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FOR THE PURPOSE OF INTERPRETING LOW-CYCLE FATIGUE DATA

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

KWO CHANG CHENG, B.Sc., M.S.

*** * ***

The Ohio State University
1960

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This investigation is a theoretical analysis of the state of stress and strain in work-hardening prismatical bars (narrow rectangular section or circular section) subjected to pure bending in the elastic-plastic region. This analysis corresponds to the treatment of one-half cycle of released loading or to one-quarter cycle of fatigue loading. The member is loaded at its ends by pure bending moments in the plane of maximum bending rigidity for the case of a narrow rectangular section. Analytical expressions for the relationship between bending moment and maximum strain or radius of curvature are derived for both the linear work-hardening and nonlinear work-hardening solids, based on the assumption of the uniaxial state of stress.

The assumptions that the neutral surface at all times coincides with the central plane of the bar, and that the radial stress is zero, are certainly admissible for moderate plastic bending, since the radius of curvature is large in relation to the thickness of the bar. For severe plastic bending, where the bar is loaded, for example, to the fully plastic state, the previous assumptions cannot be made because the neutral surface does not coincide with the central surface of the bar and the radial stresses are no longer negligible. The plastic bending problem for the case of uniaxial stress specified above has received thorough coverage in the technical literature. However, no reasonably exact solution that takes radial stresses into consideration for the case of a
work-hardening solid has been offered. The present investigation represents an exploratory attempt in this direction. The biaxial stress analysis for the rectangular prismatical bar subjected to bending is presented for several idealized solids, i.e., completely elastic, rigid perfectly plastic, and rigid strain-hardening solid. The analysis for a rigid strain-hardening solid leads to a nonlinear differential equation which defies analytical integration, and an analytical solution in elementary functions cannot be obtained. Numerical or graphical methods of solution are suggested. This biaxial stress analysis for the rigid strain-hardening solid is based on Nadai-Hencky's plasticity law. An analytical solution for a work-hardening rectangular bar subjected to bending in the elastic-plastic region cannot be obtained. In order to demonstrate the general features of the elastic-plastic solution of the problem, the solution of a rectangular section for a perfectly plastic solid in the elastic-plastic range is presented. A method of solution for a work-hardening solid in the elastic-plastic region is suggested. A method of attacking the present problem for the case of a circular cross section is developed.

A study of functional relationships between the maximum shear stress and normal stress acting on the maximum shear plane for various conventional failure theories is presented for the case of combined bending and torsion. A hypothesis of failure based on the concept of the nonlinear influence of normal stress on the critical shear stress under triaxial state of stress is derived mainly from the mathematical point of view.
CHAPTER I
INTRODUCTION

1. Low Cycle Fatigue Background

The problem of fatigue is generally concerned with lifetimes in excess of $10^5$ cycles, where the applied stress or strain is probably slightly above the initial yield of the material. However, in recent years low-cycle fatigue testing has become increasingly important, as manifested by the increasing number of research papers appearing in the technical literature. The fatigue behavior of various materials in the $10^5$ to $10^8$ cycle endurance life range is well explored and the effects of notches, surface finish, temperature, corrosion, and size on fatigue strength are now well recognized. In the last two decades a great many developments in science and engineering have called for an increase in the fatigue stresses to which metals were previously subjected, and advances in knowledge of plasticity theory, coupled with a fuller understanding of elasto-plastic states of stress, have encouraged the toleration of some degree of yielding under working conditions. The main incentive for low-cycle fatigue investigation has come from both civil and military applications. The cases falling within the former are high-pressure vessels, pipework, and parts of power plants, turbine disks, gas and high temperature steam turbines, compressor blades, boiler drums, structural parts of liquid-metal-cooled nuclear reactors, and many other pieces of heat transfer equipment in which the main variation of stress
and strain occurs on starting and stopping or by thermal transients. In the latter cases, guided missiles, heavy guns, and certain aircraft components, for example, are only required to withstand a comparatively limited number of cycles. For these applications, design for comparatively short endurances, less than, say, \(10^5\) cycles, has become extremely important. It is also of considerable interest for the designer to know the effect of a few cycles of reversed large-amplitude plastic strain on the fatigue properties of metals (cumulative fatigue damage). In addition, it may be possible in this field to obtain further fundamental information regarding the basic mechanism of fatigue. It is seen that the resistance of materials to cyclic plastic strain is of interest both from a fundamental and practical point of view. It is now generally accepted that fatigue failure in metals is a consequence of the localized slip deformation which occurs within the individual crystals of metals. Therefore the failure process should be strongly dependent upon the magnitude of the gross cyclic plastic deformation of the metal. Consequently an experimental investigation in the plastic fatigue field should also shed new light on some of the basic aspects of the fatigue problem.

Cases of the application of high stresses or strains frequently involve a cyclic operation, not in the sense of several hundred or thousand cycles per minute, but of one in an hour, a day, or even perhaps several weeks. If failure should occur, it might be attributable to one or more of several mechanisms, of which fatigue is undoubtedly a great possibility. A more accurate knowledge of operating conditions of machine parts or structural elements, combined with improved methods of
stress analysis, makes it possible to design for a desired life expectancy. As mentioned above, there are many applications in which it would be rather unreasonable to use fatigue data of millions of stress cycles where the maximum life expectancy of a part may be only a few thousand cycles. The need for low-cycle fatigue research has been recognized particularly for high-pressure vessels. An organized experimental program has been conducted under the auspices of the Pressure Vessel Research Committee of the Welding Research Council during the last decade.

Certain interesting considerations arose from low endurance fatigue. In fatigue tests, concern is normally with failures in $10^5$ to $10^8$ cycles and mostly with cycles of constant stress amplitude. For lives below $10^4$ cycles, choice could conveniently be made between tests at constant stress or constant load amplitude as opposed to constant strain amplitude. There is, therefore, a division of the field at those very short lives regarding which way to test. If constant strain amplitudes were chosen, then the load or stress values would change in successive cycles as the test progressed. That would not affect the continuity or stability of the test whether in bending or push-pull testing. However, if constant load amplitudes were chosen, the strain values in each cycle would gradually change, causing a change in the geometry of the test piece. Low-cycle fatigue tests can be valuable in throwing light upon a question of design and also can be equally valuable in offering guidance in the choice of material.
2. Experimental Methods Used for Low-Cycle Fatigue Tests

Survey of literature in the plastic-fatigue field shows that the cyclic plastic strains or stresses in the specimen are induced by either mechanical loads or thermal cycling. Fatigue testing can be performed by many different methods. Most frequently, the metal fibers are cycled between tensile and compressive stresses. These can be obtained in a well defined manner by homogeneous stressing or straining of the entire specimen in direct load cycling. In contrast, bend cycling creates non-homogeneous tensile and compressive stresses and strains which increase as the distance from the neutral plane increases. Furthermore, cyclic bending can be effected either by reversing the load from a positive to a negative limit, or by rotating the specimen under a static load. In thermal fatigue, failure can occur as a consequence of a comparatively few cycles of thermally induced mechanical deformation. In most conventional fatigue testing methods, the cyclic stresses or strains in the specimen are induced by mechanical loads. The combination of mechanical loads and enforced thermal strains apparently has not yet been completely studied.

From another point of view the testing technique employed by the investigators in the low-cycle fatigue field is either a constant amplitude-of-deflection or constant amplitude-of-load type. In each case the applied loads can be measured. In service there exist both constant load and constant deflection conditions of loading, and for this reason both are important in the laboratory. However, it is also true that while in service the magnitude of the load may vary irregularly from one cycle to the next. It is seen that constant-load testing is not neces-
sarily constant stress testing and constant-deflection testing is not necessarily constant-strain testing, especially in the vicinity of a fatigue crack or a region of slip. Conventional fatigue machines generally apply constant load or bending-moment cycles, and these may be taken as implying constant stress and strain for the lower values. However, this is no longer the case when considering low-endurance failures. In the elasto-plastic and fully plastic ranges there are distinct differences in the history of the material and in the changes that occur within it, according to whether strain cycling or load cycling is taking place. The fact that, in practice, the cycle, of whatever type, may well not be of constant amplitude throughout the life, presents an added complication.

Some particular aspects of low-endurance fatigue should be brought out in order to facilitate a better understanding. Although the life of a subject in terms of cycles may be $10^4$ or less, on a time basis this may be considerably longer than a normal fatigue life of say $10^7$ cycles, where the frequency is in hundreds or thousands per minute. Consequently the speed effects should be considered. One most important problem in low-cycle fatigue is that of hysteresis. For high stress values this will be large, and at rapid cyclic frequencies the attempted dissipation of energy will result in a considerable rise in temperature. At low frequency this problem is avoided. The shape of the load cycle is generally taken as sinusoidal, but this is unlikely to be true for the low-endurance problem. The particular shape of load cycle may well influence fatigue behavior of the material. In general, the critical cross section of the test specimen in fatigue research is either rectangular or circu-
lar. Usually the specimen is designed with a nonuniform cross section in the form of a large radius notched bar or plate. The fatigue tests may be conducted either at room temperature or at elevated temperature.

3. Review of Previous Experimental Work Regarding the Measurement of Plastic Strains

Several research papers in the low-cycle fatigue field will be reviewed from the point of view of measurement of plastic strains. In the neighborhood of the endurance limit of a metal the stresses and strains at the location of potential failure are proportional to the loads and overall distortions of the system. In the low-cycle range, however, proportionality between load, stress, distortion, and strain is lost, and various types of cycling must be considered separately. The commonly used parameter of stress in the conventional fatigue tests is unsatisfactory for comparison of data in the low-cycle fatigue field; stress is not a simple or linear function of total strain because of plastic deformation which occurs in low-cycle tests. Similarly, when the cyclic strain exceeds the yield point strain, the stress range is no longer linearly related to the strain range because of the hysteresis effect. In the plastic region, stress concentration tends to decrease with load while strain concentration increases with load.

A brief review of several methods of measuring maximum fiber strain and computing maximum fiber stress for the case of reversed or repeated bending in short-endurance fatigue will follow. In plastic-fatigue investigations at Lehigh University conducted under the auspices
of the Pressure Vessel Research Committee, Welding Research Council (1,2).
rectangular-beam specimens with throat were loaded as cantilever beams
with a cam supplying a constant amount of deflection throughout each
test. The strain was measured by means of SR-4 gages during the initial
cycle of loading (1952).

In subsequent tests (1953) at Lehigh University (3), a specimen
having a width five times its thickness was designed to develop essen-
tially 2 to 1 biaxial strains on the test surface. The biaxial bending
fatigue test was designed to simulate stress conditions occurring in a
pressure vessel shell plate during a temperature swing. It was pointed
out that when a narrow rectangular section of steel is bent into the
plastic range, the section becomes roughly trapezoidal. At the outer
fibers, the lateral movement is roughly 0.5 that of the longitudinal
strain in obedience to Poisson's ratio. As the width is increased, a
constraint to this lateral movement is developed and lateral stresses
are created. A wide plate is capable of only negligible lateral flow
and therefore the outer fibers will be placed under biaxial tension in
which the effective tensile lateral strain bears the ratio of 1 to 2 to
the longitudinal strain. Biaxial strains present in the specimen were
measured by SR-4 strain gages affixed to the mid-section in both lateral
and longitudinal directions. Readings were recorded by loading the
specimen slowly to maximum deflection. It is to be noted that a calibra-
tion on a 1-inch wide, 5/8-inch thickness specimen indicates that the
biaxiality in the narrow specimen was about 4.5 to 1.

1The numbers in parentheses refer to the Bibliography at the end
of the chapter.
In later investigations at Lehigh, 1954 (4), the strains imposed in the specimen throughout the test life of the specimen were measured to permit the best choice of a criterion for indicating the severity of plastic loading. Strains in the section were measured with a Tuckerman optical strain gage. The gage length was 1/4 inch centered at the minimum cross section. Strains were measured at small increments of deflection during the first tension cycle and then were recorded from maximum tension to maximum compression at frequent intervals until the specimen developed a crack. Loads were also recorded on several of the tests by means of a strain gage affixed to the loading arm of the fatigue machine.

Bowman and Dolan at the University of Illinois, 1953 (5,6,7), have conducted biaxial fatigue tests on pressure vessel steels using rectangular plate specimens of 3/4-inch plate, approximately 9-7/8 x 17 inches in size fitted into a fixture that permitted hydraulic pressure to be applied to the bottom face of the plate. The apparatus repeated a pressure cycle of zero to maximum at the rate of 100 cpm. A state of biaxial tension with principal stresses having a ratio of about 2 to 1 was developed in the top surface of the center zone of the plate. A careful record of the plastic action and of the cyclic strains was obtained during the progress of each experiment by means of a wire resistance type strain gage attached to the surface of the plate and by means of deflection measurements obtained from an Ames dial mounted at the center of the plate. It was pointed out that the action in the plate is primarily that of biaxial bending. A stress gradient existed throughout the thickness of the plate and only the top surface of the plate was
regarded as representative of the desired state of stress for the shell plate.

Low in 1956 (8) described a machine which was designed and constructed to apply a preset angular movement to the ends of a flat rectangular test piece with critical section 0.25 inch in thickness and 1.5 inches in width. The curvature of the test section and therefore the maximum fiber strain was determined by a spherometer. This machine was capable of applying a strain of ± 1 percent to the test piece. For higher strains a special rig was designed to be hand operated in a tensile testing machine. In both cases there was no tensile end load on the test piece as the test piece was being bent. The tests showed that while the spherometer readings remained substantially the same throughout the greater part of a test, once localized yielding or cracking of the test piece occurred, the angular movement required to give the same reading altered considerably. Low plotted the results of the plane bending fatigue test data as maximum fiber strain against endurance life. The maximum fiber stress against endurance curves were also plotted by reference to the static stress against strain curves up to strains of 5 or 6 percent, as determined by the tensile tests. Conversion of strain readings used in the fatigue test to equivalent stresses was obtained by using static stress-strain curves. The stress-against-endurance curves have not been constructed using dynamic stress-against-strain values. Low stated that these are of only academic value. Low concluded that for the five widely different materials tested, fatigue life in reversed bending was found to depend solely on the degree of strain, for maximum
fiber strains between ± 0.4 percent and ± 5 percent. In the wholly elastic range the fatigue life did not depend uniquely on strain.

Coffin and Taverneli in 1959 (9) described tests using longitudinal test specimens stressed uniaxially by the slow cyclic motion of the end grips of the test bar at room temperature to produce the required longitudinal strain. The test specimen was designed with a nonuniform cross section in the form of a large radius notched bar to prevent the lateral buckling of the test specimen as a consequence of severe cyclic strain. Complexities regarding the measurement of the longitudinal strain with a nonuniform gage length were resolved by measuring the change in the minimum diameter of the test specimen and converting this diameter change to longitudinal strain. The tests were push-pull type and the change of diameter was measured by use of a dial indicator. The method of computing longitudinal plastic-strain range was explained in their paper. Load measurements were made by wire resistance-type strain gages mounted on an accurately machined tool steel weigh bar. These gages were used in conjunction with an Instron portable recorder. From the load, a corresponding stress was calculated. The specimens were hourglass shaped, machined and polished carefully. This type of specimen concentrates the stress and, therefore, strain, at the minimum section. The test results were interpreted in terms of plastic-strain-range cycles to failure relationship.

Baldwin, Sokol, and Coffin in 1957 (10) described cyclic strain fatigue studies in which they used a specimen similar to a cylindrical tensile-type with a test region of constant diameter with accurately machined and polished gage section. The specimen was subjected to mechan-
ical strain in the gage length of the specimen. The strain in the specimen during a test was actually measured by two dial gages, an average strain resulting from the two readings. The load and subsequently stress on the specimen during the cycle was determined by using wire resistance-type strain gages together with a strain indicator. In the case of push-pull tests, since an approximately uniaxial state of stress existed in the test section, the computation of plastic-strain range and stress range were straightforward. They conclude that the plastic strain range was a fundamentally significant variable in predicting fatigue failure.

Coffin, in his work (1954) (11), described methods for calculating plastic-strain range. The thermal strain was found from the thermal deformation of the test specimen between the high and low temperatures of the cycle. In this thermal fatigue test a tubular specimen held rigidly at each end was alternately heated and cooled. Thus it was subjected to a uniaxial and uniform stress. The elastic strain was calculated from a knowledge of the stress involved. Because of the end constraint the thermal strain was equal and opposite to the total strain. The plastic-strain range was then found by subtracting the elastic portion from the total strain.

Johansson (1955) made a study of fatigue of different steels at constant strain amplitude and elevated temperature (12). A round tapered test piece was bent between two limit positions. The force required to bend the test piece between the two limit positions was measured with a strain gage dynamometer.

Vinckler and Dechaene (1959) described an experimental technique for the study of strain distribution in a metallic member during plastic
deformation using the Moire effect (13). Regularly spaced lines were
drawn on an undeformed metal test specimen and also on a transparent
screen. He then deformed the test piece into the plastic range. Moire
fringes were formed when the transparent master grid was superimposed
upon the deformed grid. This provided the basis from which the amount
of plastic deformation could be calculated. It should be possible to
measure elastic as well as plastic strains using sufficiently fine
screens. They have shown the results of the compression test on a ring,
tensile test on a mild steel bar with a central circular hole and a ten-
sile specimen with notches. It is felt that this method can be applied
to the axial push-pull test with uniform test section on the specimen
without difficulty. With some modifications, it is thought that this
method may be applied to the flat plate specimen for bending test. It
is to be expected that some difficulties would be encountered in measur-
ing the plastic strain in a small necked cylindrical test specimen sub-
jected to bending by the above method. It is to be noted that the spec-
imen will undergo a rigid body rotation due to deflection by bending in
addition to the distortion on the surface. However, it is felt that
this experimental technique might prove to be effective for measuring
plastic strains in necked cylindrical specimens subjected to bending.

Bailey (1956) used a rectangular cantilever specimen of 0.375 inch
x 0.75 inch at test section for repeated strain tests (14). The required
total strain (0.45 ~ 0.50) on the test surface was measured by a special
extensometer on a gauge length of 0.65 inch at the middle of the parallel
length. Hysteresis curves were obtained at 1, 2, 3, 500, and 1,000 cy-
cles of repetition.
Weisman and Kaplan (1949) (15) described flexural fatigue tests in which they mentioned the method of correction for bending in the plastic range by the method of Cozzone (16) using a shape factor of 1.5. The calculated stresses were corrected for both plastic bending and the deflection of the specimen. The method of calculating the deflection correction is given in their paper. Finally, the axial tension stress due to the component of force parallel to the specimen at the break was added to obtain the maximum corrected outer fiber stress at the point of failure.

4. Purpose, Scope, and Approach of Present Investigation

One of the most important problems in fatigue research is the proper interpretation and careful analysis of laboratory test data. It is seen that a large proportion of the available fatigue data have been obtained from rotating beam tests of small polished specimens to determine a value for the fatigue limit for, say, 10 million cycles of stress. At higher stresses corresponding to a fatigue life of, say, $10^2$, $10^3$, $10^4$, or $10^5$ cycles, the strains exceed the proportional limit, and hence the simple flexure formula from which the stresses are calculated is definitely not applicable. Thus nominal calculated stresses for test data are not valid for purposes of estimating the life of a particular material for given stress cycles. A question pertaining to interpretation of fatigue test results might be asked. In rotating-beam tests and in flexural tests, what allowance should be made for plastic flow in evaluating stresses? It appears that bending stresses, computed from the simple beam formula $S = Mc/I$ without allowance for plastic flow
would be unduly high. In the case of bending fatigue tests, there is the
difficulty of accurately determining the extreme fiber stress after
yielding begins. When materials are tested under nonuniform stress sys-
tems, it is necessary to revise the method by which the stress is com-
puted. It is seen that the importance of interpretation of experimental
results in the low-cycle region with respect to plastic as well as elas-
tic behavior of metals should be emphasized and explored. It is also
seen from the above discussion that one of the problems involved in fa-
tigue testing in the plastic range is to express the severity of loading
in quantitative and significant terms. Until recently, stress cycling
tests were only occasionally extended to cover the range of small numbers
of cycles to failure and very few serious attempts were made to analyze
such data. The relation between the stress range, $S$, and the number of
cycles to failure, $N$, is generally represented on either semilogarithmic
or logarithmic coordinates. In most fatigue tests, nominal stresses are
used instead of actual stresses because of some difficulties involved in
computing the equivalent actual stresses in the plastic range.

Sachs and Taber (1958) (17) stated that as the number of cycles
to failure decreases below about 1000, the bend fatigue strength becomes
increasingly higher than the direct stress fatigue strength. This dif-
ference, for any given number of cycles to failure, appears to be gov-
erned by the static bend strength ratio or by the ductility of the metal.
The condition of biaxiality in a bent circular section is not known. In
a flat rectangular specimen, biaxiality increases with the ratio of spec-
imen width to thickness. Bend tests conform more closely than direct
stress tests to the actual conditions which are encountered; for example,
in thermal fatigue which generally involves temperature gradients and also strain gradients. However, the results of bend fatigue tests are more difficult to evaluate than those of direct stress cycling. This also applies to strain cycling. In addition to the problem of transforming a controlled total strain into an average plastic strain, it is observed that the controllable strain quantity, e.g., deflection or angular distortion, does not insure a constant total strain at the section subject to maximum straining. This applies to the majority of low cycle bending fatigue work, which used either contoured round specimens or contoured flat specimens.

Findley (1956) (18) stated that in order to determine more accurately the actual stresses existing in the specimens where yielding occurred, plasticity theory should be employed requiring static test results. As corrections for yielding applied to bending, for example, tests were based on the assumption that the unloading stress-strain diagrams were linear, it was necessary to establish whether or not the stress ranges which were corrected encountered the Bauschinger effect. In instances in which the maximum stress of the cycle was above the proportional limit, the actual maximum stress was less than the nominal maximum stress calculated from elastic formula. The correction employed in bending was a semigraphical procedure based on an extension of the Herbert equation by Morkovin and Sidebottom (19). The analysis was based on relationships applicable only when the stress-strain relation was the same in tension and compression. However, it has been shown that a difference of 20 percent in the yield strength in tension and compression of steel will result in less than a 2 percent error in the computed
stresses if the average of tension and compression stress-strain diagram is employed. In addition, errors were also introduced due to several simplifications from actual material behavior.

As discussed in section 3, in low-cycle fatigue tests for a rectangular cross section plate specimen, plastic strains can be measured by various experimental techniques. It is also seen that in axial push-pull tests, plastic strains can be measured and stresses computed. However, for the rotating bending test and repeated or reversed bending test employing a circular cross section specimen with a nonuniform cross section in the form of a large radius notched bar, the method of measuring plastic strains is not yet available in the technical literature as far as the author is aware. The difficulty arises from the fact that the critical diameter of the specimen is small, say 1/4 inch, in addition to the nonuniform cross section in the form of a notched bar with suitable sweep radius. Since rectangular and circular section specimens are the most important types of specimen used in low-cycle fatigue research, the present theoretical investigation was attempted to develop analytical expressions which will show the relationship between applied bending moment and strain or stress induced for the elastic-plastic range in a narrow rectangular bar and a circular bar. The analysis corresponds to the initial 1/2 cycle for released loading or to 1/4 cycle for alternate loading cycle. Analytical stress or strain analysis on the specimen in the elastic-plastic range was attempted by using static test results (tension or compression) and plasticity theory. A knowledge of the reaction to an applied load in terms of the stress or strain existing in the specimen is required in the interpretation of low-cycle fatigue data and
in the design of fatigue specimens. It is to be noted that for bending fatigue tests there is a stress gradient between the neutral surface and the outside surface of the specimen. The experimentally measured strains are for the most stressed outside fibers only and the state of stress and strain inside the specimen is unknown. It is also for this reason that the present investigation was undertaken to determine the state of stress or strain in the elastic-plastic range for specimens subjected to bending.

It is realized that the bending of metal by an amount sufficient to cause plastic deformation of practically the entire volume of the metal produces a condition of stress and strain throughout the part which is exceedingly complex. Under certain conditions, however, it is possible to treat the problem quantitatively as will be discussed later. Pure static bending will be assumed in order to allow mathematical treatment of the problem. Time-dependent and temperature-dependent effects will be neglected in this analysis. The basic relationships between stress and strain in the plastic range under cyclic conditions are not, at present, well understood. Specifically, in this investigation a theoretical relation between bending moment and strain or stress in the most stressed extreme fibers may be determined for a given cross section.

Two approaches are proposed and examined in this analysis: first, a less rigorous but more tractable treatment of the problem by mechanics of materials; and, second, a more accurate analysis based on the classical methods of elasticity and plasticity. Throughout the present investigation the main assumptions are that (1) transverse planes remain plane during deformation; (2) the material of the member is isotropic.
and homogeneous; (3) the material yields homogeneously and no wedge or vein type yielding will occur; (4) pure bending exists.

A study of the functional relationship between the maximum shear stress and the normal stress acting on the maximum shear plane for various existing failure theories is included in Chapter IV. A hypothesis of failure based on the concept of a nonlinear influence of normal stress on the critical shear stress under general triaxial state of stress was derived primarily from the mathematical point of view. The author wishes to point out that the study in Chapter IV is not directly related to the present investigation but was a by-product of it.
BIBLIOGRAPHY


CHAPTER II

ANALYSIS OF THE STATE OF STRESS AND STRAIN IN PRISMTICAL BARS SUBJECTED TO BENDING IN THE ELASTIC-PLASTIC REGION ASSUMING UNIAXIAL STRESS

1. Review of Plastic Bending Literature

There is already in existence a fair volume of work on the subject of bending beyond the limit of proportionality, most of which requires the use of actual material stress-strain curves. It can be said that there have been a great many advancements to the pure bending theory of beams, in which the effect of nonlinear stress-strain characteristic of material is taken into account. Saint-Venant (1), Timoshenko (2), Meyer (3), Von Karman (4), Herbert (5), Bach (6), and others, have attacked this problem either analytically or graphically. These prominent investigators dealt principally with methods for determining the general solution of the problem. It is felt pertinent to review some of the results of investigations in the past two decades.

The three basic principles used throughout all the investigations on plastic bending are that

1) the equation of equilibrium for the force acting on the section,

\[ \int_A \sigma dA = 0, \]

2) the equation of equilibrium for moment,

\[ \int_A \sigma y dA = M, \]
3) the cross sections remain plane during pure bending in the plastic range. Several investigators have shown experimentally that this is true.

Beilschmidt (1942) (7) has devised a method of correlating the section geometrical constants and the material stress-strain characteristics and takes the form of an extension of the simple theory of bending for various structural sections. The effect of the material stress-strain characteristic on the position of the neutral axis as well as the method of defining the neutral-axis position for non-symmetrical sections and dissimilar tension and compression material characteristics were discussed. Cozzone (1943) (8) has developed a method for plastic bending by a simple rectangular or trapezoidal stress distribution that varies with the applied bending moment. This stress distribution approximates the distribution based on the actual stress-strain characteristics of the material and shape of the cross section. The following points are of interest in this discussion:

1) The resisting moment of any cross section corresponding to any extreme fiber stress up to the ultimate stress of the material,

2) the corresponding stress distribution over the entire cross section,

3) the corresponding shear distribution.

The proposed method is widely used for direct engineering design purpose in aircraft work.
W. R. Osgood (1944) (9) has worked out the relation between bending moment and curvature for three hypothetical materials. The general equations are applied to I, T, and rectangular sections.

Williams (1947) (10) has illustrated the possibilities of the graphical solution of Saint-Venant's method by using what is termed a plastic bending factor which is plotted against outer fiber stress. The investigation is restricted to pure bending. The method was basically sound and involved no simplifying assumptions other than that the stress-strain relationships in a beam are identical with those of simple tension and compression. Solutions for rectangular, circular and other cross sections are presented. This method seems to be one of the best available at present and it is strongly recommended particularly when the mathematical expressions for stress-strain curves fail to approximate the experimental data closely. For further detailed analysis the reader is referred to the original paper.

Wang (1947) (11) presented a method of analyzing fiber stresses of rectangular, circular and other beam sections which includes the full effect of a nonlinear stress-strain characteristic of materials throughout the elastic and plastic range of bending. Both graphical and analytical methods are presented for rectangular beams.

Morkovin and Sidebottom (1947) (12) presented a theoretical analysis of the relationship between applied moment and strain for both elastic and plastic strains in rectangular and circular beams, based on the assumption that the strain distribution in an overstrained beam is linear and that the stress in any fiber of the beam can be read from the stress-strain diagram of the material. A theoretical moment-strain dia-
gram was obtained for a beam made of material having the stress-strain diagram represented by two straight lines. A theoretical moment-strain diagram for a rectangular beam was checked by experimental data.

Phillips (1951) (13) discussed a new method of calculation of bending moment and maximum strain curve, which he claims to be more satisfactory than any of the methods proposed and this is especially so when the cross section has only one axis of symmetry. This method is based upon approximating the stress-strain curves for tension and compression by means of polygonal lines. By increasing the number of sides of these polygonal lines, it is possible to approximate the stress-strain curves with any desired degree of approximation. The equation of equilibrium for the force is used for determining the position of the neutral axis for a given maximum strain while the second equation of equilibrium for moment is used for determining the value of the bending moment corresponding to the selected maximum strain and position of the neutral axis. A graphical method of construction was illustrated.

Barrett (1953) (14) presented an analysis for the purpose of calculating the maximum permissible bending moment obtained from a form factor based on a standardized form of the stress-strain curve and working to a maximum fiber stress of the 0.5 percent proof stress for some common beam sections in the plastic range. By using a suitable mathematical expression for the stress-strain curve, it is possible to relate the bending moment sustained by the rectangular or other beam to the cross-sectional dimensions, a 0.2 percent proof stress, a function derived from the form of the stress-strain curve and the maximum fiber stress or strain.
The above brief review represents the current status of the problem.

2. The Selection of a Mathematical Expression to be Used for the Stress-Strain Relationship in the Plastic Region

The solution of problems concerning the bending of sections beyond the limit of proportionality is usually brought about by processes of arithmetical summation applied to actual stress-strain curves. Generalized relations may be set up for the bending moment if a mathematical expression is used instead of an actual stress-strain curve. The generalized mathematical form of the stress-strain curve which seems to best fit a wide range of actual ductile materials will be examined. Many analytical representations of the stress-strain curve have been proposed in the literature (15).

Some of the most important requirements that must be fulfilled by any equation for a stress-strain curve are —

a) Accuracy.

The accuracy of any analytical method used in engineering work is obviously of prime importance. In the present case it is necessary that the accuracy of the equation be at least of the same order as that to which material data is evaluated and recorded.

b) Simplicity.

The equation is wanted for analytical work and any formula containing more than two parameters is almost sure to be either unnecessarily complicated or too complicated for practical use. On the other hand, it is apparent that one parameter is not enough to cover a wide
range of materials with varying mechanical properties and stress-strain curves of different shapes.

c) Flexibility.

The wide range of basic composition and heat treatment of materials now in use imposes the requirement that the equation should be sufficiently flexible to cover a wide range of stress-strain characteristics. It is desirable that the formula give physically possible values of the stress for all values of the strain and vice versa.

d) Ease of Application.

The equation must not be sensitive to small parametric differences and it should be possible to determine these parameters with a fair degree of precision. A further desirable property of the equation would be that the process of differentiation and integration might be effected without much difficulty. It is also desirable that the parameters entering in the stress-strain formula be easily and accurately determinable for a good fit to experimental data.

The problem confronting the engineer is the efficient combination of the above several requirements. The formula should be one that, when substituted in the integrals of plastic bending is not likely to lead to integrals that cannot be expressed in closed form. For strains up to about one percent it is immaterial whether the stresses and strains are based on the original cross sectional area and the original gage length or whether they are "true stresses" and "natural strains," but for larger strains the latter quantities should be used. A single expression covering both tension and compression will not be considered; abso-
lute values of stress and of strain are assumed throughout this discussion.

Only a few of the proposed formulas which will be discussed in later analysis will be examined here.

1) Holmquist and Nadai (16) proposed a double formula that, in its most general form, may be written as,

\[
\begin{align*}
\varepsilon & = \frac{\sigma}{E} \quad \sigma \leq \sigma_0 \\
\varepsilon & = \frac{\sigma}{E} + K(\sigma - \sigma_0)^n, \quad \sigma > \sigma_0
\end{align*}
\]

where \( \varepsilon \) = the strain
\( \sigma \) = stress
\( E \) = Young's Modulus
\( \sigma_0 \) = the proportional limit
\( K, n \) = shape parameters.

It is pointed out (14) that if \( \sigma_0 \) is simply regarded as a third parameter, the formula becomes a powerful one. It could probably be made to represent accurately the stress-strain relations of materials that show a break in the lower part of the curve. To obtain the parameters for the formula, a series of values of \( \sigma_0 \) might be assumed and \( K \) and \( n \) then determined by plotting \( \log(\varepsilon - \sigma/E) \) against \( \log(\sigma - \sigma_0) \). Some values of \( \sigma_0 \) would probably result in a straight-line plot for a reasonably large interval of \( \varepsilon \), and \( n \) and \( \log K \) would be the slope and intercept of the straight line. If the corresponding value of \( \sigma_0 \) led to a poor fit outside this interval, a compromise between \( \sigma_0 \), \( n \), and \( K \) would be necessary. It is of interest to note that the formula has the further disadvantage
of being in two parts, thus adding to the complication of three parameters in using it for problems of plastic bending.

2) Ramberg and Osgood (1943) (17) showed that the formula

$$\varepsilon = \sigma/E + K(\sigma/E)^n$$

is applicable to a wide variety of materials. This formula is a special case of Homquist and Nadai's formula and this mathematical expression for stress-strain curves has found considerable favor among theoretical, particularly in the USA. It may be obtained by setting $\sigma_0 = 0$. It is easy to find out whether the above equation is a good fit to stress-strain data and to determine the parameters if it is. If log $(\varepsilon - \sigma/E)$ is plotted against log $(\sigma/E)$ and the result is a straight line, the formula will give a good fit, and the slope and the intercept of the line are $n$ and log $K$. This equation has been found to apply accurately to many structural materials, at least up to a strain of 0.01.

3) As stated by Barrett (14), Ramberg and Osgood's equation may be expressed more readily in the form

$$\varepsilon = \sigma/E + B(\sigma/\sigma_2)^n$$

where $\sigma_2 = 0.2$ percent proof stress (lb/in$^2$).

As discussed by Nicholls (1954) (18), by satisfying the condition that the above curve pass through $\sigma_2$ (the 0.2 percent proof stress) at the appropriate strain, means that the equation may further be simplified to read

$$\varepsilon = \sigma/E + 0.002 (\sigma/\sigma_2)^n$$

$n$ may be obtained by making the curve pass through stress $\sigma$ (in the higher range of interest).
The number of formulas that could be devised is of course unlimited. Some day empirical formulas may be replaced by a theoretical formula. Meanwhile, it is suggested to use Homquist and Nadai's formula in the analysis which follows. By plotting as suggested previously, it is easy to find out whether this formula may be expected to give a fit and to determine the parameters if they do give a good fit. It has been shown that Holmquist and Nadai's formula approaches the test values for ductile metals very closely (14,15,16,17,18). It is to be noted that Holmquist and Nadai's formula can be expressed as

$$\varepsilon = \varepsilon_e + \varepsilon_p = \sigma/E + K(\sigma - \sigma_0)^n$$

where

$$\varepsilon_e = \sigma/E = \text{elastic strain}$$

$$\varepsilon_p = K(\sigma - \sigma_0)^n = \text{plastic strain}.$$ 

If the material has yield point elongation \(\varepsilon_o\), then the plastic strain can be written as

$$\varepsilon_p = \varepsilon_o + K(\sigma - \sigma_0)^n$$

The above mathematical expression will be used in the analysis.

3. Rectangular Section in Pure Bending

The problem of finding the relationship between bending moment and maximum strain induced in the case of symmetrical pure bending of bars in plasticity has been discussed repeatedly in the literature. When the cross section of a beam is subjected to a pure bending moment of such magnitude that the stress in the outer fiber exceeds the limit of proportionality for the metal, the stress distribution across the beam ceases to be linear and the elementary theory of bending is no longer
applicable. It has been pointed out that the stress distribution over
the cross section tends to follow the stress-strain characteristics of
the material (8). Consequently, the conventional formula for linear
bending stress distribution no longer applies. Pure bending in the
plastic range has been discussed by a number of investigators. It seems
to the author that none of them points out specifically the important
restriction on the problem to ensure uniaxial state of stress in the
beam.

Let us consider the necessary conditions to ensure uniaxial state
of stress in the beam. Firstly, the prismatical bar should have narrow
rectangular section, i.e., the width of the beam should be at least
smaller than the height. Secondly, the loading should be moderate to
keep the radial normal stress insignificant (moderate bending). In
other words, it is supposed that the radius of curvature is so large in
relation to the height of the beam (thickness) that the induced trans-
verse normal stresses in the radial direction can be neglected. The ef-
effect of these radial stresses will be considered in a later chapter when
the problem is analyzed for biaxial state of stress. It is pertinent to
state the basic assumptions on which to base analytical expressions for
the relationship between the bending moment and the most-stressed-fiber
strain. These basic assumptions are made.

1) The variation of strain across the cross section is linear,
that is, sections plane before bending remain plane during bending.
Since this is the most important assumption which enables us to evaluate
the strain at any point of the beam (that is, $\varepsilon_x = y/R$), its validity
will be discussed further. The validity of this assumption has been
shown particularly by C. V. Bach (6) on materials which do not obey Hooke's law and possess no straight-line stress-strain curve for tension or compression. The assumption was also confirmed by tests by Eugen Meyer (19) on wrought iron above the yield point. Hill (1950) in his book (20) stated that transverse planes continue to remain plane while part of the beam is plastic, and that the state of stress is still longitudinal tension or compression. This may be justified by substitution in the Reuss equations and the equations of equilibrium. Further experimental evidence of the validity of the assumption may be found in references (12,21,22,23). Numerous investigators have compared the results of the analysis based on the assumption plane sections remain plane with the experimental data and found the validity of the assumption satisfactory. It appears to the author that the assumption is reasonably valid since there is strong evidence in the literature.

2) The stress-strain curve is identical in both tension and compression. The relaxation of this restriction will be discussed later.

3) The relationship between stress and strain is the same for beam fibers as it is for simple tension and compression and this relationship holds for the full range of the stress-strain diagram. Williams (10) pointed out that the assumption is reasonable if the total strain is less than about 10 percent. Maximum strains measured on the outer fibers of cast beams agreed fairly well with tensile specimen strains.

4) The geometry of the beam remains essentially unaltered during the bending process (true for moderate bending).
5) The neutral surface coincides at all times with the central plane of the beam (true for moderate bending).

6) The yield stresses and the modulus of elasticity have the same values in compression as in tension.

7) The material is homogeneous and isotropic in both the elastic and plastic states.

8) The stress distribution is found from the strain distribution by assuming that the stress in any fiber for a given value of strain can be read from the tensile or compressive stress-strain diagram of the material; hence it is assumed that the stress gradient does not influence the stress at which yielding starts (12). X-ray diffraction studies indicate quite strongly that, in most practical cases, the so-called restraining effect of the region under low stress does not exist. The assumption that a theory based on homogeneous stress may usually be applied to an inhomogeneous state is therefore reasonable.

Since the moment on any section can be found from a known stress distribution and the dimensions of the cross section of the beam, a theoretical relation between bending moment and strain in the extreme fibers may be determined for a beam of a given cross section. Consideration will now be given to the solid rectangular section, shown in Figure 1(a), subjected to a bending moment \( M \) about the \( x \)-axis. The stress distribution across the section is shown in Figure 1(b). It is noted that the beam is loaded by forces directed perpendicular to its longitudinal axis in one of the principal planes of inertia of the cross section. In order to determine the distribution of normal stresses over the cross section of the bar, the stress-strain curve of the material for tension
or compression must be known from tests. We shall assume that this stress-strain curve has been determined from a tension and a compression test in the form of an analytical expression as shown before.

\[ \varepsilon = \varepsilon_e + \varepsilon_p = \frac{\sigma}{E} + \varepsilon_o + K(\sigma - \sigma_o)^n \quad \text{for } \sigma > \sigma_o \]  

(1)

Several investigators have shown experimentally that cross sections remain plane during pure bending in the plastic range. With this as a basis, it follows that

\[ \varepsilon = \frac{y}{r} \]  

(2)

where \( y \) is the distance from the neutral axis to the layer of fibers in question and \( r \) is the radius of curvature of the bent bar. From Eq. (2) it can be seen that if the radius of curvature is assumed, the strain or stress at any point in the bar can be determined. For this reason

---

Fig. 1. Solid Rectangular Section Bending about an Axis of Symmetry.
we will seek the relation between the radius of curvature of a bar and the applied bending moment. In order to obtain a general solution, the yield point elongation will be considered in this analysis. Since the tensile and compressive forces acting on the bar cross section must be equal, we can write the equilibrium relationship.

\[ \int_{0}^{+C} \sigma \, dy = \int_{0}^{-C} \sigma \, dy \]  

(3)

From Eq. (2) \( y = r \xi \), \( dy = rd\xi \)

and Eq. (3) becomes

\[ rb \int_{0}^{+r} \varepsilon_m \, d\xi = rb \int_{0}^{-r} \varepsilon_m \, d\xi \]  

(4)

where \( \varepsilon_m \) is the strain at the outside fibers. This equation shows that the tension and compression areas under the stress-strain curves in Figure 1(b) must be equal. Since we have assumed identical curves in this case, it follows that the neutral axis must be at the centroid of the cross section. The moment of the force acting on the elementary area body is \( \sigma y dA = \sigma y bd y \). Hence,

\[ M = \int_{A} \sigma y dA = \int_{-C}^{+C} \gamma y bd y = 2b \int_{0}^{C} \sigma y dy \]  

(5)

The derivation thus far follows Timoshenko's presentation (2). Returning to Eq. (1) and let us investigate the physical meaning of the shape parameter \( n \). From Eq. (1) we obtain

\[ \varepsilon_p = \varepsilon_o + K(\sigma - \sigma_o)^n \]  

(6)

If \( n = 1 \), we have linear strain-hardening. If \( n > 1 \), we have nonlinear strain hardening. Each case will be considered separately.
Fig. 2. Stress or Strain Distribution across the Rectangular Cross Section.

a) Linear Strain-Hardening Solid, \( n = 1 \).

The stress and strain distributions across the section are shown in Figure 2.

\( t, t', y, C \) = distance from the neutral plane to the respective position as indicated in Fig. 2(a).
While the bar is stressed below the yield point, it is known from the elastic theory that the longitudinal stress is distributed linearly across a transverse section. Fibers above the neutral plane are extended and those below compressed. The surface fibers of the bar yield at a stress $\sigma_o$. As the bending moment is further increased, plastic regions spread inwards exactly in the same manner from both surfaces.

Let us investigate the stress distribution as indicated in Figure 2(a). Elastic deformation is observed between station 0 and station $t$, consequently the following relationship holds for this region.

$$\sigma = E\varepsilon_e = E(y/r) = E\varepsilon$$  \hspace{1cm} (7)

From station $t$ to station $t'$, the stress is constant

$$\sigma = \sigma_0 = \text{const.}$$  \hspace{1cm} (8)

From station $t'$ to outside fibers $t''$, the stress can be written as,

$$\sigma = E\varepsilon_e$$  \hspace{1cm} (9)

The plastic strain is

$$\varepsilon_p = \varepsilon_0 + K(\sigma - \sigma_0)$$  \hspace{1cm} (10)

The total strain can be written as,

$$E\varepsilon = E(\varepsilon_e + \varepsilon_p) = E(\varepsilon_0 - K\sigma_0) + (KE + 1)\sigma$$

$$= E(y/r)$$  \hspace{1cm} (11)

From the equilibrium condition of bending moment, Eq. (5), we have

$$M = 2b \int_0^C \sigma y dy = 2b \left[ \int_0^t \frac{Ey^2}{r} dy + \int_t^{t'} \sigma o y dy + \int_{t'}^{t''} \frac{E}{EK + 1} \left\{ \frac{y^2}{r} - (\varepsilon_0 - K\sigma_0)y \right\} dy \right]$$  \hspace{1cm} (12)
Next the strain at each station will be considered. At station $t$,
\[ \varepsilon = \frac{y}{r} = \frac{t}{r} = \varepsilon_{et} \]  
\[ \varepsilon_{et} = \text{elastic strain, first subscript indicates elastic} \]
\[ \text{part and second subscript indicates station position.} \]

Rewriting the above equation,
\[ t = r \varepsilon_{et} = r(\sigma_0/E) \]

Similarly, at station $t'$,
\[ \varepsilon = \frac{y}{r} = \frac{t'}{r} = \varepsilon_{et} + \varepsilon_o \]
\[ t' = r(\sigma_0/E) + \varepsilon_o r = r(\sigma_0/E + \varepsilon_o) \]  
(14)

At station $t''$,
\[ y = C. \]  
(15)

Substituting the above equations into the limits of the integral in Eq. (12),
\[ M = 2b \left[ \frac{E}{r} \int_0^r \frac{(\sigma_0/E)}{y^2} dy + \int r \left( \frac{\sigma_0}{E} + \varepsilon_o \right) \frac{1}{r} \right. \]
\[ + \left. \int_0^C \frac{E}{EK+1} \left\{ \frac{y^2}{r} - (\varepsilon_o - K\sigma_o) \right\} dy \right] \]  
(16)

After integrating,
\[ M = \frac{2b}{r} \left[ \frac{1}{3} \sigma_o \frac{r^3}{E} + \frac{1}{2} \sigma_o r^3 \left( \frac{2\sigma_o \varepsilon_o}{E} + \varepsilon_o^2 \right) \right. \]
\[ + \left. \frac{E}{KE+1} \left\{ \frac{1}{3} r^3 - \frac{1}{3} \frac{r^3}{E} (\sigma_o + \varepsilon_o)^3 - \frac{1}{2} (\varepsilon_o - K\sigma_o) \right\} \right] \]
\[ \left( r^2 - r^3 \left( \frac{\sigma_o}{E} + \varepsilon_o \right) \right) \]  
(17)
Setting

\[ \frac{2bE}{KE+1} \left( \frac{1}{3} c^3 - a \right) \]

\[ \frac{2bE}{KE+1} \left\{ - \frac{1}{2} (\varepsilon_o - a\sigma_o)c^2 \right\} = \beta \]

\[ \frac{2b}{3} \frac{\sigma_o}{E} + \frac{2b\sigma_o}{2} \left\{ \frac{2\sigma_o\varepsilon_o}{E} + \varepsilon_o^2 \right\} + \frac{2bE}{KE+1} \frac{1}{3} + \frac{2bE}{KE+1} \left\{ - \frac{1}{3} \left( \frac{\sigma_o}{E} + \varepsilon_o \right)^3 \right\} \]

\[ + \frac{2bE}{KE+1} \left\{ + \frac{1}{2} (\varepsilon_o - a\sigma_o) \left( \frac{\sigma_o}{E} + \varepsilon_o \right)^2 \right\} = \gamma \]

Eq. (17) can be written as

\[ M = \left( \frac{1}{r} \right)(a + \beta r + \frac{\gamma}{r} r^3) \quad (18) \]

It is seen from Eq. (18) that the bending moment, \( M \), is not proportional to the curvature. It is also seen that the constants \( a, \beta, \gamma \) are functions of the mechanical properties and the geometry of the cross section.

b) **Nonlinear Strain-Hardening Solid, \( n > 1 \)**

If the plastic deformation is large, the elastic strain can be neglected compared with the plastic strain. In the following discussion, the elastic strain will be neglected. The stress and strain distributions across the section are shown in Figure 3.

From station 0 to station \( t' \), the stress is constant.

\[ \sigma = \sigma_o = \text{const.} \quad (19) \]

From station \( t' \) to outside fibers \( t'' \), the following relations can be written.

\[ \sigma = E\varepsilon_e, \quad \varepsilon_p = \varepsilon_o + K(\sigma - \sigma_o)^n \approx \varepsilon = y/r \quad (20) \]
From the equilibrium condition of bending moment, Eq. (5), we obtain

\[ M = 2b \int_0^C \sigma y dy = 2b \left[ \int_0^{t'} \sigma y dy + \int_{t'}^C \sigma y dy \right] \tag{21} \]

From Eq. (20),

\[(\sigma - \sigma_0)^n = (1/K)(y/r - \varepsilon_0) = x\]

\[(1/Kr)dy = dx\]

\[\sigma = \sigma_0 + x^{1/n}\]

\[y = r(Kx + \varepsilon_0)\]
The second integral on the right-hand side of Eq. (21) can be further written as,

\[ \int_{t'}^{C} c y dy = \int_{t'}^{C} (c_y + x^{1/n}r) (Kx + \varepsilon_0)Kr \, dx \quad (22) \]

At station \( t' \),

\[ t' = r\varepsilon_0 = y \quad (23) \]

and \( x = 1/K(y/r - \varepsilon_0) = 1/K(\varepsilon_0 - \varepsilon_0) = 0 \quad (24) \)

At station \( t^m, y = \infty \)

\[ y = (1/K)(c/r - \varepsilon_0) \quad (25) \]

Substituting the above equations into the limits of Eq. (21),

\[ M = 2b \left[ a_o \int_0^{r\varepsilon_0} y \, dy + Kr^2 \int_0^{1/K(r - \varepsilon_0)} (c_y + x^{n/2})(Kx + \varepsilon_0) \, dx \right] \]

\[ M = 2b \left[ \frac{1}{2} a_o \varepsilon_0^2 + (c/r - \varepsilon_0)^2 \frac{1}{2} + \frac{n}{1 + 2n} \frac{1}{K} \left( \frac{c}{r} - \varepsilon_0 \right)^{1+2/n} \right. \]

\[ \left. + \varepsilon_0 \sigma_o \left( \frac{c}{r} - \varepsilon_0 \right) + \frac{n}{n+1} \frac{\varepsilon_0}{K} \left( \frac{c}{r} - \varepsilon_0 \right)^{1+1/n} \right] \quad (26) \]

The above method provides a straightforward procedure for the calculation of bending moment when the radius of curvature is assumed. Knowing the relationship between bending moment and radius of curvature, the relationship between bending moment and maximum strain can be obtained by using Eq. (2). Consequently the stress can be read from the stress-strain diagram. In the above analysis, we have obtained the relationship between bending moment and radius of curvature. However, the radius of curvature may be eliminated and the moment may be expressed in
terms of the strain in the outer fibers of the bar. This will be demonstrated in a later analysis for a cylindrical bar.

Although we have obtained an analytical solution, a graphical solution of the problem is possible. This method was developed by Williams (10). From Eq. (2), when \( y = C, \varepsilon = \varepsilon_m \) and \( r = C/\varepsilon_m \). Substituting in Eq. (5):

\[
M = 2b \int_0^C \sigma y \, dy = 2br^2 \int_0^{\varepsilon_m} \sigma \varepsilon \, d\varepsilon = 2bc^2 \left[ \int_0^{\varepsilon_m} \frac{\sigma \varepsilon \, d\varepsilon}{\varepsilon_m^2} \right]
\]

\[
M = (bh^2/2)K_f = 3(I/C)K_f
\]  

where \( K_f = \int_0^{\varepsilon_m} \sigma \varepsilon \, d\varepsilon / \varepsilon_m^2 \)  

\[
I = bh^3/12 = \text{moment of inertia of the cross section.}
\]

The term, \( K_f \), was referred to by Williams as the plastic bending factor. It is equal to the moment of the area under one branch of the stress-strain curve divided by the square of the strain in the outer fiber of the bar. The method of calculating the plastic bending factor for any maximum strain \( \varepsilon_m \) in the outer fibers was explained by Williams (10). For further details the reader is referred to the original paper.

If the tension and compression stress-strain curves are not identical, the neutral axis will not stay at the centroid of the cross section and can be located by the method discussed by Williams (10) or Beilschmidt (7). After finding the neutral axis, the method similar to the one developed above may be applied. For the method of analysis discussed above the material constants involved in Eq. (1) are best determined from the composite tension compression stress-strain diagrams.
4. Circular Section in Pure Bending

An expression for the plastic bending moment of a bar of circular cross section can be solved in a manner similar to that for the rectangular beam but the expression takes a somewhat different form. It is assumed that the tension and compression stress-strain curves are identical and that the strain is linear across the section.

a) Linear Strain-Hardening Solid, \( n = 1 \)

The stress and strain distribution across the circular section are as indicated in Figure 4. The moment of the force acting on the element in Figure 4(b) is

\[
y_0 dA = y_0 2x dy = 2ox \sqrt{c^2 - y^2} \ dy
\]

Hence, from the equilibrium of the bending moment

\[
M = \int_0^C cy \sqrt{c^2 - y^2} \ dy
\]

Substituting \( dA = 2 \sqrt{c^2 - y^2} \ dy \) instead of \( dA = b dy \) in Eq. (16), the following equilibrium equation can be written easily for the circular section.

\[
M = \frac{E}{r} \left[ \int_0^C \frac{r(C_0)}{E} \ y \sqrt{c^2 - y^2} \ dy + \sigma_0 \int_0^C \frac{r(C_0 + \sigma_0)}{r(C_0)} \ y \sqrt{c^2 - y^2} \ dy \right] + \frac{E}{KE + 1} \int_0^C \left[ \frac{y^2}{r} - (\sigma_0 - K\sigma_0)y \sqrt{c^2 - y^2} \right] \ dy
\]

Using the following mathematical identities,

\[
\int y^2 \sqrt{c^2 - y^2} \ dy = -\frac{1}{4} \sqrt{(c^2 - y^2)^3} + \frac{c^2}{8} \sqrt{c^2 - y^2} + \frac{c^4}{6} \sin^{-1}(y/c)
\]
Fig. 4. Stress or Strain Distribution across the Circular Cross Section

\[ \int y \sqrt{c^2 - y^2} \, dy = -\frac{1}{3} (c^2 - y^2)^{3/2} \]

and integrating, we obtain the following result.

\[
M = 4 \left[ \frac{E}{r} \left\{ -\frac{1}{4} r \frac{\sigma_0}{E} \left( c^2 - \frac{r^2 \sigma_0^2}{c^2} \right)^{3/2} + \frac{1}{8} \frac{r \sigma_0}{E} c^2 \sqrt{c^2 - r^2 \frac{\sigma_0^2}{E^2}} \right. \\
+ \frac{\sigma_0}{8} \sin^{-1} \frac{r}{c} \right\} - \frac{\sigma_0}{3} \left\{ c^2 - r^2 \left( \frac{\sigma_0}{E} + \varepsilon_o \right)^2 \right\}^{3/2} \]
\]
The above equation represents the relationship between bending moment and the radius of curvature. As an alternate solution, we will seek the relationship between bending moment and maximum strain in the outer fibers. From station 0 to station t, the stress at any point can be written as

\[ \sigma' = \left( \frac{y}{t} \right) \sigma_0 \]  

(34)

where \( \sigma_0 \) = yield point stress.

Between station t and station t', the stress is constant, and

\[ \sigma'' = \sigma_0. \]  

(35)

Between station t' and station t'', the following relationship holds,

\[ E \varepsilon_p = E(\varepsilon_0 - K \sigma_0 + K \sigma'''). \]  

(36)

Considering the total strain,

\[ E \varepsilon = E(\varepsilon_0 + \varepsilon_p) = E(\varepsilon_0 - K \sigma_0) + (KE + 1) \sigma'''. \]  

(37)

From the plane-remains-plane condition, the following relationship can be written,

\[ \varepsilon / \varepsilon_Y = y/t; \quad t/c = \varepsilon_Y / \varepsilon_m; \quad t'/c = (\varepsilon_Y + \varepsilon_0) / \varepsilon_m \]  

(38)

where \( \varepsilon_Y = \sigma_0 / E \)

\( \varepsilon_m = \) maximum total strain at extreme fibers.
For the region between station \( t' \) and \( t'' \),
\[
\varepsilon = \varepsilon_e + \varepsilon_p = \varepsilon_m (y/c)
\]
\[
E\varepsilon = E(\varepsilon_o - K\sigma_o) + (KE + 1)\sigma'' = \varepsilon_m(y/c)E
\]
Solving for \( \sigma'' \),
\[
\sigma'' = \frac{\varepsilon_m E}{KE+1} \frac{y}{c} - \frac{E}{KE+1} (\varepsilon_o - K\sigma_o) \tag{39}
\]
From equilibrium of the bending moment,
\[
M = 2\int_0^c \sigma y \, dA = 2 \left[ \int_0^T \sigma'ydA + \int_T^{t'} \sigma''ydA + \int_{t'}^{t''} \sigma''ydA \right]
\]
It is understood that \( T, T', T'' \) represent the area from the neutral plane to the plane passing through station \( t, t', t'' \), respectively.
\[
dA = 2\sqrt{c^2 - y^2} \, dy
\]
Substituting \( dA \) into the above equation,
\[
M = 4 \int_0^c \left\{ \sigma_o \sqrt{c^2 - y^2} \, dy + \sigma_o \sqrt{c^2 - y^2} \, dy + \right. \\
+ \left. \int_{t'}^{c} \left\{ \frac{\varepsilon_m E}{KE+1} \frac{y}{c} - \frac{E}{KE+1} (\varepsilon_o - K\sigma_o) \right\} \sqrt{c^2 - y^2} \, dy \right\} \tag{40}
\]
After integration, we obtain
\[
M = 4 \int_0^c \left\{ \frac{\sigma_o}{t} \left( - \frac{t}{4} (c^2 - t')^2 \right) + \frac{t}{4} (c^2 - t')^2 \right\} + \frac{E \varepsilon_m}{(KE+1)c} \left\{ \frac{\sigma_o}{8} \left( \frac{\pi}{2} \right) \right. \\
+ \left. \left( - \frac{1}{3} (c^2 - t')^2 \right) + \frac{1}{3} (c^2 - t')^2 \right\} + \frac{E \varepsilon_m}{(KE+1)c} \left\{ \frac{\sigma_o}{8} \left( \frac{\pi}{2} \right) \right. \\
+ \left. \left( \frac{t'}{4} (c^2 - t')^2 \right) - \frac{t'c^2}{8} (c^2 - t')^2 \right\} \right\} \tag{41}
\]
\[
- \frac{E(\varepsilon_o - a\sigma_o)}{KE + 1} \left\{ \frac{1}{3} (c^2 - t')^2 \right\}
\]
where the distances \( t \) and \( t' \) can be obtained by the following relationships.

\[
t = \left( \frac{\varepsilon_0}{E_m} \right) C; \quad t' = \frac{\varepsilon_y + \varepsilon_0}{E_m} C
\]

Substituting Eqs. (42) into (41), we have

\[
M = f(\varepsilon_m)
\]

where \( \varepsilon_m \) = maximum total strain at outside fiber.

The outside fibers of the bar will start to yield at the moment the bending moment reaches the following value,

\[
My = (1/4) \pi c^3 \sigma_0.
\]

It is seen that Eq. (41) represents the relationship between bending moment and maximum strain in the outer fibers.

b) **Nonlinear Strain-Hardening Solid, \( n > 1 \)**

If plastic deformation is large, the elastic component can be neglected compared with the plastic strain. The stress and strain distributions across the section are shown in Figure 5. From station 0 to station \( t' \), the stress is constant.

\[
\sigma = \sigma_0 = \text{const.}
\]

From station \( t' \) to outside fibers \( t'' \), the following relations can be written.

\[
\sigma = E \varepsilon_0; \quad \varepsilon_p = \varepsilon_0 + K(\sigma - \sigma_0)^n \Rightarrow \varepsilon = y/r
\]

From the equilibrium condition of bending moment,

\[
M = \int_A \sigma y dA = 2 \int_0^C \sigma y dA = 2 \left[ \int_0^{T_1} \sigma y dA + \int_{T_1}^{T_1} \sigma y dA \right]
\]
Fig. 5. Stress or Strain Distribution across the Circular Cross Section.

It is understood here that $T'$ and $T''$ represent the area from the neutral plane to the plane passing through stations $t'$ and $t''$, respectively.

$$dA = 2 \sqrt{c^2 - y^2} \, dy$$

Substituting $dA$ into the above equation,

$$M = h \left[ \sigma_o \int_0^{t'} y \sqrt{c^2 - y^2} \, dy + \int_{t'}^c \sigma_y \sqrt{c^2 - y^2} \, dy \right] \quad (46)$$
From Eq. (45),
\[(\sigma - \sigma_0)^n = (1/K)(y/r - \varepsilon_0) = z\]

\[dy/Kr = dz; \quad \sigma = \sigma_0 + z^{1/n}\]

and
\[y = r(Kz + \varepsilon_0); \quad y^2 = r^2(K^2z^2 + 2Kz\varepsilon_0 + \varepsilon_0^2)\]

\[dA = 2\sqrt{c^2 - y^2} dy = 2\sqrt{c^2 - r^2(K^2z^2 + 2Kz\varepsilon_0 + \varepsilon_0^2)} Kr dz\]

Since for most ductile metals except mild steel \(\varepsilon_0 = 0\), we will discuss the materials which do not exhibit yield point elongation phenomena.

For \(\varepsilon_0 = 0\), the elementary area \(dA\) can be written as
\[dA = 2Kr \sqrt{c^2 - r^2K^2z^2} dz, \quad y = rKz\]

For \(\varepsilon_0 = 0\), Eq. (46) can be written as

At station 0, \(y = 0 \quad z = 0\)

At \(y = c \quad z = 1/K(c/r)\).

For this case, Eq. (46) can be written as,

\[M = 4K^2r^2 \left[ \frac{1}{c} \int_0^c \frac{1}{r} \left( \sigma_0 z + z^n \right) \sqrt{c^2 - r^2K^2z^2} dz \right] \]

\[= 4K^3r^3 \left[ \sigma_0 \left( \int_0^c \frac{1}{Kr} z \sqrt{\frac{c^2}{K^2r^2} - z^2} dz + \int_0^c \frac{1}{Kr} \left( \frac{1}{n+1} \right) \sqrt{\frac{c^2}{K^2r^2} - z^2} dz \right) \right] (47)\]

The first integral can be evaluated easily as,

\[\int_0^c \frac{z}{Kr} \sqrt{\frac{c^2}{K^2r^2} - z^2} dz = (1/3) \frac{c}{Kr}^3 \]

Since the second integral cannot be integrated analytically, graphical
or numerical integration is suggested. Rewriting Eq. (47),

\[ M = hK^2 r^3 \left[ \frac{c}{3} \left( \frac{c}{K} \right)^3 + \int_{0}^{c} \left( \frac{c}{K} \right)^2 \left( \frac{1}{h} + 1 \right) \sqrt{\frac{c^2}{K^2 r^2} - z^2} \, dz \right]. \]  

(48)

As can be seen from the above analysis, the expression for the relationship between bending moment and maximum strain or radius of curvature is rather complicated. Williams (10) has illustrated the possibility of the graphical solution of Saint-Venant's method by using what is termed a plastic bending factor. From the rectangular beam theory the following relationship can be written:

\[ \varepsilon = \frac{y}{r}; \quad \varepsilon_m = \frac{c}{r}; \quad y = r \varepsilon; \quad dy = r d \varepsilon. \]

From Eq. (31),

\[ M = h \int_{0}^{c} \sigma_y \sqrt{c^2 - y^2} \, dy, \]

or, from the above relationship, between \( y \) and \( \varepsilon \) and between \( r \) and \( \varepsilon_m \),

\[ M = h r^2 \int_{0}^{\varepsilon_m} \sigma \varepsilon \sqrt{c^2 - r^2 \varepsilon_m^2} \, d \varepsilon = h c 3 \int_{0}^{\varepsilon_m} \sigma \varepsilon \sqrt{1 - \left( \frac{\varepsilon}{\varepsilon_m} \right)^2} \, d \varepsilon \]

\[ = (1/2) D^3 J_f \]

\[ \frac{\varepsilon_m}{\varepsilon_m^2} \int_{0}^{\varepsilon_m} \sigma \varepsilon \sqrt{1 - \left( \frac{\varepsilon}{\varepsilon_m} \right)^2} \, d \varepsilon \]

(49)

where \( J_f \) = the plastic bending factor

The plastic bending factor is equal to the integral divided by the square of the strain in the extreme fibers.

The procedure of evaluating the factor \( J_f \) is numerically illustrated by Williams in his paper mentioned above. The method discussed
above is based on the assumption that the relationship between stress and strain is the same for bar fibers as it is for simple tension and compression, and that this relationship holds for the full range of the stress-strain diagram. Cross sections for which the neutral axis is not an axis of symmetry can be solved by the method. If the tension and compression stress-strain curves are reasonably alike, the available experimental evidence indicates that fair results can be obtained by assuming the neutral axis through the centroid. Actually it constantly shifts from this position after the proportional limit stress is exceeded in the outer fibers. The slope of the stress vs. plastic-strain curve can be written as,

\[ \varepsilon_p = \varepsilon_o + K(\sigma - \sigma_o)^n \]

\[ \frac{d\sigma}{d\varepsilon_p} = \frac{K'}{n} \left( \frac{1}{\varepsilon_o} - 1 \right) \]

where \[ K' = \left( \frac{1}{K} \right)^{1/n} \] 

The extent to which the above solutions could be used in practice should be determined by tests. The assumptions that the neutral surface at all times coincides with the mid-plane of the bar and that the stress in the radial direction is zero, are certainly admissible for moderate bending, and have enabled a very neat solution. For very severe bending, where the bar was loaded to the fully plastic state, the previous assumptions could not be made, because the neutral surface will not coincide with the central surface of the bar and the radial stresses were no longer negligible. The consideration of the radial stresses in the analysis will be the subject of the next chapter.


CHAPTER III

ANALYSIS OF THE STATE OF STRESS AND STRAIN IN PRISMATICAL BARS SUBJECTED TO BENDING IN THE ELASTIC-PLASTIC REGION ASSUMING BIAXIAL STRESS

1. Review of Deformation Theories of Plasticity

In recent years many research papers have appeared on the problem of establishing a relationship between the stresses and strains in plastic bodies that have the work-hardening property. Advances in the field of work-hardening solids have been primarily in the general theory and not many specific practical problems have been solved because of mathematical complexity. A great deal of experimental work has been carried out to find laws which will predict the behavior of materials under combined stress from their action in simple tension. It is beyond the scope of this brief review to discuss the various theories concerning stress-strain relations in the plastic range or the extensive literature on the subject of plasticity. A critical survey of theories of plasticity can be found in textbooks (1,2,3,4,5,6,7,8). Many comprehensive and lucid surveys of the stress-strain relations for a plastically deforming metal are available in the technical literature (2,9,10,11,12,13). Basically the existing theories on plastic stress-strain relations can be classified into three categories: 

a) The deformation theories of plasticity for work-hardening materials postulate that the state of stress determines the state of strain uniquely as long as plastic deformation continues. 

b) On the other hand, the flow theories of plasticity
postulate that the increment of strain is uniquely determined by the existing stress and the increment of stress. Thus the flow theory takes into account the history of loading in predicting the strain at a point at any stage in the loading, whereas the deformation theories say that the strain is the same irrespective of the path of loading. This constitutes the main difference between these two types of theory and both theoretical arguments and experimental evidence have established the flow theories as conforming more to physical reality than the deformation laws. c) Batdorf and Budiansky (14) have developed a theory for the polyaxial stress-strain relations in the plastic range based on the physical mechanism of plastic deformation. At the present time, it is not clear whether a phenomenological theory such as some kind of generalized incremental theory, or a physical theory, i.e., one based on a physical model of the plastically deforming material, will first achieve success in correlating polyaxial flow phenomena properly. This represents the challenge for the engineer and physicist. The general theories on the structure of stress-strain laws in the work-hardening range are well discussed and easily available in the technical literature. The author wishes to confine himself to the review of certain parts of deformation theories which are pertinent in a later analysis.

Certain inconsistencies as a consequence of the deformation theories have been convincingly demonstrated by Drucker, Prager, Hill, and their associates. It is generally accepted by research workers that the use of incremental theories describes more completely the physical behavior of a volume element. As a consequence of this, theoretical work in this field is now based exclusively on incremental stress-strain laws
(flow theories); except for the case of proportional loading, where they can be shown to be integrated forms of incremental laws. The deformation theories (total stress-strain laws) have been discarded by the theoreticians. Because the deformation laws are mathematically more convenient than the incremental laws, many solutions of specific problems offered in the contemporary literature are still based on the total laws. It is clear that the so-called deformation, or finite, theories of plasticity, which relate total plastic strain to final stress, are applicable to proportional loading. It has also been shown repeatedly, by means of physical arguments, that a deformation theory cannot have general validity for all stress histories. It can be summarized that the two characteristics of deformation theories are validity for proportional (radial) loading and lack of general validity for all types of loading. The predictions of a deformation theory for total loading paths do not necessarily constitute a correct description of the plastic behavior of any given material. Only experiment can determine the accuracy of the predictions.

It is pertinent to compare the facility with which the two types of theory can be applied to practical engineering problems. In this respect deformation theories have a decided advantage over flow theories. Frequently, even though the conditions on stress axes and ratios are not strictly maintained, solutions to specific problems, using the deformation theories of Hencky-Nadai, vary only a small amount from those solved by the incremental relations of Prandtl-Reuss. For this reason, the adoption of deformation theories as a basis for engineering solution should not necessarily be ruled out. The errors involved in using a
deformation theory can be estimated either from what is deemed to be the
more correct incremental theories or better still from reliable experimental observations. It should be pointed out that the hypotheses underly-ing the modern theory of plasticity are of a precise formal nature,
while the experimental data are insufficiently definite and are ordinar-
ily used only for indirect verification of the theory. They admit, as a
rule, of various interpretations.

The ultimate objective of the mathematical theory of plasticity is to determine the history of the state of stress and strain at all
points in a partially or totally plastic body when the history of the
boundary loadings and displacements are specified. In order to facili-
tate analysis, some assumptions are required. Real materials are often
reasonably isotropic in the unstrained state and therefore initial iso-
tropy is assumed. Time and temperature effects, also, though very im-
portant, will be neglected in the analysis. The mechanism of isotropic
strain hardening, too, will be assumed.

The complete solution of a general problem involves the calcula-
tion of stresses and strains in both the elastic and plastic regions.
In the elastic region, the stress is directly connected with the total
strain and the equations there are fundamentally different from those
holding in the plastic region. The solutions in the two regions cannot
be found separately and independently since both depend on certain con-
ditions of continuity in the stresses and displacements across the
elastic-plastic boundary. This boundary is itself one of the unknowns
usually difficult to determine. It is pointed out by experts that the
complete solution of a plastic problem will be practicable in relatively
few cases. Complete solutions can only be expected where there is some special geometrical symmetry or other simplifying property of the problem. Reasonable approximation may be assumed which will allow solutions to be obtained for the more complicated problems of technical importance.

A theory of plastic deformation establishes nonlinear stress-strain relations between instantaneous stresses and strains. Such stress-strain relations for perfectly plastic materials were first discussed by Hencky (15), for materials with strain-hardening by Nadai (16). In the following discussion, only materials with strain-hardening will be considered (17). If the principal stresses and strains are denoted by \( \sigma_1, \sigma_2, \sigma_3, \) and \( \varepsilon_1, \varepsilon_2, \varepsilon_3, \) respectively, Hooke's law for an incompressible elastic solid can be written in the form,

\[
\frac{\sigma_1 - \sigma_2}{\varepsilon_1 - \varepsilon_2} = \frac{\sigma_2 - \sigma_3}{\varepsilon_2 - \varepsilon_3} = \frac{\sigma_3 - \sigma_1}{\varepsilon_3 - \varepsilon_1} = 2G_0 \tag{1}
\]

where \( G_0 \) denotes the shear modulus in the elastic range. For small strains the condition of incompressibility can be expressed as

\[
\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0 \tag{2}
\]

Eq. (1) implies that the principal axes of stress and strain should coincide. It is recalled that Hooke's equations embody three hypotheses:

1) Principal axes of stress and strain coincide.

2) Mohr's circle diagrams of stress and strain are similar.

3) Volume changes are proportional to mean normal stress.

The first two hypotheses are satisfied by Eq. (1). Eq. (1) is easily generalized to yield a typical stress-strain relation of the "deformation type" (18,19).
Hencky-Nadai's theory provides the logical extension to Hooke's elastic equations. Following Sokolovsky (3,13), Hencky-Nadai's equations embody three hypotheses:

1) The directions of principal stresses and strains coincide.
2) The Mohr's circle diagrams of stress and strain are similar at any stage in the flow.
3) Volume changes are elastic.

Hypotheses 1) and 2) are satisfied by taking (for example),

$$\frac{\varepsilon_1 - \varepsilon_2}{\sigma_1 - \sigma_2} = \frac{\varepsilon_2 - \varepsilon_3}{\sigma_2 - \sigma_3} = \frac{\varepsilon_3 - \varepsilon_1}{\sigma_3 - \sigma_1} = K \quad (3)$$

where $K$ is a scalar factor of proportionality dependent on the position of the volume element in the body and the stage of plastic deformation.

Constant $K$ is not a material constant. When the stresses are functions of the coordinates (but otherwise constant) $K$ becomes also a space function. This function was defined with the exception of a factor by the theory of Mises-Hencky. Nadai (20) has shown that if for the strains $\varepsilon$ the strain rates $d\varepsilon/dt$ are substituted in Eq. (3), we can distinguish a few interesting cases:

a) Constant $K$ is a true material constant. Stress is proportional to strain rate in the sense as assumed in the theory of viscous liquids. The case $K = \text{const.}$ is indeed the case of pure viscous flow, with a constant viscosity coefficient. Assuming equilibrium between the stresses, this leads to the case of creep or of the slow motion of a very viscous material.

b) If $K$ is left undetermined but assumed as not dependent on stress, we need naturally one additional condition or a new equation.
For example, a condition of yielding or of plasticity may be introduced which must be satisfied by the stress components. This leads back to the theories of St. Venant or Mises-Hencky for stationary plastic flow.

c) If \( K \) is a function of the stresses, the new interesting case of a material with a variable viscosity (dependent upon stress) may be constructed.

In our following discussion the time effect will be neglected. Eq. (3) was obtained under the assumption that for stationary plastic flow at any given instant the principal shear strains are proportional to the principal shear stresses. These relations are based upon those suggested by St. Venant and were used by Lode, Nadai, Hencky, and other investigators. Lode tested tubes in combined tension and internal pressure (21). In order to compare his test results he introduced two variables \( \mu \) and \( \nu \) which are defined as the ratios of \( \text{OB} \) to \( \text{OA} \) in Figure 6.

\[
\mu = \frac{2\sigma_2 - \sigma_1 - \sigma_3}{\sigma_1 - \sigma_3} = \frac{\sigma_2 - \left(\frac{\sigma_1 + \sigma_3}{2}\right)}{\frac{1}{2}}
\]

\[
\nu = \frac{2\varepsilon_2 - \varepsilon_1 - \varepsilon_3}{\varepsilon_1 - \varepsilon_3} = \frac{\varepsilon_2 - \left(\frac{\varepsilon_1 + \varepsilon_3}{2}\right)}{\frac{1}{2}}
\]

In order for Eq. (3) to be satisfied, it is evident that \( \mu \) must be equal to \( \nu \). Lode's test results show a deviation from the line \( \mu = \nu \) but for practical purposes this relation may be taken as representing the behavior of some materials. Taylor and Quinney (22) ran
combined tension-torsion tests on various materials and found that the departure from the line $\mathcal{M} = \mathcal{J}$ was different for each material. This would tend to indicate that the shape of the stress-strain diagram may influence the distribution of the principal strains. For the convenience of later analysis the proportionality factor $K$ will be replaced by $(3/2)\phi$. Rewriting Eq. (3) and considering only the plastic component,

$$\frac{\varepsilon_p^{1} - \varepsilon_p^{2}}{\sigma_1 - \sigma_2} = \frac{\varepsilon_p^{2} - \varepsilon_p^{3}}{\sigma_2 - \sigma_3} = \frac{\varepsilon_p^{3} - \varepsilon_p^{1}}{\sigma_3 - \sigma_1} = \frac{3}{2} \phi . \quad (5)$$

From mathematical identity, the following alternate equations can be written,

$$\frac{\varepsilon_p^{1} - \varepsilon_p^{2}}{\sigma_1 - \sigma_2} = \frac{\varepsilon_p^{2} - \varepsilon_p^{3}}{\sigma_2 - \sigma_3} = \frac{\varepsilon_p^{3} - \varepsilon_p^{1}}{\sigma_3 - \sigma_1}$$
The three principal shear stresses are defined as,

\[\tau_1 = \frac{\sigma_2 - \sigma_3}{2}, \quad \tau_2 = \frac{\sigma_3 - \sigma_1}{2}, \quad \tau_3 = \frac{\sigma_1 - \sigma_2}{2}\]  

Similarly the three principal shear strains are

\[\gamma_1 = \varepsilon_{p_2} - \varepsilon_{p_3}, \quad \gamma_2 = \varepsilon_{p_3} - \varepsilon_{p_1}, \quad \gamma_3 = \varepsilon_{p_1} - \varepsilon_{p_2}\]  

It is to be noted that the stresses and strains discussed here will be the true stresses and natural strains.

Figure 7 shows a small cubical element subjected to three principal stresses \(\sigma_1, \sigma_2, \) and \(\sigma_3\) where \(\sigma_1 > \sigma_2 > \sigma_3\).

Fig. 7. Cubical Element Subject to Combined Stresses.

By definition (23),

\[\left[ (\varepsilon_{p_1} - \varepsilon_{p_2})^2 + (\varepsilon_{p_2} - \varepsilon_{p_3})^2 + (\varepsilon_{p_3} - \varepsilon_{p_1})^2 \right]^{\frac{1}{2}} = \left[ \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 \right]^{\frac{1}{2}} = \varepsilon_m\]
\( \gamma_m \) is the root-mean-square of the three principal shear strains and it is noted that \( \gamma_m \) is proportional to octahedral shear strain or effective (significant) shear strain. \( \gamma_m \) might be called the root-mean-square shear strain. Similarly by definition,

\[
\left[ (\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 \right]^{1/2} = 2(\tau_1^2 + \tau_2^2 + \tau_3^2)^{1/2} = 2 \tau_m \tag{9}
\]

It is seen that \( \tau_m \) is proportional to the octahedral shear stress or effective stress and may be called the root-mean-square shear stress.

Substituting Eqs. (8) and (9) into Eq. (6), the following equation can be obtained.

\[
\phi = \frac{2}{3} \left[ \frac{(\varepsilon_{p1}-\varepsilon_{p2})^2 + (\varepsilon_{p2}-\varepsilon_{p3})^2 + (\varepsilon_{p3}-\varepsilon_{p1})^2}{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2} \right]^{1/2} = \frac{1}{3} \frac{\gamma_m}{\tau_m} \tag{10}
\]

It is seen that \( \phi \) should be a positive quantity. In general, the following functional relationship can be written by using test results (uniaxial or, better, biaxial).

\[
\tau_m = f(\gamma_m) \tag{11}
\]

Eq. (10) can be written as

\[
\phi = (1/3)(\gamma_m/f(\gamma_m)) = F(\gamma_m). \tag{12}
\]

It is seen that \( \phi \) can be expressed as a function of \( \gamma_m \). Experiments (13) have shown that during plastic deformation the plastic components of strain do not contribute materially to volume change. In other words the volume change is always elastic. Consequently,

\[
\varepsilon_{p1} + \varepsilon_{p2} + \varepsilon_{p3} = 0 \tag{13}
\]
Solving Eqs. (5) and (13) for $\varepsilon_{p1}, \varepsilon_{p2}, \varepsilon_{p3}$, respectively, we obtain:

$$\varepsilon_{p1} = \phi \left\{ \sigma_1 - \frac{1}{2} (\sigma_2 + \sigma_3) \right\}$$

$$\varepsilon_{p2} = \phi \left\{ \sigma_2 - \frac{1}{2} (\sigma_3 + \sigma_1) \right\}$$

$$\varepsilon_{p3} = \phi \left\{ \sigma_3 - \frac{1}{2} (\sigma_1 + \sigma_2) \right\}$$

(14)

Prager (18, 24) has given the most general form of deformation theory on the assumption of isotropy, plastic incompressibility, and the absence of influence of the sum of principal stresses.

$$\varepsilon_{ij}^p = P(J_2, J_3) S_{ij} + Q(J_2, J_3) t_{ij}$$

(15)

where $\varepsilon_{ij}^p$ is the plastic component of strain. The usual tensor notation, with repeated subscripts indicating summation was used in the above equation.

- $\sigma_{ij}$ = the stress
- $S_{ij} = \sigma_{ij} - (1/3)\sigma_{kk} \delta_{ij}$ = the deviator stress
- $\sigma_{kk}$ = the sum of principal stresses
- $\delta_{ij}$ = the Kronecker delta
- $J_2 = (1/2) S_{ij} S_{ji} = (1/2)(s_1^2 + s_2^2 + s_3^2)$
- $J_3 = (1/3) S_{ij} S_{jk} S_{ki} = (1/3)(s_1^3 + s_2^3 + s_3^3) = s_1 s_2 s_3$

where $S_1, S_2, S_3$ are the principal stress deviations. In the customary engineering notation,

- $S_1 = (1/3)(2\sigma_1 - \sigma_2 - \sigma_3)$
- $J_2 = (1/6) \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]$
- $J_3 = (1/27) \left[ (2\sigma_1 - \sigma_2 - \sigma_3)(2\sigma_2 - \sigma_3 - \sigma_1)(2\sigma_3 - \sigma_1 - \sigma_2) \right]$ or
- $\sigma = (8/27) \left[ (\tau_1 - \tau_2)(\tau_2 - \tau_3)(\tau_3 - \tau_1) \right]$
where \( \sigma_1, \sigma_2, \sigma_3 \) are principal stresses,
\( \tau_1, \tau_2, \tau_3 \) are principal shear stresses,
\( t_{ij} = S_{ik}S_{kj} = (2/3)(J_2 \bar{\sigma}_{ij}) \) = the deviation of the square of stress deviation.

The elastic component of strain is assumed to be given by Hooke's law. A special case of Eq. (15) is obtained by taking \( Q = 0 \) and \( P = P(J_2) \). In engineering terminology, this relation may be written as (18, 24),

\[
\varepsilon_{pl} = \frac{1}{c} \left[ \sigma_1 - \frac{1}{2}(\sigma_2 + \sigma_3) \right] \\
\varepsilon_{p2} = \frac{1}{c} \left[ \sigma_2 - \frac{1}{2}(\sigma_3 + \sigma_1) \right] \\
\varepsilon_{p3} = \frac{1}{c} \left[ \sigma_3 - \frac{1}{2}(\sigma_1 + \sigma_2) \right]
\]

where \( c \) is the plastic modulus, a function of \( J_2 \), or equivalently the octahedral shearing stress. It is seen that Eq. (14) and Eq. (15) have the same form by setting \( \phi = 1/c \).

The physical and theoretical backgrounds of the deformation theories of plasticity are well established. For advanced treatment, the reader is referred to the research papers appearing in the contemporary literature, particularly the work by Prager, Drucker and their associates.

2. Correlation of Behavior under Multiaxial Stress with Behavior under Simple Tension or Torsion (Functional Representation of Behavior)

A suitable method of representing the plastic behavior of metals under multiaxial stress is of importance in dealing with the problem in hand. Many investigators have studied the relation between multiaxial stresses and the resulting strains. It was pointed out above that the
Hencky-Nadai equations introduce another unknown $\phi$. Hence one more independent equation must be found. This is supplied by the loading function. Yield functions which predict the elastic breakdown of a volume element have been well discussed in the literature. It is logical to extend such functions to describe the additional plastic deformation once the material has exceeded the yield point. These are called loading functions. Two simple examples of loading function are the maximum shear stress and the Von Mises' function, the latter being the second invariant of the stress deviation. Nadai (25) has shown that the octahedral shear is proportional to the square root of the shear-strain energy (Von Mises' criterion) and so may be taken as basically the same criterion for the onset of plastic flow. The octahedral shear stress may be expressed as follows:

$$\tau = \left(\frac{1}{3}\right) \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{1/2}$$  \hspace{1cm} (17)

Nadai has extended this concept to stresses and strains in the plastic range by use of a corresponding strain function, the octahedral shear strain

$$\varepsilon = \left(\frac{2}{3}\right) \left[ (\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2 \right]^{1/2}$$  \hspace{1cm} (18)

in which $\varepsilon_1$, $\varepsilon_2$, and $\varepsilon_3$ are the principal natural strains. Diagrams of octahedral shear stress versus octahedral shear strain for a given metal are essentially of the same form regardless of the condition of multiaxial stress. Lode (21) demonstrated experimentally that in the plastic range the change in the natural shear strains of an isotropic material are proportional to the shear stresses (26). Dorn and his
Collaborators (26) have obtained a pair of functions which they term the effective stress and effective strain, respectively, as follows.

\[
\bar{\sigma} = (1/\sqrt{2}) \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{1/2}
\]

(19)

\[
\bar{\varepsilon} = (\sqrt{2}/3) \left[ (\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2 \right]^{1/2}
\]

(20)

It will be recognized that these functions differ from the octahedral shear stress and strain functions only in the values of the numerical coefficients, but they have the advantage that (theoretically) their plot is identical with the curve of true stress versus natural strain for a given material under a simple tension (13). The work-hardening function most commonly used in combination with Nadai-Hencky's equations, is

\[
\bar{\sigma} = f(\bar{\varepsilon})
\]

(21)

Nadai (1) uses as a work-hardening function, for small strains,

\[
\tau = f_1(\varepsilon^*).
\]

(22)

In relating \( \varepsilon^* \) to a tension test, Nadai also neglects elastic strains. When the ratios of the principal stresses are not constant during the loading of a metal, neither the octahedral shear stress-strain nor the effective stress-strain equations are strictly applicable. Analytical procedures are thus available for estimating the plastic behavior of metals under multiaxial loading from the data of the simple tension test or other available test. Davis and Parker (26) stated that in a majority of cases, reasonable prediction of the stress-strain behavior of the metal in the plastic range can be made for various biaxial stress conditions from the data of the simple tension test, by use of the modified
octahedral-shear concept. The work-hardening functions discussed above may be considered as plausible extensions of Von Mises' yield function. A corresponding generalization of Tresca's maximum shear stress function has received rather little discussion. Ludwik (27) in 1909 suggested that work-hardening could be described by taking the maximum shear stress as a function of the maximum shear strain,

$$\tau_{\text{max}} = F(\gamma_{\text{max}})$$  \hspace{1cm} (23)

F can either be obtained from a tension test or a pure shear test as in the previous cases. Comparison of Tresca and Von Mises' functions is well discussed elsewhere (2,13,28).

Experiments (29) on metals have shown that initial yielding and subsequent plastic flow are not affected by a moderate hydrostatic pressure or tension, either applied alone or superposed on a state of combined stress. From this consideration alone all the so-called theories of failure, except the maximum shear stress and energy of distortion, must be rejected. Therefore shear stresses are the predominant factor in plastic deformation, which is a succession of permanent shear strains. A variable which is a function of the shear stresses plotted against a function of the shear strains, can be used to correlate the results of all tests on a given material (24). Many variables which are homogeneous functions of the shear stresses will work reasonably well although one should be best. A plot of maximum shearing stress versus maximum shearing strain can be used with reasonable success. The octahedral-shearing-stress plot also shows good correlation (26,30). However, these are only two possibilities among a great number. Experimental results of
biaxial tests are too numerous to mention and the reader is referred to the technical literature (for example, Journal of Applied Mechanics, ASME, 1933-1959).

The solution of general plasticity problems by means of Eqs. (10) and (14) in their present forms is very difficult and discouraging. Apparently further simplification is required. The expression for $\phi$ (Eq. 10) will be discussed further based on the concepts of modified octahedral shear and maximum shear stress criterion. The laws of plastic deformation to be postulated are in accordance with the uniaxial stress-strain behavior which is the slightly idealized one usually adopted for the engineering purposes. The effect of reversal of loading upon the plastic behavior is not treated in this analysis. As in the uniaxial case, time dependent effects, such as creep, elastic recovery, and the effect on the stress-strain curve of rate of loading, are neglected. Plastic stress-strain diagrams of ductile metals for tension can be expressed by the following mathematical expression (see Chapter II).

$$\varepsilon_p = K(\sigma - \sigma_0)^n$$

or

$$\sigma = \sigma_0 + (\varepsilon_p/K)^{1/n}$$

where

$\sigma$ = tensile stress

$\varepsilon_p$ = plastic strain in uniaxial tension

$n, K$ = material constants

$\sigma_0$ = yield point stress in tension.

It will be assumed that the material will not exhibit a yield point elongation phenomenon. This is true for most ductile metals except mild
steel. It is not difficult to include yield point elongation $\epsilon_o$ in the analysis but it will be neglected here. Furthermore, for the simplification of the analysis, the stress-strain curve is assumed to be identical in both tension and compression. The relaxation of this restriction will make the analysis very difficult to handle.

Similarly plastic stress-strain diagrams of ductile metals for torsion can be expressed by the following mathematical expression (28),

$$\tau_p = L(\tau - \tau_o)^m$$

or

$$\tau = \tau_o + (\tau_p/L)^{1/m}$$

where

$\tau$ = shear stress

$\tau_p$ = plastic shear strain

$\tau_o$ = yield point shear stress

$m, L$ = material constants

The relationship between Eqs. (24) and (25) will be investigated next. Denoting by $\epsilon_{p1}, \epsilon_{p2}$, and $\epsilon_{p3}$, the three principal plastic strains, the following relations can be written for the uniaxial tension.

$$\epsilon_{p1} = \epsilon_{pt}, \quad \epsilon_{p2} = \epsilon_{p3} = -(1/2)\epsilon_{pt}$$

$$\epsilon_{p1} - \epsilon_{p3} = (3/2)\epsilon_{pt} = \gamma$$

where

$$\gamma = \epsilon_{p1} - \epsilon_{p3}.$$ 

Substituting Eq. (26) into Eq. (24), we obtain

$$\sigma = \sigma_o + (2/3K)^{1/n}\gamma^{1/n} = \sigma_o + K_o^{1/n}\gamma^{1/n}$$

where

$$K_o = 2/3K.$$ 

Eq. (27) shows that tensile stress $\sigma$ is a function of shearing strain $\gamma$. Consequently Eq. (27) reduces to Eq. (25) by setting $\sigma = 2\tau$ and $\sigma_o = 2\tau_o$. 
It has been repeatedly pointed out by investigators that good
correlation can be found between the modified octahedral shear stress
and octahedral natural shear strain. In the most general form the fol-
lowing functional relationship can be written (28),

\[ \tau_m = K (\tau_m - \tau_{mo})^n \]  

(28)

where

\[ \tau_m = [ (\varepsilon_{p1} - \varepsilon_{p2})^2 + (\varepsilon_{p2} - \varepsilon_{p3})^2 + (\varepsilon_{p3} - \varepsilon_{p1})^2 ]^{1/2} \]

\[ \tau_{mo} = \text{yield point root-mean-square shear stress} \]

\[ K, n = \text{material constants determined by experiment}. \]

It will be recalled that the work-hardening function based on the
concept of Von Mises' yield function can be written as follows.

\[ \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{1/2} \times \sqrt{2} \sigma_t = 2 \tau_m \]

Consequently for a tension test,

\[ \tau_m = (1/\sqrt{2}) \sigma_t, \quad \tau_{mo} = (1/\sqrt{2}) \sigma_0 \]

where the subscript \( t \) means tension.

Considering plastic strain the following relations can be written.

\[ \varepsilon_{p1} = - (1/2) \varepsilon_{p1} = - (1/2) \varepsilon_{pt} \]

\[ \varepsilon_{p2} = \varepsilon_{p3} = - (1/2) \varepsilon_{p2} = - (1/2) \varepsilon_{p3} \]

\[ \varepsilon_{p3} = - (1/2) \varepsilon_{p3} = - (1/2) \varepsilon_{p3} \]

\[ \varepsilon_{p1} = - (1/2) \varepsilon_{p1} = - (1/2) \varepsilon_{p1} \]

\[ \varepsilon_{p2} = (3/\sqrt{2}) \varepsilon_{pt} \]

\[ \varepsilon_{p3} = (3/\sqrt{2}) \varepsilon_{pt} \]

Substituting \( \tau_m, \tau_{mo}, \tau_m \) into Eq. (28), we obtain

\[ \varepsilon_{pt} = K \frac{\sqrt{2}}{3} \left( \frac{1}{\sqrt{2}} \right)^n (\sigma_t - \sigma_0)^n \]
Setting
\[ K \frac{1}{3} \left( \frac{1}{(\sqrt{2})^{n-1}} \right) = K \]
\[ \varepsilon_{pt} = K(\sigma_t - \sigma_0)^n. \] (30)

Similar manipulation will be done for the torsion test. For the torsion test the following relations between plastic strains can be written,
\[ \varepsilon_{pl} = -\varepsilon_{p3}, \quad \varepsilon_{p2} = 0 \]
\[ \tau_m = \left[ (\varepsilon_{p1} - \varepsilon_{p2})^2 + (\varepsilon_{p2} - \varepsilon_{p3})^2 + (\varepsilon_{p3} - \varepsilon_{p1})^2 \right]^{\frac{1}{2}} = \sqrt{6} \varepsilon_{pl} \] (31)

Considering the stress in simple torsion,
\[ \sigma_1 = \tau = -\sigma_3, \quad \sigma_2 = 0 \]

Substituting \( \sigma_1, \sigma_2, \sigma_3 \) into Eq. (9), we obtain
\[ \tau_m = (\sqrt{6}/2)\tau, \quad \text{and} \quad \tau_{mo} = (\sqrt{6}/2)\tau_0 \]

Substituting \( \tau_m, \tau_{mo}, \) and \( \tau_m \) into Eq. (28), we obtain
\[ \varepsilon_{pl} = K'(\tau - \tau_0)^n \] (32)

where
\[ K' = K \frac{\left( \frac{n-1}{2} \right)}{\left( \frac{n+1}{2} \right)} \]

The relationship between material constant \( K \) in tension (Eq. 24) and material constant \( L \) in torsion (Eq. 25) will be obtained next. From Eq. (25),
\[ \tau_p = L(\tau - \tau_0)^m \]

As before for the torsion test, \( \varepsilon_{pl} = -\varepsilon_{p3}, \quad \varepsilon_{p2} = 0. \)
\[ \varepsilon_{p1} - \varepsilon_{p3} = \varepsilon_p = 2\varepsilon_{pl} \]

Subsequently, we have,

\[ 2\varepsilon_{pl} = L(\tau - \tau_0)^m \]

or

\[ \varepsilon_{pl} = (L/2)(\tau - \tau_0)^m \]  \hspace{1cm} (33)

Comparing Eq. (32) and Eq. (33) shows that \( n = m \) and \( K' = L/2 \). The ratio \( K/L \) can be obtained as

\[ K/L = 1/(3^{(n+1)/2}) \]

As shown above, based on tension test the following relations were obtained.

\[ \tau_{mo} = (1/\sqrt{2}) \sigma_o \]

\[ \text{bar} = K(3)(2^{(n-1)/2}) \]

Substituting the above into Eq. (28),

\[ \varepsilon_m = (3)(2^{(n-1)/2}) (K)(\tau_m - \frac{1}{\sqrt{2}} \sigma_o)^n \]

\[ = (3/\sqrt{2}) K (\sqrt{2} \tau_m - \sigma_o)^n \]  \hspace{1cm} (34)

Substituting Eq. (34) into Eq. (10),

\[ \varphi = \frac{1}{3} \frac{(3/\sqrt{2})K(\sqrt{2} \tau_m - \sigma_o)^n}{\tau_m} = \frac{1}{\sqrt{2}} \frac{K(\sqrt{2} \tau_m - \sigma_o)^n}{\tau_m} \]  \hspace{1cm} (35)

where

\[ \tau_m = (1/2) \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{1/2} \]

The above expression for \( \varphi \) in its present form is again too complicated to be of practical value in the solution of general plasticity problems.
For plane stress, \( \sigma_2 = 0 \), and

\[
\left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{1/2} = \left[ \sigma_1^2 + \sigma_3^2 + (\sigma_1 - \sigma_3)^2 \right]^{1/2}
\]

\[
= \sqrt{2}(\sigma_1 - \sigma_3) \left[ 1 + \frac{\sigma_1 \sigma_3}{(\sigma_1 - \sigma_3)^2} \right]^{1/2}
\]

By the binomial expansion, the following equation can be written,

\[
\left[ 1 + \frac{\sigma_1 \sigma_3}{(\sigma_1 - \sigma_3)^2} \right]^{1/2} = 1 + \frac{1}{2} \left\{ \frac{\sigma_1 \sigma_3}{(\sigma_1 - \sigma_3)^2} \right\} + \frac{1}{2} \left\{ \frac{1}{2} \right\} \left\{ \frac{\sigma_1 \sigma_3}{(\sigma_1 - \sigma_3)^2} \right\}^2 + \ldots
\]

Since the second term of the series is very small, we can write,

\[
\left[ 1 + \frac{\sigma_1 \sigma_3}{(\sigma_1 - \sigma_3)^2} \right]^{1/2} = 1 + \frac{1}{2} \frac{\sigma_1 \sigma_3}{(\sigma_1 - \sigma_3)^2}
\]

Consequently the root-mean-square shear stress \( \tau_m \) becomes,

\[
\tau_m = \frac{1}{\sqrt{2}}(\sigma_1 - \sigma_3) \left[ 1 + \frac{1}{2} \frac{\sigma_1 \sigma_3}{(\sigma_1 - \sigma_3)^2} \right]
\]

Substituting \( \tau_m \) into Eq. (35),

\[
\phi = \frac{K \left[ (\sigma_1 - \sigma_3)^2 \right] \left\{ 1 + \frac{1}{2} \frac{\sigma_1 \sigma_3}{(\sigma_1 - \sigma_3)^2} \right\}^{-n}}{(\sigma_1 - \sigma_3)^2 \left\{ 1 + \frac{1}{2} \frac{\sigma_1 \sigma_3}{(\sigma_1 - \sigma_3)^2} \right\}}.
\]

The above development is based on the concept of the so-called modified octahedral shear stress. It was pointed out earlier that a plot of maximum shearing stress versus maximum shearing strain can be used with reasonable success.
The work-hardening functions based on this concept will be considered next. Considering only the maximum shear stress and the maximum shear strain, Eq. (5) can be written as

$$\phi = \frac{2}{3} \frac{\varepsilon_{pl} - \varepsilon_{p3}}{\sigma_1 - \sigma_3}$$  \hspace{1cm} (38)$$

On account of the assumed incompressibility of the material, the following relations for plastic strains can be written for tension test.

$$\varepsilon_{pl} = \varepsilon_{pt}$$,  $$\varepsilon_{p2} = \varepsilon_{p3} = - (1/2) \varepsilon_{pt}$$

$$\varepsilon_{pl} - \varepsilon_{p3} = \gamma = (3/2) \varepsilon_{pt}$$

For stress,  $$\sigma_1 = \sigma$$,  $$\sigma_2 = 0$$,  $$\sigma_3 = 0$$.

Substituting the above equations into Eq. (38),

$$\phi = \frac{\varepsilon_{pt}}{\sigma} = (2/3)(\gamma/\sigma)$$

Substituting Eqs. (27) and (30) into the above equation we obtain

$$\phi = \frac{K(\sigma - \sigma_0)^n}{\sigma} = \frac{2}{3} \frac{\gamma}{\sigma_0 + K_0^{1/n}(\gamma)_{1/n}}$$  \hspace{1cm} (39)$$

It is seen that $\phi$ is a function of maximum shear strain $\gamma$. Rewriting Eq. (38),

$$\phi = \frac{1}{3} \frac{\varepsilon_{pl} - \varepsilon_{p3}}{(\sigma_1 - \sigma_3)/2} = \frac{1}{3} \frac{\gamma}{\tau}$$  \hspace{1cm} (40)$$

where  $$\gamma$$ = maximum shear strain

$$\tau$$ = maximum shear stress.
In general, the relationship between the maximum shearing stress \( \tau \) and the maximum shearing strain \( \gamma \) can be written as
\[
\gamma = K(\tau - \tau_0)^n
\]  
where \( \tau_0 \) = yield point shearing stress
\( K, n \) = material constants.

It is understood that the elastic shear strain is neglected. Substituting Eq. (41) into Eq. (40),
\[
\phi = (1/3) \frac{K(\tau - \tau_0)^n}{\tau}
\]
where \( \tau = (\sigma_1 - \sigma_3)/2 \).

It is recalled that the modified octahedral shear stress concept establishes a relation between the root-mean-square shear stress and the root-mean-square shear strain.
\[
\tau_m = \bar{\gamma} (\bar{\tau}_m)
\]
The complexity of the expressions \( \tau_m \) in terms of the stresses and \( \bar{\tau}_m \) in terms of the strains, as well as the complexity of the function \( \bar{\gamma} \), determine the difficulty of the nonlinear problem, to which the integration of the equations of plasticity is reduced. In the equation (10), it is very advantageous to simplify the expression for \( \bar{\gamma}_m / \tau_m \). Ilyushin (28) has considered one of the possible simplifications which is essentially based on the work of Von Mises. Considering the expression
\[
R = \frac{1}{|a_{12}|} \sqrt{a_{12}^2 + a_{23}^2 + a_{31}^2}
\]
in which the real quantities \( a_{ij} \) satisfy the conditions
\[
a_{12} + a_{23} + a_{31} = 0 \quad (|a_{12}| \geq |a_{23}| \geq |a_{31}|)
\]
It can be shown that the quantity $R$ is included between the limits
\[ \sqrt{2} \geq R \geq \sqrt{3/2} \]

It differs from the value
\[ R = \frac{(2 + \sqrt{3})}{(2\sqrt{2})} \approx 1.32 \] (46)

by not more than 7.1 percent. With the same degree of precision a function of three variable $a_{ij}$ satisfying Eq. (45) can be replaced by a function of a single variable.

\[ \frac{(2 + \sqrt{3})}{(2\sqrt{2})} |a_{12}| \]

Now, the expressions for $\tau_m$ and $\gamma_m$ can be replaced by the maximum shearing stress $\tau_{\text{max}}$ and the maximum shear strain $\gamma_{\text{max}}$ multiplied by $(2 + \sqrt{3})/(2\sqrt{2})$, respectively. Accordingly, to within 7.1 percent, we have

\[ \tau_m = \frac{(2 + \sqrt{3})}{(2\sqrt{2})} \tau_{\text{max}}, \quad \gamma_m = \frac{(2 + \sqrt{3})}{(2\sqrt{2})} \gamma_{\text{max}}. \] (47)

In the general case of plane stress in a material with strain-hardening, the principal stresses $\sigma_1$, $\sigma_2$, ($\sigma_3 = 0$) are related to the principal plastic strains $\varepsilon_{p1}$, $\varepsilon_{p2}$ ($\varepsilon_{p3} = -\varepsilon_{p1} - \varepsilon_{p2}$) by equations which follow from Eq. (14):

\[ \varepsilon_{p1} = \phi (\sigma_1 - (1/2)\sigma_2) \]
\[ \varepsilon_{p2} = \phi (\sigma_2 - (1/2)\sigma_1) \]
\[ \varepsilon_{p3} = -\varepsilon_{p1} - \varepsilon_{p2} \] (48)

where \[ \phi = \frac{(1/3)(\gamma_m/\tau_m)}. \]

Further discussion on the comparison of Tresca and Von Mises' functions can be found in (2).
3. Elastic Solution for Rectangular Section

The complete solution of a general plasticity problem involves the calculation of stresses and strains in both the elastic and plastic regions. Before proceeding with the elastic-plastic solution of the problem in hand, we will investigate the elastic solution first. The equations of equilibrium and one of the compatibility equations, together with the boundary conditions, give us a system of equations which is usually sufficient for the complete solution of a two-dimensional problem. Consider the problem of pure bending of a narrow rectangular bar to which end moments are applied. As these moments are increased the stress distribution within the bar goes through three distinct phases. In the first phase, the entire bar is elastic; in the second a portion of the bar is plastic whereas the remainder is elastic; in the third phase the entire bar is plastic. It is intended here to present a solution for the stress or strain distribution within the bar during the first phase, i.e., the completely elastic solution to this problem.

A typical section of the bent bar, which is of uniform cross section and whose dimension perpendicular to the plane of curvature is small, is shown in Figure 8. The pure bending moment is uniformly distributed along the end of the bar. The bending moment in this case is constant along the width of the bar and it is natural to expect that the stress distribution is the same in all radial cross sections, and that we may calculate the stress distribution at any radial section. Let \( R \) denote the current radius to the middle plane of the bar and \( M \) the bending moment at the ends perpendicular to the plane of the paper. The stresses on a typical element within the bar, shown shaded in Figure 8,
Fig. 8. Geometry of the Bent Bar.

Fig. 9. The Stresses on a Typical Element.
are shown in Figure 9, where $\sigma_r$ denotes the radial stress and $\sigma_\theta$ the circumferential stress. These stresses do not depend on $\theta$ but only on $r$, because the bending moment is the same in all radial cross sections.

For narrow rectangular cross section the stress $\sigma_z$ in the longitudinal direction, perpendicular to the plane of curvature, can be neglected. Since the radial stress is zero at the convex and concave surfaces, these surface fibers will be in a state of uniaxial tension and uniaxial compression, respectively, in the tangential direction. Under conditions of uniaxial tension and compression there are lateral contraction and expansion, respectively, causing the cross section to distort into a figure resembling a trapezoid (Fig. 8). Let $dr$ be the thickness of the element, $d\theta$ the included infinitesimal angle between the two radial planes which bound the element. Let $w$ be the full breadth of the bar at radius $r$, and $(w + dw)$ that at radius $(r + dr)$. The condition for radial equilibrium of forces is (33),

$$\left(\sigma_r + \frac{d\sigma_r}{dr}\right)(w + dw)(r + dr)d\theta = \sigma_r r d\theta w - 2\sigma_\theta dr (w + \frac{dr}{2}) \sin(\frac{d\theta}{2}) = 0$$

Neglecting products of differentials and noting that $\sin(\frac{d\theta}{2}) \approx \frac{d\theta}{2}$, approximately, for small angles, the above equation reduces to

$$\left(\frac{d\sigma_r}{dr}\right) + \frac{(\sigma_r - \sigma_\theta)}{r} + \frac{(\sigma_r dw)}{(w dr)} = 0$$

For simplification $dw/dr$ will be neglected, i.e., $dw/dr = 0$. Consequently the above equation reduces to

$$\frac{d\sigma_r}{dr} + \frac{(\sigma_r - \sigma_\theta)}{r} = 0 \quad (49)$$

For the problem of pure bending there are no shear stresses acting on the element abcd which is shown in Figure 9. The stresses $\sigma_r$ and $\sigma_\theta$
are, therefore, the principal stresses. The stresses on a typical element within the elastic range must satisfy the equilibrium equation and the compatibility equation. From the symmetry of the problem, the only equation of equilibrium which remains to be satisfied is Eq. (49). It will be assumed that the plane transverse section remains plane during bending.

Denoting by

\[ \varepsilon_c = \text{strain in the tangential direction at central plane} \]
\[ \varepsilon_\theta = \text{strain in the tangential direction at radius (y + R)} \]

and from the assumption of plane remains plane after deformation, the following stress-strain relationship can be written (50),

\[ \varepsilon_\theta = \varepsilon_c + \frac{y}{R} = \frac{1}{E} \left( \sigma_\theta - \nu \sigma_r \right) \]

or

\[ \sigma_\theta = E(\varepsilon_c + \frac{y}{R}) + \nu \sigma_r, \quad \sigma_r - \sigma_\theta = \sigma_r(1 - \nu) - E(\varepsilon_c + \frac{y}{R}) \]

where \( E = \text{modulus of elasticity} \)
\( \nu = \text{Poisson's ratio}. \)

Substituting Eq. (50) into Eq. (49),

\[ \frac{d\sigma_r}{dr} + (1 - \nu)(\sigma_r/r) - (E/r)(\varepsilon_c + \frac{y}{R}) = 0 \]

Setting \( (1 - \nu) = k \) and noting \( y = r - R \), the above equation can be written as,

\[ \frac{d(\sigma_r^k)}{dr} = E \frac{\sigma_r^k}{r^{k-1}} \left( \varepsilon_c + \frac{r}{R} - 1 \right) \]

(51)

After integration and rearranging,

\[ \sigma_r = E \left\{ \frac{\varepsilon_c - 1}{k} + \frac{r}{R(k + 1)} + \frac{a}{r^k} \right\} \]

(52)

where \( a = \text{constant of integration} \).
From the boundary condition \( \sigma_r = 0 \) at \( r = R + c \),
\[
\alpha = (R + c)^k \left\{ \frac{1 - \varepsilon_c}{k} - \frac{R + c}{R(k + 1)} \right\} \tag{53}
\]
Similarly from the boundary condition \( \sigma_r = 0 \) at \( r = R - c \),
\[
\alpha = (R - c)^k \left\{ \frac{1 - \varepsilon_c}{k} - \frac{R - c}{R(k + 1)} \right\} \tag{54}
\]
\( \varepsilon_c \) can be found by equating Eq. (53) and Eq. (54). The result is
\[
\varepsilon_c = 1 - \frac{k}{R(k+1)} \frac{(R+c)^{k+1} - (R-c)^{k+1}}{(R+c)^k - (R-c)^k} \tag{55}
\]

The observed fact that the neutral axis will move toward the compression surface can be proved by the following mathematical argument.
It is only necessary to show that \( \varepsilon_c \) is positive with progressive bending. Let us consider the extreme case when \( R = c \). (This condition cannot be reached physically for the elastic deformation.) Then Eq. (55) reduces to
\[
\varepsilon_c = 1 - 2k/(1+k).
\]
It is easily seen that \( \varepsilon_c > 0 \) since \( k \ll 1 \).
Substituting Eq. (55) into Eq. (53),
\[
\alpha = \frac{2c(R^2 - c^2)^k}{R(k+1)} \left\{ (R+c)^k - (R-c)^k \right\} \tag{56}
\]
Substituting Eqs. (55) and (56) into Eq. (52),
\[
\sigma_r = \frac{E}{R(k+1)} \left[ - \frac{(R+c)^k - (R-c)^k}{(R+c)^k - (R-c)^k} + r + \frac{2c(R^2 - c^2)^k}{r^{k}(R+c)^k - (R-c)^k} \right] \tag{57}
\]
The position of minimum $\sigma_r$ can be found by the following condition,
\[ \frac{d\sigma_r}{dr} = 0. \]
Substituting $\varepsilon_c$ and $\sigma_r$ into Eq. (50) we obtain
\[ \sigma_\theta = \frac{E}{R^2(\frac{1}{R^2}-\frac{1}{y^2})} \left[ \left( \frac{2}{y^2} \right) \left( R+y \right) - \left( \frac{1}{R^2} \right) \left( R+c \right)^{k+1} - \left( R-c \right)^{k+1} \right] + \frac{2c(R^2 - c^2)^k}{(R+y)^k((R+c)^k - (R-c)^k)} \]
(58)

The strains $\varepsilon_\theta$, $\varepsilon_r$, and $\varepsilon_z$ at any point can be calculated from the following equations.
\[
\begin{align*}
\varepsilon_\theta &= \frac{1}{E} (\sigma_\theta - \nu \sigma_r) \\
\varepsilon_r &= \frac{1}{E} (\sigma_r - \nu \sigma_\theta) \\
\varepsilon_z &= \frac{1}{E} (-\nu \sigma_\theta - \nu \sigma_r)
\end{align*}
\]

It can be shown that $b \int_{R-c}^{R+c} \sigma_\theta \, dr = 0$, where $\sigma_\theta$ is given by Eq. (58).

The bending moment can be obtained as
\[ M = \int_{R-c}^{R+c} \sigma_\theta \, br \, dr \]
(59)

It is seen from Eq. (58) that by assuming the radius of curvature the tangential stress $\sigma_\theta$ at any point may be calculated. For obtaining bending moment $M$, the graphical or numerical integration is suggested.

It is felt that the solution of the present problem can also be obtained by using the following expression for the stress function $\phi$ (31).
\[ \phi = A \log r + B r^2 \log r + C r^2 + D \]
(60)

It is pointed out by Timoshenko (31) that the solutions of all problems of symmetrical stress distribution and no body forces can be obtained from Eq. (60). The corresponding stress components (31) are,
\[ \sigma_r = (1/r)(d\phi/dr) = (A/r^2) + B(1 + 2 \log r) + 2C \]
\[ \sigma_\theta = d^2\phi/dr^2 = -(A/r^2) + B(3 + 2 \log r) + 2C \]  
\[ \tau_{r\theta} = 0 \]

Further discussion will be omitted here and the solution can be obtained by fitting the boundary conditions. It is to be noted that Shaffer and House (39) as well as Swida (34) used the above stress function in their analyses for the elastic-plastic stress distribution within a wide curved bar subjected to pure bending.


The classical theory of plasticity assumes a perfectly plastic material, i.e., a material which does not exhibit work-hardening but flows plastically under constant stress. The rigid, perfectly plastic solid with the stress-strain diagram of Figure 10 will be discussed in the present analysis. Tresca's and Von Mises' yield functions are the most widely used in the theory of plasticity. Consequently the analysis based on these two yield functions will be considered.

a) Solution Based on Tresca's Yield Function

Plastic flow occurs whenever the stress components satisfy the Tresca yield condition, namely,

\[ \sigma_{\text{max}} - \sigma_{\text{min}} = 2\tau_0 \]  

where \( \tau_0 \) is the maximum shear stress of the material equal to one-half
the tensile yield stress. For plane stress

\[ \sigma_r \leq \sigma_z \leq \sigma_\theta \]

or

\[ \sigma_\theta \leq \sigma_r \leq \sigma_z \]

(63)

where \( \sigma_z = 0 \) is the stress in the longitudinal direction, perpendicular to the plane of curvature. From the symmetry of the problem, the stresses \( \sigma_r, \sigma_\theta \) and \( \sigma_z = 0 \) are the principal stresses.

The completely plastic solutions to this problem for the case of a very wide plate (plane strain) have been previously reported (32, 33). The case of a bar with a constant narrow rectangular cross section bent in the plane of curvature by couples \( M \) applied at the ends will be investigated here. Let \( n \) be the radius of the neutral plane (see Fig. 8). Fibers outside this surface are momentarily extended and those inside compressed. The Tresca yield condition may be written in the form
\[ \sigma_\theta - \sigma_T = 2\tau_0 \quad (n \leq r \leq R + c) \]
\[ -\sigma_\theta = 2\tau_0 \quad (R - c \leq r \leq n) \]  

Substituting Eq. (64) in the equilibrium equation (69), and integrating,

\[ \sigma_T = -2\tau_0 \ln \left( \frac{R+c}{r} \right) \quad (n \leq r \leq R + c) \]
\[ \sigma_T = -\frac{(c+y)/(R+y)} 2\tau_0 \quad (R - c \leq r \leq n) \]  

after using the boundary conditions that \( \sigma_T \) is zero on both surfaces.

The radius of the neutral surface can be obtained from the continuity of radial stress \( \sigma_T \) across the neutral surface. This condition can be written as,

\[ \frac{c + y}{R + y} = \ln \left( \frac{R + c}{R + y} \right). \]

From Eqs. (64) and (65) the tangential stress is

\[ \sigma_\theta = 2\tau_0 \left\{ 1 - \ln \left( \frac{R+c}{r} \right) \right\} \quad (n \leq r \leq R + c) \]
\[ \sigma_\theta = -2\tau_0 \quad (R - c \leq r \leq c) \]  

The bending moment is

\[ M = \int_{R-c}^{R+c} \sigma_\theta r \, (bdr) \]  

where moments have been taken about the center of curvature. It is seen from Eq. (66) that the position of the neutral surface depends only on the geometry of the problem, not on the applied load. The question of plastic strain at the outside fibers will be investigated next.

Knowing the radius of the central plane the position of neutral axis can be found by using Eq. (66). By assuming plane-remains-plane
after deformation the following expression for plastic strain can be written,

$$\varepsilon_{p\theta} = \varepsilon_c + \frac{y}{R}.$$  \hfill (69)

At the neutral surface $\varepsilon_{p\theta} = 0$ and the distance from the central plane to neutral plane $y = y_0 = -(R-n)$ can be obtained by using Eq. (66). Consequently the tangential strain at the central plane can be written as

$$\varepsilon_c = \frac{(R-n)}{R} = 1 - \frac{n}{R}.$$  \hfill (70)

Knowing $\varepsilon_c$, the plastic strain at any position can be found by Eq. (69). For example, the plastic strain at the convex surface is

$$\left(\varepsilon_{p\theta}\right)_{y=c} = \left(1 - \frac{n}{R}\right) + \frac{c}{R}$$  \hfill (71)

From Eq. (14) the following plastic stress-strain relations can be written for the present problem.

$$\varepsilon_{p\theta} = \psi\left\{\sigma_\theta - \frac{1}{2}\sigma_T\right\}$$

$$\varepsilon_{pT} = \psi\left\{\sigma_T - \frac{1}{2}\sigma_\theta\right\}$$

$$\varepsilon_{pz} = \psi\left\{-(1/2)(\sigma_\theta + \sigma_T)\right\}$$

where $\psi = (1/3)(\frac{f_m}{f_n})$. From the development below Eq. (38), the following expression for $\psi$ can be written for the rigid, perfectly plastic solid.

$$\psi = \frac{\varepsilon_T}{\sigma_0}.$$  \hfill (73)

As shown above $\varepsilon_{p\theta}$ can be calculated by Eq. (69) and $\varepsilon_p$ in Eq. (73) can be calculated by the following equation.

$$\varepsilon_{p\theta} = \left(\varepsilon_p/\sigma_0\right)\left\{\sigma_\theta - \frac{1}{2}\sigma_T\right\}$$  \hfill (74)
where \( \sigma_0 \) = yield point stress in tension. The terms \( \varepsilon_{pr} \) and \( \varepsilon_{pz} \) can be calculated by the following equations.

\[
\varepsilon_{pr} = \left( \frac{\varepsilon_p}{\sigma_0} \right) \left\{ \sigma_r - (1/2)\sigma_0 \right\}
\]
\[
\varepsilon_{pz} = \left( \frac{\varepsilon_p}{\sigma_0} \right) \left\{ - (1/2)(\sigma_0 + \sigma_r) \right\} \quad (75)
\]

b) Solution Based on Von Mises' Yield Function (51)

The problem of plastic equilibrium in a thick-walled short tube using the Von Mises-Hencky condition of plasticity has been solved first by Nadai (1). Sachs and Lubahn (53) discussed the same problem for fully plastic tube in plane-stress condition. The following analysis follows the approach given by Sachs and Lubahn for the case of thick-walled tubes. It is to be noted that the equilibrium equation is the same for a thick-walled tube under internal pressure and a prismatical bar subjected to pure bending. The analogy regarding the stress distribution between a fully plastic thick-walled tube subjected to the internal pressure and the plate in pure bending was discussed by Sachs and Hoffman (51).

Plastic flow occurs whenever the stress components satisfy the Von Mises' yield condition, namely,

\[
\sigma_0^2 - \sigma_\theta \sigma_r + \sigma_r^2 = \sigma_0^2 \quad (76)
\]

Solving for \( \sigma_\theta \),

\[
\sigma_\theta = \left(1/2\right) \left[ \sigma_r \pm \sqrt{\sigma_0^2 - 3\sigma_r^2} \right]
\]

Rearranging,

\[
\sigma_\theta - \sigma_r = \left(1/2\right) \left[ - \sigma_r \pm \sqrt{\sigma_0^2 - 3\sigma_r^2} \right]
\]
It is noted that the plus sign in front of the root applies to the region between the neutral plane and the convex surface.

Substituting the above equation into the equilibrium equation (49) gives,

\[
\frac{d\sigma_r}{dr} = \frac{\sigma_0 - \sigma_r}{r} = \frac{1}{2r} \left[ -\sigma_r \pm \sqrt{4\sigma_0^2 - 3\sigma_r^2} \right]
\]

Separating variables,

\[
\frac{(\frac{r}{2r} \sqrt{4\sigma_0^2 - 3\sigma_r^2} + \sigma_r) d\sigma_r}{4(\sigma_0^2 - \sigma_r^2)} = \frac{dr}{2r}.
\]

The first term on the left-hand side of the equation can be integrated by setting

\[
\sigma_r = (2/\sqrt{3}) \sigma_0 \sin \varphi,
\]

\[
\int \frac{\sqrt{4\sigma_0^2 - 3\sigma_r^2}}{4(\sigma_0^2 - \sigma_r^2)} d\sigma_r = \frac{\sqrt{3}}{4} \sin^{-1}\left(\frac{\sqrt{3} \sigma_r}{\sigma_0}\right) + \frac{1}{8} \ln \frac{(h\sigma_0^2 - 3\sigma_r^2)^{1/2} + \sigma_r}{(h\sigma_0^2 - 3\sigma_r^2)^{1/2} - \sigma_r}
\]

\[
\int \frac{\sigma_r d\sigma_r}{4(\sigma_0^2 - \sigma_r^2)} = -(1/8) \ln(\sigma_0^2 - \sigma_r^2).
\]

The following expressions can be written for the integration of Eq. (77).

\[
2 \sqrt{3} \sin^{-1}\left(\frac{\sqrt{3} \sigma_r}{\sigma_0}\right) = \ln \left[ a^4(\sigma_0^2 - \sigma_r^2)(\sqrt{4\sigma_0^2 - 3\sigma_r^2} - \sigma_r) \right]
\]

\[
(R - c \leq r \leq R + c)
\]

\[
-2 \sqrt{3} \sin^{-1}\left(\frac{\sqrt{3} \sigma_r}{\sigma_0}\right) = \ln \left[ \beta^4(\sigma_0^2 - \sigma_r^2)(\sqrt{4\sigma_0^2 - 3\sigma_r^2} + \sigma_r) \right]
\]

\[
(R - c \leq r \leq n)
\]

(78)  (79)
where $\alpha$ and $\beta$ are constants of integration, respectively, and can be obtained from the boundary conditions $\sigma_r = 0$ at $r = R+c$ and $\sigma_r = 0$ at $r = R-c$. The results are

$$\alpha = \frac{1}{(R+c)\sigma_0^2}, \quad \beta = \frac{1}{(R-c)\sigma_0^2}$$

Since the radial stresses $\sigma_r$ are continuous across the neutral surface, the values of radial stresses calculated from Eqs. (78) and (79) should be equal at $r = n$, i.e., neutral surface.

It is seen from Eqs. (78) and (79) that by assuming the radius of curvature the radial stress at any position can be found. The tangential stress $\sigma_\theta$ can be found by one of the following equations.

$$\sigma_\theta = \frac{1}{2} \left[ \sigma_r + \sqrt{\frac{1}{\sigma_0^2} - \frac{\sigma_r^2}{\sigma_0^2}} \right] \quad (n \leq r \leq R+c) \quad (80)$$

$$\sigma_\theta = \frac{1}{2} \left[ \sigma_r - \sqrt{\frac{1}{\sigma_0^2} - \frac{\sigma_r^2}{\sigma_0^2}} \right] \quad (R-c \leq r \leq n) \quad (81)$$

The bending moment can be obtained by integration across any section of the following equation.

$$M = \int_{R-c}^{R+c} \sigma_\theta br\, dr \quad (82)$$

The method of calculating plastic strain was discussed earlier in the case of the solution based on Tresca's yield function and will not be repeated here.

Some elastic-plastic solutions for curved beams have been given by Swida (34,35,36), Ohno (37), Phillips (38), and Shaffer and House (39,40). In addition to the practical importance of this problem, the main objective of the following analysis is to demonstrate the method of attacking plastic bending for a strain-hardening material in the elastic-plastic range. As will be shown later the analytical solution in closed form is difficult to obtain for materials with work-hardening, because the analysis leads to a nonlinear differential equation which defies integration. The complete solution of the present problem involves the calculation of stresses and strains in both the elastic and plastic regions. The solutions in the two regions cannot be found separately since both depend on certain conditions of continuity across the elastic-plastic boundary. As applied moments are increased the stress distribution within the bar goes through three distinct phases. In the first phase the entire bar is elastic; in the second a portion of the bar is plastic whereas the remainder is elastic; in the third phase the entire bar is plastic. The solutions for the first and third phases have been obtained earlier. It is our intention to present a solution for the stress or strain distribution within the bar during elastic-plastic deformation. The qualitative picture of the stress distribution inside the bar can be seen from the solution for the perfectly plastic solid. The radial stress $\sigma_r$ is always compressive whereas the circumferential (tangential) stress $\sigma_\theta$ is compressive near the inner (concave) surface but tensile near the outer surface. At a particular radius $r = n$, the
radius of the neutral surface, the tangential stress is zero. As the bending moment increases, the neutral surface will shift from the central surface towards the inner surface. It is also seen that as the bending moment increases, \( \sigma_\theta \) and \( \sigma_r \) increase, and so does the difference in their numerical value. This difference is greatest at the inner surface of the bent bar where \( \sigma_r = 0 \) and therefore yielding will start at this surface. This fact can also be seen from the consideration of the equation of equilibrium for the force acting on the cross section since the neutral surface will shift toward the compression surface. The explicit expression for the radial stress \( \sigma_r \) is difficult to obtain by using Von Mises' yield function as can be seen from Eq. (78) or Eq. (79). For this reason the Tresca yield function will be used in the present analysis. The Tresca yield condition for the inner surface (see Fig. 8) may be written in the form

\[
\sigma_z - \sigma_\theta = 2\tau_0 \quad \text{or} \quad -\sigma_\theta = 2\tau_0. \tag{82}
\]

At the inner surface \( r = R-c \), the radial stress \( \sigma_r = 0 \) and the longitudinal stress \( \sigma_z = 0 \) (plane stress). Consequently,

\[
\sigma_\theta = -2\tau_0. \tag{83}
\]

At the instant the inner surface starts to yield, the elastic solution Eq. (58) still holds. From Eq. (58) by setting \( y = -c \),

\[
\sigma_\theta = \frac{E}{R(\frac{2}{\nu}-1)} \left[ \frac{2}{\nu} (R-c) \left( \frac{1}{\nu} \right) \frac{(R+c)^{k+1}-(R-c)^{k+1}}{(R+c)^k-(R-c)^k} + \frac{2\sigma(R^2-c^2)^k}{(R-c)^k(R+c)^k-(R-c)^k} \right] \tag{84}
\]

Substituting Eq. (83) into Eq. (84) the radius of curvature \( R \) can be found. Once the radius of curvature is known, the stress distribution
along the radial cross section can be calculated and the moment can be obtained by the integration of the following equation.

\[ M = \int_A \sigma \gamma \, dA \quad \text{or} \quad M = \int_{R-c}^{R+c} \sigma \, rb \, dr \] (85)

With further increase in bending moment exceeding the above calculated numerical value, a plastic region develops around the lower fibers of the cross section and we enter upon what we will call the first stage of the elastic-plastic problem. In view of the symmetry of the problem, the stress distribution at each radial cross section is identical. Therefore it is possible to predict the shape of the elastic-plastic boundary. Its position would depend on the applied moment.

The Elastic-Plastic Deformation - First Stage

The position of the elastic-plastic boundary for this stage is shown in Figure 11. Every element within the plastic zone, \((R-c) \leq r\)

![Diagram of the First Stage of the Elastic-Plastic Deformation](image)
\( \leq (R-y_1) \), satisfies the equation of equilibrium, Eq. (49), and the yield condition, Eq. (82). Substituting Eq. (82) into Eq. (49), the solution for \( \sigma_r \) can be found by noting \( \sigma_r = 0 \) at \( r = R-c \),

\[
\sigma_r = -2\tau_0 \left( \frac{c+y}{r} \right) \quad R-c \leq r \leq R-y_1
\]

and \( \sigma_\theta \) can be written as,

\[
\sigma_\theta = -2\tau_0 \quad R-c \leq r \leq R-y_1
\]

It can be seen from Figure 10(b) that \( \sigma_\theta \) will reach yield point stress \((-2\tau_0)\) independent of \( \sigma_r \). The stresses within the elastic zone can be obtained from Eqs. (50) and (52).

\[
\sigma_r = E \left[ \frac{\varepsilon_c}{k} + \frac{r}{R(k+1)} + \frac{a}{r^k} \right]
\]

\[
\sigma_\theta = E(\varepsilon_c + \frac{y}{R}) + \nu E \left[ \frac{\varepsilon_c}{k} + \frac{r}{R(k+1)} + \frac{a}{r^k} \right]
\]

At the convex surface of the elastic zone, \( r = R+c \), the radial stress \( \sigma_r \) is equal to zero. At the elastic-plastic boundary, the radial and tangential stresses are continuous. In view of equations (86) and (87), the following equations can be written,

\[
E \left\{ \frac{\varepsilon_c - 1}{k} + \frac{R+c}{R(k+1)} + \frac{a}{(R+c)^k} \right\} = 0
\]

\[
E \left\{ \frac{\varepsilon_c - 1}{k} + \frac{R-y_1}{R(k+1)} + \frac{a}{(R-y_1)^k} \right\} = -2\tau_0 \left( \frac{c-y_1}{R-y_1} \right)
\]

\[
E(\varepsilon_c + \frac{y_1}{R}) + \nu E \left\{ \frac{\varepsilon_c - 1}{k} + \frac{R-y_1}{R(k+1)} + \frac{a}{(R-y_1)^k} \right\} = -2\tau_0
\]
\( \varepsilon_c \) and constant \( a \) can be obtained from Eqs. (90a) and (90b). The results are,

\[
\varepsilon_c = 1 + \frac{k(R-y_1)^k}{(R-y_1)^k-(R+c)^k} \left\{ -2\tau_0 \frac{(c-y_1)}{R-y_1} - \frac{R-y_1}{R(k+1)} + \frac{(R+c)^{k+1}}{R(R-y_1)^k(k+1)} \right\} \tag{91a}
\]

\[
a = (R+c)^k \left[ -\frac{(R-y_1)^k}{(R-y_1)^k-(R+c)^k} \left\{ -2\tau_0 \frac{(c-y_1)}{R-y_1} - \frac{R-y_1}{R(k+1)} + \frac{(R+c)^{k+1}}{R(R-y_1)^k(k+1)} \right\} \right] + \frac{R+c}{R(k+1)} \right\} \tag{91b}
\]

It is easily seen that Eqs. (91) automatically satisfy Eq. (90c). Substituting \( a \) and \( \varepsilon_c \) into Eqs. (88) and (89),

\[
\sigma_r = E \left[ \frac{(R-y_1)^k}{(R-y_1)^k-(R+c)^k} \left\{ -2\tau_0 \frac{(c-y_1)}{R-y_1} - \frac{R-y_1}{R(k+1)} + \frac{(R+c)^{k+1}}{R(R-y_1)^k(k+1)} \right\} + \frac{R+c}{R(k+1)} \right] \tag{92a}
\]

\[
\sigma_\theta = E \left[ \frac{\tau_0}{R+y_1} + \frac{k(R-y_1)^k}{(R-y_1)^k-(R+c)^k} \left\{ -2\tau_0 \frac{(c-y_1)}{R-y_1} - \frac{R-y_1}{R(k+1)} + \frac{(R+c)^{k+1}}{R(R-y_1)^k(k+1)} \right\} + \frac{R+c}{R(k+1)} \right] \tag{92b}
\]

The above expressions for \( \sigma_\theta \) and \( \sigma_r \) represent the stresses within the elastic region \((R-y_1) \leq r \leq (R+c)\). The position of the neutral surface, \( r = n \), corresponding to \( \sigma_\theta \) equal to zero, is given by the equation,
The bending moment acting across the section during the first stage of the elastic-plastic solution is obtained by the following equation of equilibrium.

\[
M = \left[ \int_{R-c}^{R-y_1} \sigma_\theta r \, dr + \int_{R-y_1}^{R+c} \sigma_\theta r \, dr \right] b.
\]  \hspace{1cm} (94)

The resulting equation may be used to determine the position of the elastic-plastic boundary \(R-y_1\) for a given bending moment. Its numerical value is needed to evaluate the stress components and the position of the neutral surface. The resultant force across any arbitrary radial cross section should be equal to zero, i.e.,

\[
F/b = \int_{R-c}^{R-y_1} \sigma_\theta r \, dr + \int_{R-y_1}^{R+c} \sigma_\theta r \, dr = 0.
\]  \hspace{1cm} (95)

After further increase of the bending moment, yielding will start at the convex surface of the bar. The yield condition at the convex surface of the bar can be written as

\[
\sigma_\theta - \sigma_r = 2\tau_0.
\]  \hspace{1cm} (96)

At convex surface, \(\sigma_r = 0\), consequently

\[
\sigma_\theta = 2\tau_0
\]

and

\[
\sigma_r = E \left[ \frac{\left(R-y_1\right)^k}{\left(R-y_1\right)^k - (R+c)^k} \left\{ -2\tau_0 \left( \frac{c-y_1}{R-y_1} \right) - \frac{R-y_1}{R(k+1)} + \frac{(R+c)^{k+1}}{R(R-y_1)^k(k+1)} \right\} + \frac{R+c}{R(k+1)} \right]
\]
The corresponding bending moment may be found by substituting the numerical value of \( y_i \), specified in Eq. (97), into Eq. (94). The alternate method of finding \( y_i \) is from the condition

\[
\sigma_\theta = 2\tau_0
\]  

(98)

where

\[
\sigma_\theta = E \left[ \frac{c}{R+1} + \frac{k(R-y_i)^k}{(R-y_i)^k-(R+c)^k} \left\{ \frac{-2\tau_0}{E} \left( \frac{c-y_i}{R-y_i} \right) - \frac{R-y_i}{R(k+1)} + \frac{(R+c)^{k+1}}{R(R-y_i)^k(k+1)} \right\} \right]
\]

(97)

Upon further increase of the bending moment, the upper plastic zone together with the existing lower plastic zone will advance towards the neutral surface. Thus we enter the second stage of the elastic-plastic solution.

The Elastic-Plastic Deformation - Second Stage.

At this stage of the elastic-plastic deformation, there are two elastic-plastic boundaries, one at \( y = y_i \) and the other at \( y = y_0 \). These boundaries are shown in Figure 12. The stresses within the lower plastic zone still satisfy the conditions specified for this region in the first stage of the elastic-plastic deformation and are given by Eqs. (86) and (87). Substituting Eq. (96) into the equilibrium Eq. (49) and satisfying the boundary condition at the convex surface, the stress
distribution for the upper plastic zone can be written as,

\[
\sigma_r = -2\tau_0 \ln \left\{ \frac{(R+c)}{r} \right\} \quad (R+y_0 \leq r \leq R+c) \tag{99}
\]

\[
\sigma_\theta = 2\tau_0 \left\{ 1 - \ln \left( \frac{R+c}{r} \right) \right\} \quad (R+y_0 \leq r \leq R+c) \tag{100}
\]

From the continuity of stresses across the lower elastic-plastic boundary, the following equations can be written,

\[
E \left\{ \frac{\varepsilon_c - 1}{k} + \frac{R-y_i}{R(k+1)} + \frac{a}{(R-y_i)^k} \right\} = -2\tau_0 \left( \frac{c-y_i}{R-y_i} \right) \tag{101}
\]

\[
E(\varepsilon_c - \frac{y_i}{R}) + E Y \left\{ \frac{\varepsilon_c - 1}{k} + \frac{R-y_i}{R(k+1)} + \frac{a}{(R-y_i)^k} \right\} = -2\tau_0 \tag{102}
\]

Similarly at the upper elastic-plastic boundary, we obtain

\[
E \left\{ \frac{\varepsilon_c - 1}{k} + \frac{R+y_o}{R(k+1)} + \frac{a}{(R+y_o)^k} \right\} = -2\tau_0 \ln \frac{R+c}{R+y_o} \tag{103}
\]
\[ E \left( \varepsilon_c + \frac{y_0}{R} \right) + E \psi \left( \frac{\varepsilon_c - 1}{k} + \frac{R+y_0}{R(k+1)} + \frac{\alpha}{(R+y_0)^k} \right) = 2\tau_0 \left\{ 1 - \ln \frac{R+c}{R+y_0} \right\} \]  

\[ (104) \]

\[ \varepsilon_c \text{ and constant } \alpha \text{ can be obtained by solving Eqs. (101) and (103). The result of the solution is} \]

\[ a = \frac{(R-y_1)^k(R+y_0)^k}{(R+y_0)^k-(R-y_1)^k} \left[ -\frac{2\tau_0}{E} \left( \frac{c-y_1}{R-y_1} + \ln \frac{R+y_0}{R+c} \right) - \frac{1}{R(k+1)} \left( R-y_1 - (R+y_0) \right) \right] \]  

\[ (105) \]

\[ \varepsilon_c = 1 - \frac{2k\tau_0}{E} \left( \frac{c-y_1}{R-y_1} - \frac{k(R-y_1)}{R(k+1)} - \frac{k(R+y_0)^k}{(R+y_0)^k-(R-y_1)^k} \left[ -\frac{2\tau_0}{E} \left( \frac{c-y_1}{R-y_1} + \ln \frac{R+y_0}{R+c} \right) \right] \right. \]

\[ \left. - \frac{1}{R(k+1)} \left( R-y_1 - (R+y_0) \right) \right\} \]  

\[ (106) \]

Substituting \( \varepsilon_c \) and \( a \) into Eq. (102) or Eq. (104) will give a relationship between the positions of the elastic-plastic boundaries,

\[ E \left[ 1 + \left( \frac{c-y_1}{R-y_1} - \frac{k(R-y_1)}{R(k+1)} - \frac{k(R+y_0)^k}{(R+y_0)^k-(R-y_1)^k} \left( -\frac{2\tau_0}{E} \left( \frac{c-y_1}{R-y_1} + \ln \frac{R+y_0}{R+c} \right) \right) \right. \]

\[ \left. - \frac{1}{R(k+1)} \left( R-y_1 - (R+y_0) \right) \right] + \frac{y_1}{R} \]  

\[ - \frac{2k\tau_0}{E} \left( \frac{c-y_1}{R-y_1} + \ln \frac{R+y_0}{R+c} \right) - \frac{1}{R(k+1)} \left( R-y_1 - (R+y_0) \right) + \frac{R-y_1}{R(k+1)} \]

\[ \left. + \frac{(R+y_0)^k}{(R-y_1)^k} \left( -\frac{2\tau_0}{E} \left( \frac{c-y_1}{R-y_1} + \ln \frac{R+y_0}{R+c} \right) - \frac{1}{R(k+1)} \left( R-y_1 - (R+y_0) \right) \right) \right] \]

\[ = -2\tau_0 \]  

\[ (107) \]

In view of equations (88), (89), (105), and (106), the stresses in the elastic region, \( R-y_1 \leq r \leq R+y_0 \), are given by the expressions,
\[
\sigma_r = E \left[ \frac{-2\tau_0}{E} \left( \frac{c-y_1}{R-y_1} \right) - \frac{R-y_1}{R(k+1)} - \frac{(R+y_0)^k}{(R+y_0)^k-(R-y_1)^k} \left\{ \frac{-2\tau_0}{E} \left( \frac{c-y_1}{R-y_1} + \ln \frac{R+y_0}{R+c} \right) \right\} \right]
\]

\[
- \frac{1}{R(k+1)} (R-y_1-R-y_0) + \frac{r}{R(k+1)} + \frac{(R-y_1)k(R+y_0)^k}{r^k(R+y_0)^k-(R-y_1)^k]}
\]

\[
E \left( \frac{c-y_1}{R-y_1} + \ln \frac{R+y_0}{R+c} \right) - \frac{1}{R(k+1)} (R-y_1-R-y_0) \right] \right) \right) \right)
\]

\[
(108)
\]

\[
\sigma_\theta = E(\varepsilon_c + \frac{y}{R}) + \frac{1}{\nu} \sigma_r
\]

\[
(109)
\]

where \( \varepsilon_c \) can be obtained from Eq. (106).

The position of the neutral surface, \( r = n \), found by setting \( \sigma_\theta = 0 \), is given for the second stage of the elastic-plastic deformation by the expression,

\[
E(\varepsilon_c + \frac{y}{R} - 1) + EV\left\{ \frac{\varepsilon_c - 1}{k} + \frac{r}{R(k+1)} + \frac{a}{r^k} \right\} = 0.
\]

\[
(110)
\]

The radii to two elastic-plastic boundaries \((R+y_0)\) and \((R-y_1)\) which appear in Eqs. (108), (109), and (110) are related to each other by Eq. (107). They are uniquely prescribed for a given bending moment \( M \). The relationship between bending moment \( M \) and these radii can be found from the condition of equilibrium.

\[
M = b \left[ \int_{R-c}^{R-y_1} \sigma_\theta r \, dr + \int_{R-y_1}^{R+y_0} \sigma_\theta r \, dr + \int_{R+y_0}^{R+c} \sigma_\theta r \, dr \right]
\]

where \( b = \) width of the section.

The numerical values of the radii \((R-y_1)\) and \((R+y_0)\) needed to evaluate the stress components and the position of the neutral surface may be calculated for a given bending moment from Eqs. (107) and (111).
It can be shown that the resultant force over any arbitrary radial cross section is zero.

When the bending moment is large enough, the boundary of the upper plastic zone and the boundary of the lower plastic zone meet at the neutral surface. This case was discussed earlier and it will not be repeated here. It is seen in the above discussions that as the bending moment on a bent bar increases, the bar passes through three distinct phases: the completely elastic, the elastic-plastic, and the fully plastic phase. Mathematical expressions have been derived which describe for each phase the stress distribution, the position of the elastic-plastic boundary and the position of the neutral surface as a function of the bending moment and the physical dimensions of the problem. Knowing the position of the neutral surface, the plastic strains at any place inside the bar can be evaluated by using the assumption of planes remain planes during deformation. The details were discussed earlier for the case of a rigid, perfectly plastic solid. Shaffer and House (52) demonstrated that plane sections remain plane independent of the state of stress existing in a wide curved bar made of an incompressible, perfectly plastic material subjected to pure bending in both the elastic and plastic regions.


For sufficiently large values of the bending moment, the entire bar will behave plastically. The method of solution for this case taking radial stresses into consideration will be discussed here. It was demonstrated convincingly for the case of an ideal plastic metal that the
radial stresses cannot be neglected (33). The stress-strain diagram having a constant slope for the work-hardening branch, for the rigid work-hardening solid, is shown in Figure 13.

Fig. 13(a) Rigid, Work-Hardening Solid in Uniaxial Stress. 
(b) Tresca's and Von Mises' Yield Functions.

The bar is assumed to be composed of isotropically work-hardening material. Budiansky (41) has shown that deformation theories of plasticity may be used for range of loading paths other than proportional loading without violation of the general requirements for the physical soundness of a plasticity theory. The extent to which deviations from proportional loading are admissible on this basis is calculated by Budiansky quantitatively for the simple deformation theory of Nadai. The necessity for accurate solutions to the plastic-bending problem cannot be denied. Mathematical analysis of the problem is obviously complex, and it is felt that the general problem has not yet been analyzed successfully. A number of investigators (42) have performed mathematical
analyses of the behavior of a sheet bent into a circular arc based on the assumptions of perfectly plastic materials in plane strain. In the following analysis, the deformation theories of plasticity discussed in sections 1 and 2 will be applied.

In the present analysis two separate and distinct ideal materials will be considered (43). In the first material it is assumed that the root-mean-square shear stress (the modified octahedral shearing stress) \( \tau_m \) above a yield value \( \tau_0 \) is a linear function of the root-mean-square shear strain (the modified octahedral shearing strain) \( \gamma_m \). This corresponds to the case of \( n = 1 \) in Eq. (28). In alternate form this relationship can be written as

\[
\tau_m = \tau_0 + G_1 \gamma_m.
\]  

(112)

In the second material a similar relationship exists between the maximum shearing stress \( \tau_{\text{max}} \) and the maximum shearing strain \( \gamma_{\text{max}} \). This corresponds to the case of \( n = 1 \) in Eq. (41). In alternate form this relationship can be written as

\[
\tau_{\text{max}} = \tau_0 + G_2 \gamma_{\text{max}}
\]  

(113)

The constants \( G_1 \) or \( G_2 \) give the slope of the stress-strain curve and will have a value between 0 for an ideal plastic body without strain-hardening and \( G \), the elastic modulus in shear. It is felt that either of these relations might be justified for the problem in hand. The elastic strains have been neglected and hence the whole problem of partial yielding has been neglected in this section. As will be indicated later, an understanding of partial yielding in plastic bending would require a much more complicated approach. By setting \( n = 1 \), the reciprocal
of the plastic modulus $\phi$ in Eq. (37) can be written for the present problem ($\sigma_1 = \sigma_9$, $\sigma_2 = 0$, $\sigma_3 = \sigma_r$) as

$$\phi = \frac{K_1 \left[ (\sigma_\theta - \sigma_r) \left\{ 1 + \frac{1}{2} \left( \frac{\sigma_\theta \sigma_r}{(\sigma_\theta - \sigma_r)^2} \right) \right\} - \sigma_\theta \right]}{\left( \sigma_\theta - \sigma_r \right)^2}$$  (114)

The above expression is too complicated to be of practical value. For practical purposes we can set,

$$\frac{\sigma_\theta \sigma_r}{(\sigma_\theta - \sigma_r)^2} \approx 0$$  (115)

The qualitative picture of the stress distribution inside the bar is very clear. The radial stress has maximum numerical value at the neutral surface and is zero at both convex and concave surfaces. In contrast, the tangential stress $\sigma_\theta$ is zero at the neutral surface and maximum or minimum (tension or compression) at the convex or concave surface. Consequently Eq. (115) is approximately true for the present problem because $(\sigma_\theta \sigma_r/2(\sigma_\theta - \sigma_r)^2)$ is small compared with one and can be neglected. Using Eq. (115), Eq. (114) can be written as

$$\phi = \frac{K_1 \left[ (\sigma_\theta - \sigma_r) - \sigma_\theta \right]}{(\sigma_\theta - \sigma_r)} = K - \frac{K_1 \sigma_\theta}{(\sigma_\theta - \sigma_r)}$$  (116)

Similarly for the maximum-shearing-stress law, Eq. (42) for $n = 1$ can be written as
It is recalled that Eq. (116) was obtained after some simplification.

It is possible to use either Eq. (116) or Eq. (117) for the analysis of the present problem. The analysis using Eq. (117) will be considered.

From the plastic stress-strain relationship Eq. (114) and the assumption of plane remains plane during deformation, the following equation can be written.

\[
\dot{\epsilon}_{\text{p}\theta} = \dot{\epsilon}_c + \frac{Y}{R} = \phi (\sigma_\theta - \frac{1}{2} \sigma_r) = \frac{1}{3} K_2 \left( 1 - \frac{2\tau_0}{\sigma_\theta - \sigma_r} \right) \left( \sigma_\theta - \frac{1}{2} \sigma_r \right) \tag{118}
\]

Rewriting Eq. (118),

\[
(\dot{\epsilon}_c + \frac{Y}{R})(\sigma_\theta - \sigma_r) = \frac{1}{3} K_2 (\sigma_\theta - \sigma_r) \left( \sigma_\theta - \sigma_r \right) + \frac{1}{2} \sigma_r - \frac{2}{3} K_2 \tau_0 \left( \sigma_\theta - \sigma_r \right) + \frac{1}{2} \sigma_r \tag{119}
\]

From the equilibrium equation (119), we have

\[
\sigma_\theta - \sigma_r = r \frac{d\sigma_r}{dr}
\]

Substituting the above equation into Eq. (119),

\[
(\dot{\epsilon}_c + \frac{Y}{R}) \left( r \frac{d\sigma_r}{dr} \right) = \frac{1}{3} K_2 \left( r \frac{d\sigma_r}{dr} + \frac{1}{2} \sigma_r \right) \left( r \frac{d\sigma_r}{dr} + \frac{1}{2} \sigma_r \right) - \frac{2}{3} K_2 \tau_0 \left( r \frac{d\sigma_r}{dr} + \frac{1}{2} \sigma_r \right)
\]

Rearranging,

\[
\frac{1}{3} K_2 \left( r \frac{d\sigma_r}{dr} \right)^2 + r \frac{d\sigma_r}{dr} \left( \frac{1}{6} K_2 \sigma_r - \frac{2}{3} K_2 \tau_0 - \dot{\epsilon}_c - \frac{r}{R} + 1 \right) - \frac{1}{3} K_2 \tau_0 \sigma_r = 0 \tag{120}
\]

Now setting,

\[-2\tau_0 - \frac{3\dot{\epsilon}_c}{K_2} + \frac{3}{K_2} = 0 \quad ; \quad 3/K_2R = m\]
Eq. (121) is a nonlinear equation of the first order, \( \sigma_r (d\sigma_r/dr) \), \((d\sigma_r/dr)^2\) being nonlinear terms (nonlinear in \(\sigma_r\)). Solving Eq. (121) for \((d\sigma_r/dr)\),

\[
\frac{d\sigma_r}{dr} = -\frac{1}{2r} \left( \frac{1}{2} \sigma_r + l - mr \right) + \frac{1}{2r} \left[ \left( \frac{1}{2} \sigma_r + l - mr \right)^2 + 4\tau_0 \sigma_r \right]^{1/2} \tag{122}
\]

It can be seen from Eqs. (121) and (122) that (-) sign applies to the region between neutral surface and tension surface. The application of Eq. (116) based on the modified octahedral shear concept will lead to the same form of nonlinear differential equation as Eq. (121) which defies analytical integration and a solution in elementary form cannot be obtained. By setting

\[-\frac{1}{2r} \left( \frac{1}{2} \sigma_r + l - mr \right) - \frac{1}{2r} \left[ \left( \frac{1}{2} \sigma_r + l - mr \right)^2 + 4\tau_0 \sigma_r \right]^{1/2} = f(\sigma_r, r)\]

Eq. (122) can be written as

\[d\sigma_r/dr = f(\sigma_r, r) \tag{123}\]

A family of curves defined by the following equation can be drawn,

\[f(\sigma_r, r) = \text{const.} \tag{124}\]

Obviously each of these curves connects the points in which the integral curves of Eq. (123) have the same slope. Varying the value of the constants in Eq. (124), we obtain a family of such curves, called the isoclines. The differential equation of the first order of Eq. (123) can be solved by the isocline method. It can always be used as a step-by-step method provided that \(f(\sigma_r, r)\) is a single-valued continuous
function of \( \sigma_r \) and \( r \). It is obvious that the variables \( \sigma_r \) and \( r \) in Eq. (122) cannot be separated. The result of investigation shows that 
\[
(1/2 \sigma_r + \ell - mr)^2 + h \tau_0 \sigma_r
\]
and \( h \tau_0 \sigma_r \) have the same order of magnitude for the problem in hand and consequently \( h \tau_0 \sigma_r \) cannot be neglected in comparison with \((1/2 \sigma_r + \ell - mr)^2\) inside the root in Eq. (122).

It is felt that the analytical solution of Eq. (122) is very difficult. Several methods to calculate particular solutions of any differential equation of the first order with a higher accuracy than the method of isocline have been suggested by different mathematicians. They are in general step-by-step methods; those developed by I. C. Adams and by C. Runge jointly with R. Kutta are perhaps the most practical ones. The numerical methods of solving a first-order differential equation of the type Eq. (123) are thoroughly treated in books on numerical mathematical analysis (H1, H5, H6, H7, H8, H9).

The existence and uniqueness of the solution of the differential equation (123) are guaranteed by a Lipshitz condition. The solutions sought are functions of the form \( \sigma_r = F(r) \). By setting

\[
Q = (1/r)(1/2 \sigma_r + \ell - mr), \quad R = - \tau_0(\sigma_r/r^2)
\]

an equation of first order and second degree (121) can be written as

\[
\frac{d\sigma_r}{dr}^2 + Q\frac{d\sigma_r}{dr} + R = 0 \tag{125}
\]

Eq. (125) can be solved for \( \frac{d\sigma_r}{dr} \) by the quadratic formula

\[
\frac{d\sigma_r}{dr} = -\frac{Q + \sqrt{Q^2 - 4R}}{2} \tag{126}
\]
Eq. (126) is really equivalent to two first degree equations:

\[
\begin{align*}
d\sigma_r/dr &= \frac{-Q + \sqrt{Q^2 - 4R}}{2} = f_1(\sigma_r, r) \\
d\sigma_r/dr &= \frac{-Q - \sqrt{Q^2 - 4R}}{2} = f_2(\sigma_r, r)
\end{align*}
\]

(127)

both of which can be solved graphically. The solutions form two families of curves which are, in general, unrelated, although they may fit together smoothly along a border curve. If the two families are analyzed on the same graph, we will find two line elements through each point at which \(Q^2 - 4R > 0\), one where \(Q^2 - 4R = 0\), and none where \(Q^2 - 4R < 0\).

The locus \(Q^2 - 4R = 0\) may happen to contain a solution curve, which then serves as a borderline curve. The existence theorem answers the crucial question of the conditions under which solutions can be found. The method of step-by-step integration tells how to find these solutions.

The accuracy obtained can be made as great as desired by employing sufficiently short steps. Eq. (125) illustrates another complication. The equation was replaced by the two first-order equations (127). These are meaningful, however, only when the quantity under the radical is positive; thus for some points \((\sigma_r, r)\) the equation may prescribe imaginary slopes, for others two equal slopes and for still others unequal slopes.

In addition to numerical methods various mechanical devices and other modern techniques are available for the solution of differential equation (122).

If \(f(\sigma_r, r)\) in Eq. (123) is continuous and its first derivative exists over the domain under consideration, then the solution of Eq. (123) can be written as
\( \phi(\sigma_r, r, \sigma) = \text{const.} \) \hspace{1cm} (128)

Taking derivative with respect to \( r \),

\[ \phi_r + \phi_{\sigma_r} \left( d\sigma_r/dr \right) = 0 \]

or

\[ \phi_r + f(\sigma_r, r) \phi_{\sigma_r} = 0 \] \hspace{1cm} (129)

Consequently \( z = \phi(r, \sigma_r) \) is a solution of the partial differential equation

\[ \left( \partial z/\partial r \right) + f(\sigma_r, r) \left( \partial z/\partial \sigma_r \right) = 0 \] \hspace{1cm} (130)

Conversely, solution \( z = \phi(r, \sigma_r) \) of Eq. (130) is a surface in \( r, \sigma, z \) space, i.e., an integral surface. That is, curves \( z = \text{const.} \) of any solution of Eq. (130) are obtained by solving Eq. (123). Furthermore any function \( z = \phi(r, \sigma_r) \) which is continuous and possesses a first derivative in the domain for which the projection of the curves \( z = \text{const.} \) into \( r, \sigma_r \) - plane coincides with the above curves \( \phi = \text{const.} \) is a solution of Eq. (130). Eq. (123) is called the characteristic equation of the partial differential equation (130). The curves \( z = \text{const.} \) are called "characteristic" of Eq. (130). In order to determine a solution uniquely, we need a given starting point, i.e., \( (r = R+c, \sigma_r = 0) \) at the concave boundary of the bent bar. This represents the boundary value problem in the first order ordinary differential equation. Further detailed investigation on the numerical integration of differential equations (123) is beyond the scope of the present study. The solution of the present problem must satisfy the boundary conditions \( \sigma_r = 0 \) at both \( r = R+c \) and \( r = R-c \). It will be necessary to assume the radius of curvature of the bent rectangular bar and estimate the tangential strain...
along the central surface $E_C$ beforehand. Once the radial stress $\sigma_r$ is found, the tangential stress $\sigma_\theta$ can be calculated from Eq. (118). If the tangential stress distribution is known for any cross section, the moment can be computed readily. For the method of finding the neutral surface and computing plastic strains the reader is referred to section 4 of the present chapter. It is to be expected that the numerical integration of the differential equation (122) would be tedious and time-consuming.

It is to be noted that Eq. (117) applies only to the region between the convex surface and the neutral surface. Consequently the differential equation (121) applies to this particular region only. For the region between the neutral surface and the concave surface, the maximum shear stress can be written as

$$\tau_{\text{max}} = \frac{(\sigma_z - \sigma_\theta)}{2}$$

(131)

Since we assume plane stress $\sigma_z = 0$ and $\tau_{\text{max}} = -\frac{\sigma_\theta}{2}$. Rewriting Eq. (117) for this biaxial compression region we have,

$$\phi = \frac{1}{2}K_2 \left( 1 + \frac{2\tau_0}{\sigma_\theta} \right)$$

(132)

By using Eq. (132), the following plastic stress-strain relationship can be written,

$$\sigma_\text{pl} = \varepsilon_C + \frac{y}{R} = \frac{1}{3}K_2 \left( 1 + \frac{2\tau_0}{\sigma_\theta} \right)(\sigma_\theta - \frac{1}{2}\sigma_r)$$

(133)

Substituting $\sigma_\theta = \sigma_r + r(\text{d}\sigma_r/\text{d}r)$ into Eq. (133) and setting

$$2\tau_0 - \frac{3}{K_2}(E_C - 1) = \lambda$$

$$- \frac{3}{K_2}R = m$$

$$\tau_0 - \frac{3}{K_2}(E_C - 1) = n,$$
the following differential equation can be written.

\[
\left( \frac{d\sigma_r}{dr} \right)^2 + \left( \frac{3}{2} \sigma_r + \frac{1}{r} \right) \frac{1}{r} \left( \frac{d\sigma_r}{dr} \right) + \frac{1}{r^2} \left( \frac{\sigma_r^2}{2} + n\sigma_T + m\sigma_r \right) = 0 \quad (134)
\]

It is seen that this equation is more difficult to solve than Eq. (121).

As discussed in Section 4, the radius of the neutral surface can be evaluated from the continuity of radial stress \( \sigma_r \) across the neutral surface.

7. Some Remarks Regarding the Solution for a Rectangular Section for a Strain-Hardening Solid in the Elastic-Plastic Range.

The ultimate objective of the present investigation is to seek the rigorous theoretical analysis of the state of stress and strain in a work-hardening prismatical bar (rectangular or circular) subjected to bending in the elastic-plastic region assuming biaxial stress (plane state of stress). Although up to the present there appears to be no information available in the literature regarding the question of how much of the material is subjected to plastic state in the low-cycle fatigue tests, say for \( 10^2 \) cycles endurance life, it is felt that the analysis in the elastic-plastic region is most important for low-cycle fatigue investigation. Unfortunately the analysis for a rigid, strain-hardening solid leads to a nonlinear differential equation which defies analytical integration in elementary functions. For this reason, the analytical solution in the elastic-plastic region for the work-hardening solid cannot be obtained. Even if the analytical integration of differential equation (122) were possible, the solution of the problem in the elastic-plastic range for the work-hardening solid would be involved and tedious.
Some general features of elastic-plastic problems as exemplified by the present problem will be discussed and a plausible method of attacking this problem will be suggested. The qualitative picture of the stress distribution for the problem in hand is quite clear. The difficulty exists in obtaining analytical expressions for computing stresses and strains. The calculation of stresses and strains in a general problem in plasticity involves following the history of the deformation from the initiation of plastic flow. This implies much more than a determination of the changing shape of the plastic surfaces and their influence upon the stress distribution. A complete solution of the present problem involves the calculation of stresses and strains in both the elastic and plastic regions. In the former the stress is directly connected with the total strain, and the equations there are fundamentally different from those holding in the plastic region. The solutions in the two regions cannot be found independently since both depend on certain conditions of continuity in the stresses and displacements across the plastic-elastic boundary. This boundary is itself one of the unknowns. It is clear that the complete solution of a plastic problem will be practicable in relatively few cases. Complete solutions can only be hoped for where there is some special symmetry or other simplifying property of the problem. One example of solutions in the elastic-plastic region is described in Section 5. From this it is possible to obtain a general insight into the interrelation between the states of stress and strain in the elastic and plastic regions. It has been discussed earlier that as the moments are increased the stress distribution within the bar goes through three distinct phases. In the first phase the entire bar is elastic and the
solution developed in Section 3 applies; in the second phase a portion of the bar is plastic whereas the remainder is elastic. The general method discussed in Section 5 can be applied for this phase. For the method of analysis, the reader is referred to Section 5. Since the analytical solution for the rigid, strain-hardening solid is very difficult to obtain, Section 5 is included in the present investigation to demonstrate the general features of the solution in the elastic-plastic region. The details will not be repeated here. In the third phase of deformation, the entire bar is plastic. The method of analysis for this phase is presented in Section 6. The completely elastic and the completely plastic solutions to the problem serve as the limiting case to the subject with which this investigation is primarily concerned. If the overall strain is not small, and the change in external surfaces cannot be neglected and is unknown a priori, then it is clear that the strains must be calculated simultaneously with the stresses, at any rate in the plastic region.

8. Some Remarks Regarding the Method of Solution for the Case of a Circular Cross Section.

For rotary-bending fatigue tests, circular-cross-section specimens with non-uniform cross section in the form of a large-radius notched bar are used. The difficulty of measuring plastic strains for the small circular section bar was pointed out in the Introduction. Although the fatigue specimen has a radiused test section, a uniform circular bar will be considered in the present study to simplify the analysis. The analysis of the state of stress and strain in the circular prismatical bar sub-
jected to bending in the elastic-plastic region assuming biaxial stress is the ultimate objective of the present investigation. However, some difficulties are encountered in the analysis for the case of circular cross section. In order to demonstrate the particular feature for circular section in bending, the completely elastic solution will be considered here. The equation of equilibrium and the compatibility equation together with the boundary conditions give us a system of equations for the analysis of the present problem. We assume Eq. (52) in Section 3 satisfies the equilibrium and compatibility equations for the circular section. Consequently Eq. (52) can be used for the circular section too. Rewriting Eq. (52),

\[ \sigma_r = E \left( \frac{\varepsilon_k - 1}{k} + \frac{r}{R(k+1)} + \frac{a}{r} \right) \]  \hspace{1cm} (135)

where \( a \) = constant of integration.

It is noted that the radial stress \( \sigma_r \) should be zero at the circular boundary. Furthermore, it is seen from Eq. (135) that the radial stress \( \sigma_r \) is a function of position coordinate, radius \( r \), only. Since we have a circular boundary, it is readily seen that the boundary conditions are difficult to fit. Let us consider the solution which satisfies the boundary conditions \( \sigma_r = 0 \) at points A \( (r = R+c) \) and B \( (r = R-c) \). After fitting the boundary conditions \( \sigma_r = 0 \) at points A and B, the radial stress \( \sigma_r \) can be written as,

\[ \sigma_r = \frac{E}{R(k+1)} \left[ -\frac{(R+c)^{k+1}-(R-c)^{k+1}}{(R+c)^k-(R-c)^k} + r + \frac{2c(R^2-c^2)^k}{r^k(2(R+c)^k-(R-c)^k)} \right] \]  \hspace{1cm} (136)

Although the above expression for the radial stress \( \sigma_r \) does not
vanish at the circular boundary except points A and B, it does represent the stress distribution inside the bar. It is recalled that the radial stress $\sigma_r$ depends only on the distance from the center of curvature $O$, $r$. Consequently the radial stresses $\sigma_r$ along the line $LM$ (see Fig. 14) perpendicular to the principal axis $Oy$ are all the same. From Eq. (58) the tangential stress $\sigma_\theta$ can be written as,

$$\sigma_\theta = \frac{E}{R(\frac{2}{J})-1} \left[ \frac{2}{J} \left( r \left( \frac{1}{J} \right) \left( \frac{R+c}{R-c} \right)^{k+1} - \frac{R-c}{R+c} \right) \right] + \frac{2c(r^2-c^2)^{k-1}}{r^{k+1}(R+c)^{k}-(R-c)^{k}} \right]$$

Knowing the tangential stress distribution $\sigma_\theta$, the bending moment $M$ can be obtained by the following equation.

$$M = \int_{R-c}^{R+c} \sigma_\theta 2\sqrt{c^2 - (r-R)^2} r \, dr \quad (138)$$
From Eqs. (137) and (138), it is seen that by assuming the radius of curvature, $R$, the applied bending moment $M$ can be computed. The strain $\varepsilon$ can be obtained from Eqs. (50) and (55).

From the above analysis it is seen that the method of solution for the rectangular section can be applied to the case of the circular section except for the computation of bending moment. It is evident that the above approach can be applied to the analysis of the rigid, perfectly plastic solid or rigid, strain-hardening solid without any difficulty. It is also seen that the above approach may be applied to the strain-hardening solid in the elastic-plastic region although the actual computation would be very involved and tedious. Since the general approach for the case of circular cross section is very clear, further discussion will be omitted. Although the expression for $\sigma_T$ does not vanish at the circular boundary except points A and B, no error is involved in the above analysis. It is felt that this is the outstanding feature for the case of circular cross section.

From the above investigation, the following conclusion can be drawn. The maximum stress or strain depends only on the radius of curvature and geometrical configuration of the cross section. The bending moment $M$ and the radius of curvature are related to each other by complicated functions.
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CHAPTER IV


1. Preliminary Discussion

Many failure theories for defining yield strengths of materials have been proposed during the past one hundred years. These theories have been advanced which attempt to correlate the occurrence of first yield in a tension test or some other simple laboratory test with the same phenomenon for a member subjected to a combined stress state. The common yield conditions advanced are associated with physical quantities such as maximum normal stress, maximum normal strain, maximum shear stress, total energy, energy of distortion, or internal friction reaching a critical value which is the same for all states of stress. Experiments have shown that no one theory is capable of predicting yield in all types of material. The ductile metals, however, show reasonable agreement with either the maximum shear stress or energy of distortion yield condition. It is generally accepted that at room temperature steel usually fails in a ductile manner entirely by a shear mechanism (1). Review of
the maximum shear stress and distortion energy theories indicate that principal stress difference, or principal shear stress, is by far the most important factor governing the strength of a material. It is generally accepted that deformation of ductile materials, such as steel, takes place due to gliding or sliding resulting from shearing stresses, the maximum gliding occurring on the plane of maximum shear. It has also been found that Leuders' lines consist of wedge-shaped slip layers which extend through the material and are not just a surface phenomenon. These layers nearly coincide with the planes of maximum shear stress which are theoretically at 45 degrees to the bar axis, but which have been found by various investigators to form at 47 degrees for tension and 43 degrees for compression.

Yield, flow and fracture of ductile metals have been extensively studied in the past and it is now generally believed that under most stress conditions yield and flow depend on some function of the shear stress. This conclusion has been considerably strengthened by mechanical tests carried out under large fluid pressure. Experiments (2) on metals have shown that yielding is not affected by a moderate hydrostatic pressure or tension either applied alone or superposed on a state of combined stress. From this consideration alone, all the so-called theories of failure, except the maximum shear stress and energy of distortion, must be rejected. Some of the inadequacies of the conventional theories of failure for yield strength are that they are not applicable for materials with different tensile strengths in various directions and they are not suitable for materials with different strengths in simple tension and compression. Furthermore all of the conventional theories of
failure require that the relationship between yield strength in bending and in torsion should be the same for all materials. Possibly the nature of yielding by slip, commencing as it does on a small scale at the atomic or crystal level, may in the final analysis make it impossible to define the condition of failure in terms of the usual concept of stress or strain. However, at present there is no choice but to express these conditions in terms of stress or strain.

Marin (1956) (3) indicated that the distortion energy theory can be obtained (for biaxial stress of opposite sign), if failure is assumed to be a function of internal friction.

\[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2 \sigma_0^2\]  

(1)

where \(\sigma_0\) is the yield strength in simple tension. That is, Eq. (1) is obtained if the failure shear stress \(\tau\) is considered to be a quadratic function of the normal stress \(\sigma_n\) on the critical plane — namely, 

\[\tau^2 = a \sigma_n^2 - b \sigma_n^2\].

Findley (1959) (4) developed the concept on fatigue failure of metals under combined torsion and bending that alternating shear stress is the primary cause of fatigue with the normal stress on the critical shear plane as an influencing factor. Coleman and Findley (5) have advanced a theory of nonlinear influence of normal stress on fatigue under combined stresses (1957). Although Findley’s theory is for the case of fatigue failure, it is felt of particular interest to review the existing conventional failure theories from the point of view of the influence of normal stress on the critical shear stress. It is also felt of interest to investigate the functional relationship between the normal stress on the critical shear plane and the maximum shear
stress. This will enable us to investigate all failure theories from a unified point of view since shear stress is the most important factor governing failure. The author would like to point out that the study which follows is more from the mathematical point of view and less from the physical point of view. Consequently it is less rigorous from the physical point of view. Nevertheless it does supply some failure equations which otherwise would be impossible to obtain. The validity of these failure equations may be examined by experimental data.

2. Review of Existing Failure Theories from the Point of View of the Influence of Normal Stress on the Critical Shear Stress.

The existing failure theories will be examined from the point of view of the influence of normal stress on the critical shear stress for the case of combinations of bending and torsion. It is possible to compare all the failure theories by plotting maximum shear stress $\tau_n$ versus normal stress $\sigma_n$ acting on the critical shear plane. The following table on theories of failure is taken from Findley's paper (6) adjusted for the case of static yield strength. The principal stresses for bending stress $\sigma$ and shear stress $\tau$ are,

$$
\sigma_1 = \frac{1}{2} \left[ \sigma + \sqrt{\sigma^2 + 4\tau^2} \right]
$$

$$
\sigma_2 = 0
$$

$$
\sigma_3 = \frac{1}{2} \left[ \sigma - \sqrt{\sigma^2 + 4\tau^2} \right]
$$

From Mohr's stress circle the maximum shear stress ($\tau_n$) and the normal stress ($\sigma_n$) occurring on the critical shear plane are,

$$
\tau_n = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sqrt{\sigma^2 + 4\tau^2}}{2}
$$

$$
\sigma_n = \frac{(\sigma_1 + \sigma_3)}{2} = \frac{\sigma}{2}
$$
Table 1. Theories of Failure

<table>
<thead>
<tr>
<th>Theory</th>
<th>Equation of limiting values in terms of principal stresses $\sigma_1 &gt; \sigma_2 &gt; \sigma_3$</th>
<th>Equation for combined bending $\sigma$ and twisting $\tau$</th>
<th>Required Ratio $\sigma_0/\tau_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Principal stress</td>
<td>$\sigma_1 = (1/2)\left[\sqrt{\sigma^2 + 4\tau^2} + \sigma\right]$</td>
<td>$\sigma_1 = (1/2)\sqrt{\sigma^2 + 4\tau^2}$</td>
<td>1</td>
</tr>
<tr>
<td>2. Principal shear stress</td>
<td>$\tau_1 = \frac{(\sigma_1 - \sigma_3)}{2}$</td>
<td>$\tau_1 = (1/2)\sqrt{\sigma^2 + 4\tau^2}$</td>
<td>2</td>
</tr>
<tr>
<td>3. Principal strain</td>
<td>$\varepsilon_1 = \frac{\sigma_1 - \mu(\sigma_2 + \sigma_3)}{E}$</td>
<td>$\varepsilon_1 = \frac{1+\mu}{2E} \left[\sqrt{\sigma^2 + 4\tau^2} + \frac{1-\mu}{1+\mu} \sigma\right]$</td>
<td>1 $+\mu$</td>
</tr>
<tr>
<td>4. Principal shear strain</td>
<td>$\varepsilon_1 = \frac{(1+\mu)(\sigma_1 - \sigma_3)}{E}$</td>
<td>$\varepsilon_1 = \frac{1+\mu}{E} \sqrt{\sigma^2 + 4\tau^2}$</td>
<td>2</td>
</tr>
<tr>
<td>5. Energy of distortion</td>
<td>$W_D = (1+\mu)\left[\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{6E}\right]$</td>
<td>$W_D = \frac{1+\mu}{3E} \left[\sigma^2 + 3\tau^2\right]$</td>
<td>$\sqrt{3}$</td>
</tr>
<tr>
<td>6. Octahedral shear stress</td>
<td>$\tau_{oct} = \frac{1}{3} \left[\sigma_1 - \sigma_2\right] - \frac{1}{3} \left[\sigma_2 - \sigma_3\right] + \frac{1}{3} \left[\sigma_3 - \sigma_1\right]$</td>
<td>$\tau_{oct} = \frac{\sqrt{2}}{3} \left[\sigma^2 + 3\tau^2\right]$</td>
<td>$\sqrt{3}$</td>
</tr>
<tr>
<td>7. Total energy of deformation</td>
<td>$W = (1/2E)\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]$</td>
<td>$W = \frac{1}{2E} \left[\sigma^2 + 2(1+\mu)\tau^2\right]$</td>
<td>$\sqrt{2(1+\mu)}$</td>
</tr>
<tr>
<td>8. Magnitude of state-of-stress vector</td>
<td>$S = \left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right]^{1/2}$</td>
<td>$S = \left[\sigma^2 + 2\tau^2\right]^{1/2}$</td>
<td>$\sqrt{2}$</td>
</tr>
<tr>
<td>9. Complete Guest's law</td>
<td>$\tau_0 = (1/2)\left[\frac{2\tau_0}{\sigma_0} - 1\right](\sigma_1 + \sigma_3)$</td>
<td>$\tau_0 = \frac{1}{2} \left[\sqrt{\sigma^2 + 4\tau^2} + \frac{2\tau_0}{\sigma_0} - 1\right] \sigma$</td>
<td>Any</td>
</tr>
</tbody>
</table>
Table 1. Theories of Failure (cont'd.)

<table>
<thead>
<tr>
<th>Theory</th>
<th>Equation of limiting values in terms of principal stresses $\sigma_1 &gt; \sigma_2 &gt; \sigma_3$</th>
<th>Equation for combined bending $\sigma$ and twisting $\tau$</th>
<th>Required Ratio $\sigma_0/\tau_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10. Ellipse quadrant</td>
<td>$1 = (\tau/\tau_0)^2 + (\sigma/\sigma_0)^2$</td>
<td>Any</td>
<td></td>
</tr>
<tr>
<td>11. Ellipse arc</td>
<td>$1 = (\tau/\tau_0)^2 + (\sigma/\tau_0 - 1)(\sigma/\sigma_0)^2 + 2 - (\sigma/\tau_0)\sigma/\sigma_0$</td>
<td>Any</td>
<td></td>
</tr>
</tbody>
</table>

$E$ is Young's modulus, $\mu$ is Poisson's ratio, and $\sigma_0$ and $\tau_0$ are the yield strength in bending and torsion, respectively.
The various failure theories from Table 1 will be examined by using \( \sigma_n \) and \( \tau_n \) as parameters to find the relationship between \( \tau_n \) and \( \sigma_n \), i.e., \( \tau_n = f(\sigma_n) \). By application of Eq. (3) the various equations for combined bending \( \sigma \) and twisting \( \tau \) in Table 1 can be written as follows:

1. **Principal Stress**
   \[
   \sigma_1 = \frac{1}{2} \left[ \sigma + \sqrt{\sigma^2 + 4\tau^2} \right] = \sigma_n + \tau_n = \sigma_0
   \]
   and
   \[
   \tau_n = \sigma_0 - \sigma_n
   \]

2. **Principal Shear Stress**
   \[
   \tau_1 = \frac{1}{2} \sqrt{\sigma^2 + 4\tau^2} = \tau_n = \tau_0
   \]
   \[
   \tau_n = \tau_0
   \]

3. **Principal Strain**
   \[
   \varepsilon_1 = \frac{1 + \mu}{2} \left[ \frac{\sqrt{\sigma^2 + 4\tau^2}}{1 + \mu} \right]
   \]
   \[
   = \frac{1 + \mu}{2E} \left[ 2 \tau_n + 2 \sigma_n \right] = \frac{\sigma_0}{E}
   \]
   \[
   \tau_n = \frac{\sigma_0}{1 + \mu} - \frac{1 - \mu}{1 + \mu} \sigma_n
   \]

4. **Principal Shear Strain**
   \[
   \gamma_1 = \frac{1 + \mu}{E} \sqrt{\sigma^2 + 4\tau^2} = \frac{1 + \mu}{E} \sqrt{\sigma_n^2 + \tau_n^2} = \gamma_0
   \]
   \[
   \tau_n = (1/2) \gamma_0 \left( \frac{E}{1 + \mu} \right)
   \]

5. **Energy of Distortion**
   \[
   W_D = \frac{1 + \mu}{3E} \left[ \sigma^2 + 2\tau^2 \right] = \frac{1 + \mu}{3E} \sigma_0^2
   \]
   \[
   \tau_n^2 = (1/3)(\sigma_0^2 - \sigma_n^2)
   \]

It is noted that equation of this form is suggested by Marin (3).
(6) Octahedral Shear Stress
\[ \tau_{\text{oct}} = \left(\sqrt{\frac{2}{3}}\right) \sqrt{\sigma^2 + 3\tau^2} = \left(\sqrt{\frac{2}{3}}\right) \sqrt{\sigma_n^2 + 3\tau_n^2} \]
\[ \tau_n^2 = (3/2) \tau_{\text{oct}}^2 - (1/3)\sigma_n^2 \] (9)

(7) Total Energy of Deformation
\[ W = \frac{1}{2E} \left[ \sigma^2 + 2(1+\mu)\tau^2 \right] \]
\[ = \frac{1}{2E} \left[ \mu\sigma_n^2 + 2(1+\mu)(\tau_n^2 - \sigma_n^2) \right] = (1/2E)\sigma_0^2 \]
\[ \tau_n^2 = \frac{\sigma_0^2}{2(1+\mu)} - \frac{1-\mu}{1+\mu} \sigma_n^2 \] (10)

(8) Magnitude of State-of-Stress Vector
\[ S = \sqrt{\sigma^2 + 2\tau^2} = \sqrt{2(\sigma_n^2 + \tau_n^2)} = \sqrt{\sigma_0^2} \]
\[ \tau_n^2 = (1/2) \sigma_0^2 - \sigma_n^2 \] (11)

(9) Complete Guest's Law
\[ \tau_o = \frac{1}{2} \left[ \sqrt{\sigma^2 + 4\tau^2} + \left( \frac{2\tau_o}{\sigma_0} - 1 \right) \sigma \right] \]
\[ = \tau_n + \left( \frac{2\tau_o}{\sigma_0} - 1 \right) \sigma_n = \tau_o \]
\[ \tau_n = \tau_o - \left( \frac{2\tau_o}{\sigma_0} - 1 \right) \sigma_n \] (12)

(10) Ellipse Quadrant
\[ 1 = \left( \frac{\tau}{\tau_o} \right)^2 + \left( \frac{\sigma}{\sigma_0} \right)^2 = \frac{\tau_n^2 - \sigma_n^2}{\tau_o^2} + \frac{4\sigma_n^2}{\sigma_0^2} \]
\[ \tau_n^2 = \tau_o^2 - \left( \frac{4\sigma_n^2}{\sigma_0^2} - 1 \right) \sigma_n^2 \] (13)
(11) Ellipse Arc

\[ l = \left( \frac{\tau}{\tau_0} \right)^2 + \left( \frac{\sigma}{\sigma_0} - 1 \right) \left( \frac{\tau^2}{\sigma_0^2} \right) + \left( 2 - \frac{\sigma}{\tau_0} \right) \frac{\sigma}{\sigma_0} \]

\[ \tau^2 = \tau_n^2 - \sigma_n^2 \]

\[ \sigma^2 = 4\sigma_n^2 \]

\[ \tau_n^2 = \tau_0^2 - \left( \frac{4\tau_0^2 - 2\sigma_0 \tau_0}{\sigma_0} \right) \sigma_n - \left[ \frac{4(\tau_0 \sigma_0 - \tau_0^2) - \sigma_0^2}{\sigma_0^2} \right] \sigma_n^2 \quad (14) \]

(12) Navier's Theory (7)

The maximum shear stress theory has been modified by Navier into a form

\[ \tau_n = S - m \sigma_n \quad (15) \]

where \( m \) = coefficient of internal friction.

Another interpretation of the above equation will appear in connection with Mohr's theory.

(13) Mohr Theory

Mohr (8) stated that the critical shear stress at failure is a function of the normal stress acting on this plane, i.e.,

\[ \tau_n = f(\sigma_n) \quad (16) \]

(14) Marin's Ellipse Equation

Mohr, in his formulation of a general shear theory, did not attempt to define the envelope to the Mohr circle represented by \( \tau_n = f(\sigma_n) \). Based on test data Marin (8) found that a good approximation for test results on ductile materials is an envelope expressed by the equation of an ellipse, viz.,

\[ \sigma_n^2/a^2 + \tau_n^2/b^2 = \sigma_0^2 \quad (17) \]

where \( a \) and \( b \) are the intercepts on the vertical and horizontal axes.
To provide a strength theory for materials which have different simple tensile strengths \( \sigma_t \) in different directions, Marin (9) proposed a strength theory for anisotropic materials which for the biaxial state of stress can be written as,

\[
\sigma_1^2 + \alpha \sigma_1 \sigma_3 + \beta \sigma_3^2 = \sigma_{t1}^2
\]  

(18)

where

\[
\alpha = 1 + \frac{(\sigma_{t1}/\sigma_{t3})^2 - (\sigma_{t1}/\tau_3)^2}{\tau_3}
\]

\[
\beta = \left(\frac{\sigma_{t1}}{\sigma_{t3}}\right)^2
\]

\( \sigma_{t1}, \sigma_{t3} \) = the simple tensile yield strengths in the \( \sigma_1 \) and \( \sigma_3 \) directions

\( \tau_3 \) = the simple torsional yield strength.

From Eqs. (2) and (3), the following equations can be written.

\[
\sigma_1 = \sigma_n + \tau_n
\]

\[
\sigma_3 = \sigma_n - \tau_n
\]  

(19)

Substituting Eq. (19) into (18), it can be seen that there is a non-linear relationship between \( \tau_n \) and \( \sigma_n \).

The result of the above investigation reveals that the limiting value of the maximum principal shear stress is a function of the normal stress on the critical shear plane, i.e., \( \tau_n = f(\sigma_n) \). The functional relationship between the critical shear stress and the normal stress can be written as,

a) \( \tau_n = \alpha \)

b) \( \tau_n = \alpha + \beta \sigma_n \)

c) \( \tau_n^2 = \alpha + \beta \sigma_n^2 \)

d) \( \tau_n^2 = \alpha + \beta \sigma_n^2 + \gamma \sigma_n^2 \)  

(20)
It is of interest to investigate the existing failure theories from the following several considerations.

a) Mathematical form of the failure equation
   - Linear or quadratic relation
   - Physical constants involved

b) The concept of the influence of the normal stress on the maximum shear stress \( (\tau_n = f(\sigma_n)) \)

c) The importance of shear stress

d) Parameters \( \tau_n \) and \( \sigma_n \) afford physical interpretation

e) Yield surface \( F(\tau_n, \sigma_n, \tau_o, \sigma_o) = 0 \)


If \( \sigma_o \) is the simple tensile yield strength of the material, failure is defined by Eq. (1). Eq. (1) expresses failure for an isotropic material and for a material with strength in simple tension equal to that in simple compression. The yield strength in tension and in compression for ductile steel is usually considered to be the same. The assumption is sufficiently accurate for many purposes, however, it is important to know accurately the relation between the yield tensile and compressive strength. It is found that the compressive yield strength of ductile steel is somewhat greater than the tensile yield strength except in the case of soft steel. It appears from a study that the amount of working or rolling has a marked effect upon the tensile and compressive yield strengths of steel. Eq. (1) can be obtained by one of the following concepts.

1) Failure is a function of the second invariant of the stress
deviation. This was introduced by purely mathematical arguments (Von Mises' theory).

2) Another interpretation of Eq. (1) has been given by Nadai in terms of the shear stress across the octahedral plane. Failure is assumed to be a function of the octahedral shear stress.

3) It is assumed that yield takes place when the elastic strain energy of distortion reaches a value characteristic of the material (Hencky, Huber).

4) Eq. (1) is the relation corresponding to the statistical average of the shear stress on all slip planes of a polycrystalline aggregate (Hoffman and Sachs, 1928).

5) Marin (3) suggested that Eq. (1) is also obtained for biaxial stress of opposite sign, if failure is assumed to be a function of internal friction. That is, if the failure shear stress ($\tau_n$) is considered to be a quadratic function of the normal stress ($\sigma_n$) on the critical plane, namely,

$$\tau_n^2 = a \sigma_n^2 - b \sigma_n^2$$  \hspace{1cm} (21)

It can be seen that the foregoing observations in themselves give considerable support to the selection of Eq. (1) for defining failure. The most important basis for recommending this relation is the fact that its predictions fit considerable experimental results on ductile materials. The following analysis is done according to Marin's suggestion (3). It can be shown by Mohr's circle that ($\sigma_1, \sigma_2 = 0, \sigma_3$).

$$\tau_n = \frac{\sigma_1 - \sigma_3}{2}, \quad \sigma_n = \frac{\sigma_1 + \sigma_3}{2}$$  \hspace{1cm} (22)
where \( \tau_n \) = critical shear stress
\( \sigma_n \) = normal stress acting on critical shear plane.

Substituting \( \tau_n \) and \( \sigma_n \) into Eq. (21) and noting
\[ \sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2 = \sigma_n^2 \]  
(23)

Eq. (21) can be written as
\[ \tau_n^2 = \frac{1}{3}(\sigma_n^2 - \sigma_n^2) \]  
(24)

The effect of fluid pressure on the tensile properties of metals has been extensively investigated by Bridgman. His earlier work is summarized in his book (10), and more recent work is given in a paper (11).

Various independent researchers have all led to the conclusion that pressure has little effect on the yield and flow of ductile materials. It has been demonstrated that in a ductile metal the volumetric component has no effect on the stress deviator versus strain curve. It is generally agreed that yielding of the ductile metals will not occur under moderate hydrostatic pressure. For single metallic crystals as well as for the polycrystalline ductile metals, it has been repeatedly shown that the addition of a uniform tensile or compressive stress will not change the magnitude of the principal shearing stress at a plastic limit. In the following attempt will be made to derive the distortion energy theory from Eq. (24) by using the above-mentioned concept. Since hydrostatic pressure does not have influence on the yield or flow of ductile materials, the state of stress \( (\sigma_1, \sigma_2, \sigma_3) \) will be the same as \( (\sigma_1 - \sigma_2, 0, \sigma_3 - \sigma_2) \) as far as yielding of the material is concerned. From the Mohr stress circle,
Substituting Eq. (25) into Eq. (24),

\[ \frac{1}{U} \left( \sigma_1 - \sigma_3 \right)^2 + \frac{1}{3} \left( \sigma_1 + \sigma_3 - 2\sigma_2 \right)^2 = \frac{1}{3} \sigma_t^2 \]

After some mathematical manipulation, the following equation can be obtained.

\[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_t^2 \] \hspace{1cm} (26)

Since the distortion energy theory can be derived from Eq. (25), it will be assumed here that the functional relation

\[ \tau_n = \frac{\sigma_1 - \sigma_3}{2}, \quad \sigma_n = \frac{\sigma_1 + \sigma_3 - 2\sigma_2}{2} \] \hspace{1cm} (25)

is correct. In the general form the above relation can be written as

\[ \tau_n^2 = (1/3)(\sigma_t^2 - \sigma_n^2) \] \hspace{1cm} (27)

Based on this equation, several failure theories will be proposed.

a) Biaxial Tension Failure Theory

The material considered will be anisotropic, that is, the material has directional properties. In this case no existing classical failure theories including Eq. (1) will hold. Marin (9) has proposed recently a strength theory for anisotropic materials. The biaxial tension \( (\sigma_1, \sigma_2, \sigma_3 = 0) \) is equivalent to \( (\sigma_1 - \sigma_2, 0, -\sigma_2) \) as far as the yield strength of the ductile metals is concerned. Hence the following relationship can be written,

\[ \tau_n = \sigma_1/2, \quad \sigma_n = (\sigma_1 - 2\sigma_2)/2 \] \hspace{1cm} (29)
Using Eq. (28), failure equation becomes,
\[
\left(\frac{\sigma_1}{2}\right)^2 = \alpha + \beta \left(\frac{\sigma_1 - 2\sigma_2}{2}\right)^2
\]
(30)

The values of \(\alpha\) and \(\beta\) in Eq. (30) can be obtained by two simple strength properties of the material. Various possible strengths might be considered. In the present determination, values of \(\alpha\) and \(\beta\) will be found from the simple tensile strengths in the \(\sigma_1\) and \(\sigma_2\) directions. That is, using the conditions that \(\sigma_1 = \sigma_{t1}\) for \(\sigma_2 = 0\), \(\sigma_2 = \sigma_{t2}\) for \(\sigma_1 = 0\), then values of \(\alpha\) and \(\beta\) can be determined.

\[
\alpha = -\frac{\sigma_{t1}^2 \sigma_{t2}^2}{\sigma_{t1}^2 - 4\sigma_{t2}^2}, \quad \beta = \frac{\sigma_{t1}^2}{\sigma_{t1}^2 - 4\sigma_{t2}^2}
\]
(31)

When these are substituted in Eq. (30), the failure relation becomes
\[
\frac{\sigma_1^2}{\sigma_{t1}^2} + \frac{\sigma_2^2}{\sigma_{t2}^2} - \frac{\sigma_1 \sigma_2}{\sigma_{t1}^2} = 1
\]
(32)

b) Biaxial Compression Theory

The state of stress \((0, \sigma_2, \sigma_3)\) is equivalent to \((0 - \sigma_2, 0, \sigma_3 - \sigma_2)\) since hydrostatic tension will not contribute to the yielding of ductile metals. As before the following equations can be written
\[
\tau_n = -\sigma_3/2, \quad \sigma_n = (\sigma_3 - 2\sigma_2)/2
\]
(33)

Substituting into Eq. (28),
\[
(s_3/2)^2 = \alpha + \beta \left(\frac{s_3 - 2s_2}{2}\right)^2
\]
(34)

Using boundary conditions \(s_3 = -s_{03}, s_2 = 0,\) and \(s_2 = -s_{02}, s_3 = 0,\)
the following failure equation for anisotropic material can be obtained.

\[
\left( \frac{\sigma_2}{\sigma_3} \right)^2 + \left( \frac{\sigma_2}{\sigma_3} \right) - \left( \sigma_2/\sigma_3 \right)^2 = 1 
\]

(35)

c) Combined Bending and Torsion

The state of stress in this case is \((\sigma_1, 0, \sigma_3)\). The material constants \(\alpha\) and \(\beta\) in Eq. (28) can be obtained by tension and torsion tests. The following boundary conditions may be used to determine \(\alpha\) and \(\beta\).

From tension test \(\sigma_1 = \sigma_0, \quad \sigma_3 = 0\)

From torsion test \(\sigma_1 = -\sigma_3 = \tau_0\).

The failure equation for this case can be written from Eq. (28) as,

\[
\left( \frac{\sigma_1 - \sigma_3}{2} \right)^2 = \alpha + \beta \left( \frac{\sigma_1 + \sigma_3}{2} \right)^2
\]

(36)

After applying boundary conditions the failure equation can be written as

\[
(\sigma_1 - \sigma_3)^2 = 4\tau_0^2 + \left( 1 - \frac{4\tau_0^2}{\sigma_0^2} \right) (\sigma_1 + \sigma_3)^2
\]

(37)

It is to be noted that the above equation is derived by using tension and torsion test data. Elastic properties in tension and compression were assumed to be the same for this case. It will be shown that the ellipse failure equation \(\tau^2/\tau_0^2 + \sigma^2/\sigma_0^2 = 1\) can be derived theoretically from Eq. (37). For the combination of tension (compression) and torsion, the expressions for the principal stresses are shown in Eq. (2). From Eq. (2) we have

\[
\sigma_1 + \sigma_3 = \sigma
\]

\[
\sigma_1 - \sigma_3 = \sqrt{\sigma^2 + 4\tau^2}
\]

(38)
Substituting Eq. (38) into Eq. (37), we have
\[(\tau^2/\tau_0^2) + (\sigma^2/\sigma_0^2) = 1\] (39)

According to Tresca's theory, yielding would start as soon as
\[(\sigma^2/\sigma_0^2) + 4(\tau^2/\tau_0^2) = 1\] (40)

According to Von Mises' theory, yielding would start when
\[(\sigma^2/\sigma_0^2) + 3(\tau^2/\tau_0^2) = 1\] (41)

It is easily seen that Eqs. (40) and (41) can be derived from
Eq. (39) if we set \(\tau_0 = (1/2)\sigma_0\) and \(\tau_0 = (1/\sqrt{3})\sigma_0\), respectively.

This again substantiates the validity of Eq. (37). Tension-torsion
tests by Taylor and Quinney (12) show that for some mild steel Tresca
and Von Mises' theories both do not fit the experimental data. It can
be shown that Eq. (39) will fit data very well. Eq. (39) indicates that
the ratio \(\tau_0/\sigma_0\) should be determined by tests for each material.

4. Derivation of Failure Equations Based on the Concept that the
Normal Stress Has Influence on the Critical Shear Stress.

The results of the above investigation indicate that any one of
the following functional relations between the maximum shear stress \(\tau_n\)
and the normal stress acting on the maximum shear plane \(\sigma_n\) is possible.

a) \(\tau_n = \alpha\)
b) \(\tau_n = \alpha + \beta \sigma_n\)
c) \(\tau_n^2 = \alpha + \beta \sigma_n^2\)
d) \(\tau_n^2 = \alpha + \beta \sigma_n + \gamma \sigma_n^2\)

Each case will be investigated and its physical meaning as well as the
constants involved will be determined.
Case a)

This case corresponds to maximum shear stress theory.

Case b) \( \tau_n = \alpha + \beta \sigma_n \)

Since \( \tau_n = (\sigma_1 - \sigma_3)/2, \quad \sigma_n = (\sigma_1 + \sigma_3)/2 \)

we have

\[
\frac{(\sigma_1 - \sigma_3)/2}{2} = \alpha + \beta \left(\frac{\sigma_1 + \sigma_3}{2}\right) / 2
\]

(12)

Applying the boundary conditions \( \sigma_1 = \sigma_0, \sigma_3 = 0 \) (tension test) and \( \sigma_1 = -\sigma_3 = \tau_0 \) (torsion test), Eq. (12) becomes

\[
\frac{\sigma_1 - \sigma_3}{2} = \tau_0 + \frac{2\tau_0}{\sigma_0} \left(\frac{\sigma_1 + \sigma_3}{2}\right)
\]

(13)

The above equation represents a straight line. Since for most materials \( 2\tau_0 > \sigma_0 \), it is seen from Eq. (13) that compressive normal stress will increase the critical shear stress. This is in accord with some of the facts observed in fatigue. It is to be noted that Eqs. (13) and (12) are the same.

Case c) \( \tau_n^2 = \alpha + \beta \sigma_n^2 \)

The above failure equation can be written as

\[
\left(\frac{\sigma_1 - \sigma_3}{2}\right)^2 = \alpha + \beta \frac{1}{\sigma_0^2} \left(\frac{\sigma_1 + \sigma_3}{2}\right)^2
\]

(14)

Applying boundary conditions \( \sigma_1 = \sigma_0, \sigma_3 = 0 \) (tension test) and \( \sigma_1 = -\sigma_3 = \tau_0 \) (torsion test) Eq. (14) becomes

\[
\left(\frac{\sigma_1 - \sigma_3}{2}\right)^2 = \tau_0 + \frac{1}{\sigma_0^2} \left(\frac{\sigma_1 + \sigma_3}{2}\right)^2
\]

(15)

This equation was obtained earlier (see Eq. 37). For most engineering
materials \((1 - 4\tau_0^2/\sigma_0^2) < 0\) and Eq. (45) represents ellipse. For special case \(\tau_0 = (1/\sqrt{3})\sigma_0\), Eq. (45) becomes,
\[
\left(\frac{\sigma_1 - \sigma_3}{2}\right)^2 = \frac{1}{3}\left[\sigma_0^2 - \left(\frac{\sigma_1 + \sigma_3}{2}\right)^2\right]
\]  
(46)
The above equation reduces to Eq. (24).

As far as yield is concerned the state of stress \((\sigma_1, \sigma_2, \sigma_3)\) corresponds to \((\sigma_1 - \sigma_2, 0, \sigma_3 - \sigma_2)\). Then Eq. (45) can be written as,
\[
\left(\frac{\sigma_1 - \sigma_3}{2}\right)^2 = \frac{\tau_0^2}{\sigma_0^2} + (1 - \frac{4\tau_0^2}{\sigma_0^2}) \left(\frac{\sigma_1 + \sigma_3 - 2\sigma_2}{2}\right)^2
\]  
(47)
By setting \(\tau_0 = (1/\sqrt{3})\sigma_0\), Eq. (47) can be reduced to
\[
(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_0^2
\]
Let us investigate what kind of biaxial tension failure equation we will get from Eq. (47). For the stress state of biaxial tension \(\sigma_1 > \sigma_2 > \sigma_3\) and \(\sigma_3 = 0\). Then Eq. (47) reduces to
\[
\sigma_1^2 = 4\tau_0^2 + (1 - \frac{4\tau_0^2}{\sigma_0^2}) (\sigma_1 - 2\sigma_2)^2
\]  
(48)
By setting \(\sigma_0/\tau_0 = \sqrt{3}\), Eq. (48) becomes
\[
\sigma_1^2 - \sigma_1 \sigma_2 + \sigma_2^2 = \sigma_0^2
\]
The same type of equation will be obtained for the biaxial compression.

The most general case will be discussed next.

Case d) \(\tau_n^2 = \alpha + \beta \sigma_n + \gamma \sigma_n^2\)

As before,
\[
\tau_n = (\sigma_1 - \sigma_3)/2, \quad \sigma_n = (\sigma_1 + \sigma_3)/2
\]
and \[
\left( \frac{\sigma_1 - \sigma_3}{2} \right)^2 = \alpha + \beta \left( \frac{\sigma_1 + \sigma_3}{2} \right) + \gamma \left( \frac{\sigma_1 + \sigma_3}{2} \right)^2\]

(49)

Since we have three constants to be determined tension \((\sigma_1 = \sigma_0, \sigma_2 = 0, \sigma_3 = 0)\), compression \((\sigma_1 = 0, \sigma_2 = 0, \sigma_3 = -\sigma_0^1)\), and torsion \((\sigma_1 = -\sigma_3 = \tau_0, \sigma_2 = 0)\) tests may be used as boundary conditions. After satisfying boundary conditions Eq. (49) can be written as

\[
\left( \frac{\sigma_1 - \sigma_3}{2} \right)^2 = \tau_0^2 + \frac{2\tau_0^2(\sigma_0 - \sigma_0^1)}{\sigma_0 \sigma_0^1} \left( \frac{\sigma_1 + \sigma_3}{2} \right) + \frac{\sigma_0 \sigma_0^1 - 4\tau_0^2}{\sigma_0 \sigma_0^1} \left( \frac{\sigma_1 + \sigma_3}{2} \right)^2
\]

(50)

If \(\sigma_0 = \sigma_0^1\), Eq. (50) becomes Eq. (45).

Since the state of stress \((\sigma_1, \sigma_2, \sigma_3)\) corresponds to \((\sigma_1 - \sigma_2, 0, \sigma_3 - \sigma_2)\) for yielding Eq. (50) can be written as

\[
\left( \frac{\sigma_1 - \sigma_3}{2} \right)^2 = \tau_0^2 + \frac{2\tau_0^2(\sigma_0 - \sigma_0^1)}{\sigma_0 \sigma_0^1} \left( \frac{\sigma_1 + \sigma_3 - 2\sigma_2}{2} \right) + \frac{\sigma_0 \sigma_0^1 - 4\tau_0^2}{\sigma_0 \sigma_0^1} \left( \frac{\sigma_1 + \sigma_3 - 2\sigma_2}{2} \right)^2
\]

(51)

This is a general failure equation for the triaxial state of stress. By setting \(\sigma_0 = \sigma_0^1\) and \(\tau_0 = (1/2)\sigma_0\), the maximum shear stress theory will be obtained. By setting \(\sigma_0 = \sigma_0^1, \tau_0 = (1/\sqrt{3})\sigma_0\), the distortion energy theory will be obtained.

It is to be noted that Eq. (51) suggests that \(\sigma_0, \sigma_0^1\) and \(\tau_0\) should be determined by experiments. The ratio \(\tau_0/\sigma_0\) is not constant but depends on the kind of material. It is felt that this is a more realistic approach since the ratio \(\tau_0/\sigma_0\) may be affected by the anisotropic property of the material and other complicating effects.


CHAPTER V

DISCUSSION OF THE RESULTS INCLUDING A QUALITATIVE DISCUSSION
ON ELASTIC-PLASTIC DEFORMATIONS OF METALS
WITH REPEATED OR REVERSED LOADING

The objective of the present investigation was to develop a rigorous theoretical analysis of the state of stress and strain in work-hardening prismatical bars (rectangular or circular) subjected to bending in the elastic-plastic region, assuming biaxial stress (plane stress state). The analysis of the plastic bending problem for the case of uniaxial stress has received thorough coverage in the technical literature. However, no reasonably exact solution for the work-hardening solid which takes the radial stresses into consideration has been offered. In the uniaxial stress analysis, it is assumed that the strains are so small that the radial stresses induced by the curvature can be neglected. It is supposed, too, that the neutral surface coincides with the central plane of the bar throughout the distortion. This theory is a good approximation provided the final radius of curvature is not less than, say four or five times the thickness of the bar. The extent to which approximate solutions of uniaxial stress could be used in practice should be determined by experiments. The assumption that the neutral surface at all times coincides with the mid-plane of the bar and that the radial stress $\sigma_r$ is zero, are certainly admissible for moderate bending and have enabled us to obtain a very neat solution. For very severe bending,
where the bar is loaded to the fully plastic state, the previous assumptions cannot be made, because the neutral surface does not coincide with the central surface of the bar and the radial stresses are no longer negligible.

The uniaxial stress analysis of the present investigation is based on an analytical expression for the stress-strain curve proposed by Nadai in 1939. Nadai's stress-strain equation enables us to obtain solutions for linear as well as nonlinear work-hardening solids. It is assumed that the tension and compression stress-strain curves of the material are the same. If the curves are not the same, a shift of the neutral surface of the bar will take place as it is bent. It can be shown, however, that the bending moments required to produce a given curvature will be little affected by a difference of as much as 10 percent. The method of uniaxial stress analysis for plastic bending, as proposed in Chapter II, enables us to compute the functional relationship between bending moment and the maximum stress or strain. Throughout all studies in the present investigation, it is assumed that cross sections remain plane during pure bending in the plastic range. This is the most important assumption in the whole analysis. Several investigators have shown experimentally that cross sections remain plane during pure bending in the plastic range. This assumption is indirectly verified by the fact that several investigators have demonstrated their theoretical analysis based on the above assumption checks experimental data well. The method discussed in Chapter II is also based on another assumption, that the relationship between stress and strain is the same.
for bar fibers as it is for simple tension and compression, and that this relationship holds for the full range of the stress-strain diagram.

The more rigorous analysis of the state of stress and strain in a work-hardening prismatical bar subjected to bending in the elastic-plastic range, assuming biaxial stress, is presented in Chapter III. The complete solution of the present problem involves the calculation of stresses and strains in both the elastic and plastic regions. Before proceeding to the elastic-plastic solution, biaxial stress analysis is considered for several idealized bodies, i.e., completely elastic, rigid perfectly plastic, and rigid strain-hardening solids, for the case of the rectangular section. The biaxial stress analysis for a rigid, strain-hardening solid is based on Nadai-Hencky's plasticity law. Unfortunately, this analysis leads to a nonlinear differential equation which defies analytical integration and a solution in elementary functions cannot be obtained. Numerical or graphical methods of solution are suggested. In order to demonstrate the general features of the elastic-plastic solution, the method of solution for a rectangular section for a perfectly-plastic solid in the elastic-plastic range is presented. Even for the case of a perfectly-plastic solid, the problem of finding elastic-plastic boundaries and the neutral surface is quite involved and tedious. The analytical solution in the elastic-plastic region for a work-hardening solid cannot be obtained. However, the plausible method of solving this problem by numerical analysis is suggested. Finally the method of solution for the case of a circular section is developed. The accuracy of the biaxial stress analysis based on Nadai-Hencky plasticity law should be checked by the experimental data. This investigation is
limited to the development of the rigorous method of biaxial stress analysis of prismatical bars (rectangular or circular) subjected to pure bending in the elastic-plastic range. Consequently the numerical calculation is not contemplated in the present study. The analysis presented corresponds to the initial one-half cycle for released loading and to the initial one-fourth cycle for alternate loading.

In the course of the present investigation, the author developed a hypothesis of failure (yielding) based on the concept of a nonlinear influence of normal stress on the critical shear stress under triaxial state of stress. Several failure equations may be derived from this concept mainly from the mathematical point of view. The result of this investigation is included in Chapter IV.

It is perhaps worthwhile to discuss elastic-plastic deformations of bodies with repeated or reversed loading. It is felt that this discussion would clarify some limitations of the applicability of the results of the present analysis to the interpretation of low-cycle fatigue data. The cyclic strain or stress behavior of metals will be discussed from a phenomenological point of view rather than from that of the detailed physical mechanism. Changes in properties caused by plastic cycling have been investigated by various research workers and the results are far too numerous to be discussed here, but some of the points which are pertinent to the present investigation will be brought out. Probably the sharpest division that occurs in the field of short-life fatigue testing is whether the tests are carried out under cycles of constant load (bending moment) or cycles of constant strain (bending deflection). If the tests are run at constant load, the strain limits will vary
throughout some part of the life, and probably the stress limits will change to some extent as well. If the strain amplitudes are kept constant, the load required to produce these strains will vary for a period and likewise the stress. It may be concluded in general that for any particular type of cycle (repeated, alternating) the nominal stress-cycle curve cannot be derived from the strain-cycle relationship, and vice versa, for the low endurance range. Changes in the properties of the material as a result of cyclic stress or strain are manifested by changes in some mechanical properties, i.e., hardness, hysteresis loop, energy, ductility, and strain. The changes in mechanical properties depend much on the pre-test condition of the material, i.e., whether it has been cold worked or not. When strain cycling is performed for a constant amplitude, there generally follows a change in stress amplitude. These changes are rapid and large in the first few cycles, then become very slow or cease for a great part of the life until near the growth of a fatigue crack. For cycles of constant load (nominal stress) the changes can occur in the hysteresis loop with respect to the strain axis. The true stress may also vary owing to slight permanent changes in geometry. It is seen that materials became conditioned in various ways by their strain or stress history. The behavior of material under cyclic loading conditions involves the Bauschinger effect and anisotropy which takes place during the loading process. Taking into account the Bauschinger effect and the effect of anisotropy leads to very complex plasticity laws which are scarcely applicable for practical use. The prediction of the behavior of materials under cyclic loading is apparently a very difficult subject. It is evident that great caution should
be exercised in applying the results of the present investigation to predict the maximum strain or stress of materials under cyclic loading condition in the elastic-plastic range.

The question as to the suitability of the method of biaxial stress analysis presented here must, of course, come from extensive and well-planned experimental investigations of typical materials. The possible errors will come naturally from deviation of the basic assumptions from the actual behavior of material. For example, there may be some anisotropy of the material resulting from the fabrication procedures of the material used. The biaxial stress-strain relationships for the material may not follow precisely the maximum-shearing-stress-strain law assumed in the analysis.

It is to be expected that there will be the movement of the neutral axis toward the compression surface, the decrease in the radial dimension of the bar, the gradual divergence between the tangential strains at the inside and outside surfaces and changes in the width of the cross section with increasing bending moment. The gradual change in cross sectional contour with progressive bending was not considered in the present investigation. However, it is felt that this can be done by a method of successive approximations by considering small successive intervals of increasing curvature. The method of solutions for plane stress was based on the assumption that the dimensions of the bar remain essentially unaltered in the elastic-plastic region. The conditions of plane stress and plane strain represent the extremes of zero and infinity for the breadth-to-height ratio of curved bodies. It was pointed out by other investigators that the tangential strains for plane stress at the concave
and convex surfaces are practically identical for the case of an ideal plastic metal with those obtained for plane strain. The identical results obtained for the tangential surface strains for the above two extremes suggest that the surface strains for bars of all intermediate breadths might be obtained from the plane stress solutions. The shape of the cross section can be evaluated by the distribution of lateral strains across the height of the bar.
AUTOBIOGRAPHY

I, Kwo Chang Cheng, was born in Formosa, China, February 26, 1927. I received my secondary-school education in Hsinchu, Formosa, and my undergraduate training at the Cheng Kung University (formerly Taiwan College of Engineering), Tainan, Formosa, which granted me the degree of Bachelor of Science in Mechanical Engineering in June, 1949. From July, 1949, to January, 1955, I worked as a mechanical engineer for the Kaohsiung Oil Refinery, Chinese Petroleum Corporation. From February, 1955, to May, 1956, I served as a research assistant in the Department of Mechanical Engineering at Kansas State University, which granted me the degree, Master of Science, in May, 1956. In October, 1956, I entered the Graduate School of The Ohio State University and have been employed as a research assistant by the Department of Mechanical Engineering while completing the requirements for the degree Doctor of Philosophy.