general notation for an indicator system for \( (x, D) \) with respect to \( v \) will be \( S(x, D, v) \), or in case no confusion can arise, simply \( S(x, D) \).

To simplify the notation in the sequel we shall generally write \( \sigma \) in \( S(x, D) \) in place of \( \sigma(x, D) \) in \( S(x, D) \).

Observe that if \( \sigma(x, D) \) is in \( S(x, D) \), then \(-\sigma(x, D) = \emptyset \) and \(+\sigma(x, D) = \emptyset \), so that by (ii), \( \emptyset \) is in \( S(x, D) \). Nevertheless (i) is necessary to ensure that \( S(x, D) \) contains at least one set \( \sigma(x, D) \).

III.1.6. Definition. Let \( v \) be in \( \mathcal{V} \). For each \( x \) in \( \mathbb{R}^n \) and \( D \) in \( \mathcal{O} \), let \( S(x, D) \) be an indicator system with respect to \( v \), and let

\[
M(S(x, D)) = \sup_{\sigma \in S(x, D)} \sum_{D' \in \sigma} |v(x, D')|
\]

where

\[
\sum_{D' \in \sigma} |v(x, D')| = 0 \quad \text{in case } \sigma = \emptyset \quad (1)
\]

Since \( \emptyset \) is in \( S(x, D) \), it follows from (1) that \( M(S(x, D)) \) is defined for all \( x \) in \( \mathbb{R}^n \) and all \( D \) in \( \mathcal{O} \). Hence \( M = M(S(x, D)) \) is a (non-negative) multiplicity function.

In the remainder of this chapter, unless otherwise indicated, \( v \) will be fixed in \( \mathcal{V} \).

III.1.7. Lemma. Let \( x \) be in \( \mathbb{R}^n \) and \( D \) in \( \mathcal{O} \). If \( S_1(x, D) \) and \( S_2(x, D) \) are indicator systems such that \( S_1(x, D) \subset S_2(x, D) \), then

\[
M(S_1(x, D)) \leq M(S_2(x, D))
\]

Proof. III.1.7 follows from the more general result proved in
III.1.8. Lemma. Let $x \in \mathbb{R}^n$ and $D$ in $\mathcal{D}$, and let $S(x, D)$ be an indicator system. Suppose

(i) $\{D_i\}$ is a finite collection of pairwise disjoint domains in $D$.

(ii) For each $i$, $S(x, D_i)$ is an indicator system such that $S(x, D_i) \subseteq S(x, D)$.

Then

$$\sum_{i} M(S(x, D_i)) \leq M(S(x, D)).$$

(1)

Proof. For each $i$ let $\sigma^*(x, D_i)$ be an arbitrary indicator class in $S(x, D_i)$. Since the domains $D_i$ are pairwise disjoint and contained in $D$, $\bigcup_i \sigma^*(x, D_i)$ is an indicator class $\sigma(x, D)$. By (ii) each $\sigma^*(x, D_i)$ is in $S(x, D)$. Hence $\sigma(x, D)$ is in $S(x, D)$ by III.1.5.(iii), and

$$\sum_{i} \sum_{D' \in \sigma^*(x, D_i)} |v(x, D')| = \sum_{D' \in \sigma(x, D)} |v(x, D')| \leq M(S(x, D)).$$

(2)

Since the $\sigma^*(x, D_i)$ were arbitrary in $S(x, D_i)$, we obtain with the help of (2)

$$\sum_{i} \sup_{\sigma \in S(x, D_i)} \sum_{D' \in \sigma} |v(x, D')| \leq M(S(x, D)),$$

and (1) follows.

III.1.9. Remark. It is clear that condition (i) in lemma III.1.8 could be replaced by

(i) $\{D_i\}$ is a countable collection of pairwise disjoint domains in $D$. 

**III.1.10. Definition.** Let \( x \) be in \( \mathbb{R}^n \), \( D \) in \( \mathcal{D} \), and \( D_1 \subset D \).

Let \( S(x, D) \) be an indicator system. We shall designate by \( S(x, D) | D_1 \)
the indicator system for \((x, D_1)\) consisting of all indicator classes
\( \sigma(x, D) \) in \( S(x, D) \) which are at the same time indicator classes
\( \sigma(x, D_1) \).

**Remarks.** Since \( \emptyset \) in \( S(x, D) \) and since \( \emptyset \) is a class \( \sigma(x, D) \),
we must have \( \emptyset \) in \( S(x, D) | D_1 \). Hence condition (i) of III.1.5 holds.
It is equally easy to see that (ii) and (iii) of III.1.5 hold. Hence
\( S(x, D) | D_1 \) is an indicator system and indeed an indicator system
for \((x, D_1)\) as well as for \((x, D)\).

Observe too that if \( D_2 \subset D_1 \), then clearly
\( (S(x, D) | D_1) | D_2 = S(x, D) | D_2 \) and \( S(x, D) | D_2 \subset S(x, D) | D_1 \).

Finally, we note that if \( \sigma \) is in \( S(x, D) \), then \( \sigma \) is in
\( S(x, D) | D_1 \) if and only if for each \( D' \) in \( \sigma \) we have \( \overline{D'} \subset D_1 \).

**III.1.11. Lemma.** Let \( x \) be in \( \mathbb{R}^n \) and \( D \) in \( \mathcal{D} \). Let \( S(x, D) \)
be an indicator system. If \( \{D_j\} \) is a sequence of domains filling
up \( D \) from the interior, then
\[
\lim_j M( S(x, D) | D_j ) = M( S(x, D) ). \tag{1}
\]

**Proof.** Since for each \( j \) \( S(x, D) | D_j \subset S(x, D) | D_j + 1 \), it
follows from III.1.7 that the sequence \( \{M( S(x, D) | D_j ) \}_j \) is
non-descending, and hence it has a (possibly infinite) limit. Since
\( S(x, D) | D_j \subset S(x, D) \), it follows again from III.1.7 that
\( M( S(x, D) | D_j ) \leq M( S(x, D) ) \), all \( j \). Hence
\[
\lim_j M( S(x, D) | D_j ) \leq M( S(x, D) ). \tag{2}
\]

Let \( a \) be any real number such that
a < M(S(x, D)). \tag{3}

Then there exists in S(x, D) an indicator class \( \sigma \), depending on \( a \), such that

\[
\sum_{D' \in \sigma} |v(x, D')| > a \tag{4}
\]

Let

\[
F = \bigcup_{D' \in \sigma} D' \tag{5}
\]

Evidently \( F \) is compact and contained in \( D \). Since the \( D_j \) fill up \( D \) from the interior, there exists an integer \( j' \) such that

\[
F \subseteq D_j, \quad j > j'. \tag{6}
\]

From (5) and (6) it follows that if \( D' \in \sigma \) then \( D' \subseteq D_j, \quad j > j' \). Hence by definition of \( S(x, D)|D_j, \quad \sigma \) is in \( S(x, D)|D_j \) whenever \( j > j' \). Thus

\[
M(S(x, D)|D_j) \geq \sum_{D' \in \sigma} |v(x, D')|, \quad j > j' \tag{7}
\]

(4) and (7) yield

\[
\lim_{j} M(S(x, D)|D_j) > a \tag{8}
\]

Since \( a \) was an arbitrary real number satisfying (3), (8) yields

\[
\lim_{j} M(S(x, D)|D_j) \geq M(S(x, D)) \tag{9}
\]

(2) and (9) yield (1).

III.1.12. Corollary. Let \( x \) be in \( \mathbb{R}^n \) and \( D \) in \( \mathcal{D} \). Let \( S(x, D) \) be an indicator system. If \( a \) is a real number such that \( a < M(S(x, D)) \), then there exists a compact set \( F \), depending on
a, such that $F \subset D$, and if $D_0$ is any domain with the property that $F \subset D_0 \subset D$, then

$$M(\mathcal{S}(x, D) \mid D_0) > a.$$  \hspace{1cm} (1)

**Proof.** Let $\{D_j\}$ be a sequence of domains filling up $D$ from the interior. By III.1.11 there exists an integer $j'$, depending on $a$, such that

$$M(\mathcal{S}(x, D) \mid D_{j'}) > a.$$  \hspace{1cm} (2)

Put $F = D_{j'}$. Suppose now that $F \subset D_0 \subset D$. Then

$$\mathcal{S}(x, D) \mid D_{j'} \subset \mathcal{S}(x, D) \mid D_0,$$  and by III.1.7

$$M(\mathcal{S}(x, D) \mid D_0) \geq M(\mathcal{S}(x, D) \mid D_{j'})$$  \hspace{1cm} (3)

and (2) and (3) yield (1).

**III.1.13. Definition.** Let $x$ be in $\mathbb{R}^n$, $D$ in $\mathcal{D}$, and $\mathcal{S}(x, D)$ an indicator system. We shall designate by $+\mathcal{S}(x, D)$ ( $-\mathcal{S}(x, D)$ ) the indicator system consisting of all indicator classes $+\mathcal{C}(x, D)$ ( $-\mathcal{C}(x, D)$ ) where $\mathcal{C}(x, D)$ is $\mathcal{S}(x, D)$.

If we recall that an indicator class of any of the forms $\mathcal{C}^{-}(x, D)$, $\mathcal{C}^{+}(x, D)$, $\mathcal{C}^{\ast}(x, D)$ may be empty, it is easy to check that $+\mathcal{S}(x, D)$ is indeed an indicator system, that it is contained in $\mathcal{S}(x, D)$, and that it consists of all indicator classes of the form $\mathcal{C}^{+}(x, D)$ which are in $\mathcal{S}(x, D)$. The corresponding remark holds for $-\mathcal{S}(x, D)$.

Since $+\mathcal{S}(x, D)$ and $-\mathcal{S}(x, D)$ are contained in $\mathcal{S}(x, D)$, the following lemma is a consequence of III.1.7.

**III.1.14. Lemma.** If $x$ is in $\mathbb{R}^n$, $D$ in $\mathcal{D}$, and $\mathcal{S}(x, D)$ an indicator system, then
M( S(x, D) ) ≥ M(+S(x, D) ) \quad (1)
M( S(x, D) ) ≥ M(-S(x, D) ) \quad (2)

**III.1.15. Lemma.** If $x$ is in $\mathbb{R}^n$, $D$ in $\mathcal{D}$, and $S(x, D)$ an indicator system, then

$$M( S(x, D) ) \leq M(+S(x, D) ) + M(-S(x, D) ) . \quad (1)$$

**Proof.** Let $S_1$ be an arbitrary element in $S(x, D)$. Then

$$\sum_{D' \in S_1} |v(x, D')| = \sum_{D' \in +S_1} v(x, D') + \sum_{D' \in -S_1} |v(x, D')|$$

$$\leq \sup_{S \in S(x, D)} \sum_{D' \in +S} v(x, D') + \sup_{S \in S(x, D)} \sum_{D' \in -S} |v(x, D')|$$

$$= M(+S(x, D) ) + M(-S(x, D) ) .$$

Since $S_1$ was arbitrary in $S(x, D)$

$$\sup_{S \in S(x, D)} \sum_{D' \in S} |v(x, D')| \leq M(+S(x, D) ) + M(-S(x, D) ),$$

and (1) follows.

**III.1.16. Lemma.** If $x$ in $\mathbb{R}^n$, $D$ in $\mathcal{D}$, and $S(x, D)$ an indicator system, then

(i) $M(S(x, D) ) = +\infty$ if and only if $M(+S(x, D) ) = +\infty$ or $M(-S(x, D) ) = +\infty$ .

(ii) $M(S(x, D) ) = 0$ if and only if $M(+S(x, D) ) = 0$ and $M(-S(x, D) ) = 0$ .

**Proof.** This is a corollary of III.1.14 and III.1.15.

**III.1.17. Lemma.** Let $x$ be in $\mathbb{R}^n$, $D$ in $\mathcal{D}$, and $S(x, D)$ an indicator system. Suppose $M( S(x, D) ) < +\infty$ (and hence by III.1.16.(1)
M(+S(x, D)) < +\infty and M(-S(x, D)) < +\sum v(x, D')n a necessary and sufficient condition that

\[ M(S(x, D)) = M(+S(x, D)) + M(-S(x, D)) \]  \hspace{1cm} (1)

is that for any positive number h, there exist in S(x, D) indicator classes \( \sigma^+ \) and \( \sigma^- \), depending on h, such that

\[ \sum_{D' \in \sigma^+} v(x, D') - \text{is in } S(x, D), \]  \hspace{1cm} (2)

\[ \sum_{D' \in \sigma^-} v(x, D') > M(+S(x, D)) - h \]  \hspace{1cm} (3)

and

\[ \sum_{D' \in \sigma^-} |v(x, D')| > M(-S(x, D)) - h \]  \hspace{1cm} (4)

**Proof.** Sufficiency of the condition follows by adding (3) and (4), and then taking into account (2) and III.1.15. We prove necessity. Suppose then that (1) holds. Since \( M(S(x, D)) \) is finite, there exists an indicator class \( \sigma \), depending on h, such that \( \sigma \) is in S(x, D) and

\[ \sum_{D' \in \sigma} |v(x, D')| > M(S(x, D)) - h \]

or

\[ \sum_{D' \in +\sigma} v(x, D') + \sum_{D' \in -\sigma} |v(x, D')| > M(S(x, D)) - h \]  \hspace{1cm} (5)

Since \( +\sigma \cup -\sigma = \sigma \) is in S(x, D), and since \( +\sigma \) and \( -\sigma \) are sets \( \sigma^+ \) and \( \sigma^- \) respectively, it suffices to show that

\[ \sum_{D' \in +\sigma} v(x, D') > M(+S(x, D)) - h \]  \hspace{1cm} (6)

and
\[ \sum_{D' \text{ in } -\sigma} |v(x, D')| > M(-S(x, D)) - h \]  

(7)

By (5)

\[ \sum_{D' \text{ in } +\sigma} v(x, D') > M(S(x, D)) - \sum_{D' \text{ in } -\sigma} |v(x, D')| - h. \]  

(8)

Since

\[ M(-S(x, D)) \geq \sum_{D' \text{ in } -\sigma} |v(x, D')|, \]

it follows from (8) that

\[ \sum_{D' \text{ in } +\sigma} v(x, D') > M(S(x, D)) - M(-S(x, D)) - h \]  

(9)

(1) and (9) yield (6). The proof of (7) is analogous to that of (6).

**III.1.18. Lemma.** Suppose that \((x, D)\) is an admissible pair and 
\(S(x, D)\) is an indicator system. Assume that 
\(M(S(x, D)) < +\infty\) 
(and hence by III.1.16.(1), 
\(M(S(x, D)) < +\infty\) and 
\(M(-S(x, D)) < +\infty\)). Then a necessary and sufficient condition that

\[ M(S(x, D)) = M(S(x, D)) + M(-S(x, D)) \]  

(1)

and

\[ v(x, D) = M(S(x, D)) - M(-S(x, D)) \]  

(2)

is that for any positive number \(h\) there exist in \(S(x, D)\) sets \(\sigma^+\) and \(\sigma^-\), depending on \(h\), such that

\[ \sigma^+ \cup \sigma^- \text{ is in } S(x, D) \]  

(3)

\[ \sum_{D' \text{ in } \sigma^+} v(x, D') > M(S(x, D)) - h \]  

(4)
\[
\sum_{D' \in \sigma^-} |v(x, D')| > M(-S(x, D)) - h
\] (5)

and

\[
\sum_{D' \in \sigma^+ \cup \sigma^-} v(x, D') - v(x, D) \leq h
\] (6)

\textbf{Proof. (sufficiency).} (1) holds by III.1.17. To prove that (2) holds observe that by (3)

\[
\sum_{D' \in \sigma^+ \cup \sigma^-} v(x, D') = \sum_{D' \in \sigma^+} v(x, D') + \sum_{D' \in \sigma^-} v(x, D').
\] (7)

With the help of (4) and (5) we obtain

\[
M(+S(x, D)) - h < \sum_{D' \in \sigma^+} v(x, D') \leq M(+S(x, D))
\] (8)

and

\[
-M(-S(x, D)) \leq \sum_{D' \in \sigma^-} v(x, D') \leq -M(-S(x, D)) + h
\] (9)

Adding (8) and (9) and taking into account (7) we obtain

\[
M(+S(x, D)) - M(-S(x, D)) - h < \sum_{D' \in \sigma^+ \cup \sigma^-} v(x, D')
\]

\[
\leq M(+S(x, D)) - M(-S(x, D)) + h
\] (10)

Since \(h\) was an arbitrary positive number, (6) and (10) yield (2).

\textbf{(Necessity).} Suppose (1) and (2) hold. Let \(h > 0\) be given.

By hypothesis \(M(S(x, D))\) is finite. Hence there exists a set \(\sigma^-\), depending on \(h\), such that \(\sigma^-\) is in \(S(x, D)\) and

\[
\sum_{D' \in \sigma^-} |v(x, D')| > M(S(x, D)) - h.
\]
Put

\[ \sigma^+ = +\sigma \quad \text{and} \quad \sigma^- = -\sigma \]  

(11)

Then (3) holds, and as in the proof of III.1.17, (4) and (5) hold.

Since (3), (4), and (5) hold, (10) holds. Since \( h \) was an arbitrary positive number, (10) and (2) yield (6).

In III.2.13, we shall give an example showing that neither (1) nor (2) in III.1.18 necessarily hold, and that neither (1) nor (2) implies the other.

III.2. The Multiplicity Functions \( \mathbf{K} \), \( \mathbf{K}^+ \), and \( \mathbf{K}^- \)

See III.1 for general hypotheses. Unless otherwise indicated, \( v \) in \( \mathbf{V} \) will be fixed throughout III.2.

III.2.1. Definition. For each \( x \) in \( \mathbb{R}^n \) and \( D \) in \( \mathbf{D} \) let \( \mathbf{S}(x, D) \) be the set of all indicator classes \( \sigma(x, D) \), let \( \mathbf{S}^+(x, D) \) be the set of all indicator classes \( \sigma^+(x, D) \), and let \( \mathbf{S}^-(x, D) \) be the set of all indicator classes \( \sigma^-(x, D) \).

Since we allow \( \sigma(x, D) \), \( \sigma^+(x, D) \), and \( \sigma^-(x, D) \) to be the null set, it follows that \( \mathbf{S}(x, D) \), \( \mathbf{S}^+(x, D) \), and \( \mathbf{S}^-(x, D) \) are indicator systems.

III.2.2. Remark. If \( x \) in \( \mathbb{R}^n \), \( D \) in \( \mathbf{D} \), and \( D_1 \subseteq D \), then clearly

\[ \mathbf{S}^+(x, D) \subseteq \mathbf{S}(x, D) \]
\[ \mathbf{S}^-(x, D) \subseteq \mathbf{S}(x, D) \]
\[ \mathbf{S}^+(x, D) = +\mathbf{S}(x, D) \]
\[ \mathbf{S}^-(x, D) = -\mathbf{S}(x, D) \]
\[ \mathbf{S}(x, D) \mid D_1 = \mathbf{S}(x, D_1) \subseteq \mathbf{S}(x, D) \]
\[ \mathbf{S}^+(x, D) \mid D_1 = \mathbf{S}^+(x, D_1) \subseteq \mathbf{S}^+(x, D) \]
\[ \mathbf{S}^-(x, D) \mid D_1 = \mathbf{S}^-(x, D_1) \subseteq \mathbf{S}^-(x, D) \]
III.2.3. Definition. For each $x$ in $\mathbb{R}^n$ and $D$ in $\mathcal{S}$ put

$$|K|(x, D) = \sup_{\sigma \in \mathcal{S}(x, D)} \sum_{D' \in \sigma} |v(x, D')|$$

$$K^+(x, D) = \sup_{\sigma^+ \in \mathcal{S}^+(x, D)} \sum_{D' \in \sigma^+} v(x, D')$$

$$K^-(x, D) = \sup_{\sigma^- \in \mathcal{S}^-(x, D)} \sum_{D' \in \sigma^-} |v(x, D')|$$

In case we wish to indicate that these functions also depend on $v$ in $\mathcal{V}$, we shall write $|K|(x, D, v)$, $K^+(x, D, v)$, and $K^-(x, D, v)$.

III.2.4. Remark. By the remark in III.1.2 if $v$ is in $\mathcal{V}$, then $-v$ is in $\mathcal{V}$. Clearly

$$|K|(x, D, v) = |K|(x, D, -v)$$

$$K^+(x, D, v) = K^-(x, D, -v)$$

$$K^-(x, D, v) = K^+(x, D, -v).$$

III.2.5. Remark. In the notation of III.1.

$$|K|(x, D) = M(\mathcal{S}(x, D)),$$

and in view of III.2.2

$$K^+(x, D) = M(+\mathcal{S}(x, D))$$

$$K^-(x, D) = M(-\mathcal{S}(x, D)).$$

Observe too that in general (see II.1.19)

$$K^+ \neq |K|^+$$

$$K^- \neq |K|^-. $$

Indeed we shall see in III.2.5 and III.2.9 that $|K|$ satisfies conditions (ii) and (iii) in II.1.4 with $X = \emptyset$, and hence by II.1.34, II.1.35, and II.1.6.
\[ |K| = |K^+| \]
\[ |K^-| = 0 \]

**III.2.6. Lemma.** If \( x \) in \( \mathbb{R}^n \) and \( D \) in \( \mathcal{D} \), then
\[
K^+(x, D) = \sup_{\sigma \in \overline{S}(x, D)} \sum_{D' \in \sigma} v(x, D') \tag{1}
\]
and
\[
K^-(x, D) = -\inf_{\sigma^- \in \overline{S}^-(x, D)} \sum_{D' \in \sigma^-} v(x, D') = -\inf_{\sigma \in \overline{S}(x, D)} \sum_{D' \in \sigma} v(x, D') \tag{2}
\]

**Proof.** Let \( \sigma_1 \) be an arbitrary element in \( \overline{S}(x, D) \). Then
\[
\sum_{D' \in \sigma_1} v(x, D') \leq \sum_{D' \in +\sigma_1} v(x, D') \tag{3}
\]
Since \( +\sigma_1 \) is in \( \overline{S}^+(x, D) \), it follows from (3) that
\[
\sum_{D' \in \overline{S}^+(x, D)} v(x, D') \leq \sup_{\sigma \in \overline{S}^+(x, D)} \sum_{D' \in \sigma} v(x, D') = K^+(x, D) \tag{4}
\]
Since \( \sigma_1 \) was arbitrary in \( \overline{S}(x, D) \), it follows from (4) that
\[
\sup_{\sigma \in \overline{S}(x, D)} \sum_{D' \in \sigma} v(x, D') \leq K^+(x, D). \tag{5}
\]
On the other hand \( \overline{S}(x, D) \supset \overline{S}^+(x, D) \) so that the opposite inequality holds in (5). Hence (1) holds.

In view of our convention \(-(-\infty) = +\infty\), the first equality in (2) follows readily. The second equality may be proved in a manner analogous to the proof of (1).

In view of III.2.2 and III.2.5 many of the following lemmas amount to a change in notation of results in III.1. When this is the case we shall merely place in parentheses at the end of the statement of the lemma, the number of the result from which the lemma follows.
III.2.7. Lemma. Let $x$ be in $\mathbb{R}^n$ and $D$ in $\mathcal{D}$. Then

(i) $K^+(x, D) \leq |K|(x, D)$ and $K^-(x, D) \leq |K|(x, D)$. (III.1.14).

(ii) $|K|(x, D) \leq K^+(x, D) + K^-(x, D)$. (III.1.15).

(iii) $|K|(x, D) = +\infty$ if and only if $K^+(x, D) = +\infty$

or $K^-(x, D) = +\infty$. (III.1.16).

(iv) $|K|(x, D) = 0$ if and only if $K^+(x, D) = 0$

and $K^-(x, D) = 0$. (III.1.16).

(v) $|K|(x, D) = 0$ if and only if $v(x, D') = 0$

for all $D'$ such that $(x, D')$ is admissible, and $D' \subset D$.

Proof (v). The proof of (v) follows readily from the definition of $|K|$.

III.2.8. Lemma. For each $D$ in $\mathcal{D}$, $|K|(x, D)$, $K^+(x, D)$, and $K^-(x, D)$ are subadditive on $\mathbb{R}^n$. (III.1.8).

III.2.9. Lemma. For each $D$ in $\mathcal{D}$, $|K|(x, D)$, $K^+(x, D)$, and $K^-(x, D)$ have the filling up property on $\mathbb{R}^n$. (III.1.11).

III.2.10. Lemma. Let $x$ be in $\mathbb{R}^n$ and $D$ in $\mathcal{D}$. If $a$ is a real number such that $a \leq K(x, D)$, then there exists a compact set $F$, depending on $a$, such that $F \subset D$, and if $D_0$ is a domain satisfying the condition $F \subset D_0 \subset D$, then $a \leq K(x, D_0)$. Corresponding statements hold for $K^+$ and $K^-$. (III.1.12).

III.2.11. Lemma. Let $x$ be in $\mathbb{R}^n$ and $D$ in $\mathcal{D}$. Suppose

$|K|(x, D) < +\infty$ (and hence by III.2.7 (iii), $K^+(x, D) < +\infty$ and $K^-(x, D) < +\infty$). Then a necessary and sufficient condition that

$$|K|(x, D) = K^+(x, D) + K^-(x, D)$$

is that for any positive number $h$ there exist sets $\sigma^+$ and $\sigma^-$ in $\mathcal{S}(x, D)$ such that
III.2.12. Lemma. Suppose that \((x, D)\) is an admissible pair.

Assume that \( |K|(x, D) < +\infty \) (and hence by III.2.7 (iii), \( K^+(x, D) < +\infty \) and \( K^-(x, D) < +\infty \)). Then a necessary and sufficient condition that
\[ |K|(x, D) = K^+(x, D) + K^-(x, D) \]
and
\[ v(x, D) = K^+(x, D) - K^-(x, D) \]
is that for every positive number \( h \) there exist sets \( \sigma^+ \) and \( \sigma^- \) in \( S(x, D) \) such that
\[
\sigma^+ \cup \sigma^- \text{ in } S(x, D)
\]
\[ \sum_{D' \text{ in } \sigma^+} v(x, D') > K^+(x, D) - h \]
\[ \sum_{D' \text{ in } \sigma^-} \left| v(x, D') \right| > K^-(x, D) - h \]
and
\[ \left| \sum_{D' \text{ in } \sigma^+} v(x, D') - v(x, D) \right| < h \quad (\text{III.}1.18) \]

III.2.13. Lemma. Suppose that \((x, D)\) is an admissible pair and that \( |K|(x, D) < +\infty \) (and hence \( K^+(x, D) < +\infty \) and \( K^-(x, D) < +\infty \)). Then
\[ (i) \quad K(x, D) = K^+(x, D) + K^-(x, D) \text{ does not} \]
imply $v(x, D) = K^+(x, D) - K^-(x, D)$

(ii) $v(x, D) = K^+(x, D) - K^-(x, D)$ does not imply $|K|(x, D) = K^+(x, D) + K^-(x, D)$.

Proof. Let $R^n = R^1$, $D = (-3,3)$, and let $T$ be the identity transformation. Let $D_0 = (-2,2)$ and $D_1 = (-1,1)$. Let $(x, D)$ be any admissible pair, and define

$$v(x, D) = \begin{cases} 
1 & \text{if } D = D_1 \text{ and } x \in D \\
0 & \text{otherwise.} 
\end{cases}$$

Now let $x$ be fixed in $D_1$. Then if $D \neq D_1$ and $v(x, D)$ is defined, we have $v(x, D) = 0$. Hence the only indicator classes $\mathcal{O}(x, D_0)$ are the classes consisting of the null set and of the domain $D_1$ itself. Put

$$\mathcal{O}_1(x, D_0) = \{ D_1 \}.$$ Then $\mathcal{O}_1(x, D_0) = + \mathcal{O}_1(x, D_0)$, and one readily checks that

$$v(x, D_0) = 0$$

$$|K|(x, D_0) = 1$$

$$K^+(x, D_0) = 1$$

$$K^-(x, D_0) = 0.$$ Hence

$$|K|(x, D_0) = K^+(x, D_0) + K^-(x, D_0) = 1$$

But

$$v(x, D_0) = 0 \neq 1 = K^+(x, D_0) - K^-(x, D_0).$$ Thus, (i) holds. To prove (ii) let $R^n$, $D$, $D_0$, and $T$ be as before.

Let $D_{1} = (-1, \frac{1}{2})$, $D_1 = (-\frac{1}{2}, 1)$, and $D_2 = D_1 \cap D_1 = (-\frac{1}{2}, \frac{1}{2})$. Let $(x, D)$ be any admissible pair and define
\[ v(x, D) = \begin{cases} +1 & \text{if } x \in D_2 \text{ and } D = D_{+1} \\ -1 & \text{if } x \in D_2 \text{ and } D = D_{-1} \\ 0 & \text{otherwise} \end{cases} \]

Now let \( x \) be fixed in \( D_2 \). Consider any indicator class \( \mathcal{C}(x, D_0) \).

Either \( \mathcal{C}(x, D_0) = \{ D_{+1} \} \), \( \mathcal{C}(x, D_2) = \{ D_{-1} \} \), or \( \mathcal{C}(x, D_0) = \emptyset \), and one readily checks that

\[ v(x, D_0) = 0 \]
\[ |K|(x, D_0) = 1 \]
\[ K^+(x, D_0) = 1 \]
\[ K^-(x, D_0) = 1 \]

Hence

\[ v(x, D_0) = K^+(x, D_0) - K^-(x, D_0) = 0 \]

But

\[ |K|(x, D_0) = 1 \neq 2 = K^+(x, D_0) + K^-(x, D_0). \]

Thus (2) holds.

**Remark.** In the example constructed to prove (i) in III.2.13, every indicator domain is positive. Hence every indicator class is of the form \( \mathcal{C}^+(x, D) \). It follows then for every \( x \) in \( \mathbb{R}^n \) and \( D \subset D \), that

\[ |K|(x, D) = K^+(x, D) \]

and

\[ K^-(x, D) = 0 \]

Thus

\[ |K|(x, D) = K^+(x, D) + K^-(x, D), \text{ all } x \text{ in } \mathbb{R}^n. \]

**III.2.14. Lemma.** Suppose that for each \( D \) in \( \mathcal{D} \) such that \( \overline{D} \subset D \), it follows that the absolute value \( |v| = |v(x, D)| \) of the function
$v = v(x, D)$ is a lower semicontinuous (l.s.c.) function of $x$ in $C_{TfrD}$. Then for each $D$ in $\mathcal{D}$, $|K| = |K|(x, D)$, $K^+ = K^+(x, D)$, and $K^- = K^-(x, D)$ are l.s.c. functions of $x$ in $\mathbb{R}^n$.

**Proof.** Let $x$ be fixed in $\mathbb{R}^n$ and let $D$ be fixed in $\mathcal{D}$. We shall show that $|K|$ is l.s.c. at $x$. If $|K|(x, D) = 0$, then since $|K|$ is non-negative, $|K|$ is l.s.c. at $x$. Assume that

$$|K|(x, D) > 0,$$

and let $a$ be any positive number such that

$$|K|(x, D) > a$$

Then, since $a > 0$, there exists an indicator class

$\mathcal{O}(x, D) = \{D_1\}_{i=1}^m$, depending on $a$, such that

$$\sum_i |v(x, D_i)| > a$$

Since the domains $D_i$ are indicator domains for $(x, D)$, it follows that $\overline{D_i} \subset D$ and $x$ is in $C_{TfrD_i}$. Hence $TfrD_i$ is compact and $G = \bigcap_i C_{TfrD_i}$ is an open set containing $x$. Observe that if $y$ in $G$, then $y$ is in $C_{TfrD_i}$, each $i$, so that $v(y, D_i)$ is defined. Now for each $i, i = 1, 2, \ldots, m$, let $b_i$ be a positive number such that

$$b_i < |v(x, D_i)|$$

In view of (3) we may choose the $b_i$ so that they also satisfy the condition
\[ \sum_i b_i > a \] (5)

Let
\[ \sum_i |v(x, D_i)| = b \] (6)

Then by (3), \( b - a > 0 \). For each \( i \) let
\[ h_i = \text{inf} \left\{ \frac{b-a}{m}, b_i \right\} \] (7)

that is to say, \( h_i \) is the smaller of the two (positive) numbers \( \frac{b-a}{m} \) and \( b_i \). Since \( G \) is open and \( |v| \) is l.s.c. at \( x \), there exists for each \( i \) an open neighborhood \( N_i(x) \) of \( x \) such that \( N_i(x) \subset G \), and if \( y \) in \( N_i(x) \), then
\[ |v(y, D_i)| > |v(x, D_i)| - h_i \] (8)

Let \( N(x) = \bigcap_i N_i(x) \). Then by (7) and (8)
\[ y \text{ in } N(x) \implies \sum_i v(y, D_i) > \sum_i |v(x, D_i)| - b + a \] (9)

Hence by (6)
\[ \sum_i |v(y, D_i)| > a, \quad y \text{ in } N(x). \] (10)

By (7)
\[ |v(x, D_i)| - h_i > 0, \text{ each } i. \] (11)

(8) and (11) yield
\[ |v(y, D_i)| > 0, \quad y \text{ in } N(x). \]
Thus, since \( \overline{D}_1 \subset D \), for each \( y \) in \( N(x) \), \( D_1 \) is an indicator domain for \((y, D)\). Hence by definition of \( |K|_{(y, D_1)} \) we have

\[
|K|_{(y, D_1)} \geq |v(y, D_1)|, \quad y \text{ in } N(x).
\]  

(12)

(10) and (12) yield

\[
\sum_{1} |K|_{(y, D_1)} > a, \quad y \text{ in } N(x).
\]  

(13)

Since \( |K| \) is subadditive for each \( y \) in \( \mathbb{R}^n \),

\[
\sum_{1} |K|_{(y, D_1)} \leq |K|_{(y, D)}, \quad y \text{ in } N(x).
\]  

(14)

(13) and (14) yield

\[
|K|_{(y, D)} > a, \quad y \text{ in } N(x)
\]

and it follows that \( |K| \) is l.s.c. at \( x \). Since \( x \) and \( D \) were arbitrary, the lemma follows with respect to \( |K| \). The proofs for \( K^+ \) and \( K^- \) are similar.

**III.2.15. Lemma.** Suppose that for each \( D \) in \( \mathcal{D} \) such that \( \overline{D} \subset D \), it follows that the absolute value \( |v| = |v(x, D)| \) of the function \( v = v(x, D) \) is a l.s.c. function of \( x \) in \( \text{CTfrD} \). Then \( |K| = |K|(x, D) \), \( K^+ = K^+(x, D) \), and \( K^- = K^-(x, D) \) are L-measurable functions of \( x \) in \( \mathbb{R}^n \).

**Proof.** Since lower semicontinuity implies L-measurability, III.2.15 follows immediately from III.2.14.
**III.2.16. Definition.** Designate by $\mathcal{K}$ the set $\{ |K|, K^+, K^- \}$ of functions defined in III.2.3, and let $K$ be generic for an element of $\mathcal{K}$.

**III.2.17. Definition.** To simplify the wording of the following four lemmas we shall say that the index function $v$ satisfies condition (i') provided that $v(x, D) = 0$ whenever $(x, D)$ is an admissible pair with respect to $T$ and $x$ is in CTD. Observe that condition (i') for the index function $v$ is analogous to condition (i) in II.1.4 for multiplicity functions.

**III.2.18. Lemma.** Suppose that $v$ satisfies condition (i'). Let $x$ be in $\mathbb{R}^n$ and $D$ in $\mathcal{D}$. If $x$ is in CTD, then $K(x, D) = 0$, $K$ in $\mathcal{K}$.

**Proof.** Let $D'$ be any domain in $\mathcal{D}$ such that $\overline{D'} \subset D$. Then $x$ in CTD implies $x$ in CTD'. Hence $x$ is in $\text{CTD'}$, and consequently $(x, D')$ is an admissible pair. Since $x$ in CTD' implies $x$ is in CTD', and since $v$ satisfies condition (i'), it follows that $v(x, D') = 0$. Since $D'$ was an arbitrary domain such that $\overline{D'} \subset D$, it follows from III.2.7.(v) that $|K|(x, D) = 0$. But then by III.2.7.(iv), $K^+(x, D) = 0$ and $K^-(x, D) = 0$.

**III.2.19. Lemma.** Assume that $v$ satisfies condition (i'). Let $K$ be in $\mathcal{K}$, and suppose that $K(x, D)$ is an $L$-measurable function of $x$ in $\mathbb{R}^n$, all $D$ in $\mathcal{D}$. If there exists a set $X \subset \mathbb{R}^n$ such that $LX = 0$ and $K(x, D) < +\infty$ whenever $x$ is in $OX$, then $K$ is admissible in the restricted sense.

**Proof.** We must show that conditions (i) through (v) in definition II.1.4 are satisfied. By the preceding lemma condition (i)
is satisfied. By lemmas III.2.8 and III.2.9 conditions (ii) and (iii) respectively are satisfied for all \( x \) in \( \mathbb{R}^n \). Since condition (ii) (subadditivity) is satisfied for all \( x \) in \( \mathbb{R}^n \), and since \( K(x, D) < +\infty \) for all \( x \) in \( \mathcal{C} \), it follows that \( K(x, D) < +\infty \), all \( D \) in \( \mathcal{D} \), whenever \( x \) is in \( \mathcal{C} \). Hence condition (iv) is satisfied. By hypothesis \( K(x, D) \) is an L-measurable function of \( x \) in \( \mathbb{R}^n \), all \( D \) in \( \mathcal{D} \), so that condition (v) is satisfied.

**III.2.20. Lemma.** Suppose that \( \nu \) satisfies condition (i'). Let \( K \) be in \( \mathcal{K} \), and suppose that \( K(x, D) \) is an L-measurable function of \( x \) in \( \mathbb{R}^n \), all \( D \) in \( \mathcal{D} \). If \( K(x, D) \) is L-summable over \( \mathbb{R}^n \), then \( K \) is admissible in the restricted sense.

**Proof.** Since \( K(x, D) \) is L-summable over \( \mathbb{R}^n \), it follows from I.2.6 that there exists a set \( X \subset \mathbb{R}^n \) such that \( LX = 0 \) and \( K(x, D) < +\infty \) whenever \( x \) is in \( \mathcal{C}X \). Hence by the preceding lemma, \( K \) is admissible in the restricted sense.

**III.2.21. Lemma.** Suppose that \( \nu \) satisfies condition (i''), and that for each \( D \) in \( \mathcal{D} \) such that \( \overline{D} \subset D \), it follows that the absolute value of \( \nu \) is a lower semicontinuous function of \( x \) in \( \mathcal{C}TfrD \). Let \( K \) be in \( \mathcal{K} \) and suppose that there exists a set \( X \subset \mathbb{R}^n \) such that \( LX = 0 \) and \( K(x, D) < +\infty \) whenever \( x \) is in \( \mathcal{C}X \). Then \( K \) is admissible in the restricted sense.

**Proof.** By III.2.15 the lower semicontinuity of the absolute value of \( \nu \) implies that for each \( D \) in \( \mathcal{D} \), \( K(x, D) \) is an L-measurable function of \( x \) in \( \mathbb{R}^n \). III.2.21 now follows from III.2.19.

**III.2.22. Remark.** The topological index \( \kappa = \kappa(x, T, D) \) as defined in [2;125] satisfies condition (i''), and it is a continuous
function of \( x \) in \( \mathcal{D} \) for all \( D \) in \( \mathcal{D} \) such that \( \overline{D} \subset D \). It follows from III.2.21 that if \( K \) in \( \mathcal{K} \) is generated by \( \mu \) as in definition III.2.3, and if \( K(x, D) \) is finite a. e. in \( \mathbb{R}^n \), then \( K \) is admissible in the restricted sense. Furthermore, if \( K \) in \( \mathcal{K} \) is generated by \( \mu \), then our multiplicity function \( |K| \) is the multiplicity function \( K \) in \([2;155]\) and our multiplicity functions \( K^+ \) and \( K^- \) are the multiplicity functions \( K^+ \) and \( K^- \) in \([2;155]\).

III.3. Bounded Variation and Absolute Continuity with Respect to \( K^+ \) and \( K^- \).

We shall assume throughout III.3 that if \( K \) is in \( \mathcal{K} = \{ |K|, K^+, K^- \} \), then \( K \) is admissible in the restricted sense. This amounts to assuming properties which guarantee that for all \( D \) in \( \mathcal{D} \), \( K(x, D) = 0 \) whenever \( x \) is in \( \mathcal{D} \cap \mathcal{C} \), where \( \mathcal{X} \subset \mathbb{R}^n \) is an exceptional set for \( K \), and that \( K(x, D) \) is an \( L \)-measurable function of \( x \) in \( \mathbb{R}^n \) (see II.1.4 and III.2.18 through III.2.22).

III.3.1 Lemma. Let \( D \) be a domain in \( \mathcal{D} \). Then \( T = BV|K| \) in \( D \) if and only if \( T = BVK^+ \) and \( T = BVK^- \) in \( D \).

Proof. By definition II.2.1, \( T = BV|K| \) in \( D \) implies
\[
|K|(x, D) \text{ is } L\text{-summable over } \mathbb{R}^n. \tag{1}
\]
By III.2.7
\[
K^+(x, D) \leq |K|(x, D) \text{ and } K^-(x, D) \leq |K|(x, D), \quad x \in \mathbb{R}^n. \tag{2}
\]
Since \( K \) in \( \mathcal{K} \) implies that \( K \geq 0 \) and that \( K \) is \( L \)-measurable, \( K^+(x, D) \) and \( K^-(x, D) \) are \( L \)-summable over \( \mathbb{R}^n. \tag{3} \)

Hence \( T = BVK \) in \( D \) implies \( T = BVK^+ \) and \( T = BVK^- \) in \( D \). On the other
hand, if $T - BVK^+$ and $T - BVK^-$ in $D$, then (4) holds. Hence

$$K^+(x, D) + K^-(x, D) \text{ is } L\text{-summable over } \mathbb{R}^n. \quad (5)$$

By III.2.7

$$|K|(x, D) \leq K^+(x, D) + K^-(x, D), \quad x \text{ in } \mathbb{R}^n. \quad (6)$$

(3), (5), (6), and I.2.5 yield (1). Hence $T - BVK^+$ and $T - BVK^-$ in $D$ imply $T - BV|K|$ in $D$.

**III.3.2. Lemma.** $K^+ + K^-$ is in $\mathcal{M}$, and $K^+ - K^-$ is in $\mathcal{N}$

(See II.1.4 and II.1.13 for the definitions of $\mathcal{M}$ and $\mathcal{N}$ respectively.)

**Proof.** Since $K^+$ and $K^-$ are admissible in the restricted sense, they are in $\mathcal{M}$. By II.1.10 $\mathcal{M}$ is positively linear. Hence $K^+ + K^-$ is in $\mathcal{M}$. By II.1.14 $\mathcal{M} \subseteq \mathcal{N}$, and by II.1.17 $\mathcal{N}$ is linear. Hence $K^+ - K^-$ is in $\mathcal{N}$.

**III.3.3. Lemma.** Let $D$ be a domain in $\mathcal{D}$. Then $T - BV|K|$ in $D$ if and only if $T - BV(K^+ + K^-)$ in $D$.

**Proof.** By III.3.2

$$K^+ + K^- \text{ is in } \mathcal{M} \quad (1)$$

Suppose now that $T - BV|K|$ in $D$. Then by III.3.1, $T - BVK^+$ and $T - BVK^-$ in $D$; consequently

$$(K^+ + K^-)(x, D) \text{ is } L\text{-summable over } \mathbb{R}^n. \quad (2)$$

(1) and (2) imply $T - BV(K^+ + K^-)$ in $D$. On the other hand, if $T - BV(K^+ + K^-)$ in $D$, then (2) holds. By III.2.7

$$|K|(x, D) \leq (K^+ + K^-)(x, D), \quad x \text{ in } \mathbb{R}^n. \quad (3)$$

Since $|K|$ is $L$-measurable and non-negative, (2), (3), and I.2.5 imply that $|K|(x, D)$ is $L$-summable over $\mathbb{R}^n$. Hence $T - BV(K^+ + K^-)$ in $D$ implies $T - BV|K|$ in $D$.

**III.3.4. Lemma.** If $T - BV|K|$ in a domain $D$ in $\mathcal{D}$, then $T - BV(K^+ - K^-)$ in $D$. 

Proof. By III.3.2

\[ K^+ - K^- \text{ is in } \mathcal{N}. \] (1)

By II.1.5 there exists an exceptional set \( Y \) for the pair \((K^+, K^-)\).

Clearly \((K^+, K^-; Y)\) is a representation of \( K^+ - K^- \) and (see II.1.11, II.3.1, and III.3.1).

\((K^+, K^-; Y)\) is BV in \( D \). (2)

(1), (2), and II.3.3 imply \( T - BV(K^+ - K^-) \) in \( D \).

III.3.5 Lemma. Let \( D \) be a domain in \( \mathcal{D} \). Then \( T - ACK^+ \) and \( T - ACK^- \) in \( D \) if and only if \( T - AC(K^+ + K^-) \) in \( D \).

Proof. By II.2.11 \( T - ACK^+ \) and \( T - ACK^- \) in \( D \) imply \( T - BVK^+ \) and \( T - BVK^- \) in \( D \). Hence by III.3.1 and III.3.3

\[ T - BV(K^+ + K^-) \text{ in } D. \] (1)

\( T - ACK^+ \) and \( T - ACK^- \) in \( D \) yield by definition

\[ \int_D D(u, K^+) = \int K^+(x, D) < +\infty, \]

and

\[ \int_D D(u, K^-) = \int K^-(x, D) < +\infty, \]

so that

\[ \int_D (D(u, K^+) + D(u, K^-)) = \int (K^+(x, D) + K^-(x, D)) , \]

or with the help of I.3.12

\[ \int_D D(u, K^+ + K^-) = \int (K^+ + K^-)(x, D) \]

(2)

(1) and (2) imply \( T - AC(K^+ + K^-) \) in \( D \). On the other hand, if \( T - AC(K^+ + K^-) \) in \( D \), then by II.2.11, \( T - BV(K^+ + K^-) \) in \( D \). Hence
by III.3.1 and III.3.3

(3) and II.2.3 yield

\[ D(u, K^+) \text{ and } D(u, K^-) \text{ exist a.e. in } D, \]

\[ (4) \]

\[ 0 \leq \int_D D(u, K^+) \leq \int K^+(x, D) < + \infty \]

and

\[ 0 \leq \int_D D(u, K^-) \leq \int K^-(x, D) < + \infty. \]

(6)

T - AC(K^+ + K^-) in D also yields by definition

\[ D(u, K^+ + K^-) \text{ exists a.e. in } D, \]

(7)

and

\[ \int_D D(u, K^+ + K^-) = \int (K^+ + K^-)(x, D). \]

(8)

(4), (7), and I.3.12 yield

\[ D(u, K^+) + D(u, K^-) = D(u, K^+ + K^-) \text{ a.e. in } D. \]

(9)

(5), (6), (8), and (9) yield

\[ \int_D D(u, K^+) = \int K^+(x, D), \]

(10)

and

\[ \int_D D(u, K^-) = \int K^-(x, D). \]

(11)

(3), (10), and (11) imply T - ACK^+ AND T - ACK^- in D.
**III.3.6. Lemma.** If \( T - \text{ACK}^+ \) and \( T - \text{ACK}^- \) in a domain \( D \) in \( \mathcal{D} \), then \( T - \text{AC}(K^+ - K^-) \) in \( D \).

**Proof.** \( T - \text{ACK}^+ \) and \( T - \text{ACK}^- \) in \( D \), imply \( T - \text{BVK}^+ \) and \( T - \text{BVK}^- \) in \( D \). Hence by III.3.1 and III.3.4

\[
T = \text{BV}(K^+ - K^-) \text{ in } D. \tag{1}
\]

By II.1.5 there exists an exceptional set \( Y \) for the pair \( (K^+, K^-) \).

Clearly \( (K^+, K^-; Y) \) is a representation of \( K^+ - K^- \) and (see II.1.11 and II.3.1)

\[
(K^+, K^-; Y) \text{ is AC in } D. \tag{2}
\]

(1), (2), and III.3.9 imply that \( T = \text{AC}(K^+ - K^-) \) in \( D \).

**Remark.** Suppose \( T = \text{ACK}^+ \) and \( T = \text{ACK}^- \) in a domain \( D \) in \( \mathcal{D} \).

Then by III.3.2 \( K^+ + K^- \) is in \( \mathcal{M} \), and by III.3.5 \( T = \text{AC}(K^+ + K^-) \) in \( D \). Hence, with \( M = K^+ + K^- \), the results of section II.2 apply to \( K^+ + K^- \) whenever \( T = ACM \) in \( D \). In particular we obtain from II.2.23:

**III.3.7. Theorem.** Suppose \( T = \text{ACK}^+ \) and \( T = \text{ACK}^- \) in a domain \( D \) in \( \mathcal{D} \). Let \( H(x) \) be a real-valued, \( L \)-measurable function on \( \mathbb{R}^n \).

If either

\[
H(Tu)D(u, K^+ + K^-) \text{ is } L\text{-summable over } D, \tag{1}
\]

or

\[
H(x)(K^+ + K^-)(x, D) \text{ is } L\text{-summable over } \mathbb{R}^n, \tag{2}
\]

then both (1) and (2) hold and

\[
\int_D H(Tu)D(u, K^+ + K^-) = \int H(x)(K^+ + K^-)(x, D). \]
III.3.8. Definition. Assume $T = BV|K|$ (and hence $T = BV(K^+ - K^-)$) in $D$. For each $x$ in $\mathbb{R}^n$ and for each $D$ in $\mathcal{D}$, let (see I.1.13)

$$v^*(x, D) = (K^+ - K^-)(x, D).$$

Recall that III.2.12 gives sufficient conditions that $v(x, D) = v^*(x, D)$ whenever $(x, D)$ is an admissible pair with respect to $T$.

Suppose now that $\overline{D} \subset D$. If $v(x, D)$ is the topological index $\mu(x, T, D)$ defined in [2;125], then as we noted in III.2.22 our functions $|K|$, $K^+$, and $K^-$ become the functions $K$, $K^+$, and $K^-$ respectively of [2]. Thus our assumption that $T = BV|K|$ in $D$ means that $T = eBV$ in $D$ according to [2; 249]. Consequently, the (possibly empty) set $X \subset \mathbb{R}^n$ where $|K|(x, T, D) = +\infty$ has $L$-measure zero, $v^*(x, D)$ is equal on $C_X$ to the algebraic multiplicity function $\mu_e(x, T, D)$ defined in [2;169], and $v(x, D) = v^*(x, D)$ whenever $x$ is in $CT_{fr}D \cap C_X$ (see theorems 2 and 4 in [2;170, 171]).

III.3.9. Definition. Suppose $T = BVK^+$ and $T = BVK^-$ in a domain $D$ in $\mathcal{D}$, or, equivalently by III.3.1, $T = BV|K|$ in $D$. Then by III.3.4 and III.3.8, $T = BVv^*$ in $D$. Hence by III.3.2 and II.3.6, the derivative $D(u, v^*)$ of $\int v^*(x, D')$, $D' \subset D$, exists a.e. in $D$. As a matter of notation and terminology we put

$$V(u, T) = D(u, v^*)$$

and say that $V(u, T)$ is a Jacobian on $D$ for $T$. If $D = D$, we say simply that $V(u, T)$ is a Jacobian for $T$.

Note that by III.3.8, III.3.2, and II.3.6

$$V(u, T) = D(u, K^+) - D(u, K^-)$$
and if \( D = \mathcal{D} \), then by definition, \( V(u, T) \) is the essential generalized Jacobian \( J_e(u, T) \) defined in \( [2; 258] \).

If \( T - \text{ACK}^+ \) and \( T - \text{ACK}^- \) in a domain \( D \) in \( \mathcal{D} \), then by III.3.2 \( v^* \) is in \( \mathcal{N} \) and by III.3.6, \( T - \text{AC}v^* \). Hence, with \( N = v^* \) (and \( D(u, N) = V(u, T) \)) the results of section II.3 apply to \( v^* \) whenever \( T - \text{AC}N \) in \( D \). In particular we obtain from II.3.18:

**III.3.10. Theorem.** Let \( D \) be a domain in \( \mathcal{D} \), and let \( H(x) \) be a real valued, L-measurable function on \( \mathbb{R}^n \). Suppose that \( T - \text{ACK}^+ \) and \( T - \text{ACK}^- \) in \( D \), and that \( H(Tu)V(u, T) \) is L-summable over \( D \). Then \( H(x)v^*(x, D) \) is L-summable over \( \mathbb{R}^n \), and

\[
\int_D H(Tu)V(u, T) = \int H(x)v^*(x, D),
\]

or, if \( \overline{D} \subset D \), and if \( v(x, D) = v^*(x, D) \) a.e. in \( \mathbb{R}^n \),

\[
\int_D H(Tu)V(u, T) = \int H(x)v(x, D).
\]

In what follows we shall relate III.3.10 to theorem 2 and 3 in \( [2; 262, 263] \).

Suppose that \( v \) is the topological index defined in \( [2] \). Then by III.2.22 and the lemma on page 262 of \( [2] \), \( T - \text{AC}|K| \) in \( D \) in if and only if \( T - \text{ACK}^+ \) and \( T - \text{ACK}^- \) in \( D \). Hence from III.3.10 we obtain

**III.3.11. Theorem.** Suppose that \( v \) is the topological index, and that \( H(x) \) is a real valued, L-measurable function on \( \mathbb{R}^n \). Suppose also that \( D \) is a domain in \( \mathcal{D} \), and that \( T - \text{AC}|K| \) in \( D \). If \( H(Tu)V(u, T) \) is L-summable over \( D \), then \( H(x)v^*(x, D) \) is L-summable over \( \mathbb{R}^n \), and
In view of the remark on page 264 of [2], III.3.11 is seen to be theorem 2 on page 262 of [2].

Suppose now that $T$ is defined and continuous on the closure of $D$, and that $T - BV|K|$ in $D$. Then the topological index $\nu$ is an index function $\nu = \nu(x, D)$ defined for all $D \subset \overline{D}$, and all $x$ in $\Gamma \cap D$. By lemma 4 in [2;260, 261] $\nu(x, D) = \nu^*(x, D)$ a.e. in $\mathbb{R}^n$ whenever $D \subset \overline{D}$ and $LTfrD = 0$. Thus from the first transformation formula in III.3.10 we obtain

III.3.12. Theorem. Suppose that $T$ is defined and continuous on $\overline{D}$, and that $LTfrD = 0$. Let $\nu$ be the topological index, and let $H(x)$ be a real valued, $L$-measurable function defined on $\mathbb{R}^n$. Suppose that $T - AC|K|$ in $D$. If $H(Tu)V(u, T)$ is $L$-summable over $D$, then $H(x)\nu(x, D)$ is $L$-summable over $\mathbb{R}^n$, and

$$\int_D H(Tu)V(u, T) = \int H(x)\nu(x, D).$$

This is theorem 3 in [2;263].
BIBLIOGRAPHY


AUTOBIOGRAPHY

I, George A. Craft, was born in Mahoning County, Ohio, November 16, 1916. I received my high school education at Boardman High School, Youngstown, Ohio. Miami University, Oxford, Ohio, granted me a Bachelor of Science degree in Education in 1939, and Indiana University granted me a Master of Arts degree in 1950. From 1950 to 1953 I held the position of Instructor in the Department of Mathematics at Montana State University in Missoula, Montana. In 1953 I was appointed Graduate Assistant in the Department of Mathematics at The Ohio State University. I held this position until 1956 when I was appointed Assistant Instructor. While at The Ohio State University I completed the course requirements for the degree of Doctor of Philosophy. In 1958 I was appointed Instructor in the Department of Mathematics at Denison University in Granville, Ohio, and in 1960 I became an Assistant Professor at Denison University.