Copyright by

Herbert Francis Miller

1959
THE COMBINATION OF THE GUESS-AND-CHECK AND
MULTI-EQUATION METHODS FOR DERIVING THE EQUATIONS FOR
VERBAL PROBLEMS IN ELEMENTARY ALGEBRA

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

HERBERT FRANCIS MILLER, A. B., M. A.

* * * * * *

The Ohio State University
1959

Approved by:

[Signature]
Adviser
Department of Education
ACKNOWLEDGMENTS

The writer wishes to express his sincerest appreciation to the following persons:

To Dr. Nathan Lazar, his major adviser, for the suggestion of this topic of study and for his guidance and counseling throughout the study.

To Dr. Robert Wherry for helpful advice on some statistical aspects of the study.

To the personnel of Lyons Township High School, especially to the late George Hawkins, Head of the Mathematics Department, and to Richard Ellis, LeRoy Stoldt, and Charles Stegmeir, teachers of the experimental classes, for their cooperation and willingness to undertake the experimental teaching.

To many colleagues at Northern Illinois University for their encouragement and helpful suggestions.

To his family for their patience and understanding forbearance.

To the authors, publishers, or editors of the longer passages quoted in this study for their permission to quote.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>The Principal Difficulty in Algebraic Problem Solving</td>
<td>1</td>
</tr>
<tr>
<td>Four Approaches to the Formation of Equations for Problems</td>
<td>2</td>
</tr>
<tr>
<td>The Uni-equation Method without any Technique for Discovering Implicit Relationships</td>
<td>3</td>
</tr>
<tr>
<td>The Uni-equation Method with a Guess-and-Check Procedure for Discovering the Implicit Relationship</td>
<td>4</td>
</tr>
<tr>
<td>The Multi-equation Method without the Guess-and-Check Procedure for Discovering the Implicit Relationship</td>
<td>5</td>
</tr>
<tr>
<td>The Multi-equation Method with the Guess-and-Check Procedure for Discovering the Implicit Relationship</td>
<td>5</td>
</tr>
<tr>
<td>Statement of the Problem of this Study</td>
<td>7</td>
</tr>
<tr>
<td>Importance of this Study</td>
<td>7</td>
</tr>
<tr>
<td>Definitions of Terms Used in this Study</td>
<td>9</td>
</tr>
<tr>
<td>Algebraic Verbal Problem</td>
<td>9</td>
</tr>
<tr>
<td>Examples of Algebraic Verbal Problems</td>
<td>9</td>
</tr>
<tr>
<td>The Numerical Elements</td>
<td>11</td>
</tr>
<tr>
<td>Mathematical English</td>
<td>13</td>
</tr>
<tr>
<td>The Uni-equation Method</td>
<td>14</td>
</tr>
<tr>
<td>The Multi-equation Method</td>
<td>15</td>
</tr>
<tr>
<td>The Guess-and-Check Method</td>
<td>16</td>
</tr>
<tr>
<td>The Combination Method</td>
<td>17</td>
</tr>
<tr>
<td>Converse Problem</td>
<td>17</td>
</tr>
<tr>
<td>History of the Combination Method</td>
<td>18</td>
</tr>
<tr>
<td>Proposals to Use the Guess-and-Check Procedure</td>
<td>19</td>
</tr>
<tr>
<td>Proposals to Use Multi-equation Methods for Multi-Unknown Problems</td>
<td>21</td>
</tr>
<tr>
<td>The Combination Method</td>
<td>22</td>
</tr>
<tr>
<td>The Experiment to Test the Hypothesis</td>
<td>23</td>
</tr>
<tr>
<td>Scope and Limitations of the Experiment</td>
<td>26</td>
</tr>
<tr>
<td>Outline of the Remainder of this Report</td>
<td>27</td>
</tr>
</tbody>
</table>
II. PLANNING THE EXPERIMENT

General Arrangements for the Experiment ........................................... 28
The Teachers in the Experiment ....................................................... 33
The Evaluation Instrument ............................................................. 35
The Statistical Method of Testing the Main Hypothesis of the Experiment ........ 39
The Adaptation of the Textbook to the Combination Method ...................... 41
The Specific Suggestions ................................................................. 46
Summary ......................................................................................... 52

III. ANALYSIS OF THE DATA

Testing the Homogeneity of the Thirteen Sections on the Tests for Units 1 and 2 .......................................................... 54
The Analysis of Data for the Test of Unit 1 ........................................ 55
The Analysis of Data for the Test of Unit 2 ........................................ 61
Comparison of the Control and Experimental Groups for the Tests of Units 1 and 2 ......................................................... 67
Summary ......................................................................................... 69
Estimating the Reliability of the Evaluation Instrument ......................... 70
Calculating the Estimate of Reliability ............................................... 71
Testing the Main Hypothesis ........................................................... 85
The Analysis of Variance and Covariance ............................................ 90
Relative Weights of Control Variables and the Criterion Variable in the Experiment ......................................................... 94
Factors which May Have Affected the Validity of the Experiment ............ 98
Opinions of the Experimental Teachers .............................................. 96
Summary ......................................................................................... 98

IV. CONCLUSIONS AND RECOMMENDATIONS

Conclusions from the Experiment ....................................................... 100
Recommendations for Further Research ............................................. 103

APPENDIX

A. RECOMMENDATIONS RELATED TO GUESS-AND-CHECK PROCEDURES AND TO THE USE OF MULTI-EQUATIONS IN METHODOLOGICAL LITERATURE ......................................................... 107

B. RECOMMENDATIONS IN ARTICLES IN AMERICAN PERIODICALS AND TEXTBOOKS RELATED TO THE GUESS-AND-CHECK PROCEDURE AND TO THE USE OF MULTI-EQUATIONS ......................................................... 128
### LIST OF TABLES

<table>
<thead>
<tr>
<th>TABLE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Data for the Comparison of Teachers Distribution of Scores on Test for Unit 1</td>
<td>56</td>
</tr>
<tr>
<td>2 Data for the Comparison of Teachers Calculation of the Sums of Squares for the Test of Unit 1</td>
<td>58</td>
</tr>
<tr>
<td>3 Calculation of $L_1$ for the Comparison of Teachers, Based on Test for Unit 1</td>
<td>59</td>
</tr>
<tr>
<td>4 Data for the Comparison of Teachers Distribution of Scores on Test for Unit 2</td>
<td>63</td>
</tr>
<tr>
<td>5 Data for the Comparison of Teachers Calculations of Sums of Squares for the Test of Unit 2</td>
<td>64</td>
</tr>
<tr>
<td>6 Calculations of $L_1$ for the Comparison of Teachers, Data from the Test for Unit 2</td>
<td>65</td>
</tr>
<tr>
<td>7 Data for the Comparison of Teachers Analysis of Variance for the Test of Unit 2</td>
<td>66</td>
</tr>
<tr>
<td>8 Comparison of Control and Experimental Sections on the Results of the Test for Unit 1</td>
<td>68</td>
</tr>
<tr>
<td>9 Comparison of Control and Experimental Sections on the Results of the Test for Unit 2</td>
<td>69</td>
</tr>
<tr>
<td>10 Basic Data for the Estimation of the Reliability of the Test</td>
<td>72</td>
</tr>
<tr>
<td>11 Estimation of Reliability of the Test</td>
<td>73</td>
</tr>
<tr>
<td>12 Summary of Basic Data for the Comparison of the Experimental Method to the Traditional Method</td>
<td>78</td>
</tr>
<tr>
<td>13 Data for the Test for Homogeneity of $Y$</td>
<td>82</td>
</tr>
<tr>
<td>14 Test of $H_1$: $Y_g = Y$</td>
<td>84</td>
</tr>
<tr>
<td>15 Calculations of $t_0$ for Comparison of Means of Control and Experimental Groups on $Y$</td>
<td>84</td>
</tr>
<tr>
<td>16 Summary of Deviations</td>
<td>87</td>
</tr>
<tr>
<td>Page</td>
<td>Section Title</td>
</tr>
<tr>
<td>------</td>
<td>---------------------------------------------------</td>
</tr>
<tr>
<td>17</td>
<td>Summary of Regression Coefficients for Adjustment of Y</td>
</tr>
<tr>
<td>18</td>
<td>Analysis of Variance of Adjusted Y</td>
</tr>
<tr>
<td>19</td>
<td>Data for the Calculation of r's</td>
</tr>
<tr>
<td>20</td>
<td>Correlation Coefficients</td>
</tr>
</tbody>
</table>
CHAPTER I

INTRODUCTION

The Principal Difficulty in Algebraic Problem Solving

The formulation of equations for solving verbal problems is one of the most basic and most difficult topics customarily taught in elementary algebra in the secondary schools of the United States. Because of the difficulties involved in setting up appropriate equations, verbal problems are often the most frustrating part of the course for both pupils and teachers. Thirty years ago Rudman noted that "None of the pitfalls of algebra is dreaded by the 'average' pupil more than the verbal problem."\(^1\)

Brown says that deriving equations is the "critical point" in the solution of verbal problems.

\[\ldots\] Generally, after this step is successfully completed, the remaining steps are reasonably simple. They are essentially mechanical applications of routine processes. The thinking is done in deriving the equation or equations. This may be regarded as the process of translating the problem from the verbal language of the book into the symbolic language of algebra. \[\ldots\]\(^2\)

---


Butler and Wren have also noted that formulating the equations is the student's fundamental difficulty: "The trouble lies in setting up the equations, i.e., in translating the verbal statements into algebraic language."³

Similar observations regarding the difficulty pupils have in deriving equations have been made by Minnick,¹ Young,⁵ Clem and Hendershot,⁶ and others.

Four Approaches to the Formation of Equations for Problems

Perhaps the clearest way to demonstrate the essential features of each of four approaches to formulate equations to use in solving problems algebraically is to work out a problem in each of the four ways. The following problem will be used for the four illustrations:

One of two complementary angles is four times as large as the other. What is the size of each angle?


The Uni-equation Method without any Technique for Discovering Implicit Relationships

After reading the problem and identifying mentally what he is to find the pupil using the uni-equation method writes:

Let \( x \) = the number of degrees in the smaller angle, and \( 4x \) = the number of degrees in the larger angle.

Then \( x + 4x = 90 \).

The essential steps of this method are (1) to choose a letter to represent one of the unknown numbers, (2) to represent the other unknowns in terms of this letter, and, perhaps most difficult of all, (3) to form an equation based on an unused relationship among the numbers of the problem. It will be noted that the pupil is expected to be or to become aware of the mathematical implication of the word complementary and he must be able to use this information to form an equation.

Since this method for forming equations is the one most often used in textbooks and courses throughout the United States and since only one equation is to be formed no matter how many unknowns may be involved, this method will be referred to as the uni-equation method or the traditional uni-equation method.
The Uni-equation Method with a Guess-and-Check Procedure
for Discovering the Implicit Relationship

This method is a variation of the uni-equation method; using it the pupil has the opportunity to explore for himself the implicit relationship and the form which the equation should take. He guesses a numerical value for one of the unknowns and proceeds to check whether this value is correct or not. For the sample problem, suppose that the pupil guesses 15 degrees as the size of the smaller angle. His check may take the following form:

\[ 4 \times 15 = 60, \]
\[ 4 + 60 \neq 90. \]

Since 15 + 60 does not equal 90, the guess is not correct. In this situation the "=" is crossed out.

The pupil now uses a letter, say \( s \), to replace the number 15 in the check. Thus:

\[ s + 4s = 90, \]

which is the same equation formed by means of the previous approach.

In this method there is the difficulty for the pupil of realizing that the number 60 is not given in the problem. Since it has been derived from the guessed number 15, the number 60 must also be replaced by an expression involving the letter \( s \).
The Multi-equation Method without the Guess-and-Check Procedure for Discovering the Implicit Relationship

Since the sample problem asks for the values of two unknowns, two equations may be written with a different letter representing the value of each of the two unknowns. A pupil using this method will write the following without any preliminary exploration:

Let \( S \) = the number of degrees in the smaller angle,

and \( L \) = the number of degrees in the larger angle.

Then \( L = \frac{1}{2} S \),

and \( L + S = 90 \).

Again it is expected that the pupil realize the mathematical implication of the word complementary and that he make use of it in the formation of the second equation.

The Multi-equation Method with the Guess-and-Check Procedure for Discovering the Implicit Relationship

Using the fourth approach the pupil guesses values for all the unknown numbers explicitly asked to be found. For example, suppose that the pupil guesses that one of the angles is 15 degrees and the other 60 degrees. It is helpful for the pupil to mark these guesses distinctively, with squares, circles, and the like, to facilitate keeping account of them while he checks his guesses. Two statements are now necessary for the complete check:

\[
4 \times 15 = 60
\]

and \( 15 + 60 \neq 90 \).
Since \(15 + 60\) does not equal \(90\) the guesses are not correct answers to the problem, and the "\(=\)" has been crossed out where it is not correct.

If the pupil now selects two letters to replace, respectively, the two numbers guessed as answers, two equations will be formed:

\[
L + S = L
\]

and \(L + S = 90\).

This fourth method of approaching equations uses the combination of the guess-and-check procedure and multi-equations, neither of which is used traditionally with the first multi-unknown problems which pupils meet in elementary algebra. Hence this method will be called the combination method.

Using any of the above four methods the pupil must provide the number \(90\) from his own knowledge of the word complementary or by discovering the meaning of the word. He can probably realize the full import of this number in the problem more easily when checking guessed answers, since these are specific numbers for him, than when he must produce directly an equation involving literal symbols for some of the numbers. The direct use of literal symbols in producing an equation involves a much more abstract and formidable problem than is involved when the checked work is available as a definite illustration of the relationship.
Statement of the Problem of this Study

No statistical experiment has been carried out previously to test the relative effectiveness of the uni-equation method and the combination method for deriving the equations for problems. It is the primary purpose of this study to determine the relative effectiveness of these two methods. The hypothesis to be tested may be stated as follows:

Pupils succeed significantly better in writing equations for problems when they have been taught to use the combination method than when they have been taught to use the traditional uni-equation method.

Importance of this Study

Perusal of the literature on problem solving in elementary algebra has failed to reveal that any author has advocated that the program of problem solving in the elementary algebra course be diminished or eliminated, even though many authors express dissatisfaction with the results of the teaching of this topic; for example, Young has said:

It is a common experience of teachers that pupils find great difficulty in translating into equations conditions stated in words. Yet the ability to do this well is one of the most important and valuable results of the study of algebra; the thought power so developed is one of its most useful products, and the pupil should not be allowed to end the study of algebra without a goodly measure of success in such translations.7

7Young, op. cit., p. 309.
Regarding the importance of learning to solve problems Breslich has stated:

The importance of developing power to solve problems is generally accepted. Indeed, some writers consider the verbal problem the most important and significant part of algebra. They would introduce the subject with problems. Power to solve problems would be the ultimate objective of the course. . . . 8

While the present writer does not believe that verbal problems, important as they are, should necessarily be the primary or sole emphasis in the elementary course, he agrees with Davis that "All too often the verbal problems are slighted in the course because of their difficulty or because of lack of time," 9 and with Bennett that "Surely however narrow a view we may take of Elementary Algebra, any slighting of training in casting verbal problems into algebraic form is neglecting our task. . . . 10

The brief sampling of comments by various authors here and at the opening of this chapter establishes (1) that the teaching of problem solving in algebra is extremely important and (2) that the pupil's fundamental difficulty is in framing equations for problems. These facts suggest (1) that teachers and investigators interested in the teaching of mathematics in secondary schools should attempt to improve


the teaching of algebraic solving of verbal problems, and (2) that some effort may well be concentrated on helping pupils become more effective in deriving equations for problems, especially those with implicit relationships and more than one unknown.

**Definitions of Terms Used in this Study**

Some of the terms to be defined in this section are used commonly in the literature and textbooks discussing the solving of problems, but possibly without general agreement as to their meanings. Others may be new to the reader.

**Algebraic Verbal Problem**

The term *algebraic verbal problem* is difficult to define satisfactorily. Many synonyms for the adjective *verbal* are used in the literature and in textbooks, e.g., thought, word, statement, story, concrete, descriptive, reading, prose, and stated. The existence of so many descriptive names for these problems indicates the difficulty of phrasing a succinct and clear definition and the practical necessity to resort to description and example.

**Examples of Algebraic Verbal Problems**

1. Three more than four times a number is seven less than five times the number. What is the number?

2. The difference between two complementary angles is 18 degrees. What is the size of the smaller angle?

3. The perimeter of a rectangle is 40 in. If the length is 7 in. more than the width, how many inches are there in each dimension?
1. The difference between two numbers is 6. If 10 is subtracted from five times the smaller, the result is the same as if two times the larger is subtracted from 16. What is the larger number?

5. The smallest angle of a triangle is 50 degrees less than the largest angle, and the middle-sized angle is half the largest angle. How large is each angle?

6. James is five years older than his brother. In ten years his brother will be 3/4 as old as James is then. How old is each boy now?

7. Robert has 17 coins in his bank; some are dimes and the rest are quarters. If the total value is $2.90, how many of each kind are there?

8. A man inherited $10,000 which he invested, part at 1-1/2% and the rest at 5% annual interest. His income from the investments for the first year was $480. How much money did he invest at each of the rates?

9. For a school play 388 tickets were sold, some at 35 cents and the rest at 50 cents each. There were 62 more 50-cent tickets sold than 35-cent tickets. How many of each kind were sold?

10. A chemist has 30 oz. of 20% alcohol solution which he must change to a 25% solution. How much pure alcohol should he add in order to do this?

In the following discussion, reference to these problems will be made by number.

Hereafter the term problem, unless otherwise qualified, will be used to denote algebraic verbal problems similar to those just listed. The use of the term in this study will also imply (1) that the problem might occur in a typical textbook for ninth-grade classes, and (2) that it is intended that the problem be solved by algebraic means including (a) the use of literal symbols for unknowns, (b) the use of equations containing these symbols, and (c) the use of axioms, substitutions, and other algebraic techniques for solving these equations for their roots, but not by trial and error or graphing.
The Numerical Elements

Two kinds of numerical elements are found in problems, data and unknowns.

Data: — The data are numbers which are presented verbally or numerically to the pupils in the statement of the problem or numbers for which it may be assumed that the pupils know numerical values. For example, in the statement from problem 6, "James is five years older than his brother," the word five (or the numeral 5) provides a datum which is the difference in the ages of the two boys in years.

Unknowns: — The word unknown is used in this study to refer to those numbers or quantities such as ages, sizes of angles, distances, rates, weights, amounts of money, etc., for which numerical values must be derived from other information in the problem; the working out of such numerical values provides the solution of the problem, i.e., the answer to the question posed by the problem. The word unknown will not be used in this study to refer to the literal symbol for an unknown number, as is sometimes done in textbooks; rather it will be used to refer to the generic concept of the unknown number for which the numeral, letter, or verbal name is merely the symbol.

Implicit unknowns: — Sometimes there are other quantities involved in a problem for which values are not specified but for which the problem does not ask that numerical values be derived. If it is necessary, nevertheless, that they be taken into account in deriving the values of the unknowns required to be found, then by definition
they too are unknowns. For example, in problem 10 of the examples above, the number of ounces of alcohol in the original 30 oz. of solution is a number which must be taken into account in order to determine the amount of alcohol to add to the solution. Yet no direct mention of this amount is made in the original statement of the problem. Such an unknown is an implicit unknown in contrast to those unknowns for which values are explicitly required to be derived.

Number of unknowns in a problem: Few verbal problems in algebra have only one unknown. Even when the problem asks explicitly for only one number there are usually others which must be considered and for which values could have been asked readily. Problem 2 concerning the complementary angles whose difference was given, illustrates this point, for the number of degrees in each of the two angles could have been required as easily as only the smaller.

The precise number of unknowns recognized in a problem may vary among pupils because of differences in their analysis of the description provided. Some of the implicit unknowns may become immediately obvious to a sophisticated student when he first reads the problem, while the need for taking such numbers into account may remain unrecognized by the less-experienced pupil. In this study, a problem will be considered to be a multi-unknown problem whenever the pupil recognizes at least two unknowns for which he will guess values in applying the combination method. Certainly he should guess values for all the unknowns asked explicitly to be found. If a problem asks only for one number and the pupil guesses a value only for that
number, then one equation will be produced. If the pupil recognizes an implicit unknown as one which he may treat explicitly by guessing a value for it, then the problem has become for him a multi-unknown problem and his checking of the guesses will lead to a set of two or more equations. Thus, many problems that ask for only one number may be considered multi-unknown problems when pupils deal explicitly with other unknowns implied in the problem.

**Relationships**

Relationships are usually expressed as verbal statements or algebraic formulas. They state or imply that certain numbers in a problem result from arithmetic operations based on other numbers in the problem; such operations include addition, subtraction, multiplication, division, the raising to powers, and the extraction of roots, or combinations of these.

**Explicit relationships:** Whether or not a relationship among the numbers in a problem is explicit or implicit to a pupil is a relative matter; it depends upon such factors as his familiarity with the physical situation described, his previous experiences with other problems involving the same relationship, his understanding of the vocabulary, and the like. The statements that one number is four times another, or that James is five years older than his brother may be considered to be relatively explicit relationships for ninth-grade pupils, since they will be expected to associate readily the arithmetic operations of multiplication and subtraction with these respective situations. Even though the relationships involved may
not always be as obvious to pupils as teachers might desire, such statements in problems will be called explicit relationships.

Implicit relationships: — Examination of problem 5 will reveal quite a different situation. Two statements provide explicit relationships: (1) the smallest angle is 50° less than the largest and (2) the middle-sized angle is half the largest. Since the problem asks for the size of each angle of a triangle, there are three unknowns, and another relationship is needed. The fact that the problem deals with a triangle is the only indication of this necessary relationship. Yet the relationship must be utilized if the pupil is to be successful in solving the problem in a systematic manner. Since pupils differ widely in their experience with, understanding of, and insight into such relationships, they may or may not recognize easily that the three angles should total 180° and that this fact can be used to help derive the number of degrees in each of the three angles.

Similarly, in problem 2, the relationship implied by the word complementary and in problem 3, that implied by the word perimeter must be realized and used by the pupil, i.e., they must become explicit for him. Such relationships will be called implicit relationships.

Mathematical English

In order to help make some relationships more explicit for pupils Lazar has devised a technique which makes use of an intermediate language form, called Mathematical English, to be used by pupils as a transition between the ordinary English of the words of the problem
and symbols of the algebraic equation. For example, the phrase "Two complementary angles" becomes in Mathematical English: (the number of degrees in the first angle) (increased by) (the number of degrees in the second angle) (is) (90 degrees), and "James is five years older than his brother" becomes: (the number of years in James' age) (decreased by) (5 years) (is) (the number of years in the brother's age). These statements can be translated very directly into equations.

This technique has three features: (1) A consistent language form is used for each operation, e.g., "increased by" is always used to indicate addition. (2) Verbal phrases which are to become single symbols are marked off with parentheses. (3) Phrases are placed in the sequence necessary for the sequence of symbols in the equation.

The Uni-equation Method

Most textbooks introduce pupils to problems involving more than one unknown before simultaneous equations are introduced. The only algebraic equations the pupils are prepared to solve, then, are those linear equations with one literal symbol. If pupils with such limited background are to solve multi-unknown problems it is essential that all

---

the conditions of the problem be compressed into one equation. This method of forming an equation for a problem, where the aim is (1) to express all unknown numbers in terms of one literal symbol, and consequently (2) to derive only one equation, will be referred to as the uni-equation method.

The Multi-equation Method

The method involving only one letter and one equation contrasts with the method involving different literal symbols to represent more than one of the unknowns in a multi-unknown problem; as many separate equations as symbols will be required generally in order to solve the problem. Since more than one equation will result, the term multi-equation method will be used to designate the method in which more than one letter is used.

In some other studies the term multiple-equation method has been used for this method; however, in published literature it is usually called the simultaneous-equation method.

The Guess-and-Check Method

The term guess-and-check method (or guess-and-check procedure) will be used to refer to the procedure in which the pupil (1) guesses an answer or answers (if more than one number is to be found) and explores the relationships among the numbers of the problem by checking whether or not these answers are correct and then (2) replaces the respective guessed numbers in the statements of the check by individual letters in order to obtain equations. Its primary purpose is to provide the pattern or patterns for the
equation or equations. Supplementary purposes may include: (1) to help the pupil become aware of implicit relationships, if any, (2) to help him discover implicit unknowns, and (3) to help the teacher determine when individual pupils must become more familiar with the relationships needed for solving a problem.

The Combination Method

As has been illustrated above (see page 4) the guess-and-check procedure may be used either to produce one equation for multi-unknown problems or to produce more than one equation. When the guess-and-check method and the multi-equation method are combined to produce more than one equation the method will be referred to as the combination method. Since in this study the combination method is to be tested experimentally the term experimental method will be used as a synonym for the term combination method.

Converse Problem

A special relationship exists between an algebraic verbal problem and the arithmetic problem created by the pupil when he attempts to check guessed answers. Precisely the same type of problem is created by the pupil whenever he attempts to check answers for the problem, no matter how the answers were obtained. In solving the original problem the pupil derives the answers from the data; in the checking, he derives some of the data from the remaining data and the answers.

The similarity of this situation to that of converse propositions in geometry is striking, for there data and conclusions are interchanged, while in problem solving, data and derived or guessed
answers (which may be regarded as conclusions) are interchanged. Hence, in this study, the relationship existing between the problem as given and the problem created in checking guessed answers will be called converse. Either problem is a converse of the other. In the literature the terms reverse and inverse are sometimes used with this meaning.

History of the Combination Method

Most verbal problems to be solved in ninth-grade algebra have more than one unknown and usually at least one implicit relationship. The traditional uni-equation method is advocated almost exclusively in both the methodological books and the periodical literature. It is advocated for the beginning work even though it necessitates compressing all the conditions into one equation and even though it provides no means by which the pupil can, by his own efforts, become aware of the implicit relationships. Consequently the writers who employ this method often suggest the use of boxes, or provide illustrative solutions, or classify problems by subject matter in order to help the pupil become aware of these implicit relationships. Such devices are artificial, at best, for the pupil can not have them on hand for all problems, nor can he depend on them outside the classroom.

12 For a more complete explanation of converse problems and illustrations of the variety of possible converses see Ratana Tanboonteck, "A Proposed Method of Teaching the Solution of Algebraic Verbal Problems," unpublished Master of Education Field Service Project Report, Columbus, Ohio: The Ohio State University, 1958, pp. 15-16 and 45-75.
Although the reader can sense in the literature on problem solving much uneasiness about the difficulties pupils have in writing equations, surprisingly little constructive advice is offered; most writers seem resigned to using the conventional approach to a difficult topic, even though they are dissatisfied with its results.

Proposals to Use the Guess-and-Check Procedure

One of the most constructive suggestions for revising the uni-equation method was made by Nyberg in 1932 when he proposed a guess-and-check approach for a pupil having difficulty with a coin problem.\(^\text{13}\) The illustration above (see page 4), in which the guess-and-check method is combined with the uni-equation method, is similar to Nyberg's suggestion. In 1955 two additional and similar proposals were made. One by Yeshurun was somewhat different because the checking was organized in tabular form.\(^\text{14}\) The other, in a textbook by Smith and Lankford, is the only such instance in a textbook.\(^\text{15}\) Although Yeshurun illustrates one solution in multi-equation form, in general both these proposals were variations of the uni-equation method. Likewise, in the 1957 edition of the experimental course for


ninth grade at the University of Illinois, the guess-and-check method is combined with the uni-equation method.\textsuperscript{16}

There is no evidence that Nyberg or Smith and Lankford intended that the solving of problems be introduced by the guess-and-check method, for in each instance the context in which the approach is suggested involves a pupil in difficulty when using the traditional uni-equation method. On the other hand, both Yeshurun and the University of Illinois Committee on School Mathematics believe that the study of solving problems should start with this method.

There is no treatment of the guess-and-check procedure in any of the methodology books for high school mathematics which have been examined.

Additional reviews of the proposals to use guess-and-check methods are provided in Appendix B.

Proposals to Use Multi-equation Methods for Multi-Unknown Problems

Only one writer of a book of methodology — Ligda — has advocated that the multi-equation method be introduced early in the course and used for all multi-unknown problems.17 A detailed review of his proposal is provided in Appendix A. On the other hand, seven authors of periodical articles have made similar proposals with varying degrees of persistence and vigor. Nyberg seems to have been the first to make such a proposal,18 but Lazar has been the most persistent advocate of the method. Lazar has dealt with various aspects of the method in a series of articles extending over the period from 1933 to 1951.19 Reviews of these articles are provided in Appendix B.


In 1930 Lazar began to experiment informally with the use of multi-equations for problems earlier in the course in his algebra classes at the Alexander Hamilton High School in New York City. This experimentation was continued later at the Bronx High School of Science. The pupils were left largely to their own devices in obtaining the equations but were to use as many equations as might be convenient.

The articles by Lazar making the first suggestions that multi-unknown problems be solved by the multi-equation method contained no suggestions about how to derive the equations from the words of the problem. The fact that an equation could be derived from each relationship was emphasized, but no help was provided for discovering or exploring relationships.

During the period from 1937 to 1948 Lazar incorporated into his method the technique of Mathematical English as an aid to students in transforming the words of the problem into equations. This device works well as long as all the relationships are explicit for the pupils.

The Combination Method

In searching for a technique which would be more effective than Mathematical English in teaching students to derive multi-equations for multi-unknown problems which contain implicit relationships, Lazar began to teach his students in 1949 to use the guess-and-check procedure. He has continued to advocate the use of a combination of
the guess-and-check and multi-equation methods in his lectures of the
teaching of algebra at the Ohio State University since 1950.

The combination method has not been described in the published
literature. Some of Lazar's students have tried the combination
method in their classrooms and the results have been reported in the
form of theses or projects. Reviews of these are presented in
Appendix C.

The Experiment to Test the Hypothesis

In order to test whether or not students using the combination
method succeed better in deriving equations for problems than students
using the traditional uni-equation method, the writer undertook a
comparative study in 1955.

In Lyons Township High School, LaGrange, Illinois, in the fall of
1955, thirteen sections of algebra classes for pupils of middle
ability were scheduled out of twenty-eight sections of mathematics
for ninth grade. Three teachers were willing to try the combination
method in their six sections. Hence, it was possible to compare the
effectiveness of the pupils in deriving equations by the combination
method with that of the pupils in seven other sections in which the
uni-equation method was taught.

It was arbitrarily decided that measures of I.Q., algebraic
aptitude, and reading level, which were available from tests which
had been administered to the pupils near the end of the eighth grade,
would be valid evidence of ability differences among the pupils.
It was assumed that after such differences were taken into account, the remaining variation in the ability to derive equations could be attributed to differences in method. In searching for a technique which could take such differences into account without the use of matching procedures, the writer found that the statistical method of analysis of variance and covariance was effective. This method was used to analyze the data of the experiment.

The problem of finding or devising an instrument with which to measure the final success of the pupils in deriving equations for problems was an important phase of this study. The details of this procedure will be described later.

In planning the experiment the writer kept the following considerations in mind: (1) The school in which the experiment would be conducted should be disturbed from its regular routines as little as possible. (2) Since it was desirable that the teachers who would be using the combination method should feel that the method had merit and deserved a serious trial, the participation of teachers in teaching the combination method should be completely voluntary. (3) The teachers of the other sections, referred to as control sections, would be expected to teach problem solving by the traditional uni-equation method. As a precaution that the teachers of control sections would not introduce features from the combination method the writer did not discuss the combination method with them. Of course, all teachers of algebra in the school knew that an experiment was in progress, that their pupils would be given a test in deriving equations at the close of the experiment, and that they
might have some part in devising the test. (4) Because the combination method is completely new, the likelihood of finding teachers who were experienced in using it was extremely small. Furthermore, it was unlikely that the teachers who volunteered to teach the experimental sections would find it easy to adjust quickly and completely to the combination method. Hence the writer decided to help as much as possible in planning the reorganization of the course and textbook materials to accommodate the earlier introduction of the multi-equation method. He prepared substitute explanations and worksheet materials and made other suggestions to the teachers for the revision of exercises in the text. These materials were made available to the teachers and pupils in mimeographed form, but the use of them in the classroom was not imposed nor supervised.

In the preparation of these materials, the writer was helped materially by the experiences which Arnett reported in adapting a textbook for the earlier introduction of the multi-equation method, and the experiences which Thompson reported regarding her two-year trial of the combination method.

---

20 Eleanor Arnett, "The Possibility of Adapting a Traditional Textbook in Ninth Year Algebra to the Early Use of the Multiple-Equation Method," unpublished Master of Education Field Service Project Report, Columbus, Ohio: The Ohio State University, 1952.

Scope and Limitations of the Experiment

1. The experiment has been limited to thirteen sections of ninth-grade algebra divided so that six sections were taught the combination method while seven sections were taught the traditional uni-equation method for forming equations for problems.

2. The thirteen sections were all in one school: Lyons Township High School, LaGrange, Illinois.

3. Five other sections of ninth-grade algebra exclusively for superior students were eliminated from the experiment.

4. The experiment was confined to one unit of work on "Equations and Problems" during the last five weeks of the first semester of the school year 1955-56.

5. The experiment was limited to comparing the relative effectiveness of these two groups of pupils in writing equations for problems at the close of the unit. No effort was made to measure the extent to which the methods were retained. No attempt was made to determine the relative effectiveness of the two groups of pupils in carrying problems through to final solution.

6. Only the combination and the traditional uni-equation methods were compared. No effort was made to compare the guess-and-check method combined separately with the uni-equation and multi-equation methods. Neither was the combination method compared with the multi-equation method without the guess-and-check feature.
Outline of the Remainder of this Report

In Chapter II the planning and arrangements for the experiment will be described. The details of the statistical procedures to be used will also be presented.

In Chapter III the data gathered in the experiment will be analyzed.

In Chapter IV, the final chapter, conclusions will be drawn and recommendations made.

In the Appendixes, the literature which deals with the features of the combination method will be reviewed under three headings: A. methodological books, B. periodical articles and algebra textbooks, and C. studies. Other sections of the Appendix will provide examples of the materials planned for classroom use during the experiment, copies of the test used in the experiment, copies of correspondence with the experimental teachers, and some computations used in the analysis of the data.
CHAPTER II

PLANNING THE EXPERIMENT

General Arrangements for the Experiment

In July, 1955, after it had been decided that the combination method should be tried in a statistically controlled situation in order to determine its effectiveness in comparison to the traditional uni-equation method, the writer approached the late George Hawkins, then Head of the Mathematics Department and Administrative Assistant of Lyons Township High School, LaGrange, Illinois, for permission to gather data in that school. Hawkins became interested in the proposed experiment and asked his teaching staff whether any of them were willing to try the experimental method in sections of ninth-year algebra which they would teach in the fall of 1955.

In Lyons Township High School an enrollment of approximately 650 ninth graders was expected, with pupils coming from seventh and eighth grade junior high schools and parochial schools of the Township. There were plans for twenty-eight classes of mathematics for these pupils. On the basis of three tests administered during the latter part of their eighth grade, (1) I.Q., (2) California Algebraic Aptitude Test, and (3) the California Achievement Test, Intermediate Battery, these pupils were classified into three categories. Those with the
highest achievement and aptitude formed five sections of algebra labeled "s," and those with the lowest aptitude formed ten sections of general mathematics. The remaining pupils formed thirteen sections of algebra labeled "r." The assignment of pupils to the sections within any category was random. Since the combination method was designed to aid those pupils who might experience most difficulty with algebraic problem solving, it was decided that the experiment would be confined to the thirteen "r" sections. These classes would average approximately twenty-five pupils each. It was proposed that the combination method be used in six sections taught by three teachers who were willing to use it. These six sections would be called experimental sections while the remaining seven sections would be called control sections. The text in use in all sections was Welchons and Krickenberger, *Algebra, Book One,* in which considerable work with verbal problems is conducted early in the course, starting on page twenty-three.

The writer was concerned that the three teachers willing to try the combination method might have some outstanding attributes as teachers which would tend to bias the experimental results in favor of the sections which they taught. Hawkins stated that a department-wide testing program for uniformly organized units, which had been in operation for several years, had indicated that the teachers were

---

remarkably uniform in the achievement of their pupils. Nevertheless, the writer kept searching for some way to measure this variable.

Much of the remaining part of the summer of 1955 was spent by the writer in planning in detail the statistical procedures which would be followed in the experiment; in revising the materials of the text to be used, which would be submitted to the teachers of the experimental sections for their approval; and in preparing descriptions of the experiment and the experimental procedure for the teachers of the experimental sections.

Later in the summer of 1955 the writer received a letter from Hawkins in which the participation of three teachers — LeRoy Stoldt, Richard Ellis, and Charles Stegmeir — teaching the combination method was assured, and arrangements were completed for the writer to attend sessions of the workshop to be held prior to the opening of regular classes. It was at this time that the general plan of the experiment was to be explained to all the teachers in the mathematics department, and the details of the procedure in the experimental classes were to be worked out with the three teachers.

At the workshop, the writer met first with the teachers of the experimental sections; they informed him that the order of topics in the text was not to be followed and that the work with problems early in the text would be postponed until the fourth unit and would be preceded by three units: (1) Algebraic Language, (2) Signed Numbers, (3) Fundamental Operations. The writer explained to these teachers his concept of the experimental procedure and gave them copies of a written description of the method and copies of the proposed changes.
in the materials of the textbook. These changes were designed to facilitate the use of the combination method with the textbook. (Copies of these materials are found in Appendix D.) Later in the same afternoon he met with all the teachers of the mathematics department and briefly outlined the experiment to them. No detailed description of the method proposed for the experimental classes was ever presented by the writer to any algebra teachers other than those teaching the experimental sections and the head of the department. The teachers of control sections were expected to teach the topic as they always had, to help design the final evaluation instrument, and to administer the final test in their sections. This was the extent of their participation.

During the course of the semester the writer twice visited the classes of each of the three teachers of experimental sections. He also met occasionally with the department in their departmental meetings to answer questions or to seek the advice of the teachers in various details of the experiment. On these occasions he also began compiling from the records in the Deans' offices the data which would be needed on the individual pupils of the thirteen sections.

At first the writer had assumed that the results of the regular unit test would be adequate for evaluating the effectiveness of the combination method as compared to the traditional method. Copies of the tests used for each of the first four units were furnished to the writer; a study of the test for the unit on equations and problem solving indicated that the test would be inadequate for the following reasons: (1) there were two pages of exercises in the translation of
English phrases into algebraic expressions most of which were not complete equations, (2) there were six equations to be solved, (3) there were only four verbal problems to be solved, and (4) there were only four exercises in the entire test dealing primarily with the framing of equations for verbally stated relationships; all these were to be stated with one unknown symbol.

With the help of all the teachers of the department who were teaching courses in algebra for freshmen, an instrument was developed for testing primarily the equation-producing abilities of the students. The format of the test was designed to provide equally well for students using either the multi-equation method or the uni-equation method. It was decided to administer the test to all thirteen sections on the second day of the second semester, prior to the time the students in the control sections would start their study of simultaneous equations. In this manner, each of the teachers was able to make use of the regular unit test for the purposes of grading for the first semester, and the continuity of the local testing program was not disturbed. The writer was present at the school on the day of the special test, and visited each of the sections briefly as the test was given. Test booklets were packaged for each section with the following set of instructions:
SUGGESTIONS FOR THE ADMINISTRATION OF THE TEST

1. Distribute the test booklets by rows, to be passed back in the rows, and kept face down until further instructions.

2. When each has a booklet, instruct the pupils to turn them over and write their names in the space provided, and also to indicate the class hour and the teacher's name.

3. Read the directions printed on the first sheet orally with the students following. Point out how the instructions apply to the sample problem worked out at the bottom of the page.

4. Give opportunity for questions. When there are no further questions, instruct the class to turn the page and to commence work. There are five pages of problems and sixteen problems.

5. Allow 45 minutes of work and then say, "Stop."

6. Collect the booklets immediately.

The period of 55 minutes was adequate for the instructions and the testing time.

The Teachers in the Experiment

Hawkins had assured the writer that the teachers in the department were remarkably uniform as to the achievement of their pupils. The writer was convinced that the three teachers who were to teach the experimental sections had volunteered primarily because of their willingness to participate in experimental projects, and not necessarily because they were outstanding teachers. It seemed necessary, nevertheless, that some statistical measure of the equality of the teachers in the experiment be made. Accordingly, the writer requested from Hawkins records of these teachers as
indicated by the achievement of their pupils on semester or unit tests the previous year. Such records were available and use of them was granted. One teacher of two experimental sections, however, was to be new to the school in the fall of 1955, and no comparable record of his work would be available.

When it was learned that the unit on problem solving would not occur until approximately the last quarter of the fall semester, the writer suggested that the records of the thirteen sections on previous unit tests for the same semester could be used instead. Permission to use these was granted. Test scores on the first two units were made available for this purpose. The method of treating these data statistically was to determine whether or not the sets of test grades of the various sections under these teachers had variances which were homogeneous. If the variances were homogeneous, then it could be assumed that the achievement of the teachers was uniform.

Two hypotheses were to be tested. In symbolic representation they are:

\[ \begin{align*}
H_0 : \sigma_s &= \sigma, \\
H_1 : \bar{X}_s &= \bar{X}.
\end{align*} \]

\( H_0 \) states that the standard deviation of any section does not differ significantly from the standard deviation of the whole group, while \( H_1 \) states that the arithmetic mean of any section does not differ significantly from the mean of the whole group.
To test the hypothesis $H_0$, the $L_1$ test of Welch and Mayer was selected. The acceptance of this hypothesis would mean that the standard deviations would be homogeneous, and that one might then proceed with the analysis of variance by the usual procedure. The data and the analysis are presented in the following chapter.

The Evaluation Instrument

Review of the available commercial standardized tests in algebra failed to reveal any which put as much emphasis on problem solving and the writing of equations as the writer believed was required in the instrument to be used as a final evaluation of the achievement of the pupils in the experiment. At one of the meetings of the mathematics department of the school during the semester of the experiment, the writer proposed some changes in the unit test which he thought would make it more acceptable as the evaluation instrument of the experiment. Some of the teachers of the control sections objected that the changes were too radical, so that it was decided instead to draft a completely new instrument and not to attempt to change the unit test in any way. The evaluation instrument developed for the experiment was not a factor in grading the pupils in their respective sections.

---

Since the features of the experimental method were designed to aid the pupils in producing the necessary equations for algebraic solution of verbal problems and not in solving the equations once they were produced, it was decided to focus the test on equation-producing activities. A draft of sixteen exercises was prepared and submitted to the teachers of the thirteen sections for their consideration. This first draft is presented in Appendix E. The first paragraph of the instructions set forth the purposes of the test:

This is a test of your ability to represent verbal problems by means of algebraic equations. Hence do not waste time by working beyond the stage of obtaining the equations unless you are specifically asked to do so.

The remaining paragraphs of the instructions pointed out the necessity properly to identify any literal symbols used, and pointed out the fact that one or more equations may be made. So that both the uni-equation approach with its one symbol or the multi-equation approach with more than one symbol could be used equally easily, each problem was structured to provide for this possibility. This provision is illustrated in the first problem from the draft:

1. Separate the number 315 into two parts so that one of the parts is twice as large as the other part.

   Let ______ = (and) ______ =

   Equation(s) ___________________________

____________________________
The first draft consisted of this and fifteen other problems set up in a form similar to that illustrated above. They were arranged so as to be of roughly increasing difficulty, e.g., one of the problems near the end was a three-unknown problem concerned with the angles of a triangle. (In the final form, the three-unknown problem was placed last.) Problem 9 was introduced mainly for psychological reasons: since in the other problems the pupils were required only to write the equation or equations and not to complete the solution, it was deemed desirable to require students to complete the solution of one problem, approximately midway through the test. Hence in Problem 9 pupils were required to solve the equations produced for Problem 3. Problem 3 was chosen as one simple enough so that most of the pupils would have been successful in producing correct equations for it.

The teachers' chief criticisms of the draft concerned the inclusion of certain types of problems, such as rate of work (requiring fractions), which some of the control sections would not have studied by the close of the semester. The administration of the instrument could not be postponed beyond the start of the second semester, since some of the control sections would then be ready to study simultaneous equations. The writer had hoped to include some types of problems in the test which were unfamiliar to the pupils, for he felt that the guess-and-check procedure would aid the pupils in deriving correct equations even though they had not encountered similar problems previously. He agreed, however, to replace certain problems with other more familiar types. Those which were changed were No. 10,
rate of work; No. 11 and No. 12, mixtures; No. 14, digit, and No. 16, uniform motion with different rates on two legs of a round trip.

Three other changes were suggested by the teachers and incorporated into the final form of the test: (1) A sample problem illustrated with both the uni- and the multi-equation methods was included in the instructions of the test. (2) Spaces for more literal symbols than needed were supplied in all problems, and the space for the equations was left unstructured. (3) One of the other problems for which equations were produced was to be carried through to solution.

The problems the writer agreed to change were replaced by the following types in the next draft: (1) A sum and difference problem, (2) a consecutive odd number problem, (3) a problem dealing with fractional parts of sums of money, and (4) a problem involving hours of work, but not rate of work. A copy of the second draft is found in Appendix F.

The writer decided to try this form of the test on a class of pupils as a pilot run and asked Miss Helming, a geometry teacher at DeKalb Township High School, DeKalb, Illinois, for permission to administer the test to one of her classes. The test was given to a class in geometry, for all of these pupils had been taught both the uni- and the multi-equation methods. After the test had been administered, the writer informally questioned the pupils who had just taken the test to determine whether any obscure language or unfamiliar vocabulary was involved, and whether they understood the situations in each of the problems. On the pilot run the range of correct responses was from 0 to 12, with nearly normal distribution. As a result of the
pilot run and the comments of the pupils minor changes in wording were made for the final form of the test used in the experiment. The test in its final form constitutes Appendix G.

This instrument required pupils to do precisely what they had been doing as part of their regular class work for several weeks. The types of problems were similar to those which had been included in the text materials which had been covered, and yet they provided considerable variety.

The writer felt that a measure of the reliability of the test should be undertaken. The method based on analysis of variance developed by Hoyt was chosen to estimate its reliability; the data used were the scores of the pupils in the experiment on the items of the test. This analysis is described in the following chapter.

---

cognizance of certain basic differences in the pupils, other than the difference of method in their classes, which might influence their performance in the deriving of equations for problems. The analysis of variance and covariance is such a method. In this method the influence of any number of variables other than the main factor may be determined, but it is usually unnecessary and unduly unwieldy to consider more than two or three such variables. This is approximately the same number of variables that would be used in matching pupils if such a procedure were to be undertaken.

As a result of a testing program in the eighth grade for obtaining information to be used in counseling, guidance, and sectioning in the ninth grade, recent records were available on I.Q., algebraic aptitude, and reading level, both comprehension and vocabulary, for most of the pupils who would be in the thirteen sections of "r" algebra. The writer believed that no three measures of variables which could influence a pupil's ability to derive equations for problems would be more significant than these three; the only more significant factor would be the direct experiences with verbal problems which the pupils might have in their school activities. Hence it was decided to use these three measures as the control variables in the analysis of covariance. After differences in these factors which might affect the student's ability to derive equations had been taken into consideration, it would be assumed that the residual variances were due to the difference of method used in the classroom.
The hypothesis to be tested by means of these data with the analysis of variance and covariance may be stated in null form as follows:

There is no significant difference in the achievement of the pupils in the control sections and the experimental sections in representing verbal problems as equations.

The Adaptation of the Textbook to the Combination Method

The textbook used in all of the classes which participated in the experiment was Algebra, Book One, by Welchons and Krickenberger. The approach to problem solving in this book is the typical uni-equation approach:

Solving Problems by Algebra

Some problems can be solved by either algebra or arithmetic. Some problems can be solved very easily by arithmetic and there is no need of using algebra to solve them. Then there are problems whose solutions by algebra are not difficult, but whose solutions by arithmetic are almost impossible.

Some of the problems which you will soon be asked to solve by algebra are very easy and you may wish to solve them by arithmetic. However, you are asked to solve them by algebra so that you may learn the algebraic method. You can learn the algebraic method more easily if you begin with easy problems.

Now read the following directions for solving problems by algebra. Then study the solutions of the examples on the next page. Notice that there are five steps to any solution.

1. Read the problem and determine what number (or numbers) you are asked to find.
2. Represent the unknown number (or numbers) algebraically.
   a. If you are asked to find only one number, let some letter equal it (represent it).
   b. If you are asked to find more than one number, let some letter equal one of them and then represent each of the other numbers in terms of this letter.

For another statement of the hypothesis, in positive form, see page 7.
3. From the conditions of the problem find two expressions or quantities that are equal. Then connect these two equal expressions by an equals sign, forming the equation.

4. Solve this equation for the unknown letter.
   If you are asked to find more than one number, do this from step 2b above.

5. Check by seeing that your answer (or answers) satisfies all the conditions of the problem.\(^5\)

In the illustrations which follow this explanation and list of directions, two derivations of the equation for the third example are presented:

Dick's father is three times as old as Dick, and the sum of their ages is 48 years. How old is each?

**Solution 1.**

Let \(d\) = the number of years in Dick's age

Then \(3d\) = the number of years in his father's age

Then \(d + 3d = 48\).

**Solution 2.**

Let \(f\) = the number of years in Dick's father's age

Then \((1/3)f\) = the number of years in Dick's age

Then \(f + (1/3)f = 48\).\(^6\)

Although these are only two of the several possible derivations of the equation by the uni-equation method, the authors at least recognize and attempt to make explicit for pupils the fact that there are several possibilities for compressing two or more relationships into one

---


\(^6\)Ibid., p. 24.
equation. In some textbooks the existence of more than one possibility is never suggested to the pupils.

After the pupils have seen the process for solving each of the equations they are asked which they think is the easier. It seems to be expected that the pupils will want to avoid the one involving fractions if at all possible.

In the first list of problems after the introduction and sample solutions quoted above there is additional advice in the form of hints following some of the problems. Problem 4 is an example:

One number is 5 times as large as another and their sum is 54. What are the numbers? (Why will you choose to let \( x \) represent the smaller number?)

A device like this seems also to be a means of helping pupils to avoid fractions. Such indirect advice, although not universally used, occurs quite frequently in standard textbooks.

In spite of the traditional uni-equation approach which it employs, this book has several features which help to make it more easily adaptable to the combination method than it might have been otherwise:

1. It makes a start in algebra with a brief consideration of formulas in which several letters commonly occur.

2. It provides some translation activities from algebra to English, and a greater amount of the reverse type, the latter mostly in the form of phrases to be translated into expressions.

\[7\text{Ibid., p. 25.}\]
3. It provides a variety of evaluation activities with both of the above features.

4. It provides a formal list of conditional equations in the form $ax = b$, and $ax + (or -) bx = c$ which are to be solved by combining terms and the use of the division axiom.

5. The first list of verbal problems occurs on page 25, and the fourth problem in the list is already a multi-unknown problem.

6. These first multi-unknown problems are of the type for which sets of equations may be derived in which one equation will state that one of the unknowns is a multiple of the other, and the other equation usually will provide some information about sum or difference. These are especially well adapted to the substitution method of solving multi-unknown equations for roots.

7. In the examples worked out preceding the first list of verbal problems, one two-unknown problem is solved two ways, with the literal symbol representing each of the unknowns in turn. This is an acknowledgement, at least, of alternate possibilities of procedure. Mnemonic symbols were used in this example, but $x$ was used in the other examples.

There are some other features of the book, however, which do not lend themselves quite so readily to teaching the combination method for deriving equations:

1. The translation activities are nearly all based on the use of expressions and phrases rather than on complete sentences and equations.
2. Only one table of values is set up to be filled in in these early experiences; it is based on the relationship $p = 6m$, where $m$ is the length of a side of a regular hexagon. In this table there is only one opportunity to solve for the value of $m$ from a given value of $p$, and no opportunity for the pupils to fill in corresponding values of both variables. This does not correspond to the more complete freedom of guesses expected in the combination method.

3. The framing of the equations for verbal problems is not based on the complete relationships in the problems, for the following advice is given: "From the conditions of the problem find two expressions or quantities that are equal. Then connect these two equal expressions by an equals sign, forming the equation."  

4. Several of the first verbal problems are followed by hints as to what literal symbols to use and what they are to represent. These hints suggest "$x$" exclusively. Since these problems have two or more unknowns in which some are multiples of the others, no doubt this hint is given to prevent the use of fractional coefficients.

5. After a brief introduction to the solving of verbal problems a list of steps is presented fully developed, followed by four examples solved by means of these steps. There is no opportunity for the pupils to experiment with problems and develop these steps for themselves.

---

8 Ibid., p. 23.
After the arrangements for the use of the experimental method in six classes by three teachers had been completed with the head of the mathematics department of Lyons Township High School, but prior to the time the writer had met the particular teachers who were to use the method, he outlined some suggestions for changes which would make the materials of the text correspond more nearly with the features of the combination method. These suggestions were presented to the teachers of the experimental sections at the writer's first meeting with them. The writer with the advice of his major adviser had assumed that only the minimum changes deemed necessary should be suggested. The complete written suggestions presented to the three teachers have been included as Appendix A.

The Specific Suggestions

In the first formal list of exercises in the text, the first three exercises deal with complete formulas: \( A = bh \), \( A = \frac{1}{2} bh \), and \( H = ht \); the latter is used for finding the height of a stack of four books each \( t \) millimeters thick. The first two required among other things that the formulas be stated as verbal rules, instances of "translation" from algebra to English. So far complete relationships are the basis for the pupil's thinking. In the next three exercises an entirely different approach is made. Exercise 4 states:

"If one sack of sugar weighs \( p \) pounds, what will eight of these sacks weigh?"

Obviously the answer \( 8p \) will suffice; this is an expression, a phrase only, not a complete relationship. To keep the emphasis on complete relationships, the writer suggested that this question be reworded as
follows:

One sack of sugar weighs $p$ pounds. Write the formula which will tell how to find the weight $w$ of eight of these sacks of sugar.

Similarly exercise 5 asked:

A house is twice as high as a garage. If $h$ stands for the height of the garage, how many $h$ will stand for the height of the house?

and similarly, the writer suggested that it be changed to read:

A house is twice as tall as a garage. If $h$ stands for the height of the house and $g$ stands for the height of the garage, write the formula which states the relationship of $h$ and $g$.

The writer suggested similar changes for questions 6, 11, and 14, and arranged to have a set of exercises mimeographed to replace this set in the textbook if the teachers wished to have such a substitution.

On page 9 there are sets of two kinds of exercises. First there are 16 verbal phrases such as "the sum of $h$ and $k" to be translated into algebraic expressions. All of them involve addition or subtraction. The exercises from 17 to 32 were algebraic expressions involving the addition and subtraction of various combinations of numerals and four different letters for which values were given in the instructions. The expressions contained two or three terms; among the latter a maximum of two terms were preceded by minus signs.

Since the guess-and-check idea, incorporated in the combination method, assumes that arithmetic is often a clue to the algebra and that one may make an inductive approach to algebra, the writer suggested (1) that the order of consideration of these two sets of exercises be reversed; (2) that instead of expressions each exercise
be made into a complete relationship simply by equating the expression to \( m, p \) or \( r \) in different cases; and (3) that the exercises were to be more varied than they had been. Thus the general instructions and first few exercises, which had read:

In exercises 17 - 32 the additions and subtractions are only indicated. Now perform these operations when \( a = 6, b = 2, x = 4, \) and \( k = 1/2. \)

17. \( a + b \)  21. \( a - x \)  25. \( x - k \)  29. \( a + b + x \)

now read as follows:

In the first 16 exercises below, certain additions and subtractions are indicated for finding the values of \( m \) or \( p \) or \( r \). Find the values of \( m \) or \( p \) or \( r \), as the case may be, when \( a = 6, b = 2, x = 4, \) and \( k = 1/2. \)

1. \( m = b + x \)  5. \( m = x + k \)  9. \( p = x - k \)  13. \( r = x + b - a. \)

For the other kind of exercise (now exercises 17 - 32), the writer suggested expanded instructions and greater variety among the examples:

As you know, there are many ways to express in words the ideas of addition and subtraction. For example, the formula in exercise 8 above might be stated in words as follows: "\( r \) is found by subtracting \( k \) from \( b, \)" or "\( r \) equals \( b \) diminished by \( k, \)" or "\( r \) equals \( k \) less than \( b. \)" There are many other ways we might state this same idea in words. The following exercises are planned to help you review these different ways of expressing addition and subtraction. Suppose that \( r \) is the number that results from each of the following additions or subtractions. Write the formula that represents the complete situation. For example, "\( x \) added to \( h \)" would then be completely represented as "\( r = h + x. \)" If you are not sure try letting the letters be certain numbers you choose.

17. \( x \) plus \( c. \)  23. 10 less than \( y. \)  29. \( k \) increased by \( t. \)

A complete list of the exercises recommended is presented in Appendix D.
Similarly it was recommended that all the exercises on pages 19 and 20 be changed so as to require complete relationships in the form of equations. The details of all the changes can be seen in Appendix D; only one example will be described here, exercise 6, page 20:

Original form:

Charles is 6 years older than Henry. If x stands for the number of years in Henry's age, how can you represent Charles's age?

Recommended form:

If $H$ represents Henry's age and $C$ represents Charles' age, express algebraically the fact that Charles is 6 years older than Henry.

The latter form of exercise, the writer believed, is a much better preparation for the solving of verbal problems than is the former. This is especially true of approaches using the multi-equation method but also applies to those using the uni-equation method.

The textbook is organized so that verbal problems are introduced immediately following a brief consideration of equations solvable generally by the division axiom, and the first list of problems includes some involving multi-unknowns. Hence, if the combination method is to be used, some formal consideration of sets of equations must be introduced just prior to taking up the problems. The writer prepared a few paragraphs of explanation and introduction, a few sample exercises in solving sets of equations for their roots, and a worksheet of sixteen sets of equations to be solved by the pupils. Four of these are sets of three equations. It was felt that this material in the hands of the pupils would be more satisfactory than using a similar type of explanation and exercises farther along in the text.
All sixteen exercises are of a type similar to those which would be produced in the verbal problems of the first chapter in the textbook by the multi-equation method, and all lend themselves easily to the substitution method of eliminating one of the unknowns. The material in complete form as it was prepared for the use of the pupils is found in Appendix D.

On page 23 of the text there is a brief introduction to solving problems by algebra; this has been quoted above on page 41. The writer prepared (1) a substitute introduction, (2) a set of examples worked out by the combination method, and (3) a set of steps based on the examples worked out by the combination method. It was planned that such materials would also be prepared for distribution to the pupils. These materials are given completely in Appendix D.

The verbal problems presented in the text are generally well adapted to the combination method, except that hints are included with some of them. These hints assumed the use of the uni-equation method with the formation of the equation from two expressions for the same quantity. Hence it was suggested that the same or similar problems without the hints could be prepared in mimeographed form for the use of the pupils, if the teachers agreed that this procedure was desirable.

The writer suggested another worksheet to be used after the pupils had had some experience in solving verbal problems; this worksheet required the students to draft verbal problems to fit the sets of equations which had been used previously in learning to solve sets of equations for their roots. In the instructions for
the exercises an example was worked out for the pupils. This type of experience was suggested by Thompson, and was used by her pupils to aid in the understanding of the relationships involved in verbal problems (see Appendix C). The worksheet is presented in Appendix D.

The writer provided the teachers of the experimental classes with a list of additional suggestions in connection with other verbal problems scattered throughout the text to approximately page 200. These suggestions are also listed in Appendix D.

At the first conference with the teachers of the experimental sections the writer explained the purpose of the experiment and illustrated his concept of the combination method. Each teacher and the head of the department were given copies of the statement of the general plan of this study which had been prepared for the writer's advisory committee, except that the details of the statistical analysis were omitted, and also copies of the materials included in Appendix D as described above. The writer offered to provide any or all of these materials in mimeographed form for the use of the classes. Since work with verbal problems was to be delayed until the final quarter of the semester, there was no need to decide immediately what materials would be used. Later in the semester, after the teachers had studied the materials which had been furnished them and had developed their plans for carrying out the experiment in their classrooms, it was decided that materials which made only slight changes in the text materials would not be used but that the additional materials would be provided. Accordingly, the writer provided two packets of materials for each member of the experimental
classes. All of the materials in these packets have been included in Appendix D. In the first packet there were (1) the introduction to equations with two or more letters and (2) the introduction to solving problems by algebra. In the second packet there were four worksheet plans, including:

1. A complete substitute sheet for the oral exercises on page 20 of the text.

2. A sheet of sixteen sets of equations in two or more unknowns to be solved for the roots.

3. Instructions for an assignment of selected problems to be solved by the method outlined in the explanatory materials.

4. Instructions for a worksheet in which the pupils were to phrase verbal problems for the sets of equations provided in the second worksheet.

These packets were delivered to the school prior to the start of unit four.

**Summary**

Permission was obtained to have three teachers use the combination method of teaching verbal problems in six "r" sections of ninth-year algebra of Lyons Township High School, LaGrange, Illinois, in a unit on equations and problem solving during the latter part of the first semester, 1955-56. In the remaining seven sections of "r" algebra taught by six different teachers the usual method of teaching verbal problems which these teachers had used formerly would be followed. At the close of the unit students were to be tested to determine
whether their ability to represent verbal problems by means of
equations was significantly influenced by the difference in methods
by which they were taught to approach algebraic solutions to verbal
problems. Account would be taken of differences in the pupils' I.Q., algebraic aptitude, and reading level, as determined in tests
given in the latter part of their eighth grade.

A comparative analysis of the scores made by the pupils in the
experimental and in the control sections on tests of other units of
work would be undertaken to determine whether or not the teachers
of the two groups of sections differed significantly in the
achievement of their pupils.

On the basis of the results of administering the test to the
pupils involved in the experiment, an estimate of the reliability
of the test would be made by a method involving analysis of variance.

Materials to be used for the experimental sections were prepared
to replace and revise some of the exercises in the textbook. These
revisions and replacements were necessary (1) so that some experience
with simultaneous equations could be introduced before verbal
problems were taken up and (2) so that translation experience would
deal with complete sentences and equations.
CHAPTER III

ANALYSIS OF THE DATA

In this chapter the data of the experiment will be analyzed. As indicated in the previous chapter there are three main sections to the analysis: (1) the determination of whether or not the variances of the sections taught by the teachers of the experiment are homogeneous in some algebraic topics other than the teaching of verbal problems, (2) the determination of the reliability of the evaluation instrument, and (3) the testing of the main hypothesis of the experiment, i.e., whether or not there is residual variance in the ability of the pupils in the experimental sections to represent verbal problems by means of equations which may be attributed significantly to the method of teaching verbal problem solving.

Testing the Homogeneity of the Thirteen Sections on the Tests for Units 1 and 2

Data regarding two unit tests for each of the thirteen sections were made available to the writer for the purpose of determining whether or not the teachers in the experimental and control sections were achieving equally. These were for Unit 1, on algebraic language, and Unit 2, on signed numbers. Some teachers submitted the scores in tabulated form, as used for grading purposes. Hence the intervals were not completely uniform.
The Analysis of Data for the Test of Unit 1

In the following analysis, each of the nine teachers participating in the experiment as a control or as an experiment teacher is identified by a letter. If the teacher taught two sections the sections will be identified by the teacher’s symbol with the addition of a subscript 1 or 2. Otherwise the teacher’s symbol will also identify the section. In Table 1, the scores made by the pupils of each of the teachers on the test for Unit 1 have been indicated, classified according to the marks indicated in column 1. The scores on the test ranged from approximately 27 to 60. It was known that the top scores were exactly 60; hence the classification was not carried uniformly into the top class, since it was felt that this would unduly distort the data. It is evident from the frequencies in the table that teachers B, G, H, and I had two sections each. Teachers G, H, and I were teachers of experimental sections, while the remaining all taught control sections.

In Table 2, the sums of scores for the pupils under each teacher have been indicated in column 3, and the resulting mean for the pupils of each teacher has been indicated in column 4. In column 5 the sums of squares of the scores have been indicated as an aid in calculation of the sums of squares about means which have been indicated in column 6.

The basic hypothesis to be tested by means of these data may be stated in null form as follows: There is no significant difference in the achievement of the pupils under each of the nine teachers on the test for Unit 1. Symbolically this may be stated $H_0: \bar{X}_g = \bar{X}$,
Table 1

Data for the Comparison of Teachers

Distribution of Scores on Test for Unit 1

<table>
<thead>
<tr>
<th>Class Mark*</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>57</td>
<td>3</td>
<td>6</td>
<td>6</td>
<td>3</td>
<td>11</td>
<td>3</td>
<td>14</td>
<td>3</td>
<td>12</td>
</tr>
<tr>
<td>52</td>
<td>10</td>
<td>15</td>
<td>10</td>
<td>4</td>
<td>9</td>
<td>11</td>
<td>18</td>
<td>12</td>
<td>15</td>
</tr>
<tr>
<td>47</td>
<td>4</td>
<td>14</td>
<td>7</td>
<td>3</td>
<td>3</td>
<td>9</td>
<td>9</td>
<td>18</td>
<td>11</td>
</tr>
<tr>
<td>42</td>
<td>5</td>
<td>13</td>
<td>5</td>
<td>8</td>
<td>3</td>
<td>3</td>
<td>14</td>
<td>8</td>
<td>10</td>
</tr>
<tr>
<td>37</td>
<td>3</td>
<td>8</td>
<td>6</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>7</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td>7</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>27</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>27</td>
<td>60</td>
<td>28</td>
<td>29</td>
<td>28</td>
<td>28</td>
<td>57</td>
<td>55</td>
<td>60</td>
</tr>
</tbody>
</table>

*The classification is not completely uniform.*
which simply means that the mean of each group of pupils under one teacher does not differ significantly from the mean of the whole group of 372 pupils. The method of analysis of variance was selected to test this hypothesis. This method assumes equal variability among the sections. Hence it is necessary to test the hypothesis

\( H_0: \sigma = \sigma' \). The Nayer-Welch test for \( L_1 \) was used to test this hypothesis, and the formula for the calculations is given at the foot of Table 3. The basic information in Table 3 is the number of students under each teacher and the sums of squares about means, which are taken from Table 2, and are indicated in columns 2 and 5 respectively. The remaining columns are the results of calculations which are necessary in evaluating the formula for \( L_1 \). Prior to entering a table to determine the probability of a particular \( L_1 \) it is necessary to determine the number of degrees of freedom in the situation. This is calculated as the harmonic mean of the frequencies of the groups. Hence, \( \text{d.f.} = \frac{k}{\sum_{s} \frac{1}{n_s}} \), where \( k \) is the number of groups, and \( n_s \) is the number of pupils in each group.

From the data in Table 3, \( L_1 \) is determined to be 0.8335, and the degrees of freedom are 36.31. Entering the \( L_1 \) table with \( k = 9 \) and with 36.31 degrees of freedom, it is determined that the probability of variability as great as that occurring among the groups on the test on Unit 1 is less than 0.01. Hence the hypothesis \( H_0 \) must be rejected. It may not be concluded from these data that the teachers achieved equally in Unit 1.
Table 2
Data for the Comparison of Teachers
Calculation of the Sums of Squares for the Test of Unit 1

<table>
<thead>
<tr>
<th>Teacher</th>
<th>n_s</th>
<th>ΣX</th>
<th>X</th>
<th>ΣX²</th>
<th>Σs² *</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>27</td>
<td>1264</td>
<td>46.8</td>
<td>60,598</td>
<td>6162.071</td>
</tr>
<tr>
<td>B</td>
<td>60</td>
<td>2750</td>
<td>45.8</td>
<td>128,960</td>
<td>2918.333</td>
</tr>
<tr>
<td>C</td>
<td>28</td>
<td>1401</td>
<td>50.0</td>
<td>70,817</td>
<td>716.961</td>
</tr>
<tr>
<td>D</td>
<td>29</td>
<td>1233</td>
<td>42.5</td>
<td>54,341</td>
<td>1917.241</td>
</tr>
<tr>
<td>E</td>
<td>28</td>
<td>1457</td>
<td>52.0</td>
<td>76,963</td>
<td>1146.961</td>
</tr>
<tr>
<td>F</td>
<td>28</td>
<td>1366</td>
<td>48.8</td>
<td>67,402</td>
<td>760.714</td>
</tr>
<tr>
<td>G</td>
<td>57</td>
<td>2819</td>
<td>49.5</td>
<td>141,473</td>
<td>2056.141</td>
</tr>
<tr>
<td>H</td>
<td>55</td>
<td>2460</td>
<td>44.7</td>
<td>112,820</td>
<td>2790.909</td>
</tr>
<tr>
<td>I</td>
<td>60</td>
<td>2866</td>
<td>47.8</td>
<td>140,652</td>
<td>3752.733</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>372</td>
<td>17616</td>
<td>47.4</td>
<td>854,026</td>
<td>22184.074</td>
</tr>
</tbody>
</table>

* Σs² = ΣX² - \( \frac{(\sum X)^2}{n_s} \)
### Table 3
Calculation of $L_1$ for the Comparison of Teachers, Based on Test for Unit 1

<table>
<thead>
<tr>
<th>Teacher</th>
<th>$n_s$</th>
<th>$\log n_s$</th>
<th>$n_s \log n_s$</th>
<th>$\Theta^i_s$</th>
<th>$\log \Theta^i_s$</th>
<th>$n_s \log \Theta^i_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>27</td>
<td>$1.43136$</td>
<td>$38.64672$</td>
<td>$64.210741$</td>
<td>$3.80782$</td>
<td>$102.81111$</td>
</tr>
<tr>
<td>B</td>
<td>60</td>
<td>$1.77814$</td>
<td>$106.68900$</td>
<td>$2918.3334$</td>
<td>$3.46513$</td>
<td>$207.90780$</td>
</tr>
<tr>
<td>C</td>
<td>28</td>
<td>$1.64716$</td>
<td>$40.52048$</td>
<td>$716.9514$</td>
<td>$2.85550$</td>
<td>$79.95100$</td>
</tr>
<tr>
<td>D</td>
<td>29</td>
<td>$1.646240$</td>
<td>$42.40960$</td>
<td>$1917.2414$</td>
<td>$3.28267$</td>
<td>$95.19713$</td>
</tr>
<tr>
<td>E</td>
<td>28</td>
<td>$1.64716$</td>
<td>$40.52048$</td>
<td>$1116.9563$</td>
<td>$3.05955$</td>
<td>$85.66740$</td>
</tr>
<tr>
<td>F</td>
<td>28</td>
<td>$1.64716$</td>
<td>$40.52048$</td>
<td>$760.7143$</td>
<td>$2.88122$</td>
<td>$80.67416$</td>
</tr>
<tr>
<td>G</td>
<td>57</td>
<td>$1.75587$</td>
<td>$100.08159$</td>
<td>$2056.1404$</td>
<td>$3.31305$</td>
<td>$188.84385$</td>
</tr>
<tr>
<td>H</td>
<td>55</td>
<td>$1.74036$</td>
<td>$95.71980$</td>
<td>$2790.9091$</td>
<td>$3.44561$</td>
<td>$189.50855$</td>
</tr>
<tr>
<td>I</td>
<td>60</td>
<td>$1.77815$</td>
<td>$106.68900$</td>
<td>$3752.7334$</td>
<td>$3.57435$</td>
<td>$214.61600$</td>
</tr>
</tbody>
</table>

$N = 372$  
$log N = 2.57054$  
$log \Theta^i_s = 4.35188$

$log L_1 = log N - \frac{1}{N} \sum_{s} n_s \log n_s + \frac{1}{N} \sum_{s} n_s \log \Theta^i_s - log (\sum \Theta^i_s)$

$= 2.57054 - \frac{611.80015}{372} + \frac{1215.02533}{372} - 4.35188 = 9.92088 - 10$

$L_1 = .8335$,  
$k = 9$,  
$d.f. = 36.34$.  

5
Two further attempts to examine the situation were made.

1. While theoretically $H_0$ must be fulfilled in order to test $H_1$, many authorities now recognize that strict adherence to this principle is unnecessary. For example, one group of authors states:

From theoretical considerations the foregoing assumptions [including the assumption of homogeneous variances] must be satisfied before the application of the analysis of variance is appropriate. However, it is becoming more apparent that the analysis of variance technique is sufficiently satisfactory even when there is considerable departure from the strict fulfillment of the assumptions.\(^1\)

On the basis of this statement, the writer ignored the lack of fulfillment of $H_0$ and proceeded with the analysis of variance. However, hypothesis $H_1$ still had to be rejected.

2. Since there was some evidence that one of the control sections ought to be eliminated from the statistical considerations of this study, the writer decided to repeat the text of $H_0$ without the data pertaining to teacher D, in whose class the results of the test for Unit 1 produced the lowest mean. (The section under this teacher also had the lowest mean on the main criterion and in reading level, but not in I.Q. nor aptitude. In fact it was third highest in aptitude.) The omission of these data did not improve the success of the test, and $H_0$ was still rejected.

---

The scores on the test for Unit 2 made by the various sections under each of the nine teachers have been summarized in Table 4. The range of the scores was from approximately 50 to 100. The class marks indicated in Table 4 have been reduced uniformly by 50 1/2 from the actual classification used. This in no way affects the results, since all calculations are in terms of deviations from means; the procedure was used solely to simplify calculations.

The analysis of the data has been made in the same manner as that previously described for the Test of Unit 1. In Table 5 the sums of scores and sums of squares have been used to calculate the sums of squares about means for the various groups. These are indicated in column 6. The mean scores are indicated in column 4. The testing of $H_0: \sigma_s^2 = \sigma^2$ by means of the calculation of $L_1$ is indicated in Table 6. $L_1 = 9.9655$. The degrees of freedom, as determined on the basis of the harmonic mean of the frequencies of the groups, previously described, is 35.73. Entering the appropriate table, it is found that the calculated value of $L_1$ is larger than the value for the five per cent limit; consequently, the hypothesis of equal variance may be accepted.

Since $H_0$ has been accepted, the analysis of variance may be continued. The calculations involved are indicated in Table 7. The sum of squares for total is calculated from information obtained from
the totals in columns 2, 3, and 4 of Table 5, by means of the formula:

\[ S.S. \text{ total} = \sum x^2 - \frac{(\sum x)^2}{N}. \]

The sums of squares for within is the total of column 5 of Table 6, and the sums of squares between sections is the difference of the other two sums of squares. The number of degrees of freedom for total is one less than the number of students involved, and the number of degrees of freedom for between sections is one less than the number of sections. The number of degrees of freedom for within sections is the difference of the other two.

The mean squares are the results of dividing the sums of squares by their respective degrees of freedom. F is the ratio of these mean squares. The tables of F distribution are entered with the appropriate degrees of freedom to determine whether the ratio of the mean squares indicates significant differences. In this case \( F = 4.8 \), and with 8 and 356 degrees of freedom respectively, it must be concluded that the differences are significant. Hence the hypothesis of homogeneity based on the results of the test on Unit 2 must also be rejected.

On the basis of the evidence obtained from the analysis of the results of the tests for Units 1 and 2, it seems to be a valid conclusion that these teachers do not achieve uniformly in their various sections. This is contrary to the opinion that Hawkins expressed when the writer first approached him about the possibility of conducting this experiment in his department. Of course the
Table 4

Data for the Comparison of Teachers

Distribution of Scores on Test for Unit 2

<table>
<thead>
<tr>
<th>Class Mark* $X_i$</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td>46</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>38</td>
<td>6</td>
<td>9</td>
<td>7</td>
<td>4</td>
<td>12</td>
<td>10</td>
<td>17</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>30</td>
<td>9</td>
<td>22</td>
<td>6</td>
<td>9</td>
<td>8</td>
<td>5</td>
<td>17</td>
<td>11</td>
<td>13</td>
</tr>
<tr>
<td>22</td>
<td>4</td>
<td>14</td>
<td>8</td>
<td>10</td>
<td>5</td>
<td>4</td>
<td>8</td>
<td>12</td>
<td>17</td>
</tr>
<tr>
<td>14</td>
<td>6</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>2</td>
<td>3</td>
<td>6</td>
<td>14</td>
<td>14</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Totals</td>
<td>26</td>
<td>57</td>
<td>28</td>
<td>29</td>
<td>28</td>
<td>27</td>
<td>57</td>
<td>55</td>
<td>58</td>
</tr>
<tr>
<td>Grand Total</td>
<td>365</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*The class marks listed have been obtained from the true class mark by subtracting 50 1/2. This reduction is permitted since all calculations based on class marks are in terms of deviations from means.*
Table 5

Data for the Comparison of Teachers

Calculations of Sums of Squares for the Test of Unit 2

<table>
<thead>
<tr>
<th>Teacher</th>
<th>n_s</th>
<th>ΣX</th>
<th>X</th>
<th>ΣX²</th>
<th>ΣX²</th>
<th>θ_i</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>26</td>
<td>716</td>
<td>27.5</td>
<td>21992</td>
<td>2274.63154</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>57</td>
<td>1494</td>
<td>26.2</td>
<td>14804</td>
<td>5615.97370</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>28</td>
<td>816</td>
<td>29.1</td>
<td>26512</td>
<td>2731.42858</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>29</td>
<td>742</td>
<td>25.6</td>
<td>21192</td>
<td>2507.03499</td>
<td></td>
</tr>
<tr>
<td>E</td>
<td>28</td>
<td>944</td>
<td>33.7</td>
<td>33296</td>
<td>1169.71290</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>27</td>
<td>770</td>
<td>28.5</td>
<td>25804</td>
<td>3814.71075</td>
<td></td>
</tr>
<tr>
<td>G</td>
<td>57</td>
<td>1622</td>
<td>28.5</td>
<td>53380</td>
<td>7224.11040</td>
<td></td>
</tr>
<tr>
<td>H</td>
<td>55</td>
<td>1186</td>
<td>21.6</td>
<td>33564</td>
<td>7989.52730</td>
<td></td>
</tr>
<tr>
<td>I</td>
<td>58</td>
<td>1276</td>
<td>22.0</td>
<td>35752</td>
<td>7680.00000</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>365</td>
<td>9566</td>
<td>26.2</td>
<td>296596</td>
<td>41366.53105</td>
<td></td>
</tr>
</tbody>
</table>

* θ_i = ΣX² - (ΣX)^2 / n_s
Table 6
Calculations of $I_1$ for the Comparison of Teachers, Data from the Test for Unit 2

<table>
<thead>
<tr>
<th>Teacher</th>
<th>$n_s$</th>
<th>$\log n_s$</th>
<th>$n_s \log n_s$</th>
<th>$\sum q_i$</th>
<th>$\log \sum q_i$</th>
<th>$n_s \log \sum q_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>26</td>
<td>1.41497</td>
<td>36.78922</td>
<td>2274.615</td>
<td>3.35688</td>
<td>87.27888</td>
</tr>
<tr>
<td>B</td>
<td>57</td>
<td>1.75587</td>
<td>100.08459</td>
<td>5645.737</td>
<td>3.75176</td>
<td>213.85032</td>
</tr>
<tr>
<td>C</td>
<td>28</td>
<td>1.44416</td>
<td>40.52018</td>
<td>2731.4286</td>
<td>3.3638</td>
<td>96.21864</td>
</tr>
<tr>
<td>D</td>
<td>29</td>
<td>1.46240</td>
<td>42.40960</td>
<td>2507.0345</td>
<td>3.39915</td>
<td>96.57335</td>
</tr>
<tr>
<td>E</td>
<td>28</td>
<td>1.44416</td>
<td>40.52018</td>
<td>1169.7143</td>
<td>3.16723</td>
<td>88.68214</td>
</tr>
<tr>
<td>F</td>
<td>27</td>
<td>1.43436</td>
<td>38.614672</td>
<td>3841.7108</td>
<td>3.48187</td>
<td>96.79249</td>
</tr>
<tr>
<td>G</td>
<td>57</td>
<td>1.75587</td>
<td>100.08459</td>
<td>7224.1104</td>
<td>3.85799</td>
<td>219.95303</td>
</tr>
<tr>
<td>H</td>
<td>55</td>
<td>1.74036</td>
<td>95.71980</td>
<td>7989.5273</td>
<td>3.90252</td>
<td>214.63860</td>
</tr>
<tr>
<td>I</td>
<td>58</td>
<td>1.76343</td>
<td>102.27894</td>
<td>7680.0000</td>
<td>2.88536</td>
<td>225.35088</td>
</tr>
</tbody>
</table>

$N = 365$

$\log N = 5.6229$

$\log \sum q_i = 4.61665$

$L_1 = 0.9655$, $k = 9$, $d.f. = 35.73$.

Note: Formula for $L_1$ is given with Table 3, page 59.
Table 7

Data for the Comparison of Teachers

Analysis of Variance for the Test of Unit 2

<table>
<thead>
<tr>
<th>Source</th>
<th>d.f.</th>
<th>Sums of Squares</th>
<th>Mean Square</th>
</tr>
</thead>
<tbody>
<tr>
<td>Between Sections</td>
<td>8</td>
<td>4517.65429</td>
<td>564.7</td>
</tr>
<tr>
<td>Within Sections</td>
<td>356</td>
<td>1366.5211</td>
<td>116.19</td>
</tr>
<tr>
<td>Total</td>
<td>364</td>
<td>5884.17534*</td>
<td></td>
</tr>
</tbody>
</table>

\[ F = \frac{564.7}{116.19} = 4.8 \]

* S.S. total = \( \sum X^2 - \frac{(\sum X)^2}{N} \)
variation may be due to failure to assign the pupils in the "r" category completely at random to the various teachers and sections.

Comparison of the Control and Experimental Groups for the Tests of Units 1 and 2

Because the data for the experimental and control sections will be pooled separately in the analysis of the data pertaining to the main criterion, the effect of pooling the results of the data pertaining to the tests on Units 1 and 2 in this same manner was studied. In Table 8 the effect of this pooling for Unit 1 has been indicated. As a result of the sums of scores and the numbers of pupils in the two groups, the means on the test for the two groups, as indicated in the last column of the table, indicates equality. On Unit 1, there are equal means for the control and experimental groups. It will be recalled, however, that there was unequal variance among the sections making up these two groups.

In Table 9, similar data are presented with regard to Unit 2. Here the means for the two groups differ considerably. According to Johnson the significance of this difference may be determined by $t_0$, the ratio of the difference in the means to the standard error of

---

Table 8

Comparison of Control and Experimental Sections on the Results of the Test for Unit 1

<table>
<thead>
<tr>
<th>Group</th>
<th>$n_g$</th>
<th>$\Sigma X$</th>
<th>$\bar{X}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>200</td>
<td>9471</td>
<td>47.35</td>
</tr>
<tr>
<td>Experimental</td>
<td>172</td>
<td>8145</td>
<td>47.35</td>
</tr>
<tr>
<td>Total</td>
<td>372</td>
<td>17616</td>
<td>47.35</td>
</tr>
</tbody>
</table>

Table 9

Comparison of Control and Experimental Sections on the Results of the Test for Unit 2

<table>
<thead>
<tr>
<th>Group</th>
<th>$n_g$</th>
<th>$\Sigma X$</th>
<th>$\bar{X}$</th>
<th>$\Sigma X^2$</th>
<th>$Q_g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>195</td>
<td>5482</td>
<td>28.1</td>
<td>173900</td>
<td>19785</td>
</tr>
<tr>
<td>Experimental</td>
<td>170</td>
<td>4084</td>
<td>24.0</td>
<td>122696</td>
<td>24584</td>
</tr>
<tr>
<td>Total</td>
<td>365</td>
<td>9566</td>
<td>26.2</td>
<td>296596</td>
<td>45888</td>
</tr>
</tbody>
</table>
the difference. The formula is:

\[ t_0 = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{s^2 \left(\frac{1}{N_1} + \frac{1}{N_2}\right)}} \]

where \( s^2 = \frac{\Sigma(\bar{X}_1 - \bar{X})^2 + \Sigma(\bar{X}_2 - \bar{X})^2}{N_1 + N_2 - 2} \)

From the data in Table 9, \( s^2 = 123 \), approximately, and since the difference of the means is 1.1, \( t_0 \) is 3.5, which is significant. It will be noted that the achievement in this unit is in favor of the control sections.

**Summary**

On the basis of evidence presented here it may not be concluded that the sections taught by different teachers achieve equally well. When the sections are grouped into control and experimental groups, however, there is evidence that the two groups achieved equally well on the test for Unit 1, but that their difference in achievement on the test for Unit 2 was significantly in favor of the control group. It will be shown later that the experimental group has an advantage in I.Q., aptitude, and reading level, but that the advantage in the latter is slight.
Estimating the Reliability of the Evaluation Instrument

The test by which the ability of the pupils in the experiment to frame equations for verbal problems was to be determined was administered to all the pupils in the thirteen sections of "r" algebra on the second day of the second semester, 1955-56. On that day there were many pupils absent due to an epidemic of influenza. A few teachers volunteered to give the test to the absentees on their return, but no request was made of them to do this. The tests so taken were sent to the writer within a week and probably less than ten papers were so received.

A total of 291 ninth-grade pupils in the thirteen "r" sections of algebra took the test. Of these, nine pupils' records in the other data of the experiment were incomplete, and these pupils were eliminated from the comparison of the experimental method to the traditional method. These pupils' papers were considered, however, in the calculation of the reliability of the test. In each of the sections there were also a few pupils who were classified beyond the ninth grade. The data for all such pupils were eliminated from consideration in this experiment even though they took the final test.

The test has been described in the previous chapter. It will be recalled that fourteen of the items asked only that the literal symbols used be properly identified and that an equation or set of equations to represent a problem be completed. The other two of the sixteen items required the student to complete the solution of
two of the problems. All the papers were scored and twice rescored by the writer. All papers were scored from zero to sixteen.

In Table 10, the number of ninth-grade pupils in each section who took the test is indicated in the column marked \( n_s \), and the number completing any item correctly is indicated in the column under the respective item. Subtotals for each of the groups of the seven control sections and the six experimental sections have also been indicated. The total number of correct responses for each section and group is then indicated in the column marked \( T \), and the number of pupils correctly answering each item is indicated in the row marked \( P_i \). \( P_i^2 \) and \( T^2 \) are included for purposes of calculating the reliability of the test.

Calculating the Estimate of Reliability

In Table 11, the calculations for estimating the reliability of the test according to the method outlined by Hoyt\(^3\) have been tabulated. It is a method based on analysis of variance in which the sums of squares for various sources are identified and mean squares depending upon degrees of freedom are determined. The following symbols are used: \( k \) = the number of pupils, \( n \) = the number of items on the test, \( T \) = the total number of correct responses,

| Sect. | \( n_s \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | T |
|-------|----------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| A     | 19       | 17 | 15 | 19 | 18 | 7  | 5  | 9  | 11 | 17 | 17 | 14 | 7  | 1  | 10 | 12 | 16 | 195 |
| B_1   | 24       | 19 | 17 | 20 | 21 | 2  | 4  | 9  | 6  | 18 | 19 | 7  | 4  | 0  | 4  | 8  | 18 | 176 |
| B_2   | 28       | 21 | 19 | 26 | 23 | 5  | 7  | 21 | 17 | 23 | 19 | 14 | 10 | 2  | 3  | 13 | 23 | 246 |
| C     | 24       | 21 | 19 | 22 | 21 | 7  | 2  | 17 | 10 | 21 | 18 | 14 | 7  | 2  | 8  | 10 | 19 | 218 |
| D     | 18       | 14 | 8  | 14 | 16 | 6  | 3  | 7  | 3  | 12 | 13 | 7  | 2  | 3  | 4  | 2  | 11 | 125 |
| E     | 22       | 19 | 16 | 21 | 17 | 9  | 4  | 15 | 17 | 21 | 17 | 8  | 7  | 0  | 9  | 14 | 21 | 215 |
| F     | 22       | 21 | 19 | 20 | 20 | 8  | 5  | 17 | 19 | 21 | 17 | 17 | 10 | 1  | 11 | 8  | 20 | 234 |
| Cont. | 157      | 132| 113| 142| 136| 41 | 30 | 95 | 63 | 133| 120| 81 | 47 | 9  | 49 | 67 | 128| 1409|
| G_1   | 21       | 15 | 15 | 19 | 17 | 5  | 7  | 12 | 17 | 19 | 13 | 16 | 13 | 2  | 10 | 13 | 13 | 206 |
| G_2   | 22       | 17 | 15 | 20 | 15 | 1  | 7  | 11 | 11 | 19 | 15 | 6  | 8  | 0  | 6  | 8  | 17 | 176 |
| H_1   | 19       | 16 | 16 | 17 | 17 | 8  | 4  | 9  | 7  | 16 | 15 | 11 | 7  | 3  | 8  | 4  | 15 | 173 |
| H_2   | 22       | 21 | 20 | 21 | 20 | 15 | 13 | 16 | 16 | 20 | 18 | 17 | 14 | 4  | 14 | 11 | 21 | 261 |
| I_1   | 26       | 21 | 19 | 22 | 15 | 6  | 5  | 9  | 12 | 19 | 17 | 11 | 13 | 3  | 10 | 8  | 19 | 209 |
| I_2   | 21       | 20 | 18 | 23 | 20 | 6  | 4  | 13 | 13 | 19 | 19 | 13 | 11 | 1  | 6  | 5  | 20 | 214 |
| Exp.  | 134      | 110| 103| 122| 104| 41 | 40 | 70 | 76 | 112| 97 | 71 | 69 | 13 | 54 | 19 | 105| 1239|
| P_1   | (291)    | 214| 216| 264| 210| 85 | 70 | 165| 159| 245| 217| 155| 116| 22 | 103| 116| 233| 2648|
| P_1^2 | 58564    | 6956 | 6956 | 57660 | 7285 | 23225 | 25881 | 10896 | 24225 | 13156 | 181 | 10609 | 12556 | 68289 | 22 |
Table 11

Estimation of Reliability of the Test

<table>
<thead>
<tr>
<th>Source</th>
<th>d.f.</th>
<th>Sums of Squares</th>
<th>Mean Square</th>
</tr>
</thead>
<tbody>
<tr>
<td>Among items</td>
<td>15</td>
<td>289.8</td>
<td></td>
</tr>
<tr>
<td>Among students</td>
<td>290</td>
<td>208.6</td>
<td>α = .719</td>
</tr>
<tr>
<td>Residual</td>
<td>4350</td>
<td>643.6</td>
<td>β = .148</td>
</tr>
<tr>
<td>Total</td>
<td>4655</td>
<td>1142.0</td>
<td></td>
</tr>
</tbody>
</table>

\[
F = \frac{\alpha - \beta}{\alpha} = .79.
\]
\[ \sum_{i=1}^{291} t_i^2 \] is the total of the squares of the scores made by each of the 291 pupils. The latter is the sum of the \( Y^2 \) column of Table 13 increased by the squares of the scores of the nine pupils eliminated there because of the lack of other necessary data. \[ \sum_{i=1}^{16} p_i^2 \] is the total of the sixteen squares at the bottom of Table 10.

The degrees of freedom are defined as follows: for total, d.f. = nk - 1; for items, d.f. = n - 1; for students, d.f. = k - 1; and for residual, d.f. = the difference between the d.f. for total and the sum of the other two.

Sums of squares are determined as follows:

- for total, S.S. = \( \frac{T(nk - T)}{nk} \),
- for students, S.S. = \( \frac{1}{n} \sum t_i^2 - \frac{T^2}{nk} \),
- for items, S.S. = \( \frac{1}{k} \sum p_i^2 - \frac{T^2}{nk} \).

The mean square for students is identified as \( \alpha \), and the residual mean square as \( \beta \). Then reliability, \( r \), is estimated by means of the following formula:

\[ r = \frac{\alpha - \beta}{\alpha} \]

According to the calculations in Table 11, the reliability of this test is estimated to be .79, based on the scores made by the 291 ninth-grade pupils taking the test. This may be considered high for a test of only sixteen items.
The hypothesis that no difference exists between the scores of individuals may be tested through the use of the results given in Table 11, as follows: $F = \frac{\alpha}{\beta} = \frac{719}{118} = 4.8$. An $F$ of this size with the respective degrees of freedom indicates that the hypothesis must be rejected, and we conclude that the differences are significant and that the test measures differences among individuals in a satisfactory manner.

**Testing the Main Hypothesis**

The testing of the main hypothesis of this study — in null form, that "there is no significant difference between the ability of pupils to write equations for verbal problems when taught by the experimental method and when taught by the traditional method" — was accomplished by the method of analysis of variance and covariance. It has previously been indicated that this method was chosen to eliminate the necessity of matching pupils in pairs or in groups. By means of this technique allowances may be made for the same types of variables which would have been used in matching.

The variables taken into account were I.Q., algebraic aptitude, and reading level. It was assumed that the effect of these variables on the ability of the pupils to frame equations would be linear and that these variables were fundamentally significant; it was further assumed that after the effect of the variables had been taken into account, the remaining differences would be due to direct classroom experiences, which differed primarily in basic method. The analysis
of covariance is simply the application of the idea of regression to the usual analysis of variance technique.

The writer has on file thirteen tables, one for each class section, which have been summarized in Table 12. In these tables, the score of each pupil on the final evaluation, the test of sixteen items, has been listed in the column headed $Y$. Nine fewer pupils are listed in these tables than were indicated previously as having scores for the test when the reliability of the test was being estimated. The reason for their omission here is that the data for these nine pupils were not complete in the Dean's records at the school, although the administration of the school cooperated in attempting to supply the missing information. The square of each score, $Y^2$, is also indicated for each pupil as an aid in obtaining $\Sigma Y^2$ for each section.

The I.Q. of each pupil has been indicated in column $X_1$, by means of a number which is one hundred less than the I.Q. recorded in the Dean's records. This number for each pupil is also squared and recorded in column $X_1^2$. The reduction may be made because all calculations in the analysis of variance are in terms of deviations from means.

The score made by each pupil on the California Algebraic Aptitude Test, reduced by fifty, has been indicated as $X_2$. Likewise $X_2^2$ has been indicated for each pupil, as an aid in calculating the sums of squares of deviations from means for each section.
In the column headed $X_3$, the reading level of each pupil has been recorded. The number recorded is a mean of scores on two parts of the California Achievement Test, Intermediate Battery, namely, (1) Reading Vocabulary and (2) Reading Comprehension. The score represents an estimated grade level of achievement. Thus a score of 9.70 represents a reading level appropriate for a pupil in the seventh month of the ninth grade.

Because an additional datum (termed a cross product) will be needed for the analysis, the product of each of the combinations of two scores has also been indicated in appropriately marked columns for each pupil. All columns are totaled for each section.

In Table 12 the totals for the columns of the thirteen tables have been summarized for each section. In addition the arithmetic mean of each of the four measures for each section has been indicated in columns headed $\bar{Y}$, $\bar{X}_1$, $\bar{X}_2$, and $\bar{X}_3$, respectively. In Table 12, subtotals of columns have been provided for the groups of seven control sections and the six experimental sections, as well as totals for all thirteen sections, in all columns except those for arithmetic means and squares of sums. In the columns headed $\bar{Y}$, $\bar{X}_1$, $\bar{X}_2$, and $\bar{X}_3$, means for the groups as a whole have been entered. In columns $\bar{Y}^2$, $X_1^2$, $X_2^2$, and $X_3^2$ as a further aid in the calculations of the analysis, the data $(\Sigma Y)^2$, $(\Sigma X_1)^2$, $(\Sigma X_2)^2$, and $(\Sigma X_3)^2$ have been entered for each section, for both the control and experimental groups, and for the total.
Table 12

Summary of Basic Data for the Comparison
of the Experimental Method to the Traditional Method

<table>
<thead>
<tr>
<th>Sect.</th>
<th>Method</th>
<th>( \Sigma Y )</th>
<th>( \overline{Y} )</th>
<th>( (\Sigma Y)^2 )</th>
<th>( \Sigma Y^2 )</th>
<th>( \Sigma X_1 )</th>
<th>( \overline{X_1} )</th>
<th>( (\Sigma X_1)^2 )</th>
<th>( \Sigma X_1^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>17</td>
<td>176</td>
<td>10.353</td>
<td>30976</td>
<td>1918</td>
<td>200</td>
<td>11.765</td>
<td>40000</td>
<td>2944</td>
</tr>
<tr>
<td>B_1</td>
<td>23</td>
<td>169</td>
<td>7.348</td>
<td>28561</td>
<td>1457</td>
<td>200</td>
<td>8.696</td>
<td>40000</td>
<td>2524</td>
</tr>
<tr>
<td>B_2</td>
<td>28</td>
<td>216</td>
<td>8.786</td>
<td>60516</td>
<td>2450</td>
<td>243</td>
<td>8.679</td>
<td>59049</td>
<td>3629</td>
</tr>
<tr>
<td>C</td>
<td>23</td>
<td>211</td>
<td>9.304</td>
<td>45796</td>
<td>2116</td>
<td>261</td>
<td>11.348</td>
<td>68121</td>
<td>3647</td>
</tr>
<tr>
<td>D</td>
<td>17</td>
<td>123</td>
<td>7.235</td>
<td>15129</td>
<td>1167</td>
<td>197</td>
<td>11.588</td>
<td>38809</td>
<td>2829</td>
</tr>
<tr>
<td>E</td>
<td>20</td>
<td>190</td>
<td>9.500</td>
<td>36100</td>
<td>2018</td>
<td>211</td>
<td>10.550</td>
<td>44521</td>
<td>3341</td>
</tr>
<tr>
<td>F</td>
<td>22</td>
<td>234</td>
<td>10.636</td>
<td>54756</td>
<td>2554</td>
<td>241</td>
<td>10.955</td>
<td>58081</td>
<td>3753</td>
</tr>
<tr>
<td>Cont.</td>
<td>150</td>
<td>1352</td>
<td>9.013</td>
<td>1827904</td>
<td>13680</td>
<td>1553</td>
<td>10.353</td>
<td>2511809</td>
<td>22667</td>
</tr>
<tr>
<td>G_1</td>
<td>20</td>
<td>200</td>
<td>10.000</td>
<td>40000</td>
<td>2290</td>
<td>236</td>
<td>11.800</td>
<td>55696</td>
<td>3858</td>
</tr>
<tr>
<td>G_2</td>
<td>22</td>
<td>176</td>
<td>8.000</td>
<td>30976</td>
<td>1678</td>
<td>235</td>
<td>10.682</td>
<td>55225</td>
<td>3423</td>
</tr>
<tr>
<td>H_1</td>
<td>18</td>
<td>161</td>
<td>8.914</td>
<td>25921</td>
<td>1585</td>
<td>200</td>
<td>11.111</td>
<td>40000</td>
<td>2846</td>
</tr>
<tr>
<td>H_2</td>
<td>22</td>
<td>261</td>
<td>11.864</td>
<td>68121</td>
<td>3291</td>
<td>293</td>
<td>13.318</td>
<td>85849</td>
<td>4337</td>
</tr>
<tr>
<td>I_1</td>
<td>26</td>
<td>209</td>
<td>8.238</td>
<td>43681</td>
<td>2079</td>
<td>331</td>
<td>12.731</td>
<td>109561</td>
<td>5241</td>
</tr>
<tr>
<td>I_2</td>
<td>24</td>
<td>214</td>
<td>8.917</td>
<td>45796</td>
<td>2088</td>
<td>270</td>
<td>11.250</td>
<td>72900</td>
<td>2910</td>
</tr>
<tr>
<td>Expt.</td>
<td>132</td>
<td>1221</td>
<td>9.250</td>
<td>1190841</td>
<td>13011</td>
<td>1565</td>
<td>11.856</td>
<td>2449225</td>
<td>23615</td>
</tr>
<tr>
<td>Tot.</td>
<td>282</td>
<td>2573</td>
<td>9.121</td>
<td>6620329</td>
<td>26691</td>
<td>3118</td>
<td>11.057</td>
<td>9721924</td>
<td>46282</td>
</tr>
</tbody>
</table>
Table 12
(continued)

<table>
<thead>
<tr>
<th>Section</th>
<th>Aptitude</th>
<th>Reading Level</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sum X_2$</td>
<td>$\sum X_3$</td>
</tr>
<tr>
<td>A</td>
<td>266</td>
<td>168.15</td>
</tr>
<tr>
<td>B1</td>
<td>110</td>
<td>221.50</td>
</tr>
<tr>
<td>B2</td>
<td>199</td>
<td>283.55</td>
</tr>
<tr>
<td>C</td>
<td>190</td>
<td>234.20</td>
</tr>
<tr>
<td>D</td>
<td>213</td>
<td>165.50</td>
</tr>
<tr>
<td>E</td>
<td>188</td>
<td>201.70</td>
</tr>
<tr>
<td>F</td>
<td>213</td>
<td>222.70</td>
</tr>
<tr>
<td>G1</td>
<td>1439</td>
<td>1500.30</td>
</tr>
<tr>
<td>G2</td>
<td>303</td>
<td>205.35</td>
</tr>
<tr>
<td>H1</td>
<td>255</td>
<td>212.60</td>
</tr>
<tr>
<td>H2</td>
<td>129</td>
<td>179.50</td>
</tr>
<tr>
<td>I1</td>
<td>253</td>
<td>225.05</td>
</tr>
<tr>
<td>I2</td>
<td>297</td>
<td>259.15</td>
</tr>
<tr>
<td>E</td>
<td>1515</td>
<td>1325.10</td>
</tr>
<tr>
<td>T</td>
<td>2954</td>
<td>2825.40</td>
</tr>
</tbody>
</table>
Table 12  
(continued)

<table>
<thead>
<tr>
<th>Sect.</th>
<th>(\Sigma X_1)</th>
<th>(\Sigma X_2)</th>
<th>(\Sigma X_3)</th>
<th>(\Sigma X_{12})</th>
<th>(\Sigma X_{13})</th>
<th>(\Sigma X_{23})</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2063</td>
<td>2849</td>
<td>1744.60</td>
<td>3074</td>
<td>2012.85</td>
<td>2635.95</td>
</tr>
<tr>
<td>B_1</td>
<td>1609</td>
<td>996</td>
<td>1637.05</td>
<td>1191</td>
<td>2033.30</td>
<td>1940.70</td>
</tr>
<tr>
<td>B_2</td>
<td>2316</td>
<td>2150</td>
<td>2538.90</td>
<td>2594</td>
<td>2533.95</td>
<td>2148.85</td>
</tr>
<tr>
<td>C</td>
<td>2463</td>
<td>2055</td>
<td>2192.95</td>
<td>2220</td>
<td>2701.90</td>
<td>1938.00</td>
</tr>
<tr>
<td>D</td>
<td>1680</td>
<td>2271</td>
<td>1218.10</td>
<td>2960</td>
<td>1967.80</td>
<td>2202.75</td>
</tr>
<tr>
<td>E</td>
<td>2170</td>
<td>1904</td>
<td>1928.60</td>
<td>2112</td>
<td>2245.75</td>
<td>1902.00</td>
</tr>
<tr>
<td>F</td>
<td>2609</td>
<td>2580</td>
<td>2355.75</td>
<td>2797</td>
<td>2457.65</td>
<td>2553.50</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Cont.</th>
<th>111910</th>
<th>111805</th>
<th>13643.95</th>
<th>17248</th>
<th>15953.20</th>
<th>15321.75</th>
</tr>
</thead>
<tbody>
<tr>
<td>G_1</td>
<td>2452</td>
<td>3326</td>
<td>2081.80</td>
<td>3775</td>
<td>2156.70</td>
<td>3175.35</td>
</tr>
<tr>
<td>G_2</td>
<td>2046</td>
<td>2236</td>
<td>1706.55</td>
<td>2766</td>
<td>2333.85</td>
<td>2486.80</td>
</tr>
<tr>
<td>H_1</td>
<td>1968</td>
<td>1280</td>
<td>1612.60</td>
<td>2176</td>
<td>2041.00</td>
<td>1290.50</td>
</tr>
<tr>
<td>H_2</td>
<td>3182</td>
<td>3250</td>
<td>2709.15</td>
<td>3599</td>
<td>3031.05</td>
<td>2662.40</td>
</tr>
<tr>
<td>I_1</td>
<td>2837</td>
<td>2815</td>
<td>2139.00</td>
<td>3954</td>
<td>3380.80</td>
<td>3071.15</td>
</tr>
<tr>
<td>I_2</td>
<td>2647</td>
<td>2530</td>
<td>2176.90</td>
<td>3508</td>
<td>2800.05</td>
<td>2813.75</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Exp.</th>
<th>15232</th>
<th>15365</th>
<th>12426.00</th>
<th>19778</th>
<th>16043.45</th>
<th>15529.95</th>
</tr>
</thead>
</table>

| Tot.  | 30142          | 30170          | 26069.95       | 37026          | 31996.65       | 30851.70       |
In any analysis of variance and covariance, it is necessary to test whether or not certain assumptions are met: here it is necessary to test whether or not the sections are homogeneous in the measure \( Y \).

Two hypotheses will be tested:

\[ H_0 : \sigma_s = \sigma, \]

\[ H_1 : \bar{Y}_s = \bar{Y}. \]

\( H_0 \) is tested by \( L_1 \), which is accomplished by means of the data in Table 13. Here the measures \( \Sigma Y, \Sigma Y^2, \) and \( (\Sigma Y)^2 \) for each section are taken from Table 12, and the remaining columns are determined from these basic data so that the formula for \( L_1 \) may be evaluated.

\[ L_1 = \log N - \frac{1}{N} \sum n_s \log n_s + \frac{1}{N} \sum n_s \log \Theta^2 - \log \Theta^2. \]

Only the columns \( n_s, \Sigma Y, \Sigma Y^2, \Theta^2, n_s \log n_s, \) and \( n_s \log \Theta^2 \) are totaled. The \( \Sigma n_s = N \) and the other sums are needed in the formula or other calculations.

According to these data, \( L_1 \approx 21.22 \), which with \( K = 13 \) and \( 21^+ \) degrees of freedom, indicates that the null hypothesis is valid, i.e., that the standard deviations of the sections do not vary significantly from the standard deviation of the whole group.

In order to test \( H_1 : \bar{Y}_s = \bar{Y} \), analysis of variance is used. In Table 11, the calculations of the analysis have been summarized. The sum of squares within sections is taken as \( \Sigma \Theta^2 \) from Table 13. The sum of squares for total is obtained as \( \Sigma Y^2 \frac{1}{N} - (\Sigma Y)^2 \). From Table 12, \( Y^2_1 = 26691, \) and \( (\Sigma Y)^2 = 6620329 \). Hence the sum of squares for total is \( 32114.6561 \). The sum of squares between sections is the difference, which is \( 4564.925 \). The degrees of
Table 13

Data for the Test for Homogeneity of Y

<table>
<thead>
<tr>
<th>Sect. n_s</th>
<th>$\sum Y$</th>
<th>$\sum Y^2$</th>
<th>$(\sum Y)^2/n_s$</th>
<th>$\hat{\theta}_s'$</th>
<th>log $n_s$</th>
<th>$n_s \log n_s$</th>
<th>log $\hat{\theta}_s'$</th>
<th>$n_s \log \hat{\theta}_s'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>17</td>
<td>176</td>
<td>1918</td>
<td>30976</td>
<td>95.8823</td>
<td>1.230645</td>
<td>20.91765</td>
<td>1.98174</td>
</tr>
<tr>
<td>B_1</td>
<td>23</td>
<td>169</td>
<td>1457</td>
<td>28561</td>
<td>215.2174</td>
<td>1.36173</td>
<td>31.31979</td>
<td>2.33288</td>
</tr>
<tr>
<td>B_2</td>
<td>28</td>
<td>216</td>
<td>2161</td>
<td>65016</td>
<td>2161.2857</td>
<td>1.44716</td>
<td>40.52048</td>
<td>2.46047</td>
</tr>
<tr>
<td>D</td>
<td>17</td>
<td>123</td>
<td>1167</td>
<td>889.9112</td>
<td>277.0588</td>
<td>1.230645</td>
<td>20.91765</td>
<td>2.44257</td>
</tr>
<tr>
<td>E</td>
<td>20</td>
<td>190</td>
<td>2018</td>
<td>36100</td>
<td>213.0000</td>
<td>1.30103</td>
<td>26.02060</td>
<td>2.32838</td>
</tr>
<tr>
<td>F</td>
<td>22</td>
<td>234</td>
<td>2554</td>
<td>54756</td>
<td>2188.9091</td>
<td>1.34214</td>
<td>29.53324</td>
<td>1.81352</td>
</tr>
<tr>
<td>G_1</td>
<td>20</td>
<td>200</td>
<td>2290</td>
<td>40000</td>
<td>290.0000</td>
<td>1.30103</td>
<td>26.02060</td>
<td>2.46240</td>
</tr>
<tr>
<td>G_2</td>
<td>22</td>
<td>176</td>
<td>1678</td>
<td>30976</td>
<td>270.0000</td>
<td>1.34214</td>
<td>29.53324</td>
<td>2.43336</td>
</tr>
<tr>
<td>H_1</td>
<td>18</td>
<td>161</td>
<td>1585</td>
<td>25921</td>
<td>1440.0555</td>
<td>1.25527</td>
<td>22.59486</td>
<td>2.16120</td>
</tr>
<tr>
<td>H_2</td>
<td>22</td>
<td>261</td>
<td>3291</td>
<td>68121</td>
<td>3096.4091</td>
<td>1.34214</td>
<td>29.53324</td>
<td>2.28912</td>
</tr>
<tr>
<td>I_1</td>
<td>26</td>
<td>209</td>
<td>2079</td>
<td>43681</td>
<td>1680.0385</td>
<td>398.9615</td>
<td>36.78922</td>
<td>2.60064</td>
</tr>
<tr>
<td>I_2</td>
<td>21</td>
<td>211</td>
<td>2088</td>
<td>45796</td>
<td>1908.1667</td>
<td>179.8333</td>
<td>33.12504</td>
<td>2.25127</td>
</tr>
</tbody>
</table>

$N = 282$, $2573$, $26691$, $\sum \hat{\theta}_s' = 2758.1635$, $378.14540$, $646.17514$

$log N = 2.45025$, $log \sum \hat{\theta}_s' = 3.44062$

$I_1 = .9122$, $k = 13$, d.f. = 21.23 Note: Formula for $I_1$ is given with Table 3, page 59.
freedom are one less than the number of pupils for total, and one less than the number of sections for between sections. The number of degrees of freedom for within is the difference. Mean squares are the result of dividing the sums of squares by the respective degrees of freedom.

The value of $F$ is the result of dividing the mean square between sections by the mean square within, giving the value 3.7.

This value of $F$ is too large to substantiate the null hypothesis, $H_0: \bar{Y}_s = \bar{Y}$. Therefore, it may not be concluded that the means do not differ significantly.

In Table 12, the means of each of the sections, $\bar{Y}_s$, have been listed; it will be noted that these means vary from 7.235 to 11.864. However, in Table 12, it is also noted that $\bar{Y}$ for the control group is 9.013 while $\bar{Y}$ for the experimental group is 9.124, which suggests that the difference of these means ought to be tested. The data for testing the significance of the difference of these means are presented in Table 15. The formula:

$$t_0 = \frac{\bar{Y}_1 - \bar{Y}_2}{\sqrt{s^2(\frac{1}{N_1} + \frac{1}{N_2})}}$$

where $$s^2 = \frac{\sigma_1^2 + \sigma_2^2}{N_1 + N_2 - 2}$$

is used to calculate $t_0$; $t_0$ is determined to be 0.5866, which with $n = 282$ indicates a probability of differences as large as this of more than 50 per cent. Hence the difference is not significant.
Table 14
Test of $H_1$: $\bar{Y}_g = \bar{Y}$

<table>
<thead>
<tr>
<th>Source</th>
<th>d.f.</th>
<th>Sums of Squares</th>
<th>Mean Square</th>
</tr>
</thead>
<tbody>
<tr>
<td>Between</td>
<td>12</td>
<td>456.925</td>
<td>38.0401</td>
</tr>
<tr>
<td>Within</td>
<td>269</td>
<td>2758.1635</td>
<td>10.2534</td>
</tr>
<tr>
<td>Total</td>
<td>281</td>
<td>3214.1560</td>
<td></td>
</tr>
</tbody>
</table>

$$F = \frac{38.0401}{10.2534} = 3.7$$

Table 15
Calculations of $t_o$ for Comparison of Means of Control and Experimental Groups on $Y$

<table>
<thead>
<tr>
<th>Group</th>
<th>$n_g$</th>
<th>$\Sigma Y$</th>
<th>$\bar{Y}$</th>
<th>$\Sigma Y^2$</th>
<th>$\Theta'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>150</td>
<td>1352</td>
<td>9.013</td>
<td>13680</td>
<td>1493.9734</td>
</tr>
<tr>
<td>Experimental</td>
<td>132</td>
<td>1221</td>
<td>9.250</td>
<td>13011</td>
<td>1716.7500</td>
</tr>
<tr>
<td>Total</td>
<td>282</td>
<td>2573</td>
<td>9.124</td>
<td>26691</td>
<td>3210.7234</td>
</tr>
</tbody>
</table>

$$t_o = .5866$$
The rejection of the null hypothesis as to homogeneity of variance theoretically indicates that it is inappropriate to continue with the proposed analysis of covariance in this study. However, some authorities disagree with this conclusion:

... There is increasing evidence, however, that the necessity for the homogeneity of variance is not as serious a consideration as it was formerly thought to be.\footnote{Wert et al., \textit{loc. cit.}}

This statement and the fact that the difference between $\bar{Y}$ for the control and experimental groups is not significant, even though the variance of $Y$ among the sections is not homogeneous, were considered to justify proceeding with the analysis of variance and covariance as planned. It may be anticipated from the fact that $\bar{Y}$, $\bar{X}_1$, $\bar{X}_2$, and $\bar{X}_3$ are all respectively larger for the experimental sections than for the control sections, that most of the difference in $Y$ will be accounted for by differences in the control variables.

The \textbf{Analysis of Variance and Covariance}

The first step in the analysis of covariance is to calculate sums of squares of deviations for all variables and cross products both total and within. This is done according to the following basic plan, in which the symbol for a variable which has no subscript is for total, and $c$ and $e$ are the subscripts used to denote control and experimental totals respectively:

\footnote{Wert et al., \textit{loc. cit.}}
\[ \Sigma y^2 \text{ (total)} = \Sigma y^2 - \frac{(\Sigma y)^2}{N}. \]
\[ \Sigma y^2 \text{ (within)} = \Sigma y^2 - \left[ \frac{(\Sigma y_e)^2}{N_e} + \frac{(\Sigma y_c)^2}{N_c} \right]. \]

Similar formulas are used for \(\Sigma x_1^2\), \(\Sigma x_2^2\), and \(\Sigma x_3^2\).

For cross products the deviation formulas are:
\[ \Sigma x_1y \text{ (total)} = \Sigma x_1y - \frac{(\Sigma x_1)(\Sigma y)}{N}. \]
\[ \Sigma x_1y \text{ (within)} = \Sigma x_1y - \frac{(\Sigma x_1e)(\Sigma y_c)}{N_e} + \frac{(\Sigma x_1c)(\Sigma y_e)}{N_c}. \]

Similar formulas are used for \(\Sigma x_2y\), \(\Sigma x_3y\), \(\Sigma x_1x_2\), \(\Sigma x_1x_3\), and \(\Sigma x_2x_3\), both total and within. The results of these calculations are shown in Table 16.

The next step is to use the data to calculate regression coefficients \(a_1\), \(a_2\), \(a_3\) for both total and within from the following basic formulas:
\[ \Sigma x_1y = a_1\Sigma x_1^2 + a_2\Sigma x_1x_2 + a_3\Sigma x_1x_3, \]
\[ \Sigma x_2y = a_1\Sigma x_2x_1 + a_2\Sigma x_2^2 + a_3\Sigma x_2x_3, \]
\[ \Sigma x_3y = a_1\Sigma x_3x_1 + a_2\Sigma x_3x_2 + a_3\Sigma x_3^2. \]

The respective coefficients \(a_1\), \(a_2\), \(a_3\) for both total and within were calculated according to the Doolittle method and are summarized in Table 17. The calculations are given in Appendix H.

The coefficients are used to adjust the sums of squares of deviations, \(\Sigma y^2\), for both total and within, according to the formula:
\[ \text{adjusted } \Sigma y^2 = \Sigma y^2 - \left[ a_1\Sigma x_1y + a_2\Sigma x_2y + a_3\Sigma x_3y \right]. \]
Table 16

Summary of Deviations

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>$\Sigma y^2$</th>
<th>$\Sigma x_1^2$</th>
<th>$\Sigma x_2^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total</td>
<td>3211.65603</td>
<td>11607.09220</td>
<td>28036.32625</td>
</tr>
<tr>
<td>Within</td>
<td>3210.72333</td>
<td>11648.53849</td>
<td>27787.12515</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Source</th>
<th>$\Sigma x_3^2$</th>
<th>$\Sigma x_1y$</th>
<th>$\Sigma x_2y$</th>
<th>$\Sigma x_3y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total</td>
<td>219.5016</td>
<td>1693.01119</td>
<td>3217.36880</td>
<td>290.67979</td>
</tr>
<tr>
<td>Within</td>
<td>219.40736</td>
<td>1668.04333</td>
<td>3186.06333</td>
<td>290.07100</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Source</th>
<th>$\Sigma x_1x_2$</th>
<th>$\Sigma x_1x_3$</th>
<th>$\Sigma x_2x_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total</td>
<td>14361.39717</td>
<td>756.94362</td>
<td>1255.13405</td>
</tr>
<tr>
<td>Within</td>
<td>14165.62152</td>
<td>753.07810</td>
<td>1250.28791</td>
</tr>
<tr>
<td></td>
<td>(a_1)</td>
<td>(a_2)</td>
<td>(a_3)</td>
</tr>
<tr>
<td>-------</td>
<td>---------</td>
<td>---------</td>
<td>---------</td>
</tr>
<tr>
<td>Total</td>
<td>0.0748807431</td>
<td>0.07442852</td>
<td>0.64073345</td>
</tr>
<tr>
<td>Within</td>
<td>0.0752245032</td>
<td>0.074655543</td>
<td>0.638447831</td>
</tr>
</tbody>
</table>
Table 18

Analysis of Variance of Adjusted Y

<table>
<thead>
<tr>
<th>Source</th>
<th>d.f.</th>
<th>Sums of Squares</th>
<th>Mean Square</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total</td>
<td>278</td>
<td>2662.30505</td>
<td></td>
</tr>
<tr>
<td>Within</td>
<td>277</td>
<td>2662.19311</td>
<td>9.61081</td>
</tr>
<tr>
<td>Difference</td>
<td>1</td>
<td>.11194</td>
<td>.11194</td>
</tr>
</tbody>
</table>

\[
F = \frac{.11194}{9.61081} = .011647.* \quad \text{Null hypothesis accepted.}
\]

*An F less than unity is always non-significant.*
The adjusted sums of squares of deviations have been entered in Table 18. Adjusted degrees of freedom are also entered. One additional degree of freedom is removed for each of the variables on the total, making $N - 4$ degrees of freedom. The difference has one degree of freedom for method. Therefore the degrees of freedom for within are $N - 5$. Mean squares are again calculated for difference and within, and the ratio $F$ is determined.

For these adjusted data, $F$ is .01647, which is too small to be significant. Hence the basic null hypothesis would be accepted, and it may not be concluded that when allowances were made for differences in I.Q., aptitude, and reading level, the difference in method significantly influenced the achievement of the pupils in framing equations for verbal problems.

Relative Weights of Control Variables and the Criterion Variable in the Experiment

The regression coefficients obtained in the previous section do not serve to indicate the relative weights of the different variables in their respective influence upon the success of the pupils in framing equations for problems. The reason for this is that each of the variables was measured on an independent scale, and regression was calculated upon variance from the means for each of the variables. In order to estimate the relative influence a measure called Beta, similar to a regression coefficient, but based on standard units of variance, is used. In this procedure all deviations will be measured in the same units, standard deviation.
### Table 19
Data for the Calculation of r's

<table>
<thead>
<tr>
<th></th>
<th>( \Sigma Y )</th>
<th>( \Sigma X_1 )</th>
<th>( \Sigma X_2 )</th>
<th>( \Sigma X_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>2573</td>
<td>3118</td>
<td>2954</td>
<td>2825.4</td>
</tr>
<tr>
<td></td>
<td>( \Sigma Y^2 )</td>
<td>( \Sigma X_1 Y )</td>
<td>( \Sigma Y, X_1 )</td>
<td>( \Sigma Y, X_2 )</td>
</tr>
<tr>
<td>2573</td>
<td>26691</td>
<td>30170</td>
<td>26089.95</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \Sigma X_1^2 )</td>
<td>( \Sigma X_1 X_2 )</td>
<td>( \Sigma X_1 X_3 )</td>
<td></td>
</tr>
<tr>
<td>3118</td>
<td>46282</td>
<td>37206</td>
<td>31996.65</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \Sigma X_2^2 )</td>
<td>( \Sigma X_2 X_3 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2954</td>
<td>58980</td>
<td>30851.70</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \Sigma X_3^2 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2825.4</td>
<td></td>
<td>28527.6050</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table 20
Correlation Coefficients

<table>
<thead>
<tr>
<th>( X_1 )</th>
<th>( X_2 )</th>
<th>( X_3 )</th>
<th>Y</th>
<th>Equation Getting</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_1 )</td>
<td>1</td>
<td>.214</td>
<td>.470</td>
<td>.116</td>
</tr>
<tr>
<td>( X_2 )</td>
<td>1</td>
<td></td>
<td>.432</td>
<td>.094</td>
</tr>
<tr>
<td>( X_3 )</td>
<td>1</td>
<td></td>
<td></td>
<td>.021</td>
</tr>
<tr>
<td>Y</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E.G.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Subscripts are identified as follows:

1 — I.Q.
2 ~ Aptitude
3 — Reading Level
4 — Method
U — Equation Getting

Correlation coefficients are calculated for the following array:

<table>
<thead>
<tr>
<th></th>
<th>X₁</th>
<th>X₂</th>
<th>X₃</th>
<th>Y</th>
<th>E.G.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>r₁₂</td>
<td>r₁₃</td>
<td>r₁₄</td>
<td>r₀₁</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>r₂₃</td>
<td>r₂₄</td>
<td>r₀₂</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>r₃₄</td>
<td>r₀₃</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>r₀₄</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

All of the r's excepting those with subscripts zero are calculated according to the following formulas:

\[ r_{xy} = \frac{L_{xy}}{\sqrt{L_{xx}} \sqrt{L_{yy}}} \]

where \( L_{xy} = N \Sigma xy - \Sigma x \Sigma y \)
and \( L_{xx} = N \Sigma x^2 - (\Sigma x)^2 \)

The r's with subscript zero are calculated by the following point-biserial formulas:

\[ r_{01} = \frac{x_{le} - x_{lc}}{\sigma_{x_1}} \sqrt{(pq)} \]

where \( \sigma_{x_1} = \sqrt{\frac{\Sigma x_1^2 - (\Sigma x_1)^2}{N - 1}} \)

and p and q are the proportion of N in the control and experimental groups, respectively.
The values of the various \( r \)'s have been indicated in Table 20.

The Doolittle method includes a device for calculating the Betas in the following set of simultaneous equations:

\[
\begin{align*}
\beta_1 + \beta_2 r_{12} + \beta_3 r_{13} + \beta_4 r_{14} &= r_{01} = 0 \\
\beta_1 r_{12} + \beta_2 + \beta_3 r_{23} + \beta_4 r_{24} &= r_{02} = 0 \\
\beta_1 r_{13} + \beta_2 r_{23} + \beta_3 + \beta_4 r_{34} &= r_{03} = 0 \\
\beta_1 r_{14} + \beta_2 r_{24} + \beta_3 r_{34} + \beta_4 &= r_{04} = 0
\end{align*}
\]

The calculations are given in Appendix I.

\[
\begin{align*}
\beta_1 &= .114292 \\
\beta_2 &= .183036 \\
\beta_3 &= .195514 \\
\beta_4 &= .003049
\end{align*}
\]

The interpretation of these Betas is that, of all the variables considered, the method used in framing equations had least influence on the success of the pupils in framing equations; in fact its influence was slightly negative. I.Q. was fairly influential. Reading level and aptitude were the most influential, and of those two reading was the more influential.

The equation for multiple correlation coefficient \( R_{0.123...n} \),

\[
R_{0.123...n} = \sqrt{(\beta_1 r_{01} + \beta_2 r_{02} + \cdots + \beta_n r_{0n})},
\]

can easily be solved for \( R_{0.123} \) and \( R_{0.1234} \) from the data on hand. Both \( R \)'s equal .103, to the nearest thousandth, a fact which also emphasizes that the difference between the experimental and traditional method had no significant effect on the success of the pupils in obtaining equations for problems.
In other words, the variance in ability of these pupils to form equations for problems, under the conditions of this experiment, was almost entirely accounted for in terms of differences in I.Q., aptitude, and reading level, leaving practically no variance which could be attributed to differences in method.

Factors which May Have Affected the Validity of the Experiment

In this experiment several factors were not anticipated which it would have been better to take into account, and there were other factors which, though anticipated, could not expediently be changed in the experimental situation.

1. Five sections of algebra for more able students were not used in the present experiment. If the combination method is a good tool for the middle ability group of students, it may also be a good tool for the best students as well.

2. The writer was not informed that the text was not to be followed sequentially, until the materials to replace much of the first chapter had been revised and completed. As a result the introduction to algebraic language was probably not conducted in the experimental classes in the most effective way to prepare for the combination method.

3. The prior teaching experience of all of the experimental teachers with the combination method was extremely weak.

4. The writer was the only source of information and guidance these teachers had. Although he had several years of acquaintance
with the combination method, he had not taught it himself to ninth-grade classes. He had used it briefly with college students reviewing algebra.

5. The teaching of the experimental and the control classes was not supervised during the course of the experiment because of the distance of travel which would have been required. The writer visited each of the six experimental classes twice only.

6. The writer had not anticipated that many pupils in the experimental sections would have had an introduction to solving verbal problems by the uni-equation method during their eighth-grade work; had this fact been known or discovered early enough, some adjustments in sections might have been possible to allow for it.

7. For purposes of testing whether or not the combination method makes pupils more self-reliant, it would have been desirable to have some unfamiliar types of problems on the final test. These could have been problems using such relationships as had not been used previously in algebraic problems for these pupils but with which it could be assumed that the pupils had had some experience in their earlier arithmetic. The teachers of the control sections, however, objected to including such problems in the test, and changes which they proposed were accepted.

8. In the school where the experiment was conducted it was expedient because of scheduling problems to have each of the three experimental teachers teach two experimental sections, but no control sections. These teachers could not then judge from their immediate experience the relative values of the two methods being compared in
the experiment. It might perhaps have been better to confine the experiment to six sections and three teachers, each teaching a control and an experimental section.

9. No attempt was made to measure and control the relative conscientiousness with which the nine teachers taught the unit on "Equations and Problems": the prevalent impression that more effort is put into experimental situations may or may not have been true. If extra effort was expended in the experimental sections, it may have offset the effects of unfamiliarity with the method. On the other hand, it is possible that some of the control teachers, perhaps unconsciously, knowing that their pupils would take an equation-writing test at the end of the experiment, may have expended special effort in teaching the unit in their sections.

Opinions of the Experimental Teachers

The writer wrote to each of the teachers of experimental sections at the close of the second semester, asking generally for two reactions: (1) how the teacher felt about the method at the time that he was using it and (2) what effects he had noted on the pupils' work in the second semester. There were several other questions asked for the purpose of stimulating the teachers' thinking in their answers to the two questions. While the letters to the three teachers differed slightly in wording they were essentially alike. The letter to one of the teachers and the replies of all three teachers are included in Appendix J.
In general the three teachers agreed (1) that the method helped the weaker pupils most; (2) that pupils disliked writing the detail of the combination method; and (3) that the experimental method accomplished about what the traditional method would have accomplished.

One of the teachers suggested a greater stress on simultaneous equations prior to starting verbal problems. Two teachers felt that some time was saved in the second semester as a result of introducing simultaneous equations in the first semester. Another thought that pupils found it easier to say "X is larger and Y is smaller, X-20=Y," than "Y is the larger and X-20 is the smaller."

All three teachers stated that the pupils objected to the detail and writing which is used in working out the solution by the experimental method; they had also mentioned this reaction during the course of the experiment.

Further inquiry at the time of the experiment brought out the fact that many of the pupils in the experimental sections were familiar with the traditional uni-equation method from their eighth-grade work. If the pupils had known no other method and had nevertheless suggested shortening the procedure because they felt confident that they were able to get along without some of the detail, their objections would have had greater significance. The pupils' attitude and resistance toward the experimental method could very easily have affected the results of the experiment.

Finally, the writer wishes to acknowledge a splendid spirit of cooperation on the part of all the personnel of Lyons Township High School with whom he had any contact during the course of the experiment.
Summary

To test whether or not the teachers of the experimental and control sections achieved equally well in their various sections, the writer analyzed the scores of the pupils in these sections on two unit tests. The two units were part of the class work of the various sections prior to the unit with which the experiment was concerned.

For the scores of Unit 1, it could not be determined that either the variances or the means of the various sections were alike. Hence it could not be concluded from these data that the teachers achieved equally. For the scores on Unit 2, it could be established that the variances did not vary significantly; however, it could not be established that the means did not differ significantly. Hence it could not be concluded that the achievement of the teachers was equivalent on Unit 2.

When the scores of all the pupils in the control sections and the experimental sections were pooled separately, the two groups had equal means on the test of Unit 1. On the means resulting from the pooling of the two groups of sections on Unit 2, the difference of the means was significant. The mean of the control group was significantly higher; hence, if there was any advantage involved, it was in favor of the control teachers.

The reliability of the instrument developed in the course of the study to test the equation-getting success of the pupils was determined by the Hoyt method to be .79. Since the test had only sixteen items, this estimate of reliability is high; from some points
of view, the Hoyt method might be considered to produce higher estimates than certain other methods would. The analysis indicated by means of a significant F that the test measured individual differences adequately.

It was not established that the sections to be pooled into experimental and control groups in the experiment had equal variances in the pupils' abilities to frame equations for problems. Nevertheless, the analysis of variance and covariance was undertaken, with the scores on the test as the criterion variable and the record of the individual pupil's I.Q., aptitude, and reading level as control variables. The F determined in the analysis of the Y variances due to regression was non-significant. Hence it may not be concluded that either the experimental method or the traditional method was more effective than the other; the differences in the pupils' ability to frame equations can be accounted for almost entirely by differences in their I.Q., aptitude, and reading level.

By means of Betas determined from correlation coefficients among the various variables in the experiment, it was determined that under the conditions of the experiment, reading level and aptitude were the most influential factors; aptitude was less influential than reading level, but more influential than I.Q.

The principal conclusion of this study is that under the conditions of the experiment, the experimental method and the traditional method proved to be about equally effective in teaching pupils to formulate equations for problems.
CHAPTER IV

CONCLUSIONS AND RECOMMENDATIONS

Conclusions from the Experiment

The unit on "Equations and Problems" with which the experimental part of this study was concerned was taught in all thirteen sections of "r" algebra during the last quarter of the first semester. This plan of the mathematics department at Lyons Township High School necessitated considerable rearrangement of the materials in the textbook, for the textbook introduced a considerable amount of work with verbal problems in its first chapter. The sequence of units was:

1) Algebraic Language, 2) Signed Numbers, 3) Algebraic Operations, and 4) Equations and Problems. The scores on the regular departmental tests for the first two units of work were studied in order to determine whether or not the achievement of the various sections was uniform, and from this information to deduce whether or not the group of teachers for experimental sections may have differed considerably from the group of teachers for control sections in the achievement of their pupils. The conclusions from this part of the study are:

1. From the results of the test for unit 1, when the scores of all the pupils in control sections and also those in experimental sections were pooled, the means of the two groups of sections were alike. It could not be established, however, that the variances of the sections were homogeneous.
2. From a similar pooling of the results of the departmental test for unit 2, the mean score of the students in the control sections was significantly higher than that for the experimental sections. Homogeneity of the variances of the sections also could not be established.

3. These two conclusions indicate that there is a possibility that the teachers of control sections had better achievement in their sections. The possibility that this better achievement arose from distribution of students among the sections must also be recognized. This possibility is slight, however, for the means of the combined experimental sections were higher than the means of the control sections in I.Q., algebraic aptitude, and reading level.

4. These results, although not conclusive, provided evidence that the experiment would not be biased in favor of the experimental method as a result of differences in the ability of the teachers. It must be borne in mind, however, that all the teachers of control sections excepting one had taught the uni-equation method many times, while the combination method was unfamiliar to the experimental teachers.

The second aspect of the results of the experiment concerns the test which was used to evaluate the success with which the pupils were able to derive equations for problems.

1. A study using analysis of variance indicated that the scores made by 291 students on the test produce an estimate of the reliability of the test to be .79. Since the test contained only sixteen items with scores ranging as integers from 0 to 16, a reliability of .79 is relatively high.
The test measured individual differences among the students adequately since the differences of scores proved to be significant.

When the scores of the pupils in the experimental and control sections were studied, with differences in I.Q., aptitude, and reading being taken into account, the following results were obtained:

1. It could not be established that the experimental method differed significantly from the traditional method. In other words, practically all of the variance in the ability of the pupils to write equations for problems at the close of the experiment could be accounted for in terms of original differences in I.Q., aptitude, and reading level.

2. The regression coefficients found in the analysis of variance and covariance do not serve to indicate the relative weights of the variables. After the data had been extended properly so that they were expressed in terms of standard units of variance, measures of the relative weights of the variables were determined to be:

   Reading Level .... 20
   Algebraic Aptitude ... 18
   I.Q. .............. 14
   Method ............. 00

Reading level had the greatest influence on the success of the pupils in framing equations. The difference in teaching method had practically no effect.

From letters written by the teachers of the experimental sections before they were informed of the results of the statistical analysis of the data, the following conclusions may be drawn:
1. During the course of the experiment these teachers felt that the combination method was producing in the students about the same skill and ability that would be expected from the traditional method.

2. These teachers felt that the combination method helped the weaker students most.

3. All teachers indicated that the pupils disliked writing out the steps of the solution in detail, as the combination method required. The expression of dislike for this feature must have resulted from a comparison with the more direct uni-equation method, either through prior experience with that method, or through contact with students in the control sections. The writer was not informed until the experiment was well under way that many students in the experimental sections had had some experience with algebraic problems by uni-equation methods in their eighth-grade work.

The final conclusion of the present study is that under the conditions of the experiment, the combination method proved to be about as effective in teaching pupils to derive equations for problems as the traditional uni-equation method is.

**Recommendations for Further Research**

Some of the conclusions discussed in the previous section have significant implications for further research:

1. Since psychological and methodological considerations indicate that the combination method has important possibilities in connection with minimizing some of the difficulties pupils encounter in attempting to apply the uni-equation method, further experimentation with the
2. Better design of such experiments should be possible from noting some features not incorporated in the present study: (a) The teachers using the combination method should have greater familiarity with the method; perhaps use of the method for a year or more prior to the experiment should be required. (b) Precaution should be taken that the pupils to whom the combination method is taught have not had prior acquaintance with the traditional method. Each of the three teachers in the present experiment noted resistance to the detail necessary in the early stages of application of the guess-and-check procedure and attributed the resistance to this cause. (c) If departures from the sequence of the textbook other than those necessary in providing for the earlier introduction of multi-equation methods are to be made, the planning for the experiment should take these into account. (d) Similar experiments should be carried out so as to include a wider range of student ability. (e) Perhaps it would be advisable for the same teacher to teach both traditional and experimental sections.

3. Experimentation with the preliminary guess-and-check procedure is also needed for one-unknown problems, for those leading to quadratic equations, and for those leading to sets of equations with one or more quadratic equations. The combination method should enable students to discover implicit relationships and forms for equations in such problems as effectively as in the multi-unknown linear-equation problems considered in this study.
1. The writer has on file the 291 test papers on which the conclusions of the present investigation have been based. A study of the kinds of errors made by pupils in writing equations for the problems in this test may very well reveal information which can be helpful in planning for the use of the combination method in future experiments. Relating the frequency of the various kinds of errors made to the method by which the pupils were taught may also reveal pertinent information about the methods.

5. Unfortunately, the final test omitted types of problems unfamiliar to the pupils. The writer has reason to believe that pupils who have learned to use the combination method would be more successful in deriving equations for unfamiliar types of problems than pupils who have learned only the traditional uni-equation method. Experimentation is needed to verify this hypothesis.

6. There is some evidence that many of the best students of algebra develop methods of attacking problems which are original. Perhaps some of these methods resemble or involve the combination method. An investigation is recommended dealing with the kinds of methods used by students who are most successful in attacking verbal problems; the purpose of such a study would be to determine whether there are any features common to the methods of these students to which their success may be attributed, and, if so, whether methods based on such features can be worked out for algebra classes in general.
The relative efficacy of the two methods might be tested for the students' retention of the ability to write equations after a lapse of time, e.g., at the end of the course and perhaps also after a year or so.
APPENDIX A

RECOMMENDATIONS RELATED TO GUESS-AND-CHECK PROCEDURES
AND TO THE USE OF MULTI-EQUATIONS
IN METHODOLOGICAL LITERATURE

In order to determine the extent to which some of the unique features of the combination method have been advocated the writer has endeavored to examine all the methodological books for the teaching of algebra and mathematics in the secondary schools which have been published in English since 1900. He has found no author who advocates or even recognizes the guess-and-check preliminary procedures; it follows, of course, that none is concerned with the combination method. Consequently this review will deal with those authors who advocate or at least recognize the multi-equation method for multi-unknown problems. The chapter is divided into two parts, the first dealing with the suggestions of authors of methodology books, and the second reviewing the recommendations of two writers who have advised students on the study of mathematics.

Methodology Books Recognizing the Multi-equation Method

Although there are many methodology books which discuss the formal solving of sets of simultaneous linear equations, very few illustrate the use of simultaneous equations for solving verbal problems. The recommendations of the five authors who do recognize
the technique will be reviewed in some detail. They will be considered
in chronological order.

Barber

In his book *Teaching Junior High School Mathematics*, Barber
advocates that all the topics to be studied in ninth-grade algebra
grow out of problems. Among other things, problems are the "vehicle
for the introduction of algebraic processes" as well as the source of
drill materials such as those needed for translation exercises.\(^1\) Hence
it is not surprising for him to state:

> When a problem is met in which the quantities cannot
> be expressed in terms of one unknown, we have found the
> reason for two unknowns and the linear pair. . . .\(^2\)

Although this statement has not been elaborated, Barber's attitude
regarding the use of multi-equation techniques for multi-unknown
problems is apparent in it. He recommends that the uni-equation
method be used until problems are encountered so difficult to express
in terms of one unknown symbol that the pupils find it quite impossible
to make the mental substitutions required in that method. In this
approach the use of multi-equations is based exclusively on necessity;
Barber omits any consideration of their use to increase the students'facility in understanding the relationships by which the quantities

---


\(^2\) Ibid., p. 102.
are related. This attitude ignores the fact the teachers find that many pupils encounter difficulty in making the kinds of mental substitutions needed in the uni-equation method almost from the very start of the study of problems. It does not recognize that the multi-equation method may be an aid to pupils in expressing relationships in problems and may make the correspondence of the algebraic symbolism to the verbal phrases in the problem more direct and more easily recognizable. Certainly Barber does not show any appreciation of the reasons which led Lazar to incorporate the multi-equation method into the combination method; Barber exemplifies the traditional point of view.

Ligda

Ligda is the only author of a methodology book who recommends that teachers introduce the multi-equation method early in the course so that it is available to use with multi-equation problems, and that they then use it exclusively for such problems. He says:

Tradition places the study of the simultaneous equation in the middle of the one-year course, the various methods of solution being grouped together. This arrangement was logically correct when symbolic solution was taught before any problems were presented. But this sequence causes considerable trouble in the earlier parts of the subject now that problems are presented from the first. Restriction to problems which can be expressed at once in one equation, combined with the unwise addition of difficult problems, has made teaching of these earlier parts arduous, dull, and discouraging. What teacher, when asked by the class to solve some complex problem by the one-equation method, has not wished for the simpler and more effective many-equation method! How often the time of the class has been wasted by fruitless efforts to "write down" an equation in terms of one unknown! Why is the comparison of ratios taught in the seventh grade considered too hard for high-school freshmen? Why is substituting on paper considered more difficult than
substituting mentally, as when the student is told to "express" the other unknowns in terms of one?

An experiment was recently conducted to test the soundness of the traditional sequence. Two classes of about the same average ability with the same teacher were taught, one in the usual sequence, while the other was introduced very early to simultaneous equations. It was soon found that the many-equation method was greatly superior in that it allowed the use of a larger number of illustrative problems. It was found especially valuable in justifying the use of parentheses and in illustrating the change of signs in negative quantities inside the parentheses. It facilitated the work so much that by the end of the term one class led the other by so wide a margin in amount of ground covered and in thoroughness that even superior ability could not be claimed as the determining factor in the case.3

It is unfortunate that Ligda has not published more details of the experiment to which he refers. He presents, however, four arguments for the early introduction of the multi-equation method:

(1) Most simple-equation problems are merely multi-equation problems in disguise. (2) The multi-equation method is the one used in practical formula work. (3) The early introduction of the multi-equation method will provide more opportunity for the pupils to master a process which will be more useful than the uni-equation method in practical situations. (4) The multi-equation method is inherently a part of what Ligda calls the "functional method."4

In order to explain his "functional method" Ligda analyzes the kinds of relationships which exist among the numbers in problems.


4Ibid., pp. 64-65.
The first kind are relations "between the quantities in a given situation." These are called "characteristic formulas" and are almost always implicit; for example, the formula $d = rt$ in a motion problem, in which there may be two situations, going and coming, or a slow train and a fast train. The second kind are the relations between situations: for example, that the distance in the two situations is the same, or that the time in the two situations differs by a certain amount, or that one rate is five miles an hour faster than the other rate. Most of these will very likely be given explicitly, but they may sometimes also be implicit.

The essential features of Ligda's "functional method" for the analysis of problems and the deriving of equations are: (1) recognition of a number of situations in a problem, (2) recognition of the "characteristic formula" for the situations, (3) recognition of relationships between the same quantities in the various situations, and (4) formation of equations by substitution of data into the characteristic formulas and other relationships. To illustrate:

Problem: A man has 22 minutes to get to the station, a distance of 2 miles. If he takes a car which travels at the rate of 1 mile in 8 minutes, at what distance from the station can he get out and walk, if he walks at the rate of 1 mile in 16 minutes?

---

5Ibid., pp. 112-115.
6A more complete chart of the analysis may be found ibid., p. 131.
The two situations are the man walking and the man riding; the characteristic formula for each situation is the formula $D = RT$ for the man walking and $d = rt$ for the man riding. Although it is not essential, Ligda suggests for convenience the use of a "relation diagram" for the organization of the analysis. In the diagram, the characteristic formulas are written in vertical form, one for each situation, and the various quantities are assigned their values in the problem or related to each other horizontally. The completed "relation diagram" for the sample problem is:

\[
\begin{align*}
D + d &= 2 \\
\frac{1}{16} &= R \\
x &= 1/8 \\
T + t &= 22
\end{align*}
\]

Ligda explains the formation of the simultaneous equations from the diagram in the following way:

- We eliminate the $r$'s by substituting their numerical values in the characteristic formula $r = d/t$. After this we have a choice of eliminating either the $t$'s or the $d$'s, leaving two equations and two unknowns.  

The characteristic formula is almost always implicit. How does the pupil discover it? Is it sufficient merely that the pupil expect that there be one? Ligda's functional method may provide the motivation to search for an implicit relationship, but it provides no

---


8Ligda, The Teaching of Elementary Algebra, p. 128.
specific means or technique such as the guess-and-check procedure for helping to find it.

Ligda's emphasis on relationships and his analysis of the various kinds which are found in typical problems are useful, and there is no doubt that the relation diagram shows the structure and interrelationships of the problem in a dramatically convenient form, when it is finally completed. Should it be necessary, though, for the pupils to analyze the internal structure of the problem in completely symbolic form before any use is made of the numerical data which are provided? Can a ninth-grade pupil generally be expected to ignore the numerical values provided as data and think of all the corresponding quantities by means of abstract symbols? It seems unnecessarily indirect to attempt to use these general forms in the beginning work with verbal problems, when one of the aims of the course is to relate the algebra to arithmetic. Pupils need the assurance of thinking with specific numerals. After more experience with a mixture of numerals and letters, they may be ready for the generalized solutions and the construction of relation diagrams such as Ligda illustrates.

The slight recognition of Ligda's point of view in subsequent literature on the teaching of algebra is confined to periodical literature. Perhaps the essential feature of his recommendations — the use of the multi-equation method for multi-unknown problems — was obscured by the details involved in analyzing situations and finding characteristic formulas. It may be that the functional method failed to gain attention because there was no specific technique for helping
pupils discover the characteristic formulas. The features of the combination method devised by Lazar minimize both these deficiencies.

Hassler and Smith

As a first step in the analysis of problems these authors advise pupils to list all the quantities and relationships in verbal form before attempting to express anything in algebraic symbols.

He [the pupil] is then ready to decide which one of the quantities is most suitable to be considered as independent and can then express all the other quantities in terms of it. He may find, of course, that the relationships are so complicated that it would be better to use more than one unknown [symbol].

At first it may seem that this point of view is similar to that of Barber, for it makes the choice between the multi-equation and uni-equation methods contingent upon necessity. The discussion which follows, however, seems less dogmatic. Their recommendations to teachers include the following:

Point out clearly that a condition in a problem leads to a relation between the symbols or to an equation. If $n$ unknowns are mentioned, there are $n$ statements of fact in the problem if it can be solved. If a different symbol is used for each unknown, there will be $n$ equations. If only one symbol is used, $n - 1$ statements are needed to express the $n - 1$ unknowns and the remaining statement gives rise to the equation.\footnote{Jasper O. Hassler and Rolland R. Smith, The Teaching of Secondary Mathematics (New York: The Macmillan Company, 1937), p. 287. Quoted by permission of J. O. Hassler.}

\footnote{Tbid.}
No preference between the methods is implied by this statement.

Two problems, a digit problem and one dealing with the angles of a triangle, each with three unknowns, are then put into equations, first by the multi-unknown method and then by the uni-equation method, to illustrate the application of this idea. The digit problem follows:

The sum of the three digits of a number is 20, and the digit in ten's place exceeds the digit in unit's place by 5. If 594 is subtracted from the number, the digits will be reversed. Find the number.

This problem involves three unknowns, the digits, and we find three independent statements of fact about them. If we use three independent (or arbitrary) symbols u, t, h, in stating the problem we have one equation arising from each statement, as follows:

\[ h + t + u = 20, \]
\[ t = u + 5, \]
\[ 100h + 10t + u - 594 = 100u + 10t + h. \]

Choosing arbitrary symbols for all the unknowns requires no use of the facts given in the problem. We might as well have the same symbols for any other problem involving unknowns.

If we work with but one unknown, we need the first two statements to express the other two of our unknowns in terms of the arbitrary symbol chosen for the independent unknown. We need in this case only one equation and to get it, we make use of the last of our three statements.

We then have

\[ u = \text{unit's digit}, \]
\[ t = u + 5, \]
\[ h = 20 - (u + u + 5), \]
\[ 100(15 - 2u) + 10(u + 5) + u - 594 = 100u + 10(u + 5) + (15-2u).\]

The uni-equation approach cited by these authors for this problem draws heavily on the multi-equation approach previously made and would have been practically impossible for ninth-grade pupils without it.

---

11 Ibid., pp. 287-288.
In fact, three symbols are used in setting up, though not in stating, the final equation. There is here tacit admission that the type of thinking done in the multi-equation approach is indispensable for the uni-equation approach. The expectation that pupils can make the necessary substitutions completely mentally is unrealistic for most ninth-grade students, since most of these students need to "see" these substitutions explicitly, as when substitutions are used to eliminate symbols in the solving of a set of equations for roots.

While the writer is not completely sure of the authors' intent in the two sentences:

Choosing arbitrary symbols for all the unknowns requires no use of the facts given in the problem. We might as well have the same symbols for any other problem involving unknowns.

one conclusion seems certain: they are ignoring the mnemonic use pupils may make of initial letters in such instances. If the authors mean to imply that they much prefer to have pupils state:

\[
\text{Let } u = \text{unit's digit,} \\
\text{and } u + 5 = \text{ten's digit,}
\]

than to have them state:

\[
\text{Let } u = \text{unit's digit,} \\
\text{and } t = \text{ten's digit,} \\
\text{Then } t = u + 5,
\]

then they are merely stating their preference for the uni-equation approach. The second (multi-equation) approach is much more suggestive of the context of the numerical symbolism than is the first, at least to the relatively inexperienced student.
The development of the uni-equation result for the angle problem similarly depends upon the multi-equation development which the authors present just previously; hence it is subject to the same criticisms.

Following these examples, the authors elaborate their idea of an "independent unknown" a term which they introduced in the first quotation above (see p. 111). While it may be true that there is one specific unknown in terms of which it is best to express the others when the pupil is attempting to use the uni-equation approach, no consideration need be given to the existence of an "independent unknown" during the analysis of a problem and the deriving of equations if the multi-equation method is used. This idea becomes merely a convenience in combining the equations to eliminate symbols so that roots can be found, and no attention need be given to it while the student is analyzing the problem for the formation of equations, i.e., prior to the completion of the set of equations.

Most of the relationships in algebraic problems are simple sums, differences, and ratios of unknowns. In such relationships it is often desirable that the students be able to change the role of the unknowns at will. For example, if the ages of two boys total 20 years, who can say with authority that the age of the younger is the "independent" unknown or the "dependent" one? Likewise, if a boy is one-third as old as his father, the student should be able and permitted to use this simple relationship in the form: "The father is three times as

\(^{12}\text{Ibid.},\ p.\ 289.\)
old as his son," as well. Whether a particular unknown is "independent" or "dependent" is incidental to the particular form in which the relationship happens to be symbolized; it is not a feature inherent in the nature of the problem. Hence it is the writer's conclusion that Hassler and Smith have stressed here their concept of an independent unknown merely to rationalize their advocacy of continued use of the uni-equation method. They have not recognized that a method such as the combination method makes the idea of an independent unknown completely superfluous.

Kuppuswami

In his book, published in India, Kuppuswami has also seemingly approached the multi-equation idea, one of the features of the combination method, without actually realizing its full import. In presenting a list of steps for pupils to follow in problem solving, immediately after assigning \( x \) as the symbol for one unknown, he advises the pupil to "consider the necessity or the convenience of representing any other quantity by \( y \)."\(^{13}\) Yet he never illustrates this advice nor does he resume any discussion of it.

After a complete discussion of the relationships involved in a problem like that presented by Ligda, and the complete analysis of an example, he summarizes his method. The first item in the summary

is the following statement: "Frame as many relational statements as you can, one of which at least should contain \( x \), the unknown quantity." One would expect that if as many relational statements as possible were to be listed this procedure might lead very naturally to a multi-equation situation. Yet there is no evidence in the book that this device was ever used with simultaneous equations as the result.

Whatever the reason, Kuppuswami seems to have recognized some of the potentialities of the multi-equation method without realizing its full import or capitalizing on it as Iazar did in the combination method.

Minnick

In discussing the solving of verbal problems, Minnick emphasizes the idea of a "principal" unknown, which is very similar to the idea of an "independent" unknown advanced by Hassler and Smith. He says:

Determining the principal unknown. Algebraic problems involve two or more unknowns. Even when a single equation with one unknown is used, there are two unknowns, one being expressed in terms of the other. The following problem is an illustration: William and Henry received $15.00 for doing a piece of work. William worked twice as long as Henry. How much did each receive?

Let \( h \) = the amount received by Henry.
Then \( 2h \) = the amount received by William.
\[ \therefore h + 2h = 15. \]

\(^{14}\) Ibid., pp. 388-389.
It should be noted that although apparently a single equation with one unknown has been used, the process involves two simultaneous equations with two unknowns. The thought process is equivalent to the following:

Let \( h \) = the amount received by Henry, and \( w = \) the amount received by William. Then \( h + w = 15 \), and \( w = 2h \). 
\[ \therefore h + 2h = 15 \]

If, in the first solution, the amount received by William had been chosen as the principal unknown quantity, the procedure would have been:

Let \( w = \) the amount received by William. Then \( w/2 = \) the amount received by Henry, and \( w + w/2 = 15 \).

The relation between Henry's and William's amounts is expressed in a reverse order to that used in the statement of the problem, and the resulting equation involves fractions. Clearly, the solution is simplified by the proper selection of the principal unknown. If the pupil chooses the more difficult procedure and arrives at the correct conclusion, he should receive full credit, but he should be made conscious of the simpler solution.\(^{15}\)

The choice of the word "principal" to identify merely that unknown to be represented by an arbitrary symbol is most unfortunate because for pupils this term usually carries some connotation of importance. In the usual algebraic verbal problem no one of the unknowns can be singled out as being more important than any other, for they are unknowns simultaneously, and no one of the unknowns can be found numerically without full consideration of the others. Often it is completely immaterial which of the unknowns is so selected, when the

uni-equation approach is being used; furthermore, when the selection happens to be significant, its advantage is involved in solving the equation for its root, not in deriving the equation in the first place.

Minnick clearly recognizes in the quotation cited above that the kind of thinking basic to the multi-equation approach is also essential for the uni-equation approach, except that in the latter much of it is not recorded in writing. Why then does he not recommend the multi-equation approach as the more basic and the more likely to be understood by the pupils? Again there seems to be an attempt to rationalize continued advocacy of the uni-equation method with trivial reasons. It is amazing that Minnick, and Hassler and Smith as well, in discussing the uni-equation method, should have come so close to recognizing the more fundamental nature of the multi-equation method for pupils without realizing this fact and capitalizing on it pedagogically.

Features of the Combination Method

in Books of Advice to Students

The authors of the two books to be considered in this section were writing primarily to college and university students, but much of their advice is applicable to ninth-grade algebra classes as well.

Dadourian

In his book How to Study; How to Solve, Dadourian provides a list of fifteen recommendations or directions for solving problems. One of
these, number 8, concerns the writing of equations for algebraic problems.

8. Write all of the principal equations necessary for the solution of the problem before manipulating any of them.16

This implies, of course, the multi-equation method. In the elaboration of this statement, the author points out that he considers the principal equations to consist of the mathematical definitions of the required magnitudes and the equations obtained by translating the conditions specified in the problem into the mathematical language. The remainder of this discussion on this point consists of a consideration of the difficulties of trying to make literal translations of English statements into mathematical language and two examples worked out in multi-equation form. Although the first of these two examples is perhaps too difficult for beginning ninth-grade students, the second is similar to many which they encounter:

(2) A bridge is to be built at a cost of $1,000,000; if the state is to contribute twice as much as the county, and the county 50% more than the town, what is the contribution of each?


17 Ibid., pp. 39-40.
Solution. Let $s, c, \text{ and } t,$ denote the contributions of the state, the county, and the town, respectively; then the principal equations of the problem are

$$s + c + t = \$1,000,000, \quad s = 2c, \quad \text{and} \quad c = \frac{3}{2} t.$$ 

Eliminating $s$ and $c$ from the first equation, we get

$$3t + \frac{3}{2} t + t = \$1,000,000;$$

hence

$$t = \$181,818.18$$

$$c = \$272,727.27$$

and

$$s = \$545,454.55$$

This example illustrates well a multi-unknown problem and its solution by the multi-equation method. The typical uni-equation approach to this problem would have the pupil attempt to write the equation

$$3t + \frac{3}{2} t + t = \$1,000,000$$

without the benefit of explicitly writing equations for the three prior relationships. The biggest difficulty for pupils in solving this problem probably would be the realization that it implies that the total cost is shared by the three governmental levels mentioned. Is there a better way to cause pupils to realize this than to suggest amounts for each of the agencies to contribute and ask the pupils if there is enough money to build the bridge? This is precisely the purpose of the guess-and-check technique.

In general, all the algebraic problems illustrated in this book are solved by the multi-equation method. The ninth suggestion emphasizes the author's belief in simultaneous equations:

---

18bid., p. 41.
9. Solve the equations simultaneously for the required magnitudes, and obtain an expression for each in terms of the given, and only the given, magnitudes.¹⁹

This suggestion, however, requires a general solution in completely symbolic form prior to the use of numerical values for any of the data. Such a procedure is too advanced for most ninth-graders.

As is usual with authors who recommend the multi-equation method, Dadourian suggests no supplementary technique to help the student become aware of the implicit relationships in the problems.

Polya

The advice Polya gives to students in his book How to Solve It, as far as algebraic problems are concerned, is similar to that in Dadourian's book just discussed. Algebraic examples are solved generally by the multi-equation method, and there is no need to repeat these suggestions.

Polya, however, makes another suggestion, which in a certain interpretation bears a relationship to the guess-and-check feature of the combination method. One of the suggestions he makes for students having difficulty is given in the form of a question: "Do you know a related problem?", the implication being that aid may be found in the related problem. He continues:

¹⁹Tid., p. 28.
The difficulty is that there are usually too many problems which are somewhat related to the present problem, that is, have some point in common with it. How can we choose the one, or the few which are really helpful? Polya never answers this question in a completely satisfactory manner, for his suggestions are vague and nebulous. He suggests finding a related problem that has the same or similar unknown, or one that has the same or similar data. He gives as an example a problem which concerns finding the diagonal of a rectangular solid. He states that related problems include any with unknown line segments, unknown sides of triangles, for example, and he moves from these related problems to the one of finding the diagonal of a rectangle. This in turn suggests a plan of attack for the original problem. It seems obvious that the chain of reasoning here illustrated is more mature than can be expected from most ninth-grade students. Yet the basic idea seems to be practical; it requires more explicit and more systematic development.

Polya calls each supplementary problem considered in the above chain an "auxiliary problem." It is a problem which "we consider, not for its own sake, but because we hope that its consideration may help us to solve another problem, our original problem." Although Polya clearly understood and made use of the fact that related...
auxiliary problems could be a source of help to the students and although he made many suggestions of ways to find related auxiliary problems, he did not realize that checking a set of guessed answers might often be the very auxiliary problem which could be most useful to the pupil. Furthermore this technique for finding the related problem is more definite than any he suggested, and is possible for all algebraic problems.

The need for the kind of aid which the guess-and-check procedure could provide was certainly recognized by Polya, but this particular means of achieving it seems to have eluded him.

**Summary**

In this chapter the writer has reviewed seven books by different authors insofar as the books deal with the two features of the combination method; five of these were books on methodology for teaching secondary-school mathematics, and two were books offering advice on problem solving. Only one of the authors, Ligda, advocates fully the use of the multi-equation method for multi-unknown problems. While all the others recognize the multi-equation method as a possibility, and some use it almost exclusively for illustrations, none stresses the advantages of the multi-equation method as Ligda does.

In general those authors who recognize the multi-equation method but still advocate the uni-equation approach for educational purposes have not adduced cogent arguments in support of their preference.
Rather surprisingly, in the exposition of their views they have failed to realize the full significance of the multi-equation method.

Similarly, though Polya suggested that the student might often find help from a related problem, he did not realize that the procedure of checking guessed answers was often the related problem which could be most useful to the student.

No author was found who anticipated the combination method.
Several articles in American educational journals have presented suggestions which depart from the traditional uni-equation approach and which advocate unusual procedures for deriving equations. Similarly, in spite of the traditional approach used in all textbooks which the writer has examined, their authors occasionally incorporate a suggestion or recommendation which is unusual and deserving of note. These suggestions often pertain to the use of preliminary guessing and checking or to early use of multi-equations; such recommendations will be examined in this appendix. Accordingly this discussion will be divided into four sections: (1) periodical articles which advocate the use of guess-and-check, (2) a textbook and a course of study which recommend the guess-and-check procedure, (3) periodical articles advocating the use of simultaneous equations, and (4) textbooks which introduce the method of simultaneous equations earlier than the typical book does.
Periodical Literature Dealing with
The Preliminary Guess-and-Check Procedure
for Deriving Equations

Only two articles have been found whose authors advocate the use of the guess-and-check procedure for deriving equations for verbal problems. These will be reviewed in chronological order.

Nyberg's "General Method"

In the chapter of the Seventh Yearbook of the National Council of Teachers of Mathematics which deals with the teaching of verbal problems, Nyberg has advocated what he calls a "general method." The following example will be quoted in detail to illustrate his suggestion:

Let us suppose that a pupil comes to class and says that he is unable to solve the following problem:

Mary has 3 more dimes than nickels and has a total of $1.65. How many nickels and how many dimes has she?

The teacher, instead of asking the customary questions leading to x nickels and x + 3 dimes and the value of the coins, asks the pupil to guess what the answer might be. And the pupil guesses, let us suppose that Mary has six nickels. This is, of course, an incomplete answer because the number of dimes should also be stated. But the teacher can overlook this deficiency and ask, "Can you in some way test the answer to see if your guess is correct?" The pupil will very likely do some mental arithmetic or some multiplying at the board and announce that his answer is wrong. The teacher's task now consists in making the pupil tell exactly what he did to find out if his guess is right or wrong. While this may seem easy to do, we know that pupils without proper training in talking before a class frequently cannot state just what they have done. Perhaps the pupil will say, "Well, 5 times 6 is 30, and 10 times 9 is 90, and that makes $1.20 which is wrong." The teacher must insist on having the pupil write on the blackboard exactly what he did.
T. Where did you get the number 5?

P. A nickel is 5 cents and 6 nickels are 30 cents.

T. Instead of writing the number 30 to show the value of the nickels, write $5 \times 6$ so that everyone in the class can see exactly what numbers you are using.

After the pupil has done this, the teacher continues in the following manner:

T. You said something about a number 9. How did you get the number 9?

P. If there are 6 nickels there are 9 dimes because Mary had 3 more dimes than nickels.

T. But how did you get the number 9? Did you add, subtract, multiply, or ......?

P. I added 3 to 6.

T. Instead of writing 9, write $6 + 3$ so that no one will ask later where you got the 9. Now what did you do with the number 9?

P. I got 90 cents because 9 dimes are worth 90 cents.

T. But how did you get the 90? Did you add, subtract, multiply, or ......?

P. I took 10 times 9.

T. Then write 10 times the quantity $6 + 3$. Instead of doing the additions and multiplications we will merely indicate them. After that what did you do?

P. I added 30 to 90.

T. You mean you added 5 times 6 to 10 times the quantity $6 + 3$. Write it in that way so that the class can see exactly what happens to every number. If your guess had been correct, this quantity would equal what number?

In this manner the teacher must get the pupil to write the following statement:

$$5 \times 6 + 10(6 + 3) = 165.$$
As a matter of good training the teacher should also insist that all of the pupil's analysis be written properly as shown in the following:

- The number of nickels = 6
- The number of dimes = 6 + 3
- The value of the nickels = 5 x 6
- The value of the dimes = 10(6 + 3)

We are now ready for the significant step in this method.

T. Now pick up an eraser; erase your guess, 6, wherever it appears in your work; and substitute the number x for the number 6.

After this substitution the pupil sees that his work presents the same appearance as it would have if he had worked the problem in the traditional manner, and he has found the desired equation,

\[ 5x + 10(x + 3) = 165. \]

Evidently the method consists in (1) guessing an answer, (2) indicating the arithmetic work necessary to check the guess, and (3) substituting x for the guess. But the arithmetic work must not be performed; it may only be indicated. Otherwise the pupil will be unable to substitute the number x for the number guessed. Hence when developing this method in class, it is well to precede the work with some remark about indicating arithmetic operations instead of performing them.\(^1\)

The guess-and-check procedure described and illustrated in Chapter I of this study is similar to Nyberg's illustration, with an important exception; in Nyberg's version the pupil guesses a value for only one of the unknowns and this leads to one equation only. Nyberg comments that guessing only the number of nickels rather than both the

---

number of coins is incomplete, but it seems certain, nevertheless, that had the pupil guessed both amounts, Nyberg would have had the teacher in the conversation direct the work in such a manner that one equation would have resulted. In this article, Nyberg advocated only one of the two features of the combination method developed by Lazar.

Another important difference is that Nyberg apparently had no intention of introducing pupils to the solving of problems by this means, for although he supposes a pupil having difficulty using the traditional uni-equation method in his introduction, nowhere in his article does he recognize the possibility of introducing problem solving through the guess-and-check procedure. Throughout his discussion the traditional approach is assumed to be the most desirable one, if students can master it.

Nyberg's discussion emphasizes the necessity for pupils to recognize and to use relationships. If a pupil is not sufficiently proficient in the use of relationships even to determine what arithmetic to use in checking guessed answers, how could he be expected to use these same relationships to set up equations based upon them, or to check answers obtained to the problem by any method whatsoever? Hence, another implication in Nyberg's discussion is that by means of having pupils check guessed answers for the problem the teacher may be alerted quickly to the fact that some pupils may not be sufficiently familiar with certain relationships to be able to write equations for the problems. This further implies that it may be necessary in the algebra class to provide experiences with certain relationships in problems which require only arithmetic methods.
As far as the writer has been able to determine, the guess-and-check procedure was never adopted in any textbook, even in those written by Nyberg himself, until a recent text by Smith and Lankford published in 1955. This textbook will be reviewed in a later section. No further mention of this feature of the combination method has been made in the literature until Yeshurun's article, also in 1955, which is reviewed in the following section.

Yeshurun's Tabular Version of Guess-and-Check

In an article entitled, "Let's Guess It First," Yeshurun has made a proposal similar to Nyberg's, with the exception that he records the work in a tabular form. For a typical mixture problem he uses the following example:

How much alcohol of 80% and how much alcohol of 40% must be mixed to get 80 grams of alcohol of 70%?

<table>
<thead>
<tr>
<th>Condition and its result</th>
<th>Original Form</th>
<th>Check</th>
</tr>
</thead>
<tbody>
<tr>
<td>The guessed number:</td>
<td>50 80</td>
<td>60 48</td>
</tr>
<tr>
<td>It contains 40 g. of</td>
<td>50 80</td>
<td>48</td>
</tr>
<tr>
<td>alcohol</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>There will be 30 g. of</td>
<td>80 50</td>
<td>20</td>
</tr>
<tr>
<td>40%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>It contains 12 g. of</td>
<td>(80-50) 40</td>
<td>8</td>
</tr>
<tr>
<td>alcohol</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>The composed alcohol is</td>
<td></td>
<td></td>
</tr>
<tr>
<td>80 g. of 70%</td>
<td></td>
<td>80</td>
</tr>
<tr>
<td>It contains 56 g. of</td>
<td></td>
<td>56</td>
</tr>
<tr>
<td>alcohol</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Check of the guess

\[ 40 + 12 \neq 56 \]

\[ \frac{50 \cdot 80}{100} + \frac{(80-50)40}{100} \neq 56 \]

\[ \frac{80x}{100} + \frac{(80-x)40}{100} = 56 \]

Whence \( x = 60 \).²

Check of the result

\[ 60 + 20 = 80 \]

\[ 48 + 8 = 56 \]

Perhaps the table would be improved by the addition of a column for the derived algebraic expressions between those of the original form and the final check to explain the origin of the expressions out of which the equation is finally composed.

It has been noted above that, in Nyberg's illustration, a guess was made of only one of the explicit unknowns and, as a result, only one equation was produced. In contrast, Yeshurun illustrates one digit problem with a symbol for each of the explicit unknowns. Prior to his illustration he states:

. . . Problems leading to systems of equations get almost the same treatment, except that:

1. Each unknown requires a separate guess.
2. There are as many checks involved as unknowns.
3. Several key numbers among the data, rather than one, are reserved for checking the guesses.
4. All results must check.³

A two-digit number is 8 times as great as the sum of its digits. The number with the same digits in reversed order is 18 greater than the sum. Find the number.


³Ibid., p. 19.
### Condition and its result

<table>
<thead>
<tr>
<th>Condition and its result</th>
<th>Original Form</th>
<th>Check</th>
</tr>
</thead>
<tbody>
<tr>
<td>Guess for the tens' digit: 3</td>
<td>x = 3</td>
<td>7</td>
</tr>
<tr>
<td>Guess for the units' digit: 2</td>
<td>y = 2</td>
<td>2</td>
</tr>
<tr>
<td>The original number: 32</td>
<td>3*10 + 2</td>
<td>72</td>
</tr>
<tr>
<td>The sum of the digits: 5</td>
<td>3 + 2</td>
<td>9</td>
</tr>
<tr>
<td>The tens' digit in the reversed number: 2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>The units' digit in the reversed number: 3</td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>The reversed number: 23</td>
<td>2*10 + 3</td>
<td>2(sic)</td>
</tr>
<tr>
<td>The sum of the digits: 5</td>
<td>2 + 3</td>
<td>9</td>
</tr>
</tbody>
</table>

### Check of the guess

Check of the result

\[
\begin{align*}
32 & \neq 8 \cdot 5 \\
23 & = 18 + 5 \\
3 \cdot 10 + 2 & \neq 8(3 + 2) \\
2 \cdot 10 + 3 & = 18 + (3 + 2) \\
10x + y & = 8(x + y) \\
10y + x & = 18 + (x + y) \\
\end{align*}
\]

whence \( x = 7 \) and \( y = 2 \).

### ADVANTAGES OF THE GUESS METHOD

As the foregoing examples suggest, the method of guessing furnishes the pupil with definite steps to follow. Even those pupils rating no higher than average can successfully apply the method.

It has been the experience of the writer that pupils with average and poor previous progress tend to choose the guess method almost exclusively. Pupils with good previous progress tend also to employ the method, but they use it for the more complicated cases, when the formulation of equations causes some difficulty.\(^4\)

---

\(^4\)Ibid., p. 20.
Yeshurun failed to note that the mixture problem solved earlier could have been solved just as easily, or perhaps even more easily, had the solution been permitted to lead to simultaneous equations. It is true that Yeshurun has chosen for the multi-symbol solution a problem where there are no explicit simple relationships given such as a ratio or difference relating the two explicit unknowns; yet this fact in itself is no reason to ignore the use of multi-symbols and multi-equations in a problem where such relationships do occur. Although it is undoubtedly true that the uni-equation method is more easily applied to the mixture problem than to the digit problem in Yeshurun's examples, it does not necessarily follow that the reverse is true for the multi-equation method.

Yeshurun's two solutions quoted above demonstrate the fact that pupils can often guess values for the unknowns which satisfy some of the conditions of the problems. Would it not have been just as simple for the pupil solving the mixture problem to regard both the amounts of solutions of different strengths as guesses, as to guess only the one and to obtain the other by a subtraction from 80 g? After all, a number derived from a calculation with a guessed number is likewise a guess. With the realization that both amounts were guesses, the checking should have led to a set of two equations rather than to one equation. These considerations lead to the conclusion that there is no advantage in trying to confine the work to one guess, and to one equation, and they illustrate the universal application of the multi-equation method. The combination method takes advantage of both these desirable features.
Suggestions to Pupils in a Textbook and in a Course of Study which Incorporate the Preliminary Guess-and-Check Procedure

Smith and Lankford

In a textbook by Smith and Lankford, the only textbook known to the writer which introduces the pupils to the guess-and-check procedure, this procedure is suggested for pupils who are having difficulty solving multi-unknown problems by the direct uni-equation method. The guess-and-check procedure is introduced after the pupils have studied five pages of verbal problems, including number, age, mixture, and coin and capacity problems:

Numerical Approach to Problem Solving

If verbal problems are hard for you, it is probably because you think readily in particular terms but have difficulty thinking in general terms. For this reason, we suggest first try a direct numerical approach. Instead of using a letter, such as $n$, to represent an unknown, assume some numerical value for the unknown and then work through the problem. You may find that your assumed value is not correct. However, in the process of working with this assumed value, you will have used all the relationships that exist in the problem. Once the relationships are seen by this numerical approach, it is a simple matter to use a letter for the unknown instead of the assumed number. The algebraic solution follows the pattern you used in working with your assumed value.

Two illustrations will help make the method clear.

Example 1: A boy had $2.90 in dimes, nickels and quarters. If he had 2 more dimes than quarters and 4 more nickels than dimes, how many of each kind did he have?
Numerical approach: Assume that the boy had 5 quarters. The problem states that he had two more dimes than quarters; so he had $5 + 2 = 7$ dimes. He had 4 more nickels than dimes; so he had $7 + 4 = 11$ nickels.

The value of the 5 quarters is $25 \times 5$, or 125 cents.
The value of the 7 dimes is $10 \times 7$, or 70 cents.
The value of the 11 nickels is $5 \times 11$, or 55 cents.
The total value is $25 \times 5 + 10 \times 7 + 5 \times 11$, or 250 cents.

This amount is not correct because the problem says the boy had 290 cents. However, in doing this work you have used all the relationships in the problem and the algebraic solution is now easy.

Algebraic Solution: Use $n$ for the number of quarters instead of the assumed value 5. Then you will have $n + 2$ dimes instead of $5 + 2$, and $(n + 2) + 4$ or $n + 6$ nickels instead of $7 + 4$. The values of the coins will be $25n$ for the quarters, $10(n + 2)$ for the dimes, and $5(n + 6)$ for the nickels. You get these algebraic numbers by using the same processes that you used in getting the arithmetic numbers. But the problem states that the sum of the values is 290 cents.

So $25n + 10(n + 2) + 5(n + 6) = 290$

Solving this equation, you get $n = 6$ quarters; $n + 2 = 8$ dimes; and $n + 6 = 12$ nickels.5

A solution of another problem using the "numerical approach" is provided by these authors, but its examination here would not provide any additional insight into the procedure.

The similarity of this "numerical approach" to Nyberg's discussion is most striking: (1) Guess and check is introduced by both authors as a remedial measure for pupils who are having difficulty using the uni-equation approach. (2) A guess is made for only one of the explicit unknowns. (3) Only one equation is produced, and this, because it compresses all the conditions of the problem into one statement, bears little resemblance to any clause in the problem or to any statement of relationship applicable to the problem.

The most constructive aspect of this approach seems to be that it points out specifically to the pupils that in checking the "assumed" answer all of the relationships involved in the problem must be used. Thus, if any relationship were necessary but unrecognized at first, it should be explicit after the check is completed. Again, the use of this procedure to produce only one equation does not allow for the exploitation of advantages which the multi-equation approach can give, since the pattern of the relationships is obscured in the complex uni-equation.

In the list of problems which follows the two illustrations of the "numerical approach" in Smith and Lankford's textbook, the pupil is told that he may use any method he prefers. No direct reference to this method is made again in this book, and no further illustrations of it are provided. These authors certainly do not exploit the possibilities of the guess-and-check method; nor do they relate it to the multi-equation method as is done in the combination method worked out by Lazar.

The University of Illinois Experimental Course of Study

In the 1957 Teacher's Edition of High School Mathematics, First Course, which is an experimental course developed by the University of Illinois Committee on School Mathematics, Unit Three — "Equations" — includes the study of verbal problems. This material, which is mimeographed, includes three kinds of pages: (1) white sheets, which serve as the pupil's text, numbered serially; (2) green sheets, which contain suggestions for teachers made by the committee developing the
materials, numbered to correspond to the pupil's pages, but distin-
guished from them by an additional "A"; and (3) yellow sheets, 
which present comments from teachers who have already used the material 
or additional comments from the committee, also numbered to correspond 
and designated respectively by a "B" or a "C".

Since this course introduces the pupils to the solving of verbal 
problems by the guess-and-check procedure, a brief resume of the 
opening of the unit will be helpful in understanding the procedure. 
The opening paragraphs describe a meeting of the student council in a 
school at which the chairman of a committee in charge of selling tickets 
for a school play makes a report. The report includes statements of 
the prices charged for each of two kinds of tickets and the total 
amount collected. Another member of the council complains that the 
advertising was expensive, and he wonders if the adult attendance 
warranted the expenses for advertising. The question of how many 
adults bought tickets is not answered in the text, but the pupils are 
encouraged to try to find the answer. In the suggestions for the 
teachers a very clever arithmetic solution is presented, but it is not 
intended that the teacher present any solution to the pupils.6

It seems to be assumed that some pupils will be able to solve the 
problem by various means but that few, if any, will use algebraic 
means. After advising the pupils that they will be able to solve such 

6University of Illinois Committee on School Mathematics, High 
School Mathematics, First Course (Teachers Edition, mimeographed, 
Urbana, Illinois: University High School, 1957), pp. 3-1 and 1A. 
Quoted by permission of the Committee.
problems easily after completing the unit, the committee introduces pupils to the formal solving for roots of equations of the usual sorts. Throughout the work both the usual literal symbols for unknowns and geometric symbols in the shape of circles, rectangles, triangles, etc., are used and referred to as "pronumerals." In this manner the fact that the symbol represents an unknown numeral is emphasized.

The work in solving verbal problems is introduced in the following way with only the first of three illustrative problems quoted here.

Using pronumerals to solve problems.—Now that you have learned how to solve equations, you are ready to solve problems like those that were given at the beginning of this unit. Problems like these are commonly found in high school mathematics—in fact, problems of this kind have been found in manuscripts that are thousands of years old. Most students like these problems because it is interesting to puzzle them out. Although the problems deal with such things as coins, tickets, mixtures, interest, rates, areas, perimeters, and other things that are of a practical nature, you should realize that these problems are really only puzzles. The problems themselves may not be practical, but it is important that you develop methods for solving them and that you practice using these methods. Such practice will help you solve more important and practical problems in the future.

If you want to become a good problem-solver, you will have to develop your own ways of attacking a problem. We shall give you samples of how some problems can be attacked and solved. You may find these illustrations helpful in developing your own methods.

**Problem 1.**

A jar of coins contains 3 times as many dimes as nickels and twice as many quarters as nickels. The total value of the quarters, dimes, and nickels in the jar is $21.25. How many nickels are there in the jar?

A good way to attack the problem and to see that you really understand it is to make a guess at the answer, and then to check your guess. Suppose we guess that there are 10 nickels in the jar. Now, we go through the problem using our guess.
A jar of coins contains $3 \times 10$ dimes, $2 \times 10$ quarters, and $10$ nickels. The total value of the quarters, dimes, and nickels in the jar is found in the following way:

- $2 \times 10$ quarters gives $25 \times 2 \times 10$ or $500$ cents;
- $3 \times 10$ dimes gives $10 \times 3 \times 10$ or $300$ cents;
- $10$ nickels gives $5 \times 10$ or $50$ cents;

the total value is $500 + 300 + 50$ or $850$ cents.

Since we are told that the value of the coins is $2125$ cents, our guess leads to the statement:

$$850 = 2125$$

But, this last statement is false. So our guess was wrong. (It was too low.) Now, we could continue to make guesses and, eventually, we would hit upon the right answer. (In some problems the "guess" method might not ever lead you to a correct answer.) There is another way to attack this problem which involves less work.

If you were to make another guess and work through the problem using the guess, the actual arithmetic would be quite like the arithmetic we did in checking your guess above. The arithmetic would follow the same pattern. If we follow this pattern with a pronumeral rather than with a numeral, we can check a guess merely by putting a numeral in place of the pronumeral. Look at the way in which we used the numeral '10' when we checked our guess above. Instead of using a numeral, let us use a pronumeral say, '□', and go through the problem again.

A jar of coins contains $3 □$ dimes, $2 □$ quarters, and □ nickels. The total value of the quarters, dimes, and nickels is found in the following way:

- $2 □$ quarters gives $25 \times 2 □$ or $50 □$ cents;
- $3 □$ dimes gives $10 \times 3 □$ or $30 □$ cents;
- □ nickels gives $5 □$ cents;

the total value is $50 □ + 30 □ + 5 □$ or $85 □$ cents.

Since we are told that the value of the coins is $2125$ cents, we write:

$$85 □ = 2125$$
The last expression is an equation in the pronumeral \( \Box \). If we can find a root of this equation, and put a numeral for the root in every \( \Box \) above, then we shall have hit upon a correct "guess." Thus, finding the answer to the problem is the same as solving the equation \( 85 \Box = 2125 \). A root of this equation is 25. We write '25' in each \( \Box \) and go through the problem again.

A jar of coins contains 3 \( \Box \) dimes, 2 \( \Box \) quarters, and \( \Box \) nickels. The total value of the quarters, dimes, and nickels is found in the following way:

- 2 \( \Box \) quarters gives 50 \( \Box \) or 1250 cents;
- 3 \( \Box \) dimes gives 30 \( \Box \) or 750 cents;
- \( \Box \) nickels gives 5 \( \Box \) or 125 cents;

the total value is 1250 + 750 + 125 or 2125 cents or $21.25.

So, we know that there are 25 nickels in the jar. The problem is solved.\(^7\)

Each of the three illustrative problems is solved similarly by an equation in which the pronumeral is inserted for the guessed answer for the problem in the arithmetic statements which constitute the checking of the guess. Since each of these problems asks specifically for only one of the unknowns for which values might have been asked, the pupil is prompted to guess only one number, and his checking leads only to one equation. At least two of the three illustrative problems could easily have been asked in multi-unknown form, in which case the pupil would have been prompted to guess a set of more than one number to serve as the answer for the problem. In the sample problem quoted, the number of each of the three kinds of coins could have been required; then numbers for each of the three kinds might have been guessed.

\(^7\)Ibid., pp. 3-47 to 3-49.
These illustrations of solutions are followed by a list of 47 problems for which the following instructions are given:

In solving the first few problems which follow you should guess at an answer, check your guess, and then use a pronumeral in the same manner as we did in the problems above. After a while you will be able to use the pronumeral method without guessing at all. Perhaps some problems will be so easy for you that you will not need to use the pronumeral method. However, you may want to practice the pronumeral method even on the easy problems so that you will become skillful in its use.

In the list, 34 of the problems are stated so as to ask for only one numerical value and 13 for more than one number. Of the former kind, at least 15 are potentially multi-unknown problems. For example, the first problem in the list is stated as follows:

Agnes bought 115 stamps for $2.62. Some of the stamps were 2-cent stamps and the others were 3-cent stamps. How many 2-cent stamps did she buy?

This is potentially a multi-unknown problem since similar problems in most textbooks would ask that the number of each kind of stamp be found. In such cases, the guess of only one of the numbers of stamps would be an incomplete guess of an answer to the problem.

The following suggestions made by the committee writing this experimental course are directed to teachers: (1) Excessive formality should not be required in the pupil's work; pupils are not expected to write as full explanations as those provided with the sample problems.

---

8Ibid., p. 3-55.
9Ibid.
10Ibid., p. 47A.
(2) Pupils may profit from checking several guesses for the same problem.\(^{11}\) (3) In the statements of the check of a guess, the numeral of the guess may be encircled to keep its identity in mind.\(^{12}\) (4) Some confusion may be created in the pupil's mind when the assumed answer is called a guess. Perhaps a better name would be "sample answer."\(^{13}\)

(5) The committee discourages the use of "devices" common in older textbooks. Since these devices are not specified, the committee's remarks will be quoted to indicate its point-of-view more adequately:

"As we indicated in the first commentary on page T.C. 47A, there is no easy road to solving worded problems. We think that a student's facility in solving problems is highly correlated with his native intelligence. All of us know that it is possible to get students to solve certain kinds of worded problems in a fairly mechanical manner. Elementary algebra textbooks of 30 years ago (and some of the more recent ones) are full of teaching devices which succeed in getting students to solve certain kinds of worded problems in a mechanical way. There is very little to be gained by using such techniques. They do nothing whatsoever to increase the student's power in mathematics; they simply make an "expert" of him in solving a restricted type of worded problems. If this expert is given a worded problem that varies only slightly from the type he has learned to solve mechanically, then he is completely lost.\(^{14}\)

Presumably some of the "devices" referred to in the above quotation might include boxes (tabular forms of analysis), sample solutions with lists of problems following whose solutions are quite similar, stating relationships in hints, presenting lists of problems which concentrate

\(^{11}\)Ibid., p. 48A.

\(^{12}\)Ibid.

\(^{13}\)Ibid., p. 47B.

\(^{14}\)Ibid., pp. 47A and 47B.
exclusively on one characteristic relationship, such as a list of motion problems or coin problems.

On the whole the point of view of the committee is admirable. It seems to the writer, however, that the committee limits the work of the pupils undesirably by deliberately dealing with many of the problems which are potentially multi-unknown as though they had only one explicit unknown. This in turn requires the guessing of only one number and the derivation of an equation which is apt to be more complicated than necessary. It is desirable that pupils be able to "see" the relationship in the structure of an equation, and the likelihood of this happening decreases whenever two or more simple relationships are compressed into a single more complex equation.

The multi-equation method for verbal problems is not introduced in this experimental course for ninth grade, since there is no formal work in solving simultaneous equations except by graphic methods.\(^\text{15}\) Hence there is no likelihood that a procedure comparable to the combination method will be developed in this course, structured as it is.

\(^{15}\)Ibid., pp. h-hl.
Traditionally, the study of simultaneous equations is the first topic in the second semester of the ninth-grade course. Only six authors have been found who have written periodical articles in which an earlier introduction of this topic is advocated so that pupils may have this technique to use with problems with multi-unknowns. Their recommendations will be considered in chronological order:

Nyberg

Nyberg seems to be the first writer to mention this possibility. In 1920 he said:

The first months of algebra contain many problems involving two unknowns wherein the sum of the two is given; and one of the most difficult ideas for the pupil to grasp and use is that if the sum is 60, for example, and one unknown is called x, then the other must be 60 - x. Usually the text leads the pupil to the quantity 60 - x by various questions (all of which the pupil may answer correctly, and still the next day use x - 60 instead of 60 - x) or else impresses it by writing

\[
\text{smaller number} + \text{larger number} = 60
\]
\[
X + \text{larger number} = 60
\]
\[
\text{larger number} = 60 - x
\]

which is a good method if transposition has already been considered (as the writer believes it should be). But after many attempts the writer has found that the easiest method is to use two letters x and y and let the pupil write

\[
x + y = 60
\]

This involves an early use and solution of linear simultaneous equations. . .

---

In the ensuing discussion, Nyberg points out that many of the more formal aspects of the course, such as multiplication of binomials and factoring, could be postponed to the second semester to make room for the earlier work with simultaneous equations and a greater emphasis on verbal problems in the first semester.\textsuperscript{17}

Similar suggestions are made by Nyberg in two other articles.\textsuperscript{18}

Unfortunately Nyberg seems not to have adopted these suggestions in complete form in any of the textbooks he has written. It is true that in the 1944 edition of his \textit{Fundamentals of Algebra} simultaneous equations are introduced somewhat earlier than had been the custom in standard textbooks, and some of the more formal topics are postponed as he had suggested; however, in two chapters prior to the one in which simultaneous equations are introduced, problems occur for which only the uni-equation method is available, though these problems contain the very difficulties of which he complained in 1920.\textsuperscript{19}

In view of the fact that Nyberg has advocated the guess-and-check procedure (see p.129) and also the multi-equation method, on separate occasions, it seems strange that the combination of the two procedures did not occur to him; however, there is no indication in his published writing that he ever considered this possibility. Although he saw the

\textsuperscript{17}\textit{Ibid.}
need for and the advantages of each of the two features of the
combination method, apparently Nyberg never realized the fact that
they complement each other.

Ligda

The two articles in which Ligda discusses the solving of verbal
problems were published in 1926 and 1930, and merely repeat the ideas
he presented in his book on the teaching of algebra;20 these ideas
have been presented in the previous Appendix.

Ferrar

In 1927, Ferrar made a suggestive incidental remark concerning
eyearly class work with simultaneous equations; the remark occurred in
a discussion of the mathematical understanding of pupils:

I believe that many pupils . . . can do simple problems
more readily if they are allowed to use two unknowns. For
this reason I advocate teaching some sort of solution of
simultaneous equations very early in the year, certainly
before factoring.21

Unfortunately there is no further discussion of this point in
Ferrar's article.

Iazar

The most persistent arguments for the earlier introduction of

20Paul Ligda, "Systematic Analysis and Solution of Quantitative
Problems," School Science and Mathematics, XXVI (February and March,
1926), pp. 173-180 and 211-252, and "How We Solve Problems," School

21Alice W. Ferrar, "Two Variables," Mathematics Teacher, XX
(October, 1927), p. 342.
simultaneous equations and the exclusive use of the technique of multi-equations for multi-unknown problems have been made by Lazar in a series of articles extending from 1933 to 1951. In one of his studies he analyzed the problem materials in five textbooks to determine the percentage of one-unknown problems included in these books prior to the introduction of simultaneous equations. He found these per cents to be 13, 33, 18, 23 and 28. In connection with the two textbooks in which the per cent was greater than 25, Lazar analyzed the nature of the problems; he pointed out that in the one book eighty of ninety-five such problems were nothing more than "verbalized equations," and that in the other book a large portion of the seventy-five problems were "equations in word form."

In general, Lazar has presented his arguments for the multi-equation method by identifying defects in the traditional uni-equation method and by citing the positive advantages of the multi-equation method. Among the former he cites the following: (1) The beginning student gets the "erroneous impression" that there is "but one equation and but one central thought" for the problem. (2) There are as a result of the uni-equation approach "three or four different ways of doing the same problem" and this "is certainly confusing to the beginner, and is probably one of the causes of his pitiful floundering." (3) By means

---


of formulas teachers build up a belief that equations are a superior expression for numerical relations both as an aid to memory and as a saving of energy; then "we do not keep our faith with the pupils as to the symbolic nature of algebra" when "this one-to-one correspondence between an equation and the statement and between a symbol and the unknown" is destroyed by compressing "two statements into one equation." (4) The imposing of the uni-equation method almost inevitably brings a need for crutches such as boxes or hints as to the implicit relationships necessary to complete the elements.2h

Lazar incorporated a list of positive advantages of the multi-equation technique into the following statement in 1951:

> For better or for worse the solution of verbal problems is an integral part of the teaching of algebra. Whatever value may be derivable from this topic is vitiated, however, by the blind insistence, almost general in elementary textbooks, on the use of one unknown and one equation in solving problems that really involve more than one unknown and more than one explicit or implicit statement. . . .

> Whenever a problem really contains more than one unknown use as many symbols as there are unknowns, and, of course, set up as many equations as there are unknowns.

> Some of the advantages of such an approach are:

1. It is easier to set up a set of two or three equations rather than compress all the information and relationships into one equation.

2. There is a scarcity of verbal problems that really involve only one unknown.

3. The method of solution by simultaneous equations has to be taught eventually in the course; why not begin the method at the very beginning of the course, thus giving the students not only a longer exposure to the method but all the benefit of a more powerful tool?

---

The multiple equation approach not only makes the solution of verbal problems easier but also enables the student to realize that whenever there are not as many explicit statements as there are unknowns, an implicit relationship must be hidden. He thus gets the incentive to search for the missing relationship, or the idea which might yield the necessary equation. This approach will obviate, in the main, the numerous artificial devices that have been the stock-in-trade of teachers of algebra.

Finally the student gets a feeling of confidence in the use and power of symbols and equations which is certainly one of the objectives of the teaching of algebra.

Similar advantages were listed in an article published in 1933.

Lazar summarizes his arguments in the following statement in which he points out the importance of relationships:

It is on this claim, the superior power of analysis, that I believe the strongest argument for the proposed approach [the multi-equation method] rests. For in setting up some of the equations of a problem, one often discovers that other equations are missing, and is spurred on to search for another relation which may serve as a basis for an equation. The student thus forms the habit of reasoning and analyzing in terms of symbols and relations, which is the very core of mathematical thinking, and which he may carry away with him, into his later intellectual life, long after all the puzzles, tricks and magic boxes have been forgotten.

---


Wilson has described her method of approaching the solving of verbal problems, by which the student uses letters mnemonically and uses multi-equations as well:

The main idea of this method is to learn to translate the verbal problem into algebra by one method, namely that of using only the letters of the words used in the sentence; thus doing away with the formal use of letters that have no meaning in connection with the problem... It is the solution of the problems that deal with more than one unknown which I wish to discuss, and these lend themselves to this method as readily as the others.

To begin this I would write the problem on the board as I have it here:

(1) Mr. Brown and his son together earned $42 a week. Mr. Brown earns $8 a week more than John. How much did each earn?

If $B = J = 42$; and $B = J + 8$; then $(J + B) + 42 = J$.

$J = 17$ $B = 25$

In summarizing the values of her method, Wilson states:

The value of this type of teaching of verbal problems is that all problems can be done by one method. It is an easy method, it helps the pupil with his English in that it trains him to pick out the essential parts of a sentence; it establishes a concept of the IF-THEN relation which is not only necessary for geometry but is needed in life. It strengthens the ability to substitute, a process which seems to be difficult for some, so the more they do it the better. It has used the idea of simultaneous equations all through the year so that chapter may be eliminated except

---

for the graph work that will also bring out the IF-THEN concept. This will leave more time for the interesting chapters which are finding their way into our better books, namely, the units on geometry, and trigonometry, and statistics.²⁹

Lazar has also recommended that pupils use letters mnemonically.³⁰

Wilson does not indicate in her article how she handles problems in which there are implicit relationships; for this reason her view of problems seems too limited. Translation procedures such as Wilson advocates are of little help in identifying or discovering implicit relationships, a need for which the combination method provides explicitly.

Smith

Smith has suggested a unique means of introducing simultaneous equations early in the course: she advocates that the technique of eliminating one of the letters by addition of the two simultaneous equations be introduced the day after the addition of signed numbers has been taught. Furthermore, she advocates that after a few days verbal problems be introduced which will lead to such pairs of equations.³¹ An example might be the typical "sum and difference" problem:

²⁹Ibid.


Problem: If a bottle and its contents are worth $1.05 and the contents are worth one dollar more than the bottle, how much is the bottle worth?

Another example might concern the round trip of a boat on a stream.

If the length of each leg of the trip and the time for each leg are given, the student is required to find the rate of the stream and the rate of the boat in still water.

Since the topic of signed numbers is usually taught very early in the algebra course, this recommendation is in fact also a recommendation for introducing the multi-equation method and multi-unknown problems early in the course.

This proposal to use multi-equation techniques early in the course for multi-unknown problems is subject to the same limitations as the similar proposals made by Nyberg and Wilson: the method is desirable, but it should be augmented by some device such as the guess-and-check procedure to help pupils find the implicit relationship which is so often the basis for one of the equations in the set of equations needed. The great array of devices (e.g., boxes; hints; lists of steps; and classifications of problems into categories, such as coin, motion, mixture, digit, and the like) which almost always accompany the study of verbal problems is ample evidence that pupils must have such help in exposing the hidden relationships in problems.

Ransom

Ransom's views on the use of multi-equations for the solving of problems have been taken from an article which is concerned primarily with the traditional "work" problem. In this article he advocates
working a complete literal solution first. For the typical work problem he uses five literal symbols and four equations; one of the symbols is eliminated and becomes indeterminate. This point of view — that of solving a problem in its general or literal form first — seems to be beyond the needs and abilities of most pupils just beginning a study of algebra. Certainly there is a place for this type of generalization in more advanced courses; but in the elementary course, in which a problem presents numerals as data, there seems to be no legitimate reason to use literal symbols for any but the really unknown numbers. After all the purpose of the exercise is to have the pupil solve the problem and not necessarily to generalize its solution.

In this article, Ransom makes observations which support the use of multi-equations for all multi-unknown problems:

• • In the one-unknown method, we are mentally solving one equation and substituting into the other. Is it not more in accord with the spirit of algebra to do this explicitly, rather than in part secretly?

• • The introduction of several letters is no disadvantage unless the study of simple simultaneous equations has been artificially too long postponed.


33Ibid., p. 222. Quoted by permission of the business manager of School Science and Mathematics.

34Ibid., p. 221.
Although Ransom's article deals primarily with the "work" problem, at the conclusion he makes some further statements regarding the derivation of equations in general. He gives a list of "relations" most frequently appearing in textbook problems, and then concludes:

Some one of these relations must be used throughout in formulating any equation; any number required by the relation, but not given in the problem, must be represented by a (mnemonic, if possible) letter. Every equation must represent a whole equated to the sum of its parts, or to two descriptions of the same quantity equated to each other. Until the student sees in the problem these cases of whole-and-parts or the cases of duplicate-description, he must read the problem again more understandingly. Failure to "set up" the necessary equations then reduces to a mere failure to understand what the wording of the problem means.35

It will be noted that this paragraph also implies the use of multi-equations, for each relationship is to be kept in as simple a form as possible, and letters are to be used to represent each of the numbers "required by the relationship but not given in the problem."

The writer feels that while Ransom's purpose — to make the situation easier for the pupils — is certainly commendable, Ransom has perhaps over-simplified his suggestions by stating all relationships as sums or as products; this does not allow for the possibility of some pupils using converse forms of some relationships with which they may be familiar from previous school work, for example $r = d/t$.

---

35Ibid., p. 223.
The Early Introduction of Multi-equation Techniques and Their Exclusive Use in Solving Multi-Unknown Problems in Current Textbooks

In order to determine the influence of multi-equation techniques on the writing of textbooks, the writer has examined the procedures used to introduce pupils to the solving of verbal problems and especially to the solving of multi-unknown problems in sixteen currently-used textbooks. A complete list of these texts is found in Bibliography D. The earliest publication date of any of these is 1953.

The Traditional Nature of the Textbooks

Without exception, each of these sixteen books introduces pupils to multi-unknown problems prior to any work with simultaneous equations. Obviously, then, pupils are expected to derive equations for such problems by the traditional uni-equation method. Furthermore, formal lists of directions or steps of procedure are provided for the pupils in all but two of these books. In most of the books these lists of steps are presented so early in the pupils' experiences with problems that the only possible interpretation is that they are presented as directions, not as generalizations based on the pupils' experiences with problems.

In eight of the texts the lists of steps are so worded that, at least for the beginning work with problems, the pupil is directed to form the equation by finding two expressions for the same number. Unless he ignores these instructions, the pupil is forced to think
in terms of phrases instead of in the complete sentences by which relationships among the quantities in the problem are expressed. Such a suggestion for the formation of the equation practically precludes functional thinking on the part of the pupil, for he does not write simple relational ideas in complete equation form as he does when multi-equations are permitted.

It is not the case that none of these sixteen books shows evidence of being influenced by ideas similar to those presented in the review of periodical articles in the previous section, although it is true that none has been extensively influenced by them as indicated by the traditional aspects just described. Two textbooks incorporate features with regard to the use of simultaneous equations which deserve noting at this time.

Fehr, Beberman, and Carnahan

Only one textbook has been found in which there seems to have been a deliberate attempt on the part of the authors, Fehr, Beberman, and Carnahan, to postpone most of the verbal problems with more than one unknown until after the topic of simultaneous equations has been studied. Only one list of a few consecutive-number problems has been included in multi-unknown form prior to the study of sets of equations. From the page numbers and table of contents it appears

---

to be the authors' intention that sets of equations be the final
topic of the first semester of work. There are many verbal problems
with one unknown, however, in the earlier work of the semester.

Except for the consecutive-number problems, these authors have
gone to considerable pains to ensure that the problems included in
the early work have one explicit unknown, as illustrated in the
following example:

On a vacation trip, a father and his son shared the
driving. The father drove 3-1/2 times as far as the son.
Thus far the problem reads as many similar multi-unknown problems
might read. If the total length of the trip were now provided and
if the pupil were asked to find the number of miles each drove, the
problem would be an excellent example of a typical multi-unknown
problem. In the book under consideration, however, this problem is
concluded with the following sentence:

How far did the son drive if the father drove 721
miles? 37

By the device of providing a specific number for what would
have been an unknown in the more typical problem, the authors have
converted this into a problem with only one explicit unknown. It may
be argued that the total length of the trip may be considered to be an
implicit second unknown in this converse form of the more typical
multi-unknown problem. If the problem asked that the total length of
the trip also be found, however, it should be noted that the two

37 Ibid., p. 84.
explicit unknowns would be consecutive rather than simultaneous.

It seems to the writer that these authors have risked losing the interest of pupils in algebraic techniques for solving problems by including so many problems similar to the one just described. Over-simplification of the problems may eliminate for a large portion of ninth-grade pupils the need and desire to use algebraic means. It is too easy to solve the problem by a simple arithmetic division. In fact, the authors are compelled to admonish the pupils to use algebraic means even though they could easily solve the problem by arithmetic.\(^{38}\)

After taking such elaborate pains to avoid many multi-unknown problems until after the simultaneous equation method is available to the pupils, the authors might be expected to exploit the multi-equation method fully. This is not the case, however. In the introductory remarks to "Using Systems of Equations to Solve Problems," the following statement is made: "Many problems are much easier to solve if we use two variables even though it would be possible to solve them with one variable."\(^{39}\) Following this introduction there are problems involving number relations, digits, and balanced weights, all of which are illustrated by the multi-equation method. The next type to be considered, coin problems, is illustrated by an example with three unknown numbers of coins. This example is solved by the

\(^{38}\)Ibid., p. 60.

\(^{39}\)Ibid., p. 223.
uni-equation method, though it occurs only five pages after the introduction of the multi-equation method.\textsuperscript{40} No comment is made to explain the reversion to the uni-equation method, nor to advise the student which method to use with the ensuing problems. A mixture problem is subsequently illustrated with the multi-equation method.\textsuperscript{41} The sudden unexplained intrusion of a multi-unknown example using the uni-equation method would certainly be mysterious to the pupil.

\textbf{Grove, Mullikin, and Grove}

In Chapter 2 of the textbook by Grove, Mullikin, and Grove, only problems with one unknown are considered.\textsuperscript{42} These authors postpone the consideration of problems with two unknowns until Chapter 5, where they introduce both the uni-equation and the multi-equation methods for solving such problems.\textsuperscript{43} The uni-equation method is introduced first, however, and is used with number, age, and value problems (including coin problems). The use of boxes with both age and value problems is illustrated. After twenty pages of work with the uni-equation method, formal work with simultaneous equations is introduced.

\textsuperscript{40}Ibid., p. 228.
\textsuperscript{41}Ibid., p. 230.
\textsuperscript{43}Ibid., pp. 113-162.
including solving sets by graphing, and by elimination using both substitution and addition or subtraction. Finally, thirty-five pages into the chapter the solving of verbal problems by the multi-equation method is begun, with two coin problems illustrated both ways. Of the three succeeding lists of problems in the chapter, two lists which include some digit problems are to be solved by systems of equations, while the student may choose "the most convenient method" for the problems in the third miscellaneous list.

From the manner in which verbal problems and the multi-equation method are treated in the two textbooks just described, one may conclude that the two groups of authors have been influenced by the idea that more of the work with multi-unknown problems should be approached by the multi-equation method instead of by the uni-equation method. Neither group seems to have become convinced that the multi-equation method should be used exclusively for such problems, for each group continues to plan for the solving of some multi-unknown problems with the uni-equation method both before and after the multi-equation method has been introduced. Each of these groups of authors also introduces the multi-equation method so that it is part of the work for the first semester in contrast to the traditional idea that this topic is a part of the second semester's work.

---

*Ibid., p. 156.*
Summary

In this chapter it has been noted that relatively few authors of periodical articles venture to suggest procedures which depart from the usual uni-equation approach to multi-unknown verbal problems in the elementary course. Only two authors have been found to recommend the preliminary guess-and-check procedure, and only one illustrative problem was found where guess-and-check led to two equations which were then solved simultaneously. In general the suggestions of both Nyberg and Yeshurun were in terms of a uni-equation approach.

Seven authors were found to make suggestions regarding the earlier introduction of the multi-equation method so that it could be used with a greater number of multi-unknown problems. The most persistent arguments for this proposition were advanced by Lazar over the period from 1933 to 1951, and may be summarized by stating that Lazar believed that the multi-equation method encouraged better analysis of problems by the students.

On the whole, these proposals have had comparatively little effect on textbooks. One textbook and one course of study have been found which make some use of the guess-and-check procedure in connection with deriving equations for problems. In the textbook, however, the method is illustrated so as to lead to one equation, and it is introduced as a remedial measure for pupils having difficulty with the direct uni-equation procedure. In the course of study, the technique is illustrated only with problems in which there is one explicit unknown. This greatly restricts the flexibility of the method and thus reduces its value.
APPENDIX C

STUDIES IN PROBLEM SOLVING WHICH INVOLVED
THE GUESS-AND-CHECK PROCEDURE,
THE MULTI-EQUATION METHOD,
OR THE COMBINATION METHOD

Among published research studies of problem solving, the writer has found none which has been concerned directly with either feature of the combination method or the combination method itself. In Ligda's *The Teaching of Algebra* there is mention of an experiment in which the relative efficiency of the multi-equation method in contrast to the uni-equation method was measured. Although the author claims that the early introduction of the multi-equation technique was "greatly superior," no details of the experiment have been provided.¹

Several studies which deal with the features of the combination method or the combination method itself are in the form of unpublished theses which have been conducted under the advisership of Lazar at the Ohio State University. Consequently the reviews in this chapter will be concerned with the earlier studies of which the present one is fifth in the series. These will be presented in chronological order

under the headings: (1) Studies Involving the Use of Multi-equations and (2) Studies Involving the Combination Method.

Studies Involving the Use of Multi-equations

Hopkins

The first of these studies was reported by Hopkins and investigated the possibilities of developing greater understanding of solving problems if the multi-equation method were employed. Use of Lazar's "Mathematical English" as a means of translating statements into equations was included as a part of this study.

After noting that

Most authors seem to hold that only problems in which the relation between the unknowns is not obvious should be solved with simultaneous equations.  

Hopkins asks:

Why is the method of multiple unknowns better only for complicated problems? How is the child to determine the "wise selection of an unknown"? Is there always an independent unknown? 

She points out that textbooks have improved in the past few decades in certain respects, such as in extending arithmetic ideas, principles and processes; in helping pupils understand functional variation; and in giving them help in "expressing simple relationships between literal and arithmetical numbers before asking them to solve

---


3Ibid., p. 9.
But she concludes nevertheless:

In the opinion of the writer, authors of textbooks have no practical reason for clinging to the traditional plan of solving all verbal problems during the first semester of an elementary algebra course with one unknown. The mystery of solving would be eliminated if students were trained to set up one equation for each relationship expressed or implied.\(^5\)

Hopkins' study culminated in eight conclusions as follows:

1. In order to make algebra a functional part of a student's life, he must first be taught its language.
2. Translation of Everyday English into a mathematical equation is not a direct process. Students should learn to write relationships between quantities in Mathematical English just as they learn to write foreign language idioms.
3. Substitution is not a difficult process for a child if it is graphically illustrated on paper. It is very difficult for a beginning student to substitute mentally.
4. The early solution of simultaneous equations gives the student a favorable attitude toward algebra because he sees it as a more powerful tool than arithmetic.
5. Attempting to increase the efficiency of students in solving verbal problems by classification of types is not good psychology. The tabular form predisposes the student to a mechanized response which hinders him in his attack on new problems.
6. Literature on problem solving is full of vague and futile advice on how to carry out the steps in problem solving.
7. It will be impossible for the "experts" to give this advice so long as they cling to the traditional sequence in elementary textbooks. The single-unknown method is full of mystery and confusion to the beginner.
8. There is a shortage of problems in one unknown other than the verbalized equation or puzzle types.\(^6\)

\(^4\)Ibid., p. 12.
\(^5\)Ibid., p. 45.
\(^6\)Ibid., pp. 48-50.
From these conclusions the present writer identifies the following needs of pupils as they work with problems:

1. The ability to "see" in the equations the conditions of the problem. This is one aspect of translating and of realizing that algebra is a language. The multi-equation method helps meet this need better than the uni-equation method does since the pupils can translate a set of equations back into the words of a problem more easily than this can be done with the single, more complex equation produced by the uni-equation method.

2. The confidence engendered by explicit substitutions rather than tacit ones. When multi-equations have been derived whatever substitutions are needed in eliminating symbols are made explicitly in combining the equations. In the uni-equation method, this substituting process is done mentally and is necessary in deriving the equation in the first place, not after the equations are formed.

3. The feeling that algebraic methods supplement, not merely replace, arithmetic methods for solving problems. Too many problems which are used commonly to introduce algebraic techniques with uni-equations are simple enough for many pupils to solve arithmetically. From this kind of experience, the impression may easily be created that algebraic techniques are merely another way to solve the same familiar problems. If multi-equation techniques are used, problems which can not be solved by arithmetic means easily may be attacked immediately. The addition of the guess-and-check feature would permit the introduction of problems with implicit relationships also from the beginning.
The definiteness of concrete and tangible procedures to replace the "vague and futile advice" which too often accompanies the traditional procedures. The difficulties caused by the implicit relationships in problems can not be overcome by the multi-equation method alone. The definite tangible procedure introduced in the combination method of guessing answers and checking them starts most pupils immediately on a constructive and fruitful procedure. If they can not guess and check answers, surely they are not familiar enough with the relationships among the numbers in the problem to derive equations by any method.

From these considerations it may be concluded that the two features of the combination method are together potentially better suited to meeting all of the needs which Hopkins identified and implied in the conclusions of her study of the multi-equation method than is that method alone, even with the aid of Mathematical English.

Arnett

The second of this series of studies at the Ohio State University was an investigation into the procedure necessary to adapt a standard textbook for the early use of the multi-equation method for multi-unknown problems. Since this study is a Field Service Project Report, Arnett's suggestions were derived from actual classroom experiences.

Her outline of steps necessary to adapt a typical book follows:

1. Set up the equations for the verbal problems which can be solved by the multiple-equation method. This gives the teacher a better perspective in the use of the new method.

2. Select for first consideration the problems most easily solved by the substitution method. Postpone until later problems that are more easily solved by the addition and subtraction method. After both methods have been studied, either method may be used.

3. The concept of substitution and that of the use of parentheses are not difficult ones to get students to understand at this time. For the solution of verbal problems by the multiple-equation method, substitution is practically the only new concept necessary that is not usually taught at the beginning of the course in algebra.

4. If explanations are given carefully in class and examples given for the students to solve and keep as reference, there will be little difficulty in getting the students to accept a different method than the one used in the textbook.

5. The solution of verbal problems consists of five main parts: (a) representing the unknowns, (b) translation of the verbal problem into Mathematical English, (c) setting up the sets of equations, (d) solving the equations for the unknowns, (e) checking the values of the unknowns in each of the equations and in the words of the original problem.

6. Frequent short tests are valuable in keeping a close check on the student's progress.

7. Continually reviewing the solution of verbal problems was found valuable in helping students to retain skills.

Among the procedures to use in the classroom when the simultaneous equation method is being taught, Arnett suggests that the pupils be given pairs of equations to which they may set "stories.\(^9\) In this way they may see better how equations represent the conditions and situations described in the verbal form of the problems. Arnett's steps and this suggestion were found most useful by the writer in planning

---

\(^8\)Eleanor Arnett, "The Possibility of Adapting a Traditional Textbook in Ninth Year Algebra to the Early Use of the Multiple Equation Method," unpublished Master of Education Field Service Project Report, Columbus, Ohio: The Ohio State University, 1952, pp. 42-43.

\(^9\)Ibid., p. 22.
the revision of the textbook materials which were needed to accommodate
the earlier introduction of multi-equations and to use the combination
method for problems in the experimental part of this study.

Studies Involving the Combination Method

Thompson

The one of the series of preceding studies which is most directly
related to the present study was conducted by Thompson. She tried out
the preliminary guess-and-check procedure combined with the multi-
equation method in her algebra classes in Plain City, Ohio, over a two
year period 1954-55 and 1955-56. It will be noted that Thompson's
study was inaugurated before Yeshurun's article was published (see
above, p. 134). Her report is a description of her experiences. The
procedure used by the pupils in Thompson's classes will be illustrated
by the following examples of pupils' work:

Problem: One number is five times another. Their
difference is \( \frac{1}{2} \). What are the numbers?

\[
\begin{array}{c}
\frac{1}{2} \\
5 \\
\frac{1}{2}
\end{array}
\begin{array}{c}
\frac{1}{2} \\
5 \\
\frac{1}{2}
\end{array}
\begin{array}{c}
5s - s = \frac{1}{2} \\
5s = f \\
\frac{1}{2}
\end{array}
\]

\[
\begin{array}{c}
\frac{1}{2} \\
5 \\
\frac{1}{2}
\end{array}
\begin{array}{c}
\frac{1}{2} \\
5 \\
\frac{1}{2}
\end{array}
\begin{array}{c}
f - s = \frac{1}{2} \\
5s = f \\
\frac{1}{2}
\end{array}
\]

---

10Ethel Hart Thompson, "Application of the Guess-and-check Method
to the Early Use of the Multiple-Equation Method for the Solution of
Verbal Problems in Elementary Algebra," unpublished Master of Education
Field Service Project Report, Columbus, Ohio: The Ohio State University,
1956.

11From Dick Fravel's paper of Sept. 23, [1955,] given to the
writer by Thompson. No final check is presented by this pupil for the
problem.
Problem: A 30-lb. bag of chestnuts and a 35-lb. bag of walnuts are hung from the ends of a six-foot rod. How far from the bag of chestnuts must the fulcrum be placed in order that the two weights balance?

\[
\begin{align*}
30 \text{ lb} & \quad 6 \text{ ft} & 35 \\
30c + 35w & = 6 \\
30c - 35w & = 0 \\
65w & = 180 \\
\end{align*}
\]

chestnuts walnuts

\[
\begin{align*}
\text{ans.} & \quad \frac{3}{13} \quad \frac{2}{13} \\
\end{align*}
\]

It will be noted that these pupils enclosed their guessed numbers in squares, circles, and the like for the purpose of following them through the checking process. This is advisable so that letters may more easily be substituted for the guesses later on. A vertical form of the arithmetic was performed first, and then this was transformed into horizontal statements as an aid in deriving the equations. The letters were written over (or near) the guesses they were to replace in the horizontal form of the checking.

One of the students used a "?" over the equal sign of a statement which did not check. Other ways to indicate that a computation has

---

12 From Mary Lucas' paper of Feb. 2, [1955], given to the writer by Thompson. A final check is also presented on the paper.
not checked are (1) to cross out the "="; (2) to write the word "no" after the statement; or (3) in a vertical check, to draw a diagonal line through the work. It is suggested that some means similar to these suggestions be used whenever the arithmetic of a statement does not check.

A few additional and unique features were introduced by Thompson and are deserving of note:

1. Following Arnett's advice mentioned above Thompson gave her pupils practice in translating sets of equations, as many as three equations in a set, into verbal problems. These were sometimes presented to pupils in the form of worksheets similar to the following example:

Algebra I

Make up a verbal problem to correspond to each set of equations. Then solve the sets of equations.

1. $L + s = 42$
   $L = 5s$

2. $a = 2c$
   $a + c = 90^\circ$

3. $2w + 2L = 224$
   $L = 3w$

4. $a + s = 180^\circ$
   $2s = a$

5. $m = 2s$
   $3s + m = 50$

6. $J + B = 30$
   $J = 2B$

7. $s = 3f$
   $f + s = 320$

8. $6s = L$
   $L + s = 77$

9. $f + s + t = 180^\circ$
   $s = 4f$
   $t = f$

10. $a + b - c = 154$
    $2a = b$
    $c = 4a 14$

---

13 Thompson, op. cit., p. 41.

14 From a copy of a dittoed worksheet given to the writer by Thompson.
A student's problems for two of these sets follow:

3. In a rectangle two widths and two lengths added together equal 22l0. If one length is three times one width, what are one length and one width?

8. During baseball season Larry's homeruns added to Sam's homeruns was 77. Now if Larry hit six times as many as Larry (sic) how many did each boy hit?15

It will be noted that in the latter example, the student's grammar and English were not quite on a par with his grasp of the algebra in the situation.

2. Another type of worksheet was used by Thompson to emphasize in arithmetic situation the relationships which would be required in converse algebraic problems.16 From a worksheet which was used to study "mixture" problems two examples are quoted:

Algebra I

Introduction to Mixture Problems

2. A grocer wants 200 lb. of a mixture of prunes and apricots so that he can sell the mixture at 15¢ per lb. He intends to use 75 lb. of apricots worth 20¢ per lb. What is the total value of the mixture? What is the value of the apricots? What is the value of the prunes? How many lb. of prunes will he need to add? What is the price per lb. of the prunes?

15From Robert Bishop's paper September 27, 1954, given to the writer by Thompson.

16Thompson, op. cit., p. 43.
5. A radiator contains a 20-qt. solution of alcohol and water having a strength of 10%. How many quarts of alcohol does it contain? How many quarts of water does it contain? If 4 quarts of alcohol are added, how much alcohol does the mixture now contain? What is the total number of quarts in this new mixture? What is the strength of this new solution? 17

It is instructive to compare these materials with those used in a study conducted by Hawkins for the purpose of helping students derive equations. His materials were used in an experimental class to test whether or not they enabled students to derive equations for problems more effectively. Examples of Hawkins' materials for motion problems and a mixture problem are quoted:

1. An automobile tourist sets out on a 400 mile trip. Express the time required if he goes at the rate of:
   a. 400 mi. an hour; c. (r-10) mi. an hour;
   b. m mi. an hour; d. (2r-3) mi. an hour.

2. Two trains leave Chicago at the same time, one eastbound, the other westbound. The eastbound train travels 10 miles less in an hour than the westbound train. Express algebraically:
   a. the rate of each train;
   b. the distance traveled by each in 4 hours;
   c. the fact that the trains are 4h0 miles apart at the end of 4 hours. 18

17 From a dittoed worksheet given to the writer by Thompson.

1. A farmer has two kinds of seed — clover seed and bluegrass seed. If he has 100 lb. of the two kinds together express algebraically:

a. the number of pounds of clover seed if there are \( n \) pounds of bluegrass seed;

b. the value of the clover seed at 20¢ a pound;

c. the value of the bluegrass seed at 15¢ a pound;

d. state by an equation that the value of both kinds together was $19.\textsuperscript{19}

A basic difference in point of view as to what kind of aid pupils should have in deriving equations for problems is evident in comparing these examples of worksheet materials. In Hawkins' materials the expressions with which to build the equation are built up by the pupils piece by piece by using algebraic and numerical symbols. A mystery still remains for the pupil after such work as to how it occurred to the teacher to ask these particular questions. This kind of work does not cause the pupils to deal with complete sentences and equations until the final question. In Thompson's materials, in contrast, the kinds of questions are the very ones the pupils will have to ask themselves in checking guessed answers. The computations are entirely arithmetic, and merely in writing them down, pupils are provided tangible examples of the relationships needed to form equations. Since both types of materials are indicative of the kinds of thinking and work which will have to be done by the pupils in using

\textsuperscript{19}Ibid., p. 657.
the respective methods, it is obvious that the pupils using the arithmetic kinds of question material will be able to derive the equations more readily and directly.

3. A special study of the vocabulary used in presenting problems was undertaken in Thompson's classes. She based the study on the lists of verbal expressions which imply each of the operations as compiled by Hopkins. From this study the pupils were to realize that there were many ways to express the same operation. The various verbal forms were not taught in isolation but over a long period of time. They were always used in complete sentences, and translations using multiple symbols were the rule. For example, the sentence, "The sum of two numbers if 50," is not translated into \( x \) and \( 50 - x \), but rather into \( L + s = 50 \).\(^{21}\)

Thompson required that the first written work of the pupils with the combination method show three phases: (1) the guesses were stated and checked by vertical arithmetic; (2) the checks were transformed into horizontal forms and mnemonic substitutions made; (3) the resulting equations were solved and the final results checked.\(^{22}\) A student's work on a test paper illustrates these phases:

\(^{20}\text{Hopkins, op. cit., pp. 21-22.}\)
\(^{21}\text{Thompson, op. cit., p. 46.}\)
\(^{22}\text{Ibid., pp. 10-41.}\)
2. The length of a rectangle exceeds the width by 6 feet, and the perimeter is 56 feet. Find the dimensions.

\[
\begin{align*}
2L + 2w &= 56 \\
-L + w &= -6 \\
2L + 2w &= 56 \\
-2L + 2w &= -12 \\
4w &= 4w \\
w &= 11 \\
L &= 17
\end{align*}
\]

Thompson states that the pupils asked to be permitted to omit phase I, or to go directly to the equations as soon as they were confident of their ability to write equations directly.\(^\text{24}\)

Thompson's general conclusions follow:

1. The multiple-equation approach to the solution of verbal problems may be employed by any classroom teacher who is willing to devote a small amount of extra time to the rearrangement of the material in a traditional algebra textbook.

2. The development of any new concept should be based upon the experiential background of the student, if the student is to gain real understanding through the new learning experience. Hence the solution of verbal algebraic problems should begin with use of the fundamental arithmetic operations with which the student ought to be familiar at the ninth-grade level.

\(^\text{23}\) From Mary Lucas' paper, February 2, 1955, given to the writer by Thompson.

\(^\text{24}\) Thompson, op. cit., p. 41.
3. Guessing can be made a respectable learning technique. The effectiveness of the method, of course, depends upon the manner in which it is introduced and used in the classroom situation.

4. Students actually seemed eager to use the guess-and-check method. They did not approach new types of problems with reluctance, but with great eagerness and a competitive spirit.

5. The guess-and-check method places verbal problems within the realm of understanding of the slower student in the class and provides him with a greater feeling of confidence in his ability to solve problems.

6. The guess-and-check method was found to be a great timesaver. Heretofore, time had been consumed in setting up and solving meaningless and often incorrect equations.

Among her recommendations Thompson suggested that

An experiment should be carried out in which the achievement of pupils taught by the traditional method is compared with the achievement of pupils taught by the guess-and-check method.

The present study attempts to carry out this recommendation.

Tanboonteck

In the fourth study in the series, a four-stage development for the study of problems in elementary algebra has been described. The four stages and a brief sample of each kind are quoted from the conclusions of the study:

I. The teaching begins with problems which are easily translatable into equations. The statements of the problem should, of course, be simple. For instance:

1. A number increased by 7 is equal to 29. What is the number?
2. A number is 5 times another number. If it is increased by 3 it will be equal to 43. Find the two numbers.

25 Ibid., pp. 64-65.
26 Ibid., p. 66.
II. The second stage consists of the problems which must first be translated into mathematical English before the statements of the problem can be reduced to algebraic equations. For example:

1. Jim is 5 years older than his sister. The sum of their ages is 29. How old are they?

This problem in everyday English should first be translated into mathematical English as follows:

(The number of years in Jim's age) (is equal to) (the number of years in his sister's age) (increased by) (5) and then into the equation

\[ J = s + 5 \]

The second statement is translated into the following:

(The number of years in Jim's age) (increased by) (the number of years in his sister's age) (is equal to) (29)

\[ J + s = 29 \]

III. When the student is able to solve those problems which contain no implicit relationships, problems which contain both explicit and implicit relationships should be presented; in these, the many-equation approach, again, should be used. For example:

1. The sum of two consecutive even numbers is 182. What are the numbers?

This problem should be first translated into mathematical English and then into equations. The implicit relationships lie in the meaning of the words "consecutive" and "even."
IV. The fourth stage consists of solving those problems in which implicit relationships are not as obvious as they are in the two examples illustrated in Stage III. Examples of the following types fall into this category: mixture, distance, coin, work, ratio, etc. . . .

The solutions of this stage are worked out with the help of converse forms. To illustrate, Tanboonteck uses a typical mixture problem:

Twenty pounds of 85¢ coffee are mixed with 50 pounds of 92¢ coffee. Find the total amount of mixed coffee and the cost per pound of the mixture.

This problem is a "matrix arithmetical problem"; it is analyzed into items of data and questions, and the implied relationships are stated verbally. Data and questions can be interchanged to form fourteen converse problems, which are classified into four types as follows:

1. Convereses in which simple arithmetic processes and relations are used

2. Convereses which can be solved by both arithmetic and algebra

3. Convereses which it is almost impossible to solve by arithmetic alone, but which can easily be solved by means of algebraic techniques

4. Convereses which produce indeterminate equations

The procedure in the fourth stage consists of starting with the matrix arithmetical problem and having the pupils progress through the various types of convereses. The pupils gain an understanding of the

---


28Ibid., p. 46.

29Ibid., p. 48.
basic relationships from the matrix problem; in the converses, especially those in which algebra is advisable or necessary, the pupils are encouraged to use the guess-and-check procedures and derive multi-equations by substitution of letters for the guessed numbers.30

In order to carry out this plan fully, Tanboonteck recommends that the problems in a standard textbook be rearranged, that model problems be worked out on the pupil's level of understanding, and that a separate pamphlet be prepared containing the verbal problems and explanatory materials so that the textbook will not be used during the study of verbal problems.31

In all of the algebraic solutions which Tanboonteck illustrates the guess-and-check procedure is used to derive the equations, and multi-equations are derived wherever appropriate. Hence the combination method of the present study is the basic part of the method Tanboonteck has developed.

This method adds to the combination method a feature which brings the interrelationships among groups of problems into focus for the pupils, for many different problems are based on the same relationships among the quantities, various quantities being unknowns in one problem and these same quantities being data in another problem.

30Ibid., pp. 81-82.
31Ibid., p. 82.
Tanboonteck's method was not incorporated into the experiment of the present study since it was not fully developed at that time.

Summary

In planning the experiment for this study, the writer was aided especially by the experiences reported in two of the studies reviewed in this chapter, those by Arnett and Thompson. Hopkins' study pertains less directly to the combination method and Tanboonteck's had not yet been written. For example, (1) in the reorganization of the course to accommodate the earlier introduction of multi-equation techniques, Arnett's suggestions were followed closely; (2) some worksheet materials were patterned upon examples from both Arnett and Thompson; and (3) the form of recording the solution suggested for use in the experimental sections follows closely the examples provided by Thompson.
The main criticism the writer has with the text materials as they stand is that many exercises seem to ask for answers in the form of an expression only. This is adequate for preparation for the traditional approach to problem solving, since the pupils find two expressions for the same quantity and form the equation by equating these expressions. In the experimental method one of the emphases is relationships which in algebraic form are complete thoughts or equations. Thus we should constantly encourage the pupils to use complete sentences, - equations, rather than phrases, - expressions.

Consider the exercises starting at the bottom of page 5. In each of the first three questions, complete formulas or equations are used. Not so with question 4, however. The manner of posing the question would make the answer "8p" adequate, although some teachers may insist that the answer to the question should be a complete equation. The writer suggests that the question be reworded as follows: One sack of sugar weighs p pounds. Write the formula which will tell how to find the weight w of eight of these sacks of sugar. Similarly, question 5 may be reworded as follows: A house is twice as tall as a garage. If h stands for the height of the house, and g stands for the height of the garage, write the formula which states the relationship of h and g. Similarly, questions 6, 11, and 14 can be reworded.
to encourage complete equation answers. The writer is willing to prepare a mimeographed worksheet to incorporate these changes to replace the exercises pages 5 - 8, if the experimental teachers think it is advisable.

Another principle which seems to be indicated in the experimental method is that the arithmetic of a similar situation may be the clue to the algebra. From this point of view it seems advisable that the questions in the second part of the exercises on page 9 should precede the first part. To be consistent, the directions are revised so that complete formulae are used.

EXERCISES

In the first 16 exercises below, certain additions and subtractions are indicated for finding the values of m or p or r. Find the values of m or p or r, as the case may be, when \( a = 6, b = 2, x = \frac{1}{2} \), and \( k = 1/2 \).

1. \( m = b + x \)  
2. \( p = a - x \)  
3. \( r = x - b \)  
4. \( p = a + b \)  
5. \( m = x + k \)  
6. \( r = 11 - a \)  
7. \( p = a + x \)  
8. \( r = 8 - k \)  
9. \( p = x - k \)  
10. \( r = a + b + x \)  
11. \( m = a + k - b \)  
12. \( m = 4 - x \)  
13. \( r = x + b - a \)  
14. \( p = a - 4 \)  
15. \( m = 8 - b - x \)  
16. \( p = 10 - b \)

As you know there are many ways to express in words the ideas of addition and subtraction. For example, the formula is exercise 8 above might be stated in words as follows: "\( r \) is found by subtracting \( k \) from \( 8 \)" or "\( r \) equals \( 8 \) diminished by \( k \)" or "\( r \) equals \( k \) less than \( 8 \)," etc. There are many other ways we might state this same idea in words. The following exercises are planned to help you review these different ways of expressing addition and subtraction. Suppose that \( r \) represents the number that results from each of the following additions or subtractions. Write the formula that represents the complete situation. For example "\( x \) added to \( h \)" would then be completely represented as \( r = h + x \). If you are not sure try letting the letters be certain
numbers you choose.

17. \( x + c \)  
18. \( s - b \)  
19. \( 5 - c \)  
20. \( m + x \)  
21. \( p - h \)  
22. \( h + y \)  

Suggest replacing page 19 with the following:

EXAMPLE. Find the number of inches \( n \) in \( x \) feet.

SOLUTION. 1. Find the number of inches \( n \) in 2 feet.

2. To find \( n \) in this case we multiply 12 \( \times \) 2, or \( n = 12 \times 2 \).

3. Hence to find \( n \) in the original problem multiply \( x \) by 12, or \( n = 12x \).

EXERCISES

Let \( r \) represent the resulting number in each case in the first 26 exercises and express the complete formula in the symbols of algebra.

1. seven minus two  
2. nine divided by four  
3. seven minus \( x \)  
4. the product of 2 and \( m \)  

etc.

If you have any difficulty with the following exercises, try the suggestion of making up a similar numerical problem as was illustrated at the top of this page.

27. Express the cost \( c \) of \( x \) pounds of sugar at 8 cents a pound.

28. Express the cost \( c \) of \( y \) oranges at 10 cents apiece.

29. Find the number of inches \( i \) in \( y \) yards.

30. How many feet \( f \) are there in \( m \) inches?
31. How many yards \( y \) are there in \( h \) feet?
32. How many dimes \( d \) are there in \( x \) cents?
33. At 10 cents apiece, how many grapefruit \( n \) can you buy for \( k \) cents?
34. Find the number of quarts \( q \) in \( g \) gallons.
35. Find the number of gallons \( g \) in \( q \) quarts.
36. How many inches \( i \) are there in \( (x \text{ feet} + y \text{ inches}) \)?
37. How many pounds \( p \) are there in \( o \) ounces?
38. Express the number of pints \( p \) there are in \( q \) quarts.

Suggest the following rewording for the Oral Exercises on page 20.

**WORKSHEET**

Several times previously it has been suggested that if you are having difficulty you should try letting the letters stand for some specific numbers and then see how the relationship stands. The same suggestion is made for the following problems. You will find that in algebra a similar arithmetic exercise will make an algebraic exercise clear many times.

1. If \( n \) stands for a certain number and \( m \) stands for a number which is three times as big, state the formula which tells the relationship between \( m \) and \( n \).

2. When \( h \) represents Henry's age in years and \( f \) represents Frank's age, express as a formula the fact that Frank is twice as old as Henry.

3. The length of a rectangle is five times its width. If \( w \) stands for its width, how can you represent its length \( l \) in terms of \( w \)?

4. If \( a \) stands for the number of years in Amy's present age and \( A \) stands for the number of years in her age 5 years from now, express in a formula the relationship of \( a \) and \( A \). Suppose that \( x \) represents her age eight years ago. Try to represent \( x \) two ways, once in terms of \( a \) and again in terms of \( A \).

5. Janice has six times as many stamps as Eunice has. If \( x \) represents the number of stamps that Eunice has and \( y \) represents the number of stamps that Janice has, express the relationship between the numbers \( x \) and \( y \).
6. If \( H \) represents Henry's age in years and \( C \) represents Charles's age, express algebraically the fact that Henry is six years older than Charles.

7. The difference between two numbers is 7. If \( y \) is the larger and \( x \) is the smaller, represent this fact algebraically.

8. The sum of two numbers is 63. If \( y \) is the larger and \( x \) is the smaller, represent this fact as a formula.

9. John has five times as many nickels as dimes. If \( n \) stands for the number of nickels he has, and \( D \) stands for the number of dimes, represent the relationship between the numbers \( n \) and \( D \).

10. The length of rectangle \( ABCD \) is three times its width.

\[
\begin{array}{c}
A \\
\hline
B \\
\hline
C \\
\hline
D
\end{array}
\]

If \( x \) stands for the width and \( y \) stands for the length of the rectangle, state the formula which relates the numbers \( x \) and \( y \).

The following material is planned to follow the material on the solution of equations ending with exercise 20 on page 23. If the multi-equation method is to be used then some experience with solution of pairs of equations must be provided.

EQUATIONS WITH TWO OR MORE LETTERS

Many of the equations you have already used in this chapter contain more than one letter, but the equations you have just learned to solve for the numerical value of the letter all contained only one letter. Let us consider for a moment how we might try to find numerical values for the letters in an equation when there are more than one letter in the equation, for example \( x + y = 63 \). First of all, is there only one value for each of the letters which fits? If \( x \) were 3 what would be the value of \( y \)? Could \( x \) be some other value, say 5, and \( y \) still have the same value that it had before? Of course the answer to the last question is "No" since now \( y \) needs to provide fewer of the total of 63 than it did before. Thus we see that 3 and 60 is a possible pair of values of \( x \) and \( y \) to fit the equation \( x + y = 63 \), and so is the pair 5 and 58. List several other pairs of values of \( x \) and \( y \) to fit the equation.

How do we know which one of the many possible pairs of numbers is the one we should use for an equation like this? The answer is that we do not know unless we have some more information about the situation, and this information usually takes the form of another equation. Suppose that at the same time that it is necessary that \( x + y = 63 \), it is also necessary that \( x = 5y \). Now look back over
the list of pairs of values you made so that the sum of the two
numbers in the pair was 63. Did you list a pair where one of the
numbers was five times as big as the other? Most likely you did not
have such a pair in your list, and it is also likely that you would
have to spend quite a while hunting for such a pair by trial and error.
Is there some systematic way to find these numbers without trial and
error? Yes, there is.

Now just as we may substitute the number 5 for x when we know
that $x = 5$, we may substitute $5y$ for x when we know that $x = 5y$. When
we make this substitution in the equation $x + y = 63$ the work may be
written as follows:

$$x + y = 63$$

Substitute $5y$ for $x$ in the first equation
since $x = 5y$

Combining terms on the left $6y = 63$

Divide both members by 6 $y = 10 \frac{1}{2}$

Since $x = 5y$ $x = 52 \frac{1}{2}$

We check these two numbers in both equations to see if they fit. Since
the sum of these two numbers is 63, and since $52 \frac{1}{2}$ is five times as
big as $10 \frac{1}{2}$, we may conclude that the solution of the pair of
equations

$$x + y = 63$$

$$x = 5y$$

is $y = 10 \frac{1}{2}$ and $x = 52 \frac{1}{2}$.

The written form of the check may appear like this:

$$52 \frac{1}{2} + 10 \frac{1}{2} = 63$$

$$63 = 63$$

and

$$52 \frac{1}{2} = 5 \times 10 \frac{1}{2}$$

$$52 \frac{1}{2} = 52 \frac{1}{2}$$

Another example: Let us find the values of $f$ and $g$ which satisfy
the pair of equations

$$f - g = 35,$$

$$f = 6g.$$
Now you can see that there are many pairs of numbers for $f$ and $g$ whose difference is 35, such as 40 and 5, 36 and 1, 50 and 15, etc. None of the pairs so far listed fits the situation that $f = 6g$. Likewise there are many pairs of numbers which fit the equation $f = 6g$ such as 6 and 1, 30 and 5, 15 and 2 1/2, etc. None of these pairs fits the other equation, $f - g = 35$. There will very likely be only one pair of values which will fit both equations at the same time, and we might be lucky and find such a pair by trial and error. However, there are such sets of equations so that we might not live long enough to try all the possibilities we could think of and still not find the correct pair of values. So let us use our algebra.

Substitute $6g$ for $f$ in the first equation since $f = 6g$

$$f - g = 35$$

$$6g - g = 35$$

Simplifying the left member

$$5g = 35$$

Divide both members by 5

$$g = 7$$

Substitute 7 for $g$ in the second equation

$$f = 6 \times 7 = 42$$

Check:

$$42 - 7 = 35$$

$$35 = 35$$

and

$$42 = 6 \times 7$$

$$42 = 42$$

We may now conclude that $f = 42$ and $g = 7$.

**WORKSHEET**

Each of the following exercises consists of a pair of equations. You are to find the solution which consists of statements that the letters in each pair of equations stand for certain specific numbers, as in the examples just explained.

1. $h + s = 42$
   
   $h = 5s$

2. $a = 2c$
   
   $a + c = 90$

3. $J + B = 36$
   
   $J = 3B$

4. $6s = m$
   
   $s + m = 77$

5. $3m + J = 50$
   
   $m = 2J$

6. $s = 3f$
   
   $f + s = 320$

7. $G - H = 24$
   
   $G = 9H$

8. $J = 4R$
   
   $6R - J = 28$

9. $K = 3B$
   
   $16 = K - B$

10. $2w + 2L = 224$

   $L = 3w$

11. $a = 2b$

   $5a + b = 24$

12. $2b - a = 7$

   $a = b$
If there are more than two letters whose numerical values are required, then generally we need as many equations as there are letters to be found.

1. \(a + b + c = 84\)
   \(2a = b\)
   \(c = 4a\)

2. \(f + s + t = 180\)
   \(s = 4f\)
   \(t = f\)

3. \(3p + 4q - r = 108\)
   \(q = r\)
   \(p = 2q\)

Suggest the following material to replace pp. 23 ff.

SOLVING PROBLEMS BY ALGEBRA

Some problems can be solved by either algebra or arithmetic. Some problems can be solved very easily by arithmetic, and there is no need of using algebra to solve them. Then there are problems whose solutions by algebra are not difficult, but whose solutions by arithmetic are almost impossible.

Some of the problems which you will soon be asked to solve by algebra are very easy and you may wish to solve them by arithmetic. However, you are asked to solve them by algebra so that you may learn the algebraic method. You can learn the algebraic method easily if you begin with easy problems.

One method that you may use to solve problems algebraically will be illustrated now. The problem is:

The sum of two numbers is 99. One of the numbers is five times the other number. What are the numbers?

After reading the problem you should realize that you are looking for two numbers. Now let's concentrate on the first sentence of the problem for a moment. The sum of the two numbers we are looking for must be 99. Well, \(1/2\) and \(98 1/2\) would be such a pair, and so would 90 and 9. We should all be able to name some other pairs of such numbers. Make a list of several pairs. Do any of the pairs you named solve the problem? Very likely not, since we were considering only the fact that the sum of the two numbers was to be 99, and we were for the moment ignoring the other condition our pair of numbers must meet. What is the other condition? Read the second sentence of the problem again. It states the other condition that one of the numbers must be five times as big as the other number. Of all the pairs of numbers you found whose total was 99, did you have a pair such that one of the numbers was also five times as big as the other? If you were very lucky, you may have included the correct pair, even on the first try. However, on the other hand, it may take a very long time to try enough guesses to find the correct pair. Fortunately we do not have
to find answers to such problems by trial and error. One guess of answers, even if they are wrong, is sufficient to give us the clue to the algebraic method.

Suppose that our original one-guess set of answers had been 85 and 114. It is a help to enclose each of these guess numbers in a square, triangle, or circle whenever we use them to remind us that they are only guesses. Now we check to see if our guesses were correct.

\[ \begin{align*} 85 + 114 & = 99 \\ 114 \times 5 & = 85 \end{align*} \]

The two numbers have the correct sum. Are we finished checking? No, we must check the second statement in the problem — whether or not one of the numbers is five times as big as the other.

\[ \begin{align*} 114 & \quad \text{No.} \end{align*} \]

Since this does not check we write "No." beside the arithmetic. We should be convinced that 85 and 114 are not the correct answers to the problem.

Now we write the arithmetic of the check horizontally in the form of equations, one equation for each step of the check.

\[ \begin{align*} 85 + 114 & = 99 \\ 114 \times 5 & = 85 \end{align*} \]

Note that we have placed a question mark above the "=\) sign to show that we should like the equal sign to fit, but that it does not.

Since we need two numbers and one of them will be considerably larger than the other one, let us choose \( L \) to represent the larger of the two numbers and \( s \) to represent the smaller. We often use the initial letter of words to stand for numbers in algebra such as \( A \) for area, \( b \) for base, \( n \) for number, \( F \) for Frank's age, etc. Write these letters alongside the corresponding numbers in the horizontal form of the check equations like this:

\[ \begin{align*} L + s & = 99 \\ s \times 5 & = L \end{align*} \]

Next write the equations without the numbers we guessed, since they were wrong.

\[ \begin{align*} L + s & = 99 \\ 5s & = L \]
Note that $5s$ is really the same as $s \times 5$. Now proceed with the solution of this pair of equations as you have learned to do.

Substituting $5s$ for $L$, since $5s = L$

Simplifying

$$5s + s = 99$$

$$6s = 99$$

$$s = 16\ 1/2$$

$$L = 82\ 1/2$$

Note that $D_6$ is used as a symbol for "dividing both members of the equation by 6."

Do these numbers check in both statements of the problem?

Now that you have seen the numbers which are the solution of the problem, do you agree that the likelihood of guessing them at first was very small? Yet the arithmetic you used in checking was exactly that you needed in forming the equations.

Consider another example:

Your family is considering buying a house and you go to look at some that a builder is constructing. He tells you that in his development he considers that the houses are eight times as valuable as the lots. The particular house your family is considering costs $17,100 complete. How much of the total can be apportioned to the house and to the lot separately?

Guess: House, $16,000; lot, $2,000

Check: $16,000 \times 8 = 128,000$

Yes. $17,100$ No.

Horizontal check:

$$\frac{L}{2000} \times 8 = H$$

Replace the $16,000$ by the letter $H$ to stand for house and the $2,000$ by $L$ to stand for lot.
\[ L \times 8 = H \quad \text{or} \quad 8L = H \]
\[ H + L = 17,100 \]

Solution of the pair of equations:

\[ 8L + L = 17,100 \]
\[ 9L = 17,100 \]
\[ L = 1,900 \]
\[ H = 15,200 \]

Check:

\[ 1900 \times 8 = 15,200 \]
\[ 15,200 + 1900 = 17,100 \]

Hence the value of the house is $15,200 and the value of the lot is $1,900.

By studying these two examples which have been explained in detail you should see that there are seven steps to the solution by this method which are listed below:

1. Read the problem carefully to determine what information you are given and what numbers you are to find.

2. Guess numbers which might possibly be the correct answers.

3. Check by arithmetic to see if the guesses are correct. If you write the arithmetic vertically at first, also write it horizontally. It helps keep track of the guess numbers if you enclose them in shapes such as squares, circles, or triangles.

4. Select letters to replace the guessed numbers as they were used in the horizontal check.

5. Write the resulting equations without the guessed numbers in them.

6. Solve the equations for the values of the letters.

7. Check the numbers in the statements of the verbal problem.

For the purpose of writing out your work, you may reduce the number of steps to four, as follows:

I. Guess answers and vertical check.

II. Horizontal check and replacing the guessed numbers by appropriate letters.

III. Writing the pair (or set) of equations containing letters.

IV. Solving the equations and final check.
The next example will show you how to record your work according to the four-part outline.

Problem: Jimmy's grandfather is 49 years older than Jimmy and he is also 4 1/2 times as old as Jimmy. How old is each of them?

I. 10 years and 52 years. III. \( g - J = 49 \)


II. 59 - 10 \( J = 49 \) Yes.

Check:

Another worksheet is suggested as follows.

In a previous worksheet you learned to solve for the numerical values of the letters in a pair of equations such as

\[
\begin{align*}
   f - g &= 35 \\
   f &= 6g
\end{align*}
\]

From what kind of problem might such a pair of equations come? If we let our imaginations loose it is not difficult to see that there are many situations we can construct which these equations will represent. For example, if a class is earning money by selling magazine subscriptions \( f \) can represent the number of subscriptions Frank sold, while \( g \) can represent the number of subscriptions George sold. Then \( f - g = 35 \) would tell us that Frank sold 35 more subscriptions than George sold. Likewise, \( f = 6g \) tells that Frank sold six times as many as George. Hence the two equations could represent a problem such as the following:

Frank and George are on opposite sides in a magazine-selling campaign to raise money for their class. Frank sold 35 more subscriptions than George sold, and Frank's total number of sales was 6 times as large as George's. How many subscriptions did each boy sell?
Make up some problem situations which the various pairs of equations on that previous worksheet can represent. Can you make up problems which the sets of three equations can also represent?

Following are a few brief references to changes suggested through the first 180 pages of the text in keeping with the previously explained principles.

Page
32 Keep the exercises except for the hints.
36 Reword problems 22, 25, 41, 20 so that complete formulae are implied.
38 Reword problems 14 and 20 similarly.
101 (This suggestion is not connected with the experiment) Should the fact that there is no "zero" year be taken into account in problems 25 and 26?
106 Reword problem 1 for complete formulae.

Formula, as a word, has been used throughout the earlier parts of the book, but without formal definition, as far as I have been able to determine. Perhaps this is a good place to review the use of the word and arrive at some generalizations about formulae.

132 The introduction to this page states, "In order to solve problems algebraically, it is necessary to express number relations by the use of symbols." These relations should be expressed, in my opinion by complete sentences or formulae. Why then should the following exercises require answers which in many cases are mere expressions, implying, but not really expressing relations? Suggest requiring complete formulae in every case.

134 Instructions should be revised and hints removed from the exercises.
146 Part II, reword problems 1, 3.
147 Part II, reword problems 1, 2.
172 In problems 17 and 18, expect complete formulae.
180 ff The examples worked out as samples before various sets of exercises should be reworked by the experimental multi-equation method.
APPENDIX E

FIRST DRAFT OF TEST

LaGrange Township High School

Algebra I  Name _____________

Class hour ____________

Teacher ____________

This is a test of your ability to represent verbal problems by means of algebraic equations. Hence do not waste time by working beyond the stage of obtaining equations unless you are specifically asked to do so.

For each problem, state what each letter you use is to represent such as "Let x = the number of degrees in the smallest angle of the triangle" and write the equation or set of equations which will completely represent the problem in the form of algebraic equations. Space has been provided for at least two letter-identification statements and two equations in each problem. You may need to use only one of each kind of space for each problem if you choose.

There will probably be more work here than you can complete in the time allowed. Hence do not try to finish all the problems, but try to work as fast as you can work accurately.

1. Separate the number 315 into two parts so that one of the parts is twice as large as the other part.
   Let ___ =
   (and) ___ =
   Equation(s) ________________

2. The smaller of two complementary angles is 17° less than the other angle. (Two complementary angles have a sum which is the same size as a right angle.) What is the size of each of the complementary angles?
3. Mr. Jones wishes to invest a total of $25,000 in real estate and in industrial stocks. He decides to invest 7 times as much in real estate as he invests in stocks. How much does he invest in each way?

Let __ = 
(and) __ = 
Equation(s) __________

4. A suit of clothes was sold at 20% discount for a sale price of $118. What was the original selling price of the suit?

Let __ = 
(and) __ = 
Equation(s) __________

5. Judy is 6 years older than her brother Paul. Four years ago Judy was twice as old as her brother. How old is each now?

Let __ = 
(and) __ = 
Equation(s) __________

6. A boy has some dimes and quarters which have a total value of $2.85. If there are four less quarters than there are dimes, how many of each kind of coin does he have?

Let __ = 
(and) __ = 
Equation(s) __________
7. Two boys whose weights differ by 24 lbs. find that they balance on a playground teeter board if one sits 8 feet from the fulcrum and the other sits 6 feet from the fulcrum. What is the weight of each of the boys? (Ignore the weight of the board.)

Let ___ = 

(and) ___ = 

Equation(s) ___________

8. It takes 312 feet of fence to enclose a small garden in the shape of a rectangle whose length is three feet less than twice the width. What are the dimensions of the garden?

Let ___ = 

(and) ___ = 

Equation(s) ___________

9. Copy here the equation or equations which you wrote for problem number 3 and complete the solution of the problem.

Therefore Mr. Jones invested $__________ in real estate and $__________ in stocks.

10. One pipe can fill a tank in 3 hours and another pipe can fill the same tank in five hours. How long will it take the two pipes together to fill the tank?

Let ___ = 

(and) ___ = 

Equation(s) ___________

11. A candy-shop keeper decided to mix some candy selling for 75 cents a pound with some selling for 50 cents a pound so that the mixture could be sold for 60 cents a pound. How many pounds of each kind should he use to get 50 pounds of the mixture?
12. A tank contains \( t \) gallons of a 5% alcohol solution. How many gallons of water should be added to the tank to change the solution so that it is only 3% alcohol?

Let \( ____ = ____ \)

\( \text{and} __ = ____ \)

\( \text{Equation(s):} \) \_

13. The sum of the three angles of a triangle is 180°. If the second angle is 5 times the first angle and the third angle is 48° more than the second angle, what is the size of each of the three angles?

Let \( ____ = ____ \)

\( \text{and} ____ = ____ \)

\( \text{and} ____ = ____ \)

\( \text{Equation(s):} \) \_

14. For a certain two-digit number, the sum of the digits is 12. If the digits of the number are reversed, the number becomes 18 larger than it originally was. What is the number?

Let \( ____ = ____ \)

\( \text{and} ____ = ____ \)

\( \text{Equation(s):} \) \_

15. Two trucks are to haul some freight totaling 69 tons, and the capacities of the trucks are 4 and 7 tons respectively. How many trips will each truck need to make if the larger truck makes two more trips than the smaller truck makes?
16. On a round trip between two cities a pilot found that his average speed going was 175 miles per hour and that due to a tail wind his average speed returning was 200 miles per hour. He made the return trip in 1/2 hour less time than it took him going. How far apart were the two cities?

Let ___ =

(and) ___ =

Equation(s) ____________

___

___
APPENDIX F

SECOND DRAFT OF TEST

Test on writing equations

This is a test of your ability to change verbal problems so that they are expressed in the form of algebraic equations. Therefore you should not work a problem beyond the stage of getting an equation or a group of equations to represent the problem, unless you are asked specifically to do so.

For each problem there are several lines which you may use to identify the letters or other algebraic expressions you may need in the equations. You must be sure to identify each of the letters you use by means of a statement such as the following: "Let \( f \) = the number of degrees in the smallest angle of the triangle." Hence you must use at least one of the lines for each problem, since you will need at least one letter. If you use two or more letters in a problem, be sure to identify each one. The remaining lines are for your convenience to use or not, as you choose, in identifying other expressions you may wish to use. You may add more lines of this type if you need more of them.

Next write the equation or group of equations by which the problem can be solved in the space below the lines. Remember that you do not need to complete the solution unless you are definitely asked to do so. It will probably be best for you to use the procedures for writing equations for problems which you have been using in your class work.

There is more work here than you are expected to complete in the time that you will be allotted. Therefore do not try necessarily to complete all the work, but try to work as much as you can work accurately.

Here is a sample problem worked out in two different ways:

PROBLEM: Jane was 5 years old when her brother Ralph was born. Four years ago Jane was three times as old as her brother. How old is each of them now?
Let $J$ = number of years in Jane's age now, and $R$ = number of years in Ralph's age now.

Let $J - 5$ = number of years in Jane's age now, and $J - 4$ = number of years in Ralph's age now.

Let $J - 9$ = Jane's age four years ago, and $J - 9$ = Ralph's age four years ago.

$J - 5 = R$

$J - 4 = 3(R - 4)$

$J - 4 = 3(J - 9)$
1. Separate the number 376 into two parts so that one of the parts is three times as large as the other part.

Let _____ =
and _____ =
and _____ =

2. Mr. Jones sawed a board which was twelve feet long into two pieces. He then compared the pieces and found that one of them was one and one half feet longer than the other piece. How long is each of the two pieces?

Let _____ =
and _____ =
and _____ =

3. The smaller of two complementary angles is 17° less than the other angle. (Two complementary angles must have sizes whose sum is 90°.) What is the size of each of the two angles?

Let _____ =
and _____ =
and _____ =

4. Mr. Smith inherited $25,000 which he decided to invest in two ways, in real estate and in industrial stocks. If he decided to invest 7 times as much in real estate as in the other type of investment, how much money did he invest in each type of investment?

Let _____ =
and _____ =
and _____ =
and _____ =
5. A storekeeper sold me a suit of clothes which he said was marked down with a discount of 20% from the original selling price. If I paid $18 for the suit, what should have been the original selling price?

Let \[ \text{price} = \]
and \[ \text{discount} = \]
and \[ \text{original price} = \]

6. Robert is 16 years old now and his father is 41 years old. How many years ago was it that Robert was just \( \frac{1}{6} \) as old as his father?

Let \[ \text{Robert's age} = \]
and \[ \text{father's age} = \]
and \[ \text{years ago} = \]

7. George is class treasurer. One day he took some money to be deposited in the class account which totaled $2.85. The money was in quarters and dimes only, and there were \( \frac{1}{4} \) fewer quarters than dimes. How many of each kind of coin were there to be deposited?

Let \[ \text{quarters} = \]
and \[ \text{dimes} = \]
and \[ \text{total money} = \]
and \[ \text{number of coins} = \]

8. Two girls whose weights differ by 24 lb. find that they balance on a teeterboard if one of them sits 8 ft. from the fulcrum and the other sits 6 ft. from the fulcrum. What is the weight of each of the girls? (Ignore the weight of the board.)

Let \[ \text{weight} = \]
and \[ \text{distance} = \]
and \[ \text{total weight} = \]
and \[ \text{weight difference} = \]
9. Copy here the equation or equations you wrote for problem number 3 and continue to solve the problem by solving the equation or equations.

Therefore the sizes of the two angles are _____ and ______.

10. Copy here the equation or equations you wrote for problem number 4 and continue to solve the problem by solving the equation or equations.

Therefore Mr. Smith invested $ _____ in real estate and $ _____ in stocks.

11. It takes 312 ft. of fence to enclose a small garden which is shaped like a rectangle. What are the dimensions of the garden if its length is 3 feet less than twice the width?

   Let _____ = 
   and _____ = 
   and _____ = 
   and _____ = 
   and _____ =
12. Ray was complaining about the small amount of pay he received in comparison to Ken's pay at the supermarket where they both worked some after school hours and on Saturday. Ken said, "Look, Ray, I worked three times as many hours as you did." "Yes," answered Ray, "now I see that you worked 1\frac{1}{4} more hours than I did. No wonder your pay is so much more than mine." How many hours did each of the boys work that week?

Let _____ =
and _____ =
and _____ =
and _____ =

13. Jan went shopping and bought two items. For the first item she paid $\frac{1}{5}$ of the money she took with her, and for the second item she paid $\frac{1}{3}$ of the remainder. Then she had $1.60 left. How much money did she take with her and how much did she spend for the first item?

Let _____ =
and _____ =
and _____ =
and _____ =

14. In a certain school the ninth grade has a total of 195 members. They are assigned to homerooms where it happens that the separate enrollments are consecutive odd numbers. How many ninth graders are assigned to each of the homerooms?

Let _____ =
and _____ =
and _____ =
and _____ =

15. Two trucks are assigned to haul some freight totaling 69 tons. The capacities of the two trucks are 4 tons and 7 tons respectively. What is the minimum number of trips for each truck if the larger truck makes two more trips than the smaller truck makes?

Let _____ =
and _____ =
and _____ =
and _____ =
16. The sum of the degrees in each of the three angles of any triangle is 180°. In a certain triangle the second angle is 5 times as large as the first angle, and the third angle is 48° more than the second angle. What is the size of each of the angles in this particular triangle?

Let _____ = 
and _____ =
and _____ =
and _____ =
and _____ =
APPENDIX G

FINAL FORM OF TEST

LYONS TOWNSHIP HIGH SCHOOL
LaGrange, Illinois

Algebra I

Name

Class Hour

Teacher's Name

This is a test of your ability to change verbal problems so that they are expressed in the form of algebraic equations. Therefore you should not waste time by working any problem beyond the stage of getting the equation or a group of equations to represent the problem, unless you are asked specifically to do so.

For each problem there are several lines set up which you may use to identify the letters or other algebraic expressions you need in the equations. You must be sure to identify each letter you use by means of a statement such as the following example: "Let $s$ be the number of degrees in the smallest angle of the triangle." Hence you must use at least one of the lines for each problem since you will need at least one letter. If you use two or more letters in a problem be sure to identify each one. The remaining lines are for your convenience to use or not, as you choose, in identifying other algebraic expressions you may wish to use. You may add more lines of this type if you need more of them.

Next write the equation or group of equations by which the problem can be solved in the remaining space below the lines of identification. Remember that you do not need to complete the solution of the problem unless you are definitely asked to do so. It will probably be best for you to use the procedures for writing equations for problems which you have been using in your class work.

There is more work here than you are expected to complete in the time which will be allowed. Therefore do not try necessarily to complete all the work, but try to do as much as you can work accurately. Here is a sample problem worked out in two different ways:
PROBLEM: Jane was 5 years old when her brother Ralph was born. Four years ago Jane was 3 times as old as her brother. How old is each of them now?

Let \( J \) = number of years in Jane's age now.
and \( J - 5 \) = number of years in Jane's age now.
and \( J - 4 \) = Jane's age 4 years ago.
and \( J - 9 \) = Ralph's age 4 years ago.
and \( \) = Ralph's age 4 years ago.

\[ J - 4 = 3(J - 9) \]

DO NOT TURN TO THE NEXT SHEET UNTIL YOU ARE TOLD TO DO SO.
1. Separate the number 376 into two parts so that one of the parts is 3 times as large as the other part.

Let _____ = 
and _____ = 
and _____ = 

2. Mr. Jones sawed a board which was 12 ft. long into two pieces. He then compared the pieces and found that one piece was one and one half ft. longer than the other piece. How long is each of the two pieces?

Let _____ = 
and _____ = 
and _____ = 

3. The smaller of two complementary angles is 17° less than the other angle. (Two complementary angles must have sizes whose sum is 90°.) What is the size of each of the two angles?

Let _____ = 
and _____ = 
and _____ = 
and _____ = 

4. Mr. Smith inherited $25,000 which he decided to invest in two ways, in real estate and in industrial stocks. If he decided to invest seven times as much in real estate as in the other type of investment, how much did he invest in each of the two ways?

Let _____ = 
and _____ = 
and _____ = 
and _____ =
5. A storekeeper sold me a suit of clothes which he said was marked down with a discount of 20% from the original selling price. If I paid $48 for the suit, what should have been the original selling price?

Let \( x = \) and \( y = \) and \( \frac{y}{x} = \)

6. Robert is 16 years old and his father is 41 years old. How many years ago was it that Robert was just 1/6 as old as his father?

Let \( t = \) and \( a = \) and \( b = \)

7. George is class treasurer. One day he deposited $2.85 in the class account. The money was entirely quarters and dimes and there were 4 fewer quarters than dimes. How many of each kind of coin was there to be deposited?

Let \( q = \) and \( d = \) and \( q - d = \) and \( q = \)

8. Two girls whose weights differ by 24 lb. find that they balance on a teeterboard if one of them sits 8 ft. from the fulcrum and the other sits 6 ft. from the fulcrum. What is the weight of each of the girls? (Ignore the weight of the board.)

Let \( w_1 = \) and \( w_2 = \) and \( w_1 = \) and \( w_2 = \)
9. Copy here the equation or equations you wrote for problem number 3 and continue to solve the problem by solving the equation or equations.

Therefore the sizes of the two angles are ____ and ____.

10. Copy here the equation or equations you wrote for problem number 4 and continue to solve the problem by solving the equation or equations.

Therefore Mr. Smith invested $____ in real estate and $____ in industrial stocks.

11. It takes 312 feet of fence to enclose Mr. Brown's garden which is in the shape of a rectangle with its length 3 ft. less than twice the width. What are the dimensions of the garden?

Let _____ = 
and _____ = 
and _____ = 
and _____ =

12. Ray was complaining about the small amount of pay he received in comparison to John's pay at the supermarket where they both worked some after school and on Saturdays. John said, "Look, Ray, I worked three times as many hours as you did." "Yes," answered Ray, "Now I see that you worked 1\frac{1}{4} more hours than I did. No wonder your pay is so much more than mine." How many hours did each of the boys work during the period for which they were paid that time?

Let _____ = 
and _____ =
13. Janice went shopping one day and bought two items. For the first item she paid \(\frac{1}{3}\) of the money she had taken with her, and for the second item she paid \(\frac{1}{3}\) of the remainder. Then she had $1.60 left. How much money did she take with her and how much did she spend for the first item?

Let _____ = 
and _____ = 
and _____ = 
and _____ = 

14. In a certain school the ninth grade has a total of 195 members. They are assigned to 3 homerooms where it happens that the separate enrollments are consecutive odd numbers. How many ninth graders are assigned to each of the homerooms?

Let _____ = 
and _____ = 
and _____ = 
and _____ = 
and _____ = 

15. Two trucks are assigned to haul some freight totaling 69 tons in weight. The capacities of the two trucks are 14 tons and 7 tons respectively. What is the minimum number of trips for each truck if the larger truck makes two more trips than the smaller truck makes?

Let _____ = 
and _____ = 
and _____ = 
and _____ =
16. The sum of the numbers of degrees in the three angles of any triangle is 180°. In a certain triangle the second angle is 5 times as large as the first angle, and the third angle is 48° more than the second angle. What is the size of each angle of the triangle?

Let _____ = 
and _____ = 
and _____ = 
and _____ = 
and _____ =
### APPENDIX H

**CALCULATION OF REGRESSION COEFFICIENTS (TOTAL)**

<table>
<thead>
<tr>
<th>C</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1693.01419</td>
<td>11807.09220</td>
<td>4364.39717</td>
<td>756.94362</td>
</tr>
<tr>
<td>3217.36880</td>
<td>4364.39717</td>
<td>28036.32625</td>
<td>1255.13405</td>
</tr>
<tr>
<td>290.67979</td>
<td>756.94362</td>
<td>1255.13405</td>
<td>219.50160</td>
</tr>
</tbody>
</table>

| 2.23661503 | 15.5983773  | 5.76581538 | 1.00000   |
| 2.563366677 | 3.177235893 | 22.33731628 | 1.00000  |
| 1.324271850 | 3.145865159  | 5.71810868 | 1.00000   |

| 1.239094827 | 0.028770734 | 16.61920714 | 0.477065  |
| 0.912373180 | 12.14991214 | 16.61920714 | 0.477065  |

| 0.0745579976 | 0.00173117365 | 1.00000   |
| 19.12471424 | 254.6801434 | 1.00000   |

| 19.05015624 | 254.6787028 | 0.074807431 |

| 19.12471424 | -19.05028572 | = 0.07482852 |

| 2.563366677 | -260099829 | -1.66253339 | = 0.640733457 |

**Check:**

\[
290.67979 = 56.61995 + 93.41777 + 140.64202
= 290.67974
\]
CALCULATION OF REGRESSION COEFFICIENTS (WITHIN)

<table>
<thead>
<tr>
<th></th>
<th>(a_1)</th>
<th>(a_2)</th>
<th>(a_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1668.04333</td>
<td>1168.53949</td>
<td>4165.62152</td>
<td>753.97810</td>
</tr>
<tr>
<td>3186.06333</td>
<td>4165.62152</td>
<td>2787.12515</td>
<td>1250.28791</td>
</tr>
<tr>
<td>290.07100</td>
<td>753.07810</td>
<td>1250.28791</td>
<td>219.40736</td>
</tr>
<tr>
<td>2.21196725</td>
<td>15.10679023</td>
<td>5.53116017</td>
<td>1.00000</td>
</tr>
<tr>
<td>2.548263727</td>
<td>3.331792825</td>
<td>22.22158117</td>
<td>1.00000</td>
</tr>
<tr>
<td>1.322065950</td>
<td>3.132328341</td>
<td>5.69847752</td>
<td>1.00000</td>
</tr>
<tr>
<td>1.226197778</td>
<td>-0.100598516</td>
<td>16.52610365</td>
<td>1.00000</td>
</tr>
<tr>
<td>0.89290130</td>
<td>12.03557396</td>
<td>-1.67017350</td>
<td></td>
</tr>
<tr>
<td>0.0741976332</td>
<td>0.006087249</td>
<td>1.00000</td>
<td></td>
</tr>
<tr>
<td>5.36158946</td>
<td>72.0618181</td>
<td>-1.00000</td>
<td></td>
</tr>
<tr>
<td>5.4203565792</td>
<td>72.05573115052</td>
<td>0.072245032</td>
<td></td>
</tr>
<tr>
<td>0.0741976332</td>
<td>0.0006579103</td>
<td>0.074655543</td>
<td></td>
</tr>
<tr>
<td>2.548263727</td>
<td>-2.50627721</td>
<td>-1.65918175</td>
<td>0.638447831</td>
</tr>
</tbody>
</table>

Check:

\[3186.06333 = 313.35681 + 2074.46292 + 798.24360 + 0.638447831\]
### APPENDIX I

**CALCULATION OF BETAS BY DOOLITTLE METHOD**

<table>
<thead>
<tr>
<th>Method</th>
<th>I.Q.</th>
<th>Aptitude</th>
<th>Reading</th>
<th>Equation Getting</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00000</td>
<td>.214</td>
<td>.470</td>
<td>.116</td>
<td>-.275</td>
</tr>
<tr>
<td>-1.00000</td>
<td>-.214</td>
<td>-.470</td>
<td>-.116</td>
<td>.275</td>
</tr>
<tr>
<td>1.00000</td>
<td>.452</td>
<td>.094</td>
<td>-.302</td>
<td>-.302</td>
</tr>
<tr>
<td>-.045796</td>
<td>-.100580</td>
<td>-.024824</td>
<td>.058850</td>
<td></td>
</tr>
<tr>
<td>.954204</td>
<td>.351420</td>
<td>.069176</td>
<td>-.243150</td>
<td></td>
</tr>
<tr>
<td>-1.00000</td>
<td>-.368286</td>
<td>-.072496</td>
<td>.254820</td>
<td></td>
</tr>
<tr>
<td>1.00000</td>
<td>.021</td>
<td>-.346</td>
<td>.129250</td>
<td></td>
</tr>
<tr>
<td>-.220900</td>
<td>-.054520</td>
<td>.129250</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-.129423</td>
<td>-.025477</td>
<td>.089549</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.649677</td>
<td>-.058997</td>
<td>-.127201</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1.00000</td>
<td>.090810</td>
<td>.195791</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.00000</td>
<td>-.035</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-.013456</td>
<td>.031900</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-.005015</td>
<td>.017627</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-.005358</td>
<td>-.011551</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>.976171</td>
<td>.002976</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1.00000</td>
<td>-.003049</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
\[ \beta_4 = -.003049 \]
\[ \beta_3 = (-.003049)(.090810) + .195791 = .195514\]
\[ (-.000277) \]
\[ \beta_2 = (-.003049)(-.072496) + (.195514)(-.368286) + .254820 =
(+.000221) + (-.0720051) + .254820 = .183036 \]
\[ \beta_1 = (-.003049)(-.116) + (.195514)(-.470) + (.183036)(-.214) +
.275 = (+.000353) + (-.091892) + (-.039170) + .275 =
.144292 \]
APPENDIX J

CORRESPONDENCE WITH EXPERIMENTAL TEACHERS

June 12, 1956

Mr. Leroy D. Stoldt
Lyons Township High School
LaGrange, Illinois

Dear

Now that your school year is over, or nearly over, I should like to make one additional request of you, which I hope will not be too burdensome. I feel that it will make my report sufficiently better to warrant me to ask you for an indication of your personal reactions to the method you tried in the ninth-year algebra classes for teaching the solution of verbal problems. I think it best to ask you for this prior to any indications to you or to your school of the specific results of the statistical analysis I have now made.

Certainly your comments may take any form you see fit. However, I am wondering specifically about two things. (1) How did you feel about the method as you were using it? Did you feel that the method was accomplishing less, about the same, or more than the traditional method would have done, during the problem-solving unit? Did you find the method easier, or harder to teach, considering, of course, that the method was probably relatively unfamiliar to you? (2) Could you detect any effects of the experiences with this method which carried over into the work of the second semester? For example, did the students attack new types of problems with better understanding, ease, or success? Was there any adjustment of the work load and topics considered in the second semester as a result of the experiences with simultaneous equations which the students had during the first semester? Please amplify your comments as much as you care to do, and please add any other personal reactions you may have had.

Thank you very much for all the help you have given during the course of this experiment.

Very truly yours,

Herbert F. Miller

220
Dear

I found that the introduction of simultaneous equations early in the year was helpful to the student in solving verbal problems. Also, an easier transition into the second semester's work, however the saving in time during the work on simultaneous equations was slight. (Perhaps a week.)

The "guess" method was difficult to teach because many of the students resisted the necessary writing of the steps to solve a verbal problem. Equations with three unknowns were more difficult by the "guess" method.

I definitely feel that the "guess" method was helpful to the very slow algebra learner, but that the better student could solve verbal problems as well by the "traditional" approach.

Hope this answers your questions, it was definitely an interesting experience working on this project, and I certainly enjoyed meeting and working with you.

Sincerely yours,

LeRoy V. Stoldt
June 14, 1956

Mr. Herbert F. Miller  
Northern Illinois State College  
Department of Mathematics  
DeKalb, Illinois

Dear

In response to your letter, I believe that this method was helpful to many of the students. The bright students of course would have no trouble with either method. Some of them objected to the length of the method. This was also true of the average students. They object to taking the time to write out all of the steps. This is true in the Traditional approach also—but since there is more writing necessary with the new approach they complained even more. This group as well as the slow students did quite well after mastering the use of the procedure.

In a sense this method would be harder to teach for the reasons listed above. (Student objection to its length when they can see the solution.) I think that it is an excellent approach as it promotes thinking about the factors involved. One of the most difficult processes for the student is obtaining the equation—and this method helps here.

I believe it would be best used in conjunction with the other Traditional methods. The bright student doesn't need it. Some students find it very easy and can do well. However, this is such an individual problem that several approaches are needed.

In general the method accomplished about the same results as the Traditional—with some doing better. Much less difficulty with simultaneous equations (solved by substitution) was encountered in the second semester. A specially devised course giving more practice in simultaneous equations before using this approach would help as the students made errors here while understanding the equations.

I have enjoyed working with you and this new approach. I notice you will present it at the ICTM meeting next fall. Hope to see you there.

Sincerely,

Charles R. Stegmeir
July 27, 1956

Herbert F. Miller  
Northern Illinois College  
DeKalb, Illinois  

Dear

Due to various circumstances I wasn't able to get this letter to you earlier. I'm sorry and I hope it hasn't inconvenienced you.

I do have certain comments concerning the guess method. I believe that the idea of guessing answers, checking them, and then replacing these incorrect answers with letters did help some people secure equations. The ones helped most, generally speaking, were the low average students.

I think that the two equation, two unknown, idea helped certain students with the representation of given and unknown parts of the problem. It seemed to be easier to say X is the first (a larger) and Y is the smaller, X - 20 = Y, than X is the larger and X - 20 is the smaller.

Very few of the students liked the detail involved in following your suggested outline for problem solving by the guess method. It was very difficult to get the students to do this. I don't think that much time was saved the second semester by introducing simultaneous equations the first semester. After the experiment I showed the experimental group how to use the traditional one unknown, one equation and we spent some time with this method. During the second semester, when a choice could be made, most preferred the 2 equations, 2 unknowns.

From my own observation, I would say that the guess method might have helped some students, and the idea of introducing simultaneous equations early is good, but I didn't feel that the difference in the 2 methods was too noticeable.

Very truly yours,

Richard Ellis

P.S. I did enjoy the experiment and it has been a pleasure working with you.
BIBLIOGRAPHY

A. Books


B. Periodical Articles


Ferrar, Alice W. "Two Variables," Mathematics Teacher, XX (October, 1927), 331-343.

Hawkins, George E. "Teaching Verbal Problems in First Year Algebra," School Science and Mathematics, XXXII (June, 1932), 655-60.


Wilson, Mary M. "Method of Teaching Verbal Problems," Bulletin of the Kansas Association of Teachers of Mathematics, XV (April, 1941), 62-64.


C. Unpublished Materials

Arnett, Eleanor. "The Possibility of Adapting a Traditional Textbook in Ninth Year Algebra to the Early Use of the Multiple-Equation Method." Unpublished Master of Education field service project report, Columbus, Ohio: The Ohio State University, 1952.


D. Algebra Textbooks and Courses of Study


E. References on Statistics


Jackson, Robert W. B. Application of the Analysis of Variance and Covariance Method to Educational Problems. Toronto, Ontario: Bulletin No. II of the Department of Educational Research, University of Toronto, 1940.


I, Herbert Francis Miller, was born June 11, 1909, at Olmsted Falls, Ohio. After attending both elementary and secondary school there, I graduated in 1927 as valedictorian of the class. I attended Baldwin-Wallace College, Berea, Ohio, receiving the A.B. degree, cum laude, in 1931.

From 1931 to 1942, I taught a variety of subjects, including mathematics, science, and music in the high school at Olmsted Falls. During this period I studied music and music education two summers at the National Music Camp and three summers at Northwestern University. I was also enrolled in graduate study at the Ohio State University from January to June, 1940, and the following two summers.

From 1942 to 1946 I served as an officer in the Coast Artillery Corps, and my service overseas was spent in the Philippines. After this service I returned to Olmsted Falls to continue teaching for the following three years.

In the summer of 1949 I returned to Ohio State University and received the M.A. degree in December. I continued to study in the Graduate School of the University until the fall quarter of 1952, having served as a graduate assistant, teaching in the Mathematics Department of the College of Arts and Sciences; as a graduate assistant to Dr. Earl Anderson of the Department of Education; and as a research assistant in the Bureau of Educational Research.
During the school year 1952-53, I was Head of the Mathematics Department of the Campus (laboratory) School, Iowa State Teachers College, Cedar Falls, Iowa.

Since September, 1953, I have taught as an Assistant Professor in the Mathematics Department at Northern Illinois University, DeKalb, Illinois.