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SURFACE WAVE DIFFRACTION AND ITS
RELATIONSHIP TO SURFACE WAVE ANTENNAS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

TA-SHING CHU, B.S., M.Sc.

****

The Ohio State University

1960

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CHAPTER I
INTRODUCTION

During the past decade the demand of flush-mounted antennas has created great interest in plane surface wave radiators.\(^1\)\(^{-9}\) At least two major approaches that explain the radiation mechanism of the trapped surface wave antenna have been proposed. The first approach\(^5\) attributed the radiation to an equivalent current sheet covering the surface wave structure and the radiation pattern is obtained by an integration over the assumed current sheet. This method, evidently inherited from the treatment of other traveling wave type antennas, usually achieved qualitative agreement with experimental patterns, and offered the Hansen-Woodyard condition as a design guide. However, the assumed radiation by the equivalent current distribution along the surface wave structure contradicts the requirement that there is no leakage of energy from the trapped surface wave in the propagating path, and its agreement with experiment often deteriorates rapidly when \(\beta/k > 1.04.\)\(^9\) In the second approach\(^10\) it is suggested that the field is the combination of a component due to the radiation from the feed in the presence of the surface wave structure, and a component due to the diffraction of the surface wave by the termination of the structure. This method has not been thoroughly exploited because it involves difficult boundary value problems. The launching problem
has received considerable attention,\textsuperscript{11,12,13,14} although no available surface wave feed provides one hundred percent excitation efficiency. In comparison, the effect of the geometrical configuration on the diffraction of the surface wave has received slight attention. A variational solution of the diffraction of surface waves by a semi-infinite dielectric slab has been presented by Angulo,\textsuperscript{15} and a Wiener-Hopf solution of the scattering of surface waves by a discontinuity in reactance has been obtained by Kay.\textsuperscript{16} But it is noted that both of the above problems are characterized by an infinite plane boundary which restricts the region of diffraction to the upper half space. Recently Arbel\textsuperscript{17} analyzed the dielectric disk which is essentially the cylindrical version of the semi-infinite dielectric slab.

In the present study the diffraction of TM and TE surface waves incident along a right-angled wedge is treated. The wedge is reactive on one surface and perfectly conducting on the other. An exact mathematical solution is obtained for the infinite wedge, with the reflection coefficients and the far-field patterns given in compact, closed forms. The resulting simple expressions can be evaluated readily by desk calculation, and numerical results are presented here in graphical form.

The difficulty of the mixed boundary condition in this problem is overcome by a simple transformation which was originally proposed
by Stoker\textsuperscript{18} and Lewy\textsuperscript{19} who studied water wave theory. This method has been employed by Karp and Karal for solving electromagnetic diffraction problems.\textsuperscript{20,21} They treated the scattering of a TM plane wave by a right-angled wedge when the incidence angle lies in the range $0 \leq \theta_o < 3\pi/2$.\textsuperscript{22} However, they left open the possibility that the problem has no solution for $\theta_o = 3\pi/2$. Although no homogeneous plane wave comes from this direction, a TM surface wave can propagate along a surface with an inductive surface reactance. The present work will serve to cover this important physical case. The diffraction of a TE surface wave, which requires a capacitive surface reactance, is also analyzed here.

In the effort to understand surface wave antennas, it is desirable to estimate the effect on the surface wave diffraction pattern when the perfectly conducting face of the wedge is truncated. This is done by applying the exact solution for the infinite wedge and an approximate method which was proposed by Meixner\textsuperscript{23} and has been recently applied to the finite conical antenna by Adachi\textit{ et al.}\textsuperscript{24}

The far field of a line source radiating in the presence of a right-angled wedge can be obtained by the application of the reciprocity theorem and the plane wave diffraction solution.\textsuperscript{22} When the line source is far from the edge and close to the reactive surface, an approximate expression of its far field can be shown as the sum of one
component for the diffraction of the surface wave by the wedge and another component of a geometrical optics nature corresponding to the feed radiation. Numerical patterns are calculated from the superposition of the two components and compared with measured patterns. Examples are calculated and checked by measurement. These illustrate that the Hansen-Woodyard condition in general is not valid for this type of planar surface wave antenna. Experiments are also carried out to observe the effect on the patterns of changing the wedge angle.
CHAPTER II
DIFFRACTION OF A SURFACE WAVE BY A
RIGHT-ANGLED WEDGE

A. GEOMETRY AND IMPEDANCE
BOUNDARY CONDITIONS

The right-angled wedge is defined by the surfaces \( y = 0, \ x > 0 \) and \( x = 0, \ y < 0 \) as shown in Fig. 1. Free space is defined by the

\[
\begin{array}{c}
\text{Perfectly Conducting Surface} \\
\text{Reflected Surface Wave} \\
\text{Incident Surface Wave} \\
\text{Receiving Surface}
\end{array}
\]

Fig. 1. The infinite right-angled wedge.

angular region \( 0 \leq \theta \leq 3\pi/2 \). An impedance type boundary condition which can support surface waves is prescribed on the front face of the wedge and a perfectly conducting boundary condition is prescribed on the other. The mathematical formulation of the boundary conditions is given by
\[
\begin{align*}
(2-1) & \quad \frac{\partial h}{\partial y} = 0 \quad y = 0, \ x \geq 0 \\
& \quad \frac{\partial h}{\partial x} - \alpha h = 0 \quad x = 0, \ y \leq 0
\end{align*}
\]

where \( h \) is the z-component of the magnetic vector for a TM surface wave, and

\[
(2-2) \quad \begin{align*}
& \quad e = 0 \quad y = 0, \ x \geq 0 \\
& \quad \frac{\partial e}{\partial y} - \alpha e = 0 \quad x = 0, \ y \leq 0
\end{align*}
\]

where \( e \) is the z-component of the electric vector for a TE surface wave.

\( \alpha \) is the attenuation constant of the surface wave in a direction perpendicular to the reactive surface

\[
(2-3) \quad \alpha = i \omega \varepsilon Z = i \omega \varepsilon (R - iX) \quad \text{for the TM case}
\]

\[
(2-4) \quad \alpha = i \omega \mu /Z = i \omega \mu /(R - iX) \quad \text{for the TE case}
\]

where \( \varepsilon \) is the permittivity of free space, \( \mu \) is the permeability of free space, \( e^{-i\omega t} \) time dependence is assumed, and \( Z, R, \) and \( X \) are the impedance, resistance, and reactance of the surface, respectively.

The requirement \( \text{Re} \ \alpha > 0 \) is imposed to satisfy the radiation condition; this implies that an inductive surface can support a TM surface wave and a capacitive surface can support a TE surface wave. The condition \( R \ll X \) is generally met by practical surface wave transmission systems.
because there is negligible attenuation along the reactive surface. Physical examples are a corrugated perfectly-conducting surface with grooves small compared with a wavelength and a very thin dielectric slab backed by a metal plate. Two excellent review articles on surface wave structures are presented by Zucker, and Barlow and Cullen.

B. TM SURFACE WAVE DIFFRACTION

In this section the diffraction of a TM surface wave by an infinite right-angled wedge as shown in Fig. 1 will be considered. Suppose a TM surface wave traveling in the positive y-direction is guided by a surface with surface reactance $X$ which is inductive and hence positive in sign. The only magnetic field component in this incident wave is the $z$-component, perpendicular to the paper, and it may be written as

$$h_{inc}(x, y) = e^{i\beta y + \alpha x} \quad x \leq 0, \quad \beta = \sqrt{k^2 + \alpha^2}$$

where $k$ is the free space phase constant, $\beta$ is the propagation constant of the surface wave, and $\alpha$ is a positive real quantity and is related to the reactance by $\alpha = \omega \varepsilon X$. Because of the geometry and the chosen incident field, the total electromagnetic field is independent of $z$ and hence is completely determined by the value of $H_z$. One then obtains from Maxwell's equations,
(2-6) \[ H_x = H_y = E_z = 0 \]

and

\[
\begin{align*}
(2-7) & \\
\begin{cases}
E_x = -\frac{1}{i\omega \varepsilon} \frac{\partial H_z}{\partial y} \\
E_y = \frac{1}{i\omega \varepsilon} \frac{\partial H_z}{\partial x}
\end{cases}
\]

\( H_z \equiv h \) should satisfy the homogeneous wave equation

\[
(2-8) \quad (\nabla^2 + k^2) h = 0
\]

and the boundary conditions on the wedge

\[
(2-9) \begin{cases}
\frac{\partial h}{\partial y} = 0 & \quad y = 0, \ x \geq 0 \\
\frac{\partial h}{\partial x} - \alpha h = 0 & \quad x = 0, \ y \leq 0
\end{cases}
\]

where the first equation in (2-9) corresponds to the vanishing of \( E_x \) on the perfectly conducting surface and the second corresponds to the impedance relation \( E_y = (-i\chi) H_z \) on the reactive surface. Now the mathematical problem reduces to solving the homogeneous wave equation (2-8) and the mixed boundary conditions of equations (2-9) for excitation by the incident surface wave (2-5). Furthermore, the scattered field, \( h - h_{\text{inc}} \), must satisfy the radiation condition and be finite everywhere.

The following linear combination of the magnetic field and its
cartesian derivative is introduced as an auxiliary function $g$:

\[ 2-10 \quad g = \left( \frac{\partial}{\partial x} - \alpha \right) h \]

Then $g$ satisfies the wave equation

\[ 2-11 \quad (\nabla^2 + k^2) g = 0 \]

and the simple homogeneous boundary conditions

\[ 2-12 \begin{cases} 
\frac{\partial g}{\partial y} = 0 & y = 0, \ x \geq 0 \\
g = 0 & x = 0, \ y \leq 0 
\end{cases} \]

Then $h$ can be obtained from $g$ by integrating (2-10)

\[ 2-13 \quad h(x,y) = -e^{\alpha x} \int_{-\infty}^{\infty} e^{-\alpha \xi} g(\xi,y) \, d\xi + e^{\alpha x} S(y) \]

where $e^{\alpha x} S(y)$ is chosen in such a way that $h(x,y)$ satisfies all the requirements. The nature of the transformation in Eq. (2-10) eliminates the incident surface wave in the expression for $g$; in other words, a source-excitation problem has been transformed into a source-free problem. Because the function $g$ does not have to be finite at the origin, all radiating solutions that satisfy the assigned boundary conditions may be introduced at the beginning. If one employs the method of separation of variables, the most general representation for $g$ is found to be
\( g(r, \theta) = \sum_{n=0}^{\infty} A_n H_{2n+1}^{(1)}(kr) \cos \frac{2n+1}{3} \theta. \)

In the above expression, \( r \) and \( \theta \) are the usual polar coordinates and \( 0 \leq \theta \leq \frac{3\pi}{2} \). However, only those terms that yield a finite value of the electromagnetic field everywhere in the original physical problem for \( h \) should be retained. It follows from this condition that all but the first term vanish in Eq. (2-14). Substituting (2-14) into (2-13) and introducing the incident and reflected surface wave terms for \( e^{ax} S(y) \)

\[ h(x, y) = -A_0 e^{ax} \int_{-\infty}^{\infty} e^{-\alpha \xi} H_{\frac{1}{3}}^{(1)}(kr) \cos \frac{1}{3} \theta \, d\xi \]

\[
= \begin{cases} 
   e^{ax} + i \sqrt{k^2 + \alpha^2} \, y + C_0 e^{ax} - i \sqrt{k^2 + \alpha^2} \, y & y < 0 \\
   0 & y > 0 
\end{cases}
\]

is obtained. Note that the values of \( r \) and \( \theta \) in the above equation are given by \( \sqrt{\xi^2 + y^2} \) and \( \arctan \frac{y}{\xi} \).

In Eqs. (2-13) and (2-15) when \( y \) is negative, the integration is carried into the region occupied by the wedge. This is possible because the fourth quadrant is just part of a Riemann surface. The integral in Eq. (2-15) plays a role similar to that of complementary modes in satisfying the continuity conditions along the negative x-axis.
The integral and the terms to the right of the brace in Eq. (2-15) are wave functions within the two regions which are separated by the negative x-axis, and they are individually discontinuous across this boundary. The unknown constants $A_0$ and $C_0$ are determined by imposing the continuity conditions

$$
\begin{align*}
(I) & \quad [h] = \{h(x, +0) - h(x, -0)\} = 0 \quad x < 0 \\
(II) & \quad \left[ \frac{\partial h}{\partial y} \right] = \left\{ \frac{\partial h}{\partial y} (x, +0) - \frac{\partial h}{\partial y} (x, -0) \right\} = 0 \quad x < 0
\end{align*}
$$

Differentiating Eq. (2-15) with respect to $y$, and then integrating by parts, using the relationship

$$
(2-17) \quad \frac{\partial}{\partial y} \left[ \frac{H_1^{(1)} (kr)}{3} \cos \frac{\theta}{3} \right] = \frac{\partial}{\partial x} \left[ \frac{H_1^{(1)} (kr)}{3} \sin \frac{\theta}{3} \right] + ke \frac{2}{3} \pi
$$

Then,

$$
\frac{\partial h}{\partial y} = A_0 \frac{H_1^{(1)} (kr)}{3} \sin \frac{\theta}{3} - A_0 \alpha e \int_{x}^{\infty} e^{-\alpha \xi} H_1^{(1)} (kr) \sin \frac{\theta}{3} d\xi
$$

$$
(2-18) \quad -A_0 k e \left( -\frac{2}{3} \pi \right) e \alpha x \int_{x}^{\infty} e^{-\alpha \xi} H_2^{(1)} (kr) \sin \frac{2}{3} \theta d\xi
$$

$$
\left\{ \begin{array}{l}
\left[ \frac{i k^2 + \alpha^2}{k^2 + \alpha^2} \right] e^{\alpha x} + i \left[ \sqrt{k^2 + \alpha^2} y - i \sqrt{k^2 + \alpha^2} \right] C_0 e^{\alpha x - i \sqrt{k^2 + \alpha^2} y} \\
0
\end{array} \right. \quad y < 0
$$

$$
\left\{ \begin{array}{l}
\left[ \sqrt{k^2 + \alpha^2} \right] e^{\alpha x} - i \left[ \sqrt{k^2 + \alpha^2} y - i \sqrt{k^2 + \alpha^2} \right] C_0 e^{\alpha x + i \sqrt{k^2 + \alpha^2} y} \\
0
\end{array} \right. \quad y > 0
$$
Now, imposing the continuity conditions (2-16), and noting that when Eqs. (2-15) and (2-18) are substituted into (2-16), \( \theta = 0 \) for \( y = +0 \) and \( \theta = 2\pi \) for \( y = -0 \)

\[
\begin{align*}
(2-19) \quad & \begin{cases}
\frac{3}{2} I_1 A_o + C_o = -1 \\
\frac{\sqrt{3}}{2} \left( \frac{I_1}{3} \alpha - \frac{I_2}{3} k e^{i \frac{2}{3} \pi} \right) A_o + i \sqrt{k^2 + \alpha^2} C_o = i \sqrt{k^2 + \alpha^2}
\end{cases}
\end{align*}
\]

Where \( I_\nu \) is the Laplace transform of \( H_\nu^{(1)} (k |\xi|) \) and can be evaluated explicitly as given by Magnus and Oberhettinger

\[
(2-20) \quad I_\nu = \int_0^\infty e^{-\alpha \xi} H_\nu^{(1)} (k |\xi|) \, d\xi
\]

\[
\frac{(\sqrt{k^2 + \alpha^2} - \alpha)^\nu}{k \nu \sqrt{k^2 + \alpha^2}} \left\{ 1 + \frac{i}{\sin \nu \pi} \left[ \cos \nu \pi - \frac{(\alpha + \sqrt{\alpha^2 + k^2})}{k^{2\nu}} \right] \right\}
\]

where \(-1 < \text{Re} \nu < 1\).

Solving Eq. (2-19) gives

\[
(2-21) \quad A_o = \frac{-4i \sqrt{k^2 + \alpha^2}}{3 i \sqrt{k^2 + \alpha^2} \frac{I_1}{3} - \sqrt{3} \left( \frac{I_1}{3} \alpha - \frac{I_2}{3} k e^{i \frac{2}{3} \pi} \right)}
\]

\[
(2-22) \quad C_o = \frac{3 i \sqrt{k^2 + \alpha^2} \frac{I_1}{3} + \sqrt{3} \left( \frac{I_1}{3} \alpha - \frac{I_2}{3} k e^{i \frac{2}{3} \pi} \right)}{3 i \sqrt{k^2 + \alpha^2} \frac{I_1}{3} - \sqrt{3} \left( \frac{I_1}{3} \alpha - \frac{I_2}{3} k e^{i \frac{2}{3} \pi} \right)}
\]
Substituting the above two constants into Eq. (2-15) completes the formal solution of the problem. Here $|C_0|^2$ is the power reflection coefficient for the reflection of the TM surface wave by the tip of the wedge and has been computed as a function of the relative surface reactance $\alpha/k$ in Fig. 2. It is not convenient to reduce the radiating term in the formal solution to a form amenable to the numerical calculation of the far-zone field; therefore, it is desirable to find the expression for the far field by the following simple, direct method.

![Graph](image)

Fig. 2. The power reflection coefficient of a TM incident surface wave as a function of the relative surface reactance $\alpha/k$. 
The asymptotic form of the auxiliary function $g$ is

$$
(2-23) \quad (g)_a = A_0 \sqrt{\frac{2}{\pi kr}} e^{i\left(kr - \frac{5\pi}{12}\right)} \cos \frac{1}{3} \theta.
$$

The far field expression for the original diffracted field $h^d$ is expected to have a similar form

$$
(2-24) \quad (h)_a^d = \frac{m(\theta)}{r} e^{ikr},
$$

where $m(\theta)$ is to be determined. Using the far-field approximation

$$
(2-25) \quad \frac{\partial}{\partial x} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \approx \cos \theta \frac{\partial}{\partial r},
$$

the transformation equation (2-10) becomes

$$
(2-26) \quad (g)_a = \left( \cos \theta \frac{\partial}{\partial r} - \alpha \right) (h)_a^d.
$$

Substituting Eqs. (2-23) and (2-24) into Eq. (2-26)

$$
(2-27) \quad A_0 \sqrt{\frac{2}{\pi kr}} e^{-i\frac{5\pi}{12}} \cos \frac{1}{3} \theta \approx (ik \cos \theta - \alpha) \frac{m(\theta)}{r},
$$

or

$$
(2-28) \quad m(\theta) = A_0 \sqrt{\frac{2}{\pi k}} e^{-i\frac{5\pi}{12}} \frac{\cos \frac{1}{3} \theta}{ik \cos \theta - \alpha}
$$

and

$$
(2-29) \quad |m(\theta)|^2 = \frac{2 |A_0|^2}{\pi k^3} \frac{\cos^2 \frac{1}{3} \theta}{(\alpha/k)^2 + \cos^2 \theta}.
$$

Putting $m(\theta)$ back into (2-24), the far-zone expression for the diffracted field finally becomes
(2-30) \( \frac{d}{a} = A_0 \sqrt{\frac{2}{\pi kr}} \frac{\cos \frac{1}{3} \theta}{ik \cos \theta - \alpha} e^{i(kr - \frac{5\pi}{12})} \)

\[ |m(\theta)|^2 \]

is the power pattern of the incident TM surface wave diffracted by this infinite right-angled wedge, and is plotted for several values of \( \alpha/k \) in Fig. 3. The radiation pattern becomes more directive when \( \alpha/k \) decreases and the direction of the pattern maximum always lies a little below the direction of the incident surface wave.

C. TE SURFACE WAVE DIFFRACTION

The diffraction of a TE surface wave by an infinite right-angled wedge as shown in Fig. 1 may be analyzed by a method parallel to that for the TM case. For this reason, all of the details of the method will not be repeated in the following solution. A TE surface wave traveling in the positive \( y \)-direction is guided by the reactive surface of surface reactance \( X \) which is capacitive and hence negative in sign. The only electric field component in this incident surface wave is the \( z \)-component, and it may be written as

\[ e_{\text{inc}}(x, y) = e^{i\beta y + \alpha x} \quad x \leq 0, \quad \beta = \sqrt{k^2 + \alpha^2} \]

where \( \alpha \) is a positive real quantity and is related to the reactance by \( \alpha = -\omega \mu /X \). The total electromagnetic field is independent of \( z \), and hence is completely determined by the value of \( E_z \); therefore, from
Fig. 3. Normalized power patterns for TM incident surface waves of constant power content.
Maxwell's Equations

(2-32) \[ E_x = E_y = H_z = 0 \]

and

(2-33) \[
\begin{align*}
H_x & = \frac{1}{i\omega \mu} \frac{\partial E_z}{\partial y} \\
H_y & = -\frac{1}{i\omega \mu} \frac{\partial E_z}{\partial x}
\end{align*}
\]

\( E_z = e \) should satisfy the homogeneous wave equation

(2-34) \[ (\nabla^2 + k^2) e = 0 \]

and the boundary conditions on the wedge

(2-35) \[
\begin{align*}
e & = 0 & y = 0, \ x \geq 0 \\
\frac{\partial e}{\partial x} - \alpha e & = 0 & x = 0, \ y \leq 0
\end{align*}
\]

where the second equation in (2-35) corresponds to the impedance relation \( E_z = iX H_y \) on the reactive surface. As in the preceding section, the following auxiliary function is introduced,

(2-36) \[ f = \left( \frac{\partial}{\partial x} - \alpha \right) e \]

Then \( f \) satisfies the wave equation

(2-37) \[ (\nabla^2 + k^2) f = 0 \]

and the simple homogeneous boundary conditions

(2-38) \[
\begin{align*}
f & = 0 & y = 0, \ x \geq 0 \\
f & = 0 & x = 0, \ y \leq 0
\end{align*}
\]
e can be obtained from \( f \) by integrating (2-36)

\[
(2-39) \quad e(x, y) = -e^{\alpha x} \int_0^\infty e^{-\alpha \xi} f(\xi, y) \, d\xi + e^{\alpha x} T(y)
\]

where \( T(y) \) is an arbitrary function of \( y \) which can be adjusted so that \( e(x, y) \) meets the requirements of the problem.

The most general representation for \( f \) which satisfies (2-37) and (2-38) is

\[
(2-40) \quad f(r, \theta) = \sum_{n=1}^\infty B_n H_{2n}^{(1)}(kr) \sin \frac{2n}{3} \theta .
\]

However, only those terms that yield a finite value of the electromagnetic field everywhere in the original physical problem for \( e \) should be retained. It follows that \( B_1 \) is the only non-vanishing coefficient in the above series. Substituting (2-40) into (2-39) and introducing the incident and reflected surface wave terms for \( e^{\alpha x} T(y) \),

\[
(2-41) \quad e(x, y) = -B_1 e^{\alpha x} \int_0^\infty e^{-\alpha \xi} \frac{H_2^{(1)}(kr)}{3} \sin \frac{2}{3} \theta \, d\xi
\]

\[
\begin{cases} 
  e^{\alpha x} + i \sqrt{k^2 + \alpha^2} y + D_1 e^{\alpha x} - i \sqrt{k^2 + \alpha^2} y & y < 0 \\
  0 & y > 0 
\end{cases}
\]

is obtained. The unknown constants \( B_1 \) and \( D_1 \) are determined by satisfying the following continuity conditions across the negative \( x \)-axis:
Differentiating Eq. (2-41) with respect to \( y \), and then integrating by parts, with the aid of the following equation

\[
(2-43) \quad \frac{\partial}{\partial y} \left[ \frac{H_2^{(1)} (kr) \sin \frac{2}{3} \theta}{3} \right] = -\frac{\partial}{\partial x} \left[ \frac{H_2^{(1)} (kr) \cos \frac{2}{3} \theta}{3} \right] + ke \frac{\pi}{3} \frac{H_1^{(1)} (kr)}{3} \cos \frac{1}{3} \theta.
\]

\[
(2-44) \quad \frac{\partial e}{\partial y} = -B_1 \frac{H_2^{(1)} (kr) \cos \frac{2}{3} \theta + B_1 a e^{\alpha x} \int_{x}^{\infty} e^{-\alpha \xi} \frac{H_2^{(1)} (kr)}{3} \cos \frac{1}{3} \theta d\xi}{3}
\]

Substituting (2-41) and (2-44) into (2-42) leads to

\[
(2-45) \quad \begin{cases}
\frac{3}{2} I_2 B_1 + D_1 = -1 \\
\frac{3}{2} (ke^{\frac{\pi}{3}} I_1 - \alpha I_2) B_1 - i \sqrt{k^2 + \alpha^2} D_1 = -i \sqrt{k^2 + \alpha^2}
\end{cases}
\]

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where $I_1$ and $I_2$ are two integrals which have been expressed in terms of the elementary functions in the preceding section. Solving Eqs. (2-45) gives

$$B_1 = \frac{+ 4 i \sqrt{k^2 + \alpha^2}}{-i \sqrt{3} \left[k^2 + \alpha^2 I_2 + 3(\alpha I_2 - k e^{i \frac{\pi}{3}} I_1)\right] \frac{3}{3}}$$

$$D_1 = \frac{-i \sqrt{3} \left[k^2 + \alpha^2 I_2 - 3 \left(2 I_2 - k e^{i \frac{\pi}{3}} I_1\right)\right]}{-i \sqrt{3} \left[k^2 + \alpha^2 I_2 + 3(\alpha I_2 - k e^{i \frac{\pi}{3}} I_1)\right] \frac{3}{3}}$$

The formal solution is completed by substituting (2-46) and (2-47) into (2-41). Now $|D_1|^2$ is the power reflection coefficient for the reflection of the TE surface wave by the edge of the wedge and has been computed as a function of $\alpha/k$ in Fig. 4. A comparison of Fig. 2 with Fig. 4 shows that for the same propagation constant the power reflection coefficient of the TE surface wave is always larger than that of the TM surface wave.

The far field can also be obtained in a simple form suitable for numerical computation. The asymptotic form of the auxiliary function $f$ is

$$f_a = B_1 \left\{ \frac{2}{\pi kr} e^{i(kr - \frac{7\pi}{12})} \sin \frac{2}{3} \theta \right\}$$

The far-zone form of the original diffracted field $e^d$ is expected to be
Fig. 4. The power reflection coefficient of a TE incident surface wave as a function of the relative surface reactance α/k.

\[(2-49)\quad (e)^d_a = \frac{n(θ)}{\sqrt{r}} e^{ikr}\]

where \( n(θ) \) is to be determined. As in the preceding section, the transformation equation in the far-zone can be approximated by

\[(2-50)\quad (f)^a (\cos θ \frac{∂}{∂r} - α) (e)^d_a \]

Substituting Eqs. \(2-48\) and \(2-49\) into \(2-50\)

\[(2-51)\quad B_1 \sqrt{\frac{2}{\pi k}} e^{-i \frac{7π}{12}} \sin \frac{2}{3} θ \approx (ik \cos θ - α) n(θ)\]

or
\((2-52)\)  
\[ n(\theta) = B_1 \sqrt{\frac{2}{\pi k}} e^{-i \frac{7\pi}{12}} \frac{\sin \frac{2}{3} \theta}{ik \cos \theta - \alpha} \]

and

\[ |n(\theta)|^2 = \frac{2|B_1|^2}{\pi k^3} \frac{\sin^2 \frac{2}{3} \theta}{(\alpha/k)^2 + \cos^2 \theta} \]

Substituting \((2-52)\) back into \((2-49)\),

\[(2-53) \quad (e)^d_{a} = B_1 \sqrt{\frac{2}{\pi kr}} \frac{\sin \frac{2}{3} \theta}{ik \cos \theta - \alpha} e^{i(kr - \frac{7\pi}{12})} \]

Here \(|n(\theta)|^2\) is the power pattern of the incident TE surface wave diffracted by the infinite right-angled wedge, and numerical curves are given in Fig. 5 for several values of \(\alpha/k\). As in the TM case, the pattern sharpens when \(\alpha/k\) decreases. However, the direction of the pattern maximum now lies somewhat above the direction of the incident surface wave.

D. EFFECT OF TRUNCATING THE CONDUCTING SURFACE

It is desirable to find the change in the surface wave diffraction pattern when the perfectly conducting surface of the right-angled wedge is truncated. This structure is clearly a more realistic model for the termination of a surface wave antenna, and the results of an approximate calculation for the effect of truncation will be described in this section. The original current distribution on the infinite perfectly conducting surface of the wedge is taken as an approximation to the true current distribution on the truncated surface.
Fig. 5. Normalized power patterns for TE incident surface waves of constant power content.
problem can be conveniently reformulated as follows: the radiation field due to the wedge with a finite face can be calculated as the difference between the radiation field due to the wedge with semi-infinite faces and that due to the surface which complements the finite face. The current distribution on the complementary surface is the same as that on the corresponding part of the infinite perfectly-conducting surface. This approximate method was proposed by Meixner\textsuperscript{23} for the problem of the radiation from annular slots on a finite circular disk and has also been applied successfully to the finite conical antenna by Adachi et al.\textsuperscript{24}

The configuration of the truncated wedge is shown in Fig. 6, where \( l \) is the length of the truncated surface, and the remaining part of the positive \( x \)-axis is the complementary sheet. An incident TM surface wave will be first considered. Let the far-zone magnetic fields in the cases of the complete wedge and the wedge truncated in the manner described be denoted as \( H_z^\infty \) and \( H_z \), respectively, and the far-zone magnetic field due to the infinite wedge current distribution flowing on the complementary surface be denoted as \( H_z^c \). Then the far-zone magnetic field under consideration is approximately given by

\begin{equation}
(2-54) \quad H_z \cong H_z^\infty - H_z^c,
\end{equation}

where the far-zone magnetic field \( H_z^\infty \) is given as \((h)_a^d\) in Eq. (2-30).
Fig. 6. The truncated right-angled wedge and its complementary sheet.
The original current distribution on the perfectly conducting surface of the infinite wedge has only an x-component and can be easily deduced from Eq. (2-30) when \( kx > 1 \)

\[
J_x(x) = \frac{A_0}{ik - \alpha} \sqrt{\frac{2}{\pi kx}} e^{i(kx - \frac{5\pi}{12})}. 
\]

(2-55)

The far-zone field of a line source of electric dipoles positioned along the z-axis can be shown to be (see Appendix I)

\[
H_z^0 = \frac{i}{4} \sqrt{\frac{2}{\pi kR}} \int J_x(x) \sin \theta e^{i[kR - \frac{3}{4} \pi]} \sin \theta, 
\]

(2-56)

and the far-zone magnetic field due to a current distribution (2-55) on the complementary sheet is obtained by applying Eq. (2-56) and integrating over the sheet.

\[
H_z^c = - \frac{i}{2\pi} \frac{A_0}{ik - \alpha} \sin \theta \int_{x_0}^\infty \frac{e^{ikx(1-\cos \theta)}}{\sqrt{x}} dx. 
\]

(2-57)

The integral expression on the right-hand side can be easily reduced to the well known Fresnel's Integral which is tabulated in the literature.\(^{27,28}\)

\[
H_z^c = - \frac{A_0}{ik - \alpha} \frac{\cos \theta}{\pi kR} e^{i(kR + \frac{\pi}{3})}, 
\]

\[
\left\{ \frac{1}{2} - C \left[ kx_0 (1 - \cos \theta) \right] + \frac{1}{2} - i S \left[ kx_0 (1 - \cos \theta) \right] \right\} 
\]

(2-58)

where \(C(x)\) and \(S(x)\) are the standard forms of Fresnel's Integrals, i.e.,
\[(2-59) \quad C(x) + iS(x) = \int_0^\infty \frac{\cos t}{\sqrt{2\pi t}} \, dt + i \int_0^\infty \frac{\sin t}{\sqrt{2\pi t}} \, dt.\]

Substituting Eqs. (2-30) and (2-58) into (2-54), one can immediately write down an approximate expression for the far-zone magnetic field due to the truncated wedge.

\[(2-60) \quad H_z \approx A_0 \left[ \frac{2}{\pi kr} \right] e^{i(kr - \frac{5\pi}{12})} \left\{ \frac{\cos \frac{1}{3} \theta}{ik \cos \theta - i} \right\}
\]

\[-\frac{\cos \frac{\theta}{2}}{\sqrt{2(ik - \alpha)}} e^{-\frac{i\pi}{4} \left[ \frac{1}{2} - C(kx_0 \frac{1 - \cos \theta}{1 - \cos \theta}) + i \frac{1}{2} - i S(kx_0 \frac{1 - \cos \theta}{1 - \cos \theta}) \right]} \}

Numerical patterns have been obtained from the above expression for \(kx_0 = 5\pi\). Figure 7 shows that the diffraction pattern of a tightly bound surface wave corresponding to \(\alpha/k = 1\) deteriorates considerably in the presence of the truncation. Figure 8 shows that the diffraction pattern of a more loosely bound surface wave corresponding to \(\alpha/k = 0.32\) is very little affected by the truncation.

This approximate method can also be applied to estimate the effect of the truncation on the diffraction pattern of an incident TE surface wave. However, it can be easily shown that the current distribution on the perfectly conducting face of the wedge due to a TE surface wave incident along the reactive face is inversely proportional to the three-half power of the distance from the origin when \(kx \gg 1\). This indicates that the effect of the truncation on the TE pattern is much smaller.
Fig. 7. Surface wave diffraction patterns for $\beta/k = 1.414$. 

By The Truncated Wedge 

By The Infinite Wedge 

TM Surface Wave 

Reactive Surface 

($\alpha/k = 1$, $\beta/k = 1.414$) 

Perfectly Conducting Surface 

2.5 $h$
Surface wave diffraction power patterns for $\beta/k = 1.05$. 

Fig. 8. Surface wave diffraction power patterns for $\beta/k = 1.05$. 

$\alpha_k = 0.32$, $\beta_k = 1.05$
E. DISCUSSION

It is of interest to determine the values of the parameters for very
loosely bound surface waves and very tightly bound surface waves.
These limiting cases correspond to a small $\alpha/k$ and a large $\alpha/k$.

First consider the case of small $\alpha/k$, i.e., $\alpha/k$ approaches zero.

It is easily shown that

\begin{align*}
(2-61) \quad I_1 &\approx \frac{1}{\sqrt{3}} \left(1 - \frac{i}{\sqrt{3}}\right) \\
(2-62) \quad I_2 &\approx \frac{1}{\sqrt{3}} \left(1 - i\sqrt{3}\right) .
\end{align*}

Substituting (2-61) and (2-62) into (2-21), (2-22), (2-46) and (2-47),
the pattern coefficients $A_0$ and $B_1$, and the reflection coefficients $C_0$
and $D_1$ for small $\alpha/k$ have the following limiting forms:

\begin{align*}
(2-63) \quad A_0 &\approx \frac{k}{\sqrt{3}} e^{-i \frac{5}{6} \pi} , \\
(2-64) \quad B_1 &\approx \frac{k}{\sqrt{3}} e^{-i \frac{2\pi}{3}} , \\
(2-65) \quad C_0 &\approx \frac{1}{2\sqrt{3}} \frac{\alpha}{k} e^{i \frac{3}{2} \pi} , \\
(2-66) \quad D_1 &\approx \frac{\sqrt{3}}{2} \frac{\alpha}{k} e^{i \frac{3}{2} \pi} .
\end{align*}

Both the reflection coefficients for TM and TE surface waves approach
zero as $\alpha/k$ approaches zero. It is certainly reasonable to expect a
zero reflection coefficient for the limiting case of an incident homogeneous plane wave. This fact also shows that almost all of the energy in
a loosely bound surface wave is radiated, with a very small amount being reflected at the termination of the structure.

Next consider the case of large $\alpha/k$, i.e., $\alpha/k$ approaches infinity. It is easily shown that

\[(2-67) \quad \frac{I_1}{3} \approx -i \frac{2}{\sqrt{3}} \left( \frac{2}{\alpha k} \right)^{2\beta} \]

\[(2-68) \quad \frac{I_2}{3} \approx -i \frac{2}{\sqrt{3}} \left( \frac{2}{k} \right)^{2\beta} \left( \frac{1}{\alpha} \right)^{2\beta} \]

Substituting (2-67) and (2-68) into (2-21), (2-22), (2-46) and (2-47) the limiting forms for large $\alpha/k$ become

\[(2-69) \quad A_0 \approx -\left( \frac{\alpha k}{2} \right)^{2\beta} e^{i\frac{\pi}{3}} \]

\[(2-70) \quad B_1 \approx \alpha^{2\beta} \left( \frac{k}{2} \right)^{2\beta} e^{i\frac{\pi}{6}} \]

\[(2-71) \quad C_0 \approx e^{-i\frac{\pi}{3}} \]

\[(2-72) \quad D_1 \approx e^{-i\frac{2\pi}{3}} \]

The magnitude of the reflected surface wave approaches that of the incident surface wave as $\alpha/k$ approaches infinity. Hence in the case of a very tightly bound incident surface wave, most of the energy is reflected back from the termination of the structure and there is very
little radiated. This statement can be further checked by substituting
the limiting forms (2-69) and (2-70) of the pattern coefficients for
large $\alpha/k$ back into the far-field expressions (2-30) and (2-53). It
is easily seen that the magnitude of the radiation field vanishes as
$\alpha/k$ becomes infinitely large.

Kay\textsuperscript{16} has obtained the reflection coefficient of a TM surface wave
scattered by a discontinuity in the surface reactance of a plane surface.
It is interesting to compare the magnitude of the reflected TM surface
wave for a right-angled wedge with that for a plane surface. When the
reactance is large the amplitudes of the reflected surface waves are
the same, because both approach that of the incident surface wave. It
seems reasonable to speculate that this is true for any wedge regardless
of the angle. In the case of a surface with small reactance, the
amplitude of the reflected surface wave for a wedge approaches zero
as the first power of ($\alpha/k$) while the reflected amplitude for a
discontinuity of reactance on a plane surface approaches zero as the
square of $\alpha/k$.

The problem of an incident plane wave diffracted by a perfectly
conducting wedge has been solved by Reiche\textsuperscript{29} among others. When
the direction of the incident plane wave is parallel to the surface, his
result agrees with the limiting form of the far-field expression for
very small $\alpha/k$. 

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In calculating the radiation pattern of the aperture type antenna, one can often take the unperturbed field of the incident guided wave as an approximation to the source distribution in the aperture. It is easy to calculate the approximate radiation pattern of the surface wave diffraction by this Kirchhoff type of approximation (see Appendix II).

The approximate expression for the far-zone magnetic field due to the diffraction of a TM surface wave by the wedge is

\[
\left(2-73\right) \quad (h)^d_{\text{app.}} = -\frac{\beta}{2\sqrt{\pi kr}} \frac{1}{ik \cos \theta - \alpha} e^{i(kr - \frac{\pi}{4})}
\]

\[
\left(2-74\right) \quad \left|(h)^d_{\text{app.}}\right|^2 = \frac{1}{2\pi kr} \frac{1}{(\alpha/k)^2 + \cos^2 \theta} \left(\frac{\beta}{k}\right)^2
\]

and the approximate expression for the far-zone electric field due to the diffraction of a TE surface wave by the wedge is

\[
\left(2-75\right) \quad (e)^d_{\text{app.}} = -\frac{ik}{2} \sqrt{\frac{2}{\pi kr}} \frac{\sin \theta}{ik \cos \theta - \alpha} e^{i(kr - \frac{3\pi}{4})}
\]

\[
\left(2-76\right) \quad \left|(e)^d_{\text{app.}}\right|^2 = \frac{1}{2\pi kr} \frac{\sin^2 \theta}{(\alpha/k)^2 + \cos^2 \theta}
\]

It is of value to compare the above approximate solutions with corresponding exact solutions, because the comparison checks the general accuracy of the approximate method for other surface wave diffraction problems which may be too difficult for exact solution.
Comparisons between power patterns computed from the exact solutions and approximate solutions are shown in Figs. 9 and 10. If one substitutes the limiting forms (2-63) and (2-64) of the pattern coefficients $A_0$ and $B_1$ into the far field expressions (2-30) and (2-53), it is found that for small $\alpha/k$ the far-field phases given in the approximate solutions coincide with those in the exact solutions.

In a recent brief announcement $^{31}$ Maliuzhinets presents a solution to this diffraction problem for a wedge of arbitrary angle when the wedge supports surface waves. His method is different from the one described here and is probably related to the method used by Peters$^{32}$ and Senior. $^{33}$ Numerical results and physical interpretations are not given in his paper. One should be able to obtain the expressions given here starting with his formulations, but the procedure presented in this paper seems much simpler to employ when the problem involves a right-angled wedge.
Fig. 9. Comparison between exact solution and Kirchhoff approximation for the TM case.
Fig. 10. Comparison between exact solution and Kirchhoff approximation for the TE case.
CHAPTER III
APPLICATION TO SURFACE WAVE ANTENNAS

A. A MAGNETIC CURRENT LINE SOURCE EXCITING
A RIGHT-ANGLED WEDGE

Although the excitation of surface waves on an impedance structure and their radiation from its termination, have each been investigated separately for several cases, the combined problem of a source exciting an impedance structure of finite length has not been previously described in the literature. In this section the combined problem will be considered for the case of a magnetic current line source exciting a right-angled wedge. The line source is located close to the reactive surface and far from the edge of the wedge as shown in Fig. 11. The surface wave excitation and the radiation field due to a magnetic current line source located above a reactive surface of infinite extent have been obtained by Cullen.\textsuperscript{12} The radiation field of the surface wave diffraction by a right-angled wedge is given in the preceding chapter. One may propose that the superposition of the above two radiation fields can be taken as an approximation to the total radiation field of a magnetic current line source exciting a right-angled wedge. This suggestion will be justified by using reciprocity in conjunction with an exact solution and the approximations involved in the superposition procedure will be pointed out in the course of this analysis.
Fig. 11. Geometry of a line source radiating in the presence of a right-angled wedge.
The diffraction of a plane wave by a right-angled wedge has been solved exactly by Karal and Karp.\textsuperscript{22} Applying the reciprocity theorem to this exact solution gives the far field of a line source located at any position with respect to the wedge. Nevertheless, the resulting formal expression is too involved for physical interpretation and numerical computation. Since for most surface wave antennas the exciting device is always quite a few wavelengths removed from the termination of the surface wave structure, i.e., the exciting device is often located in the far zone of the scattering from the termination, one can then proceed to simplify the exact solution by an asymptotic form, which is appropriate for this case.

The geometry of the configuration is illustrated in Fig. 11; the unit magnetic current line source A under consideration is located at a distance $R$ from the origin, and at a height $h$ above the reactive surface. It should be noticed that $\Delta \theta$ is very small and $L = \sqrt{R^2 - h^2} \approx R$. Imagine another unit magnetic current line source B which is located in the direction $\theta_0$ and its distance $r_0$ from the origin is large compared with $R$. The free space far-field of this distant line source can be written as

\begin{equation}
H_z^b = U_0 e^{-jk r \cos(\theta - \theta_0)}
\end{equation}

where
Equation (3-1) is essentially a plane wave when \( r << r_0 \). The far field of the diffraction of a plane wave (3-1) by a right-angled wedge has been obtained for region I \((x < 0, y < 0)\)

\[
(3-3) \quad H_z^b = C_2 e^\alpha x - i \beta y + \frac{m(\theta)}{\sqrt{r}} e^{ikr}
\]

where \( C_2 \) is the amplitude of the induced surface wave propagating in the negative \( y \)-direction, and \( m(\theta) \) is the coefficient of the diffraction term. Both \( C_2 \) and \( m(\theta) \) are defined by lengthy expressions in Karal and Karp's paper. The remaining terms may be interpreted in terms of geometrical optics. \( R(\theta_0) \) is the reflection coefficient of a plane wave incident upon the reactive surface in the direction \( \theta_0 \)

\[
(3-4) \quad R(\theta_0) = \frac{ik \cos \theta_0 + \alpha}{ik \cos \theta_0 - \alpha}
\]

For the convenience of the present application the reciprocity theorem is best expressed in terms of the reaction integral.\(^{34}\)
\[ (3-5) \quad \iint_{S_A} \left[ \overline{J}(A) \cdot \overline{E}(B) - \overline{K}(A) \cdot \overline{H}(B) \right] ds \]

\[ = \iint_{S_B} \left[ \overline{J}(B) \cdot \overline{E}(A) - \overline{K}(B) \cdot \overline{H}(A) \right] ds \]

where \( \overline{J}(A) \) and \( \overline{K}(A) \) are the equivalent electric and magnetic currents over the surface \( S_A \) which contains the source \( A \) and \( \overline{E}(B) \) and \( \overline{H}(B) \) are the fields on the surface \( S_A \) due to the source \( B \). \( \overline{J}(B), \overline{K}(B), \overline{E}(A), \) and \( \overline{H}(A) \) are correspondingly defined. It should be mentioned that the source \( A \) and the source \( B \) are radiating in the absence of each other with all the scatterers present.

Now in the problem under consideration \( \overline{J}(A) \) and \( \overline{J}(B) \) do not exist. \( \overline{K}(A) \) corresponds to the unit magnetic current line source \( A \) located at \( (R, \frac{3}{2} \pi - \Delta \theta) \), and \( \overline{K}(B) \) corresponds to the distant unit magnetic current line source \( B \). Since \( \overline{H}(B) \) has been given as \( H^b \) in Eq. (3-3). \( \overline{H}(A) \) is the only unknown in Eq. (3-5). Substituting all the quantities into Eq. (3-5), one obtains the far-zone magnetic field \( \overline{H}(A) = H^a_Z \) of the magnetic current line source exciting the right-angled wedge

\[ (3-6) \quad H^a_Z = C_2 e^{-\alpha h+\beta L} + \frac{m(\Delta \theta)}{\sqrt{\pi}} e^{ikR} \]

\[ + U_0 \begin{cases} \cos(\Delta \theta - \theta_o) + R(\theta_o) e^{\pm ikR \cos(\Delta \theta + \theta_o)} & \frac{\theta_o - \pi}{2} > \Delta \theta \\ \Delta \theta > \theta_o - \frac{\pi}{2} > -\Delta \theta & \frac{\theta_o - \pi}{2} < -\Delta \theta \end{cases} \]
where $C_2$ can be related to the diffraction of the surface wave by the wedge and is found to be (see Appendix III)

$$
(3-7) \quad C_2 = A_0 k Y \sqrt{\frac{2}{\pi kr_0}} \frac{\alpha}{\beta} \frac{\cos \frac{1}{3} \theta_0}{\alpha - ik \cos \theta_0} e^{i(kr_0 - \frac{5\pi}{12})}.
$$

The second term on the right-hand side of Eq. (3-6) can be neglected if the distance between the line source $A$ and the edge is large. However, one may notice that the elimination of the second term gives rise to a discontinuity which can be avoided by discarding the small sector $\Delta \theta > \theta' > -\Delta \theta$. If one introduces $\theta' = \theta_0 - \frac{\pi}{2}$, and substitutes (3-7) into (3-6), some simple algebraic manipulations lead to the following approximate expression:

$$
(3-8) \quad H_z^a \sim U_0 \left[ -4A_0 \frac{\alpha}{\beta} e^{-\alpha h} \frac{\cos \frac{1}{3} (\theta' + \frac{\pi}{2})}{ik \sin \theta' + \alpha} e^{i(\beta L - \frac{\pi}{6})} \right]
$$

$$
+ U_0 \begin{cases}
(e^{-ik \sin \theta'} + \frac{ik \sin \theta' - \alpha}{ik \sin \theta' + \alpha} e^{ik L \cos \theta'}) & \theta' > 0 \\
0 & \theta' < 0
\end{cases}
$$

It is of interest to compare this expression with Cullen's work on excitation of plane surface waves. The first term on the right hand side of Eq. (3-8) is evidently the diffraction of an incident surface wave whose magnitude coincides with that of Cullen's surface wave term. When $\theta' > 0$, the second term on the right-hand side of Eq. (3-8) is identical to Cullen's radiation field over an infinite
reactive sheet.* This comparison clearly demonstrates the separation of the radiation of a surface wave antenna into two major components and justifies the superposition of the two radiation fields as a close approximation to the total radiation field under the conditions mentioned. The minimization of the second term and the region \( \Delta \theta > \theta' > -\Delta \theta \) to the point where they do not materially effect the solution in Eq. (3-7) can be interpreted physically as the situation where the incident field near the termination of the surface wave structure is essentially a surface wave field; in other words, the guiding structure is sufficiently long to effectively launch the surface wave.

In order to find the phase relationship between the two components, Eq. (3-8) for \( \theta' > 0 \) may be rearranged as

\[
H_x^z \approx U_0 \left\{ -4A_0 \frac{A}{B} e^{-\alpha h} \frac{\cos \frac{1}{3} (\theta' + \frac{\pi}{2})}{ik \sin \theta' + \alpha} e^{i(\beta L - \frac{\pi}{6})} 
+ \frac{2i[k \sin \theta' \cos(kh \sin \theta') - \alpha \sin(kh \sin \theta')]}{ik \sin \theta' + \alpha} e^{ik L \cos \theta'} \right\}.
\]

In Eq. (2-63) it is noticed that \( A_0 \) has a phase angle \(- \frac{5}{6} \pi\) when \( \alpha/k \) approaches zero; furthermore, numerical calculation shows that the phase angle of \( A_0 \) deviates less than 0.1 radians from \(- \frac{5}{6} \pi\) so

*An error in sign is found in Cullen's expression for radiation field. This sign error has no effect on the excitation efficiency, but it is significant in pattern synthesis.*
long as \( \alpha' < 0.5 \). Then if \( \sin \theta' > \alpha' \tan(kh \sin \theta') \) and \( \alpha' < 0.5 \), the phase difference between the two components can be obtained as

\[
\Delta \phi = \beta L - kL \cos \theta' - \frac{\pi}{2} \quad (\theta' > 0).
\]

The above equation gives the approximate locations of the first minima which corresponds to \( \Delta \phi = \pi \) and the first side-lobe maxima which corresponds to \( \Delta \phi = 2\pi \).

\[
\theta'(\text{first min.}) = \cos^{-1} \left( \frac{\beta}{k} - \frac{3\pi}{2kL} \right)
\]

\[
\theta'(\text{first side-lobe max.}) = \cos^{-1} \left( \frac{\beta}{k} - \frac{5\pi}{2kL} \right).
\]

These locations are approximate in the sense that the variations in the amplitudes of the two components are not taken into account. In Eq. (3-10) it is also noticed that \( \Delta \phi \) vanishes when \( \theta' = 0 \), if \( \beta L - kL = \frac{\pi}{2} \). However, a small upward tilt of the pattern peak is still possible, because the second component in Eq. (3-9) vanishes, when \( \theta' \) is zero, and its magnitude increases rapidly as \( \theta' \) increases.

The Hansen-Woodyard condition i.e., \( \beta L - kL = \pi \) has frequently served as a design criteria for the trapped surface wave antennas. Zucker first proposed the breaking down of the Hansen-Woodyard pattern expression into two integrals which may be interpreted as the launching and diffraction patterns of identical form.
These two hypothetical radiation components reinforce each other in the endfire direction under the Hansen-Woodyard condition. The success of the Hansen-Woodyard condition with uniform surface wave structures of the axial cylindrical type implies the validity of Zucker's interpretation in those cases. However, Eq. (3-10) indicates that the phase difference between the two components never vanishes for $0 < \theta' < \theta'$ (first min.) when the Hansen-Woodyard condition is satisfied. Their patterns are also by no means similar to each other.

It follows that the Hansen-Woodyard condition is in general not applicable for the trapped surface wave antenna of the plane type.

The above statement is further illustrated by the pattern calculated from Eq. (3-8), which is shown in Fig. 12. The phase velocity of the

![Fig. 12. The amplitude pattern of a magnetic current line source exciting a right-angled wedge ($\beta/k = 1.05$, $L = 10\lambda$, $kh = 1.5625$).](image-url)
surface wave and the length of the reactive surface are so related that the Hansen-Woodyard condition is fulfilled. The line source is located at a height which gives an excitation efficiency of 84%. The excitation efficiency $\varepsilon$ is defined as

$$
\varepsilon = \frac{P_S}{P_S + P_R}
$$

where $P_S = \text{the surface wave power excited by a line source above an infinite reactive surface.}$

$P_R = \text{the power radiated by the line source above the infinite reactive surface.}$

The high side-lobe level in Fig. 12 might be surprising at first sight. The reason for this high side-lobe level can be explained by a close inspection of the two components in Eq. (3-8). The first component due to the surface wave diffraction drops to about half-power in the direction of the first side-lobe peak where the second component due to the direct radiation of the line source just rises to a nearly maximum value. The pattern shown in Fig. 13 is calculated for the same configuration but at a lower frequency. Although at this lower frequency the Hansen-Woodyard condition is not met, i.e., $\beta L - kL < \pi$, and the excitation efficiency is lower, a reduction in side-lobe level is obtained in Fig. 13. It is also noted that the phase difference between the two components at the peak of the major lobe in Fig. 13 is smaller.
Fig. 13. The amplitude pattern of a magnetic current line source exciting a right-angled wedge ($\beta/k = 1.033$, $L = 8.34 \lambda$, $kh = 1.309$).

than that in Fig. 12. Therefore, it is evident that the Hansen-Woodyard condition is not valid for the examples treated and the phase relationship between the two radiation components and their individual patterns should be relied upon as the more basic information. Also, it has been shown that the pure surface wave diffraction pattern has zero side-lobes, therefore very low side-lobe level can be obtained if the radiation from the primary source is kept small enough.

The theoretical model analyzed in this section is chosen for the availability of a rigorous solution. The study is mainly concerned with the justification and the implication of the superposition
method. No efforts have been made to obtain the lowest side-lobe level which can be achieved by this structure. Although the analysis presented here is restricted to a magnetic current line source exciting a right-angled wedge. Its extension to other sources and terminations is clear.

B. EXPERIMENTS

Experiments have been made to verify the calculated patterns described in the preceding section. Since the pattern measurement is already a standardized experiment, the details of the equipment and the procedure will not be given here. The sketch of the experimental model is given in Fig. 14. The magnetic current line source is simulated by a slot (0.159 cm wide; 13 cm long) in the metal plate AC. The slot is fed by a regular H-plane horn through a tapered transition section. A 0.159 cm thick polystyrene sheet backed by a metal plate is taken as a simple physical approximation to the reactive surface. DE is another metal plate which forms the perfectly conducting face of the wedge. The transverse dimension perpendicular to the page is 30 cm. This experimental model would be ideal if the ground planes AC and DE could be extended very far beyond the whole measuring system and if the reactive surface could be approximated by a corrugated conducting surface with narrow grooves. However, the available pattern range in the Antenna Laboratory and the
Fig. 14. Sketch of the experimental model.
commercial polystyrene sheet made the arrangement described in Fig. 14 very convenient. The patterns obtained with this structure in K\textsubscript{u} band are characterized by many high spikes which may be attributed to the finite size of the upper ground plane AC. An attempt was made to remove the spikes by adding two curved sections as shown by the dotted curves CF and EG. The sharp spikes are somewhat reduced but not eliminated. It was then decided to remove the section BC which is almost the entire upper ground plane. Now the exciting source is essentially a regular horn which is very narrow in the E-plane and very wide in the H-plane. This feed can still be taken as a crude approximation to a magnetic current line source, because owing to the small terminal admittance, the radiation by its aperture electric current distribution should be much smaller than that by its aperture magnetic current distribution. The amplitude patterns of this final model are measured at 15 kmca and 12.5 kmca which correspond to the electromagnetic dimensions of the calculated patterns in Figs. 12 and 13. Considering the approximations made in the calculation and measurement, the comparisons shown in Figs. 12 and 13 can be regarded as reasonably good agreement. The directivity of the slot pattern over that of a true line source probably caused the discrepancy in the side-lobe levels.
The theoretical analysis in this work is restricted to the wedge with a right angle. It is of interest to observe experimentally the change in the surface wave diffraction pattern when the wedge angle deviates from a right angle. Since no perfect surface wave launcher is available, a series of patterns of the line source excitation structure of Fig. 14 have been measured for various values of the wedge angle \( \psi \). These experimental patterns measured at 12.5 km are shown in Fig. 15. No significant changes in the pattern occurs until the wedge angle \( \psi \) increases to 150°. It is believed that the low side-lobe level in the case when \( \psi = 180° \) results from a possible 6 db enhancement of the pattern peak caused by the influence of the conducting surface.
Fig. 15. The measured amplitude patterns for various wedge angles $\psi$ at 12.5 KMC ($\beta/k = 1.033$, $L = 8.34\lambda$, $kh = 1.309$).
CHAPTER IV
CONCLUSIONS

An exact mathematical solution has been obtained for the diffraction of a surface wave by an infinite right-angled wedge which sustains the surface wave on one face and is perfectly conducting on the other face. The difficulty of the mixed boundary condition in this problem is overcome by a simple transformation which was originally proposed by Stoker and Lewy for certain problems involving water waves. The diffraction of both TM and TE incident surface waves are treated. In each case the reflection coefficients and the far-field patterns are given in simple closed forms. The resulting compact expressions can be readily evaluated by desk calculations and numerical curves are presented for various values of $\alpha/k$ which correspond to surface reactances. When the reactive surface has a small $\alpha/k$, which implies a loosely bound surface wave, the power reflection coefficient is very small, and most of the energy is radiated in a forward directive beam without side-lobes. The smaller the value of $\alpha/k$, the narrower becomes the single lobe of the radiation pattern. When the reactive surface has a large $\alpha/k$, which implies a tightly bound surface wave, more energy will be reflected back and the remaining energy is radiated in a broad lobe. Consideration of the limiting cases shows that when the value of $\alpha/k$ approaches infinity,
the magnitude of the reflection coefficient approaches unity; when the value of $\alpha/k$ approaches zero, the magnitude of the reflection coefficient approaches zero as the first power of $\alpha/k$. With the same value of $\alpha/k$ there is more reflection for the TE surface wave than for the TM surface wave. A small downward beam tilt is observed in the TM surface wave diffraction pattern, while a small upward beam tilt is observed in the TE surface wave diffraction pattern.

When the perfectly conducting face of the wedge is truncated, the effect on the diffraction pattern is considered by an approximate method which has been proposed by Meixner. The original current distribution on the infinite, perfectly conducting face is taken as an approximation to the current distribution on the truncated, perfectly conducting face. Numerical computations using the approximate results have been made for a truncation of the perfectly conducting surface at 2.5 wavelengths from the edge of the wedge. The diffraction pattern for a tightly bound TM surface wave is greatly distorted, while that for a loosely bound TM surface is not appreciably affected. This approximate method reveals very little effect on the TE surface wave diffraction pattern when the truncation is a few wavelengths away from the edge of the wedge.

A comparison is made between the diffraction patterns obtained from the exact solution and those from Kirchhoff approximation. It
indicates that a Kirchhoff type approximation is quite accurate in the calculation of diffraction patterns resulting from loosely bound incident surface waves.

Using the reciprocity theorem, the field is calculated in the case of a magnetic current line source exciting a right-angled wedge which is reactive on one face and perfectly conducting on the other face. It is shown that when the line source is far from the edge and is close to the reactive surface, its radiation field can be approximately divided into one component due to the diffraction of the surface wave by the edge of the wedge and a second component due to the original radiation of the line source above a reactive surface of infinite extent. The Hansen-Woodyard condition, which implies launching and diffraction patterns of identical form with opposite sign, is in general not valid for the type of planar surface wave antenna under consideration. Two examples are calculated and checked by measurement to illustrate the above statement. The phase relationship between the two radiation fields and their individual patterns should be relied upon as the more basic information. A radiation pattern without side-lobes could be achieved with 100% excitation efficiency, hence very low side lobe levels with planar type surface wave structures are possible if a very high excitation efficiency can be achieved.
A series of pattern measurements have also been made for the line source exciting wedges of angles other than the right-angle; no significant change in the pattern occurs until the wedge angle increases to 150°. The lower side-lobe level observed for the case when the wedge angle equals 180° probably results from a 6 db enhancement of the pattern peak. This enhancement would appear to result from a partial blocking of the radiation into the first quadrant \((0 \leq \theta \leq \pi/2)\) and the resulting imperfect imaging effect in the finite ground plane.

Although the analysis presented here is restricted to a magnetic current line source exciting a right-angled wedge, its extension to structures involving other sources and terminations is clear. In this connection the Kirchhoff-type pattern calculation and the approximate method for truncation may often be found useful.

This work indicates that a loosely bound surface wave at the termination of the surface wave antenna is essential not only for the higher gain but also for least sensitivity to diffraction geometry; and a very high excitation efficiency is required to achieve a low side-lobe level. It is also well known that a tightly bound surface wave can be more easily launched with a high excitation efficiency. These two conflicting requirements may be reconciled by a variable impedance structure. Plane surface wave antennas with tapered surface wave structures were investigated experimentally with some moderate
success. A theoretical analysis is recommended for the surface wave antenna employing a variable impedance structure, in this connection the work of Felsen would probably be of value.
CHAPTER V
REFERENCES


APPENDIX I
LINE SOURCES

The field of a magnetic line current $M_z$ flowing in the $z$-direction

$$F_z = \frac{\epsilon M_z}{4\pi} \int_{z_1}^{z_2} \frac{e^{ikR}}{R} \, dz'$$

where $R = \sqrt{x^2 + y^2 + z'^2}$

$$\overline{E} = -\frac{1}{\epsilon} \nabla \times F$$

$$\overline{H} = \frac{1}{i\omega \mu} \nabla \times \overline{E} = -\frac{1}{i\omega \mu \epsilon} \nabla \times \nabla \times F$$

$$H_z = -\frac{M_z}{4\pi \omega \mu} \int_{z_1}^{z_2} \nabla \times \nabla \times \hat{z} \frac{e^{ikR}}{R} \, dz'$$

$$= -\frac{M_z k^2}{4\pi \omega \mu} \int_{z_1}^{z_2} \frac{e^{ikR}}{R} \, dz'$$

$$= -\frac{M_z k^2}{4\pi \omega \mu} \left[ \pi i H_O^{(1)} (kr) \right] \quad \text{when} \quad z_2 \to +\infty$$

$$\quad \text{and} \quad z_1 \to -\infty$$

$$= -\frac{kYM_z}{4} \overset{1}{H_O} (kr) \quad \text{where} \quad r = \sqrt{x^2 + y^2}$$

where $r$ is the distance from the line source to the field point. When $kr \gg 1$, the far-zone form of the magnetic field becomes
\[ H_z = -\frac{kYM_z}{4} \sqrt{\frac{2}{\pi kr}} e^{i(kr - \frac{\pi}{4})}. \]

Similarly, the field of an electric line current \( I_z \) flowing in the \( z \)-direction is

\[ E_z = -\frac{kYI_z}{4} \mathcal{H}_0^{(1)}(kr). \]

When \( kr \gg 1 \),

\[ E_z = -\frac{kYI_z}{4} \sqrt{\frac{2}{\pi kr}} e^{i(kr - \frac{\pi}{4})}. \]

The field of a line of magnetic dipoles oriented in the \( x \)-direction and continuously distributed with a density \( M_x \Delta x \) in the \( z \)-direction.

\[ F_x = \frac{\varepsilon M_x \Delta x}{4\pi} \int_{z_1}^{z_2} e^{iKR} \frac{dz'}{R} \]

\[ E = -\frac{1}{\varepsilon} \nabla \times F = -\frac{M_x \Delta x}{4\pi} \int_{z_1}^{z_2} \nabla \times \frac{\mathbf{A} e^{iKR}}{R} dz' \]

\[ E_z = \frac{M_x \Delta x}{4\pi} \frac{\partial}{\partial y} \int_{z_1}^{z_2} \frac{e^{iKR}}{R} dz' \]

\[ = \frac{M_x \Delta x}{4\pi} \frac{\partial}{\partial y} \left[ \pi i \mathcal{H}_0^{(1)}(kr) \right] \quad \text{when } z_2 \to +\infty \text{ and } z_1 \to -\infty \]

\[ = -\frac{M_x \Delta x}{4\pi} \mathcal{H}_1^{(1)}(kr) \pi i \frac{V}{r} \]

\[ = -\frac{ikM_x \Delta x}{4\pi} \mathcal{H}_1^{(1)}(kr) \sin \theta \]

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where \( r \) is the distance from the line source to the field point and \( \theta \) is measured from the positive \( x \)-axis. When \( kr \gg 1 \), the far-zone form of the electric field becomes

\[
E_z = -\frac{ikM_x \Delta x}{4} \frac{2}{\sqrt{\pi kr}} e^{i(kr - \frac{3}{4} \pi)} \sin \theta.
\]

Similarly, the field of a line source of electric dipoles oriented in the \( x \)-direction and continuously distributed with a density \( J_x \Delta x \) in the \( z \)-direction is

\[
H_z = \frac{ikJ_x \Delta x}{4} H_1^{(1)}(kr) \sin \theta.
\]

When \( kr \gg 1 \)

\[
H_z = \frac{ikJ_x \Delta x}{4} \frac{2}{\sqrt{\pi kr}} e^{i(kr - \frac{3}{4} \pi)} \sin \theta.
\]
APPENDIX II

DIFFRACTION PATTERNS BY KIRCHHOFF APPROXIMATION

The aperture distribution along the negative x-axis in Fig. 1 is assumed to be the field of the incident TM surface wave. Using the special Green's function for the infinite perfectly conducting screen along the positive x-axis and the far-field expression of a magnetic current line source in Appendix I, the approximate diffraction field associated with the incident TM surface wave can be found as follows

\[ H_z = H_{inc} = e^{i\beta y + \alpha x} \]

and

\[ E_x = -\frac{1}{i\omega} \frac{\partial H_z}{\partial y} = -\frac{\beta}{\omega} e^{i\beta y + \alpha x}. \]

Now since,

\[ \mathbf{M} = \mathbf{E} \times \hat{n} = \mathbf{E} \times \hat{y}, \]

\[ M_z = E_x = -\frac{\beta}{\omega} e^{i\beta y + \alpha x}. \]

Therefore

\[ H_z \sim \frac{-kY}{4} \left( -\frac{2\beta}{\omega} \right) \sqrt{\frac{2}{\pi kr}} e^{i(kr - \frac{\pi}{4})} \int_0^\alpha e^{-ik\cos \theta} \cos \theta d\theta \]

\[ = -\frac{\beta}{2} \sqrt{\frac{2}{\pi kr}} \frac{1}{ik \cos \theta - \alpha} e^{i(kr - \frac{\pi}{4})}. \]
The incident TE surface wave diffraction pattern by Kirchhoff approximation can be found accordingly with the aid of the far-zone expression of a line source of magnetic dipoles derived in Appendix I.

\[ E_z = E_{inc} = e^{i\beta y + \alpha x}. \]

Now since

\[ \overline{M} = \overline{E} \times \hat{n} = \overline{E} \times \hat{y} \]

\[ M_x = -E_z = -e^{i\beta y + \alpha x}. \]

Therefore

\[ E_z^d \sim -\frac{ik}{4} (-2)\sqrt{\frac{2}{\pi kr}} \sin \theta e^{i(kr - \frac{3}{4} \pi)} \int_{-\infty}^{0} e^{\alpha x - ik \cos \theta x} dx \]

\[ = -\frac{ik}{2} \sqrt{\frac{2}{\pi kr}} \frac{\sin \theta}{ik \cos \theta - \alpha} e^{i(kr - \frac{3}{4} \pi)}. \]
Here the reciprocity theorem in the form of Eq. (3-4) is applied. Let $\bar{K}(B)$ be a unit magnetic current line source of infinite extent in the z-direction and at a large distance $r_o$ from the origin in the direction $\theta_o$, and $\bar{J}(B)$ is set equal to zero. Imagine a surface wave incident from the negative y-direction with an equivalent source distribution $\bar{J}(A)$ and $\bar{K}(A)$ located very far from the origin

$$H_z = h_{inc} = e^{i\beta y + \alpha x}$$

$$E_x = -\frac{1}{i\omega \varepsilon} \frac{\partial H_z}{\partial y} = -\frac{\beta}{\omega \varepsilon} e^{i\beta y + \alpha x}$$

$$\bar{J}(A) = H_z \hat{x} = \hat{x} e^{i\beta y + \alpha x}$$

$$\bar{K}(A) = E_x \hat{z} = -z \frac{\beta}{\omega \varepsilon} e^{i\beta y + \alpha x}.$$

$\bar{E}(B)$ and $\bar{H}(B)$ can be easily obtained from Eq. (3-3). $\bar{H}(A)$ is given in Eq. (2-30). Substituting all the quantities into Eq. (3-4) and performing the necessary integrations, $C_2$ is expressed as

$$C_2 = A_o k \sqrt{\frac{2}{\pi kr}} \frac{\alpha}{\beta} \frac{\cos \frac{1}{3} \theta_o}{\beta - i \kappa \cos \theta_o} e^{i(kr_o - \frac{5\pi}{12})}.$$
Note that since the equivalent source distribution of the incident surface wave is located very far from the origin, the second term in Eq. (3-3) is neglected and the region $\Delta \theta > \theta_o - \pi/2 > -\Delta \theta$ is vanishingly small. The integrals resulting from the third term in Eq. (3-3) also vanish.
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