ACCURACY OF TRACKING RADAR SYSTEMS

DISSERTATION

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*****

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CHAPTER I
DESCRIPTION OF TRACKING RADAR SYSTEMS

The problem to be considered is the accuracy with which the angular position of a complex radar target can be determined. The most obvious approach is the use of a narrow-beam antenna. In this case if the target lies within the beam width of the antenna, it will be seen; otherwise it will not be observed. This is the problem of determining position by means of a search system, and it has been shown¹ that bounds can be established for the determination of the target position in the presence of receiver noise for two types of targets, the point scatterer and the scatterer for which the reflected signal is Rayleigh distributed. The author states that for the point target, the position could be determined exactly if the antenna gain pattern is known and if no receiver noise is present. Otherwise, the accuracy is dependent on the beamwidth of the radar antenna. Since the antenna beamwidth is related to the physical aperture of the antenna, this is restricted by practical considerations.

In order more accurately to determine the angular position of a radar target, some means of generating an error signal must be used. The operation of various systems that perform this function is described in the following introductory paragraphs. This description is

in terms of the point target. The analysis of these systems for more complex targets is given in the report.

Perhaps the earliest of such systems consisted of lobe switching, i.e., phasing of antenna elements to cause the antenna beams to switch about an equivalent tracking axis. The two positions of the antenna beam are shown in Fig. 1. In such a system, a human operator moved the antenna to match the return from each lobe in order to determine the target position.

Next in chronological order the conical scan system appeared. In this system the antenna was offset from the equivalent tracking axis and made to nutate about it. If a point target is located off the tracking axis, the reflected signal is modulated by a sinusoidal error voltage at the scan frequency. The phase of this error voltage is dependent on the direction of the target from the equivalent tracking axis, and its magnitude is dependent on the distance from the tracking axis. The voltage from a reference generator attached to the nutating antenna is used in a phase-sensitive detector to select the in-phase component of the error voltage (after video detection) generated by the target.
Two such systems (generator and phase-sensitive detectors) are required to determine the direction of the target from the equivalent tracking axis.

Finally a group of systems appeared that have the property of instantaneously generating directional information. Since each pulse contains the required information, these systems are known as monopulse radar.

Perhaps the first of the monopulse systems is the one known as the amplitude-comparison or more simply as the amplitude-monopulse system, which was developed simultaneously at the Naval Research Laboratory and the Bell Telephone Laboratories.² This system requires two separate antennas whose patterns diverge (Fig. 2a) as in the lobe-switching method in order to determine the direction of the target in the plane in which the patterns diverge. Since another pair of antennas is required to locate the target in the orthogonal plane, this system has also been called the four-lobe amplitude monopulse system. Because of symmetry only one pair of antennas need be discussed. A hybrid T is used to form the sum and difference of the signals received by the two antennas. The antenna patterns obtained at the terminals of the hybrid T are shown in Figs. 2b and 2c. As in all monopulse systems,

the difference signal contains the direction information while the sum signal is a reference signal. The phase of the difference signal (either in- or out-of-phase with the sum signal) indicates the sense of directional information, while its relative magnitude indicates the distance from the tracking axis. The sum and difference signals are amplified separately and finally combined in a product detector. Page 2 described such a system at the 1955 National IRE Convention.

The second monopulse system, known as the phase-comparison monopulse or more simply as the phase-monopulse system, again uses
two separate receiving antennas which have in this case coincident beams to obtain directional information in a single plane. As a point scatterer is moved off the equivalent tracking axis, the reflected signal seen by the two antennas differs in phase by an amount proportional to the difference in respective path lengths as shown in Fig. 3.

![Fig. 3. Antenna geometry for a phase monopulse system.](image)

Again the sum and difference of the signals from the two antennas are formed. For the point scatterer, it can readily be demonstrated that the difference or error signal is in phase quadrature with the sum signal.

As before, the sum signal is used as a reference and the difference signal indicates the position of the target. Again these two signals (sum and difference) are fed through two completely separate receiving channels until they are combined in the detection process. Since
another pair of antennas is required to obtain directional information in the orthogonal plane, this system has also been called the four-lobe phase-monopulse system.

The third monopulse that has been used, to be called here the phase amplitude-monopulse system, is a hybrid combination of the two systems described previously. Since a single pair of antennas is used in this case to obtain directional information in two orthogonal directions, it is also called the two-lobe phase amplitude system. The antennas are displaced physically in one direction and their beams are coincident in this direction, thus introducing phase information related to this direction. In the orthogonal direction their beams are caused to diverge, thus introducing amplitude information in this direction.

Again a hybrid T is used to form the sum and difference of the signals received by the two antennas. In this case, however, the component of the difference signal in phase with the sum signal contains the information for the plane in which the antenna beams diverge, while the component of the difference signal in phase quadrature with the sum signal contains directional information for the plane in which the antennas are physically separated.

In all of the preceding discussion of tracking systems, the target has been assumed to be a point target. In tracking such a point scatterer all of these systems would work very well. Indeed, in theory
they would work perfectly. However, most radar targets are not point targets by any manner of speaking. The most accurate description of a target such as an aircraft would be that it is a target which is finite in extent. In general, a target such as an aircraft does not conform to any simple geometrical shape and cannot be handled by any convenient mathematical technique. Thus it becomes necessary to approximate the radar target. An approximation that appears to be generally in use is that of a multipoint target. Here the assumption is that the signal from an aircraft, for example, is reflected from a group of points or dominant scatterers, and the rest of the aircraft can be neglected. Since the results are not dependent on any particular configuration of target points, however, they may be applied to any shape whatsoever simply by taking a sufficient number of points.

In general, the work that has been done using the above approximation is of a statistical nature. The radar target is assumed to be in a random motion, and because of the phasing between the various scatterers, target-generated noises appear in the radar. These noises have in general been classified as servo noise or jitter, receiver noise, noise due to the amplitude variations of the received signal, and noise due to the motion of the apparent angle of arrival of the signal. Brockner\(^3\) discusses the relative relationship of these

various noises and the total noise as a function of range. Similar discussions are given by both NRL and General Electric in unpublished reports. These relationships are illustrated in Fig. 4. It may be seen from these curves that at short ranges the angular noise is dominant while at very long ranges the receiver noise is dominant.

The approach used in this report deviates from the above in that it is not statistical in nature. The target is considered in quasistatic motion, i.e., the radar servo is assumed to be sufficiently fast so that it follows the balance point, * and thus no electrical noise is generated. This leads to a study of the position of the balance point. Feagin and Watson of the University of Texas have obtained an equation for the balance point of a conical-scan system and a phase-amplitude-monopulse system. The results of the present analysis can be reduced to those of this earlier work. Since the balance point moves as the target aspect changes, angular noise is generated. This phenomenon is particularly significant, then, at short ranges (see Fig. 4).

*The balance point is defined as the point on a complex target toward which the radar must point in order that no error signals are generated.
Fig. 4. Magnitude of tracking noise components as a function of range.
Much of the work described in this paper is contained in a series of earlier reports.\textsuperscript{4,5,6} This present paper summarizes much of the previous work, adds some additional material, and attempts to present a more complete picture.

\textsuperscript{4} Weimer, F. C. and Peters, Leon, Jr., \textit{An Alternate Solution for the Balance Point of a Conical Scan Tracking Radar}, Report 601-11, 31 July 1956, Antenna Laboratory, The Ohio State University Research Foundation; prepared under Contract AF 33(616)-2546, Air Research and Development Command, Wright Air Development Center, Wright-Patterson Air Force Base, Ohio.

\textsuperscript{5} Weimer, F. C. and Peters, Leon, Jr., \textit{Study of Pointing Errors in Conically Scanning and Monopulse Tracking Radar for Multipoint Targets}, Report 601-12, 15 October 1956, Antenna Laboratory, The Ohio State University Research Foundation; prepared under Contract AF 33(616)-2546, Air Research and Development Command, Wright Air Development Center, Wright-Patterson Air Force Base, Ohio.

\textsuperscript{6} Peters, Leon, Jr. and Weimer, F. C., \textit{Comparison of Balance Point and Static Error Voltages of a Complex Target for Various Tracking Radar Systems}, Report 601-23, 31 August 1957, Antenna Laboratory, The Ohio State University Research Foundation; prepared under Contract AF 33(616)-2546, Air Research and Development Command, Wright Air Development Center, Wright-Patterson Air Force Base, Ohio.

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CHAPTER II
ANALYSIS OF TRACKING RADAR SYSTEMS

Introduction

The purpose of this chapter is to derive the equations for the balance-point and the static-error voltages when tracking a multipoint target. Conical-scan, amplitude-monopulse, phase-monopulse, and phase amplitude-monopulse systems are treated.

Target Geometry

An equation has been derived for the balance position of a conically scanning radar when tracking an n-point target. The same geometry and initial assumptions are to be used in the derivation of another equation for this same case. Consequently the following paragraphs are taken verbatim from the previous work.\(^7\)

It is assumed that the points which constitute a radar target are situated in a small region in space so that the lines of sight from the illuminating radar set at some range, \(R\), are nearly parallel. A reference plane is set up perpendicular to the line joining the radar antenna to a given reference point in this plane. This reference point could be one of the points in question but in general is not. Each point \(P_T\) has a projection \(P'_T\) on this plane. The radar

\(^7\) Feagin, R.B., Watson, R.B., and Norwood, J.M., Detailed Data Reduction Procedure for Airborne N-Point Target Backscattering, 30 March 1955, Military Physics Research Laboratory, University of Texas, Austin, Texas.
center line crosses this plane at a point $P$. These points and the plane are illustrated in Fig. 5.

![Fig. 5. Geometry of reference plane.](image)

The coordinates in the reference plane are $x_r$ and $y_r$ for any projected point $P'_r$. It is assumed that the $x$ dimension is horizontal. The point $P_r$ scatters radar energy from the incident beam from the radar set; the portion of this scattered energy of interest is that which passes through the point $P'_r$, i.e., the back-scattered energy. Therefore with the back-scattering from a point $P_r$ there will be associated at $P'_r$ an amplitude $A_r$ and
an electrical phase angle \( \theta_r = (4\pi/\lambda) \frac{\vec{P}_T - \vec{P}_r'}{P_T - P_r'} \)

**Conical Scan Tracking System**

The signal received by a conically-scanning radar set due to back scattering is proportional to

\[
(2-1) \quad S_T = \sum_{i=1}^{n} A_i \left[ 1 + m_i \sin(\omega_s t + \phi_i) \right] \cos(\omega_c t + \theta_i)
\]

where \( \phi_r \) is determined by the line joining \( P \) and \( P_r \) as shown in Fig. 6, and the coefficient \( m_r \) is proportional to the small angle \( (\xi_r) \) between the lines of sight from

---

**Fig. 6. Geometry in reference plane.**
the radar set to $P$ and $P_R$; this angle must be sufficiently small to allow the earlier assumption of parallel lines of sight throughout the region of the $n$ point scatterers.

Note that $\omega_S$ is the scan frequency of the antenna and $\omega_0$ is the radar frequency.

Since $m$ is proportional to the small angle $\xi$, it is shown in the previous work\textsuperscript{1} that

$$m_r \sin \phi_r = \frac{K}{R} (y_r - y)$$

(2-2)

$$m_r \cos \phi_r = \frac{K}{R} (x_r - x).$$

The material contained above forms the foundation for the solution to be presented here. The assumptions made are examined in Appendix A.

Expanding the $\sin(\omega_S t + \phi)$ term of eq. (2-1) gives

$$S_T = \sum_{i=1}^{n} A_i \cos(\omega_0 t + \theta_i)$$

$$+ \sum_{i=1}^{n} A_i m_i \cos \phi_i \sin \omega_S t \cos(\omega_0 t + \theta_i)$$

$$+ \sum_{i=1}^{n} A_i m_i \sin \phi_i \cos \omega_S t \cos(\omega_0 t + \theta_i).$$
Substituting eqs. (2.2) into the above equation gives

$$S_T = \sum_{i=1}^{n} A_i \cos(\omega_0 t + \theta_i)$$

(2-3)

$$+ \frac{K}{R} \sum_{i=1}^{n} A_i \left[(x_i-x) \sin \omega_s t + (y_i-y) \cos \omega_s t \right] \cos(\omega_0 t + \theta_i).$$

If $S$ is defined so that

$$S_T = \text{Re} \left[ S e^{j\omega_0 t} \right]$$

then eq. (2-3) becomes

$$S = \sum_{i=1}^{n} A_i e^{j\theta_i} + \frac{K}{R} \sum_{i=1}^{n} A_i (x_i-x) e^{j\theta_i} \sin \omega_s t$$

(2-5)

$$+ \frac{K}{R} \sum_{i=1}^{n} A_i (y_i-y) e^{j\theta_i} \cos \omega_s t$$

where $\text{Re}$ denotes "real part of" and $\text{Im}$ denotes "imaginary part of".

The balance position of the radar is taken as the values of $x$ and $y$ that make the coefficients of the $\sin \omega_s t$ and $\cos \omega_s t$ terms respectively equal to zero. The coefficient of the $\sin \omega_s t$ will vanish when

$$\sum_{i=1}^{n} x_i A_i e^{j\theta_i} = x \sum_{i=1}^{n} A_i e^{j\theta_i}.$$

If this particular value of $x$ is defined as $\bar{x}$, then

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i A_i e^{j\theta_i}}{\sum_{i=1}^{n} A_i e^{j\theta_i}}.$$
In the same manner,

\[ y = \frac{\sum_{i=1}^{n} y_i A_i e^{j\theta_i}}{\sum_{i=1}^{n} A_i e^{j\theta_i}}. \]  

(2-7)

Obviously eqs. (2-6) and (2-7) cannot always be satisfied since a
dimension can never possess an imaginary part.

Defining

\[ A e^{j\theta} = \sum_{i=1}^{n} A_i e^{j\theta_i} \]

(2-8)

gives

\[ A \bar{x} e^{j\theta} = \sum_{i=1}^{n} x_i A_i e^{j\theta_i} \]

(2-9)

and

\[ A \bar{y} e^{j\theta} = \sum_{i=1}^{n} y_i A_i e^{j\theta_i}. \]

(2-10)

Then eq. (2-5) becomes

\[ \bar{S} = A e^{j\theta} + \frac{K}{R} (\bar{x} - x) A e^{j\theta} \sin \omega t \]

\[ + \frac{K}{R} (y - y) A e^{j\theta} \cos \omega t. \]  

(2-11)
From eq. (2-4)

\[ S_T = \Re \left( \overline{S} e^{j\omega_0 t} \right) \]

\[ = \Re \left[ A e^{j(\omega_0 t+\theta)} \left( 1 + \frac{K}{R} (\overline{x}-x) \sin \omega_s t + \frac{K}{R} (\overline{y}-y) \cos \omega_s t \right) \right] \]

\[ + \frac{K}{R} (\overline{y}-y) \cos \omega_s t \]

\[ = A \cos(\omega_0 t+\theta) \left( 1+\frac{K}{R} \Re(\overline{x}) - x \right) \sin \omega_s t + \frac{K}{R} \Re(\overline{y}) - y \right) \cos \omega_s t \]

\[ - \frac{K}{R} \quad A \sin(\omega_0 t+\theta) \left( \Re(\overline{x}) - x \right) \sin \omega_s t + \Re(\overline{y}) - y \right) \cos \omega_s t \].

To consider the action of a linear detector eq. (2-12) may be re-written as

\[ S_T = A \cos(\omega_0 t+\theta+\xi) \sqrt{1+\frac{K}{R} \Re(\overline{x}) - x \right) \sin \omega_s t}

\[ + \frac{K}{R} \Re(\overline{y}) - y \right) \cos \omega_s t \]}

\[ + \left( \frac{K}{R} \right)^2 \left( \Re(\overline{x}) - x \right) \sin \omega_s t + \Re(\overline{y}) - y \right) \cos \omega_s t \]

or

\[ S_T = \left[ 1 + \frac{2K}{R} \Re(\overline{x}) - x \right] \sin \omega_s t + \frac{2K}{R} \Re(\overline{y}) - y \right] \cos \omega_s t \]

\[ + 2 \left( \frac{K}{R} \right)^2 \left( \Re(\overline{x}) - x \right) \Re(\overline{y}) - y \right] \sin \omega_s t \cos \omega_s t \]

\[ + \left( \frac{K}{R} \right)^2 \left( \Re(\overline{x}) - x \right)^2 \sin^2 \omega_s t + \left( \frac{K}{R} \right)^2 \Re(\overline{y}) - y \right]^2 \cos^2 \omega_s t \]

\[ + \left( \frac{K}{R} \right)^2 \left( \Re(\overline{x}) - x \right) \sin \omega_s t + \Re(\overline{y}) - y \right] \cos \omega_s t \]

\[ + \left( \frac{K}{R} \right)^2 \left( \Re(\overline{x}) - x \right) \sin \omega_s t + \Re(\overline{y}) - y \right] \cos \omega_s t \]

where \( \theta' = \theta + \xi \).
The quantity in the brackets may be raised to the $\frac{1}{2}$ power by means of the binomial theorem. Since $K/R << 1$, higher order terms may be neglected and

\[(2-15)\quad S_T \approx \left[1 + \frac{K}{R} \left\{ \text{Re}(\bar{x}) - x \right\} \sin \omega_st + \frac{K}{R} \left\{ \text{Re}(\bar{y}) - y \right\} \cos \omega_st\right.
\]
\[\quad + \frac{(K/R)^2}{2} \left\{ \text{Re}(\bar{x}) - x \right\}^2 \sin^2 \omega_st + \frac{K}{R} \left\{ \text{Re}(\bar{y}) - y \right\}^2 \cos^2 \omega_st\]
\[\quad + \frac{1}{2} \frac{(K/R)^2}{2} \left[ \text{Im}(\bar{x}) \sin \omega_st + \text{Im}(\bar{y}) \cos \omega_st \right]^2 A \cos(\omega_0t + \phi).\]

Because

\[\sin^2 \omega_st = \frac{1}{2} - \frac{1}{2}\cos 2\omega_st\]
\[\cos^2 \omega_st = \frac{1}{2} + \frac{1}{2}\cos 2\omega_st, \quad \text{and}\]
\[\sin \omega_st \cos \omega_st = \frac{1}{2}\sin 2\omega_st,\]

only the second and third terms of the above equation involve the fundamental of the scan frequency.

If the balance point $x_b, y_b$ is defined as the values of $x$ and $y$ for which the coefficients of $\sin \omega_st$ and $\cos \omega_st$ vanish, then from eq. (2-15)

\[(2-16)\quad x_b = \text{Re} \left( \bar{x} \right) = \text{Re} \left[ \frac{\sum_{i=1}^{n} x_i A_i e^{j\theta_i}}{\sum_{i=1}^{n} A_i e^{j\theta_i}} \right].\]
If now a linear video detector and a bandpass filter of center frequency $\omega_s$ are used, then

$$S(\text{det}) \propto \frac{KA}{R} \left[ \Re(\bar{x}) - x \right] \sin \omega_s t + \left[ \Re(\bar{y}) - y \right] \cos \omega_s t.$$ 

To consider the action of a square-law detector, eq. (2-14) is squared giving

$$S_T^2 = \frac{1}{2} U A^2 \left[ 1 + \cos 2(\omega_s t + \delta) \right]$$

where $U$ is the bracketed term of eq. (2-14). If this signal is passed through a band-pass filter of center frequency $\omega_s$, then

$$S_T^2(\text{det}) \propto \frac{KA^2}{R} \left[ \left[ \Re(\bar{x}) - x \right] \sin \omega_s t + \left[ \Re(\bar{y}) - y \right] \cos \omega_s t \right]$$

and coefficients of $\sin \omega_s t$ and $\cos \omega_s t$ go to zero for

$$x = x_b = \Re(\bar{x}) = \Re \frac{\sum x_i A_i e^{j\theta_i}}{\sum A_i e^{j\theta_i}}$$

and

$$y = y_b = \Re(\bar{y}) = \Re \frac{\sum y_i A_i e^{j\theta_i}}{\sum A_i e^{j\theta_i}}.$$
It is to be noted that the only difference in the two cases is that the radar signal appears as the first power for the linear detector and as the second power for the square-law detector.

Extension of Conical Scan Balance-Point Equation to Include a Finite Scatterer

The signal received by a conically scanning system for a collection of point sources may be represented as in eq. (2-5) by

\[ S = \sum_{i=1}^{n} A_i e^{j\theta_i} + \frac{K}{R} \sum_{i=1}^{n} A_i (x_i-x) e^{j\theta_i} \sin\omega_st \]

\[ + \frac{K}{R} \sum_{i=1}^{n} A_i (y_i-y) e^{j\theta_i} \cos\omega_st. \]

While the concept of a point scatterer is convenient, scattering does not usually occur from a point, but instead from an element of area. Thus the magnitude \( A_i \) might better be represented by

\[ A_i = A_{jk} \Delta x_j \Delta y_k \]

where \( \Delta x_j \Delta y_k \) is the effective area of the scatterer. The phase \( \theta_{jk} \) of the signal reflected from the element of area \( \Delta x_j \Delta y_k \) is constant over the element of area provided that \( \Delta x_j \Delta y_k \) is sufficiently small. Thus eq. (2-5) may be rewritten as
\[ S = \sum_{j=1}^{N} \sum_{k=1}^{N} A_{jk} \Delta x_j \Delta y_k e^{i\theta_{jk}} \]
\[ + \frac{K}{R} \sin \omega_s t \sum_{j=1}^{N} \sum_{k=1}^{N} A_{jk} \Delta x_j \Delta y_k e^{i\theta_{jk}} (x_j - x) \]
\[ + \frac{K}{R} \cos \omega_s t \sum_{j=1}^{N} \sum_{k=1}^{N} A_{jk} \Delta x_j \Delta y_k e^{i\theta_{jk}} (y_k - y). \]

If now \( N \to \infty \) in such a way that all \( \Delta x_i \) and all \( \Delta y_i \) approach zero, then the summations become integrals in the Riemann sense, i.e.,

\[ \lim_{N \to \infty} S = \int \int A'(x^*, y^*) e^{i\theta(x^*, y^*)} \, dx^* dy^* \]
\[ + \frac{K}{R} \sin \omega_s t \int \int A'(x^*, y^*) e^{i\theta(x^*, y^*)} (x^*-x) \, dy^* dx^* \]
\[ + \frac{K}{R} \cos \omega_s t \int \int A'(x^*, y^*) e^{i\theta(x^*, y^*)} (y^*-y) \, dx^* dy^* \]

where the limits of the integrals are the extreme ends of the target. This may be put into a more convenient form by performing some of the indicated integrations, giving

\[ (2-19) \quad S = A e^{i\theta} + \frac{K}{R} \sin \omega_s t \int A''(x^*) (x^*-x) e^{i\theta''(x^*)} \, dx^* \]
\[ + \frac{K}{R} \cos \omega_s t \int A''(y^*) (y^*-y) e^{i\theta''(y^*)} \, dy^* \]
where

\[ A e^{i\theta} = \lim_{i \to \infty} \sum_{i} A_i e^{i\theta} \]

\[ = \int \int A(x^*, y^*) e^{i\theta(x^*, y^*)} \, dx^* dy^* \]

(2-21) \[ A''(x^*) e^{i\theta''(x^*)} = \int A'(x^*, y^*) e^{i\theta'(x^*, y^*)} \, dy^* \]

and

(2-22) \[ A''(y^*) e^{i\theta''(y^*)} = \int A'(x^*, y^*) e^{i\theta'(x^*, y^*)} \, dx^*. \]

The coefficients of \( \sin \omega_s t \) and \( \cos \omega_s t \) vanish for

\[ x = \overline{x'} = \frac{\int A''(x^*) x^* e^{i\theta''(x^*)} \, dx^*}{\int A''(x^*) e^{i\theta''(x^*)} \, dx^*} \]

and

\[ y = \overline{y'} = \frac{\int A''(y^*) y^* e^{i\theta''(y^*)} \, dy^*}{\int A''(y^*) e^{i\theta''(y^*)} \, dy^*} \]

Since

\[ A e^{i\theta} = \int A''(x^*) e^{i\theta''(x^*)} \, dx^* \]

\[ = \int A''(y^*) e^{i\theta''(y^*)} \, dy^*, \]

then
\begin{align*}
(2-23) \quad A\bar{x}' e^{j\theta} &= \int A''(x^*) x^* e^{j\theta''(x^*)} dx^* \\
\text{and} \\
(2-24) \quad A\bar{y}' e^{j\theta} &= \int A'''(y^*) y^* e^{j\theta'''(y^*)} dy^*.
\end{align*}

Substitution of eqs. (2-20), (2-23), and (2-24) into eq. (2-19) gives

\begin{align*}
\bar{S} &= A e^{j\theta} + \frac{K}{R} (\bar{x}' - x) A e^{j\theta} \sin \omega_t \\
&\quad + \frac{K}{R} (\bar{y}' - x) A e^{j\theta} \cos \omega_t,
\end{align*}

which is identical in form to eq. (2-11). With the same procedure as for the multipoint target the balance point equations are

\begin{align*}
(2-25) \quad x_b &= \text{Re} (\bar{x}') = \text{Re} \left\{ \frac{\int A''(x^*) x^* e^{j\theta''(x^*)} dx^*}{\int A''(x^*) e^{j\theta''(x^*)} dx^*} \right\} \\
\text{and} \\
(2-26) \quad y_b &= \text{Re} (\bar{y}') = \text{Re} \left\{ \frac{\int A'''(y^*) y^* e^{j\theta'''(y^*)} dy^*}{\int A'''(y^*) e^{j\theta'''(y^*)} dy^*} \right\}.
\end{align*}

The * may be dropped in the last two equations since there is no possibility of confusion at this point. These results could have been obtained directly from eqs. (2-16) and (2-17) by taking the limit of the summations involved in these equations in the same manner as in the preceding paragraphs to obtain
\[ x_b = \Re \frac{\int x A(x) e^{j\theta(x)} \, dx}{\int A(x) e^{j\theta(x)} \, dx} \]

and

\[ y_b = \Re \frac{\int y A(y) e^{j\theta(y)} \, dy}{\int A(y) e^{j\theta(y)} \, dy} . \]

These equations are identical to eqs. (2-25) and (2-26), thus indicating that the results obtained for the balance point from the N-point target can be made completely general simply by allowing N to approach infinity.

**Phase Monopulse Tracking System**

This monopulse system receives signals simultaneously on four antennas whose beams coincide but whose electrical phase centers are displaced normal to the equivalent tracking axis. Two antennas are required to obtain directional information in the plane in which these centers are displaced. Thus two pairs of antennas whose electrical centers are displaced in orthogonal planes are needed to locate the target. Only the single pair of antennas shown in Fig. 3 needs to be analyzed since the results apply equally well to the other pair.

For a point target at long range and near the equivalent radar axis, the signal received on antenna number one is

\[(2-27) \quad B_{i1} = A_i \sin(\omega_0 t + \theta_i + \phi_i)\]
and on antenna number two the received signal is

\[(2-28) \quad B_{12} = A_1 \sin(\omega_0 t + \theta + \phi_i)\]

where \(\phi_i\) is the phase deviation caused by the difference in range due
to the separation of the antennas.

In order to combine the signals from \(N\) target points, it is convenient
to define phasors (vectors) like those used in alternating-
current circuit theory such that

\[B_{11} = \Delta m \left[ \overline{B_{11}} e^{j\omega_0 t} \right] \quad \text{and} \quad B_{12} = \Delta m \left[ \overline{B_{12}} e^{j\omega_0 t} \right].\]

Then for a multipoint target

\[
\overline{B_1} = \sum_{i=1}^{n} \overline{B_{1i}} = \sum_{i=1}^{n} A_i e^{j(\theta_i + \phi_i)}
\]

and

\[
\overline{B_2} = \sum_{i=1}^{n} \overline{B_{2i}} = \sum_{i=1}^{n} A_i e^{j(\theta_i - \phi_i)}.
\]

The receiving apparatus forms the phasor sum and difference of
these two signals. The sum is

\[
\overline{S} = \overline{B_1} + \overline{B_2} = \sum_{i=1}^{n} A_i (e^{j(\theta_i + \phi_i)} + e^{j(\theta_i - \phi_i)}) = 2 \sum_{i=1}^{n} A_i e^{j\theta_i} \cos \phi_i
\]
and the difference is

\[ \bar{D} = \bar{B}_1 - \bar{B}_2 = 2j \sum_{i=1}^{n} A_i e^{j\theta_i} \sin \phi_i. \]

From the geometry of Fig. 3 it is seen that

\[ \phi_i = \frac{\pi d \sin E_{pi}}{\lambda} \approx \frac{\pi d E_{pi}}{\lambda} \approx K_p E_{pi} \]

where \( d \) is the distance between the antennas and \( \lambda \) is the wavelength.

If the angle \( E_{pi} \) is sufficiently small, the approximations

\[ \cos \phi_i \approx 1 \quad \text{and} \]

\[ \sin \phi_i \approx \phi_i = K_p E_{pi} \approx \frac{K_p}{R} (x_i - x) \]

may be made. Then

\[ (2-29) \quad \bar{S} = 2 \sum_{i=1}^{n} A_i e^{j\theta_i} \quad \text{and} \]

\[ \bar{D} = 2j \frac{K_p}{R} \sum_{i=1}^{n} A_i e^{j\theta_i} (x_i - x). \]

Noting that the difference term goes to zero for

\[ x = \bar{x} = \frac{\sum x_i A_i e^{j\theta_i}}{\sum A_i e^{j\theta_i}}, \]
we again define

\[(2-8) \quad \sum A_i e^{j0_i} = A e^{j0}\]

and obtain

\[(2-9) \quad A \bar{x} e^{j0} = \sum x_i A_i e^{j0_i}. \]

Then

\[\bar{S} = A e^{j0}\]

and

\[\bar{D} = 2j \frac{K_p}{R} A e^{j0} (\bar{x} - x).\]

Now

\[S = j_m (\bar{S} e^{j\omega_0 t}) = 2A \sin(\omega_0 t + \theta)\]

and

\[D = j_m (\bar{D} e^{j\omega_0 t}) = \frac{2K_p A}{R} \left[ (\bar{R}_0 x) - x \right] \cos(\omega_0 t + \theta) - j_m \bar{x} \sin(\omega_0 t + \theta).\]

It may now be noted that the component of the difference signal in phase quadrature with the sum term contains the directional information, and that the component in phase with the sum term is independent of the direction in which the equivalent radar axis is pointing. This "in phase" term is due to the weighting function \((x_i - x)\) as may be seen in eq. \((2-29)\).

The required information may be extracted by use of a phase-sensitive detector. When such a device is used, the error signal is
\[ \epsilon \propto \frac{Kp}{R} A^2 (\Re (\bar{x}) - x) \]

and the balance point is

\[ x_b = \Re (\bar{x}). \]

**Amplitude-Monopulse Tracking System**

This monopulse system receives signals simultaneously on four antennas so placed that the beams of one pair diverge in a given plane about the equivalent tracking axis and coincide in a plane orthogonal to the first as shown in Fig. 2. The other pair of antennas are oriented so that these two planes are interchanged. The first part of this analysis assumes that the antennas occupy the same physical position ("true" amplitude-monopulse system). If the antennas are physically displaced, then a phase factor will appear. This configuration will be treated later in this section.

A single pair of these antennas will yield directional information in the plane in which their beams diverge. Thus only a single pair is to be treated here. However, the results will apply equally well to the other pair.

For a point target at long range and near the equivalent tracking axis, the signal received on one of these antennas is

\[ (2-30) \quad B_{i1} = A_i (1 + K_a E_{ai}) \sin(\omega_0 t + \theta_i) \]
and on the other antenna is

\[(2-31) \quad B_{12} = A_i (1 - K_a E_{ai}) \sin(\omega_0 t + \theta_i)\]

where \(E_{ai}\) is the angle between the line of sight to the target and the equivalent radar axis, \(K_a\) is a constant, and the other symbols have been defined earlier.

Again we define phasors such that

\[B_{i1} = \Im \left[ \bar{B}_{i1} e^{j\omega_0 t} \right]\]

and

\[B_{i2} = \Im \left[ \bar{B}_{i2} e^{j\omega_0 t} \right]\]

where \(\Im [\ ]\) indicates the imaginary part of a complex quantity.

Then the total signal received on each antenna can be written as

\[\bar{B}_1 = \sum_{i=1}^{n} \bar{B}_{i1} = \sum_{i=1}^{n} A_i (1 + K_a E_{ai}) e^{j\theta_i}\]

and

\[\bar{B}_2 = \sum_{i=1}^{n} \bar{B}_{i2} = \sum_{i=1}^{n} A_i (1 - K_a E_{ai}) e^{j\theta_i} .\]

Again the receiving apparatus forms the sum and the difference of these two signals. The sum is

\[\bar{S} = \bar{B}_1 + \bar{B}_2 = 2 \sum_{i=1}^{n} A_i e^{j\theta_i}\]
and the difference is

\[ \bar{D} = B_1 - B_2 = 2 \sum_{i=1}^{n} A_i K_a E_{ai} e^{j\theta_i}. \]

Since at large distances the angle \( E_{ai} \approx (x_i - x)/R \), the difference becomes

\[ (2-32) \quad D \approx 2 \sum_{i=1}^{n} A_i \frac{K_a}{R} (x_i - x) e^{j\theta_i}. \]

This expression goes to zero when

\[ x = \bar{x} = \frac{\sum_{i=1}^{n} x_i A_i e^{j\theta_i}}{\sum_{i=1}^{n} A_i e^{j\theta_i}}. \]

Recalling that

\[ \psi = (2-8) \quad A e^{j\theta} = \sum_{i=1}^{n} A_i e^{j\theta_i} \]

and

\[ (2-9) \quad A e^{j\theta} \bar{x} = \sum_{i=1}^{n} x_i A_i e^{j\theta_i}, \]

we obtain

\[ \bar{D} \approx \frac{2K_a}{R} A e^{j\theta} (\bar{x} - x). \]
Now

\[ S = \Im\left( \overline{S} e^{j\omega_0 t} \right) = 2A \sin(\omega_0 t + \theta) \]

and

\[ D = \Im\left( \overline{D} e^{j\omega_0 t} \right) \]

\[ = \frac{2K_a A}{R} \left\{ \sin(\omega_0 t + \theta) [\Re(\overline{x}) - x] + \cos(\omega_0 t + \theta) [\Im(\overline{x})] \right\}. \]

Note that the component of the difference signal in phase with the sum signal goes to zero when the equivalent tracking axis (defined as the direction in which the radar must point to receive zero signal when tracking a point target) is pointed such that \( x = \Re(\overline{x}) \) while the component of the difference signal in quadrature with the sum signal is independent of the direction in which the equivalent tracking axis is pointing. This phase quadrature component of the difference is due to the weighting factor \((x_1 - x)\) of eq. (2-32). It would indeed be a coincidence if \( \overline{S} \) and \( \overline{D} \) were even in phase.

We define the balance point \( x_b \) as the position toward which the equivalent tracking axis must point so that the error voltage goes to zero. For the case of the phase-sensitive detector, the sum and difference signals are simply multiplied together, and the high-frequency term is filtered out to get

\[ \varepsilon \propto \frac{2K_a A^2}{R} \left[ \Re(\overline{x}) - x \right]. \]
In this case $x_b = \Re(\overline{x})$.

If the phase centers of the antennas of the amplitude-monopulse system are displaced in space, the phasing due to their displacement becomes significant. The configuration to be treated is that in which the antenna displacement occurs in the plane for which their patterns diverge (see Fig. 2). The other possible configuration is to be treated later in the section on the phase amplitude-monopulse system.

In this case eqs. (2-30) and (2-31) become

\[ B_{i1} = A_i (1 + K_a E_{ai}) \sin(\omega_0 t + \phi_i + \theta_i) \]

\[ B_{i2} = A_i (1 - K_a E_{ai}) \sin(\omega_0 t - \phi_i + \theta_i) \]

where $\phi_i$ is the phase deviation caused by the difference in range due to the separation of the two antennas. Following the same procedure as before, eqs. (2-33) and (2-34) become

\[ \overline{S} = \sum_{i=1}^{n} A_i e^{j\theta_i} \left[ e^{j\phi_i + e^{-j\phi_i}} + K_a E_{ai} (e^{+j\phi_i} - e^{-j\phi_i}) \right] \]

and

\[ \overline{D} = \sum_{i=1}^{n} A_i e^{j\theta_i} \left[ e^{j\phi_i - e^{-j\phi_i}} + K_a E_{ai} (e^{j\phi_i} + e^{-j\phi_i}) \right]. \]

The subscripts $a$ and $p$ now refer to amplitude and phase. Noting that $\phi_i$ is the same quantity that was defined in the phase-monopulse
case and that the approximations
\[ \cos \phi_i = 1 \quad \text{and} \quad \sin \phi_i = \phi_i \]

apply here as well, we have
\[ \overline{S} \approx 2 \sum_{i=1}^{n} A_i e^{j\theta_i} \]

and
\[ \overline{D} \approx 2 \sum_{i=1}^{n} A_i e^{j\theta_i} \left[ j \phi_i + K_a E_{ai} \right] \]

\[ \approx 2 \sum_{i=1}^{n} A_i e^{j\theta_i} \left[ j \left( \frac{K_p}{R} (x_i - x) \right) + \frac{K_a}{R} (x_i - x) \right] . \]

Recalling that
\[ (2-8) \quad A e^{j\theta} = \sum_{i=1}^{n} A_i e^{j\theta_i} \quad \text{and} \quad \]
\[ (2-9) \quad \overline{x A e^{j\theta}} = \sum_{i=1}^{n} x_i A_i e^{j\theta_i} , \]

we obtain
\[ \overline{S} = 2A e^{j\theta} \]

and
\[ \overline{D} = \frac{2A e^{j\theta}}{R} \left[ jK_p (x - x) + K_a (x - x) \right] . \]
Now
\[ S = A \sin(\omega_0 t + \theta) \]
and
\[ D = \frac{2A}{R} \left[ \sin(\omega_0 t + \theta) \left( K_a \delta_e(\overline{x}) - K_p \delta_m(\overline{x}) \right) \right] \]
\[ + \cos(\omega_0 t + \theta) \left( K_p \delta_e(\overline{x}) - K_a \delta_m(\overline{x}) \right) \].

If a phase-sensitive detector that selects the component of difference signal in phase with the sum signal is used, the error signal is
\[ \varepsilon_p \propto \frac{4A^2}{R} \left[ K_a \delta_e(\overline{x}) - K_p \delta_m(\overline{x}) \right] \]
and the balance point is
\[ x_b = \delta_e(\overline{x}) - \frac{K_p}{K_a} \delta_m(\overline{x}). \]

If, on the other hand, the detector selects the component of the difference term in phase quadrature with the sum signal, the error signal is
\[ \varepsilon_q \propto \frac{4A^2}{R} \left[ K_p \delta_e(\overline{x}) - \delta_m(\overline{x}) \right] \]
and the balance point is
\[ x_b = \delta_e(\overline{x}) + \frac{K_a}{K_p} \delta_m(\overline{x}). \]
Phase Amplitude-Monopulse Tracking System

This monopulse uses only two receiving antennas combining the features of the amplitude-monopulse system and phase-monopulse system to obtain the required directional information.

In the analysis which follows, the antenna patterns are taken as divergent in the y direction, and the antennas are displaced in the x direction. Then the angular displacement of a point from the equivalent radar axis becomes

$$E_{pi} \propto \frac{x_i - x}{R}$$

and

$$E_{ai} \propto \frac{y_i - y}{R}$$.

Again subscripts a and p refer to amplitude and phase.

For a point target at long range and near the equivalent tracking axis, the signal received on antenna number 1 is

$$B_{i1} = A_i (1 + K_a E_{ai}) \sin(\omega_0 t + \phi_i + \theta_i)$$

and that received on antenna number 2 is

$$B_{i2} = A_i (1 - K_a E_{ai}) \sin(\omega_0 t - \phi_i + \theta_i)$$,

where terms have the same meaning as for the amplitude- and phase-monopulse systems.
To combine signals from several target points we again define phasors such that

\[ B_{i1} = \delta_m \left[ \bar{B}_{i1} e^{j\omega_0 t} \right] \]

and

\[ B_{i2} = \delta_m \left[ \bar{B}_{i2} e^{j\omega_0 t} \right]. \]

The total signal on each antenna can then be written as

\[ \bar{B}_1 = \sum_{i=1}^{n} \bar{B}_{i1} = \sum_{i=1}^{n} A_i (1 + K_a E_{ai}) e^{j(\phi_i + \theta_i)} \]

\[ \bar{B}_2 = \sum_{i=1}^{n} \bar{B}_{i2} = \sum_{i=1}^{n} A_i (1 - K_a E_{ai}) e^{j(-\phi_i + \theta_i)}. \]

The receiving apparatus now forms the sum and difference of these two signals. The sum is

\[ \bar{S} = \bar{B}_1 + \bar{B}_2 = \sum_{i=1}^{n} A_i \left\{ (1 + K_a E_{ai}) e^{j(\phi_i + \theta_i)} + (1 - K_a E_{ai}) e^{j(-\phi_i + \theta_i)} \right\} \]

(2-35)

\[ = 2 \sum_{i=1}^{n} A_i e^{j\theta_i} (\cos \phi_i + j K_a E_{ai} \sin \phi_i). \]

The difference is
\[
\bar{D} = (\bar{E}_1 - \bar{E}_2) = \sum_{i=1}^{n} A_i \left[(1 + K_a E_{ai}) e^{i(\phi_i + \theta_i)} - (1 + K_a E_{ai}) e^{i(-\phi_i + \theta_i)}\right]
\]

(2-36)

\[
= 2 \sum_{i=1}^{n} A_i e^{j\theta_i} (K_a E_{ai} \cos \phi_i + j \sin \phi_i).
\]

Recalling from the phase-monopulse system that

\[
\cos \phi_i \approx 1
\]

and

\[
\sin \phi_i \approx \frac{K_p}{R} E_{pi} \approx \frac{K_p}{R} (x_i - x),
\]

eqs. (2-35) and (2-36) can now be written as

\[
\bar{S} \approx 2 \sum_{i=1}^{n} A_i e^{j\theta_i} (1 + j K_a E_{ai} K_p E_{pi}) \approx 2 \sum_{i=1}^{n} A_i e^{j\theta_i}
\]

\[
\bar{D} \approx 2 \sum_{i=1}^{n} A_i e^{j\theta_i} (K_a E_{ai} + j K_p E_{pi})
\]

\[
\approx \frac{2}{R} \sum_{i=1}^{n} A_i e^{j\theta_i} \left[K_a (y_i - y) + j K_p (x_i - x)\right].
\]

Recalling that

\[
A e^{j\theta} = \sum A_i e^{j\theta_i},
\]

(2-8)

\[
A x e^{j\theta} = \sum x_i A_i e^{j\theta_i}
\]

(2-9)
and from symmetry that

\[ A\gamma e^{j\theta} = \sum y_i A_i e^{j\theta_i}, \]

then

\[ S = 2A e^{j\theta} \]

\[ D = \frac{2}{R} A e^{j\theta} \left[ K_a (\overline{y} - y) + j K_p (\overline{x} - x) \right]. \]

Now

\[ S = \delta_m (S e^{j\omega t}) = A \sin(\omega t + \theta) \]

and

\[ D = \delta_m (D e^{j\omega t}) \]

\[ = \frac{2A}{R} \left[ \cos(\omega t + \theta) \left[ K_a \delta_m (\overline{y}) + K_p (\Re(\overline{x}) - \overline{x}) \right] + \sin(\omega t + \theta) \left[ K_a (\Re(\overline{y}) - y) - K_p \delta_m (\overline{x}) \right] \right]. \]

If, by analogy with the amplitude-monopulse system and phase-monopulse system, the component of the difference signal in phase with the sum signal is taken as the error voltage in one plane (amplitude) and the component in phase quadrature is taken as the error voltage in the other plane (phase), the balance point is

\[ y_b = \Re(\overline{y}) - \frac{K_p}{K_a} \delta_m (\overline{x}) \]

\[ x_b = \Re(\overline{x}) + \frac{K_a}{K_p} \delta_m (\overline{y}). \]
The error voltage for the in-phase component after detection (similar to that discussed in the amplitude-monopulse system) is

\[ \epsilon \approx \frac{A^2}{R} \left[ K_a \left( \mathcal{R}_{\alpha} (\bar{y}) - y \right) - K_p \delta_m (\bar{x}) \right] \]

and that of the phase-quadrature component after detection (similar to that discussed in the phase-monopulse system) is

\[ \epsilon \approx \frac{A^2}{R} \left[ K_a \delta_m \bar{y} + K_p \mathcal{R}_{\alpha} (\bar{x}) - x \right] . \]

Thus a cross-talk term appears, i.e., the distribution of scatterers in the y direction produces an error signal in the x direction.

It has been stated earlier that this error is a fundamental one in these monopulse systems and is due to the weighting factor \( x_1 - x \) that appears in the difference signal.

**Extension of Balance-Point Equation for Monopulse Systems to Include Finite Targets**

The balance-point equation of any of the monopulse systems may be extended to include finite targets simply by taking the limit in the manner described earlier to obtain

\[ \lim_{n \to \infty} \left( \sum x_i A_i e^{j\theta_i} = \int x A(x) e^{j\theta(x)} \, dx \right) ; \]

it has been shown previously (eq. (2-20)) that

\[ A(x) e^{j\theta(x)} = \int A(x, y) e^{j\theta(x, y)} \, dy . \]
Thus

\[ \overline{x} = \frac{\int x A(x) e^{j\theta(x)} \, dx}{\int A(x) e^{j\theta(x)} \, dx} \]

and

\[ \overline{y} = \frac{\int y A(y) e^{j\theta(y)} \, dy}{\int A(y) e^{j\theta(y)} \, dy} \]

Substitution of these last two equations in the balance-point equations of any system will yield the balance-point equation of that system.

Alternate Form of the Balance-Point Equation

The balance-point equation may also be written in terms of the pattern of the received radar signal and its first derivative. Since static echo area data are usually obtained in the form of a pattern, this alternate form should be a convenient representation. In addition to the usually recorded amplitude pattern \( A \) of eq. (2-8), the phase pattern \( \theta \) of eq. (2-8) is also required.

Now from Fig. 5 \( \theta_i = \frac{4\pi}{\lambda} \overline{P_i - P_i'} \), which may be written as

\[ \theta_i = \frac{4\pi}{\lambda} z_i = \frac{4\pi}{\lambda} r_i \sin \gamma_i \]

where \( r_i \) and \( \gamma_i \) are shown in Fig. 7. Figure 7 is simply a view of the reference geometry of Fig. 5 projected to the \( x, z \) plane. Thus
\( r_i \) is the distance of the \( i^{th} \) scatterer from the origin of coordinates projected to the \( x, z \) plane, and \( \gamma_i \) is the angle the projected line from the \( i^{th} \) scatterer to the origin makes with the \( x \) axis.

**Fig. 7.** Geometry of reference plane (top view).

The signal as observed by a nontracking radar system is

\[
S = A e^{j\theta} = \sum_{i=1}^{n} A_i e^{j \frac{4\pi}{\lambda} r_i \sin \gamma_i}
\]

If it is assumed that

\[
\frac{\partial A_i}{\partial \gamma} = 0 \quad \text{and}
\]
\[ \frac{\partial r_i}{\partial \gamma} = 0, \]

and the target is rotated about the y axis, then

\[
\frac{dS_s}{d\gamma} = \sum_{i=1}^{n} j \frac{4\pi}{\lambda} r_i A_i \cos \gamma_i e^{j \frac{4\pi}{\lambda} r_i \sin \gamma_i}
\]

\[
= j \frac{4\pi}{\lambda} \sum_{i=1}^{n} x_i A_i e^{j \theta_i},
\]

or

\[
\sum_{i=1}^{n} x_i A_i e^{j \theta_i} = \frac{dS_s}{d\gamma j \frac{4\pi}{\lambda}}.
\]

Thus eq. (2-6) may be rewritten as

\[
\bar{x} = \frac{\frac{dS_s}{d\gamma}}{j \frac{4\pi}{\lambda} S_s} = \frac{d}{d\gamma} \frac{A e^{j \theta}}{j \frac{4\pi}{\lambda}}.
\]

Similarly if \( \eta_i \) is defined as the angle the projected line in the y, z plane from the \( i \)th scatterer to the origin makes with the y axis, then eq. (2-7) may be written as

\[
\bar{y} = \frac{\frac{d}{d\eta}}{j \frac{4\pi}{\lambda} A e^{j \theta}}.
\]

Thus the balance-point equations for all systems considered can be written in terms of the echo pattern, generalized to include phase.
and the first derivative of this echo area pattern with respect to the target rotation. The following chart compares these new balance-point equations with those given previously for the systems considered.

**CHART I**

<table>
<thead>
<tr>
<th>System</th>
<th>Scatterers</th>
<th>Pattern</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conical Scan and Phase Monopulse</td>
<td>$x_b = \Re (\bar{x})$</td>
<td>$x_b = \Re \bar{x}$</td>
</tr>
<tr>
<td></td>
<td>$= \Re \sum x_i e^{j\theta_i}$</td>
<td>$= \Re \left( \frac{d}{dy} A e^{j\theta} \right)$</td>
</tr>
<tr>
<td></td>
<td>$\sum A_i e^{j\theta_i}$</td>
<td>$= \Re \left( \frac{d}{j \frac{4\pi}{\lambda}} A e^{j\theta} \right)$</td>
</tr>
<tr>
<td>Amplitude Monopulse</td>
<td>$x_b = \Re \bar{x} - \frac{K_p}{K_a} \delta_m \bar{x}$</td>
<td>$x_b = \Re (\bar{x}) - \frac{K_p}{K_a} \delta_m (\bar{x})$</td>
</tr>
<tr>
<td></td>
<td>$= \Re \sum x_i A_i e^{j\theta_i}$</td>
<td>$= \Re \left( \frac{d}{dy} A e^{j\theta} \right)$</td>
</tr>
<tr>
<td></td>
<td>$\sum A_i e^{j\theta_i}$</td>
<td>$= \Re \left( \frac{d}{j \frac{4\pi}{\lambda}} A e^{j\theta} \right)$</td>
</tr>
<tr>
<td></td>
<td>$- \frac{K_p}{K_a} \delta_m \sum y_i A_i e^{j\theta_i}$</td>
<td>$- \frac{K_p}{K_a} \delta_m \left( \frac{d}{j \frac{4\pi}{\lambda}} A e^{j\theta} \right)$</td>
</tr>
<tr>
<td>Phase-Amplitude Monopulse</td>
<td>$x_b = \Re \bar{x} + \frac{K_p}{K_a} \delta_m (\bar{y})$</td>
<td>$x_b = \Re (\bar{x}) + \frac{K_p}{K_a} \delta_m (\bar{y})$</td>
</tr>
<tr>
<td></td>
<td>$= \Re \sum x_i A_i e^{j\theta_i}$</td>
<td>$= \Re \left( \frac{d}{dy} A e^{j\theta} \right)$</td>
</tr>
<tr>
<td></td>
<td>$\sum A_i e^{j\theta_i}$</td>
<td>$= \Re \left( \frac{d}{j \frac{4\pi}{\lambda}} A e^{j\theta} \right)$</td>
</tr>
<tr>
<td></td>
<td>$+ \frac{K_p}{K_a} \delta_m \sum y_i A_i e^{j\theta_i}$</td>
<td>$+ \frac{K_p}{K_a} \delta_m \left( \frac{d}{j \frac{4\pi}{\lambda}} A e^{j\theta} \right)$</td>
</tr>
</tbody>
</table>
Summary

The results of this chapter are summarized in Chart II so that the various systems may be readily compared. Except for the phase amplitude-monopulse system, only one channel is considered, since the results from the other channel are the same. The symbols are those used in the text. All of the conclusions that have been made in text of the derivations of the equations of the various systems are reviewed here making use of this chart.

It can be immediately noted that the balance-point equations are identical for the conical-scan, phase-monopulse system and the "true" amplitude-monopulse systems. An additional term involving $\delta_m (\bar{X})$ is introduced in the case of the amplitude-monopulse system. The magnitude of this term is of course dependent on the spacing of the antennas and can probably be made relatively small in many cases. However, in the case of the phase-amplitude-monopulse system a cross-talk term is introduced. It has been shown that this term can lead to serious errors.

From the chart, it may be noted that the error-voltage equations of the conical-scan system differ for the case of the linear and square-law detector only in that the power of the radar signal is changed. The error-voltage equations of the phase-monopulse and the true amplitude-monopulse systems are identical to that of the conical-scan system.
using a square-law video detector. As in the case of the balance-point equation, an additional term is introduced for the case of the amplitude-monopulse system with separated antennas. Its magnitude is dependent on the antenna separation. Also as in the case of the balance-point equations, a cross-talk term is introduced for the case of the phase-amplitude monopulse system.

Thus it can be concluded from this quasi-static analysis of radar systems tracking complex targets that:

a) the conical-scan system, the four-lobes phase-monopulse system and the "true" amplitude system are equally accurate tracking systems.

b) the separation of the antennas for the actual four-lobes amplitude-monopulse system should be minimized as much as possible to approximate the "true" amplitude-monopulse system.

c) the phase-amplitude-monopulse system should not be used for tracking complex targets.
<table>
<thead>
<tr>
<th>System</th>
<th>Source and use of Reference Voltage for Phase-Sensitive Detector</th>
<th>Balance Point</th>
<th>RF Signal</th>
<th>Error Voltage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conical Scan Linearity Video Detector</td>
<td>From a Generator Mechanically Coupled to Antenna Which Generates c sin ωat</td>
<td>Re( PHPUnit)</td>
<td>A cos(ω0t+θ) + \left(1 + \frac{K}{R} \right) (Re( PHPUnit) - x)sin(ωat) + \frac{K}{R} (Re( PHPUnit) - y)cos(ωat)</td>
<td>\frac{KA}{R} \left[Re( PHPUnit) - x\right]</td>
</tr>
<tr>
<td>Square-Law Video Detector</td>
<td>Sum Signal to Select Quadrature Component of Difference Signal</td>
<td>Re( PHPUnit)</td>
<td>S = 2A sin(ω0t + θ) + \left(\frac{2K_A}{R}\right) (Re( PHPUnit) - x)cos(ω0t+θ) - Im( PHPUnit)sin(ω0t+θ)</td>
<td>\frac{KA^2}{R} \left[Re( PHPUnit) - x\right]</td>
</tr>
<tr>
<td>Four-Lobe Phase-Monopulse</td>
<td>Sum Signal to Select &quot;in-phase&quot; Component of Difference Signal</td>
<td>Re( PHPUnit)</td>
<td>S = 2A sin(ω0t + θ) + \left(\frac{2K_A}{R}\right) (Re( PHPUnit) - x)sin(ω0t+θ) + Im( PHPUnit)cos(ω0t+θ)</td>
<td>\frac{KA^2}{R} \left[Re( PHPUnit) - x\right]</td>
</tr>
<tr>
<td>Four-Lobe Amplitude-Monopulse with Separated Antennas</td>
<td>Sum Signal to Select in-phase Component of Difference Signal</td>
<td>Re( PHPUnit) - \frac{KA}{K_p} Im( PHPUnit)</td>
<td>S = 2A sin(ω0t + θ) + \left(\frac{2A}{R}\right) \left[sin(ω0t+θ) \left(K_a \right \right)(Re( PHPUnit) - x) - K_p Im( PHPUnit) + \cos(ω0t+θ) \left(K_a \right \right)(Re( PHPUnit) - x) + K_p Im( PHPUnit)]</td>
<td>\frac{A^2}{R} \left[KA \right \right](Re( PHPUnit) - x) - K_p Im( PHPUnit)</td>
</tr>
<tr>
<td>Two-Lobe Phase Amplitude-Monopulse</td>
<td>Sum Signal to Select Quadrature Component of Difference Signal</td>
<td>Re( PHPUnit) + \frac{KA}{K_p} Im( PHPUnit)</td>
<td>S = 2A sin(ω0t + θ) + \left(\frac{2A}{R}\right) \left[sin(ω0t+θ) \left(K_a \right \right)(Im(ーター) + K_p (Re( PHPUnit) - x)) + \cos(ω0t+θ) \left(K_a \right \right)(Im(ーター) + K_p (Re( PHPUnit) - x)) + K_p Im(_Utils)]</td>
<td>\frac{A^2}{R} \left[KA Im(ーター) + K_p (Re(_Utils) - x)\right]</td>
</tr>
<tr>
<td>Two-Lobe Phase Amplitude-Monopulse</td>
<td>Sum Signal to Select in-phase Component of Difference Signal</td>
<td>Re(_Utils) - \frac{KA}{K_p} Im(_Utils)</td>
<td>S = 2A sin(ω0t + θ) + \left(\frac{2A}{R}\right) \left[sin(ω0t+θ) \left(K_a \right \right)(Re(_Utils) - y) - K_p Im(_Utils)]</td>
<td>\frac{A^2}{R} \left[KA (Re(_Utils) - y) - K_p Im(_Utils)\right]</td>
</tr>
</tbody>
</table>
CHAPTER III
POINTING ERRORS IN TRACKING RADARS
FOR COMPLEX TARGETS

The balance-point for various tracking radar systems has been derived in Chapter II. These equations are given here for convenience of the reader:

a) conical-scan and phase-monopulse system

\[ x_b = \Re (\overline{x}) - \sum \frac{x_i A_i e^{j\theta_i}}{\sum A_i e^{j\theta_i}} \]

b) amplitude-monopulse system

\[ x_b = \Re (\overline{x}) - \frac{K_p}{K_a} \Im (\overline{x}) \]

c) phase amplitude-monopulse system

\[ x_b = \Re (\overline{x}) + \frac{K_a}{K_p} \Im (\overline{x}) \]

\[ y_b = \Re (\overline{y}) - \frac{K_p}{K_a} \Im (\overline{x}) \]

If the signals from all target points are in phase, the balance-point equation for all systems reduces to

47
\[ x_b = \frac{\sum_{i=1}^{n} x_i A_i}{\sum_{i=1}^{n} A_i} \]  
\[ y_b = \frac{\sum_{i=1}^{n} y_i A_i}{\sum_{i=1}^{n} A_i} \]

The balance point given by eqs. (3-4) and (3-5) will be regarded as the desired balance point or radar centroid, and designated as \((x_d, y_d)\) throughout the remainder of this paper. Thus

\[ x_d = \frac{\sum_{i=1}^{n} x_i A_i}{\sum_{i=1}^{n} A_i} \]
\[ y_d = \frac{\sum_{i=1}^{n} y_i A_i}{\sum_{i=1}^{n} A_i} \]

The pointing error is defined as having the following \(x\) and \(y\) components:
(3.8) \[ x_e = x_b - x_d \]

(3.9) \[ y_e = y_b - y_d \]

In general, \( x_e \) and \( y_e \) include terms involving \( \text{Re} \overline{x} \), \( \delta m \overline{x} \), \( \text{Re} \overline{y} \) and \( \delta m \overline{y} \). In the following sections bounds will be established for these quantities in terms of the relative signal strength.

**Bounds for Pointing Error Contained in \( \text{Re} (\overline{x}) \)**

Substituting the complex ratio

\[
E e^{j\psi} = \frac{\sum_{i=1}^{n} A_i e^{j\theta_i}}{\sum_{i=1}^{n} A_i}
\]

(3.10) into eq. (2-6) gives

\[
\text{Re} (\overline{x}) = \text{Re} \left[ \frac{\sum_{i=1}^{n} x_i A_i e^{j(\theta_i - \psi)}}{E \sum_{i=1}^{n} A_i} \right],
\]

or

\[
\text{Re} (\overline{x}) = \frac{\sum_{i=1}^{n} x_i A_i \cos(\theta_i - \psi)}{E \sum_{i=1}^{n} A_i}.
\]
If the origin is chosen so that the smallest \( x_i \) is zero then all other \( x_i \)'s are positive (or zero), and

\[
x_{b1} < \frac{\sum_{i=1}^{n} x_i A_i}{E \sum_{i=1}^{n} A_i}
\]

or by eq. (3-6)

\[
(3-11) \quad x_{b1} < \frac{x_{d1}}{E}
\]

where the subscript 1 indicates this choice of origin.

Similarly if the origin is chosen so that the smallest \( |x_i| \) is zero and all other \( x_i \)'s are negative, \( x_d \) will be negative (or zero) and

\[
(3-12) \quad x_{b2} > \frac{x_{d2}}{E}
\]

where the subscript 2 indicates this choice of origin.

Now let the origin be chosen as any point within the target and let the subscript 0 indicate this choice. The following relations will be clear from Fig. 8.

\[
(3-13) \quad x_{d0} = x_{d1} - a = b - |x_{d2}| = b + x_{d2}
\]

\[
(3-14) \quad x_{b0} = x_{b1} - a = b + x_{b2}
\]
where \( a \) is the distance of the origin from the left side of the target and \( b \) is the distance from the right side. Substituting the values of \( x_{d1} \) and \( x_{b1} \) from eqs. (3-13) and (3-14) into inequality (3-11) gives

\[
a + x_{bo} \leq \frac{a + x_{do}}{E}.
\]

or

\[
(3-15) \quad x_{bo} \leq -a + \frac{a + x_{do}}{E}.
\]

Also substituting the values of \( x_{d2} \) and \( x_{b2} \) from eqs. (3-13) and (3-14) into inequality (3-12) gives

\[
x_{bo} - b \geq \frac{x_{do} - b}{E}.
\]
or

\[(3-16) \quad x_{bo} \geq b + \frac{x_{do} - b}{E}.\]

Inequalities (3-15) and (3-16) may be written as

\[b + \frac{x_{do} - b}{E} \leq x_{bo} \leq -a + \frac{a + x_{do}}{E}\]

or

\[\frac{x_{do} - (1 - E)b}{E} \leq x_{bo} \leq \frac{x_{do} + (1 - E)a}{E}.\]

Subtracting \(x_{do}\) from each term gives

\[(3-17) \quad \frac{(1 - E)(x_{do} - b)}{E} \leq x_{bo} - x_{do} \leq \frac{(1 - E)(x_{do} + a)}{E}.\]

Since the pointing error defined in eq. (3-8) is independent of the choice of origin, inequality (3-17) may be written as

\[(3-18) \quad \left(1 - \frac{1}{E}\right)(x_{do} - b) \leq x_{e} \leq \left(1 - \frac{1}{E}\right)(x_{do} + a),\]

and by eq. (3-13)

\[(3-19) \quad \left(1 - \frac{1}{E}\right)x_{d2} \leq x_{e} \leq \left(1 - \frac{1}{E}\right)x_{d1}.\]

If the origin is chosen midway between the extremes of the target so that \(b = a = w/2\), where \(w\) is the total target width, inequality (3-19) becomes
\begin{equation}
\left( \frac{1}{E} - 1 \right) \left( x_{dc} - \frac{w}{2} \right) \leq x_e \leq \left( \frac{1}{E} - 1 \right) \left( x_{dc} + \frac{w}{2} \right)
\end{equation}

where $x_{dc}$ is the position of the radar centroid measured from the midpoint of the target. Inequality (3-20) may now be written as

$$\left( \frac{1}{E} - 1 \right) \frac{w}{2} \leq x_e - \left( \frac{1}{E} - 1 \right) x_{dc} \leq \left( \frac{1}{E} - 1 \right) \frac{w}{2}$$

or

\begin{equation}
| x_e - \left( \frac{1}{E} - 1 \right) x_{dc} | \leq \left( \frac{1}{E} - 1 \right) \frac{w}{2}.
\end{equation}

Universal bounds may be established for the pointing error by noting that the radar centroid must lie within the target so that

$$- \frac{w}{2} \leq x_{dc} \leq \frac{w}{2}.$$

Hence from inequality (3-20),

\begin{equation}
| x_e | \leq \left( \frac{1}{E} - 1 \right) w.
\end{equation}

Inequality (3-22) gives universal bounds for all targets without requiring knowledge of the position of the radar centroid. Inequality (3-21) gives closer bounds but requires knowledge of the position of the radar centroid. These bounds are shown in Figs. 9, 10, and 11.
Fig. 9. Bounds of $\bar{E}(\bar{x})$. 
Fig. 10. Bounds for pointing error of Example 1.
Fig. 11. Bounds for pointing error of Example 2.

Example 2
3 Target Points
X = -a  X = 0  X = a
A/0  A/8  1.5A/180°
X_d = \frac{2}{7} = \frac{\pi}{14}
Examples

The following two examples are used to illustrate the bounds given in inequalities (3-21) and (3-22).

Example 1 - Two target points

<table>
<thead>
<tr>
<th>Target point no.</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Magnitude</td>
<td>A</td>
<td>1.5A</td>
</tr>
<tr>
<td>Phase</td>
<td>0</td>
<td>θ</td>
</tr>
<tr>
<td>Location</td>
<td>(-a, 0)</td>
<td>(+a, 0)</td>
</tr>
</tbody>
</table>

As the phase angle is varied from 0° to 180° the pointing error involved in the $\Re (\overline{x})$ is calculated by eqs. (3-1), (3-4), and (3-8) and the ratio $E$ is calculated by eq. (3-10). The results are plotted in Fig. 10.

Example 2 - Three target points

<table>
<thead>
<tr>
<th>Target point no.</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Magnitude</td>
<td>A</td>
<td>A</td>
<td>1.5A</td>
</tr>
<tr>
<td>Phase</td>
<td>0</td>
<td>θ</td>
<td>180°</td>
</tr>
<tr>
<td>Location</td>
<td>(-a, 0)</td>
<td>(0, 0)</td>
<td>(a, 0)</td>
</tr>
</tbody>
</table>

As the phase angle is varied from 0 to 180°, the pointing error and the ratio $E$ are calculated as in Example 1. The results are plotted in Fig. 11.
Bounds for Pointing Error Contained in Term $\delta m (\bar{x})$

The balance-point equations of the amplitude-monopulse system and the phase amplitude-monopulse contain terms involving

\[
(3-23a) \quad \delta m (\bar{x}) = \delta m \frac{\sum x_i A_i e^{j\theta_i}}{\sum A_i e^{j\theta_i}}
\]

where $\bar{x}$ is defined by eq. (2-6). Since this term goes to zero for the prescribed condition of all points in phase, it is to be treated as an additional error. The limits that are to be derived for this expression are to be multiplied by an appropriate constant dependent on the monopulse system involved, and added to the limits obtained in the preceding section.

Substituting eq. (3-10) into eq. (3-23a) gives

\[
(3-23b) \quad \delta m (\bar{x}) = \delta m \frac{\sum x_i A_i e^{j\theta_i}}{\left(\sum A_i\right) Re^{j\psi}}
\]

\[
= \frac{\sum x_i A_i \sin(\theta_i - \psi)}{R \sum A_i}.
\]

To find bounds for this quantity it is helpful to define a first-quadrant angle such that
(3-24) \[ \cos \rho = E. \]

This is always possible since \( E \) is never greater than unity. Consider now the vector sum

\[ \sum_{i=1}^{n} A_i e^{i(\theta_i - \psi)}. \]

For any angular position of an individual vector, a value of \( (\theta_i - \psi) \) may always be chosen that lies between \(-\pi\) and \(+\pi\) so that \( \sin |\theta_i - \psi| \) is always positive and equal to \( |\sin(\theta_i - \psi)| \). With this choice, now consider the vector sum

\[ \sum_{i=1}^{n} A_i e^{j|\theta_i - \psi|}. \]

Since the length of a resultant vector can never exceed the sum of the lengths of the individual vectors,

(3-25) \[ \left( \sum_{i=1}^{n} A_i \right)^2 \geq \left( \sum_{i=1}^{n} A_i \cos |\theta_i - \psi| \right)^2 + \left( \sum_{i=1}^{n} A_i \sin |\theta_i - \psi| \right)^2. \]

But

\[ \sum_{i=1}^{n} A_i \cos |\theta_i - \psi| = \sum_{i=1}^{n} A_i \cos (\theta_i - \psi) = E \sum_{i=1}^{n} A_i. \]

Also since \(|\theta_i - \psi|\) does not exceed \(\pi\),

\[ \sin |\theta_i - \psi| = |\sin(\theta_i - \psi)|. \]
Since \( \cos^2 \rho + \sin^2 \rho = 1 \), inequality (3-25) may be written

\[
(\cos^2 \rho + \sin^2 \rho) \left( \sum_{i=1}^{n} A_i \right)^2 \geq \left( \sum_{i=1}^{n} A_i \right)^2 + \left( \sum_{i=1}^{n} A_i |\sin \{\theta_i - \psi\}| \right)^2,
\]

and since

(3-24) \( \cos \rho = E \),

\[
(\sin^2 \rho) \left( \sum_{i=1}^{n} A_i \right)^2 \geq \left( \sum_{i=1}^{n} A_i |\sin \{\theta_i - \psi\}| \right)^2
\]
or

\[
(\sin \rho) \sum_{i=1}^{n} A_i \geq \sum_{i=1}^{n} A_i |\sin \{\theta_i - \psi\}|.
\]

Now from eq. (3-23b)

\[
|\delta_m (\overline{x})| \leq \sum_{i=1}^{n} \left( |x_i| |A_i| |\sin \{\theta_i - \psi\}| \right)
\]

\[
|\delta_m (\overline{x})| \leq \frac{E \sum_{i=1}^{n} A_i}{x_{i \max} \sum_{i=1}^{n} A_i |\sin \{\theta_i - \psi\}|}
\]

60
Since \( \delta_m \bar{x} \) is independent of the choice of the origin of \( x \), let the origin be chosen at the midpoint of the target. Then

\[
|x_i|_{\text{max}} = \frac{w}{2}
\]

where \( w \) is the total target width. Then

\[
|\delta_m (\bar{x})| \leq \frac{w}{2} \left( \sin \rho \right) \sum_{i=1}^{n} A_i
\]

or

\[
|\delta_m \bar{x}| \leq \frac{w}{2} \tan \rho
\]

or

\[
(3-26) \quad |\delta_m \bar{x}| \leq \frac{w}{2} \sqrt{1 - \frac{E^2}{1}} = \frac{w}{2} \sqrt{\frac{1}{E^2} - 1}.
\]

The bounds indicated in inequality (3-26) are plotted in Fig. 12.

Example 3 - Two target points

<table>
<thead>
<tr>
<th>Target point no.</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Magnitude</td>
<td>A</td>
<td>1.5A</td>
</tr>
<tr>
<td>Phase</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Location</td>
<td>(-a, 0)</td>
<td>(a, 0)</td>
</tr>
</tbody>
</table>

This is the same target configuration as that of Example 1 for \( \Re_e (\bar{x}) \).

The component of error due to \( \delta_m (\bar{x}) \) is calculated by eq. (3-26) and is plotted versus the ratio \( E \) in Fig. 12.
Example 3:
\[
\frac{A/L}{Q} \text{ At } (0, -a)
\]
\[
1.5A/L \text{ At } (0, a)
\]

\(\theta = 180^\circ\)

Fig. 12. Bounds of \(\hat{\eta}(x)\).
Limits of Pointing Error for Systems

In the previous sections, limits have been derived for the pointing error involved in the $R_e(\bar{x})$ and the $\delta_m(\bar{x})$. The purpose of this section is to use these limits to obtain the limits of the pointing error for the various systems.

A. Conical-Scan and Phase-Monopulse Systems

Both the conical-scan system and the phase-monopulse systems have

\[
(3-1) \quad x_b = R_e(\bar{x}) = R_e \left[ \sum x_i A_i e^{j\theta_i} \right] \frac{\sum A_i e^{j\theta_i}}{\sum A_i} \]

as the equation for the balance point. Thus the limit for pointing error for these systems is

\[
|x_e| \leq \left( \frac{1}{E} - 1 \right) w
\]

and these limits are shown in the curve of Fig. 10.

B. Amplitude-Monopulse Systems

The amplitude-monopulse system has

\[
(3-2) \quad x_b = R_e(\bar{x}) + \frac{K_P}{K_a} \delta_m(\bar{x})
\]

as the equation for the balance point, where

\[
K_p = \pi d_\lambda
\]
is the separation of the antennas in wavelengths as shown in Fig. 3, $e_\theta(\theta, \phi)$ is antenna-pattern equation adjusted to unity at the crossover angle, $\theta_c$ is the crossover angle defined as the angle measured from the pattern measurement at which the antenna pattern and the tracking axis intersect, $\phi_c$ defines the plane in which the antenna beams diverge, and $\theta, \phi$ are the usual spherical coordinates.

The bounds of pointing error for this case become the sum of eqs. (3-22) and (3-26) or

$$
\begin{align*}
(3-27) \quad \left| \frac{x_e}{\omega} \right| & \leq \left| \frac{1}{E} - 1 \right| + \frac{K_p}{2K_a} \sqrt{\frac{1}{E^2} - 1}.
\end{align*}
$$

These limits are shown in Fig. 13 for $K_p/K_a = 0, 1, 2, 3$. The constant

$$
K_a = \left. \frac{\partial e(\theta, \phi_c)}{\partial \theta} \right|_{\theta = \theta_c}
$$

The denominator of the above equation may be determined either experimentally or theoretically. Since the antenna pattern is nearly
Fig. 13. Pointing error for amplitude-monopulse system.
linear in the region of the crossover angle, it can be obtained directly from a measured antenna pattern. If the antenna is one of the many antennas whose pattern can be computed, it can be obtained analytically.

It is apparent from the curves of Fig. 13 that for an amplitude-monopulse system, the quantity $K_p/K_a$ should be minimized. Thus the antennas should be placed as close together as is physically possible. Minimizing the spacing of the antennas ($d_\lambda$) is, however, limited by the size of the antenna aperture which in turn is limited in its minimum size by the beamwidth of the antenna.

As an example, let us consider the antenna to be the continuous broadside array of length $L$ whose field pattern is

\begin{equation}
(3-28) \quad e = K \frac{\sin(\beta L \sin \theta)}{\beta L \sin \theta}
\end{equation}

where $\theta$ is the usual polar axis,
\[ \beta = \frac{2\pi}{\lambda} \] is the propagation factor.

At the crossover angle $\theta_c$

\[ e = 1 = K \frac{\sin(\beta L \sin \theta_c)}{\beta L \sin \theta_c} \]

or

\begin{equation}
(3-29) \quad K = \frac{\beta L \sin \theta_c}{\sin(\beta L \sin \theta_c)}
\end{equation}

Note that $K$ may be readily evaluated in this case by means of $\sin x/x$ tables.\(^9\)

Differentiating eq. (3-28) gives

$$\frac{de}{d\theta} = \frac{K \cos \theta}{\beta L \sin^2 \theta} \{\beta L \sin \theta \cos (\beta L \sin \theta) - \sin (\beta L \sin \theta)\}$$

and by means of eq.(3-29)

$$\left.\frac{de}{d\theta}\right|_{\theta=\theta_c} = \cotn \theta_c \{K \cos (\beta L \sin \theta_c) - 1\}.$$

Assuming that the crossover angle is located at the half-power point of the antenna pattern

$$\beta L \sin \theta_c = 1.39 \text{ radians} \quad \text{and} \quad K = 1.414.$$  

Then the minimum value of $K_p$ for a pair of such antennas is

$$K_p = \pi d_\lambda = \frac{0.695}{\sin \theta_c}$$

since the minimum value of $d_\lambda$ is the length of the array. Also eq. (3-30) becomes

$$K_a = \left.\frac{de}{d\theta}\right|_{\theta=\theta_c} = -0.745 \cotn \theta_c.$$ 

---

The ratio $K_p/K_a$ is

$$\frac{K_p}{K_a} = \frac{0.935}{\cos \theta_c}$$

where $2\theta_c$ is the half-power beamwidth of the antenna. Since $\cos \theta_c \approx 1$ for all reasonable antenna beamwidths, it is concluded that changing the dimensions of the aperture will not alter the ratio $K_p/K_a$ appreciably.

Let us now assume that the half-power beamwidth is fixed at an arbitrary value of $20^\circ$. Then:

$$\beta L \sin 10^\circ = 1.39$$

$$\beta L = 8$$

and

the minimum value of the constant $K_p$ is

$$K_p = \pi d_\lambda = 4.$$ 

The ratio

$$\frac{K_p}{K_a} = \frac{4}{\text{ctn} \theta_c \cdot \{K \cos (8 \sin \theta_c) - 1\}}$$

is plotted in Fig. 14. The antenna field strength pattern ($e(\theta)$) is also shown in Fig. 14. Observing that the radar range equation$^{10}$ states that the radar range varies as the square root of the antenna gain pattern or as the fourth root of the antenna field pattern, then this curve

---

Fig. 14. Curves for evaluating $K_p/K_a$ for a uniform source.
also gives the relation of the effective tracking range versus cross-over angle for this example. From these curves it may be observed that $K_p/K_a$ increases rapidly as the crossover angle decreases for $\theta_c \leq 8^\circ$. At this point the effective tracking range is 81% of the maximum. Note, however, that if $\theta_c = 0$, no error signal is generated, but this figure is taken as a maximum to establish a convenient reference. At this angle $(\theta_c = 8^\circ) \frac{K_p}{K_a} \approx 1.1$. Therefore, if this antenna is used in an amplitude-monopulse system, the crossover angle ($\theta_c$) should be equal to or greater than $8^\circ$, depending on the maximum tracking range of the radar. It is suggested that a similar investigation be performed with any antenna system that might be used.

Page\textsuperscript{11} suggests that a convenient method of generating the overlapping antenna patterns is to use a common aperture with two feed systems. This could be, for example, a parabola with two primary radiators displaced an equal amount from the focus, or perhaps a Luneberg lens with two separate feeds. The concept of using a common aperture is a much more effective method of reducing the magnitude of the constant $K_p/K_a$ in an amplitude-monopulse system than that described above.

C. Phase Amplitude-Monopulse System

The phase amplitude-monopulse system has

\[ x_b = \Re (x) + \frac{K_a}{K_p} \delta m (y) \]

and

\[ y_b = \Re (y) - \frac{K_p}{K_a} \delta m (x) \]

as the equations for the balance points where the symbols have all been identified previously.

The bounds for pointing error are for this case

\[ \frac{|x_e|}{W} \leq \left| \frac{1}{E} - 1 \right| + \frac{K_a}{2K_p} \sqrt{\frac{1}{E^2} - 1} \frac{h}{W} \]

and

\[ \frac{|y_e|}{h} \leq \left| \frac{1}{E} - 1 \right| + \frac{K_p}{2K_a} \sqrt{\frac{1}{E^2} - 1} \frac{W}{h} \]

where \( W \) is the extent of the target in the \( x \) direction and

\( h \) is the extent of the target in the \( y \) direction with the coordinate system defined as in Fig. 5. It would appear desirable to set the constants \( K_a \) and \( K_p \) equal to one so that the sensitivity of the system would be the same for either channel, as may be seen from eq. (3-3).
However, one can not add the two bounds directly even when these quantities are defined, because of the dependence of the limits on the target configuration. Consider, for example, the target to an aircraft at the broadside aspect with the coordinates arranged as in Fig. 15 and with \( W = 8h \). Then the bounds add to give the limits shown in Figs. 16 and 17 if \( K_a = K_p \). Note that the pointing error is much greater for the \( y \) channel than it is for the \( x \) channel. It has already been recommended that this system not be used. Figure 16 verifies this recommendation rather dramatically.

Fig. 15. Aircraft target at broadside incidence.
Fig. 16. Limits for elevation pointing error for phase amplitude-monopulse system tracking target of Fig. 15.

Fig. 17. Limits for azimuth pointing error for phase amplitude-monopulse system tracking target of Fig. 15.
Possible Methods of Reducing Pointing Error

It is apparent from Figs. 11 and 12 that the pointing error becomes less as $E$ approaches unity. To emphasize this concept, distributions of $\Re (\bar{x})$ in terms of the dimension $W$ of Fig. 8 have been plotted in Fig. 18 for minimum values of $E$ equal to 0.05, 0.1, 0.2, and 0.4. To obtain these distributions, it is assumed that all values lying between the universal curves are equally likely. These curves suggest that any method of maximizing the ratio $E$ will in turn reduce the pointing error. This suggests that the target must be arranged so that the radar can properly track it. Since this is impossible, it is suggested that the radar be altered so that the target appears to be changed.

Two possible ways of maximizing $E$ consists of:

a) Servo-controlling the radar frequency so that the echo area ($E^2$) is held within certain limits, and

b) Accepting and clamping the error signal to the servo system only when the echo area ($E^2$) is within required limits.

The frequency range required to accomplish (a) is dependent on the difference in distance of the actual scattering center from the reference plane (see Fig. 5). The inherent signal variation required to accomplish (b) is dependent on the separation of the scattering
Fig. 18. Distributions of pointing error for conical scan system.
centers in the reference plane (see Fig. 5) and the motion of the air-
craft that the radar is tracking.

There are several objections to be considered before any such systems can be accepted as practical. Some of these objections are discussed below. First of all, it is physically impossible to measure the absolute value of $E$ since there is no way of knowing when all of the scattering centers are in phase. However, a value proportional to $E$ is readily available in all systems at the following places:

a. at the output of the video detector and prior to any filtering of the scan frequency, since the modulation coefficient ($m$ of eq. (2-1)) is a very small number in the conical-scan system

b. at the output of the sum channel for any of the monopulse systems.

As a final requirement, because of the high speeds of present-day aircraft, any such system must not appreciably delay the response of the tracker servo. Also, no great increase in the weight of the radar is acceptable, for obvious reasons.
CHAPTER IV

INSTRUMENTATION FOR THE MEASUREMENT
OF BALANCE-POINT PATTERNS

Introduction

Equipment capable of measuring both the echoing area and the balance point simultaneously is described in this chapter. At The Ohio State University a "K-band" pulsed radar set has been used for some time for the measurement of the echo area of various targets.\(^{12, 13, 14}\) This radar is constructed so that it is readily adapted to simultaneous measurement of the balance-point by the addition of certain components. By nutating the feed of the receiving parabola to modulate the received voltage and by adding the proper detecting

\(^{12}\)Bacon, Jack, K-Band Radar Modifications, Report 475-12, 30 June 1954, Antenna Laboratory, The Ohio State University Research Foundation; prepared under Contract AF 18(600)-19, Air Research and Development Command, Wright Air Development Center, Wright-Patterson Air Force Base, Ohio.

\(^{13}\)Rhodes, D.R., An Investigation of Pulsed Radar Systems for Model Measurements, Report 475-6, 1 December 1953, Antenna Laboratory, The Ohio State University Research Foundation; prepared under Contract AF 18(600)-19, Air Research and Development Command, Wright Air Development Center, Wright-Patterson Air Force Base, Ohio.

\(^{14}\)Modifications of a Radar System for Automatic Recording of Back-Scattering Patterns, Report 406-1, 2 July 1951, Antenna Laboratory, The Ohio State University Research Foundation; prepared under Contract AF 33(038)-10101, Air Research and Development Command, Wright Air Development Center, Wright-Patterson Air Force Base, Ohio.
elements to demodulate this signal, the conical-scan error voltage is obtained. If this system is calibrated properly, it also gives the required balance-point.

It is to be pointed out that the conical-scan system is not the best system to use for measurement of the balance-point. As argued in the text of this chapter, the balance-point is of interest for relatively low signal levels, and consequently reflections from surrounding objects must be minimized. Usually in echo area measurements, the antenna beam is sufficiently narrow so that with the target in the center of the beam only the signal from the target is significant.

When the conical-scan system is used, the target is no longer kept in the center of the beam and it is more difficult to minimize the signal from the surrounding objects. The only tracking system that avoids this problem is the phase-monopulse system, and it is probably the system best suited for the purpose of the measurement of the balance-point.

Since the existing system was so simply modified, this was done in order to expedite the measurement of the balance-points. The modifications and the measuring technique are described in the following sections of this chapter.
The Radar Equipment

Figure 19 is a block diagram of the pulsed radar echo area measuring system showing modifications that have been made so that the echo area and the balance-point may be measured simultaneously. This system is built around a standard APS-32 pulsed radar set. A 3J31 magnetron with a pulse length of 1/4 microsecond is used as a transmitter. Separate antennas are used for transmission and reception. In order to generate an error voltage, the feed of the receiving parabola is nutated. The principal element of the echo-area recording system is the attenuator\(^{15}\) between the receiving antenna and the mixer. The attenuator, calibrated in db, is servo-controlled to keep the receiver operating at a constant level. A recording pen is mechanically linked to the attenuator shaft to give echo area data in db. A major advantage of this recording system is that receiver nonlinearities do not affect the data, which are as accurate as the calibration of the attenuator.

In the receiver, a 2K50 klystron local oscillator beats against the received echo signal in the crystal mixer, passing the beat frequency through a 60-mc IF amplifier with a 10-mc pass band. An afc system is incorporated in the receiver to hold the klystron on

\(^{15}\)Bacon, Jack, op. cit.
Fig. 19. Radar system block diagram.
frequency. After video detection, the signal passes thru a range gate synchronized with the transmitted pulses such that only the pulses reflected from the vicinity of the target are amplified. The output of the gate is detected to give a dc error voltage that is fed to the servo system used to drive the attenuator. At the output of this same detector, the conical-scan error voltage is available at the scan frequency. This output is fed to the phase-sensitive detector where it is combined with a signal from the reference generator. The voltage output from this detector, which is proportional to the balance point in a target coordinate system (see Fig. 6) selected by proper positioning of the reference generator, is recorded by a standard Speedo- max recorder. The paper drives of both the Speedo- max recorder chart and the echo-area data chart are connected to the target-rotating mechanism to give positive correlation for all data.

The final part of the radar system is the equipment used to support and rotate the target. One of the most successful means of support is by means of dacron line suspended from an "A" frame (see Fig. 20). To reduce the reflected signal from the rotating pedestal, an absorbing screen is placed in front of it. Reflections from the "A" frame are minimized by orienting it slightly off normal incidence.

Description of Modifications to Existing Radar

The mechanism used to nutate the feed of the receiving parabola is shown in Fig. 21. Since it was desirable to maintain constant
Fig. 20. Photograph showing method of target suspension.

Fig. 21. Nutating mechanism.
polarization, the waveguide feed section is passed through a hard rubber grommet inserted in the dish. Before entering the dish, the waveguide passes through a bearing which is offset from the center of a rotating wheel. Since the center of the wheel is placed on the axis of the parabola, rotating the wheel introduces the desired nutation. A dipole-type feed is used in this case instead of the shepherd hook feed used previously for the purposes of rigidity. However, since the waveguide is rectangular (commercial K-band guide) the end of the feed does not travel in a circular but rather in an elliptical path. The elliptical motion causes harmonics of the scan frequency to be generated, but these harmonics can be eliminated by proper filtering if necessary.

The phase-sensitive detector used (see Fig. 22) is a standard circuit\(^\text{16}\) whose dc output is proportional to the product of the magnitude of a reference voltage and the magnitude of the component of the error voltage in phase with the reference voltage. For the measurements to be discussed, the phase of the reference generator is adjusted to select the position of the balance-point in the horizontal plane.

Phase-Sensitive Detector

Fig. 22. Phase sensitive detector.
This reference signal is generated, as may be seen in Fig. 21, by an iron section placed on the rotating aluminum wheel which causes the feed section to nutate. A coil is so placed that the reluctance of its magnetic path is changed as the iron section passes it, thus generating a pulse for each revolution of the wheel. These pulses are passed through an amplifier tuned to select the sinusoidal component at the scan frequency, which is used as the reference signal. Phase adjustment of the error signal is made by moving the coil.

The output voltage of the phase-sensitive detector is fed into a cathode-follower stage designed to drive the Speedomax recorder.

The system described above is certainly rich in harmonics of the scan frequency, since the reference voltage is generated by means of pulses and the error signal is generated by a feed section which travels in an elliptical path. It was felt that the effect of these harmonics must be investigated.

To determine the harmonic content of the reference signal, an audio oscillator was substituted for the error voltage. As the frequency of the audio oscillator was varied, the maximum readings of the recorder were noted. To determine the harmonic content introduced in the error voltage, a point target was suspended off the tracking axis (to generate an error voltage) and the audio oscillator replaced the reference signal. Again as the frequency of the audio
Since the output of the phase-sensitive detector is the product of these magnitudes at each frequency (assuming them to be in phase), this product was formed. The magnitudes so determined are given in the following chart and indicate that the harmonics would not affect the operation of the system. Higher-order harmonics than those given were negligible.

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Magnitude of Harmonics in Reference Signal</th>
<th>Magnitude of Harmonics in Error Signal</th>
<th>Product of these Magnitudes</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>$2\frac{1}{2}$</td>
<td>22</td>
<td>55</td>
</tr>
<tr>
<td>45</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>60</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>90</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
</tr>
</tbody>
</table>

Adjustment and Calibration of the Error-Voltage Recorder

Adjustments of this system are relatively simple. First the "S" potentiometer (see Fig. 22) is adjusted so that the Speedomax pen is in the center of the chart with the error signal removed. Then an
audio oscillator whose frequency is approximately equal to the scan frequency is introduced in lieu of the error voltage. Under these conditions, the dc balance potentiometer (see Fig. 22) is adjusted so that the recorder pen swings an equal distance in either direction from the center of the chart as seen in Fig. 23. These adjustments are checked periodically.

Fig. 23. Recorded output with a 30-cps signal input to the discriminator.
In order to adjust the phase of the reference voltage, a point target is displaced from the tracking axis in the vertical direction with the system in operation. The position of the coil used to generate the reference signal is then adjusted so that the error voltage recorder reading is zero. Under these conditions, any recorder reading is due to the horizontal displacement of the target.

To establish a target coordinate system accurately, the bearing of the rotating pedestal and the string-supporting bearing in the "A" frame are positioned so that their common center line is vertical. A surveyor's level is used to insure accuracy of at least 1/8 inch. The position of the receiving parabola is adjusted so that the tracking axis intersects the center line connecting these two bearings at a convenient height. Again accuracy of at least an 1/8 inch is required. This point of intersection shown in Fig. 24 is the origin of coordinates of Fig. 6.

Calibration of the error-voltage recorder, so that the balance point can be obtained directly from the chart, is accomplished by suspending a point target a distance $L$ from the center line of the bearings and rotating the pedestal. Then the distance $x_b$ in the coordinate plane (normal to the tracking axis) of the center line from the target is

$$x_b = L \cos \xi$$
Fig. 24. Target coordinate system.

where ξ is the angular position of the pedestal taken as zero when the point target lies on the tracking axis. A typical error-voltage pattern is shown in Fig. 25. The calibration obtained from this record is shown in Fig. 26. It will be noted that the straight-line portion of this curve covers a region of about 6.5 inches from the tracking axis. This is the region for which the approximations made in Chapter 2 are valid, i.e., the target size should be no greater than 13 inches. The slope of this line is constant for long periods of time. However,
Fig. 25. Error voltage pattern of a point target.

it is checked periodically to be certain that no changes occur. Note that if the target is restricted to the region where the curve of Fig. 26 is linear, only that linear portion of the curve is significant. If the preceding statement seems absurd, the reader should recall that for a multipoint target, the balance point can appear to be completely outside the target.

The region over which the calibration is kept linear should be restricted to the physical size of the target for a reason to be explained later. This is accomplished by restricting the amount of offset of the feed in the receiving parabola since the linear portion of this curve is restricted by the increased curvature in the region of the beam maximum.
Fig. 26. Calibration curve.
A Measurement Problem

The major problem existing in the measurement of echo area, occurring for small echo area, is that of reducing the return from the supporting structure to the extent that the measurement is truly a function of the target and not the supporting structure. Fortunately the echo area is usually small for targets which have a small physical size, and consequently these targets can be supported by smaller diameter strings.

Furthermore, for most targets, one is chiefly interested in the values in the vicinity of the maximum echo area. However, in the case of measurement of the balance-point, one is very much interested in the nature of the signal return in the vicinity of the echo area nulls since this is the region for which the pointing error is greatest. Therefore, when measuring the balance-point, the problem of the signal reflected from the supporting structure becomes more severe than in the usual echo-area measurements. Thus the background signal must be reduced as much as is possible before accurate measurements can be obtained.

This is the reason for restricting the linear portion of the calibration to approximately the physical size of the target being measured. As the beam maximum of the antenna is moved farther from the tracking axis (to increase the linear portion of the calibration
curved), the scanning beam sweeps out a wider area. Thus it becomes more difficult to minimize reflections from surrounding objects. Consequently the beam maximum is maintained as close to the tracking axis as good calibration will permit.
CHAPTER V
COMPARISON OF THEORETICAL
AND EXPERIMENTAL RESULTS

Introduction

Theoretical and measured balance-point patterns obtained using a conical-scan system are compared in this chapter for various targets. Theoretical patterns are obtained using the method of Chapter 2 and experimental patterns are measured using the system described in Chapter 4.

Ogive

The first target to be discussed is the ogive of Fig. 27. The echo area of this configuration has been reported in detail.\(^\text{17}\) In the

\[\text{Fig. 27. Ogive.}\]

\(^{17}\) Peters, L., Jr., Memorandum on the Echo Area of Ogives, Report 601-7, 30 January 1956, Antenna Laboratory, The Ohio State University Research Foundation; prepared under Contract AF 33(616)-2546, Air Research and Development Command, Wright Air Development Center, Wright-Patterson Air Force Base, Ohio.
neighborhood of the broadside aspect, the theory of geometrical optics has been used successfully to predict its echo area. On the basis of this concept, it may be stated that all back-scattering appears to originate from those surfaces normal to the incident wave. Only one point on the surface of the ogive is normal to the incident wave in the vicinity of the broadside aspect. As may be seen from the geometry of Fig. 28, the x coordinate of that point may be written simply

![Diagram](image)

Fig. 28. Coordinates of the balance-point on the ogive surface.
\[ x_n = \left( R - \frac{d}{2} \right) \sin \phi \]
\[ = 37.2 \sin \phi \]

where

- \( R \) is the radius of curvature of the ogive,
- \( d \) is the maximum diameter of the ogive, and
- \( \phi \) is the angle of incidence measured from the broadside aspect as shown in Fig. 28.

Measured and computed patterns of the balance-point of the ogive are shown in Fig. 29b for the region of broadside incidence. The agreement between theory and experiment is as good as is expected since the use of the theory of geometrical optics does not precisely predict the echo area of this body, as may be seen in Fig. 29a.

However, the computed and measured values of echo area would usually be considered in quite good agreement. The agreement obtained above indicates that a tracking system would provide a good technique for locating an isolated scattering center on a body. The location of such a center is of interest since it would enhance the knowledge of the scattering mechanism from a given body.

**Two-Point Targets**

Next, two-point targets are considered. The first of the two-point targets is the one shown in Fig. 30. Reflections from the styrofoam slab are very low compared to those from the metallic
Fig. 29. Patterns of the ogive of Fig. 27.
Fig. 30. A two point target.

strips. Thus this technique permits targets to be rigidly connected without disturbing the target configuration in so far as the radar set is concerned. In the particular target of Fig. 30, the flat strips were used to increase the signal level. The strips were one wavelength in width and the echo area of such a strip in the broadside region is essentially a constant whose magnitude is that magnitude obtained if all elements of the strip are in phase. By the limits of Chapter 3 these strips are, for all practical purposes, point scatterers with the ratio \( E \geq 1 \) for each strip. Also, since it is assumed that all elements of a given strip are in phase, the signal reflected from a given strip is proportional to its length. Thus \( A_1 = 4 \) and \( A_2 = 1 \).
From the coordinate system shown in Fig. 30,

\[ \theta_1 = \frac{4\pi}{\lambda} \cdot 3.25 \sin \phi \]
\[ = 81.6 \sin \phi, \]

and

\[ \theta_2 = -\frac{4\pi}{\lambda} \cdot 3.25 \sin \phi = -81.6 \sin \phi \]

\[ x_1 = 3.25 \cos \phi \]
\[ x_2 = -3.25 \cos \phi \]

where \( \theta_1 \), \( \theta_2 \), \( x_1 \) and \( x_2 \) are defined in Chapter 2. Substituting these values in eq. (2-16) of Chapter 2 gives

\[ x_b = \Re (\overline{x}) = \Re \left( \frac{3.25 \cos \phi e^{j81.6\sin \phi} - 4(3.25)\cos \phi e^{-j81.6\sin \phi}}{e^{j81.6\sin \phi} + 4e^{-j81.6\sin \phi}} \right). \]

Figure 31a is the relative echo-area pattern of this target. Computed pattern minima are obtained by assuming that \( A_1 = 4 \), \( A_2 = 1 \) and taking the measured pattern maxima as the standard. Deviations of these points from the measured values indicate that the assumptions made are reasonable. Good agreement is obtained between the measured and computed balance-point patterns shown in Fig. 31b. Also given in Fig. 31b is the limiting value that the balance-point pattern could have attained in the region of broadside incidence based on the echo-area pattern and the theory of Chapter 3. The balance-point is
Fig. 31. Patterns of the two point target of Fig. 30.
always within this limit, thus giving some experimental verification of these limits.

The next target treated is that of Fig. 30 with the one-inch strip increased in length to three inches. If the same assumptions are made, the equation for the balance-point is

\[ x_b = R_e \cos \phi \left[ \frac{3e^{j81.6\sin\phi} - 4e^{-j81.6\sin\phi}}{3e^{j81.6\sin\phi} + 4e^{j81.6\sin\phi}} \right]. \]

The agreement between the measured and computed echo area patterns of Fig. 32a indicates the relative validity of the initial assumptions. Equally good agreement is obtained in the case of the measured and computed balance-point patterns of Fig. 32b with the exception of the balance-point when the two-point targets are nearly in phase opposition. Among the possibilities of error in this region are background reflections and coupling between the scatterers. It if therefore expected that agreement would be poor in this region.

**Corner Reflectors**

Another group of targets are those for which no point can be designated as the major contributor of the reflected signal. One such target is the triple-bounce corner reflector for which all of the "effective" aperture of the corner contributes equally to the reflected signal. It is well known that such a corner reflector can be considered as a flat plate normal to the incident wave. Cohen and
Fig. 32a. Relative Echo Area Pattern

Fig. 32b. Balance-Point Pattern

Fig. 32. Patterns of two point target for increased $A_2$. 

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Fisher\textsuperscript{18} have shown that this equivalent plate is centered at the apex of the corner reflector. A corner sphere (consisting of a ten-inch diameter sphere with one octant removed) based on their analysis has long been in use at this laboratory as a standard radar target. The corner sphere with the coordinate system used is shown in Fig. 33.

\begin{center}
\includegraphics[width=0.5\textwidth]{corner_sphere.png}
\end{center}

**Fig. 33.** Corner sphere.

Since the triple-bounce corner appears to be a plate at normal incidence with its center located at the apex of the corner, then the balance-point is the apex in the region where the triple-bounce type

of return dominates. At another aspect (see Fig. 34a) a flat plate forming a side of the cone is at normal incidence. Since the projected area of the sphere is \(25\pi\), the area of this plate is \(25\pi/4\). For a flat plate at broadside incidence the signals reflected from all elements of the plate are in phase. Thus eq. (2-26) becomes

\[
\begin{align*}
(5-1) \quad \chi_b &= \Re \left( \frac{\int_{\Omega} A(x) \, dx}{\int_{\Omega} A(x) \, dx} \right)
\end{align*}
\]

where

\[
A(x) = \int_{-y_1(x)}^{y_2(x)} A \, dy
\]

with \(y_1(x)\) and \(y_2(x)\) forming the boundaries of the target in the orthogonal direction.

The denominator of eq. (5-1) is

\[
\int_{\Omega} A \, dA = A \cdot (\text{area})
\]

\[= \frac{25\pi}{4} A.\]

The integration indicated in the numerator of eq. (5-1) is most easily carried out if polar coordinates as shown in Fig. 34a are used. Thus
Fig. 34. Other views of the corner sphere.
\[ \int_{0}^{5} \int_{y_{1}(x)}^{y_{2}(x)} A \, dy \, dx \]

\[ = \int_{0}^{\frac{\pi}{4}} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} A(-p \cos \phi) \, (\rho \, d\rho \, d\phi) \]

\[ = -\frac{2}{3} \times 125 \, A. \]

Then

\[ x_{b} = -\frac{\frac{\sqrt{2}}{3} \times 125 \, A}{\frac{25\pi}{4} \, A} = -3''. \]

This is the balance point of the corner sphere since the signals reflected from all other elements of the target are so low, compared to those of this flat plate, that they may be neglected.

At yet another aspect (see Fig. 34b) a double-bounce corner is the only scatterer that needs to be considered since the effect of all other scatterers is negligible. Since there are two plates forming the corner oriented at 45° to the incident wave, the projected area of this corner is \( \sqrt{2} \times \frac{25\pi}{4} \). Such a corner can be treated as a flat plate normal to the incident wave with an area equal to the projected area. Consequently the denominator of eq. (5-1) becomes \( A \sqrt{2} \times \frac{25\pi}{4} \), and the numerator becomes

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\[ \int_0^5 \int_{-\sqrt{25-x^2}/2}^{\sqrt{25-x^2}/2} A \, dy \, dx \]

= \[ A \int_0^5 \sqrt{2} \times \sqrt{25-x^2} \, dx. \]

This last integral may be evaluated by means of integral tables\(^1\) giving

\[ A \int_0^5 \sqrt{2} \times \sqrt{25-x^2} \, dx = -\sqrt{2} \frac{1}{3} \left( \sqrt{25-x^2} \right)^3 \bigg|_0^5 \]

\[ = \frac{1}{3} \sqrt{2} \left( 25 \right)^{3/2} A = \frac{2}{3} 125 A. \]

Therefore

\[ x_b = \Re \left( \frac{\sqrt{2}}{4} \frac{(125)A}{25\pi} \right) = 2.12'' \]

The theoretical and experimental values of the balance point

are given in Fig. 35. The echo area pattern is also shown. No attempt was made to correlate data for aspects other than those shown since the cause of these reflections are not known precisely. Again the agreement between experiment and theory is very good, thus showing experimentally that the theories are valid for targets that

\(\text{---} \)

Fig. 35. Patterns of the corner sphere.
are finite in extent. Since such targets may be considered as the result of a large number of point targets as the number of points approaches infinity, the theory may be considered valid for all targets.

Flat Plate

However, it is as yet not always possible to compute the balance-point position for all targets since not enough is known about the reflection phenomena. As an example of this, the final target to be considered is a square flat plate whose sides are 6\(\frac{5}{8}\) inches in length.

The reflected signal from such a target may be computed approximately by the well-known methods of physical optics. The assumptions involved are that the plate is constructed of a perfect conductor and that its dimensions are large in terms of wavelengths.

With the coordinate system of Fig. 36, the induced current \(K\) is then parallel to the \(x\) axis. The magnitude of the induced current is twice the component of \(n \times H^i\) parallel to the tangential electric field \(E^t\) or
\[ K = \frac{2(nz^i) \cdot E^t}{E^t} = -2H^i \cos \phi \]

where \( E^i, H^i \) are the incident electric and magnetic fields, 
\( n \) is the unit positive normal, and 
the coordinate system used is shown in Fig. 36.

The phase of the induced current varies as \( x \sin \phi \). The radiation
field \( E^s \) due to the induced current distribution is

\[ E^s = -j\omega \mu \frac{e^{-j\beta R}}{4\pi} \left( \frac{E^i}{\mu} \cos \phi \right) \int_{-a}^{a} \int_{-a}^{a} e^{j\beta x \sin \phi} \, dy \, dx \]

where \( R \) is the range and 
\( \beta \) is the propagation factor.

The usual assumption is made that variations due to range are neg-
lected in the magnitude term but are included in the phase term.

Substituting the value of \( K \) gives

\[ E^s = \frac{j\omega \mu}{4\pi} \frac{e^{-j\beta R}}{R} \left( 2 \frac{e^{i}}{\mu} E^i \cos \phi \right) \int_{-a}^{a} \int_{-a}^{a} e^{j\beta x \sin \phi} \, dy \, dx \]

\[ = j \frac{a e^{-j\beta R}}{\pi R} E^i \cot \phi \sin(\beta a \sin \phi). \]

The echo area \( (\sigma) \) of the plate may be obtained by substituting \( E^s \)
in the equation

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The measured echo area pattern is given in Fig. 37. Also given are computed maxima.

Quite obviously, either the measurements are in error or eq. (5-2) is in error, since it predicts that echo area is zero for \( \beta a \sin \phi = n\pi \). Equation (5-2) gives an approximate value for the maxima as may be seen from the computed points of Fig. 37. Deviations of as much as 2 db exist between theory and experiment. For the usual echo-area computation, this discrepancy is negligible since a change of 2 db would represent only a small change in minimum range. However, for the computation of the balance-point position, a better understanding of the reflection phenomena is needed than is available.

To illustrate this need, the balance-point pattern for this flat plate is to be computed using eq. (2-26), or

\[
(2-26) \quad x_b = \Re \left\{ \int_{-a}^{a} x A(x) e^{i\theta(x)} \, dx \right\}
\]

Equation (2-26) becomes
Fig. 37. Patterns of a square flat plate.
(5-3) \[ x_b = \Re \{ x \} = \Re \frac{\int_{-a}^{a} x e^{j\beta x \sin \phi} \, dx}{\int_{-a}^{a} e^{-j\beta x \sin \phi} \, dx} \]

since

\[ A(x) e^{j\theta(x)} = \int_{-a}^{a} A(x, y) e^{j\theta(x, y)} \, dy \]
\[ = \frac{2a j\omega \mu}{4\pi} \frac{e^{-j\beta R}}{R} \left( 2 \sqrt{\frac{e}{\mu}} e^{i \cos \phi} \right) e^{-j\beta x \sin \phi}. \]

The denominator becomes

\[ \int_{-a}^{a} e^{-j\beta x \sin \phi} \, dx = \frac{2 \sin(\beta a \sin \phi)}{\beta \sin \phi}. \]

On the other hand, the numerator becomes

\[ \int_{-a}^{a} x e^{-j\beta x \sin \phi} \, dx \]
\[ = \frac{e^{-j\beta x \sin \phi}}{(-j\beta \sin \phi)^2} \left( -j\beta x \sin \phi - 1 \right) \bigg|_{-a}^{a} \]
\[ = \frac{-j2}{\beta^2 \sin \phi} \left[ \sin(\beta a \sin \phi) - \beta a \sin \phi \cos(\beta a \sin \phi) \right]. \]

Since the denominator of eq. (5-3) is real and its numerator is imaginary, the result is

\[ x_b = \Re \{ \bar{x} \} \equiv 0. \]

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By comparing this with the experimental value given in Fig. 37, it may be seen that there is no agreement between theory and experiment for this case; thus it is concluded that the initial assumptions are not adequate for this computation.

Note that the balance-point pattern appears similar to the balance-point pattern of a pair of point scatterers; and that if the balance point is accepted only when the echo area is a maximum, the radar would track very near the center of the plate (see Fig. 37). In this case the location of the scatterers is at each edge of the plates with the edge more distant from the radar being the larger scatterer (see Fig. 38). The deviations of this pattern from that of a pair of scatterers may be due to the varying amplitude of signal reflected from the individual scatterers. Note also that the equation for the echo area of a flat plate is quite similar to that of a pair of point sources. Its most significant difference is a factor $\text{ctn}^2 \phi$.

![Diagram of radar and target](image)

Fig. 38. Illustration showing possible scatterers on the flat plate.
Therefore, it is hypothesized that the back-scattering from this flat plate is due to these edges, and that the strongest signal is from the edge most distant from the radar set. A more complete understanding of this phenomenon is required before additional computations can be made.
APPENDIX A

The purpose of this section is to verify the validity of the assumptions made in eqs. (2-1) and (2-2). An antenna is assumed that is dependent only on the polar angle measured from the center of the beam. Such an antenna pattern may be expressed as

$$f(\gamma) = \sum_{n=1}^{N} a_n P_n(\cos \gamma)$$

where $P_n(\cos \gamma)$ is the Legendre polynomial and the coefficients $a_n$ can be determined in the usual manner. The addition formula of Legendre polynomials may be used to express the instantaneous field pattern of a conical scan radar when the above antenna is used. Thus

$$f(\gamma) = \sum_{n=1}^{N} a_n P_n(\cos \gamma)$$

$$= \sum_{n=1}^{N} \left[ a_n P_n(\cos \theta) P_n(\cos \alpha) \right.$$

$$\left. + 2a_n \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos \alpha) P_n^m(\cos \theta) \cos m(\omega t - \beta) \right]$$

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where $P_n(\cos \alpha)$ is the associated Legendre function,

$\omega_s$ is the scan frequency of the radar, and

the other parameters are as shown in the coordinate system

of Fig. 39.

Fig. 39. Coordinate system for a conical scan radar.

A target located at the angular position $\alpha_i$, $\beta_i$ is illuminated

by a radar of frequency $\omega_0$. For K such targets, the voltage observed
at the radar receiver is

\[ (A-1) \quad e = \sum_{i=1}^{K} B \sigma_i^{1/2} \left[ f(\gamma_i) \right]^2 \cos(\omega_0 + \xi_i) \]

where \( \sigma_i \) is the echo area of the point,

\[ B \] is a constant, and

\[ \xi_i \] is the same phase angle defined earlier as \( \theta_i \).

Squaring \( f(\gamma_i) \) gives

\[ \left[ f(\gamma_i) \right]^2 = \left[ \sum_{n=1}^{N} a_n P_n(\cos \vartheta) P_n(\cos \alpha_i) \right]^2 \]

\[ + 4 \left[ \sum_{n=1}^{N} a_n P_n(\cos \vartheta) P_n(\cos \alpha_i) \right] \]

\[ \left[ \sum_{n=1}^{N} \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} a_n^m P_n(\cos \alpha_i) P_n(\cos \vartheta) \cos m(\omega_{st} - \beta_i) \right] \]

\[ + 4 \left[ \sum_{n=1}^{N} \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} a_n^m P_n(\cos \alpha_i) P_n(\cos \vartheta) \cos m(\omega_{st} - \beta_i) \right]^2 \]

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\[ T = \sum \left[ \begin{array}{c} T \cos \theta \\ \cos \theta \\ \cos \theta \end{array} \right] \epsilon_p(\theta) \frac{u_0^p}{u_0^p(\theta)} \epsilon_p \frac{u_0^p}{u_0^p(\theta)} \frac{i(u+1)}{i(u+1)} \frac{1}{N} \]
and

\[
(A-3) \quad m_i = \frac{4 \sum_{n=1}^{N} \frac{(n-1)!}{(n+1)!} a_n P_n^i (\cos \theta) P_n^1 (\cos \alpha_i)}{\sum_{n=1}^{N} a_n P_n (\cos \theta) P_n (\cos \alpha_i)}
\]

and

\[
q_{mn1} = \frac{(n-m)!}{(n+m)!} a_n^m P_n (\cos \alpha_i) P_n (\cos \theta),
\]

then

\[
(A-4) \quad e_i = B \sum_{i=1}^{K} A_i \left[ 1 + m_i \cos(\omega_s t - \beta_i) \cos(\omega_0 t + \xi) \right]
\]

\[
+ 4 \left[ \frac{1}{4} \sigma_i \frac{1}{2} A_i \sum_{n=2}^{N} \sum_{m=2}^{n} q_{mn1} \cos m(\omega_s t - \beta_i) \right]
\]

\[
+ \sigma_i \left[ \sum_{n=1}^{N} \sum_{m=1}^{n} q_{mn1} \cos m(\omega_s t - \beta_i) \right]^2 \cos(\omega_0 t + \xi)
\]

\[
e \approx B \sum_{i=1}^{K} A_i \left[ 1 + m_i \cos(\omega_s t - \beta_i) \cos(\omega_0 t + \xi) \right]
\]

for small \( \alpha_i \).

The approximations made in eq. (A-4) consist first of neglecting

terms of the nature of
\[
\frac{(n-m)!}{(n+m)!} P_n^m(\cos \varrho) P_n^m(\cos \alpha_i) \cos m(\omega_s t - \beta_i)
\]

when compared with terms of the nature of

\[
P_n(\cos \varrho) P_n(\cos \alpha_i),
\]

and second of neglecting terms of the nature of

\[
\frac{(n-m)!}{(n+m)!} P_n^m(\cos \alpha_i) P_n^m(\cos \varrho)
\]

for \(m > 1\)

when compared with terms of the nature of

\[
\frac{(n-1)!}{(n+1)!} P_n^1(\cos \alpha_i) P_n^1(\cos \varrho)
\]

for small \(\alpha_i\).

The validity of these approximations is most easily established

by considering the series expression of the associated Legendre func-
tions of the following form:\(^{21}\)

\[ p_m^n(\cos \alpha) = \frac{(n+m)!}{2^m m!(n-m)!} \left(1 - \cos^2 \alpha \right)^{m/2} \left[ 1 - \frac{(n-m)(n+m+1)}{1(m+1)} \left( \frac{1 - \cos \alpha}{2} \right) \right. \]
\[ + \frac{(n-m)(n-m-1)(n+m+1)(n+m+2)}{1 \cdot 2(m+1)(m+2)} \left( \frac{1 - \cos \alpha}{2} \right)^2 \left. \right] \]

For \( \alpha \) sufficiently small,

\[ (A-5) \quad \frac{(n-m)!}{(n+m)!} p_m^n(\cos \alpha) = \frac{\sin \frac{m}{2}}{2^m m!} \approx \frac{\alpha}{m!} \frac{m}{2} \]

Therefore for \( \alpha \) sufficiently small,

\[ \frac{(n-m)!}{(n+m)!} P_n^m(\cos \alpha_i) \ll P_n^m(\cos \alpha) \]

and

\[ \frac{(n-m)!}{(n+m)!} P_n^m(\cos \alpha_i) \ll \frac{(n-1)!}{(n+1)!} P_n^1(\cos \alpha_i) \]

for \( m \geq 2 \)

and consequently the approximations made in eqs. (A-4) are valid.

The approximation that is made when \( A_i \) is set equal to a constant and \( m \) is a linear function of the angle \( \alpha \) can now be estimated.

The largest error in the magnitude term \( A_i \) when this assumption is
made will be due to the square of the highest-order Legendre polynomial that is used in the expression for the field pattern of the antenna. The term for which the greatest variation will occur is the \( (\cos N\alpha)^2 \) term. Therefore the poorest approximation involved is that \( \cos^2 N\alpha \approx 1 \), or from the series expansion for the cosine function, that \( (N\alpha)^2 << 1 \). The approximation that \( m_i \) is a linear function of the angle \( \alpha_i \) is satisfied by the requirement that \( \sin N\alpha_i \approx N\alpha_i \), or from the series expansion for the sine function, that \( (N\alpha)^3 << 6 \).

If it is assumed that the radar antenna has a half power beam-width of about \( 12^\circ \), it is possible to represent the antenna pattern using Legendre polynomials of degree 10 or less. In this case \( N = 10 \).

For this case the following table gives the required distance of a target from the center of scan to introduce the indicated error in the approximation \( \cos^2 N\alpha_i = 1 \) when the range is 1000 yards.

<table>
<thead>
<tr>
<th>Error (%)</th>
<th>Target distance from center of scan (feet)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30</td>
</tr>
<tr>
<td>5</td>
<td>67</td>
</tr>
<tr>
<td>10</td>
<td>95</td>
</tr>
</tbody>
</table>

It is to be pointed out that the indicated percentage errors will be reduced by the other terms in the series of eqs. \( (A-2) \) and \( (A-3) \).
Consequently they are upper limits of the errors that are introduced for the case considered here. Therefore the approximations made in eqs. (A-2) and (A-3) are valid when the point targets are not at extremely large distances from the center of scan of the radar antenna.
APPENDIX B
DERIVATION OF PREVIOUS RESULTS

A solution of the balance-point of a conical-scan tracking radar has been obtained previously by Delano\textsuperscript{22} and by Watson and Feagin.\textsuperscript{23} It is to be shown here that all three solutions lead to the same result. However, the solution that is presented here appears to be in the most convenient form since the other solutions may be obtained from it, and the solutions of the balance-point for the monopulse systems can be obtained in the same manner. Also the analysis of Chapter 3 is made possible by the form of the present solution.

Delano gives as the equation for an apparent radar center $\epsilon_o$

$$\epsilon_o = \frac{E_s \cos(\theta_r - \theta_s)}{b_o E}$$

where the apparent radar center is defined as the position of the antenna axis for which the error signal is instantaneously zero. These symbols may be defined in terms of the present notation as


\textsuperscript{23}Feagin, R.B., Watson, R.B. and Norwood, J.M., \textit{Detailed Data Reduction Procedure for Airborne N-Point Target Backscattering}, 30 March 1955, Military Physics Research Laboratory, University of Texas, Austin, Texas.
Also Delano's equation may be written as

\[
E_S e^{j\theta_s} = b_o \sum_{i=1}^{N} A_i \frac{x_i}{R} e^{j\theta_i}
\]

and

\[
E e^{j\theta} = a e^{j\theta}.
\]

Noting that \( \theta_o \) is in this case an angle which equals \( x_o/R \), we obtain

\[
\epsilon_o = \frac{\sum_{i=1}^{n} A_i \frac{x_i}{R} e^{j\theta_i}}{A e^{j\theta}}
\]

Substituting eqs. (B-1) and (B-2) into this equation gives

\[
\epsilon_o = \frac{\sum_{i=1}^{n} A_i \frac{x_i}{R} e^{j\theta_i}}{A e^{j\theta}}
\]

Noting that \( \epsilon_o \) is in this case an angle which equals \( x_o/R \), we obtain

\[
x_o = \frac{\sum_{i=1}^{n} A_i x_i e^{j\theta_i}}{A e^{j\theta}}
\]

Substitution of eq. (2-8) gives

\[
x_o = \frac{\sum_{i=1}^{n} A_i x_i e^{j\theta_i}}{\sum_{i=1}^{n} A_i e^{j\theta_i}}
\]
which is identical to eq. (2-16), thus demonstrating that the two solutions are compatible.

The solution of Watson and Feagin may be obtained by taking the real part of eq. (2-6). Expanding

\[ x_b = \Re \left( \sum_{i=1}^{n} x_i A_i e^{j \theta_i} \right) = \Re \left( \sum_{i=1}^{n} A_i e^{j \theta_i} \right), \]

one obtains

\[ x_b = \Re \left( \sum_{i=1}^{n} x_i A_i \cos \theta_i + j \sum_{i=1}^{n} x_i A_i \sin \theta_i \right) \]

\[ = \Re \left( \sum_{i=1}^{n} A_i \cos \theta_i + j \sum_{i=1}^{n} A_i \sin \theta_i \right)^2 \]

\[ = \sum_{p=1}^{n} \sum_{q=1}^{n} x_p A_p A_q \cos \theta_p \cos \theta_q + \sum_{p=1}^{n} \sum_{q=1}^{n} x_p A_p A_q \sin \theta_p \sin \theta_q \]

\[ \sum_{p=1}^{n} \sum_{q=1}^{n} A_p A_q \cos \theta_p \cos \theta_q + \sum_{p=1}^{n} \sum_{q=1}^{n} A_p A_q \sin \theta_p \sin \theta_q \]

\[ = \sum_{p=1}^{n} \sum_{q=1}^{n} x_p A_p A_q \cos (\theta_q - \theta_p) \]

\[ \sum_{p=1}^{n} \sum_{q=1}^{n} A_p A_q \cos (\theta_q - \theta_p) \]
Now noting that

\[
\sum_{p=1}^{n} \sum_{q=1}^{n} x_{p} A_{p} A_{q} \cos(\theta_{q} - \theta_{p})
\]

\[
= \sum_{q=1}^{n} \sum_{p=1}^{n} x_{q} A_{q} A_{p} \cos(\theta_{p} - \theta_{q})
\]

\[
= \sum_{p=1}^{n} \sum_{q=1}^{n} \frac{x_{p} + x_{q}}{2} A_{p} A_{q} \cos(\theta_{q} - \theta_{p}),
\]

then

\[
x_{b} = \Re e(\overline{x}) = \frac{\sum_{p=1}^{n} \sum_{q=1}^{n} A_{p} A_{q} \cos(\theta_{q} - \theta_{p}) \frac{x_{p} + x_{q}}{2}}{\sum_{p=1}^{n} \sum_{q=1}^{n} A_{p} A_{q} \cos(\theta_{q} - \theta_{p})}
\]

In the same manner

\[
y_{b} = \Re e(\overline{y}) = \frac{\sum_{p=1}^{n} \sum_{q=1}^{n} A_{p} A_{q} \cos(\theta_{q} - \theta_{p}) \frac{y_{p} + y_{q}}{2}}{\sum_{p=1}^{n} \sum_{q=1}^{n} A_{p} A_{q} \cos(\theta_{q} - \theta_{p})}
\]

These are the same solutions that were obtained previously by Watson et al.
Watson and Feagin\textsuperscript{24} have also derived equations of the balance point of the phase amplitude-monopulse system. In the following it will be shown that their equations and the equations derived here are identical.

It has already been shown that

\[
\text{Re}(\bar{x}) = \text{Re} \left( \frac{\sum_{i=1}^{n} x_i A_i e^{j\theta_i}}{\sum_{i=1}^{n} A_i e^{j\theta_i}} \right) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} A_i A_j \frac{x_i + x_j}{2} \cos(\theta_i - \theta_j)}{\sum_{i=1}^{n} \sum_{j=1}^{n} A_i A_j \cos(\theta_i - \theta_j)}.
\]

Taking the imaginary part of $\bar{x}$ gives

\[
\text{Im}(\bar{x}) = \text{Im} \left( \frac{\sum_{i=1}^{n} x_i A_i e^{j\theta_i}}{\sum_{i=1}^{n} A_i e^{j\theta_i}} \right) = \text{Im} \left( \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} x_i A_i \cos \theta_i + j \sum_{i=1}^{n} x_i A_i \sin \theta_i}{\sum_{i=1}^{n} A_i \cos \theta_i + j \sum_{i=1}^{n} A_i \sin \theta_i} \right)
\]

\[
= \text{Im} \left( \frac{\left( \sum_{i=1}^{n} x_i A_i \cos \theta_i \right) + j \left( \sum_{i=1}^{n} x_i A_i \sin \theta_i \right)}{\left( \sum_{i=1}^{n} A_i \cos \theta_i \right)^2 + \left( \sum_{i=1}^{n} A_i \sin \theta_i \right)^2} \right)
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{n} x_i A_i A_j \sin \theta_i \cos \theta_j - \sum_{i=1}^{n} \sum_{j=1}^{n} x_i A_i A_j \sin \theta_j \cos \theta_i
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{n} A_i A_j \cos(\theta_i - \theta_j)
\]

\textsuperscript{24} Watson, R.B. and Feagin, R.B., 1 February 1956, Military Physics Research Laboratory, University of Texas, Austin, Texas.
\[
\sum_{i=1}^{n} \sum_{j=1}^{n} x_i A_i A_j \sin(\theta_i - \theta_j) \frac{x_i - x_j}{2} A_i A_j \sin(\theta_i - \theta_j) \cos(\theta_i - \theta_j) \cdot
\]

Noting that

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} x_i A_i A_j \sin(\theta_i - \theta_j) = - \sum_{j=1}^{n} x_j A_j A_i \sin(\theta_i - \theta_j)
\]

\[
= \sum_{i} \sum_{j} \frac{x_i - x_j}{2} A_i A_j \sin(\theta_i - \theta_j),
\]

\[\delta_m (\bar{x})\] may be written as

\[
\delta_m (\bar{x}) = \frac{\sum_{i} \sum_{j} \frac{x_i - x_j}{2} A_i A_j \sin(\theta_i - \theta_j)}{\sum_{i} \sum_{j} A_i A_j \cos(\theta_i - \theta_j)}.
\]

The equations for the balance point of the phase amplitude-monopulse radar system are

\[
x_b = \Re (\bar{x}) + \frac{K_a}{K_p} \delta_m (\bar{y})
\]
The above expressions are those obtained by Watson and Feagin for the phase amplitude-monopulse system.
AUTOBIOGRAPHY

I, Leon Peters, Jr., was born in Columbus, Ohio, May 28, 1923. I received my secondary school education in the public schools of Columbus, Ohio, and my undergraduate training at Ohio State University which granted me the Bachelor of Electrical Engineering degree in 1950. I received the Master of Science degree from Ohio State University in 1954.

I joined the staff of the Ohio State University in September of 1950 as a Research Associate in the Department of Electrical Engineering. I held this position, and later that of Assistant Supervisor while completing requirements for the degree Doctor of Philosophy.