A THEORY FOR CUMULATIVE FATIGUE DAMAGE
IN METALS

DISSERTATION
Presented in Partial Fulfillment of the Requirements for
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* * * * * *

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# CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>SUMMARY</td>
<td>1</td>
</tr>
<tr>
<td>INTRODUCTION</td>
<td>5</td>
</tr>
<tr>
<td>1. Fatigue</td>
<td>5</td>
</tr>
<tr>
<td>2. S-N Curves</td>
<td>6</td>
</tr>
<tr>
<td>3. Scatter</td>
<td>8</td>
</tr>
<tr>
<td>4. Cumulative Damage</td>
<td>10</td>
</tr>
<tr>
<td>5. Object and Scope</td>
<td>14</td>
</tr>
<tr>
<td>Chapter</td>
<td></td>
</tr>
<tr>
<td>I. REVIEW OF CUMULATIVE DAMAGE LITERATURE</td>
<td>17</td>
</tr>
<tr>
<td>1. The Linear Hypothesis</td>
<td>17</td>
</tr>
<tr>
<td>2. Non-Linear Theories</td>
<td>22</td>
</tr>
<tr>
<td>3. Random Loading</td>
<td>32</td>
</tr>
<tr>
<td>II. A NEW THEORY FOR CUMULATIVE DAMAGE</td>
<td>36</td>
</tr>
<tr>
<td>1. Basic Assumptions</td>
<td>36</td>
</tr>
<tr>
<td>2. Constant Stress-Amplitude</td>
<td>43</td>
</tr>
<tr>
<td>3. Uniformly Increasing Stress-Amplitude</td>
<td>49</td>
</tr>
<tr>
<td>4. Random Stress-Amplitude</td>
<td>55</td>
</tr>
<tr>
<td>III. EVALUATION OF THE THEORY FOR CONSTANT STRESS-AMPLITUDE</td>
<td>66</td>
</tr>
<tr>
<td>1. The Relationship Between Life and Stress-Amplitude</td>
<td>66</td>
</tr>
<tr>
<td>2. The Relationship Between Endurance Limit and Cycle-Ratio</td>
<td>83</td>
</tr>
<tr>
<td>3. Effect of Cycles at One Stress-Amplitude on Life at a Second Stress-Amplitude</td>
<td>93</td>
</tr>
</tbody>
</table>

---

iii
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4. Effect of Damage on the Relationship Between Life and Stress-Amplitude</td>
<td>106</td>
</tr>
<tr>
<td>5. The Sum of the Cycle-Ratios at Failure</td>
<td>113</td>
</tr>
<tr>
<td>IV. EVALUATION OF THE THEORY FOR CONTINUOUSLY CHANGING STRESS-AMPLITUDE</td>
<td>123</td>
</tr>
<tr>
<td>1. Uniformly Increasing Stress-Amplitude</td>
<td>123</td>
</tr>
<tr>
<td>2. Randomly Varying Stress-Amplitude</td>
<td>132</td>
</tr>
<tr>
<td>V. CONCLUSIONS AND DESIGN EQUATIONS</td>
<td>146</td>
</tr>
<tr>
<td>1. Conclusions Pertaining to the Evaluation of the Theory</td>
<td>146</td>
</tr>
<tr>
<td>2. Conclusions Pertaining to the Application of the Theory</td>
<td>149</td>
</tr>
<tr>
<td>3. Design Equations</td>
<td>151</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>159</td>
</tr>
<tr>
<td>AUTOBIOGRAPHY</td>
<td>166</td>
</tr>
</tbody>
</table>
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Experimental and Theoretical Lives with Random Stress-Amplitude</td>
<td>138</td>
</tr>
<tr>
<td>2</td>
<td>Probabilities Corresponding to Values of Stress-Amplitude Occurring in Several Different Random Sequences</td>
<td>141</td>
</tr>
<tr>
<td>3</td>
<td>Correction Factors for Random Loading</td>
<td>158</td>
</tr>
</tbody>
</table>
### LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Typical Relationship Between Constant Stress-Amplitude and Average Cycles to Failure.</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>Representative Damage Curves</td>
<td>25</td>
</tr>
<tr>
<td>3</td>
<td>Theoretical S-N Curve and Data from Constant Amplitude Tests on SAE 1030 Steel</td>
<td>68</td>
</tr>
<tr>
<td>4</td>
<td>Theoretical S-N Curve and Data from Constant Amplitude Tests on Steel X</td>
<td>69</td>
</tr>
<tr>
<td>5</td>
<td>Theoretical S-N Curve and Data from Constant Amplitude Tests on Steel Y</td>
<td>70</td>
</tr>
<tr>
<td>6</td>
<td>Theoretical S-N Curve and Data from Constant Amplitude Tests on Steel Z</td>
<td>71</td>
</tr>
<tr>
<td>7</td>
<td>Theoretical S-N Curve and Data from Constant Amplitude Tests on A-7 Steel</td>
<td>72</td>
</tr>
<tr>
<td>8</td>
<td>Theoretical S-N Curve and Data from Constant Amplitude Tests on SAE U3U0 Steel</td>
<td>73</td>
</tr>
<tr>
<td>9</td>
<td>Theoretical S-N Curve and Data from Constant Amplitude Tests on XU130 Steel</td>
<td>74</td>
</tr>
<tr>
<td>10</td>
<td>Theoretical S-N Curve and Data from Constant Amplitude Tests on SAE 1340 Steel</td>
<td>75</td>
</tr>
<tr>
<td>11</td>
<td>Theoretical γ-L Curve and Composite Data from Constant Amplitude Tests on Steel</td>
<td>76</td>
</tr>
<tr>
<td>12</td>
<td>Reduction of Endurance Limit Ratio Due to Load Cycles at Constant Amplitude on Steel X</td>
<td>64</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>13</td>
<td>Reduction of Endurance Limit Ratio Due to Load Cycles at Constant Amplitude on Steel Y</td>
<td>85</td>
</tr>
<tr>
<td>14</td>
<td>Reduction of Endurance Limit Ratio Due to Load Cycles at Constant Amplitude on Steel Z</td>
<td>86</td>
</tr>
<tr>
<td>15</td>
<td>Reduction of Endurance Limit Ratio Due to Load Cycles at Constant Amplitude on SAE 1030 Steel</td>
<td>87</td>
</tr>
<tr>
<td>16</td>
<td>Reduction of Endurance Limit Ratio Due to Load Cycles at Constant Amplitude on X1130 Steel</td>
<td>88</td>
</tr>
<tr>
<td>17</td>
<td>Composite Results for Reduction of Endurance Limit Ratio Due to Load Cycles at Constant Amplitude</td>
<td>89</td>
</tr>
<tr>
<td>18</td>
<td>Effect of Cycles at One Amplitude on Life at a Second Amplitude for A-7 Steel</td>
<td>95</td>
</tr>
<tr>
<td>19</td>
<td>Effect of Cycles at One Amplitude on Life at a Second Amplitude for SAE 1130 Steel</td>
<td>96</td>
</tr>
<tr>
<td>20</td>
<td>Effect of Cycles at One Amplitude on Life at a Second Amplitude for A-7 Steel</td>
<td>97</td>
</tr>
<tr>
<td>21</td>
<td>Effect of Cycles at One Amplitude on Life at a Second Amplitude for Steel B</td>
<td>98</td>
</tr>
<tr>
<td>22</td>
<td>Effect of Cycles at One Amplitude on Life at a Second Amplitude for Steel B</td>
<td>99</td>
</tr>
<tr>
<td>23</td>
<td>Effect of Cycles at One Amplitude on Life at a Second Amplitude for Steel B</td>
<td>100</td>
</tr>
<tr>
<td>24</td>
<td>Effect of Cycles at One Amplitude on Life at a Second Amplitude for X1130 Steel</td>
<td>101</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>25</td>
<td>Effect of Cycles at One Amplitude on Life at a Second Amplitude for XH130 Steel</td>
<td>102</td>
</tr>
<tr>
<td>26</td>
<td>Reduction of Endurance Limit Due to Load Cycles at Constant Amplitude on XH130 Steel</td>
<td>107</td>
</tr>
<tr>
<td>27</td>
<td>Theoretical S-N Curves and Corresponding Data for Damaged Specimens of XH130 Steel</td>
<td>108</td>
</tr>
<tr>
<td>28</td>
<td>Theoretical S-N Curves and Corresponding Data for Damaged Specimens of XH130 Steel</td>
<td>109</td>
</tr>
<tr>
<td>29</td>
<td>Reduction of Endurance Limit Due to Load Cycles at Constant Amplitude on SAE 1020 Steel</td>
<td>110</td>
</tr>
<tr>
<td>30</td>
<td>Theoretical S-N Curves and Corresponding Data for Damaged Specimens of SAE 1020 Steel</td>
<td>111</td>
</tr>
<tr>
<td>31</td>
<td>Theoretical Curve for the Relationship Between the Endurance Limit Ratio and Cycle-Ratio</td>
<td>115</td>
</tr>
<tr>
<td>32</td>
<td>Effect of Individual Cycle-Ratio Size on the Sum of the Cycle-Ratios with Two Amplitudes Acting</td>
<td>116</td>
</tr>
<tr>
<td>33</td>
<td>Effect of the Number of Stress Levels Acting on the Sum of the Cycle-Ratios</td>
<td>117</td>
</tr>
<tr>
<td>34</td>
<td>Theoretical Relationship Between Rate of Amplitude Increase and Amplitude at Failure</td>
<td>125</td>
</tr>
<tr>
<td>35</td>
<td>Results from Prot-Type Tests on SAE 2340 Steel</td>
<td>128</td>
</tr>
<tr>
<td>36</td>
<td>S-N Curve Calculated from Prot-Type Data for SAE 2340 Steel</td>
<td>129</td>
</tr>
<tr>
<td>37</td>
<td>Results from Prot-Type Tests on 1M50 Steel</td>
<td>130</td>
</tr>
<tr>
<td>38</td>
<td>S-N Curve Calculated from Prot-Type Data for 1M50 Steel</td>
<td>131</td>
</tr>
<tr>
<td>39</td>
<td>S-N Data for the SAE 4340 Steel Used for Random Loading Tests</td>
<td>135</td>
</tr>
</tbody>
</table>
**NOTATIONS***

- \( \alpha \) : Rate of increase of stress amplitude, psi per cycle (Prot rate)
- \( \beta \) : Cycle ratio, \( n/N \)
- \( \gamma \) : Overstress ratio, \( S/E_0 \)
- \( \gamma^* \) : A reference value of \( \gamma \)
- \( \rho \) : Exponent in damage function
- \( b \) : Characteristic constant for random stress-amplitude
- \( b_m \) : Characteristic constant for random moment amplitude
- \( c \) : Characteristic constant for random stress-amplitude
- \( c_m \) : Characteristic constant for random moment amplitude
- \( C \) : Constant specifying failure condition, \( E_\infty/S_u \)
- \( D \) : Fatigue damage
- \( \Delta D \) : An increment of fatigue damage
- \( E \) : Instantaneous value of the endurance limit
- \( E_N \) : Value of the endurance limit at failure
- \( E_0 \) : Virgin endurance limit
- \( k \) : Proportionality constant for the damage function
- \( K \) : \( k \times E_0 \)
- \( L \) : Life ratio, \( N/N^* \)
- \( M \) : A large arbitrary constant or bending moment as specified where used
- \( n \) : Number of cycles applied under a given loading condition

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*These notations are not necessarily strictly adhered to in the Introduction and in Chapter I where the work of other authors is reviewed.
\( \Delta n \) One cycle

\( N \) Number of cycles which will cause failure under a given loading condition

\( N_{\gamma^*} \) Number of cycles to failure at a constant stress-amplitude corresponding to \( \gamma^* \)

\( S \) Stress-amplitude

\( Z \) A constant, a variable, or a section modulus as specified where used
SUMMARY

The phenomenon of fatigue damage of metals is discussed with particular emphasis on cumulative or progressive aspects. Published literature pertinent to the development of quantitative mathematical methods for evaluating the accumulation of fatigue damage is reviewed. This includes primarily contributions due to Palmgren, Miner, Langer, Prot, Thum and Bautz, Kommers, Bennett, Richart and Newmark, Marco and Starkey, Corten and Dolan, Henry, Head and Hooke, Lundberg, and Freudenthal and Heller.

Experimental evidence is cited to establish that the stress-amplitude associated with any life, including the highest stress-amplitude associated with infinite life (i.e., the endurance limit), is reduced as a consequence of the accumulation of fatigue damage. The following hypotheses are then proposed as the basis of a quantitative cumulative damage theory.

1. The fatigue damage accumulated during any load cycle can be measured uniquely by the difference between the endurance limit before the application of that cycle and the endurance limit after that cycle.

2. The amount of fatigue damage contributed by any cycle is proportional to a power of the amount by which the stress amplitude exceeds the instantaneous value of the endurance limit.

3. The amount of damage accumulated in one cycle can be considered to represent the rate of change of damage with cycles,
where cycles are treated as a continuous independent variable and damage as the dependent variable.

4. The damage done in any cycle during the action of random loading is the expected value of damage for that cycle. (The expected value is defined in statistics as the weighted sum of all possible values. Each value is weighted by being multiplied by its associated probability.)

These hypotheses provide the basis for the derivation of the following differential equation and its application to situations where the stress-amplitude is representable either as a function of cycles or as a random variable.

\[
\frac{dE}{dn} = -k(S - F)^{p+1}
\]

The associated boundary conditions are assumed to be that the endurance limit is equal to the conventional or virgin endurance limit when \( n = 0 \) and that the endurance limit has a terminal value greater than zero at rupture when \( n = N \). It is assumed that the ratio of this terminal value of the endurance limit to the stress-amplitude at rupture is of the same magnitude as the ratio of the virgin endurance limit to the static ultimate tensile strength.

The solutions of the differential equation are obtained for constant amplitude, uniformly increasing amplitude, and random amplitude loading. The resulting expressions are compared with experimental data available in the published literature and with empirical equations which have been found compatible with such data by various investigators. The data considered were obtained by many different investigators on a wide
A variety of steels and consist principally of the results of rotating-bending tests on smooth, polished specimens with critical sections between one-eighth and one-third of an inch in diameter.

A substantial agreement is demonstrated to exist between the data and the theory and on this basis the suitability of the results of the theory for application to design are considered.

Some of the more important conclusions and results are as follows:

1. A basically phenomenological approach can be used as a basis for more extensive correlation of cumulative damage data than has heretofore been supposed.

2. Composite S-N curves for different steels can be plotted if suitable non-dimensional coordinates are used, i.e., Over-stress ratio vs. Life ratio.

3. Relationships suitable for design calculations are developed which are in many cases superior to those otherwise available.

The dependence of stress-amplitude on cycles, the basic differential equations, and the theoretical relationships between the number of cycles required to cause rupture and the parameters characteristic of the loading are summarized below for the three types of loading considered. In these expressions the constant $\rho$, which appears in the formal solutions, has been replaced with the value unity, because this value was found to give reasonable results for a wide variety of steels.

A. Constant stress-amplitude:

1. $S = f(n) = a$ constant

2. $\frac{dF}{dn} = -k(S - E)^2$
3. \[ kN = \frac{1}{S - E_0} \frac{1}{S(1-C)} \]

B. Uniformly increasing stress-amplitude:

1. \[ S = E_0 + \alpha n \]

2. \[ \frac{dF}{dn} = -k(E_0 - E + \alpha n)^2 \]

3. \[ S_f = E_0 + \alpha N = \frac{\alpha k}{1-C} \tan \sqrt{ak} \cdot N \]

or from an approximate solution

\[ S_f = E_0 + \frac{\alpha}{2\sqrt{k}} \cdot N \]

C. Random stress-amplitude:

1. Two types of statistical representation were considered. The equations in 2 and 3 below are for the first type only.

   a. A finite number of possible values of stress-amplitude, \( S_i \), each associated with a specified probability, \( p_i \)

   b. A finite range of possible values of stress-amplitude \( S \), associated with a continuous frequency function, \( p(S) \)

2. \[ \frac{dF}{dn} = -k \sum_{i=1}^{j} p_i(S_i - E)^2 \]

3. \[ kN = \frac{1}{\sqrt{c-b^2}} \left[ \tan^{-1} \frac{b - CS_m}{\sqrt{c-b^2}} - \tan^{-1} \frac{b - E_0}{\sqrt{c-b^2}} \right] \]

4
INTRODUCTION

1. Fatigue.- The term fatigue, in the areas of machine design and strength of materials, applies to the behavior of metals under repeated stress as contrasted to their behavior under static stress. Since at least the beginning of the nineteenth century, machine and structural components subjected to repeated or varying loads have been observed to fail at stresses which were a fraction of the stresses these same components were known to support when loads were held constant.

With the development and increased use of high-speed power sources, not only in transportation but in the mechanization of almost every human endeavor, the number of situations where repeated or varying loading occurs has increased to the extent that now this is the usual loading situation rather than the exceptional. Particularly is this true in those areas where weight must be kept to a minimum and where consequently highly stressed parts are most commonly used. As a result, for the last several decades the phenomenon of fatigue has been one of the prominent factors in the failure of metal parts in machines and structures.

The many aspects of fatigue and the consequences of inadequate allowances for its effects have been the subject of an extensive and comprehensive literature. Predominant in this literature are several books which either are devoted entirely to fatigue or contain large sections on the subject (references 1 through 14, and 65). The interested
reader will find in these books not only extensive discussions of the
many aspects of fatigue but also, through their respective bibliogra-
phies, an excellent introduction to the rest of the literature. For
those persons particularly interested in fatigue testing and the analysis
of fatigue data, several publications of the American Society for Testing
Materials will prove most useful (references 15, 16, and 17), and an in-
troduction to design methods where fatigue is a factor can be found in
any of several texts on machine design (references 18 and 19).

2. S-N Curves.— The outstanding method of portraying the be-
havior of metals under repeated stresses is the S-N diagram. This dia-
gram usually consists of a plot of stress-amplitude versus the number of
load repetitions or cycles required to cause failure at each stress-
amplitude. The data are plotted on logarithmic or semi-logarithmic
graph paper. If semi-logarithmic graph paper is used, stress-amplitude
is plotted on the linear ordinate and cycles until failure on the loga-
rithmic abscissa. An example of an S-N diagram is shown in Figure 1.
This curve is typical for steel and exhibits the characteristic bend or
"knee" just above that stress-amplitude for which the curve becomes
completely flat.

The stress-amplitude identified by the flat portion of the curve
is usually designated as the endurance limit and is that stress-ampli-
tude below which an indefinite number of cycles can be withstood. Some
nonferrous metals are not known to have an endurance limit significantly
larger than zero and experimental curves for those metals do not exhibit
the knee shown in Figure 1.
Fig. 1. Typical Relationship Between Constant Stress-Amplitude and Average Cycles to Failure
A considerable amount of S-N data is available for most of the more common metals and alloys used in the construction of machines and structures. Rather extensive collections of such data have been put together from many different sources and different kinds of tests (references 5 and 20).

3. Scatter.-- One of the outstanding characteristics of the behavior of metals under repeated stress is the variability in that behavior. For example, a group of steel specimens, when tested at a stress-amplitude for which their average life is 90,000 cycles, might typically include specimens with lives as short as 50,000 cycles and lives as long as 130,000 cycles (cf. reference 25).

All measurements exhibit some variation but the variation in fatigue data is distinctly larger than that encountered in connection with most other phenomena. The reasons for this large variation or scatter are first, the type of test used, and more important, the nature of the phenomenon.

Fatigue testing is usually destructive testing. If a specimen is subjected to a repeated load until failure, obviously that specimen is no longer available to test again. Therefore each test must be conducted on a new specimen and the inherent variations in specimen manufacture will affect the succeeding tests. This is in contrast to many other types of testing arrangements where tests can be repeated on the same set-up.

Probably most of the scatter experienced in fatigue testing is inherent in the nature of fatigue damage in the materials tested. When a member is subjected to a repeated load the weakest spot in the critical
section fails first, and this weakens the section. Therefore the actual stress-amplitude is increased causing local failure of stronger areas. The important point is that the specimen is no stronger than its weakest point.

This can be contrasted to, say, a measurement of the pressure of a gas in a cylinder. The pressure might be measured by the force which the gas exerts on a piston in the cylinder. It is true that from a microscopic point of view the pressure will not be uniform over the piston. However, the pressure measurement will not show the high or low pressure but will record only the average. In fatigue testing this automatic averaging does not occur. The strength of the weakest spot in the critical section always exercises considerable influence on the measured value.

Quite often fatigue data are not extensive enough to provide any comprehensive measure of the scatter involved. Perhaps 5 or 10 specimens may provide all the data which is available to establish an S-N curve. The specimens are tested, the data are plotted, and a "best" curve is drawn by eye through the data.

Such data may then be used as a basis for design. Scatter is taken into consideration in a qualitative manner depending on the experience of the person making the design. This is not necessarily an undesirable approach. Unless the available data have been gathered from tests which are very well correlated with the design situation there may be enough effect from other factors to introduce uncertainties in the design calculations in excess of those due to scatter in the data.

In recent years, particularly when fatigue data have been used to evaluate fatigue theories, or to measure the effect of various factors
on fatigue strength, there has been an increasing attempt to make quantitative measurements of scatter. This has lead to an increasing application of the mathematical methods of statistics (references 16, 17, 29, 30, 31, 32, 33, 34, 35, and 36).

When sufficient data are available the single S–N curve can be replaced with a family of S–N curves where each curve is associated with a certain probability of failure. The curves are then often referred to as S–N–P curves where P stands for probability. In this sense the conventional S–N curve, which is actually a best estimate of the "average" S–N curve is a 50 percent probability curve. That is, 50 percent of a large group of specimens could be expected to last as long as the life value indicated by the curve for a given stress-amplitude.

4. Cumulative Damage.— It has been conclusively shown that the damage due to cyclic stressing is cumulative. No one cycle is completely responsible for the total damaging effect but rather each cycle makes a small contribution. The evidence which supports this statement consists of two types. First, direct quantitative tests results, and second, qualitative observations of the growth of fatigue cracks.

Quantitative tests designed to show the cumulative nature of fatigue damage consist of series of experiments which measure the progressive decrease in capacity to resist cyclic loading as a result of cyclic loading.

In such tests the first step is to choose some stress-amplitude which is above the endurance limit. Then one group of specimens is subjected to a number of cycles of that stress-amplitude, a second group of specimens is subjected to some larger number of cycles of the same
stress-amplitude, a third group of specimens is subjected to a still larger number of cycles of the same stress-amplitude, and so on. Then each of these damaged groups can be used to establish an S-N curve or some selected portion of an S-N curve. Comparisons of S-N data obtained in this manner have shown a progressive deterioration of the fatigue properties of the material tested (references 21, 22, 23, 24, 25, and Chapter IV of 5). Both the endurance limit and the life at any stress amplitude above the endurance limit are decreased as a consequence of repeated loads above the endurance limit. The longer this repeated loading is continued the greater are these reductions.

In any member which has failed in fatigue at stress levels where general plastic deformation is small a fatigue crack will be found to have extended across an appreciable portion of the critical section before final rupture occurs. The existence of the fatigue crack for many cycles before failure is evidenced by the smooth appearance of the opposing crack faces due to repeated contact and rubbing during the load cycles. Furthermore fatigue cracks have been detected before final rupture and their growth observed (references 25, 26, and 27).

Such examinations have generally established that fatigue cracks are initiated as very small cracks, perhaps submicroscopic in size. At least one of these cracks gradually grows to an extent sufficient to have considerable influence on the geometry and therefore the stresses in the section where rupture occurs (references 25, 26, 27, 28, and p. 300 of 6). To the extent that fatigue damage is represented by the size of the fatigue cracks the progression of crack size is indicative of the accumulation of fatigue damage.
In some cases experimental evidence has indicated that cyclic stressing just under the endurance limit may under certain conditions improve the capacity to resist cyclic stresses (references 6, p. 280; 5, p. 50). In other cases experimental evidence has indicated a minimum number of cycles required to initiate damage at stress-amplitudes above the endurance limit. Certain investigators have attempted to establish "damage lines" which are a plot of stress-amplitudes versus cycles required to initiate damage at each stress-amplitude (reference 5, p. 39). However it remains to be proved whether these so-called "damage lines" define combinations of stress-amplitude and cycles which cause no damage or whether they define combinations of stress-amplitude and cycles which cause statistically insignificant damage. From a practical point of view it makes little difference whether the damage is non-existent or is statistically undetectable, but in terms of an understanding of the phenomenon of fatigue this is an important question.

Many different aspects of the condition and geometry of the member, the method of load application, and the environment in which the cycles of load are applied, as well as the material of the member, have been found to affect the accumulation of fatigue damage. Some of the more prominent of these include the effects due to

1. Size of the member.
2. Surface finish.
3. Residual stresses.
4. Stress raisers.
5. Stress gradient.
8. Temperature.
10. Fretting.

Still another primary factor in the evaluation of fatigue damage of machine and structural components is the variation of the amplitude of the cyclic stress encountered in many applications. Many machines and structures may be subjected to periods of varying degrees of duty and consequently to periods of different stress-amplitude. These fluctuations in stress-amplitude may be due to such things as the operation of motors or engines at different power settings. In such a case the load amplitude may have some known value for a definite number of cycles and then may subsequently change to a second magnitude at which it may remain for another period of cycles, and so on. A study of such an application may establish a fairly definite sequence of loading and it may then be possible to specify the number of cycles at each of several stress-amplitudes and their sequence of application.

On the other hand the variation of stress-amplitude may be due to something as unpredictable as the action of gusts on an airplane. In such cases it is impossible to specify any precise sequence of cycles of definite stress amplitudes, and recourse must be had to a statistical presentation. This may consist of establishing the frequency of occurrence at each of several stress-amplitudes or it may require the determination of a continuous frequency distribution of stress-amplitudes.

Although fatigue damage is always acquired by a cumulative process, such terms as "cumulative damage" and "accumulation of fatigue
damage have been restricted almost exclusively in the fatigue literature to those situations in which the stress-amplitude is not constant. One reason for this is that if the stress-amplitude is to be constant for the entire life of a member, damage can be evaluated in terms of cycles and fractional damage can be readily expressed in terms of the fraction of cycles to failure which has been applied. However, if the stress-amplitude varies with cycles no such ready-made method of defining and evaluating damage exists.

The development of a general definition for damage and a method for evaluating the accumulation of such damage has been the object of a considerable amount of study and research. However, as yet no generally satisfactory theory is available for evaluating the accumulation of fatigue damage and for predicting the lives of machines and structural members subjected to cyclic loading.

5. Object and Scope.- The object of this dissertation is the development of mathematical methods for evaluating the accumulation of fatigue damage in metals due to cyclic loading. These methods must necessarily consist of mathematical procedures by which test data can be correlated with the situation to be evaluated.

Very often when a design situation arises in which the accumulation of fatigue damage must be evaluated, the best data available have been obtained under conditions considerably different from those of the application. Therefore it is necessary to consider all of the factors known to have a significant effect on fatigue and to interpret and modify the available data before it can be correlated with the application. As previously indicated there are many factors which exercise a significant
effect on fatigue. Furthermore, experience has proven that the correlation of fatigue data where any of one of these factors varies between test data and application is a difficult and complex problem.

Since the size of the problem, the variability of fatigue data, and the limited data available render the evaluation of any fatigue theory particularly difficult, it seemed advisable to limit the scope of the present investigation. Therefore this dissertation was restricted to the development of a quantitative hypothesis for the accumulation of fatigue damage which would account for the effect of variations of stress-amplitude under conditions of fully-reversed, one-dimensional loading. All of the other factors affecting fatigue are isolated from the description of the basic phenomenon. It is assumed that the defining relationships are those corresponding to the average of the group or population considered, and it is further assumed that the effects of scatter can be superimposed on the resulting evaluations. The problem is approached from a phenomenological point of view. The basic assumptions concern parameters which are familiar to a design engineer, such as stress and cycles, and the theory is completely developed in terms of these quantities.

Some of the outstanding contributions to the development of cumulative damage concepts are briefly reviewed in Chapter I. Then in Chapter II a new theory is developed and applied to three types of loading. These are constant stress-amplitude loading, uniformly increasing stress-amplitude or Protr-type loading, and random stress-amplitude loading. In this chapter some of the results of the new theory, for example, an equation for the S-N curve, are compared with existing empirical relation-
ships, but the detailed quantitative appraisals of the theory are reserved for the following chapters.

In Chapter III the theoretical results are compared with published data from tests in which the stress-amplitude was kept constant or was changed, at most, a few times during the course of a test. The theoretical results for constantly increasing stress-amplitude and for random load amplitude are compared with published data in Chapter IV. The results of these comparisons are then discussed in Chapter V and the most promising of the derived relationships are incorporated in sample design equations.
Chapter I

REVIEW OF CUMULATIVE DAMAGE LITERATURE

1. The Linear Hypothesis.— Perhaps the first method for quantitatively evaluating the effect of varying stress-amplitude on the accumulation of fatigue damage was developed in Germany in connection with the application of roller bearings. In 1921, a paper by A. Palmgren was published in which he stated that if a bearing had a life of \( N \) cycles at a given stress-amplitude then it was conceivable that after \( n \) cycles \( n/N \) of its endurance might be used up (reference 37). He then suggested that the total damage, in case a number of different stress-amplitudes were applied, could be evaluated by adding the fractions of the endurance used up at each stress level, failure occurring when the sum reached one. That is, at failure

\[
\frac{n_1}{N_1} + \frac{n_2}{N_2} + \frac{n_3}{N_3} + \ldots \ldots = 1.
\]  

It should be noted that this theory is linear because it states that damage accumulates at a constant cycle-rate for a given stress-amplitude. The cycle-rate is assumed to be inversely proportional to the fatigue life at that stress-amplitude.

In 1937 B. F. Langer presented a paper in which he proposed a method "for estimating the life of a machine part which is subjected to repeated applications of various stresses" (reference 38). Although Langer was apparently unaware of Palmgren's suggestion and did not state the linear cumulative rule explicitly in the form which Palmgren had
used, nevertheless it can readily be shown that his method included the assumption of the linear rule.

Finally, in 1945 the well-known paper by M. A. Miner was published in which he independently arrived at the hypothesis of the linear rule for the accumulation of fatigue damage (reference 23). Apparently Miner was unaware of Palmgren's suggestion in 1924 and of some of the implications of Langer's work in 1937.

Miner assumed that cumulative damage was related to the net work absorbed by the specimen and that there was a critical amount of work which the specimen could absorb before failure. He also assumed that the fraction of the critical work absorbed in $m_1$ cycles at a stress amplitude $S_1$ was equal to the fraction $m_i/N_i$, where $N_i$ represented the total number of cycles which a virgin specimen could withstand at stress amplitude $S_1$. He then derived the linear rule directly from these two assumptions.

Following publication of Miner's paper the linear cumulative damage rule received widespread recognition and use. Not only was considerable success achieved in its use but the theory was convenient to use. Also there was no other generally known method for evaluating the effect of varying stress-amplitude.

A paper was published in France in 1948 by E. M. Prot in which the linear rule was generalized to apply to a constantly varying stress-amplitude (reference 39). The object of that particular paper was the introduction of an accelerated method of fatigue testing (the Prot Method), and the generalization of the linear hypothesis was only discussed in connection with that method.
Prot assumed that the shape of a mean S-N curve could be approximated by an equation of the form

\[(S - E_0)N = K\]  \hspace{1cm} (b)

He then assumed that this equation could be written as

\[\int_0^N (S - E_0)dN = K\]  \hspace{1cm} (c)

and that this equation would hold for \((S - E_0)\) not a constant.

That is he considered the relationship in equation (c) to be a definition of fatigue damage and assumed that \((S - E_0)dN\) represented the damage per cycle.

It should be noted that if equation (c) is rewritten in the form

\[
\frac{dD}{dn} = (S - E_0),
\]  \hspace{1cm} (d)

where \(D\) represents damage and \(n\) is cycles, the result is a differential equation expressing the cycle-rate of increase of damage as a function of \((S - E_0)\). With the introduction of the initial and terminal conditions indicated in equation (c) this equation might be solved for \(S\) as any arbitrary function of \(n\). These conditions from equation (c) are,

\[
D = 0, \text{ when } n = 0
\]
\[
D = K, \text{ when } n = N.
\]  \hspace{1cm} (e)

To demonstrate that equation (c) is equivalent to the linear rule stated in equation (a), consider a sequence of \(m\) stress amplitudes, \(S_1\), to each of which corresponds a fatigue life, \(N_1\). Let each stress amplitude be applied for \(n_1\) cycles and evaluate the total fatigue damage by equation (d). Let \(D_1\) represent the damage due to \(S_1\) when applied for \(n_1\) cycles.
Then,

$$D_{(\text{total})} = \sum_{i=1}^{m} D_i,$$  \hspace{1cm} (f)

but from (d)

$$D_i = \int_0^{n_i} (S_1 - E_0)dn = (S_1 - E_0)n_i . \hspace{1cm} (g)$$

Therefore

$$D_{(\text{total})} = \sum_{i=1}^{m} (S_1 - E_0)n_i . \hspace{1cm} (h)$$

But at failure from equation (c) we have

$$D_{(\text{total})} = K . \hspace{1cm} (i)$$

Therefore at failure

$$K = \sum_{i=1}^{m} (S_1 - E_0)n_i . \hspace{1cm} (j)$$

Now from (b) it follows

$$(S_1 - E_0) = K/N_i . \hspace{1cm} (k)$$

Substituting equation (k) into equation (j) gives

$$K = \sum_{i=1}^{m} (K/N_i)n_i$$

or at failure

$$\sum_{i=1}^{m} n_i/N_i = 1 . \hspace{1cm} (l)$$

This is exactly the linear rule as originally suggested by Palmgren.

If the linear hypothesis were correct and if Prot's interpretation also held, then equation (d) would indeed be a solution to the
cumulative damage problem. However, there is considerable evidence that the linear rule is seriously in error.

It has been thoroughly demonstrated that one of the aspects of fatigue damage is a reduction in the endurance limit (references 21, 22, 25, 40, 41). The linear rule provides no method for estimating the magnitude of this effect. In case the repeated loading contains stress-amplitudes below the virgin endurance limit the linear rule neglects them. Yet if the endurance limit is reduced below these stress-amplitudes they may contribute a large amount of fatigue damage.

Furthermore, at least for many different loading sequences, it has been demonstrated that the basic hypothesis is in error. That is, the sum of the ratios $n_i/N_i$ does not equal one at failure but may be very much more or less than one depending on the loading sequence (references 21, 22, 24, 25, 42, 43, 44).

This collective evidence is sufficient to establish that the linear rule is not in general an accurate description of the phenomenon of fatigue. However, this does not obviate the fact that much successful design has been accomplished by means of this rule. Practical application has shown that the linear rule can be used for predicting fatigue damage under conditions where sufficient test or service data exist to establish its applicability. In such circumstances a new type or model of a proven design may be successfully established by application of the linear rule. Such applications must be made with caution and any attempt to design for new and novel loading sequences may well lead to a non-conservative design.

21
In case the stress-amplitudes are applied randomly or with very few cycles consecutively at any one stress level there is some evidence that the linear rule may be applicable (reference 42 and p. 49 of 5). An example of such an application is the general formula proposed by Lundberg in 1955 for calculating the cumulative fatigue damage in aircraft structures when subjected to random gust loading (reference 45). Nevertheless, it appears highly desirable to develop a theory of cumulative damage which will successfully predict damage for any arbitrary load sequence and to establish a method of evaluating random loading which is not based on the dubious assumption of the linear rule.

2. Non-Linear Theories.— Contemporary with the development of the linear theory a considerable amount of research and testing was carried out in the area of cumulative fatigue damage. The discussion on the linear rule in the preceding section was based on the results of those researches and in Chapters III and IV much of the published data is compared with a proposed new theory. Of immediate interest, however, are the non-linear, cumulative-fatigue-damage theories which were developed as a result of those investigations.

In a discussion of the paper by Langer, which is referenced above, A. Thum and W. Bautz reviewed a cumulative damage theory which they had suggested (reference 46). Three types of information are required for their method. First a sum-of-frequencies curve is prepared from the loading sequence. This curve consists of a plot of the values of stress-amplitude which will be applied during the desired life versus the total number of cycles for which the stress-amplitude will be equal to or greater than each value. Thus if all stress-amplitudes will have either
the values $S_1$, $S_2$, or $S_3$, and if $S_1 < S_2 < S_3$, then the plotted points would be $(S_1, n_1)$, $(S_2, n_1 + n_2)$, and $(S_3, n_1 + n_2 + n_3)$, where $n_1$, $n_2$, and $n_3$ are respectively the total numbers of cycles for which the stress-amplitude will be $S_1$, $S_2$, and $S_3$.

Then the sum-of-frequencies curve is corrected to account for the fact that a given number of cycles at a high stress will cause more damage than the same number of cycles at a lower stress. Thum and Bautz assumed that the necessary correction factor could be determined as a function of the ratio of high stress to low stress. For each stress level the number of cycles for the corrected or equivalent frequency-summation curve is obtained by multiplying that part of the total cycles due to each higher stress by the corresponding value of the correction factor and then adding the resultant parts.

Finally the corrected frequency-summation curve is plotted and drawn on the same coordinates with a conventional S-N curve. If the corrected frequency-summation curve lies to the left and below the S-N curve then failure will not occur during the desired life, while if the two curves touch or intersect failure will occur.

In their original paper Thum and Bautz noted that the information required to transform the simple sum-of-frequencies curve into the corrected frequency-summation curve was not available (reference 47). They then stated that until this information became available the method could be used for rough approximations by using the simple sum-of-frequency curve in conjunction with the conventional S-N curve. This latter method has been used to some extent and is sometimes referred to as the "intercept" method.

23
It would appear that this "intercept" method is a non-conservative method because cycles of high stress are assumed to be no more damaging than the same number of cycles at a lower stress. A corrected frequency-summation curve in conjunction with an S-N curve would provide a method of evaluating cumulative damage if the sequence of loading were unimportant. However, as pointed out above, one of the principal results of the investigations into cumulative damage has been the discovery that loading sequence may have a very significant effect.

One of the difficulties which became evident in the attempts to develop cumulative-damage theories was the lack of a satisfactory mathematical definition of damage. Several different suggestions for a workable definition of damage were made and several corresponding damage theories were developed based on these definitions (references 22, 24, 42, 48, 49). Although the suggested definitions of damage are varied, the associated theories predict similar results and their primary features can be illustrated as in Figure 2.

In Figure 2 damage is plotted versus cycle-ratio for several different stress-amplitudes (cycle-ratio is the ratio of cycles applied at a given stress-amplitude to the total number of cycles required to cause failure of a virgin specimen at that stress-amplitude). From Figure 2 it appears that damage is zero for virgin specimens and that damage reaches a value of one at failure. According to the damage theories which can be represented on this type of a plot sequential repeated-stressing is divided into two processes. While a given stress-amplitude is being repeated the process can be represented as moving along the corresponding constant stress-amplitude line. When the stress-amplitude
Fig. 2. Representative Damage Curves
is changed the process can be represented by a constant damage line extending from the constant stress-amplitude line corresponding to the preceding value of stress-amplitude to the one corresponding to the new stress-amplitude.

Perhaps the first damage versus cycle-ratio plot of this type was presented by Kommers (reference 21). Although Kommers did not suggest a quantitative cumulative-damage theory in the published literature, he presented, over a period of several years, some very important contributions to the concept of cumulative damage. The principal use which he made of this type of graph was to show the relationship between cycle-ratio at one overstress and resultant damage at a second overstress. For this purpose the definition of damage was percentage reduction in the total life at the second overstress.

Bennett used the same type of graph in conjunction with a more complicated definition of damage (reference 22). From some testing he had done it appeared that the effect of fatigue damage could be represented by a series of S-N curves. He selected a damaging stress-amplitude and then determined an S-N curve for specimens which had previously been subjected to various cycle-ratios at that stress-amplitude. An S-N curve was determined for specimens which had previously been subjected to a cycle ratio of 1/10 at the damaging stress, another S-N curve was determined for specimens which had been subjected to a cycle ratio of 1/3 at the damaging stress level, etc.

From his testing it appeared that each of the members of such a family of S-N curves passed through a common point at high overstress and had a different endurance limit. Each of the curves could also be repre-
sented by a straight line in the over stress region but each such line had a different slope. S-N curves with larger slopes in the over stress region and with lower endurance limits were associated with larger numbers of cycle-ratios.

Bennett assumed that each curve could be characterized by its slope in the over stress region and its endurance limit. He then plotted the slope of the upper section versus the associated endurance limit and drew a straight line, which he called the "strength loss line", through the points. One end of this line represented virgin specimens and the other end represented completely damaged specimens. He concluded that this line was the same for all damaging stresses and he evaluated damage as percentage of distance along this line.

Bennett presented some test results with this definition of damage in the form of a damage versus cycle-ratio graph and suggested that cumulative damage could be evaluated by tracing the damaging process along constant stress-amplitude and constant damage lines as discussed above.

Richart and Newmark also made use of the damage versus cycle-ratio graph to trace the progress of damage along constant damage and constant stress lines (reference 42). They assumed that fatigue damage could be "measured by a quantity D, called 'degree of damage', which depends on β, the 'cycle-ratio', but the dependence is different at different stress levels." They expressed this dependency in the form

\[ D = \beta^x \]  

(m)

where \( x \) is a function of the stress-amplitude.

They concluded that the actual values of the exponent in equa-
tion \( (m) \) are not unique and that the relative positions rather than the actual positions of the curves on the damage versus cycle-ratio graph are significant for the evaluation of damage. They also presented a mathematical derivation to show that for infinitely small increments of cycle-ratios applied alternately at two stress-amplitudes the sum of the cycle-ratios at failure should be always less than one.

Marco and Starkey similarly concluded that damage could be expressed as an exponential function of cycle-ratio (reference 24). They found that large exponents corresponded to large values of stress-amplitude. They also had success with the use of damage versus cycle-ratio curves of the type shown in Figure 2 for predicting endurance lives for sequential loading by tracing the progress of damage along constant damage and constant stress-amplitude lines.

Corten and Dolan introduced a variation in the application of Figure 2 (reference 48). They defined damage as an exponential function of cycles of the form,

\[
D = mrN^n.
\]

In this expression \( D \) is damage, \( N \) is cycles, and \( r \) and \( a \) are constants for a given stress-amplitude, and \( m \) is the number of points at which damage is nucleating. Corten and Dolan traced the progression of damage on constant damage and constant stress lines similar to those in Figure 2. From an analytical treatment of this graphical interpretation they developed a relationship between the cycles to failure, \( N_1 \), at a constant stress-amplitude, \( S_1 \), and the cycles to failure, \( N_g \), when two stress-amplitudes, \( S_1 \) and \( S_2 \), are alternately applied for \( \alpha \) and \( (1-\alpha) \) cycles. They compared their analytical results with tests using differ-
ent values of $S_1$ and $S_2$ and of $a$. For these tests $n$ was 10,000 cycles. After the coefficients and exponents in their analytical expressions were evaluated to fit the data they found that for each pair of stress-amplitudes, $S_1$ and $S_2$, the life for constant stress-amplitude $S_1$ was related to the life for the combination of $S_1$ and $S_2$ by an equation of the form

$$N_g \left[ a + C(1 - a) \right] = N_1.$$  \hfill (c)

It is interesting to note that this expression can be written in the following manner by algebraic manipulation.

$$N_g a/N_1 + \left( N_2 k/N_1 \right) \left[ N_g (1 - a) / N_2 \right] = 1,$$  \hfill (p)

where $N_1$ and $N_2$ are the total lives which virgin specimens would have respectively at the constant stress-amplitudes $S_1$ and $S_2$. $(N_g a)$ is the total number of cycles, $n_1$, applied at the stress-amplitude $S_1$ during tests alternately at $S_1$ and then at $S_2$. Likewise $N_g (1 - a)$ is the total cycles, $n_2$, which would be applied at $S_2$. Therefore (p) can be written

$$n_1/N_1 + k(n_2/N_2) = 1.$$  \hfill (q)

This is a modification of the linear damage-rule in which $k$ could be evaluated from conventional $S-N$ data.

The linear and non-linear theories reviewed to this point have one particular inadequacy in common. They do not provide any method for evaluating the effect of damage on the endurance limit, but are concerned only with the effect on endurance life at stress levels above the endurance limit. Nevertheless as previously mentioned it has been clearly shown that reduction of the endurance limit is one consequence of fatigue damage.
The first theory to evaluate the total effect of fatigue damage on the fatigue characteristics was suggested by Henry (reference 49). He also suggested a definite, physical definition for damage.

Henry defined damage as the ratio of the reduction of the endurance limit to the endurance limit before the damaging treatment. That is,

$$D = \frac{(E_0 - E)}{E_0} \quad (r)$$

where $D$ is damage, $E_0$ is the endurance limit before the damage is applied and $E$ is the endurance limit after damage is applied. He further assumed that the conventional S-N curve could be represented by the equation,

$$N = \frac{k_0}{(S - E_0)} \quad (s)$$

where $N$ is cycles to cause failure, $S$ is stress-amplitude, and $k_0$ is a constant for the material and testing conditions. He related endurance-limit and endurance-life damage by assuming that $k_0$, in this equation, changes in proportion to the change in $E$ as damage accumulates. That is, after some repeated stressing above the original endurance limit the material would have an S-N curve,

$$N = \frac{k}{(S - E)} \quad (t)$$

where $k$ and $E$ are reduced from the original values and

$$\frac{k}{k_0} = \frac{E}{E_0} \quad (u)$$

From these assumptions he derived the relationship,

$$D = \frac{\beta}{1 - (1/\gamma)(1 - \beta)} \quad (v)$$

where $\beta$ is the "cycle-ratio" and $\gamma$ is the "stress ratio", that is,

$$\beta = \frac{n}{N}$$

and

$$\gamma = \frac{(S - E_0)}{E_0} \quad (w)$$
Equation (v) provides a means of calculating the reduction of the endurance limit due to repeated stressing at a stress-amplitude $S$ for $n$ cycles if the endurance limit and the life at the damaging stress-amplitude are known. If the constant, $k_0$, for the S-N curve is known before damage then the new value, $k$, can be calculated from (t) and the new S-N curve is then known.

Equation (v) when plotted for various values of $\gamma$ and $\beta$ provides a family of damage curves with the same general configuration as that used in the formation of most of the other non-linear damage theories. Henry drew curves calculated by this equation through some of the data presented by Kommers and by Bennett. There was remarkably good agreement with these data and this method appeared to provide a highly useful method of evaluating fatigue damage for sequences of repeated loading. The method requires that the sequential loading be divided into periods of constant-amplitude stressing and then equation (v) can be successively applied to each period.

Equation (v) predicts failure when the "damage" becomes unity which is equivalent to the endurance limit having the value zero. In most cases, as in the data presented in Henry's paper, the endurance limit does not appear to approach zero as the cycle-ratio approaches one. This is an important aspect of the problem and is discussed further in later chapters. It should also be noted that Henry's equation provides no method for evaluating fatigue damage due to a continuously varying stress-amplitude except by a step-wise approximation which would require many applications of equation (v).
3. Random Loading.— In many situations where fatigue is the usual mode of failure the load amplitude may vary in a highly erratic and unpredictable manner. For example, the load amplitude in many parts of an aircraft depends on the magnitude of the gust velocity. The amplitude of such a load cannot be expressed as a function of the number of gusts encountered. However, this kind of loading may be approximated by assuming that each successive value of load amplitude represents a random selection from a population composed of the possible values which can occur in proportion to their observed relative frequency of occurrence.

Two different approaches have been used in evaluating the accumulation of fatigue damage under this kind of loading. One approach is to develop a test procedure which applies a repeated load approximating the distribution of the random loading to be evaluated. The results of these tests can then be used to evaluate fatigue damage under similar random loading in much the same manner as conventional S-N data are used to evaluate fatigue damage under constant amplitude loading. The work of Gassner and that of Head and Hooke are examples of this approach (references 50, 51, 52).

A second approach is to develop a theory of cumulative fatigue damage which makes it possible to evaluate the accumulation of fatigue damage under random loading from data acquired by means of constant amplitude tests. This second method, if successful, is by far the more desirable whether the design requires the crudest of estimates or as accurate an evaluation as possible. In the former case a rational routine design method would permit design on the basis of existing constant am-
plitude data. In the latter case the required testing of scale or full size models would be possible at a great reduction in time and cost if only constant amplitude tests were required.

An example of this second approach is the work of Lundberg, referred to previously, in which he accepts the linear damage rule and builds a design method based on it (reference 45). Further discussion and application of this method is contained in a paper by Lundberg and Eggwerts (reference 3, p. 255).

Several recent publications have described the extensive work which Freudenthal has carried out in an attempt to develop a correlation between the fatigue damage due to random loading and that due to constant amplitude loading (references 53, 54, 55, 56, and 67). Freudenthal and Heller have considered the accumulation of fatigue damage from several different points of view and in each case have found it was plausible that a relationship of the form

$$(1/V_R)^a = \sum_{i=1}^{i=k} (\omega_i p_i/V_{s1})$$

might exist, where:

- $V_R$ = the number of cycles to failure under random loading,
- $a$ = an exponent designed to produce an approximation of the non-linearity of damage,
- $\omega_i$ = a weight function accounting for the stress interaction effects,
- $p_i$ = the probability of occurrence of the stress-amplitude $S_i$,
- $V_{s1}$ = the life for stress-amplitude $S_i$,
- $k$ = the number of different stress amplitudes which can occur.
Equation (x) is a convenient approximation of a much more complicated expression. The form of equation (x) was suggested on the basis of a probability approach to damage in which it was assumed that a certain probability of failure exists for each cycle and that the probability of failure at a stress level $S_i$ may be increased by cycles of amplitudes greater than $S_i$ (reference 53, p. 56). Equation (x) is also suggested as a generalization of the linear law because if the exponent and the weight function are set equal to one, and if it is noted that $p_i V_R$ is the total number of cycles, $n_i$, for which $S_i$ is applied then (x) becomes

$$\sum_{i=1}^{i=k} n_i/N_i = 1.$$

Equation (y) is identically the linear damage rule.

The derivation of the damage relation of which equation (x) is an approximation was based on the consideration of slip bands occurring within the material. Recent investigations into the mechanism of fatigue were cited to support the assumption that heat, released during individual slip processes, accumulates until local temperature differentials sufficient to cause cracking are induced. It was then assumed that the probability of such crack-producing temperature differentials occurring in a cycle is proportional to $S(S - S_0)$, where $S$ is the stress-amplitude of the cycle and $S_0$ is a stress level below which there is a negligible probability of such heating occurring. Next it was assumed that the number of cycles necessary to cause a given amount of damage is directly proportional to that amount of damage and inversely proportional to $S (S - S_0)$. Then from a consideration of the probability that a crack-causing temperature differential would be produced within a given
number of cycles the probability of the accumulation of a particular amount of damage was evaluated (reference 54).

The test work which Freudenthal is carrying on to evaluate this random loading theory consists of rotating bending tests in which the stress-amplitude is randomly altered between six different values. This testing program is very extensive and some of the results are compared with a method for evaluating damage developed later in this dissertation.

With regard to equation (x) it is interesting to note that Freudenthal has had considerable success in correlating his experimental data with the same value of \( \omega_1 \) for all stress levels (reference 54). While equation (x) represents an approach which appears to provide good agreement with experimental data, nevertheless it requires special testing to evaluate the weight functions and the de-linearizing exponent. A cumulative damage theory for which all coefficients and exponents could be evaluated from constant amplitudes tests for subsequent evaluation of random loading would have many advantages.
Chapter II

A NEW THEORY FOR CUMULATIVE DAMAGE

1. Basic Assumptions.— Fatigue damage in metals is recognized as being both cumulative and progressive in nature. Experimental evidence (cited in the introduction) clearly establishes that many cycles each contribute a small amount of damage which accumulates until failure. Furthermore, the effect of the accumulating damage is concentrated in a relatively small portion of the material until local ruptures occur. Continued accumulation of damage causes these local ruptures to be extended until the net area is insufficient to support the load amplitude. Then sudden rupture of the net area in a single cycle completes failure.

An exact explanation for these characteristics is not known but several different theories have been proposed which are plausible, at least with some materials and under some conditions. Fundamental to all such theories is the realization that actual metals are not homogeneous but are crystalline and include a wide variety of crystal faults, voids, inclusions, and other variations. As a result some points are more unfavorably loaded than others. This may be because a given point is weaker than average or it may be because the transmission of force is unevenly distributed such that a particular point is required to support a large load.

Orowan proposed that the actual mechanism which leads to local failure could be explained on the basis of progressive cold working.
More recent investigations have shown that all materials do not cold work under repeated loading. In fact, Coffin, Coffin and Read, and Polakowski and Palchoudhuri have found that some materials, particularly previously cold-worked materials, soften under repeated loading (references 58, 59, and 60). These and other experimental evidence have lead to the development of alternative explanations for the mechanism of fatigue. The theory proposed by Freudenthal and Heller (referred to earlier) attributes fatigue cracking to high local temperatures and associated thermal stress gradients in slip planes (p. 166 of reference 3).

Whatever the exact mechanism, it may not be overly optimistic to assume that the amount of damage done in any cycle can be evaluated as a function of the grossly observable macroscopic parameters. Fundamental to such an approach is a specific definition of damage. However, from a phenomenological point of view fatigue damage is nothing more than reduction in fatigue strength. Therefore, what is really required is a specific definition of fatigue strength.

There are many different strengths which have been defined and used to advantage in engineering, particularly in machine design calculations; for example, ultimate tensile strength, yield strength, etc. In each case the word strength denotes the magnitude of a force which will cause a particular phenomenon to occur. Usually it has been found convenient to express the numerical values of such strengths on an intensive basis. That is, the ultimate tensile strength of a bar is most usually expressed in terms of the pounds of force per square inch of cross-section rather than in terms of total pounds force. This same ap-
proach has been followed in relation to fatigue strength. In this case
the phenomenon is mechanical failure after a specified number of cycles
of the load.

A load may be cycled in any number of ways and failure may be by
definition associated with complete rupture, the existence of a crack of
specified size, or some other indicator. It is therefore necessary to
specify these details if there is any doubt what conditions were used.
Fatigue strengths measured in laboratory tests are usually specified for
fully reversed loading with failure considered to be complete rupture.
Also, as with strengths for static conditions, the intensive values of
fatigue strengths are usually expressed in terms of maximum nominal
stresses calculated from the equations of strength of materials.

If, then, fatigue strength is defined as the nominal stress which
will cause failure in a specified number of fully-reversed cycles, it is
apparent that a different strength may be associated with each different
number of cycles. A plot of all such strengths versus the corresponding
lives is, of course, the S-N curve.

As discussed in section 4 of the Introduction it has been experi-
mentally established that the strengths associated with all lengths of
life, including the highest stress-amplitude associated with infinite
life are reduced by accumulation of fatigue damage. Therefore each of
these fatigue strengths could plausibly be suggested as a measure of the
damage accumulated. The highest nominal stress-amplitude associated with
an infinite life has, however, particular significance in that it is the
stress level below which no damage is done, i.e., the endurance limit.
Therefore, it is particularly plausible and expedient to define damage
as reduction in the endurance limit.
This is very similar to the definition proposed by Henry (reference 49). He defined a "damage ratio" as the ratio of the reduction in the endurance limit to the virgin endurance limit. For purposes of clarity and reference the definition of damage which is assumed in the following work is stated as follows:

First Hypothesis - The fatigue damage accumulated during any load cycle can be measured uniquely by the difference between the endurance limit before the application of that cycle and the endurance limit after the application of that cycle. This may be stated mathematically as

\[ \Delta D = -\Delta E = E_1 - E_2 \]  

where

\[ D \] = damage,

\[ \Delta D \] = an increment of damage,

\[ E_1 \] = the endurance limit before the damaging is done,

\[ E_2 \] = the endurance limit after the damaging is done.

It should be emphasized that in accord with this hypothesis the endurance limit is a variable.

In considering the parameters which should appear in a mathematical expression to evaluate \( \Delta E \) for any cycle several factors are pertinent.

1. By definition of the endurance limit, stress-amplitudes smaller than the endurance limit do not cause damage.
2. In general, shorter lives, and therefore more accumulation of damage per cycle is associated with higher stress-amplitudes.
3. If a function is to measure the damage accumulated in any cycle and if equal amounts of damage accumulated at different
stress-amplitudes are to be equivalent, then the function can not contain cycles as an explicit parameter.

4. The function should if possible contain parameters which are easily evaluated.

In accordance with these factors a function of the amount by which the stress-amplitude exceeds the instantaneous value of the endurance limit appears plausible. A function is desired which is simple but which provides some degree of generality in terms of constants to be evaluated from experimental data. The assumption of such a function for the present development provides the second hypothesis.

Second Hypothesis - The amount of fatigue damage accumulated in any cycle is proportional to a power of the amount by which the stress-amplitude in that cycle exceeds the current value of the endurance limit. That is

\[ \Delta D \propto (S - E)^{p+1}, \]  

where \( p \) is a constant.

The first and second hypotheses deal with the damage accumulated in one cycle, and equation (2) states a proportionality involving an increment of damage per cycle. It follows that as each successive cycle occurs an additional increment of damage is accumulated. Therefore the total amount of damage will increase with the independent variable, cycles. Clearly one cycle is an increment, \( \Delta n \), of the total cycles that occur.

Therefore it follows that equation (2) can be rewritten as,

\[ \frac{\Delta D}{\Delta n} = \frac{\Delta E}{\Delta n} = k(S - E)^{p+1}, \]  

in which \( k \) is a constant of proportionality and \( \Delta n \) is one cycle.
The number of cycles required to cause fatigue failure in metals and alloys is usually large. Under typical conditions failure may occur after 100 to 10,000,000 cycles depending on the stress-amplitude. Furthermore the endurance limit of typical metal alloys clearly can be reduced thousands of psi before failure. Therefore the ratio $\Delta D/\Delta n$ represents a ratio of two relatively very small quantities. This ratio is suggestive of a derivative and not too great an amount of optimism is required to consider this as a possibility. This forms the basis of the third hypothesis.

**Third Hypothesis** - The amount of damage accumulated in one cycle can be considered to represent the rate of change of damage with cycles, where "cycles" is considered the independent variable and damage is the dependent variable.

From equation (3) and the third hypothesis it follows that

$$\frac{dE}{dn} = -k(S - E)^{p+1}.$$  \hfill (4)

This is a differential equation which expresses the rate of change of the endurance limit as a function of the amount that the stress-amplitude exceeds the endurance limit. It should be noted that in the derivation of equation (4) the phenomenon of coacting has been neglected. Therefore, the increase in the endurance limit indicated by equation (4) when $S < E$ must be disregarded. Furthermore, since the equation is only valid when the applied stress-amplitude is greater than the current value of the endurance limit a "cycle" in the sense of equation (4) only occurs when $S > E$.

The solution of equation (4) also requires that the stress-ampli-
tude be known as a function of "cycles". That is, a function

$$S = f(n),$$

(5)

must be substituted into (4) corresponding to particular loading condition to be considered. Also consideration must be given to the initial and terminal conditions associated with equation (4).

When \( n = 0 \), and no damaging cycles have been applied, there has been no reduction in the endurance limit. Therefore the initial condition is that when \( n = 0 \), \( E = E_0 \) = the virgin endurance limit.

The terminal condition may at first thought also appear to be clearly defined. When \( n = N \) = the number of cycles until failure, the structure has failed and manifestly the endurance limit will be zero. However, the final cycle which results in failure represents a discontinuity to which equation (4) does not apply.

Typically a fatigue failure is terminated by sudden rupture of a large portion of the cross-section in the last cycle. Clearly the assumption of a differential for this last \( \Delta D \) is not warranted. In fact the damage accumulated in each of the last several cycles may be too large to justify this assumption. However, even several cycles will usually be small compared with \( N \), so that no serious error may be involved in assuming that the equation holds until \( n = N \), if a value for \( E_N \), the final value of the endurance limit, is selected which is somewhat larger than zero.

The order of magnitude of \( E_N \) may be estimated by comparing the last cycle with the first cycle. At the last cycle the nominal stress-amplitude, \( S \), evidently represents the instantaneous value of the nominal ultimate strength because rupture occurs in that cycle. Also at the last cycle \( E_N \) is by definition the endurance limit.
At the first cycle the endurance limit is $E_0$ and the ultimate strength is equal to the usual static tensile strength, $S_u$. Although several indeterminate effects, such as stress concentration and speed effects are present, there is likely some correlation between the ratio of the endurance limit to ultimate strength at the first cycle with the same ratio at the last cycle. If it is assumed that this ratio is a constant the terminal boundary condition is when $n = N$, $E = E_N = S(E_0/S_u)$.

2. **Constant Stress-Amplitude.**—Much of the fatigue data available has been obtained by constant stress-amplitude tests. Therefore, the case of $S = f(n) = a$ constant is of particular interest. When $S$ is a constant, equation (4) can be solved by separating variables and integrating as follows.

$$\frac{dE}{dn} = -k(S - E)^{\rho+1}$$

$$\int (S - E)^{-\rho} dE = -k \int dr + C_1$$

$$\frac{(S - E)^{-\rho}}{\rho} = -kn + C_1 \quad (6)$$

Then considering the initial condition that $E = E_0$ = the virgin endurance limit when $n = 0$, the constant of integration, $C_1$, can be evaluated from equation (6).

$$\frac{(S - E_0)^{-\rho}}{\rho} = C_1 \quad (7)$$

When equation (7) is substituted into equation (6) the result provides a functional relationship between the endurance limit and cycles of damage at stress-amplitude $S$.

$$kn = \frac{(S - E_0)^{-\rho}}{\rho} - \frac{(S - E)^{-\rho}}{\rho} \quad (8)$$
or,

\[ \rho kn = (S - E_0)^{-\beta} - (S - E)^{-\beta}. \] (9)

The use of equation (9) to evaluate the number of cycles required
to produce a given damage or the amount of damage done by a given number
of cycles requires that the constants \( E_0, k, \) and \( \rho \) all be previously de-
termined. \( E_0 \) can be evaluated by conventional endurance limit tests
such as the "Up and Down" method (reference 61).

The constants \( \rho \) and \( k \) could be evaluated by applying the terminal
boundary condition to equation (9) and comparing the resulting equation
with conventional \( S-N \) data. Substituting the terminal condition that
\[ E = E_N = \frac{E_0}{S_u}S \text{ when } n = N \text{ into equation (9) yields equation (10).} \n
\[ \rho kn = (S - E_0)^{-\beta} - (S - E_N)^{-\beta}. \] (10)

This equation provides a functional relationship between any constant
stress-amplitude and the number of cycles to failure at that amplitude.
That is, it has the form of an equation for the conventional \( S-N \) curve.

When two pairs of associated values of \( S \) and \( N \) from an experi-
mental \( S-N \) curve are alternately substituted into equation (10), two
equations result. The two unknown constants \( \rho \) and \( k \) could be evaluated
from these two equations if the magnitudes of \( E_0 \) and \( E_N \) were known. If
\( \rho \) and \( k \) were evaluated the respective values could be substituted into
equation (9) to provide a complete equation for evaluating reduction in
the endurance limit resulting from cycles of overstress.

Alternatively, equation (10) can be solved for \( k \) and the solution
substituted into equation (9). This will eliminate \( k \) from equation (9)
and the resulting equation provides a relation between damage and cycles
in terms of the constants, \( E_0, E_N, \) and \( \rho \).
If this substitution is carried out the result is as follows.

\[
\left[ (S - E_0)^{-\rho} - (S - E_N)^{-\rho} \right] n/N = \left( S - E_0 \right)^{-\rho} - \left( S - E \right)^{-\rho} .
\]  

(11)

A small amount of rearranging puts this equation in a form convenient for solving for \( n \) when a value of \( E \) is specified.

\[
n = N \left[ \frac{(S - E_0)^{-\rho} - (S - E)^{-\rho}}{\left( S - E_0 \right)^{-\rho} - (S - E_N)^{-\rho}} \right] .
\]

(12)

Or the equation can be solved for \( E \) to provide an expression for calculating the endurance limit after some arbitrary number of cycles have been applied.

\[
E = S \left[ \frac{1}{(1 - n/N)(S - E_0)^{-\rho} + n/N(S - E_N)^{-\rho}} \right]^{1/\rho}
\]

(13)

Most testing programs designed to evaluate the damaging effects of constant stress-amplitude include one or more of the following three types of tests:

1. Determination of conventional S-N curves,
2. Determination of the decrease in the endurance limit with cycles of constant stress-amplitude,
3. Determination of the life at one stress-amplitude after damaging for a certain number of cycles at another stress-amplitude.

Assuming that all three types of data are available for the same material and loading situation, the theory can be checked in the following manner. Equation (10) can be compared directly with the S-N data (type 1) and the constants determined. Also the range of fit of equation (10) with the experimental curve can be evaluated.
Then using the values of $p$, $k$, and $E_o$ found from the S-N data, equation (13) can be used to calculate $E$ vs. $n$ curves to compare with data from the type 2 tests. Finally repeated application of equation (13) can be used to determine the effect of damage at one stress level on the life at a second stress level.

Although the data available leaves much to be desired an evaluation of this type is carried out in Chapter III. It is possible, however, to gain a general appraisal of the theory by comparing equations (10) and (13) with some of the more empirical equations which have been used to correlate constant stress-amplitude fatigue data in the past.

For example, if the constant $P$ is assumed to be equal to one, that is, if the differential equation (14) can be written

$$\frac{dE}{dn} = -k(S - E)^2,$$  (14)

then equation (10) becomes

$$kN = \frac{1}{S - E_0} - \frac{1}{S - E_N}.$$  (15)

From the form of equation (14) it appears that the endurance limit will be decreasing rapidly as $n$ approaches $N$. This is consistent with experimental observations of fatigue crack growth just prior to failure. Therefore a relatively large error in the value of $E_N$ may not have a large effect on the predicted variation of $E$. An approximation for the terminal boundary condition may therefore be that $E_N = 0$. If this value of $E_N$ is substituted into equation (15) the results can be arranged as follows:

$$(S - E_0)N = (1/k)(E_0/S).$$  (16)
Equation (16) can be compared to
\[(S - E_0)N = K,\]
which has often been used to approximate S-N curves (for example, references 33 and 17). Equation (17) has two primary faults. First, it calls for fantastically large stress-amplitudes to be associated with short lives. Second, even at relatively low stress-amplitudes the curves calculated by equation (17) tend to be steeper than many experimental S-N curves for metals.

Equation (16) also has the first fault and even if \(\rho\) is evaluated from data this characteristic of equation (16) will not be altered. However, if the equation fits the data through the range considered it may be satisfactory. The effect of the term \((E_0/S)\) on the right side of equation (16) tends to flatten S-N curves calculated from equation (16) as compared to equation (17) so that somewhat better agreement with data may be expected for low and moderate overstress.

A comparison of the curves defined by equations (15) and (16) indicates one of the consequences of replacing the terminal boundary condition \(E = E_N = C_S\) when \(n = N\) with the more approximate condition \(E = 0\) when \(n = N\). As mentioned, equation (16) indicates infinitely large forces may be supported for very short lives, whereas equation (15) is asymptotic to a line parallel to the life axis at \(S = E_0/C\). The stipulation that \(E = 0\) when \(n = N\) means that the endurance limit must be reduced from \(E_0\) to 0 in \(N\) cycles. Necessarily if \(N\) is very small, a very large amplitude would be required. However, a much more logical condition is that sudden rupture will result if a very large stress is applied without any substantial occurrence of the mechanisms of fatigue
damage. The terminal boundary condition, \( E = E_N = CS \) when \( n = N \) is compatible with that reasoning. Therefore, the condition \( E = 0 \) when \( n = N \) should be considered as an approximation suitable for only relatively low overstress.

The assumptions of \( \rho = 1 \) and \( E_N = 0 \) can also be used to simplify equation (13) as follows:

\[
E = S - \frac{1}{\frac{1 - n/N}{S - E_0} + (n/N)S},
\]

\[
E = \frac{S(1 - n/N)}{(S/E_0) - n/N}.
\]

Equation (19) can be written in a somewhat more convenient form by introducing the following notation. Let the ratio of the number of cycles applied to the number of cycles required for failure be called the "cycle-ratio" and represent this ratio by \( \beta \). (The term "cycle-ratio" was probably first introduced by Kommers in reference 21.) Also let the ratio of the stress-amplitude to the virgin endurance limit be called the "overstress ratio" and represent this ratio by \( \gamma \). (The term "overstress ratio" was used by Henry with a somewhat different definition in reference 49.) Finally let the ratio of the endurance limit to the virgin endurance limit be called the "endurance limit ratio" and represent this ratio by \( R \). Summarizing the above terms,

\[
\begin{align*}
\beta &= n/N, \\
\gamma &= S/E_0, \\
R &= E/E_0.
\end{align*}
\]

If the expressions in equations (20) are introduced into equa-
tion (19) the resulting expression is

\[
R = \frac{\gamma(1-\beta)}{\gamma - \beta}.
\]  \hspace{1cm} (21)

This equation is the same equation which Henry developed in an entirely different and independent manner (reference 49). His definitions are somewhat different so that a slight rearrangement is necessary to show the identity of equation (20) and Henry's equation.

A close comparison of equation (21) with experimental data for steel will suggest certain inadequacies in the equation. Some of these may be improved by evaluating \( \rho \) from the data and by revising the terminal boundary condition. However, consistent with the equation for the S-N curve, it would be expected that the equation might be applicable for low overstress.

Evaluation of the consequences of the three basic hypotheses will depend on a comparison of the solutions of equation (4) with experimental data. As mentioned above such a comparison is carried out in Chapters III and IV. However, even on the basis of the general results obtained for constant stress-amplitude loading a consideration of the solutions of equation (4) for other types of loading is indicated.

3. Uniformly Increasing Stress-Amplitude.-- Due to the interest in the Prot accelerated method of measuring the endurance limit, considerable data are available from tests in which the stress-amplitude was increased uniformly with cycles. This kind of loading can be expressed in terms of equation (5) as

\[
S = f(n) = A + \alpha n, \hspace{1cm} (22)
\]

where \( A \), the initial amplitude, is less than \( E_0 \) and \( \alpha \) is the rate at which the amplitude is increased.
Substituting this function of \( n \) into equation (14) for \( S \), the result is

\[
\frac{dE}{dn} = -k(A + an - E)^{\rho+1}.
\] (23)

From the second hypothesis it follows that no damage will occur until \( S > E_0 \). Therefore the expression \( A + an \) can be replaced by \( E_0 + an \) with the understanding that \( n \) is measured from the time that \( S = E_0 \).

Therefore equation (23) can be rewritten as

\[
\frac{dE}{dn} = -k(E_0 + an - E)^{\rho+1}.
\] (24)

As a first step in the solution of equation (24) let

\[
(E_0 + an - E) = Z.
\] (25)

Then

\[
\frac{dE}{dn} = a - \frac{dZ}{dn}.
\] (26)

Substituting equations (26) and (25) into equation (24) gives

\[
\frac{dZ}{dn} = a + k(Z)^{\rho+1}.
\] (27)

The variables are now separated and the equation is in a form which can be integrated.

The results for the case of constant stress-amplitude indicate that the value of \( \rho \) is probably close to one. Therefore if \( \rho \) is assumed equal to one in equation (27) a solution for Prot loading will be obtained which is comparable with equations (16) and (21) for constant stress-amplitude.

When this substitution is made and the variables in (27) are separated, the result is

\[
\int \frac{dZ}{(a + kz^2)} = \int dn + C.
\] (28)
This can be written
\[ \frac{1}{k} \int \frac{dz}{d/k + z^2} = \int dn + C , \]
and directly integrated to
\[ \sqrt{\frac{1}{ak}} \tan^{-1} Z \sqrt{k/a} = n + C . \] (29)

The constant C can be evaluated from the initial condition that
\[ E = E_0 \] when \( n = 0 \) and equation (25). Substituting \( E = E_0 \) and \( n = 0 \) into equation (25) gives
\[ Z = E_0 + 0 - E_0 = 0 . \] (30)
Therefore \( Z = 0 \) when \( n = 0 \), and replacing Z and n by zero in equation (29) gives
\[ C = 0 . \] (31)
Then replacing C and Z in equation (29) with their values from equations (25) and (31) the solution becomes
\[ (E_0 + an - E) \sqrt{k/a} = \tan \sqrt{ak} n . \] (32)
Solving this equation for \( E \) yields,
\[ E = E_0 + an - \sqrt{a/k} \tan \sqrt{ak} n . \] (33)

Now the approximate terminal condition that \( E = 0 \) when \( n = N \) at failure can be applied to give
\[ 0 = E_0 + aN - \sqrt{a/k} \tan \sqrt{ak} N . \] (34)
For the type of loading considered the stress-amplitude at failure, \( S_f \), will be equal to \( E_0 +aN \). Therefore equation (34) can be written
\[ S_f = \sqrt{a/k} \tan \sqrt{ak} N . \] (35)

In the development of his accelerated method of testing, Prot had concluded that the failure stress could be given by an equation of the
Equations (35) and (36) can be compared with each other and with data by plotting them on rectangular coordinates with $S_f$ measured along the ordinate and $\sqrt{a}$ measured along the abscissa. This is done in Chapter IV where considerable attention is also given to the comparison of values of $k$ obtained from both constant amplitude and Prot type data. However, a rough comparison of equations (35) and (36) can be obtained for at least one material as follows.

Typically Prot type tests are conducted with the stress-amplitude increasing about 0.01 psi per cycle so that $\sqrt{a} = 0.1$. For materials such as high strength steel, the endurance limit may be approximately 80,000 to 90,000 psi and the Prot failure stress has been found to be approximately 8,000 psi above the endurance limit for $a = 0.01$. Therefore, the failure stress in a Prot test might be about 100,000 psi. Such a material may have a fatigue life of about $10^5$ cycles for an overstress ratio, $S/E_0$, of around 1.2.

A typical value of $k$ can be estimated from equation (16) by solving for $k$ to give

$$k = \frac{E_0}{S} \frac{1}{N(S - E_0)} = \frac{1}{\gamma N E_0 (\gamma - 1)}$$

(37)

where

$$\gamma = \frac{S}{E_0}.$$

Then substituting the estimated values for $\gamma$, $N$, and $E_0$ into equation (37)
Now in equation (33) the endurance limit is given as a function of \( n \).

If the approximate terminal condition \( E = 0 \) at failure is still assumed then as \( n \) increases the term \( \sqrt{\alpha/k} \tan \sqrt{\alpha/k} n \) increases until at failure it becomes as large as \( E_0 + \infty \). The value of \( \tan (\sqrt{\alpha/k} n) \) at failure for a material corresponding to typical values of test data can be found by solving equation (35) for \( \tan \sqrt{\alpha/k} N \).

\[
\tan \sqrt{\alpha/k} N = \sqrt{k/\alpha} S_f
\]

Then substituting the typical values given above into the right side of equation (39),

\[
\tan \sqrt{\alpha/k} N = \frac{\sqrt{2 \times 10^{-10}}}{0.1} \times 100,000 \approx 450.
\]

When the function \( \tan x \) is as large as 450, the argument, \( x \), is very close to \( \pi/2 \). From a table of tangents

\[
\tan^{-1} 450 \approx 1.5699 \approx 1.57
\]

and also \( \pi/2 \approx 1.57 \).

Referring to equation (33), failure would surely occur by the time \( \sqrt{\alpha/k} n = \pi/2 \) because \( \tan \pi/2 = \infty \) and therefore \( E \) calculated by equation (33) would certainly have decreased to zero. However the value of \( n \) required to make \( \sqrt{\alpha/k} n = \pi/2 \) is negligibly greater than the value required for \( \sqrt{\alpha/k} n \) to be equal to 1.5699. Therefore a reasonable approximation to the number of cycles, \( N \), until failure can be obtained by assuming that,

\[
k = \frac{1}{1.2 \times 10^5 \times 85,000 \times (1.2 - 1)} = 2 \times 10^{-10}.
\]
\[ \sqrt{ak} N = \pi/2 \]  

at failure.

Multiplying both sides of equation (42) by \( \sqrt{ak} \) gives

\[ aN = \sqrt{ak} \pi/2 . \]  

(43)

Then since \( S_f = E_o + aN \), the relation between \( S_f \) and \( a \) which is approximately predicted by equation (34) is

\[ S_f = E_o + \frac{\sqrt{a} \pi}{2\sqrt{k}} . \]  

(44)

Equation (44) is in exactly the form of equation (36) which indicates that the present theory, at least for some high strength steels, gives results which are approximately in agreement with Prot's assumptions.

Alternatively the more accurate terminal condition that \( E = E_N = C S \) when \( n = N \) at failure can be substituted into equation (33) to obtain the number of cycles required to cause failure.

\[ E_N = E_o + aN - \sqrt{ak} \tan \sqrt{ak} N . \]  

(45)

In the expression \( E_N = C S \) the value for \( S \) is the stress-amplitude at failure, \( S_f \). Therefore

\[ E_N = C S_f = C(E_o + aN) \]  

(46)

since \( (E_o + aN) \) is equal to the stress-amplitude for any number of cycles.

Substituting from equation (46) into equation (45)

\[ C(E_o + aN) = (E_o + aN) - \sqrt{ak} \tan \sqrt{ak} N \]

or

\[ S_f = E_o + aN = \left(\frac{\sqrt{ak}}{1 - C}\right) \tan \sqrt{ak} N . \]  

(47)
Equation (1*7) contains only the material constants \( k \), \( E_0 \), and \( C \); the loading constant, \( \alpha \); and the cycles to failure. Therefore if the material constants have been evaluated, perhaps from constant amplitude tests, and if the rate of load increase is known, equation (1*7) presents a formal method of predicting the number of cycles until failure.

In Chapter IV this solution and the more approximate solution in equation (1*4) are compared with each other and with published data.

4. Random Stress-Amplitude.- It is of particular interest to investigate the possibility of applying equation (4) to those situations where the stress-amplitude is not known as a function of the cycles, \( n \), but can be described as a random selection from a specified population of possible values.

If the specified population consists of a finite number of possible values then that population can be described by a statement of those values and their associated relative frequencies of occurrence. For example, if in a particular loading situation only six values of stress-amplitude, \( S_1 \), \( S_2 \), \( S_3 \), \( S_4 \), \( S_5 \), and \( S_6 \) can occur and if on the average \( S_1 \) occurs \( p_1 \times 100 \) times in every one hundred cycles, \( S_2 \) occurs \( p_2 \times 100 \) times in every one hundred cycles, and in general \( S_i \) occurs \( p_i \times 100 \) times in every one hundred cycles, then this loading situation can be characterized by stating the six \( S_i \) and the six \( p_i \). Similarly any other such random loading may be characterized by stating the \( S_i \) and the associated \( p_i \) so long as the number of distinct \( S_i \) is finite.

This concept of random loading may be extended by considering a situation where all values of \( S \) between \( S_1 \) and \( S_2 \) may occur. In this case a frequency function \( p(S) \) may be defined such that the probability
of any cycle having a value of \( S \) between \( S_1 \) and \( S_2 \) is equal to

\[
\int_{S_1}^{S_2} p(S) \, dS
\]

where \( S_A < S_1 < S_2 < S_Z \). It follows from this definition of the frequency function that

\[
\int_{S_A}^{S_Z} p(S) \, dS = 1.
\]

The development of equation (4) in section 1 of this chapter is based on the assumption that the damage done in any cycle could be expressed as a function of the stress-amplitude and the endurance limit at the time of that cycle. To extend this development to cover random loading requires an additional hypothesis regarding the magnitude of the stress to be substituted into the function for each cycle.

Suppose that the stress-amplitude could have any of \( j \) different magnitudes. These magnitudes can be designated as \( S_1, S_2, S_3, \ldots, S_j \). Any particular magnitude can then be referred to as \( S_i \) where \( i \) can take on all the values 1 through \( j \), inclusive. Then each of the stress-amplitudes, \( S_i \), will have associated with it a probability, \( p_i \). \( p_i \) designates not only the fractional probability that a particular cycle will have the amplitude \( S_i \), but also the fraction of any large group of cycles which will have the magnitude \( S_i \).

According to equation (4) a cycle at the stress-amplitude \( S_i \), when it occurs, will do an amount of damage \( \Delta D_i \) such that

\[
\Delta D_i = k(S_i - E)^{p+1}
\]
with $E$ representing the instantaneous value of the endurance limit. Let $M$ be any large number. Then consider a situation where $M$ machine parts are each subjected to the random loading sequence described by the $S_1$ and the $p_1$. What amount of damage will occur on the first cycle of the sequence if all the $S_1$ are greater than $E_0$? On about $p_1 x M$ of the machine parts the damage will be $k(S_1 - E_0)^{p+1}$, on about $p_2 x M$ of the parts the damage will be $k(S_2 - E_0)^{p+1}$, and in general on about $p_1 x M$ of the parts the damage will be $k(S_1 - E_0)^{p+1}$. Now what will the average damage, $\overline{\Delta D}$, be for the first cycle for all the machine parts? An average damage can be found by adding the damage done on each specimen and dividing by the total number of specimens.

This statement can be represented by an equation in terms of the above notation as follows,

$$
\overline{\Delta D_1} = \frac{\sum_{i=1}^{J} k p_1 M(S_1 - E_0)^{p+1}}{M},
$$

(48)

where the subscript on $\overline{\Delta D}$ designates that this is for the first cycle. Since $M$ is independent of the value of $i$, it can be divided from the top and bottom of the right side of equation (48) with the consequence that,

$$
\overline{\Delta D_1} = \sum_{i=1}^{J} k p_1 (S_1 - E_0)^{p+1}.
$$

In a similar manner the average damage for the second cycle can be calculated from the equation

$$
\overline{\Delta D_2} = \sum_{i=1}^{J} k p_1 (S_1 - E_2)^{p+1},
$$

where the subscript on $E$ designates the average value of $E$ which exists
after the first cycle. This can be repeated for the third and fourth and all succeeding cycles. Therefore a general equation,

$$\Delta D = \sum_{i=1}^{j} k p_i (S_i - E)^{p+1}, \quad (49)$$

can be written which states that the average damage done in any cycle is equal to a weighted average of the damages which would be caused by all the possible stress magnitudes, the weight factor being the probability of occurrence of each stress-amplitude. Such a weighted average is usually referred to as the expected value (reference 62).

This then is the special hypothesis necessary for the case of random loading.

**Fourth Hypothesis** - The damage done in any cycle during the action of a random load is the expected value of damage for that cycle.

In case the frequency distribution of stress-amplitude is known as a function, \( p(S) \), for the full range of possible stress-amplitudes, the expected value can be calculated from the equation,

$$\Delta D = k \int_{S_A}^{S_Z} p(S) x (S - E)^{p+1} dS, \quad (50)$$

where \( S_A \) is the lowest stress-amplitude greater than \( E \) which will occur and \( S_Z \) is the highest stress-amplitude which will occur.

It must be particularly noted that according to the assumptions upon which equation (44) was based

$$\frac{dE}{dn} = - k(S - E)^{p+1}, \quad \text{for } S > E,$$
but
\[ \frac{dE}{dn} = 0, \text{ for } S \leq E. \]

Therefore in the finite sum of equation (49) all terms for which \( S_i \leq E \)
must be neglected and the integral in equation (50) becomes
\[
\int_{E}^{S_Z} p(S) \times (S - E)^{p+1} dS, \text{ for } S_A < E. \quad (51)
\]

If the case of a finite number of possible load amplitudes is con-
sidered, the failure equation for random loading can be developed from
equation (49). Again, because of the results obtained for constant
stress-amplitude and for uniformly increasing stress-amplitude, the con-
stant \( \rho \) is assumed to be equal to one.

From equation (49) and the original definitions it follows that
\[
\frac{dE}{dn} = -k \sum_{i=1}^{j} p_i (S_i - E)^2. \quad (52)
\]

Equation (52) can be expanded and rewritten to yield,
\[
\frac{dE}{dn} = -k \left[ \sum_{i=1}^{j} p_i S_i^2 - 2 \sum_{i=1}^{j} p_i S_i E + \sum_{i=1}^{j} p_i E^2 \right]. \quad (53)
\]

Since the value of \( E \) does not depend on the index of summation, the
right side of equation (53) can be multiplied and divided by \( \sum_{i=1}^{j} p_i \)
to give,
\[
\frac{dE}{dn} = -k \sum_{i=1}^{j} p_i \left[ \sum_{i=1}^{j} \frac{p_i S_i^2}{p_i} - \frac{2 \sum_{i=1}^{j} p_i S_i}{p_i} E + \frac{E^2}{p_i} \right]. \quad (54)
\]

A convenient way to take account of the fact that \((S_i - E)^2\) is
assumed to be zero for $S_i < E$ is to set the $p_i = 0$ corresponding to all $S_i < E$. Therefore if any of the $S_i$ are less than $E$ the summation of the $p_i$ will not equal one. In case all $S_i < E$ then the sum of all the $p_i$ is equal to one and equation (54) is somewhat simplified. In either case equation (54) can be written in the form

$$\frac{dE}{dn} = -k'(c - 2bE + E^2) ,$$

with

$$k' = k\sum_{i=1}^{J} p_i$$

$$c = \frac{\sum_{i=1}^{J} p_i S_i^2}{\sum_{i=1}^{J} p_i}$$

$$b = \frac{\sum_{i=1}^{J} p_i S_i}{\sum_{i=1}^{J} p_i} .$$

For a continuous frequency distribution of stress-amplitude, equation (50) and the definition of damage lead in an exactly analogous manner to equation (55) with the following substitutions of integrals for finite sums

$$\sum_{i=1}^{J} p_i \rightarrow \int_{S_A}^{S} p(S) \, dS ,$$

$$\sum_{i=1}^{J} p_i S_i \rightarrow \int_{S_A}^{S} S \, p(S) \, dS ,$$

60
\[
\sum_{i=1}^{j} p_i s_i^2 \rightarrow \int_{S_1}^{S} s^2 p(s) \, ds.
\]

If equation (55) is rearranged by dividing both sides by \((c - 2bE + E^2)\) and multiplying both sides by \(dn\), the result is directly integrable and may be written

\[
\int \frac{dE}{E^2 - 2bE + c} = - \int k'dn + C_1.
\] (56)

The left integral depends on the relative magnitudes of \(b\) and \(c\). If \(b^2 > c\) the integral will be a logarithmic function while if \(b^2 < c\) the integral will be an inverse tangent function. The expression for \(c\) is proportional to the expected value of the square of the stress-amplitude and the expression for \(b\) is proportional to the expected value of the stress-amplitude. Since the proportionality constant is the same in both cases, i.e., \(\sum_{i=1}^{j} p_i\), the comparison of \(b^2\) with \(c\) is equivalent to comparing the square of the expected value of the stress-amplitude with the expected value of the square of the stress-amplitude. Since \(S\) is a large number the expected value of the square will be larger than the square of the expected value. This can be made plausible by noting that the \(p_i\) are all less than one and occur only to the first power in the expected value of the square while they occur to the second power in the square of the expected value.

Then since \(b^2 < c\), equation (56) integrates to

\[
\frac{1}{\sqrt{c - b^2}} \tan^{-1} \frac{E - b}{\sqrt{c - b^2}} = - k'n + C_1.
\] (57)
The initial condition that \( E = E_0 \) when \( n = 0 \) can be used to evaluate \( C_1 \).

\[
\frac{1}{\sqrt{c - b^2}} \tan^{-1} \frac{E_0 - b}{\sqrt{c - b^2}} = C_1. \tag{58}
\]

When this value of \( C_1 \) is substituted into equation (57) the following relationship between endurance limit and cycles of random load amplitude results.

\[
\frac{1}{\sqrt{c - b^2}} \left[ \tan^{-1} \frac{E_0 - b}{\sqrt{c - b^2}} - \tan^{-1} \frac{E - b}{\sqrt{c - b^2}} \right] = k'n. \tag{59}
\]

If this equation is solved for \( E \) the result is

\[
E = \sqrt{c - b^2} \tan \left[ \tan^{-1} \frac{E_0 - b}{\sqrt{c - b^2}} - k'n \sqrt{c - b^2} \right] + b. \tag{60}
\]

Application of the terminal boundary condition, \( E = E_N \) where \( n = N \), to equation (59) produces the equation for the relationship between the cycles to failure and the characteristic constants for the random loading.

\[
k'N = \frac{1}{\sqrt{c - b^2}} \left[ \tan^{-1} \frac{E_0 - b}{\sqrt{c - b^2}} - \tan^{-1} \frac{E_N - b}{\sqrt{c - b^2}} \right]. \tag{61}
\]

Since \( b > E_0 > E_N \) and \( \tan^{-1} (x) = -\tan^{-1} (-x) \), equation (61) is equivalent to

\[
k'N = \frac{1}{\sqrt{c - b^2}} \left[ \tan^{-1} \frac{b - E_N}{\sqrt{c - b^2}} - \tan^{-1} \frac{b - E_0}{\sqrt{c - b^2}} \right]. \tag{62}
\]

For finite numbers of stress-amplitudes the calculations for \( k' \), \( c \), and \( b \) would be carried out according to the definitions following equation (55). If any of the \( S_i \) were less than \( E_0 \) it would be necessary
to apply equation (59) to determine the number of cycles required to reduce E to the first value of $S_1$ below $E_0$. Then a new set of constants would be determined to calculate the number of cycles required to reduce E to the next lower value of $S_1$. This process would be repeated until E were reduced to $E_N$. The number of cycles required to cause failure of a virgin specimen would then be the total for all the intermediate calculations.

If the stress-amplitude can take any value of S between $S_A$ and $S_Z$, and if therefore the constants c and b must be calculated from the frequency function $p(S)$, considerable complications will result if $S_A$ is less than $E_0$. In the first place $p(S)$ may be a function such that the integrations indicated in the corresponding definitions for c and b can not be performed analytically. In the second place if $S_A$ is less than E the lower limit on these integrals becomes E instead of $S_A$. This may introduce the variable E into the differential equation in a complicated manner such that an analytical solution to the differential equation will not be possible even if the constants can be evaluated as functions of E.

If numerical methods are to be used it is more convenient to go back to equation (50) and the original definitions to obtain, analogous to equation (52)

$$\frac{dE}{dn} = -k \int_{S_A}^{S} p(S) x (S - E)^{p+1} dS .$$

This equation could then be used in conjunction with the initial and terminal boundary conditions to determine the number of cycles required until failure. In this calculation it would also be possible to use a value of $p$ different from one if desired.
The method of solution would be to select a sequence of values of $E$ between $E_0$ and $E_N$ and graphically evaluate the integral for each value of $E$. This would provide pairs of values of $dE/dn$ and $E$. These values could be plotted and a second graphical integration performed to determine $N$.

In every instance it would be necessary to determine the value of the endurance limit at failure, $E_N$. For non-random loading it was assumed that $E_N = C \times S \approx (E_o/S_u) \times S$.

The terminal condition was originally arrived at by considering that the ultimate strength as well as the endurance limit would be reduced by fatigue damage. Then whenever a load amplitude exceeded the ultimate strength sudden failure would occur. That is, when the stress-amplitude exceeds $E \times 1/C$ failure would be predicted. Therefore for random loading the maximum value of $S$ which occurs must be used to calculate $E_N$.

Although not directly stated it has been assumed in the development that random loading can be approximated by a complete cycle of each random amplitude. This is an approximation of considerable importance and must be considered in a final evaluation of the present method.

When assuming that the average effect of random loading can be approximated by considering the expected value of damage to occur at each cycle, it must be recognized that some specimens will be subjected to a much worse load sequence and some to a much easier sequence than the average. Therefore, not only will the usual variability in the material tend to cause scatter but variations in the load sequences actually applied to different machine parts or specimen will also result in substantial scatter.
For that reason it will be necessary to statistically analyze not only the usual variability of the material but also the variability of the random load sequence before predictions of life other than the average life can be made.

In Chapter IV equation (62) is compared with some published data from tests designed to approximate random loading.
Chapter III

EVALUATION OF THE THEORY FOR CONSTANT STRESS-AMPLITUDE

1. The Relationship Between Life and Stress-Amplitude. - As previously mentioned most testing programs designed to provide fundamental information on fatigue damage for constant amplitude loading provide one or more of the following types of information:

1. The relationship between stress-amplitude and the number of cycles to cause failure,
2. The reduction of the endurance limit due to cycles of damage at a given stress-amplitude,
3. The life at one stress-amplitude subsequent to a certain number of cycles of damage at some other stress-amplitude.

In the published literature some data are available for each of these three types of investigations. For the most part these data have been obtained by tests of smooth, polished specimens subjected to rotating bending. Most commonly the specimens tested are made of steel and have a critical section which is between one-eighth and one-third inch in diameter. Except where otherwise noted all of the data considered in this evaluation were obtained from such specimens and tests.

To permit uniformity of presentation and for convenient discussion and reference these data will, as far as possible, be presented in terms of non-dimensional parameters. Type (1) data will usually be presented on semi-logarithmic coordinates. However, the linear ordinate will be
the overstress ratio, $\gamma$, instead of the stress-amplitude. (The overstress ratio is equal to the magnitude of the stress-amplitude divided by the virgin endurance limit, and the virgin endurance limit is the endurance limit at the beginning of the constant stress-amplitude test, i.e. when $n = 0$. ) The cycles to failure will be measured along the logarithmic ordinate. These data will be referred to as "$\gamma$-N" data.

All type (2) data will be presented on linear coordinates in terms of the endurance ratio, $R$, and the cycle-ratio, $\beta$. (The endurance ratio is equal to the current value of the endurance limit divided by the virgin endurance limit, and the cycle-ratio is equal to the number of cycles which have been applied at a given stress-amplitude divided by the number of cycles required to cause failure at the same stress-amplitude.) These data will be referred to as "$R-\beta$" data.

All type (3) data will also be presented on linear coordinates. The value of the cycle-ratio, $\beta$, corresponding to the damage treatment at the first stress-amplitude will be called $\beta_1$ and plotted along the abscissa. The value of $\beta$ at which failure occurs at the second stress-amplitude will be called $\beta_2$ and plotted along the ordinate. (With this definition the number of cycles, $n_2$, applied at the second stress-amplitude will be equal to $\beta_2 \times N_2$ where $N_2$ is the average number of cycles required to cause failure of virgin specimens when only the second stress-amplitude is applied.) These data will be referred to as "$\beta_1-\beta_2$" data.

Figures 3 through 10 present $\gamma$-N data for materials which were also used for other types of damage evaluation tests. Therefore these curves provide a chance to evaluate theoretical $\gamma$-N curves in conjunction with the evaluation of $R-\beta$, $\beta_1-\beta_2$, and other theoretical curves.
Theoretical S-N Curve and Data from Constant Amplitude Tests on SAE 1030 Steel

From reference 41

- $E_0 = 37,200$ psi
- $E_0/S_u = 0.46$
- $K = 2.4 \times 10^{-5}$
- $\rho = 1$
- $C = 0.4$
Fig. 4. Theoretical S-N Curve and Data from Constant Amplitude Tests on Steel X
Fig. 5. Theoretical S-N Curve and Data from Constar: Amplitude Tests on Steel Y.
$E_c = 34,000 \text{ psi}$

$E_0/S_u = 0.73$

$k = 2.83 \times 10^{-5}$

$\rho = 1$

$C = 0.4$

From reference 40

Fig. 6. Theoretical and Data from Constant Amplitude Tests on Steel Z
**Fig. 7.** Theoretical S N Curve and Data from Constant Amplitude Tests on A 7 Steel
Fig. 8. Theoretical S-N Curve and Data from Constant Amplitude Tests on SAE 4340 Steel
Overstress ratio, $\gamma$

$KN = \frac{1}{\gamma - 1} \cdot \frac{1}{\gamma(1-C)}$

$E_0 = 39,000$ psi

$E_0/S_u = 0.37$

$K = 1.44 \times 10^{-5}$

$\rho = 1$

$C = 0.4$

Notched specimens.

Failure defined when crack area was about 12% of nominal area.

Each point represents median of 11-17 tests.

From reference 22.

Fig. 9. Theoretical S-N Curve and Data for Constant Amplitude Tests on X4130 Steel
Fig. 10. Theoretical S-N Curve and Data from Constant Amplitude Tests on SAE 4140 Steel
Fig. 11. Theoretical \( \gamma \) L Curves and Composite Data from Constant Amplitude Tests on Steel
In each of Figures 3 through 10 the curves are theoretical curves of the form

\[ KN = \frac{1}{\gamma - 1} - \frac{1}{\gamma(1-c)} \]  

(63)

This equation is obtained by multiplying both sides of equation (15) by the virgin endurance limit, \( E_0 \). Therefore equation (63) results from the solution of the basic damage relationship given in equation (14) with the assumptions that \( \rho = 1 \) and \( E_N = CS \). The constant \( K \) was chosen by simple inspection in each case to give a "best" fit over the entire range.

The effect on the curve of changing the value of the constant, \( K \), can be illustrated by considering two values of the constant, \( K_1 \) and \( K_2 \). Necessarily there is another constant, \( r \), such that \( K_1 = rK_2 \). Then for a given value of \( \gamma \) and \( C \) the value of \( N \) corresponding to \( K_1 \) is

\[ N_1 = \frac{1}{K_1} \left( \frac{1}{\gamma - 1} - \frac{1}{\gamma(1-c)} \right) \]  

(64)

and the value of \( N \) corresponding to \( K_2 \) is

\[ N_2 = \frac{1}{K_2} \left( \frac{1}{\gamma - 1} - \frac{1}{\gamma(1-c)} \right) \]  

(65)

But since \( K_2 = K_1/r \), equation (65) can be written

\[ N_2 = \frac{r}{K_1} \left( \frac{1}{\gamma - 1} - \frac{1}{\gamma(1-c)} \right) \]  

(66)

From this it follows that \( N_2 = rN_1 \).

Now since \( N \) is plotted to a logarithmic scale the actual distance along the life axis from the origin to any point \( N \) is proportional to the logarithm of \( N \). Therefore the distance between the two points \( N_1 \) and \( N_2 \) will be proportional to the difference in the logarithms of the two
lives. Then if \( x \) is the actual distance between the location of the two points,
\[
x = A(\log N_2 - \log N_1),
\]
where \( A \) is a proportionality constant depending on the life scale. Substituting \( N_2 = rN_1 \), this equation can be rewritten
\[
x = A(\log r - \log N_1)
\]
\[
= A \log N_1 + A \log r - A \log N_1
\]
\[
= A \log r. \tag{67}
\]
Since this is true for any value of \( \gamma \), the only effect on the theoretical \( \gamma-N \) curve due to changing \( K \) is a uniform shift of the curve parallel to the life axis. It is therefore easy to pick a value of \( K \) to suit any particular data. A \( \gamma-N \) curve can be calculated for any convenient value of \( K \). If this curve is plotted to the same scales on vellum, the vellum sheet can be placed on top of the plot of the experimental data. Thus the theoretical \( \gamma-N \) curve will be superimposed on the data. The vellum sheet can be shifted to the right or left as required to determine a good average fit over the range of the data. Then corresponding values of \( K \) can be calculated from the values of \( \gamma \) and \( N \) corresponding to any one point on the curve. These values will, of course, be read from the graph on which the data are plotted.

Inspection of Figures 3 through 10 indicate a reasonable fit in all cases. Therefore if all the data from these figures were superimposed on the same graph but with each set of data shifted horizontally as required to make the theoretical curves coincide a composite graph would result.
It therefore appears that if a life parameter is selected which will produce an equivalent shift, all the data could be plotted on one coordinate system. This has been done and is presented in Figure 11.

For each set of data the average values of \( y \) and \( N \) are related by the equation

\[
KN = \frac{1}{\gamma - 1} - \frac{1}{\gamma(1-c)} , \tag{63}
\]

where \( K \) has the appropriate value for that data. Now consider an arbitrary value of overstress, say \( \gamma^* \). The life corresponding to that overstress can be found by substituting \( \gamma^* \) into equation (63),

\[
KN\gamma^* = \frac{1}{\gamma^* - 1} - \frac{1}{\gamma^*(1-c)} , \tag{68}
\]

where the subscript on \( N \) simply designates that it corresponds to \( \gamma^* \).

If equation (63) is divided by equation (68), the result is

\[
\frac{N}{N\gamma^*} = \frac{1}{\gamma - 1} - \frac{1}{\gamma(1-c)} \quad \frac{1}{\gamma^* - 1} - \frac{1}{\gamma^*(1-c)} . \tag{69}
\]

The denominator of the right side of equation (69) is a constant for a given value of \( \gamma^* \). Let this constant be \( k^* \). Furthermore the ratio on the left side of equation (69) can be replaced by a single symbol. Since this ratio is a life ratio it can be fittingly referred to as the "Life Ratio" and represented by \( L \). With this nomenclature equation (69) can be written

\[
K^*L = \frac{1}{\gamma - 1} - \frac{1}{\gamma(1-c)} . \tag{70}
\]

Equation (70) has exactly the same form as equation (63). Further-
more the constant $K^*$ depends only on the reference overstress ratio, $\gamma^*$. Therefore if the same value of $\gamma^*$ is chosen for all of the sets of data $K^*$ will have the same value in each case. If $L$ is plotted along the abscissa on the same semi-logarithmic paper as used for $N$ in Figures 3 through 10 the relative horizontal spacing of all the data in each set will remain unchanged.

In Figure 11 the data from Figures 3 through 9 are plotted on the Overstress ratio vs. Life ratio coordinates, with $\gamma^* = 1.3$. It is evident from the figure that these dimensionless coordinates provide a very successful basis for combining $\gamma$-N data. The range of scatter in Figure 11 is no worse than that in several of the individual sets of data.

Because the composite data defines the $\gamma$-L curve more accurately than the individual sets of $\gamma$-N data, a better evaluation of the theoretical curve is possible. The solid curve in Figure 11 is calculated from equation (70) which is identical in form with equation (63). Although this curve fits the composite data fairly well there is some deviation from what would be a best fit simply drawn by eye through the data.

Equation (63) represents a special case of the more general equation (10). Equation (10) can be put in terms of $\gamma$ by multiplying both sides by $(E_0)^P$. If this is done and if $\rho k(E_0)^P$ is set equal to $K'$ the result is

$$K'N = \frac{1}{(\gamma - 1)^P} \left( \frac{1}{\gamma^P (1 - C)^P} \right). \quad (71)$$

The effect of changing $K'$ is still simply to shift the curve defined by equation (71) horizontally along the life axis. Therefore the
curve can be calculated for any convenient value of $K'$ and the resulting curve can be superimposed on the data and shifted to obtain a best fit.

All of the data plotted in Figures 3 through 11 are based on the original investigator's estimate of the virgin endurance limit, $E_0$. In most cases the testing procedure was apparently to test specimens at successively lower stress amplitudes until one or at most a few run-outs occurred. Then a smooth curve was drawn through the test points. It is very likely that this test procedure will result in a somewhat high estimate of the endurance limit.

Usually the endurance limits determined in such a manner are intended to be average values. That is, if the endurance limits for a large number of specimens were determined, it would be expected that the endurance limits of one-half of them would be above the average value while the endurance limits of the other half would be below the average value. The values of endurance limit would be expected to be scattered above and below the average with a distribution approaching a normal distribution. The total range to include 95 percent of the group would probably extend from at least $0.91E_0$ to $1.06E_0$. (This estimate is based on data reported in reference 31.)

Therefore, if the increment in stress-amplitude were of the order of one or two percent of the endurance limit it might well be expected that a run-out would be encountered at a stress-amplitude somewhat above the actual average endurance limit. (If the actual step size were specified this probability could be calculated for a given endurance limit distribution.)

The possibility of variations in the average virgin endurance-limits from the values used should therefore be considered. If the actual
average endurance limit were 0.99 times the value used to determine $\gamma$ for purposes of plotting the data, then all the plotted values of $\gamma$ would be too low. For example a point plotted at $\gamma = 1.00$ should have been plotted at $\gamma = 1.01$ and a point plotted at $\gamma = 1.40$ should have been plotted at $\gamma = 1.414$. If such data were replotted for the correct lower value of $E_o$ the result would be to shift the data vertically and to stretch the relative vertical spacing progressively more for higher values of $\gamma$. For a range of $\gamma$ as high as 1.4 and for small errors in the estimate of $E_o$ this change might be approximated by providing for just the vertical shift and neglecting the change in the relative spacing. If a $\gamma-N$ curve is calculated by equation (63) or by equation (71) the effect of this approximation can be achieved by shifting the curve vertically without disturbing the actual curve shape.

The shape of the dashed curve in Figure 11 was determined by calculating a $\gamma-N$ curve from equation (71) with $\rho$ taken as 1.5 and $C$ as 0.4. This curve was then superimposed on Figure 11 and shifted horizontally and vertically to obtain the best fit with the composite data. Horizontal shift was considered equivalent to selecting a proper $K'$ value and vertical shift was equivalent to correcting the original estimate of the virgin endurance limit. The position of the curve shown dashed in Figure 11 was obtained by a shift of the virgin endurance limit equal to one percent of the original estimates.

The dashed curve fits the data as well as any curve can be drawn by eye. This indicates that equation (71) with $\rho = 1.5$ and $C = 0.4$ provides a very satisfactory $\gamma-N$ curve for the steels represented by these data. It is extremely interesting to note that the scatter in the com-
p o s it*  d a ta  in  F ig u re 11 i s  n o worse th an  th e  s c a tte r  in  some of th e
individual graphs, for example in Figure 3. At least for these data an
estimate of the $\gamma$-$N$ curve could best have been determined by the use of
special tests designed to measure the virgin endurance limit, locating
one point on the curve at say $\gamma = 1.3$, with constant stress-amplitude
tests, and then applying equation (71) to define the rest of the curve.

2. The Relationship Between Endurance Limit and Cycle Ratio--
The results of several attempts to experimentally evaluate the effect of
fatigue damage on the endurance limit are plotted in Figures 12 through
17 and in Figure 26. In these figures the coordinates are the endurance
limit ratio and the cycle ratio, previously defined and referred to as
$R$ and $\beta$.

The curves drawn through the data are theoretical curves based on
equation (13) with the assumptions that $E_N = CS = 0.4 S$ and that $\rho = 1.$
With these substitutions equation (13) becomes,

$$E = S - \frac{1}{\frac{1-\beta}{S-E_0} + \frac{\beta}{S(1-C)}}.$$  \hspace{1cm} (72)

Dividing both sides by $E_0$ and rearranging gives

$$R = \gamma \left[ 1 - \frac{1}{\frac{1}{(1-C)^\beta} + \frac{\gamma}{\gamma-1} (1-\beta)} \right].$$  \hspace{1cm} (73)

As previously defined $R$ is the ratio of the endurance limit after damage
to the virgin endurance limit, $\gamma$ is the ratio of the damaging stress-
amplitude to the virgin endurance limit, and $\beta$ is the ratio of the number
of cycles of damage to the number of cycles required to cause failure of
Curves calculated from equation (73)

Fig. 12. Reduction of Endurance Limit Ratio Due to Load Cycles at Constant Amplitude on Steel X
Fig. 13. Reduction of Endurance Limit Ratio Due to Local Cycles at Constant Amplitude on Steel Y
Curves calculated from equation (73).

**Fig. 14.** Reduction of Endurance Limit Ratio Due to Load Cycles at Constant Amplitude on Steel Z

- $E_0 = 34,000$ psi
- $E_0/S_u = 0.43$
- $\rho = 1$
- $\epsilon = 0.4$
- $\circ - \gamma = 1.1$
- $\square - \gamma = 1.2$
- $\triangle - \gamma = 1.3$

From reference 40
Curves calculated from equation (73).

Fig. 15. Reduction of Endurance Limit Ratio Due to Load Cycles at Constant Amplitude on SAE 1030 Steel

- $E_0 = 37,200$ psi
- $E_0/S_u = 0.46$
- $\sigma = 1$
- $C = 0.4$
- $\bigcirc - \gamma = 1.1$
- $\square - \gamma = 1.2$
- $\triangle - \gamma = 1.3$

From reference 41
Curves calculated from equation (73)

Endurance limit ratio, R

Cycle-ratio, $\beta$

$E_0 = 39,000$ psi

$E_0/S_u = 0.37$

$\rho = 1$

$C = 0.4$

$\bigcirc - \gamma = 1.08$

$\square - \gamma = 1.54$

Notched specimens

From reference 22

Fig. 16. Reduction of Endurance Limit Ratio Due to Load Cycles at Constant Amplitude on X4130 Steel
Curves calculated from equation (73).

From Figs. 12 through 16.

Fig. 17. Composite Results for Reduction of Endurance Limit Ratio Due to Load Cycles at Constant Amplitude
an average virgin specimen when tested at the stress level used for the
damage treatment.

Before discussing the agreement of theory and experiment for these
R-β data the effects of scatter should be considered. The experimental
points shown in Figures 12 through 15 and therefore most of the points
in Figure 17 are each based on two specimens, one which failed on retest
after damaging and one which did not fail on retest. In the experimental
procedure the damaging stress-amplitude was applied for the desired num-
ber of cycles. Then the damaged specimen was tested at the stress-ampli-
tude estimated to be the new endurance limit. If the specimen failed in
less than 10 million cycles it was assumed that the estimate was high.
In that case a second specimen was subjected to the same initial dam-
aging treatment and then tested at a slightly lower stress-amplitude.
If the second specimen also failed, a third specimen was damaged and
tested at a still lower amplitude. This was repeated until a specimen
lasted 10 million cycles. In each case the new test amplitude was less
than the preceding test amplitude by an amount equal to one percent of
the virgin endurance limit.

If the first damaged specimen did not fail in 10 million cycles
when tested it was assumed that the initial estimate was too low and
more specimens were damaged and tested until a failure occurred. In
this case each successive test amplitude was higher than the preceding
one by an amount equal to one percent of the virgin endurance limit. In
either case the endurance limit was assumed to be between the last two
test amplitudes (reference 41).

This test procedure may result in estimates which are very much
different from the average value. This is particularly true because the

scatter in endurance limits for damaged specimens is much worse than the scatter for virgin specimens (reference 25). Furthermore, this error will be biased in favor of the first estimate. That is, if the first estimate is too high the final test result will usually be too high and if the first estimate is too low the final test result will usually be too low.

This can be readily explained qualitatively by considering several specimens tested in sequence starting well above the endurance limit and testing each successive specimen at a slightly lower stress. The first few specimens may break but it is highly unlikely that such a sequence would reach the average endurance limit without a specimen lasting 10 million cycles.

The endurance limit of approximately 50 percent of the specimens will be above the average. For damaged specimens a scatter band which includes 95 percent of the specimens may well extend more than 10 percent above the average. If the initial estimate were 10 percent high, ten successive specimens would have to be tested to failure in this scatter range for the test to give the correct value of the endurance limit. The probability of 10 such test results occurring is very small, and could be calculated if more information were available. Therefore, it is likely that the scatter in values of $R$ extends to plus or minus 0.05 from the values plotted.

Additional uncertainty in the $R$-$\beta$ data results from possible error in the initial determination of the number of cycles required to cause failure at the damaging stress. This determination was based on the corresponding data in Figures 3, 4, 5, 6, and 9. From these curves it
is evident that such life determination might easily be in error by 
+ 10 percent.

An estimate of the resulting composite error range due to both R
and β uncertainties is shown on one of the points in Figure 12. It
should be noted that the R error will vary from point to point but the
β error will result in a uniform shift of all the data for each dam-
aging stress-amplitude.

Because of the uncertainties in the data of Figures 12 through 17
it is not possible to make a very accurate appraisal of the correspond-
ing theoretical curves. In general the data appears to show a somewhat
larger decrease in R at small β and less decrease in R for large β than
indicated by the theoretical curves. However, it would be possible to
manipulate the data within the range of uncertainty such that a very
close agreement would apparently exist. No attempt has been made to do
this for Figures 12 through 17.

Figure 17 is a composite of the data in Figures 12 through 16 for
results from damage done at \( \gamma = 1.3 \) and \( \gamma = 1.1 \). There is evidently
some general agreement between theory and experiment and a somewhat bet-
ter correlation might be possible if optimum values of \( \rho \) and C were used
in the R-β equation. Because of the uncertainties in the data this was
not attempted and the curves were calculated for \( \rho = 1 \) and \( C = 0.1t \).

Rather interesting results were found for the R-β data plotted in
Figure 26. These data were originally based on failure defined as the
growth of a crack to an arbitrary size. Since the R-β equation is based
on a terminal boundary condition which assumes rupture it was decided to
adjust the data rather than revise the equation. In adjusting the data
it was found that if the values of \( \beta \) based on the crack failure were

92
multiplied by 0.666 for $\gamma = 1.17$ and by 0.788 for $\gamma = 1.04$; a very close agreement between the data and the theoretical curves resulted. This was equivalent to assuming that at $\gamma = 1.17$ the life based on the formation of the arbitrary crack was 0.666 times the life based on rupture and at $\gamma = 1.04$ the life based on formation of the arbitrary crack was 0.788 times the life based on rupture. These assumed values were within the scatter band of the experimental results. The values of $R$ corresponding to specified damage treatments had been determined by testing damaged specimens such that the $Y-N$ curve could be estimated. Eight or ten specimens were used to determine the $Y-N$ curves so considerable uncertainty existed in the estimate of the average endurance limit but the values determined this way are probably more accurate than the data in Figures 12 through 17.

A surprisingly good agreement exists in Figure 26 between the theoretical curves and the adjusted data. However, it must be pointed out that different ratios of life for the arbitrary crack formation to life until rupture could be assumed within the scatter of the data which would result in considerable variation. Again the data is not extensive enough to indicate just what the average ratios are. The $Y-N$ curves associated with Figure 26 are discussed further in section 4 of this chapter.

3. **Effect of Cycles at One Stress-Amplitude on Life at a Second Stress-Amplitude.**—Considerable data are available from tests designed to evaluate the effect of damaging at one stress-amplitude on the life at a second stress-amplitude. The test procedure used for these tests consisted of applying a given number of cycles at one stress-amplitude,
adjusting the load to produce the second stress-amplitude, and then applying the second stress-amplitude until failure occurred. Data from tests of this kind are plotted in Figures 18 through 25.

Each of these figures contains data from tests for two different combinations of stress-amplitudes. In every case the number of cycles applied is expressed in terms of the cycle-ratio, $\beta$. Where, again, the cycle-ratio is the number of cycles applied at a given stress-amplitude divided by the total number of cycles required to cause failure of a virgin specimen when tested at that stress-amplitude. The overstress ratio corresponding to the initial damaging stress-amplitude is designated $\gamma_1$ and the corresponding cycle-ratio is $\beta_1$. Then the second or test stress-amplitude is designated $\gamma_2$ and the corresponding cycle-ratio $\beta_2$. In each case linear scales are used with values of $\beta_1$ along the abscissa and $\beta_2$ along the ordinate.

The symbols used to locate pairs of values of $\beta_1$ and $\beta_2$ on the graphs also designate the values of $\gamma_1$ and $\gamma_2$ to which each point corresponds. In each of Figures 18 through 24 the two combinations of $\gamma_1$ and $\gamma_2$ are such that $\gamma_1$ for one combination is equal to $\gamma_2$ for the other combination.

In Figures 18, 19, and 20 each point is the average of several tests while in Figures 21 through 25 each point corresponds to a single test. As would be expected a more uniform relationship is indicated by the average points than by the individual points. It is also interesting to note that more consistent agreement between the data and the theoretical curves is evident for those figures which have averages plotted.
Fig. 18. Effect of Cycles at One Amplitude on Life at a Second Amplitude for A-7 Steel

From reference 42

\[ E_0 = 43,000 \text{ psi} \]
\[ E_0/S_{tu} = 0.67 \]
\[ \rho = 1, \quad C = 0.4 \]

- \( \gamma_1 = 1.3 \) average of 4 tests
- \( \gamma_2 = 1.12 \) average of 5 tests

Range of uncertainty

Theoretical curves based on equation (73)
$E_0 = 93,500 \text{ psi}$
$E_0/S_y = 0.62$
$\rho = 1, \ C = 0.4$

$\gamma_1 = 1.39$ \text{ average of 4 to } 10 \text{ tests}$
$\gamma_2 = 1.12$ \text{ average of 3 to } 4 \text{ tests}$

From reference 12

**Fig. 19. Effect of Cycles at One Amplitude on Life at a Second Amplitude for SAE 4340 Steel**
Cycle-ratio at the second stress-amplitude, $\beta_2$

Cycle-ratio at the first stress-amplitude, $\beta_1$

Theoretical curves based on equation (73)

$E_0 = 43,000$ psi

$E_0/S_u = ()$

$\rho = 1, \ C = 0.4$

$\{\gamma_1 = 1.23\}$ average of 4 tests

$\{\gamma_2 = 1.05\}$ of 2 tests

From reference 12

Fig. 20. Effect of Cycles at One Amplitude on Life at a Second Amplitude for A-7 Steel
Fig. 21. Effect of Cycles at One Amplitude on Life at a Second Amplitude for Steel B

Theoretical curves based on equation (73)

From reference 21

\[ \varepsilon_0 = 45,000 \text{ psi} \]
\[ \frac{\varepsilon_0}{S_u} = 0.48 \]
\[ \xi = 1, \quad c = 0.4 \]

\[ \gamma_1 = 1.2 \]
\[ \gamma_2 = 1.3 \]
Theoretical curves based on equation (73).

Fig. 22. Effect of Cycles at One Amplitude on Life at a Second Amplitude for Steel B
From reference 21

$E_0 = 45,000 \text{ psi}$

$E_0/S_u = 0.48$

$\rho = 1. \quad C = 0.4$

$
\begin{align*}
\gamma_1 &= 1.3 \\
\gamma_2 &= 1.1 \\
\gamma_3 &= 0.4 \\
\gamma_2 &= 0.3
\end{align*}$

Theoretical curves based on equation (73)

Fig. 23. Effect of Cycles at One Amplitude on Life at a Second Amplitude for Steel
Fig. 24. Effect of Cycles at One Amplitude on Life at a Second Amplitude for X4130 Steel
Fig. 12: Effect of Cycles at One Amplitude on Life at a Second Amplitude for X4130 Steel
The curves drawn in Figures 18 through 25 are theoretical curves and are entirely independent of the data. These curves were determined from equation (73) with $C = 0.9$. The method of determining the $\beta_1 - \beta_2$ curves from equation (73) was as follows. First an $R-\beta$ curve was calculated and drawn for each of the values of $\gamma_1$ and $\gamma_2$, for which a $\beta_1 - \beta_2$ curve was desired. These $R-\beta$ curves are the same type of curves as those in Figures 12 through 17 and in Figures 26 and 31.

The next step in determining the $\beta_1 - \beta_2$ curves is illustrated in Figure 31. Suppose that the value of $\beta_2$ is desired for a retest at $\gamma_2 = 1.1$ following an initial damage specified by $\beta_1 = 1.6$ and $\gamma_1 = 0.5$. First the $R-\beta$ curve for $\gamma_1 = 1.6$ is followed to point A where $\beta_1 = 0.5$. According to the basic assumptions in the development of the theory the change in stress-amplitude from $\gamma_1$ to $\gamma_2$ represents a constant damage change, that is, the endurance limit is not affected by the change in $\gamma$. Therefore, the horizontal line from point A on the $R-\beta$ curve for $\gamma = 1.6$ to the point B on the $R-\beta$ curve for $\gamma = 1.1$ represents the process of changing the magnitude of the load amplitude. Then as cycling of the load is resumed at $\gamma = 1.1$ the change in endurance limit can be followed from point B along the $R-\beta$ curve for $\gamma = 1.1$.

A direct mathematical expression for $\beta_2$ as a function of $\beta_1$, $\gamma_2$, and $\gamma_1$ can, of course, be developed by carrying out the above process algebraically. However, the resulting expression is cumbersome and presents no advantages in time saved if more than one or two points are desired for a $\beta_1 - \beta_2$ curve. Furthermore, the graphical method is of distinct advantage in terms of the overall picture which it presents.
Before discussing the correlation between the theoretical curves and the data the effect of the usual scatter in fatigue data should be considered. This effect will be manifested by two different types of errors in the \( \beta_1 - \beta_2 \) data. In the first place if any given test is repeated there will be considerable variation from specimen to specimen. This is the most obvious effect of scatter and can only be controlled by testing enough specimens to provide a reasonably accurate estimate of the average result for each type of test. In Figures 18, 19, and 20 each experimental point represents the average from several tests, in some cases as many as 10 tests. However, in Figures 21 through 25 each experimental point represents but one test.

Another less obvious effect of scatter results in considerable error in the location of the points on the \( \beta_1 - \beta_2 \) coordinates. This is due to the possible error in estimating the total life of virgin specimens corresponding to either \( \gamma_1 \) or \( \gamma_2 \). These lives, \( N_1 \) and \( N_2 \), are used to calculate the nondimensional cycle ratios, \( \beta_1 \) and \( \beta_2 \). As mentioned in section 2 of this chapter an estimate of \( N \) from the usual S-N or \( \gamma\)-N curve may easily be in error by \( \pm 10 \) percent.

The effect of such an error on \( \beta_1 - \beta_2 \) data is illustrated in Figure 18 where a range is indicated on one of the points. The effect on this error is uniform for all \( \beta_1 \) values and for all \( \beta_2 \) values. The net effect is to shift the curves, with more shift occurring for large values than for small values of \( \beta \).

For example, consider an experimental value of \( \beta = 0.5 \). Recall that \( \beta = n/N \) where \( n \) is the actual number of cycles applied and \( N \) is the number of cycles required to cause failure of a virgin specimen at the
stress-amplitude in question. Suppose that the value of $N$ used to calculate $\beta$ was 10 percent less than the true average value, $N'$. Then $N = 0.9 \times N'$ and $\beta = \frac{n}{N} = \frac{n}{0.9 N'}$. But the correct value of the cycle ratio, $\beta'$, would be $\frac{n}{N'}$. Therefore $\frac{\beta'}{\beta} = \frac{n}{N'} \times \frac{0.9 N'}{n} = 0.9$. Then if $\beta = 0.5$ the true value $\beta'$ would be $0.9 \times 0.5 = 0.45$.

The qualitative agreement between the data and the theoretical curves is very reasonable in Figures 18, 19, and 20 for those points which represent the average of at least three tests. In Figures 21, 22, and 23 each point represents but one test and very extensive scatter is evident with but a very general qualitative correlation between experiment and theory.

In Figures 24 and 25 each point represents the median of the results of from six to ten tests. Here the qualitative agreement is not very good considering the number of tests. However, these data were based on "failure" defined to be the growth of a crack to approximately 12 percent of the nominal specimen area. This crack size was detected by measuring the associated change in deflection during the test. It is very possible that this introduced additional uncertainty into the determination of the number of cycles at which "failure" occurred. However, there is still a general qualitative correlation between the data and the theoretical curves.

It may be possible to select particular values of the constants $\rho$ and $C$ in the basic theoretical equations such that a better agreement between experiment and theory can be obtained for any one set of data. However, the accuracy of the data seriously limits the advisability of selecting special values of constants for particular types of tests.
Therefore, the values $p = 1$ and $C = 0.4$ were used throughout the $\beta_1 - \beta_2$ evaluation.

1. **Effect of Damage on the Relationship Between Life and Stress-Amplitude**—A limited amount of data are available from tests in which a number of specimens were all subjected to the same damage treatment and then used to determine the S-N or $\gamma$-N curve. Data of this type are presented in Figures 27, 28, and 29.

The $\gamma$-N curves in Figures 27 and 28 are based on failure defined to be rupture of the test specimen. However, the amount of damage corresponding to each curve was originally specified in terms of a cycle ratio based on the number of cycles required to cause a fatigue crack of specified size, $N_c$. This has already been discussed in section 2 of this chapter in connection with the $R-\beta$ data in Figure 26. The values of endurance limit used for Figure 26 were estimated from the data in Figures 27 and 28.

To convert the cycle ratios based on $N_c$ to cycle ratios based on the number of cycles at rupture, $N$, it was necessary to estimate the ratio $N_c/N$. This ratio was selected for each stress-amplitude within the range of the scatter such that the $R-\beta$ data in Figure 26 would fit the theoretical curves. The resulting values of cycle ratio based on $N$ are given in Figures 27 and 28 to specify the amount of damage corresponding to each of the theoretical $\gamma$-N curves.

Assuming that equation (9) correctly represents the relation between the endurance limit, $E$, and the number of cycles, $n$, applied at any constant stress-amplitude, $S$, then the $\gamma$-N curve for any amount of damage can be determined in the following manner. From equation (9)
Curves calculated from equation (73)

\[ E_0 = 46,000 \text{ psi} \]
\[ \frac{E_0}{S_u} = 0.47 \]
\[ \alpha = 1, \ \beta = 0.4 \]
\[ \omega = 1.17, \ \varepsilon = 0.67C_c \]
\[ L - \gamma = 1.04, \ \varepsilon = 0.79C_c \]

Original data was based on failure defined as a crack of a given size. Data was extrapolated to rupture.

From reference 22

Fig. 26. Reduction of Endurance Limit Due to Load Cycles at Constant Amplitude on X4130 Steel
Fig. 27. Theoretical S-N Curves and Corresponding Data for Damaged Specimens of X-1140 Steel
Overstress ratio

Theoretical curve for virgin specimen

Cycles to failure, N

Fig. 28: Theoretical S-N Curves and Corresponding Data for Damaged Specimens of X4130 Steel

\[ E_0 = 46,000 \text{ psi} \]
\[ E_0/S_u = 0.47 \]
\[ K = 4.31 \times 10^{-5} \]

Pre-damage at \( \gamma = 1.04 \)

From reference...
Theoretical curve for $n = 1$

$C = 0.5$

$\alpha = 1.31$

For use with the data in Fig. 30

Fig. 29 Reduction of Endurance Limit Due to Load Cycles at Constant Amplitude on SAE 1020 Steel
Fig. 30. Theoretical S-N Curves and Corresponding Data. Larger Specimens of SAE 4130 Steel.
\[ \rho kn = \frac{1}{(S - E_0)^\rho} - \frac{1}{(S - E)^\rho} , \]  

(9)
or if \( \rho = 1 \), then

\[ kn = \frac{1}{S - E_0} - \frac{1}{S - E} . \]  

(74)

If some amount of damage has been done such that the endurance limit is reduced from \( E_0 \) to \( E' \), then the number of cycles, \( n' \), required to cause that damage at any stress-amplitude, \( S \), can be calculated from equation (74) as follows,

\[ n' = \frac{1}{k} \left[ \frac{1}{S - E_0} - \frac{1}{S - E'} \right] . \]  

(75)

Now, if \( N' \) is the total number of cycles required to cause failure at the stress-amplitude, \( S \), after the damage treatment has been applied, then

\[ N' = N - n' , \]  

(76)

where \( N \) is the number of cycles required to cause failure of a virgin specimen at the stress-amplitude, \( S \).

Then if the endurance limit is equal to \( C \times S \) at failure, equation (74) becomes

\[ N = \frac{1}{k} \left[ \frac{1}{S - E_0} - \frac{1}{S - CS} \right] . \]  

(77)

Substituting equations (77) and (75) into equation (76) gives

\[ N' = \frac{1}{k} \left[ \frac{1}{S - E'} - \frac{1}{S - CS} \right] . \]  

(78)

Comparing equation (78) with equation (77) it is apparent that the \( S-N \) relationship is general in terms of the current endurance limit.
of the material considered. That is equation (78) expresses the relationship between the number of cycles required to cause failure in a test at any stress-amplitude in terms of the endurance limit at the start of the test.

Equation (78) is changed into the $\gamma$-$N$ relation by multiplying and dividing the right side of the equation by the virgin endurance limit, $E_0$. This gives the following relation between $N'$ and $\gamma$.

$$N' = \frac{1}{K} \left[ \frac{1}{\gamma - R'} - \frac{1}{\gamma(1 - C)} \right], \quad (79)$$

where $R'$ is the endurance limit ratio corresponding to the initial damage.

Equation (79) was used to calculate the $\gamma$-$N$ curves corresponding to different degrees of damage for Figures 27, 28, and 30. In each case the values of $R'$ used in equation (79) were calculated from the theoretical $R$-$\beta$ relation, equation (73). In the calculations for Figures 27 and 28 the constant $C$ was equal to 0.4 while the curves in Figure 30 correspond to $C = 0.5$.

The theoretical curves are in general qualitative agreement with the data but again the scatter makes a positive evaluation impossible. In Figure 30 the theoretical values of the endurance limit appear to be somewhat lower than indicated by the data. However, the data does not define these endurance limits very accurately and the curves are in very reasonable agreement for the overstress tests.

5. The Sum of the Cycle-Ratios at Failure.—The linear hypothesis discussed in section 1 of Chapter I has provided the basis for most design calculations in which an attempt has been made to account for
cumulative fatigue damage. Therefore, the question of the accuracy of the linear rule has been the basis of several experimental investigations (references 42 and 65).

The linear rule is most usually applied where the load history consists of a number of cycles at each of two or more stress-amplitudes. As given in Chapter I the usual mathematical formulation of the linear rule used to predict failure is that at failure

\[ \frac{n_1}{N_1} + \frac{n_2}{N_2} + \frac{n_3}{N_3} + \ldots = 1. \]  

(a)

In this equation \( n_1 \) is the number of cycles applied at stress-amplitude \( S_1 \) and \( N_1 \) is the number of cycles which would independently cause failure if applied at \( S_1 \).

Because of the importance of the linear rule in design it is extremely important to determine whether or not the sum of the cycle-ratios is always equal to the same constant at failure. As discussed in Chapter I this problem has been investigated experimentally and test results have indicated that the sum of the cycle-ratios may be equal to one at failure for some sequences and in other cases the sum may differ considerably from one. Therefore, it is of particular interest to see what values of the sum of the cycle-ratios are predicted by the present theory for several different types of load sequences.

Two different types of sequences have been considered and the results are given in Figures 32 and 33. In Figure 32 the points are calculated from the new cumulative damage theory and no experimental data is presented. The calculations for this figure were based on a hypothetical situation where the stress-amplitude was either at a value corresponding to \( \gamma = 1.6 \) or at a value corresponding to \( \gamma = 1.1 \). The
Curve calculated from equation (73)

Theoretical curve for $\beta = 1$

\( \gamma = 0.4 \)

\( \gamma = 0.6 \)

\( \gamma = 1.1 \)

Fig. 31. Theoretical Curve for the Relationship between the Endurance Limit Ratio and Cycle-Ratio
Fig. 32  Effect of Individual Cycle-Ratio Size on the Sum of the Cycle-Ratios with Two Amplitudes Acting
Fig. 33. Effect of the Number of Stress Levels acting on the Sum of the Cycle-Ratios.
stress-amplitude was alternately at the high value and at the low value for a number of cycles which in each case corresponded to a specified increment in the cycle-ratio.

For example, consider the circle plotted directly above the value 0.4 on the abscissa scale. This point indicates that the sum of the cycle-ratios for a certain sequence was 1.05. Since the point is enclosed with a circle the first cycles were applied at $\gamma = 1.1$. The abscissa value of $\beta = 0.4$ means that the number of cycles equivalent to $\beta = 0.4$ was applied at $\gamma = 1.1$. Then the stress-amplitude was changed and the number of cycles corresponding to $\beta = 0.4$ was applied at $\gamma = 1.6$. This was repeated applying cycles equivalent to $\beta = 0.4$ first at $\gamma = 1.1$ and then at $\gamma = 1.6$ until failure occurred.

The triangle directly above 0.4 on the abscissa indicates the sum of the cycle-ratios for a hypothetical loading sequence identical to the one just discussed except that the initial stress-amplitude corresponds to $\gamma = 1.6$.

Although it is possible to develop direct formulae for such calculations it is often more convenient to do them graphically as discussed in section 3 of this chapter in connection with the $\beta_1-\beta_2$ curves in Figures 18 through 25. To determine the points plotted in Figure 32 the $R-\beta$ curves were calculated for $\gamma = 1.6$ and $\gamma = 1.1$. These two curves are given in Figure 31. For any specified sequence involving specified increments in cycle-ratio at $\gamma = 1.1$ and 1.6 the sum of the cycle-ratios can be determined from these curves. The procedure is simply to start at the upper left-hand corner of the figure, follow down the proper curve until a value of $\beta$ is reached corresponding to the first
increment of cycles, then jump horizontally to the other curve and follow it through the next increment of cycles, then jump horizontally back to the first curve and so on. The sum of the cycle-ratios will consist of the sum of all the increments along the abscissa which were cumulated while following both curves but neglecting the horizontal jumps from curve to curve.

From Figure 32 it can be seen that according to the theory the sum of the cycle-ratios as calculated for the linear rule will vary from slightly less than 0.8 to almost 1.3 depending on the size of the cycle increment and which stress-amplitude is applied first. It is interesting to note that for very small increments in cycle-ratio, that is for a sequence which jumps back and forth between $Y = 1.1$ and $Y = 1.6$ very frequently, the sum of the cycle-ratios appears to approach about 0.9. Although this trend appears to be somewhat erratic it would seem that for frequently changing stress-amplitude the linear rule might give quite consistent agreement with this theory if the sum of the cycle-ratios were specified at 0.9 at failure instead of 1.

A direct mathematical expression giving the sum of the cycle-ratio at failure for this type of sequence was not developed. The apparent complexity of such an expression would seem to limit its usefulness. However, it might be noted that in this case the problem is ideally suited for solution on a digital computer and could easily be programmed to give the sum of the cycle-ratios for a wide selection of cycle-ratio increments for many different combinations of stress-amplitudes. Furthermore, different sequences could be considered to establish more definite bounds on the sum of the cycle-ratios at failure.
Figure 33 shows theoretical curves and plotted points for experimental data corresponding to another type of sequential loading. This type of loading again consisted of two cases. In one case the stress-amplitude initially corresponded to \( \gamma = 1.11 \) and was increased in steps during the test to \( \gamma = 1.67 \). In the second case the stress-amplitude initially corresponded to \( \gamma = 1.67 \) and was decreased in steps during the test to \( \gamma = 1.11 \). In both cases the exact sequence was specified by announcing the number of different values of stress-amplitude at which cycles were applied. These different values of stress-amplitude were spaced such that the range from \( \gamma = 1.11 \) to \( \gamma = 1.67 \) was divided into equal segments by their corresponding values of \( \gamma \).

The number of cycles applied at each stress-amplitude was such that when the corresponding value of cycle-ratio, \( \beta \), was multiplied times the number of different values of stress-amplitude the product was unity. This schedule was maintained until interrupted by failure of a test specimen or until the sum of the cycle-ratios reached unity. In the latter case the stress-amplitude being applied when the sum of the cycle-ratios reached unity was continued until failure occurred.

For example, for five different values of stress-amplitude each stress-amplitude was applied for a number of cycles corresponding to \( \beta = 0.2 \) since \( 5 \times 0.2 = 1 \) as specified. Furthermore, the five values of stress-amplitude corresponded to values of \( \gamma \) of 1.11, 1.25, 1.39, 1.53, and 1.67, since these five levels are equally spaced.

From Figure 33 the theoretical curve for increasing sequences shows a value of the sum of the cycle-ratios of about 1.13 for two different stress-amplitudes. As the number of different stress-amplitudes
increases the sum of the cycle-ratios decreases. When the number of different stress-amplitudes reaches about 12, the sum of the cycle-ratios approaches unity where it remains for larger numbers of different stress-amplitudes.

For sequences of stress-amplitudes decreasing from \( \gamma = 1.67 \) to \( \gamma = 1.11 \) the sum of the cycle-ratios is about 0.85 for two different stress-amplitudes and increases as the number of different stress-amplitudes is increased. When the number of different stress-amplitudes reaches 12 or 14, the sum of the cycle-ratios is between 0.93 and 0.94 where it remains for increasing numbers of different stress-amplitudes.

The experimental data in Figure 33 indicates a trend in the sum of the cycle-ratios which is qualitatively similar to the theoretical results. However, the data indicates a larger deviation from unity for the sum of the cycle-ratios than does the theory. Again the data contains the usual amount of scatter and in addition the uncertainty in the basic S-N data could be responsible for very considerable errors in the sum of the cycle-ratios calculated for the data. However, there are several test points available for each type of sequence which helps to minimize the error due to scatter in the final data. Also errors due to incorrect estimates of the basic S-N data would tend to cause a vertical shift of both the "going up" and the "going down" results in the same direction. Therefore, this type of error cannot be used to readily explain the wider spread from the value of unity for the sum of the cycle-ratios which exists in the data as compared with the theory.

This represents one of the larger variations between the theory and experimental data for constant stress-amplitude. Further testing
and refinement of the theory should be aimed at more accurately developing the relation shown in Figure 33.

For relatively large numbers of changes in the stress-amplitude the theory suggests that the linear rule may conservatively be used to predict failures of the two types of sequences considered in Figures 32 and 33. The only change in the usual calculation would be to assume that a sum of cycle-ratios equal to 0.9 would result in failure.
Chapter IV

EVALUATION OF THE THEORY FOR CONTINUOUSLY CHANGING STRESS-AMPLITUDE

1. **Uniformly Increasing Stress-Amplitude.**— In the third section of Chapter II two solutions were developed which define the relationship between the stress-amplitude and the number of cycles at failure for the case of uniformly increasing stress-amplitude. These solutions were based on the assumption that the stress-amplitude was initially just equal to the virgin endurance limit or that no damage was done before the endurance limit was reached. Mathematically these two assumptions are equivalent in that in either case cycles are counted from the time when the stress-amplitude is equal to the virgin endurance limit. A further assumption common to both solutions was that $\rho = 1$ in the basic differential equation. This value was found to provide reasonable agreement with available data for constant amplitude tests. Furthermore a relatively simple analytical solution of the basic differential equation for the case of uniformly increasing stress-amplitude was possible for this value of $\rho$. Such a solution was particularly desirable for evaluating the effects of the various parameters.

The two solutions for uniformly increasing stress-amplitude are different in two respects. Equation (I\(\alpha\)) was obtained by using the terminal condition that $E = 0$ at failure when $n = N$. Furthermore equation (I\(\alpha\)) is an approximation to the actual solution. (For convenience uniformly increasing stress-amplitude tests are referred to as Prot tests.)
in honor of the man who first suggested this method of testing and the rate of stress-amplitude increase in psi per cycle is called the Prot rate and represented by the symbol \( \alpha \). Equation (47) results from a direct solution of the differential equation with the terminal condition that \( E = E_N = CS_f \) at failure when \( n = N \).

The two solutions are

\[
S_f = E_0 + \frac{n}{2}(\sqrt{a/k}) \quad (44)\]

and

\[
S_f = E_0 + \alpha N = \frac{\sqrt{a/k}}{1 - C} \tan \sqrt{ak} N \quad (47)
\]

As the first step in evaluating these two equations both were plotted on a common coordinate system for comparison. For the calculations values of the parameter \( k \) and \( E_0 \) were determined from the constant stress-amplitude data for a typical SAE 4340 steel given in Figure 39. This material had a virgin endurance limit of 73,000 psi and at 112,000 psi the average life was \( 2.4 \times 10^4 \) cycles.

Using this information the parameter \( k \) was evaluated from equation (15) for \( E_N = 0 \) and for \( E_N = CS = 0.5 S \). These two values of \( k \) were then substituted respectively into equations (44) and (47) and associated values of \( S_f \) and \( \sqrt{a} \) were determined for each equation. The corresponding curves are presented in Figure 34. For convenience the \( S_f \) values were divided by \( E_0 \), and values of the resulting ratio, \( \gamma_f \), are shown on the ordinate while the abscissa is \( \sqrt{a} \).

Inspection of these curves show that the two curves stay quite close for values of \( \sqrt{a} \) from 0 to 0.6. The accuracy of the approximation will vary somewhat depending on the particular material considered.
Fig. 34. Theoretical Relationship between Rate of Amplitude Increase and Amplitude at Failure.
but for relatively high strength steels of which the sample considered is typical the approximation may be considered reasonable to at least \( \sqrt{a} = 0.5 \).

When experimental data from Prot type tests are available they can be plotted on \( S_f \) vs. \( \sqrt{a} \) coordinates. Then a best straight line can be drawn through the data and the intercept of that line at \( \sqrt{a} = 0 \) will, according to the theory, provide a measure of the average virgin endurance limit. This is exactly the procedure originally suggested by Prot (reference 39). However, equations (44) and (47) also provide a method of obtaining considerably more information from this type of data.

These equations contain the parameters, \( S_f \), \( a \), \( N \), \( E_0 \), \( k \), and \( C \). \( S_f \), \( N \), and \( a \) are directly known from Prot-type test data. Furthermore, if these data are for \( \sqrt{a} \leq 0.5 \) they can be plotted as in Figure 35 and \( E_0 \) can be evaluated by drawing the best straight line and determining its ordinate intercept. Also, since \( C \) is approximately equal to the ratio \( E_0 / S_u \), it can be evaluated if the ultimate tensile strength, \( S_u \), is known. Thus all the parameters except \( k \) can be evaluated for use in either equation (44) or (47). The solution of either of these equations then provides a value for the characteristic constant \( k \).

With the values of \( E_0 \), \( C \), and \( k \) known equation (15) can then be used to calculate the conventional S-N curve.

\[
kN = \frac{1}{S - E_0} - \frac{1}{S(1 - C)} \tag{15}
\]

It should be noted that the symbol \( N \) designates the number of cycles required to cause failure by whatever type of loading is being considered.

126
Therefore in equation (15) \( N \) is the number of cycles to failure at a constant stress-amplitude, \( S \). In equation (17), on the other hand, \( N \) designates the number of cycles required to cause failure when the stress-amplitude is initially equal to \( E_0 \) and then subsequently increases at the rate of \( \alpha \) psi per cycle.

The fact that all of the parameters can be evaluated from data resulting either from constant stress-amplitude tests or from uniformly increasing stress-amplitude tests provides a very convenient method of comparing the theoretical equations with experimental results when both types of tests have been carried out with similar specimens from the same material. Unfortunately most investigators who have done Prot type testing have been primarily interested in the virgin endurance limit. Therefore they have limited their constant amplitude tests to values close to the endurance limit. The resulting data does not provide much information for locating the position of the S-N curve. Fortunately some data by Corten, Dimoff, Dolan, and Sugi has been published which does permit some comparison of the two types of tests (reference 63). This data was in general taken on specimens and with test procedures similar to those described in the second paragraph of Chapter III.

In Figures 35, 36, 37, and 38 the results from constant amplitude tests and Prot type tests are plotted respectively on conventional S-N coordinates and on \( S_f / \sqrt{\pi} \) coordinates. Figures 35 and 36 are for an SAE 2340 steel and Figures 37 and 38 are for a 1\% B50 high Boron steel. The straight lines in Figures 35 and 37 are "best" straight lines drawn by eye through the Prot type data. The S-N curves in Figures 36 and 38 are calculated curves which were determined from the Prot type data by means of equations (17) and (15) as discussed above.

127
"Best" straight line drawn by eye through the data.

SAE 2340 steel

From reference 63

\[ \sqrt{\alpha} = \left(\text{Rate of amplitude increase, psi/cycle}\right)^{1/2} \]

Fig. 36. Results from Prot Type Tests on SAE 2340 Steel
Fig. 35. S-N Curve Calculated from Prot Type Data for SAE 2340 Steel
Stress-amplitude at failure, $S_f \sim 1000$ psi

Best straight line drawn by eye through the data.

From reference 63

$\sqrt{\alpha} = (\text{Rate of stress-amplitude increase, psi/cycle})^{1/2}$

Fig. 37. Results from Prot-Type Tests on 14B5C Steel
$S = \text{Stress-amplitude, 1000 psi}$

$$E_0 = 61,000 \text{ psi from Fig. 37}$$

$$E_0/S_u = 0.225$$

$$\Phi = 1, \ C = 0.225$$

$$k = 1.54 \times 10^{-10} \text{ evaluated from Fig. 37.}$$

From reference 63.

**Fig. 38**  S-N Curve Calculated from Prot-Type Data for 14B50 Steel
The points shown as squares in Figure 36 are for tests carried out concurrently with the tests reported in Figure 35. The points shown as circles in Figure 36 are for tests of "the same material" carried out some five years previously. It is not clear whether test conditions had been altered somewhat in the intervening period or whether the data simply represents the usual scatter. In any event the theoretical average S-N curve provides a good approximation of the test results. In Figure 38 the S-N curve which was calculated on the basis of the Prot type data in Figure 37 again is in good agreement with the constant amplitude data.

There is considerable variability evident in the data for both the 2340 and the 14850 steels. As a result the straight line through the Prot-type data could be drawn equally well at somewhat different positions and the S-N curve is not too well defined by the constant stress-amplitude data. However, in both cases no attempt was made to determine the optimum position of the straight line in the Prot-type data. The analysis was carried out and the S-N curves were drawn and then the constant amplitude data were plotted. Therefore the curves in Figures 36 and 38 represent the kind of an estimate which would have resulted if the S-N data had not been taken. The result is a reasonable approximation in both cases.

2. Randomly Varying Stress-Amplitude.— Considerable data from tests where the stress-amplitude was randomly varied have been published as a result of the very extensive work carried out by Freudenthal and Heller (references 53, 54, 55, 56, and 57). In particular the data in
reference 57 on SAE 4340 steel were selected to be compared with the theoretical relationships for random loading.

The specimen geometry and method of loading used for the tests from which these data were taken were generally similar to the corresponding testing conditions used to obtain the data considered elsewhere in Chapters III and IV. Therefore no effects other than those due to random loading were expected to influence the agreement between the theoretical predictions and experimental results.

The theoretical prediction of the number of cycles to failure were calculated for nine different test sequences. Twenty specimens had been tested with each sequence so that the average life for each sequence was well established. The method of determining the random sequences and the test machines used to apply those sequences are described in the references cited. It is sufficient here to observe that there does not appear to be any reason for doubting that the stress-amplitudes in each sequence were equivalent to a succession of random selections from a population containing the specified values of stress-amplitude in proportion to specified probabilities.

This is exactly the condition for which equation (62) was derived.

\[
k'N = \frac{1}{\sqrt{c-b^2}} \left[ \tan^{-1} \frac{b - E_N}{\sqrt{c-b^2}} - \tan^{-1} \frac{b - E_0}{\sqrt{c-b^2}} \right]
\]

(62)

where

\[
k' = k \sum_{i=1}^{j} p_i
\]
To apply equation (62) it is necessary to first determine $k'$, $E_o$, $E_N$, $b$, and $c$. The constants $b$ and $c$ depend only on the random loading and can be calculated from the above equations using the values of stress-amplitude, $S_i$, and the associated probabilities, $p_i$. However, the other constants must be determined from conventional S-N or other data. The S-N data reported in reference 57 for the same material and general test conditions used for the random tests are plotted in Figure 39.

The curve drawn in Figure 39 was calculated from equation (10) with $E_o = 78,000$ psi, $E_N = C \times S = 0.5 \times S$, $\rho = 1$, and $k = 4.8 \times 10^{-10}$.

$$p \cdot k N = (S - E_o)^{-\rho} - (S - E_N)^{-\rho}.$$  \hspace{1cm} (10)

The value unity was assigned to $\rho$ to be consistent with the random loading equation and $C = 0.5$ was selected as approximately equal to $E_o/S_u$ and in the general range previously found to give reasonable results for this type of material. The value of $k$ was selected to give a good general correlation between the data and the calculated curve. The selection of $E_o = 78,000$ was arrived at as follows.

At 82,000 psi 20 specimens had been tested. Four of them did not fail and therefore their endurance limit was above 82,000 psi. Sixteen
Fig. 39: S-N Data for the SAE 4340 Steel Used for Random Loading Tests

\[ kN = \frac{1}{S - E_o} = \frac{1}{S(1 - C)} \]

\( E_o = 78,000 \text{ psi} \)
\( E_o/S_u = 0.56, \ P = 1 \)
\( C = 0.5, \ k = 4.81 \times 10^{-10} \)

Each point represents the average of 20 specimens except at 82,000 psi where 16 failures and 4 runouts occurred.

From reference 66.
of the specimens tested at 82,000 psi failed and therefore their endurance limit was below 82,000. If the distribution of specimens with endurance limit were normal about 20 percent of such a group would be expected to have virgin endurance limits above $0.9\sigma + E_0$. Where $E_0$ is the average of the virgin endurance limits and $\sigma$ is the standard deviation of the virgin endurance limits (cf. Appendix, Table 1 of reference 61). On the basis of other tests designed to measure variation in the endurance limit, $\sigma$ might be expected to be about 5,000 psi (cf. reference 31). Therefore, since the four run-outs represent 20 percent of the specimens tested at 82,000 psi a reasonable assumption is that

$$E_0 + 0.9 \times 5,000 = 82,000$$
or

$$E_0 \approx 78,000$$

If all the $S_1$ are greater than $E_0$ the use of equation (62) is completely straightforward once the necessary constants have been determined. However, in all of the nine random sequences considered at least one of the $S_1$ was less than $E_0$.

To understand how this situation should be treated it is helpful to go back to equation (59).

$$k'n = \frac{1}{\sqrt{c - b^2}} \left[ \tan^{-1} \frac{b - E}{\sqrt{c - b^2}} - \tan^{-1} \frac{b - E_0}{\sqrt{c - b^2}} \right].$$

Equation (59) yields equation (60) if the terminal condition that $n = N$ when $E = E_N$ is considered. Equation (59) also expresses a functional relationship between the endurance limit before a random sequence is applied, $E_0$; the endurance limit after the sequence is applied, $E$; the number of cycles in the sequence, $n$; the characteristic constants of the random sequence, $b$ and $c$; and the proportionality constant, $k'$. 136
Therefore this equation, according to the theory, can be used to calculate the number of cycles required to reduce the endurance limit from $E_0$ to any desired value greater than or equal to $E_N$. Then if one of the $S_i$, say $S_1$, in a random sequence is less than $E_0$ but greater than $E_N$, equation (59) can be used to calculate the number of cycles required to reduce $E$ from $E_0$ to $S_1$. Then equation (59) can again be used to calculate the number of cycles required to reduce $E$ from $S_1$ to $E_N$. The sum of these two numbers of cycles is the total number required to reduce $E_0$ to $E_N$, that is, to cause failure of a virgin specimen.

For the last step $E_N$ should be taken equal to $C$ times $S_{\text{max}}$ where $S_{\text{max}}$ is the largest stress-amplitude in the sequence. It should be noted also that $b$, $c$, and $k'$ will have different values for the different steps in the calculation since $p_1$ is assumed to be zero for all $S_i$ less than $E$.

If several of the $S_i$ are below $E_0$ then several steps are required in the calculation. Each part of the calculation will determine the number of cycles required to reduce the endurance limit to the next lowest $S_i$ and then finally to $E_N$ and the total of all these numbers of cycles represents the total cycles which the sequence will require to cause failure of a virgin specimen.

These calculations were carried out for sequences one through nine in reference 56 and the results are listed in Table 1 along with the experimental results. In addition the numbers of cycles to failure as predicted by the linear damage rule were calculated and are presented in the same table.

The linear rule is usually applied to random loading as follows.
TABLE 1

Experimental and Theoretical Lives with Random Stress-Amplitude

<table>
<thead>
<tr>
<th>Test Sequence No.</th>
<th>Test Data $\times 10^4$ cycles</th>
<th>Linear Rule $\times 10^4$ cycles</th>
<th>Equation (62) $\times 10^4$ cycles</th>
<th>Ratio (2) A</th>
<th>Ratio (3) B</th>
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<td>28.2</td>
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</table>

(1) Reference 66.

(2) Ratio of life predicted by equation (62) to experimental life.

(3) Ratio of life predicted by equation (62) to life predicted by the linear rule.
If failure is defined by the expression

\[ \sum_{i=1}^{J} \frac{n_i}{N_i} = 1 \]

and if \( N_t \) is the total number of cycles to failure with a given random loading sequence then

\[ n_i = p_i \times N_t \]

where, as before, \( p_i \) is the probability of occurrence of \( S_i \) and \( n_i \) is the number of cycles applied with an amplitude \( S_i \). \( N_i \) is the total life associated with \( S_i \) for constant stress-amplitude loading. Then,

\[ \sum_{i=1}^{J} \frac{p_i \times N_t}{N_i} = 1, \]

or

\[ N_t \sum_{i=1}^{J} \frac{p_i}{N_i} = 1, \]

and

\[ N_t = \frac{1}{\sum_{i=1}^{J} \frac{p_i}{N_i}}. \]

For all \( S_i \) less than \( E_0 \) the associated \( p_i \) is assumed to be zero consistent with the assumption that stress-amplitudes less than the endurance limit do not cause damage.

To provide a ready comparison of the two theories and the experimental results, two ratios are also presented in Table 1. These ratios are the life predicted by equation (62) divided by the average experimental life and the life predicted by equation (62) divided by the life predicted by the linear rule. An inspection of this table shows that, as was expected, the linear rule predicted much longer lives than were
<table>
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<th>Stress Amplitude $10^3$ psi</th>
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</table>

(1) Reference 66
One possible form of interaction would be consistent with the concept of endurance limit used for the development of the theory. This kind of interaction would cause cycles with an amplitude only a small amount above the endurance limit to do more damage when a few high-amplitude cycles were interspersed among them than when they were occurring by themselves. This interaction might have an appreciable effect even if the reduction in the endurance limit due directly to the cycles of high amplitude were negligible.

A mechanism which makes this kind of interaction plausible was suggested by Corten and Dolan (Paper 2 of Session 3 in reference 12). They concluded that "the number of damage nuclei (submicroscopic voids) that form increases as the stress increases". Then with random loading the number of damage nuclei might depend on the few high-amplitude cycles while the rate of growth of these nuclei might be determined by the more numerous low amplitude cycles. The result would be the kind of an interaction suggested.

This concept would also suggest that the endurance limit would be determined by the worst damage nuclei while the rate of damage accumulation would depend on the number of damage nuclei as well as the extent of damage at each nuclei. Some such consideration as this might serve as a starting point in modifying the damage theory developed in Chapter XI to account for interaction.

Freudenthal and Heller have used a concept of interaction in the development of their cumulative damage theory for random loading (references 53, 54, 55, 56, and 66). They use a weight function on the cycle ratio which occurs at each stress-amplitude and thus obtain a modified sum of the cycle ratios as a generalization of the linear rule. The
weight functions are designed to account for the extra damage which each cycle does because cycles of higher stress-amplitude are occurring.

The possibility that cycles at an amplitude below the current value of the endurance limit may cause damage as a result of interaction should also be considered. That is, if a certain value of stress-amplitude will not cause failure when applied by itself for a very large number of cycles it is by definition below the endurance limit. However, this same value of stress-amplitude might cause damage if it was applied in combination with a larger amplitude even though the endurance limit had not yet been reduced to the value of the smaller amplitude. The plausibility of this type of interaction is not so readily demonstrated in terms of a physical mechanism. In light of the extreme complexity of the actual failure mechanism (whatever it is), this form of interaction should not be brushed aside simply because it is not immediately compatible with existing concepts.

Whether or not such interaction exists can be determined by a relatively simple two amplitude testing procedure with one amplitude well above the virgin endurance limit and the other well below the virgin endurance limit. Any appreciable damage caused by the low amplitude cycles because of interaction could be detected by comparing the number of cycles of the large amplitude which occurred before failure when the high and the low amplitudes were both applied with the number of cycles of high amplitude required to cause failure when applied without the cycles of low amplitude.

During such a test it would be necessary to keep the small amplitude below the endurance limit, taking into account that the endurance
limit will decrease during the test. For this purpose it might be necessary to reduce the magnitude of the low amplitude during the test. This would permit the use of a low amplitude which was less than the endurance limit all through the test without being too much smaller than the endurance limit at the start of the test. If this "below-the-endurance-limit" interaction occurs it might be restricted to a range of stress-amplitudes which are only some small amount less than the endurance limit.

The relative numbers of cycles of low stress-amplitude and of high stress-amplitude would also be very important in such a test. If approximately equal numbers of high amplitude and low amplitude cycles were applied the possible effect of the low amplitude cycles would be restricted. If there were no interaction a certain number of high amplitude cycles would occur before failure. If on the other hand the interaction were so extensive that the low amplitude cycles did as much damage as the high amplitude cycles then one-half as many cycles of high amplitude would occur before failure. This ratio of two to one is hardly large enough to conclusively show any interaction effect unless large numbers of specimens are tested because normal scatter may result in that much variation in the data. Therefore a large number of low amplitude cycles, perhaps 100 or 1000 should occur for every high amplitude cycle.

Some random loading data have been published by Head and Hooke in which the load sequence was generated proportional to the random voltage known as random noise which occurs in an electronic circuit (reference 52). This load sequence contains all possible amplitudes within the
range of the loading mechanism. Using derived approximations to the
frequency function for such loading and assuming that each amplitude
is applied for a complete cycle it would be possible to apply the theory
to this type of random loading.
Chapter V

CONCLUSIONS AND DESIGN EQUATIONS


General conclusions regarding the evaluation of the theory were based on the comparisons carried out in Chapters III and IV. In many cases the limited amount of data available, the scatter in the data, and the nature of the comparisons made a quantitative statement in terms of statistical parameters no more useful than a careful qualitative statement. Therefore the method used to compare the theoretical results with experimental data was to both plot the data and draw the theoretical curve for a given situation on the same appropriate sheet of graph paper. The agreement between the theoretical and experimental relationship was then classified as either good, fair, or poor, according to the following definitions.

1. Good - the theoretical curve passed through the data in close approximation to a "best" curve as drawn by eye through the data.

2. Fair - the theoretical curve was generally within the range of scatter of the data but varied noticeably from a "best" curve drawn by eye through that data.

3. Poor - the theoretical curve showed the same qualitative features as the data but was outside the range of scatter of the data.
The following conclusions were outstanding:

1. The theory was in good agreement with the basic S-N relationship for constant amplitude tests.

2. For damage short of failure accumulated at a constant stress-amplitude the theoretical results were in fair agreement with the data.

3. When damage was accumulated in several different periods during each of which the stress-amplitude was constant but between which the stress-amplitude was changed, the theoretical results were found to be in fair to poor agreement with the data.

4. When the stress-amplitude was uniformly increased while damage was accumulated the theoretical results were in good agreement with experimental data.

5. When the stress-amplitude was varied from cycle to cycle in a random manner the agreement between theory and experimental data was poor.

In addition to direct comparison with appropriate experimental data it is also important to compare a new theory with other theories dealing with the same phenomenon. If agreement with experimental observations is as good or better for the new theory than for existing theories, then in one sense the new theory is successful. On this basis the present theory was shown to rate very well.

Not only did the results of the present theory compare favorably with results from other theories for particular types of loading but they also provide a common basis for evaluating damage. A very desirable
feature for a cumulative damage theory is a mathematical method of evaluating fatigue damage accumulated under conditions of varying amplitude when data is available only from constant amplitude tests. When most cumulative damage theories are examined with regard to this feature they are found to contain constants which require evaluation by tests performed with loading histories of the same type as that to be evaluated.

In this sense then it was concluded that the present theory is successful because it presents a formal method for evaluating the accumulation of fatigue damage on the basis of constant amplitude data. For the wide variety of types of variable loading which satisfy the conditions under which the basic differential equation was derived, the only requirement is that the differential equation be solvable. In terms of a closed analytical solution this may present an insurmountable difficulty. However, the use of numerical methods particularly in conjunction with digital computers, or the use of analog computers will always provide some kind of a solution.

These conclusions suggest that the optimistic beginnings in Chapter II are to a considerable extent justified. It does now appear possible that the heretofore disconnected relationships between cycles to failure and the nature of the amplitude sequence can be correlated by a fairly straightforward phenomenological approach. To a considerable extent this correlation is achieved by the equations developed in Chapter II.

One of the outstanding conclusions resulting from the comparisons in Chapters III and IV was that interaction, as discussed in the second section of Chapter IV, does occur. In the cases where the theory devi-
ated most from the experimental results the deviation was compatible
with a concept of interaction. If this concept can be incorporated into
the basic theory the resulting relationships may well provide a fully
satisfactory mathematical treatment of the phenomenon of cumulative dam-
age.

2. **Conclusions Pertaining to the Application of the Theory.**— On
the basis of the above conclusions the suitability of the new theoretical
relationships for design calculations was considered. The factors which
particularly indicated the potential usefulness of the theoretical re-
sults included:

1. The theoretical equations are in forms relatively convenient
   for calculation.

2. A small number of experimental constants are required.

3. The experimental constants for use with all types of loading
can be evaluated from constant amplitude tests.

4. A composite plot of conventional S-N data for a wide variety
of steels is possible when the proper non-dimensional parame-
ters are used.

The significance of the first two of these factors is largely
self-evident. The third factor is of extreme importance because it pro-
vides a basis for evaluating cumulative damage in situations of variable
stress-amplitude without the necessity of testing under those same con-
ditions. That is, the required constants can be evaluated under simpler
conditions of constant stress-amplitude.
The fourth factor is particularly significant because it opens a way for combining vast amounts of data. Such composite plots provide a method of accurately determining $S$-$N$ curves without the large amount of testing heretofore required. The two parameters which make the composite plots possible, the Overstress Ratio and the Life Ratio, are discussed in section 1 of Chapter III.

The accuracy of the theoretical relationships in cases of varying stress-amplitude was shown to be very reasonable for a wide range of steels and for many different stress-amplitude sequences. The case of random loading should be particularly considered for in this case the theoretical predictions and the experimental results were the furthest apart. However, when the theoretical results were compared with predictions made with the linear rule (Miner's equation) they were found to be considerably closer to the experimental results in every case. These comparisons were made for several very different types of random loading.

The full implication of this comparison is apparent when it is recognized that the linear rule is today the basis for the usual design procedures when structure and machine components must be designed to support such loads. Therefore the equations resulting from the theory developed in Chapter II can be expected to provide useful design methods. Applications of these equations to design calculations are considered in the remainder of this chapter.
3. Design Equations.- The design of a machine part for short life, that is for loads above the endurance limit, may require calculations to determine the average life for established dimensions. Conversely, if a required life is specified it will be necessary to determine the dimensions to produce that life.

The theoretical developments of Chapter II and the evaluations in Chapters III and IV have been in terms of average or 50% probability levels. In the following applications 50% probability lives are also considered. The specification or structure size producing a 50% probability of survival for a given life may not be adequate in many instances. Because of this the statistical description of the scatter about the average is an important and integral part of the total problem.

Although methods for estimating the scatter under conditions of varying load amplitude on the basis of constant amplitude data can be developed through consideration of the same theoretical relationships used for the 50% probability estimates, their development is not within the scope of the present treatment.

A bending moment was chosen as a typical load. The amplitude of the bending moment is represented by $M$. The section modulus is used as the measure of structure size and is represented by $Z$. Therefore, from the basic relations of strength of materials and in the usual notation,

$$S = \frac{M}{Z}.$$  

(60)

The applications are based on the assumption that equation (15) represents the S-N curve with $E_N = CS = (E_0/S_0)S$.

$$\frac{kN}{S - E_0} = \frac{1}{S(1 - C)}.$$  

(81)
It is further assumed that data are available for the material to be used specifying the ultimate tensile strength, \( S_U \); the average virgin endurance limit, \( E_0 \); and the average life, \( N \), for at least one stress-amplitude, \( S \), above the endurance limit.

One case of interest requires the determination of the average number of cycles which a structure of a given size will resist when a known reversed moment of constant amplitude is applied. If the structure size is predetermined, then \( Z \) is fixed and the stress-amplitude due to an applied moment can be calculated. From the known data the constant \( k \) and \( C \) in equation (81) can be determined. Then using this value of \( k \), and the values of \( E_0 \) and \( C \) the life \( N \) corresponding to the stress-amplitude due to a constant amplitude moment is easily calculated from:

\[
N = \frac{1}{k} \left[ \frac{1}{S - E_0} - \frac{1}{S(1 - C)} \right]
\]

or

\[
N = \frac{Z}{k} \left[ \frac{1}{M - ZE_0} - \frac{1}{M(1 - C)} \right].
\]  

(82)

If the section modulus is to be determined to provide a specified average life then this equation must be solved for \( Z \). Although equation (82) is relatively simple it yields the following quadratic equation in \( Z \) when rearranged.

\[
\frac{E_0}{M(1 - C)} Z^2 + (E_0 kN - \frac{C}{1 - C})Z - MnN = 0.
\]

(83)

This equation and the resulting solution can be simplified somewhat for those cases where \( C = E_0 / S_U = 1/2 \). The expression for \( Z \) is then,
When many calculations are to be done, particularly where some control can be exercised on the data which will provide the basis of the calculations, convenient use can be made of the general $\gamma$-L relation discussed in section 1 of Chapter III and expressed in equation (70).

$$K^*L = \frac{1}{\gamma - 1} - \frac{1}{\gamma(1 - c)}.$$  \hfill (70)

In this expression $L$ is the life ratio and $\gamma$ is the overstress ratio corresponding to any stress-amplitude, $S$. (The life ratio is the ratio of the cycles of life at a stress-amplitude, $S$, to the cycles of life at a particular reference stress-amplitude, $S^*$. The overstress ratio is $S^*/E_0$.) The constant $K^*$ is a function of $S^*$, $E_0$, and $S_u$.

$$K^* = \frac{1}{\gamma^* - 1} - \frac{1}{\gamma^*(1 - c)}.$$  

The reference overstress ratio, $\gamma^*$, is equal to $S^*/E_0$.

Equation (70) was found to fit a large amount of data from widely varying steels if a common value of $\gamma^*$ was used. Therefore if the life, $\gamma^*$, corresponding to the same value of $\gamma^*$ for each of the materials to be considered is known, a plot of equation (70) can be used in a graphical solution as follows.

If a life $N$ is specified for a given design the corresponding value of $L = N/N_{\gamma^*}$ can be calculated and the associated value of $\gamma$ can be read from the curve defined by equation (70). Then since $S = \gamma x E_0$ and $Z = W/S$ the required section modulus can be calculated from

$$Z = W/(\gamma E_0).$$  \hfill (85)
Under certain conditions it may be necessary to estimate the value of the endurance limit for a structure with a section modulus $Z$ after a constant load amplitude, $M$, has been applied for a given number of cycles, $n$. Again assuming that the endurance limit and at least one other point on the $S$-$N$ curve are known so that $k$ can be evaluated the calculation could be based on equation (9). Assuming that $k = 1$ and substituting $S = M/Z$ this equation can be solved for $E$.

$$E = \frac{M}{Z} - \frac{1}{\frac{M}{Z} - E_0 - kn} .$$

(86)

Alternatively it may be required to determine the number of cycles which will reduce the endurance limit to $E_1$ when the moment amplitude and section size are known. In this case the same equation, solved for $n$ and with $E_1$ substituted for $E$ can be used.

$$n = \frac{1}{k} \left[ \frac{1}{\frac{M}{Z} - E_0} - \frac{1}{\frac{M}{Z} - E_1} \right] .$$

(87)

Still another situation may require the determination of the section size when a given moment amplitude must be resisted for a certain number of cycles without reducing the endurance limit below a minimum magnitude, $E_2$. Again the same equation can be applied by solving for $Z$ and substituting the minimum level for the endurance limit.

$$Z = \frac{M}{E_0 + E_2 + \sqrt{\frac{E_0 - E_2}{2}} \frac{E_0 - E_2}{kn}} .$$

(88)

Similar design and analysis problems may arise in connection with loading of other than constant amplitude. In each case the solution of
equation (4) with $p = 1$ for steel, may be converted into a design equation by substituting the respective expression for the stress-amplitude in terms of the load amplitude and the parameter characteristic of the section size. The resulting equation will provide a design or analysis equation for the number of cycles to failure, reduction in the endurance limit, or the section size depending on the form in which the equation is written.

In many cases the solution of equation (4) may not be possible in closed analytical form or if such solution can be obtained the expression resulting from the substitution of the load and size parameters for the stress-amplitude may be unwieldy. In such instances it may be necessary or convenient to separate the problem into two steps. The first step would be to determine the desired solution in terms of stress-amplitude and the second step would be to relate the stress-amplitude to the section modulus and the load amplitude.

Random loading is a case of varying amplitude for which the composite design equation can best be used with a trial and error solution. The relationship at failure between the parameters for this type of loading is given by equation (62) in which $E_N$ is the endurance limit at failure.

$$kN = \frac{1}{\sqrt{c-b^2}} \left[ \tan^{-1} \left( \frac{b-E_N}{\sqrt{c-b^2}} \right) - \tan^{-1} \left( \frac{b-E_0}{\sqrt{c-b^2}} \right) \right].$$  \hspace{1cm} (62)

For a constant stress-amplitude, $S$, the terminal value of the endurance limit was found to be $CS$ where $C$ was approximately $S_u/E_0$. For random loading final rupture will very likely occur during a cycle of
the largest stress amplitude. The terminal condition then becomes

$$E_N = C S_{\text{max}}$$

where $S_{\text{max}}$ is the maximum stress-amplitude.

In equation (62) the constants $c$ and $b$ are characteristic of the random stress-amplitude and are given by the following expressions.

$$b = \frac{\sum_{i=1}^{J} p_i S_i}{\sum_{i=1}^{J} p_i}$$

$$c = \frac{\sum_{i=1}^{J} p_i S_i^2}{\sum_{i=1}^{J} p_i}$$

$S_1, S_2, S_3, \ldots S_j$ represent all the values which the stress-amplitude will attain with a given random sequence and $p_1, p_2, p_3, \ldots p_j$ are the associated probabilities. If a case of simple bending is again considered with $S_i = M_i/Z$, then the constants $b$ and $c$ can be expressed in terms of the $M_i$ and $Z$.

$$b = \frac{\sum_{i=1}^{J} p_i (M_i/Z)}{Z} = \frac{\sum_{i=1}^{J} p_i M_i}{Z^2} = \frac{b_m}{Z}$$

$$c = \frac{\sum_{i=1}^{J} p_i (M_i^2/Z^2)}{Z} = \frac{\sum_{i=1}^{J} p_i M_i^2}{Z^2} = \frac{c_m}{Z^2}$$

156
When \( b \) and \( c \) are replaced by \( b_m \) and \( c_m \) in equation (62) the result is

\[
Kn = \frac{Z}{\sqrt{c_m - b_m^2}} \left[ \tan^{-1} \frac{b_m - c M_{\text{max}}}{\sqrt{c_m - b_m^2}} - \tan^{-1} \frac{b - ZE_0}{\sqrt{c_m - b_m^2}} \right] \tag{89}
\]

where \( M_{\text{max}} = Z S_{\text{max}} \).

Equation (89) cannot be conveniently solved for \( Z \). However, a trial and error solution can be obtained by assuming a value for \( Z \), substituting this value into the \( \tan^{-1} \) at the extreme right of the bracketed term, and then solving for the \( Z \) term appearing in the fraction \( \frac{Z}{\sqrt{c_m - b_m^2}} \). If the assumed value is too small for the desired life, \( N \), then the calculated value will be larger than both the necessary value and the assumed value. A second assumption about midway between the first assumption and the first calculated value can then be used to calculate a second value. A few repetitions of this process will lead to the solution. This process might be shortened by plotting the calculated values versus the assumed values on rectangular coordinates. As soon as enough points are calculated to define the curve in the region where the assumed value is equal to the calculated value the curve can be drawn and the desired value read from the curve.

In section 2 of Chapter IV a method is described for calculating the total theoretical life for random stress-amplitude sequences in which some of the stress-amplitudes are below \( E_0 \). These life predictions were found to be longer than experimental results would indicate. This error was seriously large but considerably less than the corresponding error in similar predictions by the linear rule. Based on the information in Tables 1 and 2 correction factors in Table 3 below are tentatively recommended to compensate for apparent error in the theoretical predictions.
TABLE 3
Correction Factors for Random Loading

<table>
<thead>
<tr>
<th>Probability that $S &lt; E_0$</th>
<th>Correction Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50</td>
<td>2</td>
</tr>
<tr>
<td>0.56</td>
<td>4</td>
</tr>
<tr>
<td>0.75</td>
<td>5</td>
</tr>
<tr>
<td>0.9</td>
<td>6</td>
</tr>
</tbody>
</table>

The correction factor is designed so that the total calculated life values are to be divided by the factor to give actual expected lives. If a life is specified and the theory is to be used to determine the required section modulus, then the specified life should be multiplied by the correction factor before the calculation for $Z$ is carried out. Since the value of the correction factor depends on the stress-amplitude and therefore on $Z$, it will be necessary to initially assume an intermediate value of the correction factor. After the section modulus has been selected the assumed value of the correction factor can be checked. If the assumed value is found to be incorrect, a new value based on the calculated stress-amplitudes can be used to redetermine a new required value of section modulus.

The probability that $S < E_0$ is determined by adding the probabilities, $p_i$, for all $S_i < E_0$ when a finite number of discrete values of amplitude can occur, while for a continuous frequency distribution the frequency function $p(S)$ is integrated from the minimum value of the stress-amplitude to the virgin endurance limit.

158


I, Robert Roswell Catts, was born in Berlin Heights, Ohio, March 2, 1925. I received my secondary education in the public schools of Berlin Heights, Ohio, and my undergraduate training at Kent State University and The Ohio State University. I was granted the Bachelor of Mechanical Engineering and Master of Science Degrees by The Ohio State University in 1951. While in residence there I was a research assistant to Dr. W. L. Starkey during the year 1950-51. In January, 1953, after working in industry for one and one-half years, I was appointed Research Associate in the Department of Mechanical Engineering at The Ohio State University. In October 1956 I was appointed Research Associate and part-time Instructor and in October 1958 I received an appointment as Assistant Professor. During the period of six years subsequent to January, 1953, I was concurrently completing the requirements for the degree Doctor of Philosophy.