ANALYTICAL STUDY OF THE ENERGY-WEIGHT AND ENERGY-VOLUME CHARACTERISTICS OF ENERGY STORAGE SYSTEMS

Dissertation
Presented in Partial Fulfillment of the Requirements
for the Degree Doctor of Philosophy in the
Graduate School of The Ohio State University

By

LEO VIRGIL KLINE, B.M.E., M.Sc.
The Ohio State University
1954

Approved by:

S.M. Murco
Adviser
ACKNOWLEDGMENTS

The author wishes to express his appreciation to Professor S. M. Marco, of the Mechanical Engineering Department of The Ohio State University, for his guidance and encouragement throughout the planning, development and writing of this dissertation.

The author also wishes to express his appreciation to various other individuals for their suggestions and assistance during the period that this dissertation was being developed. These individuals are Dr. W. L. Starkey, R. O. Niemi, R. H. Zimmerman and E. Roberts.
<table>
<thead>
<tr>
<th>No.</th>
<th>Figure Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Sketch Showing the Three Configurations of Flywheels Evaluated</td>
<td>13</td>
</tr>
<tr>
<td>2</td>
<td>Energy-Weight Relation for Steel Flywheel with Eight Arms</td>
<td>16</td>
</tr>
<tr>
<td>3</td>
<td>Energy-Volume Relation for Steel Flywheel with Eight Arms</td>
<td>19</td>
</tr>
<tr>
<td>4</td>
<td>Energy-Weight Relation for a Cylindrical Steel Flywheel</td>
<td>22</td>
</tr>
<tr>
<td>5</td>
<td>Energy-Volume Relation for a Steel Cylindrical Flywheel</td>
<td>24</td>
</tr>
<tr>
<td>6</td>
<td>Maximum Allowable Speed in Terms of Maximum Allowable Stress for a Cylindrical Steel Flywheel</td>
<td>25</td>
</tr>
<tr>
<td>7</td>
<td>Maximum Allowable Speed Per Unit Stress for a Steel Uniform Stress Type Flywheel</td>
<td>37</td>
</tr>
<tr>
<td>8</td>
<td>Energy Storage Capacity Per Unit Stress for a Steel Uniform Stress Type Flywheel</td>
<td>38</td>
</tr>
<tr>
<td>9</td>
<td>Energy Per Unit Weight for a Steel Uniform Stress Type Flywheel</td>
<td>39</td>
</tr>
<tr>
<td>10</td>
<td>Energy Per Unit Volume for a Steel Uniform Stress Type Flywheel</td>
<td>44</td>
</tr>
<tr>
<td>11</td>
<td>Variation of Drag Coefficient with Reynolds Number</td>
<td>47</td>
</tr>
<tr>
<td>12</td>
<td>Time-Energy Relation for a Cylindrical Steel Flywheel Rotating in Air</td>
<td>54</td>
</tr>
<tr>
<td>13</td>
<td>Time-Energy Relation for a Cylindrical Steel Flywheel Rotating in Low Pressure Hydrogen</td>
<td>55</td>
</tr>
<tr>
<td>14</td>
<td>Energy Per Unit Volume for a Coil Spring</td>
<td>66</td>
</tr>
<tr>
<td>15</td>
<td>Energy-Per-Unit-Weight Function vs. Diameter Ratio for a Torsion Bar Spring</td>
<td>68</td>
</tr>
<tr>
<td>16</td>
<td>Relation Between Energy-Per-Unit-Volume Function and Diameter Ratio for a Torsion Bar Spring</td>
<td>70</td>
</tr>
<tr>
<td>17</td>
<td>Sketch of Spiral-Wound Spring</td>
<td>72</td>
</tr>
<tr>
<td>18</td>
<td>Energy-Per-Unit-Volume Function vs. Radius Ratio for a Clock Type Spring</td>
<td>75</td>
</tr>
<tr>
<td>19</td>
<td>Sketches Showing the Various Means of Stacking Belleville Springs</td>
<td>77</td>
</tr>
<tr>
<td>No.</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>-----</td>
<td>------------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>20</td>
<td>Energy Per Unit Weight for a Belleville Spring</td>
<td>81</td>
</tr>
<tr>
<td>21</td>
<td>Energy Per Unit Volume for a Single Belleville Spring</td>
<td>83</td>
</tr>
<tr>
<td>22</td>
<td>Pressure-Relative Volume Relation for Ether at 68°F</td>
<td>92</td>
</tr>
<tr>
<td>23</td>
<td>Energy Per Unit Weight and Energy Per Unit Original Volume vs. Final Pressure for Ether Isothermally Compressed from One Atmosphere</td>
<td>93</td>
</tr>
<tr>
<td>24</td>
<td>Energy Storage Capacity Per Unit Weight for Ether at 68°F</td>
<td>98</td>
</tr>
<tr>
<td>25</td>
<td>Energy Storage Capacity Per Unit Total Volume for Ether at 68°F</td>
<td>100</td>
</tr>
<tr>
<td>26</td>
<td>Relation Between Thermal Expansion and Pressure Ratio for Ether</td>
<td>103</td>
</tr>
<tr>
<td>27</td>
<td>Relation Between Compressibility and Pressure of Ether</td>
<td>104</td>
</tr>
<tr>
<td>28</td>
<td>Relation Between the Quantity (\frac{\partial V_r}{\partial p}/(\frac{\partial V_r}{\partial T})) and Pressure for Ether</td>
<td>105</td>
</tr>
<tr>
<td>29</td>
<td>Variation of Energy-Weight Relation with Temperature for Ether</td>
<td>106</td>
</tr>
<tr>
<td>30</td>
<td>Variation of Energy-Volume Relation with Temperature for Ether</td>
<td>108</td>
</tr>
<tr>
<td>31</td>
<td>Energy-Weight Relations for Expansion of Gases</td>
<td>115</td>
</tr>
<tr>
<td>32</td>
<td>Energy-Weight Relations for Expanding Compressed Air</td>
<td>122</td>
</tr>
<tr>
<td>33</td>
<td>Energy Volume Relations for Compressed Air Expansion</td>
<td>127</td>
</tr>
<tr>
<td>34</td>
<td>Ratio of Isothermal Work to Adiabatic Work vs. Pressure Ratio</td>
<td>131</td>
</tr>
<tr>
<td>35</td>
<td>Correction Factor for Work Against Atmosphere</td>
<td>134</td>
</tr>
<tr>
<td>36</td>
<td>Ratio of Energy Storage Capacity Before Cooling to Energy Storage Capacity After Cooling vs. Pressure Ratio</td>
<td>137</td>
</tr>
</tbody>
</table>
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td>ii</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>iii</td>
</tr>
<tr>
<td>TABLE OF CONTENTS</td>
<td>v</td>
</tr>
</tbody>
</table>

## Chapter I

**INTRODUCTION**

1.1 Background 1
1.2 Statement of Problem 2
1.3 Purpose of Problem 2

## Chapter II

**SUMMARY OF RESULTS** 3

## Chapter III

**ENERGY STORAGE CAPACITY OF A FLYWHEEL**

3.1 Introduction 12
3.2 Energy Storage Capacity As a Function of Weight for the Rim-Arm Type of Flywheel 12
3.3 Energy Storage Capacity As a Function of Volume for the Rim-Arm Type Flywheel 17
3.4 Energy Storage Capacity As a Function of Weight for a Cylindrical Flywheel 20
3.5 Energy Storage Capacity As a Function of Volume for a Cylindrical Flywheel 21
3.6 Angular Velocity of Cylinder 23
3.7 Optimum Thickness of Cylinder 26
Chapter IV

ENERGY STORAGE CAPACITY OF A SPRING

4.1 Introduction 63
4.2 Energy Storage Capacity As a Function of Weight for Coil Springs 63
4.3 Energy Storage Capacity As a Function of Volume for Coil Springs 64
4.4 Energy Storage Capacity As a Function of Weight for Torsion Bar Springs 67
4.5 Energy Storage Capacity As a Function of Volume for Torsion Bar Springs 69
4.6 Energy Storage Capacity of Spiral Wound Springs As a Function of Weight 71
4.7 Energy Storage Capacity of Spiral Wound Springs As a Function of Volume 73
4.8 Energy Storage Capacity As a Function of Weight for Belleville Springs 76
4.9 Energy Storage Capacity As a Function of Volume for Belleville Springs 80
4.10 Consideration of Losses 84
4.11 Summary 84
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Chapter V</strong></td>
<td><strong>ENERGY STORAGE CAPACITY OF A COMPRESSED LIQUID</strong></td>
<td></td>
</tr>
<tr>
<td>5.1</td>
<td>Introduction</td>
<td>90</td>
</tr>
<tr>
<td>5.2</td>
<td>Energy Storage Capacity As a Function of the Weight of the Liquid</td>
<td>90</td>
</tr>
<tr>
<td>5.3</td>
<td>Energy Storage Capacity As a Function of the Volume of the Liquid</td>
<td>94</td>
</tr>
<tr>
<td>5.4</td>
<td>Energy Storage Capacity As a Function of the Weight of the Liquid and Container</td>
<td>94</td>
</tr>
<tr>
<td>5.5</td>
<td>Energy Storage Capacity As a Function of the Volume of the Liquid and Container</td>
<td>99</td>
</tr>
<tr>
<td>5.6</td>
<td>Consideration of Losses</td>
<td>101</td>
</tr>
<tr>
<td>5.7</td>
<td>Summary</td>
<td>107</td>
</tr>
</tbody>
</table>

| **Chapter VI** | **ENERGY STORAGE CAPACITY OF A COMPRESSED GAS** | |
| 6.1 | Introduction | 113 |
| 6.2 | Energy Storage Capacity As a Function of the Weight of the Gas Only | 113 |
| 6.3 | Energy Storage Capacity As a Function of the Volume of the Gas Only | 117 |
| 6.4 | Energy Storage Capacity As a Function of the Weight of the Gas and Container | 117 |
| 6.5 | Energy Storage Capacity As a Function of the Volume of Space Occupied by the Container | 125 |
| 6.6 | Ratio of Isothermal Work to Isentropic Work | 129 |
| 6.7 | Consideration of Work Done Against the Atmosphere | 132 |
| 6.8 | Consideration of Losses | 135 |
| 6.9 | Summary | 138 |
### Chapter VII

**ENERGY STORAGE CAPACITY OF AN ELECTROCHEMICAL CELL**

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.1 Introduction</td>
<td>141</td>
</tr>
<tr>
<td>7.2 Lead-Acid and Edison Cells</td>
<td>141</td>
</tr>
<tr>
<td>7.3 Miscellaneous Cells</td>
<td>142</td>
</tr>
</tbody>
</table>

### Chapter VIII

**MISCELLANEOUS ENERGY STORAGE SYSTEMS**

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.1 Introduction</td>
<td>144</td>
</tr>
<tr>
<td>8.2 Energy Storage Capacity of a Translating Medium</td>
<td>144</td>
</tr>
<tr>
<td>8.3 Energy Storage Capacity of a Raised Medium</td>
<td>144</td>
</tr>
<tr>
<td>8.4 Energy Storage Capacity of an Electrostatic Capacitor</td>
<td>144</td>
</tr>
</tbody>
</table>

**AUTOBIOGRAPHY**

146
CHAPTER I

INTRODUCTION

1.1 BACKGROUND - In the field of mechanics, the energy storage capacity of a system is defined to be the system's ability to do work. The understanding of this definition can be facilitated by an examination of the first law of thermodynamics. This law can be written mathematically as

$$W = Q - \Delta U,$$  \hspace{1cm} (1.1)

where $Q$, $W$, and $U$ represent heat added to the system, work done by the system, and internal energy of the system, respectively. Heat can be defined as energy transferred, without transfer of mass, across the boundary of a system because of a temperature difference between system and surroundings. Work can be defined as energy transferred, without transfer of mass, across the boundary of a system because of an intensive property difference other than temperature that exists between system and surroundings. Internal energy can be defined as any energy, other than heat or work. This energy can be considered as any energy stored within a system. Equation (1.1) indicates that a system's ability to do work is dependent on the amount of heat which crosses the boundaries of the system and on the change in the internal energy of the system. The amount of heat which crosses the boundaries and the change in internal energy are, in
turn, dependent on the process which is used for extracting work from the system. Thus, according to the mechanics concept of energy, the energy storage capacity of a system is dependent on the process used to extract work from the system. This is the concept of energy which will be used in this dissertation.

1.2 STATEMENT OF PROBLEM - For any energy-storage system, the following statements can be made:

(1) It may be possible to extract work from the system. According to the statements made in section 1.1, the energy-storage capacity of the system is evaluated by its ability to do work.

(2) The system has some weight.

(3) The system has some volume.

The problem of this dissertation is to evaluate the energy-storage capacity per unit weight and the energy-storage capacity per unit volume for various systems.

1.3 PURPOSE OF PROBLEM - For most engineering applications, especially those applications which deal with aircraft, increasing emphasis is being placed on weight and volume conservation. Thus, two of the primary items which need consideration in any design are the amount of work which can be obtained from a given system per unit weight and the amount of work which can be obtained from a given system per unit volume. At the present time, there is no systematic evaluation of energy storage systems which can be used in selecting the best system for a given application. It is the purpose of this dissertation to supply such an evaluation.
CHAPTER II

SUMMARY OF RESULTS

The results of this dissertation are summarized in the following four categories:

(1) Graphs which are pictorial representations of various analyses.

(2) Detailed summaries of the most significant equations which have arisen from various analyses.

(3) Detailed summaries of the most significant conclusions which have arisen from various analyses.

(4) General summary showing the relative suitability of using various energy storage systems.

The graphs which have been included in this work are located in the sections to which they pertain. A list of these graphs is included in the prefatory pages. A detailed summary of the most significant equations and conclusions which have arisen from the analyses of any given chapter is included in the summary section for that chapter.

This chapter is devoted to a general summary as described by item (4) above. The following are lists of the most common energy storage systems in order of their decreasing energy-weight and energy-volume characteristics.

<table>
<thead>
<tr>
<th>Energy Storage Capacity per Unit Weight</th>
<th>@</th>
<th>ft lb/lb</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Ag-Zn electrochemical cell</td>
<td>@</td>
<td>146,000</td>
</tr>
<tr>
<td>(2) Lead-acid electrochemical cell</td>
<td>@</td>
<td>38,000</td>
</tr>
<tr>
<td>(3) Edison electrochemical cell</td>
<td>@</td>
<td>35,000</td>
</tr>
<tr>
<td>(4) Uniform stress disk with rim</td>
<td>@</td>
<td>27,350</td>
</tr>
<tr>
<td>(5) Compressed gas (spherical container)</td>
<td>@</td>
<td>22,600</td>
</tr>
<tr>
<td>(6) Compressed gas (cylindrical container)</td>
<td>@</td>
<td>18,400</td>
</tr>
</tbody>
</table>

@ The notes which are referred to in this list are included at the end of Chapter II.
<table>
<thead>
<tr>
<th>No.</th>
<th>Energy Storage System</th>
<th>Value (ft lb/ft³)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Ag-Zn electrochemical cell</td>
<td>14,800</td>
</tr>
<tr>
<td>2</td>
<td>Uniform stress disk with rim</td>
<td>9,000</td>
</tr>
<tr>
<td>3</td>
<td>Cylindrical flywheel</td>
<td>8,750</td>
</tr>
<tr>
<td>4</td>
<td>Compressed gas (spherical container)</td>
<td>5,200</td>
</tr>
<tr>
<td>5</td>
<td>Lead-acid electrochemical cell</td>
<td>5,000</td>
</tr>
<tr>
<td>6</td>
<td>Edison electrochemical cell</td>
<td>4,800</td>
</tr>
<tr>
<td>7</td>
<td>Compressed gas (cylindrical container)</td>
<td>4,000</td>
</tr>
<tr>
<td>8</td>
<td>Rim-arm flywheel</td>
<td>1,440</td>
</tr>
<tr>
<td>9</td>
<td>Compressed liquid (ether)</td>
<td>90</td>
</tr>
<tr>
<td>10</td>
<td>Compressed solid (torsion spring)</td>
<td>8</td>
</tr>
<tr>
<td>11</td>
<td>Compressed solid (coil spring)</td>
<td>5</td>
</tr>
<tr>
<td>12</td>
<td>Compressed solid (spiral-wound spring)</td>
<td>4</td>
</tr>
<tr>
<td>13</td>
<td>Compressed solid (Belleville spring)</td>
<td>2</td>
</tr>
</tbody>
</table>

The values which are included in these lists are the largest values which are felt to be physically practical. Thus, although an infinite amount of energy can theoretically be stored in a flywheel, the practical maximum energy storage capacity is fixed by physical limitations.

For any given energy storage means, it should be emphasized that the design which produces the best energy-weight characteristics will not necessarily produce the best energy-volume characteristics. For example, for a uniform stress disk with a rim, the combination of dimensions which produces the maximum energy per unit weight is not the combination of dimensions which produces the maximum energy per unit volume.

For applications where it is unnecessary to extract energy from a system more rapidly than it can be extracted from a Ag-Zn electrochemical cell, this cell is the best of the energy storage systems which have been
considered in this dissertation.

For applications where it is necessary to extract energy from a system more rapidly than it can be extracted from an electrochemical cell, the uniform stress disk with a rim is the best of the energy storage systems considered.

If it is desired to store energy for a longer period of time than is practical in a flywheel, and it is necessary to extract energy from the system more rapidly than it can be extracted from an electrochemical cell, then a compressed gas energy storage system is best. Storage in a spherical container is superior to storage in a cylindrical container.

Although springs are one of the most commonly used means of storing energy, they are the poorest of the means which have been considered. The magnitudes of the energy storage capacity per unit weight and the energy storage capacity per unit volume for a compressed liquid system are approximately ten times the magnitudes for a spring system.
See Chapter VII.

See Fig. 9 where $C = 1$, $k = 0.4$, $\rho = 7.31\times10^{-4}$ lb sec$^2$/in$^4$, and $S_d = 100,000$ psi.

See Fig. 32 where $V = 1, k = 0.283$ lb/in$^3$, $S_{yp} = 100,000$ psi, $T_1 = 520^\circ R$, $c_v = 13.5$ ft lb/ft$^2 \cdot^\circ F$, $R = 53.3$ ft lb/ft$^2 \cdot^\circ R$, $p_0 = 6,000$ psi, $p_f = 1.7$ psi and the process is isentropic.

See Fig. 4 where $t/r = 0.2$, $\mu = 0.288$, $\rho = 7.31\times10^{-4}$ lb sec$^2$/in$^4$ and $S_m = 100,000$ psi.

See Fig. 2 where $t/r_m = 0.3$, $\rho = 7.31\times10^{-4}$ lb sec$^2$/in$^4$, and $S_m = 25,000$ psi. The allowable stress in the rim-arm type flywheel has been chosen lower than in other types of flywheels due to the greater stress concentration in the rim-arm flywheel.

See Fig. 25 where $p_f = 2$ atmospheres, $L_{r_0}/r_{1r} = 10$, $S_{yp} = 100,000$ psi, ether is the working medium, and the process is isothermal.

See equation (4.31) where $E = 30\times10^6$ psi, $\rho = 7.31\times10^{-4}$ lb sec$^2$/in$^4$ and $S_m = 50,000$ psi.

See equation (4.6) where $G = 11.5\times10^6$ psi, $\rho = 7.31\times10^{-4}$ lb sec$^2$/in$^4$ and $S_m = 50,000$ psi.

See Fig. 20 where $t/r_2 = 0.1$, $E = 30\times10^6$ psi, $\mu = 0.288$, $\rho = 3.71\times10^{-4}$ lb sec$^2$/in$^4$, $r_1/r_2 = 0.8$ and $S_m = 100,000$ psi.

See Fig. 10 where $C = 1.5$, $k = 0.75$, and $S_d = 100,000$ psi.

See Fig. 5 where $t/r = 0.2$, $\mu = 0.288$, and $S_m = 100,000$ psi.

See Fig. 33 where $S_{yp} = 100,000$ psi, $T_1 = 520^\circ R$, $c_v = 13.5$ ft lb/ft$^2 \cdot^\circ F$, $R = 53.3$ ft lb/ft$^2 \cdot^\circ R$, $p_0 = 30,000$ psi, $p_f = 1.7$ psi and the process is isentropic.

See Fig. 3 where $t/r_m = 0.4$, $S_m = 25,000$ psi and $n = 8$. See note 2 for explanation concerning $S_m$.

See Fig. 25 where $L_{r_0}/r_{1r} = 10$, $p_f = 2$ atmospheres, $S_{yp} = 100,000$ psi and the process is isothermal.

See Fig. 16 where $d_1/d_0 = 0.1$, $G = 11.5\times10^6$ psi and $S_{sm} = 50,000$ psi.
\( \circ \) See Fig. 18 where \( r_a/r_c = 0.1 \), \( E = 30 \times 10^6 \) psi and \( S_b = 100,000 \) psi.

\( \circ \) See Fig. 14 where \( D/d = 2 \), \( G = 11.5 \times 10^6 \) psi and \( S_g = 50,000 \) psi.

\( \bigcirc \) See Fig. 21 where \( t/r_2 = 0.1 \), \( r_1/r_2 = 0.625 \), \( \mu = 0.288 \), \( E = 30 \times 10^6 \) psi and \( S_m = 100,000 \) psi.
NOTATION FOR CHAPTER III

b  width of the flywheel rim measured along an axis which is parallel to the axis of rotation

c  dimensionless thickness function = b/x₂

c_D  drag coefficient

c_M  moment coefficient

e  constant (base of natural logarithms) = 2.718

E  modulus of elasticity

E_f  energy storage capacity of a flywheel

F_a  normal force acting on cross-section of rim

F_c  centrifugal force

G  constant equal to acceleration of a freely falling body

I  polar moment of inertia

I_d  polar moment of inertia of disk

I_r  polar moment of inertia of rim

k  radius function = (r₂r - r₁r)/r₁r

K_h  radius function = r₂r/r₁r

ln  natural logarithm

log  common logarithm

L  length of cylindrical shaped flywheel measured parallel to the axis of revolution

M  mass or moment

M_d  mass of disk

M_f  mass of flywheel

M_r  mass of rim

n  number of arms in rim-arm type flywheel
uniform force per unit area due to the pull of a uniform stress disk on the rim

power loss

total power loss due to friction of bearings

power loss due to friction for a single bearing

power loss due to windage for the surfaces which are perpendicular to the axis of revolution

power loss due to windage for the surface which is not perpendicular to the axis of revolution

total power loss

power loss due to windage

radius

mean radius

inside radius of rim

outside radius of rim

Reynolds number

bending stress

combined stress

stress in uniform stress disk

maximum stress

radial stress

radial stress in rim due to centrifugal force

radial stress in rim due to uniform pull p on inside of rim

tangential stress

tangential stress in rim due to centrifugal force

maximum tangential stress in rim for any given value of C

maximum tangential stress in rim

tangential stress in rim due to uniform pull p on inside of rim
thickness of rim measured along an axis perpendicular to the axis of rotation or thickness of cylindrical shaped disk

volume of space occupied by flywheel

weight which one flywheel bearing must withstand

total bearing load

weight of flywheel

thickness of uniform stress disk measured along an axis parallel to the axis of rotation

thickness of uniform stress disk at zero radius

thickness of uniform stress disk at \( r = r_1 \)

circumferential deformation

displacement of outside radius of the disk

constant = 3.1416

displacement of inside radius of rim

displacement of inside radius of rim due to centrifugal force

displacement of inside radius of rim due to pressure

unit strain

Poisson’s ratio

absolute viscosity of air at \( 60^\circ F \) and 29.92 inches of mercury

\[ \mu_a = 3.78 \times 10^{-1} \text{ lb-sec/ft}^2 \]

absolute viscosity of hydrogen at \( 60^\circ F \) and one inch of mercury pressure

\[ \mu_h = 1.817 \times 10^{-7} \text{ lb-sec/ft}^2 \]

absolute viscosity of medium in which flywheel is operating (see \( \mu_a \) and \( \mu_h \))

density

density of air at \( 60^\circ F \) and 29.92 inches of mercury

\[ \rho_a = 2.737 \times 10^{-3} \text{ slugs/ft}^3 \]

density of hydrogen at \( 60^\circ F \) and one inch of mercury

\[ \rho_h = 5.49 \times 10^{-6} \text{ slugs/ft}^3 \]

density of medium in which flywheel is operating (see \( \rho_a \) and \( \rho_h \))
\( \rho_s \) density of steel

\( \phi \) radius function = \( \frac{r_{2r}^2 - r_{1r}^2}{[C + \mu(1-C)] (r_{2r}^2 - r_{1r}^2) + (r_{2r}^2 + r_{1r}^2)} \)

\( \omega \) angular velocity

\( \tau \) constant = \( 2 \left[ 1 + \frac{1}{(1+k)^2} + \left[ 1 - \frac{1}{(1+k)^2} \right] \left[ (1-\mu)C + \mu \right] \right] / \left[ (k+1)^2 - 1 \right] \left[ (3+\mu) + (1-\mu)/(1+k)^2 \right] \)

\( \tau \) time
CHAPTER III

ENERGY STORAGE CAPACITY OF A FLYWHEEL

3.1 INTRODUCTION - As was pointed out in section 1.1, the energy storage capacity of any system is a function of the \( Q \) and \( \Delta U \) which are associated with a given process. In this chapter, the process which will be used for evaluating the energy storage capacity of a flywheel will be the ideal process where \( Q = 0 \) and \( \Delta U = M \omega^2/2 \). This process is commonly referred to as the process involving a conversion of kinetic energy into work. In this chapter, three configurations of flywheels are considered. The energy storage capacity per unit weight and the energy storage capacity per unit volume are evaluated for each configuration. A consideration of the power losses which are associated with using a flywheel-energy-storage system is also included in this chapter.

3.2 ENERGY STORAGE CAPACITY AS A FUNCTION OF WEIGHT FOR THE RIM-ARM TYPE OF FLYWHEEL - The rim-arm type flywheel is the type of flywheel sketched in Fig. 1-A. In order to carry out the analysis of the energy-weight relation for this type of flywheel, the following assumptions have been made.

3.2.1 ASSUMPTIONS

3.2.1.1 To find the combined tensile stress in the rim due to bending and direct load, it is assumed that the stretch of the arms is three-fourths of that necessary for free expansion of the rim 1/.

---

FIG. 1 SKETCH SHOWING THE THREE CONFIGURATIONS OF FLYWHEELS EVALUATED
3.2.1.2 To find the energy storage capacity of the rim-arm type flywheel, it is assumed that all of the energy is stored in the rim.

3.2.1.3 To find the moment of inertia of the rim, it is assumed that the thickness of the rim is small compared with the mean radius of the rim.

3.2.1.4 To calculate the stress due to bending in the rim, it is assumed that the portion of the rim between the arms is a straight beam, uniformly loaded with the centrifugal force load, and fixed at the ends.

3.2.1.5 The total mass of the flywheel is assumed to be 1.5 times the mass of the rim.

3.2.1.6 The direct stress in the rim due to the centrifugal force is assumed to be uniformly distributed over the cross-section of the rim.

3.2.2 ANALYSIS - According to assumption 3.2.1.1, the combined tensile stress in the rim due to bending and direct loading is given by

\[ S_c = 0.75 S_t + 0.25 S_b. \]  \hspace{1cm} (3.1)

The symbol \( S_t \) represents the direct stress in the rim which would be produced by the centrifugal force if the rim were not restrained by the arms. The symbol \( S_b \) represents the maximum bending stress produced in the rim by the centrifugal force. The expression for the centrifugal force per unit length of rim is

\[ F_c = t b \rho \omega^2 r_m. \]  \hspace{1cm} (3.2)

Writing the equilibrium equation for a unit length of the rim with \( F_c \) acting at the center of gravity reveals that the force acting on the cross-section of the rim can be expressed by

\[ F_a = F_c r_m. \]  \hspace{1cm} (3.3)

Using assumption 3.2.1.6, the stress on the cross-section is given by

\[ S_t = F_c r_m / t b. \]  \hspace{1cm} (3.4)
Combining equations (3.2) and (3.4) gives
\[ S_t = \rho \omega^2 r_m^2. \]  
(3.5)

Using assumption 3.2.1.4, the stress in the rim due to bending is
\[ S_b = 2 \rho \omega^2 \pi^2 r_m^3/n^2 t. \]  
(3.6)

Combining equations (3.1), (3.5) and (3.6) gives
\[ S_c = \rho \omega^2 r_m^2 (0.75 + n^2 r_m/2n^2 t). \]  
(3.7)

Solving this for \( \omega^2 \) gives
\[ \omega^2 = S_c/\rho r_m^2 (0.75 + n^2 r_m/2n^2 t). \]  
(3.8)

Using assumption 3.2.1.2, the energy storage capacity of the flywheel is given by
\[ E_f = I_T \omega^2/2. \]  
(3.9)

Using assumption 3.2.1.3, the moment of inertia can be expressed by
\[ I_T = M_T r_m^2. \]  
(3.10)

Using equations (3.8) and (3.10) in equation (3.9) gives
\[ E_f = M_T S_c/2 \rho (0.75 + n^2 r_m/2n^2 t). \]  
(3.11)

Using assumption 3.2.1.5 permits expression of the energy per unit mass as
\[ E_f/M_f = S_c/3 \rho g (0.75 + n^2 r_m/2n^2 t). \]  
(3.12)

The energy storage capacity per unit of weight of flywheel can be obtained from equation (3.12) by dividing both sides of the equation by \( g \). Thus,
\[ E_f/W_f = S_c/3 \rho g (0.75 + n^2 r_m/2n^2 t). \]  
(3.13)

Fig. 2 is a plot of \( E_f/W_f S_c \) vs. \( t/r_m \) for the case of a steel flywheel with eight arms.

3.2.3 CONCLUSIONS - The following conclusions can be reached from the analysis of section 3.2.2.

3.2.3.1 The energy per unit weight for the rim-arm type flywheel increases
linearly with the value of the combined stress in the rim.

3.2.3.2 Examination of equation (3.13) reveals that \( \frac{E_f}{W_f} \) increases as \( r_m \) decreases. For any given thickness \( t \) the limit to which \( r_m \) is allowed to decrease is governed by assumption 3.2.1.3. Thus, to obtain the maximum energy per unit weight for any given thickness of flywheel, \( r_m \) should be as small as assumption 3.2.1.3 permits.

3.2.3.3 Examination of equation (3.13) reveals that the energy per unit weight increases as \( t \) increases. Thus, for any given mean radius, the thickness \( t \) should be as large as assumption 3.2.1.3 permits. This will make the energy per unit weight maximum.

3.2.3.4 Examination of equation (3.13) reveals that the larger the number of arms in the rim-arm type flywheel, the greater the energy per unit weight. This results because the larger number of arms permits a higher speed to be attained before the rim becomes overstressed.

3.2.3.5 The analysis of section 3.2.2 has revealed that the maximum energy storage capacity per unit weight for a rim-arm type flywheel is in the vicinity of 2,450 ft lb/lb. This is the energy storage capacity per unit weight for a steel flywheel with eight arms where \( t/r_m = 0.3 \) and \( S_{yp} = 25,000 \text{ psi} \). The relatively low value for \( S_{yp} \) has been selected in view of the high stress concentrations in this type of flywheel.

3.3 ENERGY STORAGE CAPACITY AS A FUNCTION OF VOLUME FOR THE RIM-ARM TYPE FLYWHEEL

3.3.1 ASSUMPTION - In order to complete the analysis of section 3.3.2, it has been assumed that the thickness of the rim is small compared with the mean radius of the rim.

3.3.2 ANALYSIS - The approximate volume of space occupied by the rim-arm flywheel can be expressed by
\( V_f = n r_m^2 b. \)  \hspace{1cm} (3.14)

Combination of equations (3.11) and (3.14) gives

\[
\frac{E_f}{V_f} = M_T S_c/2 \rho n r_m^2 b \left( 0.75 + n^2 r_m/2n^2 t \right). \hspace{1cm} (3.15)
\]

Using assumption 3.3.1, the mass of the rim is given by

\[
M_T = 2 \pi n r_m \rho t b. \hspace{1cm} (3.16)
\]

Using this in equation (3.15) gives

\[
\frac{E_f}{V_f} = S_c/(r_m/t)(0.75 + n^2 r_m/2n^2 t). \hspace{1cm} (3.17)
\]

Fig. 3 is a plot of \( \frac{E_f}{V_f} S_c \) vs. \( t/r_m \) for a steel flywheel with eight arms.

3.3.3 CONCLUSIONS - The following conclusions can be reached from the analysis of section 3.3.2.

3.3.3.1 The energy per unit volume of the rim-arm type flywheel increases linearly with the value of the combined stress in the rim.

3.3.3.2 Examination of equation (3.17) reveals that \( \frac{E_f}{V_f} \) increases as \( r_m \) decreases. The limit to which \( r_m \) can decrease is governed by assumption 3.2.1.3.

3.3.3.3 Examination of equation (3.17) reveals that \( \frac{E_f}{V_f} \) increases as \( t \) increases. The limit to which \( t \) can increase, for any given \( r_m \), is governed by assumption 3.2.1.3.

3.3.3.4 Examination of equation (3.17) reveals that the larger the number of arms in the flywheel, the greater the energy per unit volume. This results because the larger number of arms permits a higher speed to be attained before the rim becomes overstressed.

3.3.3.5 The analysis of section 3.3.2 has shown that the maximum energy storage capacity per unit volume for a rim-arm type flywheel is in the vicinity of 1,400,000 ft lb/ft\(^3\). This value is for the case of a steel
flywheel with eight arms where \( t/r_m = 0.4 \) and \( S_{yp} = 25,000 \) psi. The relatively low value for \( S_{yp} \) was chosen due to the large stress concentrations in this type of flywheel.

3.4 ENERGY STORAGE CAPACITY AS A FUNCTION OF WEIGHT FOR A CYLINDRICAL FLYWHEEL - The cylindrical flywheel is of the type shown in (B) of Fig. 1. The energy storage capacity as a function of weight for such a flywheel will be studied in this section.

3.4.1 ANALYSIS - The kinetic energy of a rotating cylinder is given by

\[
E_f = I \omega^2/2. \tag{3.18}
\]

For a cylinder

\[
I = \pi \rho t r^2/2. \tag{3.19}
\]

The maximum stress in the disk \( 2/ \) is given by

\[
S_m = \rho \omega^2 \left[ (3+\mu)r^2/8 + \mu(1+\mu)t^2/2h (1-\mu) \right]. \tag{3.20}
\]

Solving this for \( \omega^2 \) gives

\[
\omega^2 = S_m/\rho \left[ (3+\mu)r^2/8 + \mu(1+\mu)t^2/2h (1-\mu) \right]. \tag{3.21}
\]

Combination of equations (3.18), (3.19) and (3.21) gives

\[
E_f = 2\pi t S_m r^2 \left[ 3 + \mu + \mu(1+\mu) t^2/3 (1-\mu)r^2 \right]. \tag{3.22}
\]

The mass of the cylinder is given by

\[
M_f = \rho t \pi r^2. \tag{3.23}
\]

Combining equations (3.22) and (3.23) gives the energy storage capacity per unit mass as

\[
E_f/M_f = 2S_m/\rho \left[ 3 + \mu + \mu(1+\mu)(t/r)^2/3(1-\mu) \right]. \tag{3.24}
\]

Using a value of 0.288 for \( \mu \) and dividing both sides of equation (3.24) by \( g \) gives

\[
\]

20
\[
\frac{E_f/W_f}{Sm/P_s} = 1.644 + 0.0867(t/r)^2. \tag{3.25}
\]

Fig. 4 is a plot of \(E_f/W_f\) vs. \(Sm\) vs. \(t/r\) for a steel flywheel.

3.4.2 CONCLUSIONS - The following conclusions can be reached on the basis of the analysis of section 3.4.1.

3.4.2.1 The energy storage capacity per unit weight for the cylindrical flywheel increases linearly with \(Sm\).

3.4.2.2 Examination of equation (3.25) reveals that if \(r\) and \(Sm\) are fixed, \(E_f/W_f\) increases as \(t\) decreases. Thus it is desirable to use the thinnest possible disk. As the disk becomes very thin, the value of \(E_f/W_f\) approaches a value given by

\[
\frac{E_f/W_f}{Sm/P_s} = 1.644. \tag{3.26}
\]

3.4.2.3 Examination of equation (3.25) reveals that if \(t\) and \(Sm\) are fixed, \(E_f/W_f\) increases as \(r\) increases. Thus, for maximum energy storage capacity per unit weight, \(r\) should be as large as possible. As \(r\) approaches infinity, \(E_f/W_f\) approaches a value given by equation (3.26).

3.4.2.4 The analysis of section 3.4.1 has shown that the maximum energy storage capacity per unit weight for a cylindrical flywheel is in the vicinity of 18,000 ft lb/lb. This value is for a steel flywheel where \(t/r = 0.2\) and \(S_{YP} = 100,000\) psi.

3.5 ENERGY STORAGE CAPACITY AS A FUNCTION OF VOLUME FOR A CYLINDRICAL FLYWHEEL - The volume of a cylindrical disk can be expressed as

\[
V_f = \pi r^2 t. \tag{3.27}
\]

Combination of equations (3.22) and (3.27) gives

\[
\frac{E_f/V_f}{2Sm/3\mu (1+\mu)(t/r)^2/3(1-\mu)}. \tag{3.28}
\]

Using a value of 0.288 for \(\mu\) gives

\[
\frac{E_f/V_f}{Sm/[1.644 + 0.0867(t/r)^2]} \tag{3.29}
\]
FIG. 4: ENERGY-WEIGHT RELATION FOR A CYLINDRICAL STEEL FLYWHEEL

\[ \mu = 0.288 \]
\[ \rho = 7.31 \times 10^{-4} \text{ lb sec}^2/\text{in}^4 \]

RATIO OF FIBER THICKNESS TO RADIUS, \( t/r \)
Fig. 5 is a plot of $E_f/V_f$ vs. $t/r$ for a steel flywheel.

3.5.1 CONCLUSIONS - The following conclusions are obtained from an examination of equation (3.28).

3.5.1.1 The energy storage capacity per unit volume for the cylindrical flywheel increases linearly with the stress in the flywheel.

3.5.1.2 If $r$ and $S_m$ are fixed, $E_f/V_f$ increases with a decrease in $t$. $E_f/V_f$ approaches a value of

$$E_f/V_f = \frac{2S_m}{(3 + \mu)}$$

as $t$ approaches zero.

3.5.1.3 If $t$ and $S_m$ are fixed, $E_f/V_f$ increases as $r$ increases. As $r$ approaches infinity, $E_f/V_f$ approaches the value

$$E_f/V_f = \frac{2S_m}{(3 + \mu)}.$$  

3.5.1.4 The analysis of section 3.5 has shown that the maximum energy storage capacity per unit volume for a cylindrical flywheel is in the vicinity of $8,750,000$ ft lb/ft$^3$. This is the value for a steel flywheel where $t/r = 0.2$ and $S_{yp} = 100,000$ psi.

3.6 ANGULAR VELOCITY OF CYLINDER - In the preceding sections, the energy storage capacity per unit weight and the energy storage capacity per unit volume were discussed as functions of the maximum stress which occurs in the flywheel. It is of value to have information concerning the relation between the angular velocity of the flywheel and this stress. Equation (3.21) indicates the desired relation between $\omega$ and $S_m$. Fig. 6 is a plot of $\omega/\sqrt{S_m}$ vs. $r$ for various values of $t$.

3.6.1 CONCLUSIONS - The following conclusions can be reached by examination of equation (3.21).

3.6.1.1 For any given $r$, $\omega/\sqrt{S_m}$ increases as $t$ decreases and approaches
FIG. 5  ENERGY-VOLUME RELATION FOR A STEEL CYLINDRICAL FLYWHEEL

\[ \frac{\Pi}{M} = 0.288 \]

RATIO OF CYLINDER THICKNESS TO RADIUS \( \frac{t}{r} \)
a value of $2/\sqrt{(3+\mu)}\rho r^2/2$ as $t$ approaches zero.

3.6.1.2 For any given $t$, $\omega/\sqrt{S_m}$ increases as $r$ decreases and approaches a value of $2/\sqrt{\rho\mu(1+\mu)}\ t^2/b\ (1-\mu)$ as $r$ approaches zero.

3.7 OPTIMUM THICKNESS OF CYLINDER - Examination of equation (3.22) indicates that if $r$ and $S_m$ are fixed, there may be some value of $t$ which makes the energy storage capacity of the flywheel a maximum. This section is devoted to determining this maximum.

3.7.1 ANALYSIS - Setting the partial derivative of equation (3.22) with respect to $t$ equal to zero and solving for $t$ gives

$$t = r \left[3(1-\mu)(3+\mu)/\mu(1+\mu)\right]^{1/2}. \quad (3.32)$$

This is the value of $t$ which makes $E_F$ maximum. Using a value of 0.288 for $\mu$ gives

$$t = 4.24 r. \quad (3.33)$$

Combination of equations (3.22) and (3.32) gives

$$E_F = \pi S_m r^3 \left[3(1-\mu)(3+\mu)/\mu(1+\mu)\right]^{1/2}/(3+\mu). \quad (3.34)$$

Combination of equations (3.22) and (3.33) gives

$$E_F = 4.16 S_m r^3. \quad (3.35)$$

This is the energy storage capacity of a steel cylindrical flywheel of optimum thickness.

3.8 ENERGY STORAGE CAPACITY AS A FUNCTION OF WEIGHT FOR A UNIFORM STRESS DISK WITH A RIM ATTACHED - The uniform stress disk is the type of flywheel shown in (C) of Fig. 1. The energy storage capacity of this type of device will be evaluated in this section as a function of the weight of the device.

3.8.1 ASSUMPTIONS - In order to complete the analysis of section 3.8.2 the following assumptions have been made.
3.8.1.1 In order to calculate the moment of inertia and mass of the uniform stress disk with a rim, it is assumed that the junction between the rim and disk is abrupt.

3.8.1.2 It is assumed that at the junction of the disk and rim, the radial force on the rim and the radial force on the disk are equal in magnitude and opposite in direction. It is assumed that these forces are uniformly distributed over the inside surface of the rim and the outside surface of the disk respectively.

3.8.1.3 It is assumed that at all points in the disk, the radial and the tangential stresses are equal and that they have some constant value for all points in the disk.

3.8.1.4 It is assumed that the maximum-normal-stress theory predicts the critical stress condition in the flywheel.

3.8.2 ANALYSIS - The kinetic energy of any rotating flywheel is given as

\[ E_f = \frac{1}{2} I \omega^2. \]  

(3.36)

The moment of inertia of a uniform stress disk with a rim is given by

\[ I = I_r + I_d. \]  

(3.37)

Using assumption 3.8.1.1 the moment of inertia of the rim can be expressed by

\[ I_r = 2\pi ptb (r_{1r} + t/2) \left[ (r_{1r} + t/2)^2 + t^2/4 \right]. \]  

(3.38)

In order that the stress in the disk portion of the flywheel be uniform it is necessary that the dimensions, speed, and stress have a particular relation to each other. This relation is given by

\[ J/ \]  

\[ x = x_2 e^{\rho \omega^2 (r_1^3 - r_2^3)/2S_d} \]  

(3.39)

The moment of inertia of the uniform stress disk is given by

\[ I_d = \int_0^{r_1 r} r^2 \, dM. \]  

(3.40)

The increment of mass is given by

\[ dM = 2 \pi r x \rho \, dr. \]  

(3.41)

Combination of equations (3.39) and (3.41) gives

\[ dM = 2\pi \rho x_2 e^{\rho \omega^2 (r_1^3 - r_2^3)/2S_d} \, dr. \]  

(3.42)

Substituting this in equation (3.40) gives

\[ I_d = 2\pi \rho x_2 \int_0^{r_1 r} r^3 e^{\rho \omega^2 (r_1^3 - r_2^3)/2S_d} \, dr. \]  

(3.43)

The integration can be carried out by parts if a change of variable is made by letting

\[ y = r^2. \]  

(3.44)

If the integration is carried out, the moment of inertia of the uniform stress disk becomes

\[ I_d = 4\pi S_d x_2 (e^{\rho \omega^2 r_1^3/2S_d} - e^{\rho \omega^2 r_1^3/2S_d} - 1)/\rho \omega^4. \]  

(3.45)

From equation (3.39)

\[ x_1 = x_2 e^{\rho \omega^2 r_1^3/2S_d}. \]  

(3.46)

Combining equations (3.36), (3.37), (3.38), (3.45), and (3.46) gives

\[ E_T = \left[ \omega^2/2 \left[ \ln(x_1/x_2) \right]^2 \right] \left[ n x_1 r_1^4 \left[ 1 - e^{\ln(x_1/x_2)} \right] \left[ \ln(x_1/x_2) + 1 \right]\right] 
+ \omega^2 n \rho C k x_2 r_1^4 (1 + k/2)(1 + k + k^2/2), \]  

(3.47)

where

\[ k = (r_2 - r_1)/r_1. \]  

(3.48)
The mass of a uniform stress disk with a rim is given by
\[ M_d = M_r + M_r. \]  \hspace{1cm} (3.50)

The mass of the rim can be expressed by
\[ M_r = \rho b n(r_2^2 - r_1^2). \]  \hspace{1cm} (3.51)

The mass of the disk is given by
\[ M_d = \int_0^{r_{1r}} dM. \]  \hspace{1cm} (3.52)

Using equation (3.52) this can be written as
\[ M_d = 2\pi \rho x_2 \int_0^{r_{1r}} r e^{\omega^2(r_{1r}^2 - r^2)/2S_d} dr. \]  \hspace{1cm} (3.53)

By adjusting the constants, this integral can be made of the form
\[ \int e^x dx. \]

Performing the integration and using equation (3.46) gives
\[ M_d = 2\pi S_d x_2(1 - e^{-\rho \omega^2 r_{1r}^2/2S_d})/\omega^2. \]  \hspace{1cm} (3.54)

Combination of equations (3.50), (3.51) and (3.54) gives
\[ M_r = \rho b n(r_2^2 - r_1^2) + 2\pi S_d x_2(1 - e^{-\rho \omega^2 r_{1r}^2/2S_d})/\omega^2. \]  \hspace{1cm} (3.55)

Combination of equations (3.47) and (3.55) gives the energy mass relation as a function of the angular velocity of the flywheel. The maximum permissible angular velocity is a function of the maximum allowable stress which is permitted in the flywheel. Since the purpose of this study is to determine the maximum energy which can be stored in a flywheel, it is desirable to find the energy-weight relation as a function of the maximum permissible angular velocity and consequently of the maximum allowable stress rather than as a function of the general angular velocity. To do this it is necessary to determine the maximum stress in the flywheel and
where this stress occurs. The following paragraphs are devoted to a stress analysis of the uniform stress type of flywheel.

The stresses in the rim are produced by a combination of the centrifugal force of the rim and the inward pull of the disk on the rim. The tangential stress in the rim due to the centrifugal force is given by

$$S_{tc} = (3+\mu)\rho \omega^2 \left[ r_{2r}^2 + r_{1r}^2 + r_{2r}^2 r_{1r}^2/r^2 - (1+3\mu) r^2/(3+\mu) \right] / 8. \quad (3.56)$$

The tangential stress due to a uniform pull p per unit area on the inside of the rim is given by

$$S_{tp} = -p r_{2r}^2 \left[ 1 + r_{2r}^2/r^2 \right] / [r_{2r}^2 - r_{1r}^2]. \quad (3.57)$$

The resultant tangential stress in the rim is given by the sum of $S_{tc}$ and $S_{tp}$, thus

$$S_t = [(3+\mu)\rho \omega^2/8] \left[ r_{2r}^2 + r_{1r}^2 + r_{2r}^2 r_{1r}^2/r^2 - (1+3\mu) r^2/(3+\mu) \right] - p r_{1r}^2 \left[ 1 + r_{2r}^2/r^2 \right] / [r_{2r}^2 - r_{1r}^2]. \quad (3.58)$$

The radial stress in the rim due to the centrifugal force is given by

$$S_{rc} = (3+\mu)\rho \omega^2 \left[ r_{2r}^2 - r_{2r}^2 r_{1r}^2/r^2 - r^2 \right]/8. \quad (3.59)$$

The radial stress in the rim due to the restraining force of the disk is given by

$$S_{rp} = -p r_{1r}^2 \left[ 1 - r_{2r}^2/r^2 \right] / [r_{2r}^2 - r_{1r}^2]. \quad (3.60)$$

The resultant radial stress is given by

$$S_r = S_{rc} + S_{rp} \quad (3.61)$$

or

---

5/ Ibid., p. 60.
6/ Ibid., p. 71.
7/ Ibid., p. 60.
Equations (3.58) and (3.62) give the desired relations between the rim stresses and the angular velocity.

In order to find the relation between the disk stresses and the angular velocity, the radial displacements of the outside radius of the disk and the inside radius of the rim are examined. Using assumption 3.8.1.3, the circumferential unit strain in the disk at \( r = r_{1r} \) for the biaxial stress condition is given by

\[
\varepsilon = \frac{S_d(1-\mu)}{E}. \tag{3.63}
\]

The total circumferential deformation is given by

\[
\delta_c = 2\pi r_{1r} S_d(1-\mu)/E. \tag{3.64}
\]

A circumferential deformation of the magnitude given by equation (3.64) results in a radial displacement of the outside radius of the disk of magnitude

\[
\delta_d = r_{1r} S_d(1-\mu)/E. \tag{3.65}
\]

The radial displacement of the inside radius of the rim due to the centrifugal force is given \( \delta_r \) by

\[
\delta_{rc} = (1-\mu^2)\rho \omega^2 \left\{ \frac{[(3+\mu)(r_{1r}^3 + r_{1r}^2 r_{2r})/(1+\mu)] - r_{1r}^3}{\rho} + \frac{(3+\mu)}{r_{1r}^2/(1-\mu)} \right\} \tag{3.66}
\]

The radial displacement of the inside of the rim due to the pressure \( p \) is given \( \delta_{rp} \) by

\[
\delta_{rp} = -r_{1r} p \left\{ \left( \frac{r_{1r}^2 + r_{2r}^2}{r_{2r}^2 - r_{1r}^2} \right) + \frac{\mu^2}{E} \right\}. \tag{3.67}
\]

---


2/ Ibid., p. 211.
The total radial displacement of the inside radius of the rim is given by the sum of equations (3.66) and (3.67). Thus

\[ \delta_r = \frac{1}{\pi} \left\{ \left[ (1-\mu)r_{1r}^3/4 + (3+\mu)r_{1r}^2r_{2r}/4 \right] \rho \omega^2 - \pi r_{1r} \left[ (r_{1r}^3 + r_{2r}^3)/(r_{2r}^3 - r_{1r}^3) + \mu \right] \right\}. \]  

(3.68)

Now when

\[ r = r_{1r}, \]  

(3.69)

\[ \delta_r = \delta_d. \]  

(3.70)

Also, assumption 3.8.1.2 can be written mathematically as

\[ 2\pi r_{1r} x_2 \cdot S_d = 2\pi r_{1r} b \cdot p \]  

(3.71)

or

\[ p = S_d/C, \]  

(3.72)

where

\[ C = b/x_2. \]  

(3.73)

Combining equations (3.65), (3.68), (3.70), (3.72), (3.73) and solving for \( S_d \) gives

\[ S_d = \left\{ \rho \omega^2/4 \left[ r_{1r}^2 + \left( 3+\mu \right)/(1-\mu) \right] r_{2r}^3 \right\}/\left[ 1 + \left[ 1/(1-\mu) \right] \left[ 1/C \right] \right] \]

\[ \left[ (r_{2r}^3 + r_{1r}^3)/(r_{2r}^3 - r_{1r}^3) + \mu \right]. \]  

(3.74)

Equations (3.58), (3.62) and (3.74) give the stress-angular velocity relations which are desired. It remains yet to be determined which of the three stresses has the largest magnitude. This is important since the greater of the three will be the controlling stress according to assumption 3.8.1.4. The following paragraphs are devoted to determining which stress in the flywheel is the controlling stress.

Setting the derivative of equation (3.58) with respect to \( r \) equal to zero and solving for \( r^4 \) gives
It can be seen from this equation that in order to have a real solution for \( r \), the following condition must hold.

\[
8p > (3+\mu)(r_{2r}^2 - r_{1r}^2)\rho \omega^2.
\] (3.76)

Equation (3.75) defines the value of \( r \) which makes \( S_\tau \) either maximum or minimum. The second derivative of equation (3.78) with respect to \( r \) is given by

\[
\frac{\partial^2 S_\tau}{\partial r^2} = -\rho \omega^2(1+3\mu)r^3.
\] (3.77)

Since this quantity is always negative, equation (3.75) defines the value of \( r \) which makes \( S_\tau \) maximum. This value of \( r \) can be further defined by the following procedure. Solving equation (3.74) for \( \rho \omega^2 \) gives

\[
\rho \omega^2 = \left[ hS_q/C \left( r_{2r}^2 - r_{1r}^2 \right) \right] \left\{ c(r_{2r}^2 - r_{1r}^2)(1-\mu) + (r_{2r}^2 - r_{1r}^2) \right\}
\]

\[
+ \mu (r_{2r}^2 - r_{1r}^2)\left/ \left[ (1-\mu)r_{1r}^2 + (3+\mu)r_{2r}^2 \right] \right. \]
\] (3.78)

Combining equations (3.72), (3.75) and (3.78) gives

\[
r^4 = \left\{ 2K_4^2 \frac{r_{1r}}{r_{2r}^2} + (1+3\mu) \right\} \left\{ [K_4^2(3+\mu) + (1-\mu)]/[K_4^2(1+\mu + C(1-\mu))] \right\}
\]

\[
+ (1 - \mu - C(1-\mu))\frac{1}{r_{1r}^2} \left/ \left[ (3+\mu)/2 \right] \right. \]
\] (3.79)

where

\[
K_4 = \frac{r_{2r}}{r_{1r}}.
\] (3.80)

If the value of \( \mu \) is taken as 0.288 and equation (3.79) is solved for \( C \), the result is

\[
C = \left\{ 1/0.712(K_4^2 - 1) \right\} \left\{ [3.288 K_4^2 + 0.712] / [1.072 K_4^2 r_{1r}^4] \right\}
\]

\[
\left[ r^4 + 1.763 K_4^2 r_{1r}^4 \right] - [1.288 K_4^2 + 0.712] \right\}.
\] (3.81)

It will be assumed that the ranges of values which are of interest in this investigation are defined by \( 1 \leq C \leq 3 \), and \( 1.1 \leq K_4 \leq 2 \). If equation (3.81) is solved for \( r = r_{1r} \) and \( K_4 = 2 \), it is determined that
Now as \( r \) increases, equation (3.81) reveals that \( C \) must decrease. Thus, there is no solution of equation (3.79) for \( C \geq 1 \) and \( r_{1r} \leq r \leq r_{2r} \) when \( K_l = 2 \). If equation (3.81) is solved for \( r = r_{1r} \) and \( K_l = 1.1 \), it is determined that

\[
C = 0.715. \tag{3.82}
\]

By reasoning similar to the above, there is no solution of equation (3.79) for \( C \geq 1 \) and \( r_{1r} \leq r \leq r_{2r} \) when \( K_l = 1.1 \). Also, it can be shown by plotting \( K_l \) vs. \( C \) from equation (3.81) where \( r = r_{1r} \), that \( C \) increases monotonically with \( K_l \). Thus, it can be concluded that for \( 1 \leq C \leq 3 \), and \( 1.1 \leq K_l \leq 2 \), equation (3.79) has no solution. This means that there is no point of relative maximum or minimum tangential stress between \( r = r_{1r} \) and \( r = r_{2r} \) for the range of values with which this investigation is concerned. It follows then that the tangential stress is maximum at \( r = r_{1r} \) or at \( r = r_{2r} \). Equation (3.82) indicates that for \( K_l = 2 \), \( r = r_{1r} \) is a solution of equation (3.79) for the case where \( C = 0.715 \).

Thus, \( S_t \) is maximum at \( r = r_{1r} \) for these conditions. As \( C \) increases to a value which is of interest in this investigation, \( r \) must decrease as is shown by equation (3.81). Thus, the point of maximum \( S_t \) falls inside of the inside radius of the rim. It can be concluded, therefore, that physically, the maximum value of the tangential rim stress occurs at \( r = r_{1r} \).

Differentiating equation (3.62) with respect to \( r \), setting the result equal to zero and solving for \( r^4 \) gives

\[
r^4 = r_{1r}^2 r_{2r}^2 \left[ (r_{2r}^2 - r_{1r}^2)(3 + \mu) \rho \omega^2 - 6p \right] / \left[ (r_{2r}^2 - r_{1r}^2)(3 + \mu) \rho \omega^2 \right]. \tag{3.84}
\]

Equation (3.84) defines the value of \( r \) which makes \( S_t \) maximum or minimum. Inspection of this equation reveals that \( r \) can have a real solution only if \( \mu \)
However, it has been shown that there is a real solution to equation (3.79), hence there is a real solution to equation (3.75). The condition for this to be true is given by inequality (3.76). It follows then, that condition (3.85) cannot hold. Thus, there is no real solution to equation (3.84) and consequently, there is no relative maximum or relative minimum point on the curve of $S_r$ vs. $r$. It can be concluded then that $S_r$ must be a maximum at $r = r_{1r}$ or at $r = r_{2r}$. $S_r$ is obviously zero at $r = r_{2r}$. At $r = r_{1r}$, $S_r$ must be equal to $p$. Thus from equation (3.72)

$$S_r = \frac{S_d}{C}$$

(3.86)

at $r = r_{1r}$. Since $S_d$ and $C$ are both positive, it follows that $S_r$ is positive at $r = r_{1r}$. It was shown above that $S_r$ is monotonic with $r$. It follows from these facts that $S_r$ is maximum at $r = r_{1r}$ and has a value as given by equation (3.86) at this point. Thus it has been determined that both $S_r$ and $S_t$ attain their maximum values at $r = r_{1r}$.

The maximum value which $S_t$ can attain is found by substituting $r = r_{1r}$ and $\rho \omega^2$ as defined by equation (3.78) into equation (3.58). If this is done and the resulting expression is simplified, the result is

$$S_{tm} = \frac{(S_d/C) [(1-C)\mu + C]}{(3.87)}$$

Inspection of this equation reveals that as $C$ increases, $S_{tm}$ decreases. Thus, the maximum value which $S_{tm}$ can attain is when $C$ attains a value of unity. Thus

$$S_{tmm} = S_d$$

(3.88)

Finally, comparison of equations (3.86) and (3.88) reveals that the stress in the disk is always equal to or greater than the largest stress which can occur in the flywheel and hence is the controlling stress. It
follows then that equation (3.78) is the equation which defines the maximum permissible angular velocity of the flywheel in terms of the maximum allowable stress. This equation can be rewritten as

$$\omega^2 = 2S_d \Omega / \rho r_{ir}^2$$  \hspace{1cm} (3.89)

where

$$\Omega = 2 \left\{ 1 + 1/(1+k)^2 + \left[ 1 - 1/(1+k)^2 \right] \right\} \left[ (1 - \mu)C + \mu \right] /
\left\{ (k+1)^2 - 1 \right\} \left[ (3 + \mu) + (1 - \mu)/(1+k)^2 \right] \right\} .$$  \hspace{1cm} (3.90)

With the information which is now available it is possible to complete the derivation of the energy expression. From equation (3.46)

$$\text{ln}(x_1/x_2) = \rho \omega^2 r_{ir}^2 / 2S_d.$$  \hspace{1cm} (3.91)

Combining this with equation (3.89) gives

$$\text{ln}(x_1/x_2) = \Omega.$$  \hspace{1cm} (3.92)

Combining equations (3.47), (3.89) and (3.92) gives

$$E_f = \text{ln} S_d x_1 r_{ir}^2 \left\{ \left[ 1 - e^{-\Omega/(1+\Omega)} \right] \left[ 1/(1+\Omega) \right] + \right\} \left\{ k \Omega e^{-\Omega/2} (1 + k/2)(1 + k + k^2/2)/2 \right\} .$$  \hspace{1cm} (3.93)

Combining equations (3.55), (3.89) and (3.46) gives

$$M_f = r_{ir}^2 x_1 \rho \left\{ C e^{-\Omega} (2k + k^2) + (1 - e^{-\Omega})/\Omega \right\} .$$  \hspace{1cm} (3.94)

From this the weight of the flywheel is given by

$$W_f = r_{ir}^2 x_1 \rho \left\{ C e^{-\Omega} (2k + k^2) + (1 - e^{-\Omega})/\Omega \right\} / g.$$  \hspace{1cm} (3.95)

Combination of equations (3.93) and (3.95) gives the desired energy-per-unit-weight relation.

The relation between \( \Omega \) and \( k \) can be found from equation (3.90). This relation is plotted in Fig. 7 for various values of \( C \) for the case of a steel flywheel. Fig. 8 is a plot of \( E_f/S_d x_1 r_{ir}^2 \) vs. \( k \) for various values of \( C \). This plot was obtained by using equation (3.93). By using Figs. 7 and 8 and equation (3.95), it is possible to plot the quantity
$E_p/W_f S_d$ vs. $k$ for various values of $C$. This plot is shown as Fig. 9.

In order to give an idea of the magnitude of the angular velocity for various specific flywheels, Table 1 has been compiled for a steel flywheel operating at a stress of 100,000 psi. This table has been compiled by using equation (3.89) and Fig. 7.

### TABLE 1

**Speed Which Is Necessary to Produce a Maximum Stress of 100,000 psi in Several Specific Flywheels**

<table>
<thead>
<tr>
<th>Disk Radius $r_{1r}$ (inches)</th>
<th>3</th>
<th>10</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$</td>
<td>$k$</td>
<td>rpm</td>
<td>rpm</td>
</tr>
<tr>
<td>1.0</td>
<td>0.1</td>
<td>4.90</td>
<td>116,000</td>
</tr>
<tr>
<td>0.5</td>
<td>0.897</td>
<td>49,700</td>
<td>11,900</td>
</tr>
<tr>
<td>1.0</td>
<td>0.384</td>
<td>32,600</td>
<td>9,760</td>
</tr>
<tr>
<td>2.0</td>
<td>0.1</td>
<td>2.60</td>
<td>84,800</td>
</tr>
<tr>
<td>0.5</td>
<td>0.527</td>
<td>38,200</td>
<td>11,400</td>
</tr>
<tr>
<td>1.0</td>
<td>0.244</td>
<td>26,000</td>
<td>7,800</td>
</tr>
<tr>
<td>3.0</td>
<td>0.1</td>
<td>1.85</td>
<td>71,600</td>
</tr>
<tr>
<td>0.5</td>
<td>0.112</td>
<td>33,800</td>
<td>10,100</td>
</tr>
<tr>
<td>1.0</td>
<td>0.197</td>
<td>23,400</td>
<td>7,000</td>
</tr>
</tbody>
</table>

In order to give an idea of the magnitudes of the energy storage capacity of various specific flywheels, Table 2 has been compiled for a steel flywheel operating at a stress of 100,000 psi.

#### 3.8.3 CONCLUSIONS

- **3.8.3.1** It was shown in the analysis of section 3.8.2 that the stress in the disk section of a uniform stress disk with a rim is always greater than or equal to the maximum stress in the rim section.
<table>
<thead>
<tr>
<th>Max. Disk Thickness x₁ (inches)</th>
<th>1</th>
<th>6</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Disk Radius r₁ (inches)</td>
<td>3</td>
<td>10</td>
<td>15</td>
</tr>
<tr>
<td>C</td>
<td>k</td>
<td>E₀ (ft·lb·x10⁻⁶)</td>
<td>E₀ (ft·lb·x10⁻⁶)</td>
</tr>
<tr>
<td>1</td>
<td>0.1</td>
<td>0.0480</td>
<td>0.533</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.234</td>
<td>2.60</td>
</tr>
<tr>
<td>1</td>
<td>1.0</td>
<td>0.499</td>
<td>5.54</td>
</tr>
<tr>
<td>2</td>
<td>0.1</td>
<td>0.0876</td>
<td>0.947</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>0.350</td>
<td>3.84</td>
</tr>
<tr>
<td>2</td>
<td>1.0</td>
<td>0.696</td>
<td>7.75</td>
</tr>
<tr>
<td>3</td>
<td>0.1</td>
<td>0.117</td>
<td>1.30</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
<td>0.433</td>
<td>4.81</td>
</tr>
<tr>
<td>3</td>
<td>1.0</td>
<td>0.869</td>
<td>9.65</td>
</tr>
</tbody>
</table>
3.8.3.2 Inspection of Fig. 9 reveals that for given values of \( \rho, S_d, C, \) and \( r_1r, \) the energy storage capacity per unit weight increases as the thickness of the rim, \( r_2r-r_1r, \) decreases. However, for fixed values of \( \rho, S_d, C, \) and \( r_1r, \) a decrease in the rim thickness necessitates a higher angular velocity in order to maintain the constant stress which was assumed in plotting Fig. 9. This is revealed by the combination of Fig. 7 and equation (3.89).

3.8.3.3 Inspection of Fig. 9 reveals that for given values of \( S_d, r_2r, r_1r, \) and \( \rho, \) the energy storage capacity per unit weight increases as \( C \) decreases. The increase of energy per unit weight necessitates an angular velocity increase to maintain the stress in the disk constant. This is revealed by the combination of Fig. 7 and equation (3.89).

3.8.3.4 The energy per unit weight for the type of flywheel under consideration is a linear function of the stress in the disk section of the flywheel. This is revealed by the combination of equations (3.93) and (3.95).

3.8.3.5 The analysis of section 3.8.2 has shown that the maximum energy storage capacity per unit weight for a uniform stress disk with rim is in the vicinity of 27,350 ft lb/lb. This is the value for a steel flywheel where \( C = 1, k = 0.4 \) and \( S_{yp} = 100,000 \) psi.

3.9 ENERGY STORAGE CAPACITY AS A FUNCTION OF VOLUME FOR A UNIFORM STRESS DISK WITH A RIM ATTACHED

3.9.1 ANALYSIS - The volume of space which is occupied by a uniform stress disk with a rim can be expressed as

\[
V_r = \pi r_2^2 b 
\]

if \( b > x_1, \) or as

\[
V_r = \pi r_2^2 b 
\]
\[ V_f = \pi r^2 x_l \quad (3.97) \]

if \( b < x_l \). The point where \( b = x_l \) can be determined from equations (3.49) and (3.92). It is found that when \( b = x_l \),

\[ C = e^{-\alpha}. \quad (3.98) \]

Figs. 7 and 8 show the curve for the case where \( b = x_l \). Combination of equations (3.93) and (3.96) or of equations (3.93) and (3.97) gives the desired energy-volume relations. They are as follows:

\[
\frac{E_f}{V_f} = 2S_d \left\{ \left[ 1 - e^{-\alpha (1+\alpha)} \right] / 2 \alpha + C k \alpha e^{-\alpha} \right\} / (1+k/2)(1+k+k^2/2)/(1+k)^2 \quad (3.99)
\]

for \( b < x_l \), or

\[
\frac{E_f}{V_f} = 2S_d \left\{ \left[ 1 - e^{-\alpha (1+\alpha)} \right] / 2 \alpha + C k \alpha e^{-\alpha} \right\} / (1+k)^2 C e^{-\alpha} \quad (3.100)
\]

for \( b > x_l \).

Fig. 10 is a plot of \( \frac{E_f}{V_f} S_d \) vs. \( k \) for various values of \( C \) for a steel flywheel. Fig. 7 has been used in making this plot.

3.9.2 CONCLUSIONS - The following conclusions can be reached on the basis of the analysis of section 3.9.1.

3.9.2.1 Inspection of Fig. 10 reveals that for given values of \( C, r_{1r}, \) and \( S_d \), there is some thickness of rim which makes the energy storage capacity per unit volume a maximum.

3.9.2.2 The curve of \( b = x_l \) defines the curve of maximum energy storage capacity per unit volume. This is revealed by inspection of Fig. 10.

3.9.2.3 Inspection of Fig. 10 reveals that for the case where \( b = x_l \) and for given values of \( S_d \) and \( r_{1r} \), the energy storage capacity per unit volume increases as the thickness of the rim increases. However, the energy per unit volume increases at a decreasing rate.
3.9.2.1 The analysis of section 3.9.1 has shown that the maximum energy storage capacity per unit volume for a uniform stress disk with a rim is in the vicinity of 9,000,000 ft lb/ft³. This is the value for a steel flywheel where \( C = 1.5, k = 0.75 \) and \( S_{yp} = 100,000 \) psi.

3.10 CONSIDERATION OF LOSSES - In order to fully evaluate a flywheel as a means of storing energy, it is necessary to consider the losses which will occur when this means of energy storage is used. The two principal power losses which occur in a flywheel are the power loss due to bearing friction and the power loss due to windage. These losses will be evaluated in this section for the case of a cylindrical shaped flywheel.

3.10.1 ASSUMPTIONS - In order to calculate the bearing loads for the rotating flywheel, it has been assumed that the flywheel is perfectly balanced.

3.10.2 ANALYSIS - As mentioned previously, the primary losses associated with operating a flywheel are the losses due to windage and the losses due to bearing friction. The windage losses can be divided into two components. The first component is the windage loss due to the drag of the atmosphere on the surfaces of the cylindrical shaped flywheel which are perpendicular to the axis of revolution. The second component is the windage loss due to the drag of the atmosphere on the surface of the flywheel which is not perpendicular to the axis of revolution. Finally, the windage losses depend on whether the flywheel is operating under turbulent conditions or laminar conditions. The losses associated with operation of a flywheel will be evaluated in the following sections. As a matter of convenience, the surface of the flywheel which is not perpendicular to the axis of revolution will be called
surface A. The combination of the surfaces which are perpendicular to the axis of revolution will be called surface B.

3.10.2.1 Power Loss Due to Windage for Surface A (Turbulent Conditions) - Theodorsen and Regier 10/ indicate that the windage power loss for a smooth cylinder (not including the ends) is given by

\[ P = C_D n \rho_m r^4 \omega^3 t. \]  

(3.101)

The symbol \( C_D \) represents the drag coefficient. For a smooth cylinder operating in the turbulent region this coefficient is defined by

\[ \frac{1}{(C_D)^{1/2}} = -0.6 + 4.07 \log_{10} R(C_D)^{1/2}. \]  

(3.102)

Fig. 11 is a plot of \( \sqrt[4]{C_D} \) vs. \( R \) as obtained from equation (3.102). This curve can be represented very well by the straight line

\[ \frac{1}{\sqrt{C_D}} = 0.354 - 0.020 \log_{10} R. \]  

(3.103)

The Reynolds number which is used in equations (3.102) and (3.103) is given by

\[ R = \omega r^2 \mu_m^{-1} \rho_m. \]  

(3.104)

Combination of equations (3.101), (3.103) and (3.104) gives

\[ P = n \rho_m r^4 \omega^3 t \left[0.354 - 0.020 \log_{10} (\omega r^2 \mu_m^{-1} \rho_m)^4. \right] \]  

(3.105)

If operating conditions are turbulent, equation (3.105) is the expression for the windage power loss for surface A. The units in the equation are as follows:

- \( P \) ft lbs/sec
- \( \mu_m \) slugs/ft³
- \( r \) ft
- \( \omega \) rad/sec
- \( t \) ft
- \( \rho_m \) lb sec/ft²

FIG. 1. VARIATION OF DRAG COEFFICIENT
WITH Reynolds Number
3.10.2.2 Power Loss Due to Windage for Surface A (Laminar Conditions) -

Theodorsen and Regier indicate that the laminar drag coefficient is given by

\[ c_D = 4 \mu_m \rho_m^{-1} \omega^2 r^{-2}. \quad (3.106) \]

Combination of equations (3.101) and (3.106) gives

\[ P = 4 \mu_m \pi r^2 \omega^2. \quad (3.107) \]

This is the expression for the windage power loss for surface A if operating conditions are laminar. The units in equation (3.107) are as indicated in section 3.10.2.1.

3.10.2.3 Power Loss Due to Windage for Surface B (Turbulent Conditions) -

Theodorsen and Regier define a moment coefficient as

\[ c_M = 2M/\rho_m \omega^2 r^5. \quad (3.108) \]

These authors indicate that the moment coefficient for a disk (very thin cylinder) operating in the turbulent region is given by

\[ c_M = 0.146 \omega^{-1/5} r^{-2/5} \rho_m^{-1/5} \mu_m^{1/5}. \quad (3.109) \]

The power loss in general can be written as

\[ P = M \omega, \quad (3.110) \]

where \( M \) is the torque caused by a drag and \( \omega \) is the angular velocity of the flywheel. Combination of equations (3.108), (3.109) and (3.110) gives

\[ P = 0.073 \rho_m^{1/8} r^{1.6} \omega^{2.8} \mu_m^{0.2}. \quad (3.111) \]

If operating conditions are turbulent, equation (3.111) can be considered as the expression for the windage power loss due to the drag of the atmosphere on surface B. The units are as indicated in section 3.10.2.1.

\[ 11/ \text{Ibid.} \]
3.10.2.4 Power Loss Due to Windage for Surface B (Laminar Conditions) -

Theodorsen and Regier indicate that the moment coefficient for a disk operating in the laminar region is given by

$$C_M = 3.87 \omega^{-1/2} r^{-1} \rho_m^{-1/2} \mu_m^{1/2}.$$  \hspace{1cm} (3.112)

Combination of equations (3.108), (3.110) and (3.112) gives

$$P = 1.93 \omega^{2.5} r^{4} \mu_m^{0.5} \rho_m^{0.5}.$$  \hspace{1cm} (3.113)

If operating conditions are laminar equation (3.113) is the expression for the windage power loss due to the drag of the atmosphere on surface B. The units are as indicated in section 3.10.2.1.

3.10.2.5 Power Loss Due to Bearing Friction - Using assumption 3.10.1, the load on a single bearing is given by

$$W_b = W_f/2.$$  \hspace{1cm} (3.114)

Shaw[12] gives some experimental curves for the friction torque due to bearing friction. It is assumed in this analysis that the curve which is shown in Fig. 10-29 for cylinder roller bearings[13] can be extrapolated to any desired load. The equation for this curve can be written as

$$M = 4.76 \times 10^{-5} W_{bT} + 0.167,$$  \hspace{1cm} (3.115)

where $M$ is in units of ft-lbs and $W_{bT}$ is in units of lbs. Combining equations (3.110) and (3.115) gives

$$P_{bs} = \omega \left[ 4.76 \times 10^{-5} W_{bT} + 0.167 \right].$$  \hspace{1cm} (3.116)

This is the expression for the power loss due to bearing friction for a single bearing. Assuming that the flywheel is supported on two bear-


ings, the loss becomes

$$P_b = \omega \left[ 4.76 \times 10^{-5} \cdot 2 \bar{W}_b + 0.167 \right]. \quad (3.117)$$

3.10.2.6 Total Power Loss Due to the Combination of Bearing Friction and Windage - The total power loss due to the combination of bearing friction and windage is

$$P_T = P_b + P_w. \quad (3.118)$$

Combining equations (3.105), (3.111), (3.117) and (3.118) gives

$$P_T = \omega \left[ 4.76 \times 10^{-5} \cdot 2 \bar{W}_b + 0.167 \right] + 0.073 \rho_m^{0.8} r^{1.6} \omega^{2.8} \mu_m^{0.2}$$

$$+ n \rho_m r^4 \omega^3 t \left[ 0.356 - 0.020 \log_{10} (\omega r^2 \mu_m^{-1} \rho_m) \right]. \quad (3.119)$$

This is the power loss relation which holds for the case of operation of a cylindrical shaped flywheel under turbulent conditions.

For laminar conditions, the combination of equations (3.107), (3.113), (3.117) and (3.118) gives

$$P_T = \omega \left[ 4.76 \times 10^{-5} \cdot 2 \bar{W}_b + 0.167 \right] + 1.93 \rho_m^{0.5} r^4 \omega^{2.5} \mu_m^{0.5}$$

$$+ t \pi r t \omega^2 \mu_m. \quad (3.120)$$

The magnitude of the power loss can be evaluated by selecting some particular flywheel operating under some given set of conditions. This study will include examination of the following two cases:

Case 1: A flywheel operating in air at 60°F and 29.92 inches of Hg.

Case 2: A flywheel operating in hydrogen at 60°F and one inch of mercury pressure.

Examination of the Reynolds number reveals that for the values of \( \omega \) and \( r \) which are used in this numerical example, the equation for turbulent flow should be used for case 1. Theodorsen and Regier indicate that the critical Reynolds number is approximately 310,000. Substituting numerical values for viscosity and density of air into equation
(3.119), the power loss expression becomes
\[ P_T = 3.25 \times 10^{-10} r^4 \omega^2 t^3 + 3.0 \times 10^{-8} \left( 0.354 - 0.020 \log_{10}(9950 \omega) \right) r^4 \omega t + \omega \left[ 4.76 \times 10^{-5} W_b + 0.167 \right]. \]

The values of density and viscosity of air are stipulated in the notation section. The units in equation (3.121) are as follows:

- \( P_T \) ft-lb/sec
- \( r \) inches
- \( \omega \) rad/sec
- \( t \) inches.

Examination of the Reynolds number reveals that for the values of \( \omega \) and \( r \) which are used in this numerical example, the equations for laminar flow should be used for case 2. Substitution of numerical values for viscosity and density of hydrogen into equation (3.120) gives
\[ P_T = 9.32 \left( 10^{-11} \right) \omega^2 r^4 + 1.325 \left( 10^{-9} \right) \omega^2 r^2 t \]
\[ + \omega \left[ 4.76 \times 10^{-5} W_b + 0.167 \right]. \]

The numerical values of the viscosity and density of hydrogen are stipulated in the notation section. The units in equation (3.122) are as specified for equation (3.121). The steel flywheel which is used in this numerical example is assumed to have a radius of 15 inches and a thickness of 5 inches. For this particular flywheel
\[ W_f = 1000 \text{ lbs.} \]

Using the specified values of radius and thickness along with equations (3.114) and (3.123) in equation (3.121) gives
\[ P_T = 8.12 \times 10^{-5} \omega^2 t^3 + 7.59 \times 10^{-3} \left[ 0.354 - 0.020 \log_{10}(9950 \omega) \right] r^4 \omega t^3 \]
\[ + 0.2146 \omega. \]

This is the expression for the total power loss for case 1.

Using the specified values of radius and thickness along with equa-
tions (3.114) and (3.123) in equation (3.122) gives

$$P_T = 4.72 \times 10^{-6} \omega^2 + 1.49 \times 10^{-6} \omega^2 + 0.21 \omega$$  \hspace{1cm} (3.125)

This is the expression for the total power loss for case 2. Now since

$$E_f = I\omega^2/2$$  \hspace{1cm} (3.126)

and

$$P_T = -\Delta E_f / \Delta \tau$$  \hspace{1cm} (3.127)

it is possible to find the time-energy relation for this particular flywheel. This can be done as follows:

1. Pick some initial value of $E_f$ with a corresponding $\omega$.
2. Pick some value of $\Delta \tau$.
3. Guess what the value of $\omega$ will be at the end of the time interval $\Delta \tau$.
4. Find the mean value of $\omega$ for the time interval $\Delta \tau$. This is the mean of the values selected in (1) and (3).
5. Calculate the value of the energy at the end of the time interval $\Delta \tau$. This is done by using equation (3.126) and the value of $\omega$ selected in (3).
6. Calculate the difference between the energy at the beginning of the time interval and the energy at the end of the time interval. This is done by taking the difference between the values obtained in (1) and (5).
7. Calculate $\Delta E_f$ from equation (3.127) using the value of $\Delta \tau$ from (2) and the value of $P_T$ as found from equation (3.124) or (3.125). The value of $\omega$ which is used in these equations will be that found in (4).
8. Repeat process (1)-(7) until the $\Delta E_f$ from (7) is the same as the $\Delta E_f$ from (6). This is the correct value of $\Delta E_f$ for the time interval chosen.
9. Select another time interval. The initial value of $E_f$ for the new time interval is taken as the final value of $E_f$ as calculated for the previous time interval. Likewise, the initial value of $\omega$ for the new time interval is taken as the final value of $\omega$ as found for the previous time interval.
10. Repeat (1)-(8) for the new time interval.
(11) Repeat (1)-(11) for other time intervals.
Figs. 12 and 13 are plots of \( E_f \) vs. \( \tau \) as obtained by this procedure. These plots are for cases 1 and 2 respectively.

3.10.3 CONCLUSIONS - Inspection of Figs. 12 and 13 reveals that the power losses can be decreased significantly by enclosing the flywheel in a low viscosity atmosphere. The kinetic energy decreases from \( 7 \times 10^6 \) ft lb to \( 5 \times 10^6 \) ft lb in approximately three hours when the flywheel is operating in low pressure hydrogen. When the flywheel is operating in air, a corresponding decrease in kinetic energy occurs in approximately three minutes.

3.11 SUMMARY - In this chapter, three types of flywheels have been evaluated. This section is devoted to a summary of the most significant equations and conclusions which have arisen from the analyses of this chapter. Graphs which are pictorial representations of the analyses have been included in previous sections.

3.11.1 SUMMARY OF EQUATIONS

Rim-Arm Type Flywheel

\[
\begin{align*}
& E_f / W_f = S_c / 3 \rho g \left( 0.75 + r_m^2 n^2 / 2 n^2 t \right) \\
& E_f / V_f = S_c / (r_m / t)(0.75 + r_m^2 / 2 n^2 t)
\end{align*}
\]

Cylindrical Type Flywheel

\[
\begin{align*}
& E_f / W_f = 2 S_m / \rho g \left[ 3 + \mu + \mu(1+\mu)(t / r)^2 / 3(1-\mu) \right] \\
& E_f / V_f = 2 S_m / \left[ 3 + \mu + \mu(1+\mu)(t / r)^2 / 3(1-\mu) \right]
\end{align*}
\]

Uniform Stress Type Flywheel

\[
\begin{align*}
& E_f = 4 \pi S_d x_1 r^2_{fr} \left[ \left[ 1 - e^{-\alpha} (1+\alpha) \right] \left[ 1/4 - \alpha \right] + \\
& + c k \alpha e^{-\alpha}(1 + k/2)(1 + k + k^2/2)/2 \right] \\
& W_f = r^2_{fr} \pi x_1 \rho [c e^{-\alpha} (2k + k^2) + (1 - \alpha)/\alpha] / g
\end{align*}
\]
FIG. 12. TIME-ENERGY RELATION FOR A CYLINDRICAL STEEL FLYWHEEL ROTATING IN AIR.
FIG. 13 TIME-ENERGY RELATION FOR A CYLINDRICAL STEEL PLANEanel ROTATING IN LOW PRESSURE HYDROGEN
\[
\frac{E_f}{V_f} = 2S_d \left[ \frac{1 - e^{-\alpha(1+\alpha)}}{2\alpha + c\alpha e^{-\alpha}} \right] \frac{(1 + k/2)(1 + k + k^2/2)}{(1+k)^2} b < x_l
\]
\[
\frac{E_f}{V_f} = 2S_d \left[ \frac{1 - e^{-\alpha(1+\alpha)}}{2\alpha + c\alpha e^{-\alpha}} \right] \frac{(1 + k/2)(1 + k + k^2/2)}{(1+k)^2} c e^{-\alpha} b > x_l
\]
\[
\alpha = 2 \left\{ 1 + 1/(1+k)^2 + \left[ 1 - 1/(1+k)^2 \right] \left[ (1-\mu)c + \mu \right] \right\} \frac{1}{c \left[ \left( (k+1)^2 - 1 \right) \left[ (3+\mu) + (1-\mu)/(1+k)^2 \right] \right]}
\]

**Power Loss Under Turbulent Conditions**

\[
P_T = \omega \left[ 4.76 \times 10^{-5} \cdot \frac{\omega}{b} + 0.167 \right] + 0.073 \cdot \frac{\rho_m}{\omega^3} \cdot 0.09 \cdot 1.6 \cdot \omega^2 \cdot \mu_m^0.2
\]
\[
+ \pi \cdot \rho_m \cdot r^4 \cdot \omega^3 \cdot t \left[ 0.35 \cdot \mu_m + 0.020 \cdot \log_{10} \left( \frac{\omega^2 \cdot \mu_m}{\rho_m} \right) \right]^{1/4}
\]

**Power Loss Under Laminar Conditions**

\[
P_T = \omega \left[ 4.76 \times 10^{-5} \cdot \frac{2W_b}{b} + 0.167 \right] + 1.93 \cdot \rho_m \cdot \omega^5 \cdot \mu_m^0.5
\]
\[
+ \pi \cdot r \cdot t \cdot \omega^2 \cdot \mu_m
\]

### 3.11.2 SUMMARY OF CONCLUSIONS

The most significant conclusions arrived at in this chapter are as follows:

**Rim-Arm Type Flywheel**

3.11.2.1 The energy storage capacity per unit weight and the energy storage capacity per unit volume both increase linearly with the value of the combined stress in the rim.

3.11.2.2 The energy storage capacity per unit weight and the energy storage capacity per unit volume both increase as the mean radius of the rim decreases.

3.11.2.3 The energy storage capacity per unit weight and the energy storage capacity per unit volume both increase as the radial thickness of the rim increases.

3.11.2.4 The energy storage capacity per unit weight and the energy
storage capacity per unit volume both increase as the number of arms increases.

3.11.2.5 The maximum energy storage capacity per unit weight is in the vicinity of 2,450 ft lb/lb. This is the energy storage capacity per unit weight for a steel flywheel with eight arms where \( t/r_m = 0.3 \) and \( S_{yp} = 25,000 \text{ psi} \).

3.11.2.6 The maximum energy storage capacity per unit volume for a rim-arm type flywheel is in the vicinity of \( 1,400,000 \) ft lb/ft\(^3\). This is the value for a steel flywheel where \( t/r_m = 0.4 \) and \( S_{yp} = 25,000 \text{ psi} \).

**Cylindrical Type Flywheel**

3.11.2.7 The energy storage capacity per unit volume and the energy storage capacity per unit weight both increase linearly with the maximum stress in the flywheel.

3.11.2.8 The energy storage capacity per unit volume and the energy storage capacity per unit weight both increase as the cylinder thickness decreases.

3.11.2.9 The energy storage capacity per unit volume and the energy storage capacity per unit weight both increase as the cylinder radius increases.

3.11.2.10 The angular velocity must increase as the cylinder thickness decreases in order to maintain a constant maximum stress.

3.11.2.11 The angular velocity must increase as the cylinder radius decreases in order to maintain a constant maximum stress.

3.11.2.12 There is some thickness which maximizes the energy which can be stored in the flywheel. This thickness is given by

\[
t = r \left[ \frac{3(1-\mu)(3+\mu)}{\mu(1+\mu)} \right]^{1/2}.
\]
A flywheel of this thickness has an energy storage capacity given by

\[ E_r = \pi S_m r^3 \left[ \frac{3(1-\mu)(3+\mu)}{\mu(1+\mu)} \right]^{1/2} / (3+\mu). \]

3.11.2.13 The maximum energy storage capacity per unit weight for a cylindrical flywheel is in the vicinity of 18,000 ft lb/lb. This value is for a steel flywheel where \( t/r = 0.2 \) and \( S_Y = 100,000 \text{ psi} \).

3.11.2.14 The maximum energy storage capacity per unit volume for a cylindrical flywheel is in the vicinity of 8,750,000 ft lb/ft\(^3\). This is the value for a steel flywheel where \( t/r = 0.2 \) and \( S_Y = 100,000 \text{ psi} \).

**Uniform-Stress Disk With a Rim**

3.11.2.15 The stress in the disk section is always greater than or equal to the maximum stress in the rim section.

3.11.2.16 The maximum radial stress in the rim is given by

\[ S_{rm} = S_d/C. \]

3.11.2.17 The maximum tangential stress in the rim is given by

\[ S_{tm} = (S_d/C) \left[ (1-C)\mu + C \right]. \]

3.11.2.18 The energy storage capacity per unit weight increases as the radial thickness of the rim decreases. The increase of energy per unit weight necessitates an increase in angular velocity in order to keep the stress constant.

3.11.2.19 There is some radial thickness of the rim which makes the energy storage capacity per unit volume a maximum.

3.11.2.20 The energy storage capacity per unit weight increases as \( C \) decreases. The increase in the energy storage capacity necessitates an angular velocity increase to maintain the disk stress constant.

3.11.2.21 The energy storage capacity per unit weight is a linear function of the stress in the disk section of the flywheel.
3.11.2.22 The maximum energy storage capacity per unit weight is in the vicinity of 27,350 ft lb/lb. This is the value for a steel flywheel where $C = 1$, $k = 0.4$, and $S_{yp} = 100,000$ psi.

3.11.2.23 The maximum energy storage capacity per unit volume is in the vicinity of 9,000,000 ft lb/ft$^3$. This is the value for a steel flywheel where $C = 1.5$, $k = 0.75$, and $S_{yp} = 100,000$ psi.

**Power Loss**

3.11.2.24 The power losses for a rotating flywheel can be decreased significantly by enclosing the flywheel in a low viscosity atmosphere.

**General**

3.11.2.25 The three configurations of flywheels studied are listed below in the order of decreasing energy storage capacity:

- Uniform-stress disk with rim
- Cylindrical shaped flywheel
- Rim-arm type flywheel.
A  area
b  width of spring
C  constant = $6 \left[ (\beta-1)/\beta \right]^2 / \pi \ln \beta$
C_1 constant = $6 \left[ (\beta-1)/\ln \beta - 1 \right] / \pi \ln \beta$
C_2 constant = $6 \left[ (\beta-1)/2 \right] / \pi \ln \beta$
d  wire diameter
d_i inside diameter of hollow torsion bar spring
d_o outside diameter of hollow torsion bar spring
D  mean coil diameter
E  modulus of elasticity
E_s energy storage capacity of a spring
E_{sb} energy storage capacity in a single Belleville spring
E_{ss} energy storage capacity of a stack of Belleville springs
g  constant equal to acceleration of freely falling body
G  modulus of elasticity in shear
h  height of a single Belleville spring
h_s solid height of coil spring
I_s moment of inertia of cross section of spring
l  length
ln natural logarithm
l_s length of spring
l_v length of wire in coil spring
L_f free length of coil spring
L_t length of torsion bar spring
\( M_0 \) bending moment
\( M_s \) mass of a spring
\( n \) number of coils in a coil spring
\( N_1 \) number of groups of springs which are stacked in series in a stack of Belleville springs
\( N_2 \) number of springs which are stacked in parallel in any one group in a stack of Belleville springs
\( r_a \) radius of arbor on which clock type spring is wound
\( r_c \) radius of casing which houses clock type spring
\( r_1 \) inside radius
\( r_2 \) outside radius
\( S \) stress
\( S_b \) bending stress
\( S_m \) maximum bending stress
\( S_s \) shearing stress
\( S_{sm} \) maximum shearing stress
\( t \) thickness
\( \tan \) tangent
\( V_i \) initial volume
\( V_f \) final volume
\( V_s \) volume of space occupied by spring
\( V_{ss} \) volume of space occupied by a stack of Belleville springs
\( W \) load
\( W_s \) weight of spring
\( W_{ss} \) weight of a stack of Belleville springs
\( y_m \) maximum deflection of a spring
\( \beta \) radius ratio = \( r_1 / r_2 \)
\( \delta \)  deflection

\( \varepsilon \)  unit strain

\( \vartheta \)  angle of deformation of a piece of rubber loaded in simple shear

\( \mu \)  Poisson's ratio for spring material

\( \pi \)  constant = 3.1416

\( \rho \)  density of spring material
CHAPTER IV

ENERGY STORAGE CAPACITY OF A SPRING

4.1 INTRODUCTION - As was pointed out in section 1.1, the energy storage capacity of any system is a function of the \( Q \) and \( \Delta U \) which are associated with a given process. In this chapter, the process which will be used for evaluating the energy storage capacity of a spring will be the ideal process where \( Q = 0 \) and \( \Delta U \) is equal to the resilience of the spring. This process is commonly referred to as the process involving a conversion of strain energy into work. In this chapter, four configurations of springs are considered. The energy storage capacity per unit weight and the energy storage capacity per unit volume are evaluated for each configuration.

4.2 ENERGY STORAGE CAPACITY AS A FUNCTION OF WEIGHT FOR COIL SPRINGS

4.2.1 ASSUMPTION - For the purposes of the analysis of section 4.2.2, it has been assumed that when a coil spring is loaded, the only load on the wire is a torsional moment.

4.2.2 ANALYSIS - The resilience of a round-wire coil spring is given by the well-known equation

\[
E_s = S_s^2 \pi^2 d^2 D n/16 G. \tag{4.1}
\]

The mass of a round wire coil spring can be expressed by

\[
M_s = \rho \pi d^2 D f_w/4. \tag{4.2}
\]

The length of wire in such a spring is given approximately by

\[
f_w = \pi D n. \tag{4.3}
\]

Using this value in equation (4.2), the mass expression becomes
\[ M_g = \rho \pi^2 d^2 D n/4. \] (4.4)

Combining equations (4.1) and (4.4) gives the energy storage capacity per unit mass as

\[ E_s/M_g = S_g^2/h \sigma G \rho. \] (4.5)

Dividing both sides of equation (4.5) by \( g \) gives the energy weight relation. Thus

\[ E_s/W_g = S_g^2/h \sigma G \rho. \] (4.6)

4.2.3 CONCLUSION - Equation (4.6) indicates that for any given material and any given allowable stress, the energy storage capacity per unit weight of the round-wire coil spring is a constant. For a steel spring with an allowable shear stress of 30,000 psi,

\[ E_s/W_g = 69.2 \text{ in.lb/lb}. \] (4.7)

4.3 ENERGY STORAGE CAPACITY AS A FUNCTION OF VOLUME FOR COIL SPRINGS

4.3.1 ASSUMPTION - In order to calculate the energy storage capacity as a function of the volume of space occupied, it has been assumed that the spring is used as a compressive spring and that the maximum stress is attained when the spring is compressed solid.

4.3.2 ANALYSIS - According to equation (4.1) the energy storage capacity of a round-wire coil spring is given by

\[ E_s = S_g^2 \pi^2 d^2 D n/16 G. \] (4.8)

Using assumption 4.3.1, the maximum volume of space occupied by such a spring is

\[ V_g = \pi(D-d)^2 L/4. \] (4.9)

The free length of the spring can be expressed as the sum of the solid height and the maximum deflection, thus
The maximum deflection of the spring is given by the well-known equation

\[ y_m = S_s \pi D^2 \frac{n}{d} G. \]  

The solid height can be expressed as

\[ h_s = n d. \]  

Combining equations (4.10), (4.11) and (4.12) gives

\[ L_T = n \frac{(d^2 G + S_s \pi D^2)}{d} G. \]  

Using this in equation (4.9) gives

\[ V_S = \pi (D+d)^2 n \frac{(d^2 G + S_s \pi D^2)}{4 d} G. \]  

Combination of equations (4.8) and (4.11) gives the energy per unit of volume of occupied space as

\[ \frac{E_g}{V_S} = S_s^2 \frac{n}{4} \frac{(D/d)}{D/d + 1} \left[ \frac{G(d/D)^2 + S_s \pi}{1} \right]. \]  

Fig. 14 is a plot of \( \frac{E_g}{V_S} \) vs. \( d/D \) for two values of \( S_s \).

4.3.3 CONCLUSIONS — The following conclusions can be reached on the basis of the analysis of section 4.3.2.

4.3.3.1 Examination of equation (4.15) reveals that as \( S_s \) increases, \( \frac{E_g}{V_S} \) increases. Thus the material which permits the largest allowable stress should be used to obtain the maximum energy per unit volume of occupied space.

4.3.3.2 Examination of Fig. 14 reveals that for a given value of stress, the energy per unit volume increases as the ratio \( d/D \) increases.

4.3.3.3 The analysis of section 4.3.2 has shown that the maximum energy storage capacity per unit volume for a coil spring is in the vicinity of 5000 ft lb/ft³. This is the value for a steel spring where \( D/d = 2 \) and \( S_s = 50,000 \) psi.
$G = 11.5 \times 10^6 \text{ p.s.i.}$

$S = 50,000 \text{ p.s.i.}$

Fig. 14: Energy per Unit Volume for a Coil Spring
4.4 ENERGY STORAGE CAPACITY AS A FUNCTION OF WEIGHT FOR TORSION BAR
SPRINGS

4.4.1 ANALYSIS - The resilience of a hollow torsion bar can be expressed as

\[ E_s = S_{sm}^2 n L_t (d_i^4 - d_o^4)/16 \rho d_o^2. \] (4.16)

The mass of a hollow torsion bar is given by

\[ M_s = \rho n L_t (d_o^2 - d_i^2)/4. \] (4.17)

Combining equations (4.16) and (4.17) gives the energy per unit mass as

\[ E_s/M_s = S_{sm}^2 (d_i^2 + d_o^4)/4 \rho G d_o^2. \] (4.18)

Dividing both sides of equation (4.16) by g and rewriting gives

\[ E_s/\dot{W}_{s} = S_{sm}^2 [1 + (d_i/d_o)^2]/4 \rho g G. \] (4.19)

Fig. 15 is a plot of \( E_{sp}G/\dot{W}_{s}S_{sm}^2 \) vs. \( d_i/d_o \).

4.4.2 CONCLUSIONS - The following conclusions can be reached on the basis of the analysis of section 4.4.1.

4.4.2.1 Inspection of equation (4.19) reveals that the ratio of \( d_i/d_o \) should be as large as possible in order to obtain the maximum energy per unit weight. The physical configuration demands that \( d_i < d_o \). The energy per unit weight approaches a value of

\[ E_s/\dot{W}_{s} = S_{sm}^2/2 \rho G, \] (4.20)

as \( d_i/d_o \) approaches its maximum physical limit of unity.

4.4.2.2 Examination of equation (4.19) reveals that the energy storage capacity per unit weight for a torsion bar spring is proportional to the square of the allowable shearing stress of the spring material.

4.4.2.3 The analysis of section 4.4.1 has shown that the maximum energy storage capacity per unit weight for a torsion bar spring is
FIG. 19. ENERGY PER UNIT WEIGHT FUNCTION VS. DIAMETER RATIO FOR A TORSION BAR SPRING.
in the vicinity of 29 ft lb/lb. This is the value for a steel spring where \( \frac{d_1}{d_0} = 1 \) and \( S_{sm} = 50,000 \text{ psi} \).

4.5 ENERGY STORAGE CAPACITY AS A FUNCTION OF VOLUME FOR TORSION BAR SPRINGS

4.5.1 ANALYSIS - The volume of space occupied by a torsion bar spring can be calculated by

\[
V_s = L_t \pi \frac{d_0^2}{4}.
\]

(4.21)

Combination of equations (4.16) and (4.21) gives the energy per unit of volume occupied as

\[
E_g/V_s = \frac{S_{sm}^2}{4} \left[ 1 - \left( \frac{d_1}{d_0} \right)^2 \right]/l_G.
\]

(4.22)

Fig. 16 is a plot of \( E_g/V_s \) vs. \( d_1/d_o \).

4.5.2 CONCLUSIONS - The following conclusions can be reached on the basis of the analysis of section 4.5.1.

4.5.2.1 Examination of equation (4.22) reveals that the energy per unit volume decreases as \( d_1/d_0 \) increases. Physically, \( d_1 \) must always be greater than zero. Thus, \( d_1/d_0 \) has a physical minimum of zero. The energy per unit of volume of occupied space approaches a value of

\[
E_g/V_s = \frac{S_{sm}^2}{4} l_G
\]

(4.23)

as \( d_1/d_0 \) approaches zero. This is the case of a solid torsion bar.

4.5.2.2 Examination of equation (4.22) reveals that the energy storage capacity per unit volume for the case of a torsion bar is proportional to the square of the allowable shearing stress of the spring material.

4.5.2.3 The analysis of section 4.5.1 has shown that the maximum energy storage capacity per unit volume for a torsion bar spring is in the vicinity of 8,000 ft lb/ft^3. This is the value for a steel spring where \( d_1/d_0 = 0.1 \) and \( S_{sm} = 50,000 \text{ psi} \).
4.6 ENERGY STORAGE CAPACITY OF SPIRAL WOUND SPRINGS AS A FUNCTION OF WEIGHT — The type of spring which will be considered in this section will be a flat spring which is wound on an arbor with its end rigidly fixed to an outside casing. Fig. 17 is a sketch of such a spring.

4.6.1 ANALYSIS — According to the simple beam theory, the energy stored in a spring subjected to the moment \( M_0 \) is given by

\[
E_s = \int_{l_0}^{L} \frac{M_0^2}{2EI_s} \, dl.
\]

Wahl indicates that in the type of spring being considered, the bending moment is uniform along the length of the spring. Thus, for a spring of uniform cross section, equation (4.24) becomes

\[
E_s = \frac{M_0^2\ell_s}{2EI_s}.
\]  

(4.25)

For a rectangular cross section the stress in the spring is related to the moment by

\[
S_b = \frac{M_0}{2I_s}. 
\]  

(4.26)

Solving equation (4.26) for \( M_0 \) and substituting the result into equation (4.25) gives

\[
E_s = 2S_b^2I_s\frac{\ell_s}{t^2}E.
\]  

(4.27)

For a rectangular cross section

\[
I_s = \frac{bt^3}{12}.
\]  

(4.28)

Using this value in equation (4.27) gives

\[
E_s = S_b^2\frac{b}{6}t\ell_s/E.
\]  

(4.29)

The mass of the spring is given by

\[
M_s = \rho bt\ell_s.
\]  

(4.30)

FIG. 17  SKETCH OF CLOCK-TYPE SPRING
(SPIRAL-WOUND)
Combining equations (4.29) and (4.30) and dividing both sides of the result by \( g \) gives the energy per unit weight as

\[
\frac{E_g}{W_s} = \frac{S_b^2}{6} \sigma \rho E. \tag{4.31}
\]

This analysis neglects the weight of the casing.

4.6.2 CONCLUSIONS - The following conclusions can be arrived at on the basis of the analysis of section 4.6.1.

4.6.2.1 For any given material and any given allowable stress, the energy storage capacity per unit weight of a spiral wound spring is a constant independent of the dimensions of the spring. For a steel spring with an allowable bending stress of 50,000 psi

\[
\frac{E_g}{W_s} = 19.1 \text{ in. lb/lb.} \tag{4.32}
\]

4.6.2.2 Examination of equation (4.31) reveals that the energy storage capacity per unit weight for a spiral wound spring is proportional to the square of the allowable bending stress for the spring material.

4.6.2.3 The analysis of section 4.6.1 has shown that the maximum energy per unit weight for a spiral wound spring is in the vicinity of 16 ft lb/lb. This is the value for a steel spring where \( S_b = 100,000 \) psi.

4.7 ENERGY STORAGE CAPACITY OF SPIRAL WOUND SPRINGS AS A FUNCTION OF VOLUME

4.7.1 ANALYSIS - The volume of space occupied by the spiral wound spring is equal to the volume of the casing containing it, thus

\[
V_s = \pi r_c^2 b. \tag{4.33}
\]

Combining equations (4.29) and (4.33) gives the energy storage capacity per unit volume of space occupied as

\[
\frac{E_g}{V_s} = \frac{S_b^2}{6} \tau / \rho E \pi r_c^2. \tag{4.34}
\]
According to Barnes, Gibson and Raymond 15/, in order that the spring may deliver the maximum number of turns in a given space, it is necessary that the space occupied by the spring shall be one-half the available space in the drum. This can be expressed mathematically by

\[ 2 \ell_s t = \pi (r_c^2 - r_a^2). \]  (4.35)

This condition will be assumed to hold for this analysis. Solving this for \( \ell_s t \) and substituting the result into equation (4.34) gives

\[ \frac{E_s}{V_s} = S_0^2 \left[ 1 - \left( \frac{r_a}{r_c} \right)^2 \right] /12 E. \]  (4.36)

Fig. 18 is a plot of \( \frac{E_s}{V_s} S_0^2 \) vs. \( \frac{r_a}{r_c} \).

11.7.2 CONCLUSIONS - The following conclusions can be reached on the basis of the analysis of section 11.7.1.

11.7.2.1 The energy storage capacity per unit volume of space occupied increases as \( \frac{r_a}{r_c} \) decreases. As \( \frac{r_a}{r_c} \) approaches zero, \( \frac{E_s}{V_s} \) approaches a value of

\[ \frac{E_s}{V_s} = \frac{S_0^2}{12} E \]  (4.37)

as a limit.

11.7.2.2 Examination of equation (4.37) reveals that the energy storage capacity per unit volume is proportional to the square of the allowable bending stress for the spring material.

11.7.2.3 The analysis of section 11.7.1 has shown that the maximum energy storage capacity per unit volume for a spiral wound spring is in the vicinity of 4,000 ft lb/ft³. This is the value for a steel spring where \( \frac{r_a}{r_c} = 0.1 \) and \( S_{yp} = 100,000 \) psi.

---

FIG. 18 ENERGY PER UNIT-VOLUME FUNCTION VS. RADIUS RATIO FOR A SPIRAL TYPE SPRING
4.8 ENERGY STORAGE CAPACITY AS A FUNCTION OF WEIGHT FOR BELLEVILLE SPRINGS - The type of spring which will be studied in this section is sketched in (A) of Fig. 19. Figs. 19-B and 19-C show means of stacking these springs which can be used rather than using a single spring. The type of stacking which is shown as (B) is known as parallel stacking while that shown in (C) is known as series stacking. It is also possible to use a combination of the types of stacking as shown in (D) of Fig. 19. In this case, two or more parallel groups of springs are stacked in series. This is the type of spring which will be studied in the following sections.

4.8.1 ASSUMPTIONS - In order to complete the analysis of the following section, the following assumptions have been made.

4.8.1.1 It is assumed that the load \( W \) is uniformly applied around the inner radius of the Belleville spring.

4.8.1.2 It has been assumed \[16/\] that when \( N_2 \) springs are stacked in parallel, the load capacity is \( N_2 \) times the load capacity of a single spring for a given deflection.

4.8.1.3 It has been assumed \[16/\] that when \( N_1 \) groups of springs are stacked in series, the deflection is \( N_1 \) times the deflection of a single spring for any given load.

4.8.1.4 It is assumed that the number of springs which are included in a parallel group is sufficiently small so that the developed theory holds. Due to friction between the springs and digging of one spring into another, the theory does not hold for \( N_2 > 4 \).

\[16/\] Almen, J. O. and Lazzlo, A. The Uniform-Section Disk Spring, Transactions of the American Society of Mechanical Engineers, Vol. 58.
FIG. 19 SKETCHES SHOWING THE VARIOUS MEANS OF STACKING BELLEVILLE SPRINGS
4.8.1.5 It is assumed that the maximum energy is stored in a Belleville spring when it undergoes a deflection of magnitude equal to h. Correspondingly the maximum stress will be attained when the deflection has a magnitude of h.

4.8.2 ANALYSIS - According to assumptions 4.8.1.2 and 4.8.1.3, the energy storage capacity of a stack of Belleville springs is given by

\[ E_{ss} = N_1 N_2 E_{sb} \]  \hspace{1cm} (4.38)

According to Almen and Lazzlo, the load deflection relation for a single Belleville type spring is given by

\[ W = E \delta \left[ (h-\delta)(h-\delta/2)t + t^3 \right]/(1-\mu^2)C r_2^2 \]  \hspace{1cm} (4.39)

where

\[ C = 6 \left[ (\beta-1)/\beta \right]^2/ n \ln \beta. \]  \hspace{1cm} (4.40)

The energy storage capacity of the spring is given basically by

\[ E_{sb} = \int_{\delta_1}^{\delta_2} W \, d\delta. \]  \hspace{1cm} (4.41)

Combination of equations (4.39) and (4.41) gives

\[ E_{sb} = E \int_{\delta_1}^{\delta_2} \left\{ \left[ (h^2t + t^3)\delta - 3ht\delta^2/2 + t\delta^3/2 \right] d\delta \right\} / (1-\mu^2)C r_2^2 \]  \hspace{1cm} (4.42)

Using assumption 4.8.1.5, the limits of integration on equation (4.42) become zero to h. Integrating equation (4.42) between these limits gives

\[ E_{sb} = E t h^2(h^2 + lt^2)/8(1-\mu^2)C r_2^2. \]  \hspace{1cm} (4.43)

Combination of equations (4.38) and (4.43) gives

\[ E_{ss} = E N_1 N_2 t h^2(h^2 + lt^2)/8(1-\mu^2)C r_2^2. \]  \hspace{1cm} (4.44)

17/ Ibid.
The mass of a stack of Belleville springs is given by

$$M_{ss} = N_1 N_2 M_{sb}.$$  \((4.45)\)

The mass of a single spring is given approximately by

$$M_{sb} = t n (r_2^2 - r_1^2) \rho.$$  \((4.46)\)

Combination of equations \((4.45)\) and \((4.46)\) gives

$$M_{ss} = N_1 N_2 t n (r_2^2 - r_1^2) \rho.$$  \((4.47)\)

Combining equations \((4.44)\) and \((4.46)\) and dividing both sides of the result by \(g\) gives the energy per unit weight as

$$E_{ss}/W_{ss} = E (h^2 + 4t^2)/8(1-\mu^2)c r_2 n(r_2^2 - r_1^2)\rho.$$

This is the energy-weight relation as a function of the dimensions of the spring.

It is also desirable to express the energy storage capacity as a function of the maximum allowable stress in the Belleville spring. The stress-deflection relation is given \(^{18/}\) by

$$S = E \delta [C_1 (h - h/2) + C_2 t]/(1-\mu^2) c r_2^2 .$$  \((4.49)\)

where

$$C_1 = 6 \left[(\beta-1)/\ln \beta - 1\right]/n \ln \beta$$  \((4.50)\)

and

$$C_2 = 6 \left[(\beta-1)/2\right]/n \ln \beta.$$  \((4.51)\)

Using assumption \(4.81.5\) and equation \((4.49)\) gives

$$S_m = E h(C_1 h/2 + C_2 t)/(1-\mu^2) c r_2^2 .$$  \((4.52)\)

Solving this for \(h/t\) gives

$$h/t = C_2 \left[1 + 2S_m(1-\mu^2)c C_1 (r_2/t)^2/E c_2^2\right]^{1/2} - 1\right]^{1/2}/C_1.$$  \((4.53)\)

Equation \((4.48)\) can be written as

\(^{18/}\) Ibid.
\[ \frac{E_{ss}}{W_{ss}} = (t/r_2)^4 \frac{E(h/t)^2 [(h/t)^2 + h]}{8g(1-\mu^2)} \pi \rho \left[ 1 - (r_1/r_2)^2 \right]. \]  

Combination of equations (4.53) and (4.54) gives the energy-weight relation as a function of the maximum allowable stress in the spring.

Fig. 20 is a plot of \( \frac{E_{ss}}{W_{ss}} \) vs. \( r_1/r_2 \) for the case of a steel spring with a maximum allowable stress of 100,000 psi. This plot was obtained by using equations (4.53) and (4.54).

4.8.3 CONCLUSIONS - The following conclusions can be reached on the basis of the analysis of section 4.8.2.

4.8.3.1 Examination of equation (4.48) reveals that \( \frac{E_{ss}}{W_{ss}} \) is independent of \( N_1 \) and \( N_2 \). It can therefore be concluded that the energy storage capacity per unit weight is independent of the number of springs involved or the method used for stacking the springs.

4.8.3.2 Examination of Fig. 20 reveals that for given values of radii, the energy storage capacity per unit weight for Belleville springs increases as the thickness of the springs increase.

4.8.3.3 Examination of Fig. 20 reveals that for given values of thickness and outside radius, the energy per unit weight increases as the inside radius increases.

4.8.3.4 The analysis of section 4.8.2 has shown that the maximum energy storage capacity per unit weight for a Belleville type spring is in the vicinity of 11 ft lb/lb. This is the value for a steel spring where \( t/r_2 = 0.1, r_1/r_2 = 0.8 \) and \( S_{yp} = 100,000 \) psi.

4.9 ENERGY STORAGE CAPACITY AS A FUNCTION OF VOLUME FOR BELLEVILLE SPRINGS

4.9.1 ANALYSIS - The volume of space occupied by a stack of Belleville
FIG. 20: ENERGY PER UNIT WEIGHT FOR A SINGLE BELLEVILLE SPRING

\[ \frac{t}{r_2} = 0.1 \]

- \( E = 3 	imes 10^3 \text{ ksi} \)
- \( \lambda = 0.48 \)
- \( s_u = 100,000 \text{ psi} \)
- \( \rho = 7.31 \times 10^{-9} \) in. \( \text{s}^2 / \text{in}^4 \)

MATERIAL: STEEL
springs is

\[ V_{ss} = \pi r_2^2 N_1 [h + N_2 t] \]  \hspace{1cm} (4.55)

Combination of equations (4.44) and (4.55) gives the energy per unit volume of space occupied as

\[ E_{ss}/V_{ss} = E(t/r_2)^4(h/t)^2 \left[ (h/t)^2 + h \right] / 8(1-\mu^2)C \pi(h/t N_2 + 1). \]  \hspace{1cm} (4.56)

When \( N_2 = 1 \), this equation becomes the energy-volume relation for one individual spring. Fig. 21 is a plot of \( E_{ss}/V_{ss} \) vs. \( r_1/r_2 \) for the case of one steel spring with a maximum allowable stress of 100,000 psi. This plot was obtained by using equations (4.53) and (4.56).

4.9.2 CONCLUSIONS - The following conclusions can be reached on the basis of the preceding analysis.

4.9.2.1 Examination of equation (4.56) reveals that the energy-volume relation for a stack of Belleville springs is independent of \( N_1 \). Thus the energy-volume relation is the same for a series stack of springs as for an individual parallel group of springs.

4.9.2.2 Examination of equation (4.56) reveals that the energy storage capacity per unit of volume of space occupied increases as the number of parallel springs in any group increases.

4.9.2.3 Examination of Fig. 21 reveals that for given values of thickness and outside radius, there is some value of inside radius which maximizes the energy storage capacity per unit volume.

4.9.2.4 Examination of Fig. 21 reveals that for given values of radii, the energy storage capacity per unit volume increases as the thickness increases.

4.9.2.5 The analysis of section 4.9.1 has shown that the maximum energy storage capacity per unit volume for a Belleville type spring is in the
vicinity of 2,000 ft lb/ft$^3$. This is the value for a steel spring where $t/r_2 = 0.1$ and $S_{yp} = 100,000$ psi.

4.10 CONSIDERATION OF LOSSES - The losses which are encountered in any device which uses a spring for storing energy are primarily due to hysteresis and Coulomb friction. The hysteresis losses for steel springs are negligible for low frequency operation. Coulomb friction losses are negligible in most springs. However, for Belleville springs stacked in parallel, this loss may be 30 to 40 per cent of the input energy.

4.11 SUMMARY - In this chapter, four types of springs have been evaluated. This section is a summary of the most significant equations and conclusions which have arisen from the analyses of this chapter. Graphs which are pictorial representations of the analyses have been included in previous sections.

4.11.1 SUMMARY OF EQUATIONS

Coil Springs

$$\frac{E_g}{W_g} = \frac{S_g^2}{h \gamma G \rho}$$

$$\frac{E_g}{V_g} = \frac{S_g^2}{h} \eta \frac{h}{d_1 (d/d)} \left[ \frac{(d/d)^2 + 2(d/d) + 1}{G(d/d)^2 + S_g \eta} \right]$$

Torsion Bar Springs

$$\frac{E_g}{W_g} = \frac{S_m^2}{h} \left[ 1 + (d_1/d_2)^2 \right] / h \gamma G \rho$$

$$\frac{E_g}{V_g} = \frac{S_m^2}{h} \left[ 1 - (d_1/d_2)^4 \right] / h G$$

Clock Type Springs

$$\frac{E_g}{W_g} = \frac{S_b^2}{6} \gamma G \rho$$

$$\frac{E_g}{V_g} = \frac{S_b^2}{12} \left[ 1 - (r_a/r_c)^2 \right] / 12 G$$
**Belleville Springs**

$$E_{ss}/V_{ss} = (t/r_2)^4 E(h/t)^2 \left[ (h/t)^2 + h_1 \right]/8g(1-\mu^2)C [1-(r_1/r_2)^2]$$

$$E_{ss}/V_{ss} = E(t/r_2)^4 (h/t)^2 \left[ (h/t)^2 + h_1 \right]/8(1-\mu^2)Cn(h/t N_2 + 1)$$

$$(h/t) = C_2 \left[ \left[ 1 + 2\mu (1-\mu^2)C \right] c_1 (r_2/t)^2/E C_2 \right]^{1/2} - 1)/c_1$$

**SUMMARY OF CONCLUSIONS** - The following conclusions are the most significant conclusions which have arisen from the analyses of this chapter.

**Round Wire Coil Spring**

4.11.2.1 The energy storage capacity per unit weight is independent of the dimensions of the spring.

4.11.2.2 The energy storage capacity per unit volume increases as the allowable shearing stress in the spring material increases.

4.11.2.3 The energy storage capacity per unit volume increases as the diameter ratio d/D increases.

4.11.2.4 The maximum energy storage capacity per unit weight is in the vicinity of 15 ft lb/lb. This is the value for a steel spring where $S_s = 50,000$ psi.

4.11.2.5 The maximum energy storage capacity per unit volume is in the vicinity of 5,000 ft lb/ft$^3$. This value is for a steel spring where D/d = 2 and $S_s = 50,000$ psi.

**Torsion Bar Spring**

4.11.2.6 The energy storage capacity per unit weight increases as the diameter ratio $d_1/d_0$ increases.

4.11.2.7 The energy storage capacity per unit weight and the energy storage capacity per unit volume are both directly proportional to the square of the allowable shearing stress of the spring material.
11.2.8 The energy storage capacity per unit volume decreases as the diameter ratio \( d_i/d_o \) increases.

11.2.9 The maximum energy storage capacity per unit weight is in the vicinity of 29 ft lb/lb. This value is for a steel spring where \( d_i/d_o = 1 \) and \( S_{sm} = 50,000 \) psi.

11.2.10 The maximum energy storage capacity per unit volume is in the vicinity of 8,000 ft lb/ft\(^3\). This is the value for a steel spring where \( d_i/d_o = 1 \) and \( S_{sm} = 50,000 \) psi.

**Spiral Wound Spring**

11.2.11 The energy storage capacity per unit weight is independent of the dimensions of the spring.

11.2.12 The energy storage capacity per unit weight and the energy storage capacity per unit volume are both directly proportional to the square of the allowable bending stress for the spring material.

11.2.13 The energy storage capacity per unit volume increases as the radius ratio \( r_a/r_c \) decreases.

11.2.14 The maximum energy storage capacity per unit weight is in the vicinity of 16 ft lb/lb. This is the value for a steel spring where \( S_b = 100,000 \) psi.

11.2.15 The maximum energy storage capacity per unit volume is in the vicinity of 4,000 ft lb/ft\(^3\). This is the value for a steel spring where \( r_a/r_c = 0.1 \) and \( S_{yp} = 100,000 \) psi.

**Belleville Spring**

11.2.16 The energy storage capacity per unit weight is independent of the number of springs which are stacked in series or in parallel.

11.2.17 The energy storage capacity per unit weight and the energy storage capacity per unit volume both increase as the thickness of the
spring increases.

11.11.2.18 The energy storage capacity per unit weight increases as the inside radius increases.

11.11.2.19 The energy storage capacity per unit volume is independent of the number of springs which are stacked in series in a stack of Belleville springs.

11.11.2.20 The energy storage capacity per unit volume increases as the number of parallel springs in a group increases.

11.11.2.21 There is some combination of spring dimensions which maximizes the energy storage capacity per unit volume.

11.11.2.22 The maximum energy storage capacity per unit weight is in the vicinity of 11 ft lb/lb. This is the value for a steel spring where \( t/r_2 = 0.1 \), \( r_1/r_2 = 0.8 \) and \( S_{yp} = 100,000 \) psi.

11.11.2.23 The maximum energy storage capacity per unit volume is in the vicinity of 2000 ft lb/ft\(^3\). This is the value for a steel spring where \( t/r_2 = 0.1 \) and \( S_{yp} = 100,000 \) psi.

**General**

11.11.2.24 The four types of springs studied in this chapter are listed below in order of their decreasing energy storage capacity.

<table>
<thead>
<tr>
<th>Energy Storage Capacity per Unit Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>Torsion spring</td>
</tr>
<tr>
<td>Spiral wound spring</td>
</tr>
<tr>
<td>Belleville spring</td>
</tr>
<tr>
<td>Coil spring</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Energy Storage Capacity per Unit Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>Torsion and Coil springs</td>
</tr>
<tr>
<td>Belleville and Spiral wound springs</td>
</tr>
</tbody>
</table>

87
### NOTATION FOR CHAPTER V

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>area</td>
</tr>
<tr>
<td>( E )</td>
<td>modulus of elasticity of container material</td>
</tr>
<tr>
<td>( E_{\text{s,\ell}} )</td>
<td>strain energy of a liquid</td>
</tr>
<tr>
<td>( F )</td>
<td>force</td>
</tr>
<tr>
<td>( k )</td>
<td>radius ratio ( r_{1r}/r_{2r} )</td>
</tr>
<tr>
<td>( L_c )</td>
<td>length of cylindrical portion of container</td>
</tr>
<tr>
<td>( M_c )</td>
<td>mass of container</td>
</tr>
<tr>
<td>( M )</td>
<td>mass of liquid</td>
</tr>
<tr>
<td>( p )</td>
<td>pressure</td>
</tr>
<tr>
<td>( P_f )</td>
<td>final pressure</td>
</tr>
<tr>
<td>( r_{1r} )</td>
<td>inside radius</td>
</tr>
<tr>
<td>( r_{2r} )</td>
<td>outside radius</td>
</tr>
<tr>
<td>( S_{\text{sm}} )</td>
<td>maximum shearing stress</td>
</tr>
<tr>
<td>( T )</td>
<td>temperature</td>
</tr>
<tr>
<td>( V )</td>
<td>volume</td>
</tr>
<tr>
<td>( V_c )</td>
<td>volume of container material</td>
</tr>
<tr>
<td>( V )</td>
<td>volume of liquid</td>
</tr>
<tr>
<td>( V_r )</td>
<td>relative volume ratio</td>
</tr>
<tr>
<td>( V_R )</td>
<td>reference volume</td>
</tr>
<tr>
<td>( V_t )</td>
<td>total volume</td>
</tr>
<tr>
<td>( V_l )</td>
<td>contained volume</td>
</tr>
<tr>
<td>( W_c )</td>
<td>weight of container</td>
</tr>
<tr>
<td>( W_{\ell} )</td>
<td>weight of liquid</td>
</tr>
<tr>
<td>( W_t )</td>
<td>total weight</td>
</tr>
</tbody>
</table>
$\delta$  displacement  
$\mu$  Poisson's ratio for container material  
$\rho_c$  density of container material  
$\rho_L$  density of liquid
CHAPTER V

ENERGY STORAGE CAPACITY OF A COMPRESSED LIQUID

5.1 INTRODUCTION - As was pointed out in section 1.1, the energy storage capacity of any system is a function of the \( Q \) and \( \Delta U \) which are associated with a given process. In this chapter, the process which will be used for evaluating the energy storage capacity of a compressed liquid will be an isothermal process. In this chapter the energy storage capacity per unit weight and the energy storage capacity per unit volume have been evaluated for a system which employs ether as the working medium. The losses which are associated with a compressed liquid energy storage system have also been evaluated.

5.2 ENERGY STORAGE CAPACITY AS A FUNCTION OF THE WEIGHT OF THE LIQUID

5.2.1 ANALYSIS - Early in this investigation, two unsuccessful attempts were made to derive a general analytical expression for the work obtained when expanding a liquid. The first attempt was to integrate the expression for work by using Van der Waals' equation to give the expression for pressure as a function of volume. It was found that the resulting equation predicted results which deviated from experimentally observed results to such an extent as to make the theoretical equation useless. The second attempt was to write an empirical equation for the pressure-volume relation by using experimentally observed results and Lagrange's interpolation method. It was found that a third order equation could be made to pass through four of the experimentally observed points but that the equation was completely unreliable between these points. The final approach which was used is explained in the
following paragraphs.

Since the attempt at deriving a general analytical expression for the energy which can be stored in any liquid was unsuccessful, it has been necessary to select a particular liquid in order to proceed with the study. Since ether is one of the most compressible liquids available, it has been chosen as the liquid which is to be used in the energy storage device. Fig. 22 is a plot of experimental data given by Bridgman 19/ which show the pressure-relative volume relation for ether at $68^\circ F$. The energy which is obtained from a liquid when it is expanded reversibly between two pressure levels is given by

$$E_s = \int_{V_1}^{V_2} p \, dV.$$  \hspace{1cm} (5.1)

A relative volume ratio can be defined by

$$V_R = \frac{V}{V_R},$$ \hspace{1cm} (5.2)

where $V_R$ is some reference volume. Combination of equations (5.1) and (5.2) gives

$$\frac{E_s}{V_R} = \int_{V_{R1}}^{V_{R2}} p \, dV_R.$$ \hspace{1cm} (5.3)

The integral of equation (5.3) is given by the area under the curve of Fig. 22. Thus, for any given reference volume, initial pressure, and final pressure, the energy stored in the liquid, as evaluated by a reversible isothermal expansion, can be evaluated by using equation (5.3) and Fig. 22. Fig. 23 shows a plot of $E_s/V_R$ vs. $p_f$ where the reference volume has been chosen as the volume of one pound of ether at $68^\circ F$ sub-

jected to a pressure of one atmosphere and where the initial pressure is assumed to be atmospheric pressure. This plot was obtained by using equation (5.3) and Fig. 22. This is the desired energy-weight relation.

5.2.2 CONCLUSION - Inspection of Fig. 23 reveals that the energy storage capacity per unit weight for ether is approximately a linear function of the final pressure.

5.3 ENERGY STORAGE CAPACITY AS A FUNCTION OF THE VOLUME OF THE LIQUID

In order to determine the energy-volume relation, it is necessary only to include the density of ether and the constant g into the energy-weight relation which was developed above. Fig. 23 shows a plot of the energy per unit volume of liquid versus the final pressure. The energy storage capacity per unit volume for ether is approximately a linear function of the final pressure.

5.4 ENERGY STORAGE CAPACITY AS A FUNCTION OF THE WEIGHT OF THE LIQUID AND CONTAINER

5.4.1 ASSUMPTIONS - In order to complete the analysis of section 5.4.2, the following assumptions are made.

5.4.1.1 It is assumed that the liquid in which energy is to be stored is compressed in a piston-cylinder device which has a hemispherical shaped end. It is further assumed that the thickness of the end is the same as the thickness of the cylindrical portion of the piston-cylinder device.

5.4.1.2 It is assumed that the maximum-shearing-stress theory of failure predicts the critical stress in the container.

5.4.1.3 It is assumed that the temperatures which are considered in this chapter are sufficiently low so that any creep effect in the metal container can be neglected.
5.4.2 ANALYSIS – Since relatively high pressures are being considered in connection with the liquid energy storage problem, it is possible that the energy stored in the container is a significant amount of energy relative to the energy which is stored in the contained liquid.

The approximate expression for the energy stored in the container is derived as follows. Murphy 20/ gives the displacement of the inside radius of a thick-walled cylinder subjected to an internal pressure as

\[ \delta = p r_{1r} \left[ \frac{r_{1r}^2 (1-2\mu) + r_{2r}^2 (1+\mu)}{E(r_{2r}^2 - r_{1r}^2)} \right]. \]  \hfill (5.4)

The energy which is stored in the cylindrical portion of the container when this displacement occurs is given by

\[ E_{sc} = \int_{\delta_1}^{\delta_2} F \, d\delta \]  \hfill (5.5)

or

\[ E_{sc} = \int_{\delta_1}^{\delta_2} Ap \, d\delta. \]  \hfill (5.6)

Considering the area to remain constant permits equation (5.6) to be written as

\[ E_{sc} = A \int_{P_1}^{P_2} p(d\delta / dp) \, dp. \]  \hfill (5.7)

Differentiating equation (5.4), putting the result into equation (5.7) and integrating gives

\[ E_{sc} = A r_{1r} p^2 \left[ k^2 (1-2\mu) + (1+\mu) \right] / 2E(1-k^2) \]  \hfill (5.8)

where

\[ k = r_{1r} / r_{2r}. \]  \hfill (5.9)

Expressing this as the energy stored in the cylindrical portion of the

---

20/ Murphy, G. Advanced Mechanics of Materials. McGraw-Hill, New York, 1946, p. 120.
container per unit of volume contained in this section of the container gives

\[ E_{sc}/V = p^2 \left[ k^2(1-2\mu) + (1+\mu) \right]/B(1-k^2). \]  
(5.10)

Equation (5.10) is now in a form such that the numerical results obtained from it can be compared with Fig. 22. If this is done, it is found that the ratio of the energy stored in the container to the energy stored in the liquid is approximately 1/20. Thus the energy stored in the container can be neglected in determining the energy-weight and energy-volume relations.

The mass of liquid in a cylindrical container with a hemispherical end is given by

\[ M_c = \rho_l \pi r_1^2 h + \rho_l \pi r_1^3/3. \]  
(5.11)

Using assumption 5.4.1.1 the mass of the container is given by

\[ M_c = \rho_c \pi r_1^2 h + 2 \rho_c \pi (r_2^3 - r_1^3)/3. \]  
(5.12)

Combining equations (5.11) and (5.12) gives the weight ratio

\[ \frac{W_c}{W_t} = \frac{(\rho_c/\rho_l)\left[ L_c(1-k^2)/k^2 r_1^2 h + 2(1-k^3)/3k^3 \right]/\left( L_c/r_1^2 h + 2/3 \right) \}}{W_t}. \]  
(5.13)

The energy per unit weight can be written as

\[ \frac{E_{sc}/W_t}{W_t} = \frac{E_{sc}}{W_t}(W_c + \omega_c) \]  
(5.14)

or

\[ \frac{E_{sc}/W_t}{W_t} = \frac{E_{sc}}{W_t}(1 + \omega_c/W_t). \]  
(5.15)

Combining equations (5.13) and (5.15) gives

\[ \frac{E_{sc}/W_t}{W_t} = \left( \frac{E_{sc}/W_t}{W_t} \right) \left[ \frac{L_c/r_1^2 h + 2/3}{\left( L_c/r_1^2 h + 2/3 \right) + (\rho_c/\rho_l)} \right] \left[ L_c(1-k^2)/k^2 r_1^2 h + 2(1-k^3)/3k^3 \right]. \]  
(5.16)

Using assumption 5.4.1.2 the quantity \( k \) must be related to the pressure by the following equation.

96
Using this relation in equation (5.16) gives the desired energy-weight relation. For ether, the quantity $E_{eg}/W_t$ in equation (5.16) can be obtained from Fig. 23. Fig. 24 is a plot of $E_{eg}/W_t$ vs. $p_f$ for two values of $S_{yp}$. This plot was obtained by using Fig. 23 and equation (5.16). For this plot ether is the working medium, steel is the material used for the container and 10 is the value of $L_o/r_{lr}$. The high value of $S_{yp}$ has been chosen in light of the fact that thick cylinders can be prestressed. This prestressing has the effect of permitting higher than usual values to be used for the allowable stress in equation (5.17).

5.4.3 CONCLUSIONS - The following conclusions can be reached on the basis of the analysis of section 5.4.2.

5.4.3.1 It was shown in the previous section that when ether is compressed, the energy which is stored in the container is sufficiently small relative to the energy which is stored in the ether so that the container energy can be neglected.

5.4.3.2 Examination of Fig. 24 reveals that for given values of $L_o/r_{lr}$ and $S_{yp}$, there is some value of final pressure which maximizes the energy storage capacity per unit weight. As the allowable stress in the container increases, the maximum point on these curves apparently shifts in the direction of higher pressures.

5.4.3.3 Examination of Fig. 24 reveals that for given values of $L_o/r_{lr}$ and $p_f$, the energy storage capacity per unit weight increases as the allowable stress of the container material increases.

5.4.3.4 The analysis of section 5.4.2 has shown that the maximum energy storage capacity per unit weight for a compressed liquid system
FIG. 24. ENERGY STORAGE CAPACITY PER UNIT WEIGHT FOR FIBER AT 600°F.
is in the vicinity of 300 ft lb/lb. This is the value for ether where
the container is composed of steel with an allowable stress of
100,000 psi.

5.5 ENERGY STORAGE CAPACITY AS A FUNCTION OF THE VOLUME OF THE LIQUID
AND CONTAINER

5.5.1 ANALYSIS - The total volume occupied by a cylindrical container
with a hemispherical end is given by

\[ V_t = V_\ell + V_c \]  \hspace{1cm} (5.18)

or

\[ V_t = V_\ell (1 + V_c/V_\ell). \]  \hspace{1cm} (5.19)

Thus

\[ \frac{E_{st}/V_t}{E_{st}/V_\ell} = \frac{E_s/V_c}{E_s} (1 + V_c/V_\ell). \]  \hspace{1cm} (5.20)

The ratio \( V_c/V_\ell \) is given by

\[ \frac{V_c}{V_\ell} = (w_c/w_\ell)(\rho_s/\rho_c). \]  \hspace{1cm} (5.21)

Combining equations (5.13) and (5.21) gives

\[ V_c/V_\ell = \left[ \frac{L_c(1-k^2)/k^2 r_{1r} + 2(1-k^3)/3k^3]}{(L_c/r_{1r} + 2/3)} \right]. \]  \hspace{1cm} (5.22)

Combining equations (5.20) and (5.22) gives

\[ \frac{E_{st}/V_t}{E_{st}/V_\ell} = \left( \frac{E_{st}/V_\ell}{E_s} \right) \left( \frac{L_c/r_{1r} + 2/3}{L_c/r_{1r} + 2/3} \right) \left[ \frac{L_c(1-k^2)/k^2 r_{1r} + 2(1-k^3)/3k^3]}{(L_c/r_{1r} + 2/3)} \right]. \]  \hspace{1cm} (5.23)

As explained in the above paragraph, \( k \) is related to the internal pres-
sure by equation (5.17). Combining equations (5.17) and (5.23) gives
the desired energy-volume relation. For ether, the quantity \( E_{st}/V_\ell \) in
equation (5.23) can be obtained from Fig. 23. Fig. 25 is a plot of
\( E_{st}/V_t \) vs. \( p_f \) for two values of \( S_{yp} \), where ether is the working medium,
steel is the material used for the container and 10 has been taken as
the value of $L_{0}/r_{1R}$. This plot was obtained by using equation (5.23) and Fig. 23.

5.5.2 CONCLUSIONS - The following conclusions can be reached on the basis of the analysis of section 5.5.1.

5.5.2.1 Examination of Fig. 25 reveals that for given values of $L_{0}/r_{1R}$ and $S_{yp}$, there is some value of final pressure which maximizes the energy storage capacity per unit volume. As the allowable stress in the container increases, the maximum point on these curves apparently shifts in the direction of higher pressures.

5.5.2.2 Examination of Fig. 25 reveals that for given values of $L_{0}/r_{1R}$ and $p_{f}$, the energy storage capacity per unit volume increases as the allowable stress of the container material increases.

5.5.2.3 The analysis of section 5.5.1 has shown that the maximum energy storage capacity per unit volume for a compressed liquid system is in the vicinity of 90,000 ft lb/ft$^3$. This value is for ether which is contained in a steel container where the allowable stress in the container is 100,000 psi.

5.6 CONSIDERATION OF LOSSES - If a fluid is confined in a closed volume and is then subjected to a temperature change, the internal energy of the liquid changes. A decrease in temperature represents a decrease in internal energy. This loss of energy is probably the most serious of the losses which are encountered in a compressed liquid energy storage system. This section is devoted to the determination of the relation between the temperature change and the internal energy change. Since the ratio of the coefficient of thermal expansion of ether to the coefficient of thermal expansion of steel is greater than 50 to one, tem-
perature has relatively little effect on the container volume. Thus, for the following analysis, the container volume is assumed to be constant. In general

\[ dV = \frac{\partial V}{\partial p} \, dp + \frac{\partial V}{\partial T} \, dT. \] (5.24)

Since the container volume is assumed to be constant

\[ dV = 0. \] (5.25)

Combining equations (5.24) and (5.25) gives

\[ dT = - \frac{\partial V}{\partial p} \frac{\partial V}{\partial T} \, dp. \] (5.26)

By using equation (5.2) this can be written as

\[ dT = - \frac{\partial V}{\partial p} \frac{\partial V}{\partial T} \, dp. \] (5.27)

Bridgman \(^{21/}\) gives experimental data for the relation between \(\frac{\partial V}{\partial T} \frac{\partial T}{\partial p}\) and pressure. These have been plotted as Fig. 26. For any pressure the quantity \(\frac{\partial V}{\partial T} \frac{\partial T}{\partial p}\) can be found by measuring the slope of the curve given in Fig. 22. Fig. 27 is a plot of \(\frac{\partial V}{\partial p}(\partial V/\partial T)\) vs. \(p_f\) as found by using Fig. 22. Fig. 28 is a plot of \(\frac{\partial V}{\partial p}(\partial V/\partial T)\) vs. \(p_f\). This plot has been obtained by using the values indicated in Figs. 26 and 27. Now inspection of equation (5.27) reveals that the area under the curve of Fig. 28, between any two pressures, represents the temperature change necessary to cause this change in pressure. Thus, the pressure change which is associated with any given temperature change can be found by trial from Fig. 28. Consider, for example, a volume of ether which is initially at a temperature of 68°F. Assume that the stress in the container is 185,000 psi. The energy-weight-vs.-pressure relation for these conditions is shown in Fig. 24. This plot is repeated as the upper curve of Fig. 29. Now if the temperature drops to -35°F

\(^{21/}\) Bridgman, op. cit., p. 133.
Fig. 10: Relation between the quantity \( \frac{(V_y - V_0)}{V_y - V_{01}} \) and pressure for fiber.
then the associated pressure drop can be found from Fig. 28, as explained above. With the new value of pressure, equations (5.16) and (5.17) can be used to obtain a new value of $E/\dot{W}_T$. This procedure has been carried out for various values of pressure before cooling and the results are shown as the lower curve of Fig. 29. Thus, if ether is contained at an initial pressure of 3000 atmospheres and an initial temperature of $68^\circ F$, the energy per unit weight is shown by Fig. 29 to be 790 ft lb/lb. If the temperature drops to $-35^\circ F$ the energy per unit weight becomes 480 ft lb/lb.

A similar procedure can be carried out for the energy-volume relation. This has been done and the results are shown in Fig. 30. These plots have been obtained by assuming that the temperature-pressure relation represented by the top curve is such as to produce a stress of 185,000 psi in the container. $L_0/T_1$ has been taken as 10 for the purpose of these plots.

Other losses which will be encountered in a compressed-liquid-energy-storage system are the losses due to leakage and the losses due to friction in the piping and valves.

5.6.1 CONCLUSION - Examination of Figs. 29 and 30 reveals that the energy loss per unit volume and per unit weight decreases as the pressure $p_t$ increases. This is shown by the fact that the curves converge as the pressure increases.

5.7 SUMMARY - In this chapter the suitability of using a liquid as an energy storage medium has been investigated. This section is a summary of the most significant equations and conclusions which have arisen from the analyses of this chapter. Graphs which are pictorial representations of the analyses have been included in previous sections.
5.7.1 SUMMARY OF EQUATIONS

\[
E_{st}/\bar{W}_t = (E_{st}/\bar{V}_t) \left\{ \frac{L_c/r_{1r} + 2/3}{\left[ (L_c/r_{1r} + 2/3) + \left( \rho_c/\rho_t \right) \left( L_c(1-k^2)/k^2 r_{1r} + 2(1-k^3)/3k^3 \right) \right]} \right. \\
E_{st}/\bar{V}_t = (E_{st}/\bar{V}_t) \left\{ \frac{L_c/r_{1r} + 2/3}{\left[ (L_c/r_{1r} + 2/3) + \left( L_c(1-k^2)/k^2 r_{1r} + 2(1-k^3)/3k^3 \right) \right]} \right. \\
\left. k^2 = 1 - 2 \frac{p_f}{S_y} \right\}
\]

5.7.2 SUMMARY OF CONCLUSIONS - The following conclusions can be arrived at on the basis of the analyses of this chapter. These conclusions are based on the assumption that ether is used as the working medium.

5.7.2.1 The energy storage capacity per unit weight and the energy storage capacity per unit volume (not including the container) are approximately linear functions of the storage pressure \( p_f \).

5.7.2.2 When ether is compressed, the energy which is stored in the container is small relative to the energy which is stored in the working medium.

5.7.2.3 There is some value of storage pressure \( p_f \) which maximizes the energy storage capacity per unit weight and the energy storage capacity per unit volume.

5.7.2.4 The energy storage capacity per unit weight and the energy storage capacity per unit volume both increase as the allowable stress of the container material increases.

5.7.2.5 The energy loss per unit weight and the energy loss per unit volume both decrease as the storage pressure \( p_f \) increases.

5.7.2.6 The maximum energy storage capacity per unit weight is in the
vicinity of 300 ft lb/lb.

5.7.2.7 The maximum energy storage capacity per unit volume is in the vicinity of 90,000 ft lb/ft$^3$. 
NOTATION FOR CHAPTER VI

BTU British thermal unit

c\text{\textsubscript{V}} specific heat at constant volume

E\text{\textsubscript{a}} energy expended in doing work against the atmosphere

E\text{\textsubscript{g}} energy stored in a compressed gas

E\text{\textsubscript{gc}} energy which can be extracted from a gas which has been cooled to the temperature which it had before compression

E\text{\textsubscript{gi}} energy of isothermal expansion

E\text{\textsubscript{n}} net energy

f(c\text{\textsubscript{V}}) function of c\text{\textsubscript{V}}

f'(c\text{\textsubscript{V}}) first derivative of function of c\text{\textsubscript{V}}

g constant equal to the acceleration of a freely falling body

k pressure function = 1 - P\text{\textsubscript{f}}/P\text{\textsubscript{o}}

lb pound

L\text{\textsubscript{c}} length of cylindrical portion of container

Lim limit

M\text{\textsubscript{g}} mass of gas

M\text{\textsubscript{t}} total mass

P\text{\textsubscript{a}} atmospheric pressure

P\text{\textsubscript{f}} pressure after expansion has occurred

P\text{\textsubscript{o}} pressure before expansion has occurred

P\text{\textsubscript{l}} pressure before compression has taken place

P\text{\textsubscript{2}} pressure after compression has taken place

P\text{\textsubscript{3}} pressure after cooling has taken place

P\text{\textsubscript{4}} pressure after cooling and expansion has taken place

r\text{\textsubscript{lc}} inside radius of cylindrical portion of container
$r_{2c}$ outside radius of cylindrical portion of container

$r_{1s}$ inside radius of hemispherical ends

$r_{2s}$ outside radius of hemispherical ends

$R$ universal gas constant

$o_R$ degrees Rankine

$(S_{sm})_c$ maximum shearing stress in cylindrical portion of container

$(S_{sm})_s$ maximum shearing stress in spherical portion of container

$S_{yp}$ yield point stress for material of container

$T_c$ temperature after cooling has taken place

$T_{ce}$ temperature after cooling and expansion has taken place

$T_f$ temperature after compression has taken place

$T_o$ temperature before compression has taken place

$T_1$ temperature of gas before expansion has occurred

$T_2$ temperature of gas after expansion has occurred

$V_c$ volume of space occupied by container

$V_g$ volume of gas

$V_i$ initial volume

$V_f$ final volume

$W_c$ weight of container

$W_g$ weight of gas

$W_t$ total weight

$\rho_c$ density of container

$\rho_g$ density of gas

$n$ constant $= 3.1416$

$\gamma$ ratio of specific heats
CHAPTER VI

ENERGY STORAGE CAPACITY OF A COMPRESSED GAS

6.1 INTRODUCTION - As was pointed out in section 1.1, the energy storage capacity of any system is a function of the $Q$ and $\Delta U$ which are associated with a given process. In this chapter, two different processes will be considered. One will be an isentropic process while the other will be an isothermal process. The energy storage capacity per unit weight and the energy storage capacity per unit volume have been evaluated for systems involving various container shapes. A consideration of the losses which are associated with a compressed gas energy storage system is also included in this chapter.

6.2 ENERGY STORAGE CAPACITY AS A FUNCTION OF THE WEIGHT OF THE GAS ONLY

6.2.1 ASSUMPTIONS - In order to complete the analysis of section 6.2.2, the following assumptions have been made.

6.2.1.1 The energy storage capacity of a compressed gas is evaluated by assuming that an isentropic process is used to extract work from the system.

6.2.1.2 It is assumed that the gas which is used as the energy storage medium obeys the perfect gas law.

6.2.2 ANALYSIS - Using assumption 6.2.1.1, the energy storage capacity of a compressed gas is equal to the change in internal energy which occurs when the gas expands. Thus

$$E_g = W_g c_v (T_1 - T_2). \quad (6.1)$$

This can be expressed as a function of the initial and final pressures
by using assumptions 6.2.1.1 and 6.2.1.2. Equation (6.1) then becomes

$$E_g = w_g c_V T_1 \left[ 1 - \frac{(p_f/p_0)^{(\gamma - 1)/\gamma}} \right]. \quad (6.2)$$

Dividing both sides of equation (6.2) by $w_g$ gives the energy per unit weight as

$$\frac{E_g}{w_g} = c_V T_1 \left[ 1 - \frac{(p_f/p_0)^{(\gamma - 1)/\gamma}} \right]. \quad (6.3)$$

Fig. 31 is a plot of $E_g/(w_g c_V T_1)$ vs. $(p_f/p_0)$ for various values of $\gamma$.

6.2.3 DETERMINATION OF OPTIMUM VALUE OF $c_V$ - Equation (6.3) can also be written as

$$\frac{E_g}{w_g} = c_V T_1 \left[ 1 - \frac{(p_f/p_0)^{R/(R+c_V)}} \right]. \quad (6.4)$$

Inspection of this equation reveals that there may be some value of $c_V$ which makes $E_g/w_g$ a maximum. If this is true it is desirable to determine what this value is, since $c_V$ can vary from $34.25$ to $584$ ft lb/lb of medium depending on the gas used. Thus, an optimum $c_V$ would indicate what medium to use in order to obtain maximum energy storage capacity per unit weight. In order to investigate the existence of an optimum $c_V$, equation (6.4) is written as

$$\frac{E_g}{w_g} = c_V T_1 \left[ 1 - (1 - k) \frac{R}{R+c_V} \right], \quad (6.5)$$

where

$$k = 1 - \frac{p_f}{p_0}. \quad (6.6)$$

Expanding equation (6.5) by means of the binomial theorem gives

$$\frac{E_g}{w_g} = T_1 \left[ \frac{R}{1 + R/c_V} \right] k + \left[ \frac{R}{1 + R/c_V} \right] \left[ \frac{1}{1 + R/c_V} \right] k^2/2!$$

$$+ \left[ \frac{R}{1 + R/c_V} \right] \left[ \frac{1}{1 + R/c_V} \right] \left[ \cdots \right]$$

$$\left[ \frac{(n-1)c_V + (n-2)R}{c_V + R} \right] k^n/n! + \cdots \right] \quad (6.7)$$

All of the terms in equation (6.7) obviously increase as $c_V$ increases.
with the exception of the term

\[ f(c_V) = \left[ (n - 1)c_V + (n - 2)R \right]/(c_V + R). \]  \hspace{1cm} (6.8)

How this term varies when \( c_V \) increases needs to be investigated. The investigation can be carried out by examining the derivation of \( f(c_V) \).

Taking the derivative of both sides of equation (6.8) and simplifying gives

\[ f'(c_V) = R/(c_V + R)^2. \]  \hspace{1cm} (6.9)

This quantity, which is the slope of the \( f(c_V) \) vs. \( c_V \) curve, is always positive. Since the slope of this curve is always positive, \( f(c_V) \) is a monotonically increasing function with \( c_V \). Hence all terms in equation (6.7) increase as \( c_V \) increases. Thus, the optimum value of \( c_V \) is the maximum possible value of \( c_V \).

6.2.4 CONCLUSIONS - The following conclusions can be arrived at on the basis of the analyses of sections 6.2.2 and 6.2.3.

6.2.4.1 Inspection of equation (6.3) reveals that the energy which can be stored in a gas per unit weight of gas is independent of the pressure level at which it is stored. It is rather a function of the pressure ratio through which the gas expands.

6.2.4.2 The analysis of section 6.2.3 led to the conclusion that in order to obtain the maximum energy storage capacity per unit weight, the gas with the maximum value of \( c_V \) should be used.

6.2.4.3 Inspection of equation (6.3) reveals that the energy storage capacity per unit weight is a linear function of the temperature \( T_1 \).

6.2.4.4 Inspection of Fig. 31 reveals that for given values of \( T_1 \), \( c_V \), and \( \gamma \), the energy storage capacity per unit weight increases as the pressure ratio \( p_f/p_o \) decreases.
6.3 ENERGY STORAGE CAPACITY AS A FUNCTION OF THE VOLUME OF THE GAS ONLY

6.3.1 ANALYSIS - The energy storage capacity as a function of the volume of the gas is obtained simply from equation (6.3) by using the perfect gas law. Thus

\[ \frac{E_g}{V_g} = \left[ \frac{P_0 c_p}{R} \right] \left[ 1 - \left( \frac{P_f}{P_0} \right)^{(y-1)/y} \right]. \] (6.10)

6.3.2 CONCLUSIONS - The following conclusions can be reached from the analysis of section 6.3.1.

6.3.2.1 Inspection of equation (6.10) reveals that the energy storage capacity per unit volume is independent of the temperature at which the gas is stored.

6.3.2.2 For a given value of \( P_0 \) and for any given working medium, the energy storage capacity per unit volume increases as the pressure ratio \( P_f/P_0 \) decreases.

6.4 ENERGY STORAGE CAPACITY AS A FUNCTION OF THE WEIGHT OF THE GAS AND CONTAINER - For high pressure energy storage, the weight of the container becomes a very important part of the total weight of the energy storage device. This section is devoted to an energy-weight relation analysis which includes the weight of the container.

6.4.1 ASSUMPTIONS - In order to complete the analysis of section 6.4.2, the following assumptions have been made.

6.4.1.1 It is assumed that the maximum-shearing-stress theory of failure controls the maximum permissible pressure in the container.

6.4.1.2 In order to calculate the mass of a cylindrical container with hemispherical ends and the mass of the gas contained, it has been assumed that the junction between the hemispherical end and the cylindrical portion of the container is an abrupt change from the inside radius.
of the cylinder to the inside radius of the hemisphere.

6.4.2 ANALYSIS FOR CASE OF CYLINDRICAL CONTAINER WITH HEMISPHERICAL ENDS-

Using assumption 6.4.1.2 the weight of a cylindrical container with hemi-
spherical ends can be expressed by

\[ w_c = \rho_c g \left[ \frac{\ln(r_{2s}^3 - r_{ls}^3)}{3} + \pi(r_{2c}^2 - r_{1c}^2) L_c \right]. \]  

(6.11)

Using assumption 6.4.1.2, the weight of the contained gas is given by

\[ w_g = (p_o/RT_1)\left(\ln r_{ls}^3/3 + \pi r_{1c}^2 L_c \right). \]  

(6.12)

Combining equations (6.11) and (6.12) gives the total weight as

\[ w_t = g \rho_c \left[ \frac{\ln(r_{2s}^3 - r_{ls}^3)}{3} + \pi(r_{2c}^2 - r_{1c}^2) L_c \right] 
+ (p_o/RT_1)\left(\ln r_{ls}^3/3 + \pi r_{1c}^2 L_c \right). \]  

(6.13)

Combining equations (6.2), (6.12) and (6.13) and rearranging gives the
energy storage capacity per unit total weight as

\[ \frac{E_w}{w_t} = \left(\frac{c_v p_o}{R} \right) \left(\frac{r_{ls}^3}{3} + r_{1c}^2 L_c \right) \left[1 - \left(\frac{p_f}{p_o}\right)^{(\gamma-1)/\gamma}\right] \Bigg/ \left(p_o/RT_1\right) 
+ \rho_c \frac{\ln(r_{2s}^3 - r_{ls}^3)/3 + \rho_c L_c(r_{2c}^2 - r_{1c}^2)}{\gamma} \left(6.14\right) \]

This is the general expression for the energy storage capacity per unit
weight of container and gas combined for the case of a cylindrical con-
tainer with hemispherical ends.

6.4.3 ANALYSIS FOR CASE OF SPHERICAL CONTAINER - The energy-weight re-
lation for the case of energy storage in a spherical container can be
found from equation (6.14) simply by letting \( L_c = 0 \). This gives

\[ \frac{E_w}{w_t} = \left(\frac{c_v p_o}{R} \right) \left[1 - \left(\frac{p_f}{p_o}\right)^{(\gamma-1)/\gamma}\right] \Bigg/ \left(p_o/RT_1\right) 
+ \rho_c \left[ (r_{2s}/r_{ls})^3 - 1 \right]. \]  

(6.15)
6.4.4 ANALYSIS FOR CASE OF CYLINDRICAL CONTAINER WITH HEMISPHERICAL ENDS WHERE THE THICKNESS OF THE ENDS IS THE SAME AS THE THICKNESS OF THE CYLINDRICAL PORTION OF THE CONTAINER — The energy-weight relation for the case of a cylindrical container with hemispherical ends where the end thickness is equal to the cylinder thickness is found from equation (6.14) by letting \( r_{1s} = r_{1c} \) and \( r_{2s} = r_{2c} \). This gives

\[
\frac{E_g}{W_t}_{cs} = \left( c_v \rho c/R \right) \left[ \frac{4/3 + L_c/r_{1c}}{1 - \left( p_f/p_o \right)^{(\gamma - 1)/\gamma}} \right] /
\]

\[
\left\{ \left( p_o/RT_1 \right) \left( \frac{4}{3} + L_c/r_{1c} \right) + \left( \varepsilon \rho_c L_c/r_{1c} \right) \left[ \left( r_{2c}/r_{1c} \right)^2 - 1 \right] \right\}.
\] (6.16)

6.4.5 ANALYSIS FOR CASE OF CYLINDRICAL CONTAINER WITH FLAT ENDS — The energy-weight relation for the case of a cylindrical shaped container with flat ends where the weight of the ends is neglected is found from equation (6.14) by letting \( r_{1s} = r_{2s} = 0 \). Making these substitutions in equation (6.14) and rearranging gives the energy-weight relation as

\[
\frac{E_g}{W_t}_c = \left( p_o c_v/R \right) \left[ 1 - \left( p_f/p_o \right)^{(\gamma - 1)/\gamma} \right] /
\]

\[
\left( p_o/RT_1 + \varepsilon \rho_c \left[ \left( r_{2c}/r_{1c} \right)^2 - 1 \right] \right).
\] (6.17)

For any actual design it would be necessary to investigate the mass of the ends to see if it can be neglected.

6.4.6 ENERGY-WEIGHT RELATION AS A FUNCTION OF MAXIMUM ALLOWABLE STRESS FOR CASE OF CYLINDRICAL CONTAINER WITH HEMISPHERICAL ENDS WHERE THE MAXIMUM STRESS IN THE ENDS IS THE SAME AS THE MAXIMUM STRESS IN THE CYLINDRICAL PORTION OF THE CONTAINER — Section 6.4.2 was devoted to an analysis of the energy storage capacity as a function of radius ratios for the case of a cylindrical container with hemispherical ends. The radius ratios can be considered as functions of the initial pres-
sure and the maximum stress which is allowed in the container. This is true since for any given initial pressure, the radius ratios must be such as to keep the stress in the container from exceeding the maximum allowable stress. This section is devoted to an analysis of the energy-weight relation as a function of the initial pressure and the maximum allowable stress rather than as a function of the radius ratios.

The maximum shearing stress in the hemispherical ends of the container is given by

\[(S_{sm})_s = 3 p_o r_2^3 / (r_2^3 - r_1^3)\]. \hspace{1cm} (6.18)

The maximum shearing stress in the cylindrical portion of the container is given by

\[(S_{sm})_c = p_o / \left[1 - (r_{1c}/r_{2c})^2\right].\] \hspace{1cm} (6.19)

Now consider the case where the thickness of the ends is related to the thickness of the cylindrical portion of the container in such a way that

\[(S_{sm})_s = (S_{sm})_c = S_{yp}/2.\] \hspace{1cm} (6.20)

Solving for \((r_{ls}/r_{2s})\) from equation (6.18) and using equation (6.20) gives

\[r_{ls}/r_{2s} = (1 - 3 p_o / S_{yp})^{1/3}.\] \hspace{1cm} (6.21)

Solving for \((r_{1c}/r_{2c})\) from equation (6.19) and using equation (6.20) gives

\[r_{1c}/r_{2c} = (1 - 2 p_o / S_{yp})^{1/2}.\] \hspace{1cm} (6.22)

For the case where \(r_{2s} = r_{2c}\), combination of equations (6.21) and (6.22) gives

\[r_{ls}/r_{1c} = (1 - 3 p_o / 2 S_{yp})^{1/3} / (1 - 2 p_o / S_{yp})^{1/2}.\] \hspace{1cm} (6.23)

Combination of equations (6.14), (6.21), (6.22) and (6.23) gives the
desired energy-weight relation as a function of the yield point stress of the container and the inside radius of the cylinder. Since the resulting general expression is somewhat complicated, various special cases will be studied in the following sections.

6.4.7 ENERGY-WEIGHT RELATION AS A FUNCTION OF MAXIMUM ALLOWABLE STRESS FOR CASE OF SPHERICAL CONTAINER

6.4.7.1 Analysis - Section 6.4.3 was devoted to an analysis of the energy storage capacity as a function of the radius ratio for the case of a spherical container. In order to obtain this relation as a function of the maximum allowable stress in the container it is necessary to combine equations (6.15) and (6.21). Such combination gives

\[
\frac{E_g}{W_t} = \left( c_v \frac{p_o}{R} \right) \left[ 1 - \left( \frac{p_f}{p_o} \right)^{(\gamma - 1)}/\gamma \right] \left\{ p_o/RT_1 \right. \\
+ p_c s \left( 3 p_o/2 s_y \right)/(1 - 3 p_o/2 s_y) \right\} .
\]

Fig. 32 shows a plot of \( \frac{E_g}{W_t} \) vs. \( p_o \) for various pressure ratios where air has been used as the working medium. The volume ratios which correspond with the various pressure ratios are also shown on this plot.

The container has been assumed to be of steel which has a yield point stress of 100,000 psi.

6.4.7.2 Conclusions - The following conclusions can be arrived at on the basis of the analysis of section 6.4.7.1.

6.4.7.2.1 Inspection of Fig. 32 reveals that when the final pressure is atmospheric, there is some value of initial pressure which maximizes the energy storage capacity per unit weight. This is true for any given combination of working medium and allowable container stress.

6.4.7.2.2 Inspection of Fig. 32 reveals that for a given volume ratio, the energy storage capacity per unit weight decreases as the initial
pressure increases.

6.1.7.2.3 Examination of equation (6.24) reveals that for some given working medium and for given values of initial pressure, initial temperature, and pressure ratio, the energy storage capacity per unit weight increases as the allowable stress in the container increases.

6.1.7.2.4 The analysis of section 6.1.7.1 has shown that the maximum energy storage capacity per unit volume for a spherical-container-compressed-gas system is in the vicinity of 22,600 ft lb/lb. This is the value for a steel container with an allowable stress of 100,000 psi where air is used as the working medium.

6.1.8 ENERGY-WEIGHT RELATION AS A FUNCTION OF MAXIMUM ALLOWABLE STRESS FOR CASE OF CYLINDRICAL CONTAINER WITH HEMISPHERICAL ENDS WHERE THE THICKNESS OF THE ENDS IS EQUAL TO THE THICKNESS OF THE CYLINDRICAL PORTION OF THE CONTAINER — The maximum shearing stress in the type of container which is being considered in this section exists in the cylindrical portion of the container. Thus, the value of the maximum shearing stress for this case then is defined by equation (6.19). The ratio \( r_{1c}/r_{2c} \) is therefore defined by equation (6.22). Since the end thickness is the same as the thickness of the cylindrical portion of the container,

\[
\frac{r_{1c}}{r_{2c}} = \frac{r_{1s}}{r_{2s}} \quad (6.25)
\]

and

\[
\frac{r_{2s}}{r_{2c}} = r_{2c} \quad (6.26)
\]

Combining equations (6.16), (6.22), (6.25) and (6.26) gives the desired energy-weight relation for this particular case.
6.4.9 ENERGY-WEIGHT RELATION AS A FUNCTION OF MAXIMUM ALLOWABLE STRESS
FOR CASE OF CYLINDRICAL CONTAINER WITH FLAT ENDS

6.4.9.1 Analysis - As in the above section, the radius ratio is related
to the maximum allowable stress by equation (6.22). Combination of
equations (6.17) and (6.22) gives

$$\frac{E_g}{W_t} = \left( \frac{p_o}{R} \right) \left[ 1 - \left( \frac{p_f}{p_o} \right)^{\gamma-1} \right] / \left[ \left( \frac{p_o}{R T_1} \right) \right]
+ \gamma p_c \left[ \frac{(2 p_o / S_{yp})}{(1 - 2 p_o / S_{yp})} \right].$$  (6.27)

Fig. 32 shows a plot of $E_g/W_t$ vs. $p_0$ for various pressure ratios where
air has been used as the working medium. The container has been as-
sumed to be of steel which has a yield point stress of 100,000 psi.
Since equation (6.17) has been used in this analysis, equation (6.27)
neglects the weight of the ends of the cylindrical container. For any
actual design it would be necessary to investigate the weight of the
ends to see if it can be neglected.

6.4.9.2 Conclusions - The following conclusions can be reached on
the basis of the analysis of section 6.4.9.1.

6.4.9.2.1 The conclusions of sections 6.4.7.2.1 and 6.4.7.2.2 apply for
the case of a cylindrical container as well as for the case of a
spherical container.

6.4.9.2.2 Inspection of Fig. 32 reveals that the spherical shaped
container utilizes material more effectively than does the cylindrical
shaped container. This is illustrated by the fact that the curves for
the sphere lie above the curves for the cylinder.

6.4.9.2.3 Examination of equation (6.27) reveals that for any given
working medium and for given values of initial pressure, initial tem-
perature, and pressure ratio, the energy storage capacity per unit
weight increases as the allowable stress in the container increases.

6.1.9.2.4 The analysis of section 6.1.9.1 has shown that the maximum energy storage capacity per unit weight for a cylindrical-container-compressed-gas system is in the vicinity of 18,400 ft lb/lb. This is the value for a steel container with an allowable stress of 100,000 psi where air is the working medium.

6.5 ENERGY STORAGE CAPACITY AS A FUNCTION OF THE VOLUME OF SPACE OCCUPIED BY THE CONTAINER

6.5.1 ANALYSIS FOR CASE OF CYLINDRICAL CONTAINER WITH HEMISPHERICAL ENDS - The volume of space occupied by a cylindrical container with hemispherical ends is given by

\[ V_c = \frac{4}{3} \pi r_{ls}^3 + \pi r_{2c} L_c \]  \hspace{1cm} (6.28)

Combination of equations (6.2), (6.12) and (6.28) gives

\[ \frac{E_g}{V_c} = \left( \frac{c}{R} \right) \frac{1}{2} \left[ \frac{1}{3} r_{ls} L_c + \frac{1}{r_{1c}} \right] \left[ 1 - \left( \frac{p_f}{p_o} \right)^{\frac{1}{k}} \right] \left[ \left( \frac{r_{2s}}{r_{1c}} \right)^3 + \left( \frac{r_{ls}}{r_{1c}} \right)^2 \left( \frac{L_c}{r_{1c}} \right) \right] . \]  \hspace{1cm} (6.29)

This is the energy-volume relation for the case of a cylindrical container with hemispherical ends.

6.5.2 ANALYSIS FOR CASE OF SPHERICAL CONTAINER - The energy-volume relation for the case of a spherical container can be found from equation (6.29) by letting \( L_c = 0 \). This gives

\[ \frac{E_g}{V_c} = \left( \frac{c}{R} \right) \frac{1}{2} \left[ \frac{1}{3} r_{ls} L_c + \frac{1}{r_{1c}} \right] \left( \frac{r_{ls}}{r_{2s}} \right)^3 . \]  \hspace{1cm} (6.30)

6.5.3 ANALYSIS FOR CASE OF CYLINDRICAL CONTAINER WITH A HEMISPHERICAL ENDS WHICH HAVE THE SAME THICKNESS AS THE CYLINDER - The energy-volume relation for a cylindrical container with hemispherical ends where the ends have the same thickness as does the cylindrical portion of the
container can be found from equation (6.29) by letting \( r_{1c} = r_{ls} \) and \( r_{2s} = r_{2c} \). This gives

\[
E_g/V_c = \left( p_o c_v / R \right) \left[ \frac{1}{3} - \frac{C}{L_c} \right] \left( 1 - \left( \frac{p_f}{p_o} \right)^{\gamma-1} / \gamma \right) \left( \frac{r_{lc}}{r_{2c}} \right)^2
\]

(6.31)

6.5.4 ANALYSIS FOR CASE OF CYLINDRICAL CONTAINER WITH FLAT ENDS - The energy-volume relation for the case of a cylindrical container with flat ends can be found from equation (6.29) by letting \( r_{2s} = r_{ls} = 0 \). This gives

\[
E_g/V_c = \left( p_o c_v / R \right) \left[ \frac{1}{3} - \frac{C}{L_c} \right] \left( 1 - \left( \frac{p_f}{p_o} \right)^{\gamma-1} / \gamma \right) \left( \frac{r_{lc}}{r_{2c}} \right)^2.
\]

(6.32)

6.5.5 ENERGY-VOLUME RELATION AS A FUNCTION OF MAXIMUM ALLOWABLE STRESS FOR THE CASE OF A SPHERICAL CONTAINER - The energy-volume relation as a function of the maximum allowable stress for the case of a spherical container can be found by combining equations (6.21) and (6.30). This gives

\[
E_g/V_c = \left( p_o c_v / R \right) \left[ \frac{1}{3} - \frac{C}{L_c} \right] \left( 1 - 3 \frac{p_o}{\gamma p_f} \right) \left[ 1 - 3 \frac{p_o}{\gamma p_f} \right]
\]

(6.33)

Fig. 33 shows a plot of \( E_g/V_c \) vs. \( p_o \) for various pressure ratios. Air has been used as the working medium and the allowable stress in the container has been taken as 100,000 psi.

6.5.5.1 Conclusions - The following conclusions can be arrived at on the basis of the analysis of section 6.5.5.

6.5.5.1.1 Inspection of Fig. 33 reveals that for a given working medium and for given values of expansion ratio and maximum allowable stress in the container, there is some value of initial pressure which maximizes the energy storage capacity per unit volume.

6.5.5.1.2 Examination of equation (6.33) reveals that for a given
FIG. 33 ENERGY-VOLUME RELATION FOR COMPRESSED AIR EXPANSION
working medium and for given values of initial pressure and pressure ratio, the energy storage capacity per unit volume increases as the allowable stress in the container increases.

6.5.1.3 The analysis of section 6.5.5 has shown that the maximum energy storage capacity per unit volume for a spherical-container-compressed-gas system is in the vicinity of 5,200,000 ft lb/ft$^3$. This is the value for a steel container with an allowable stress of 100,000 psi where air is the working medium.

6.5.6 ENERGY-VOLUME RELATION AS A FUNCTION OF MAXIMUM ALLOWABLE STRESS FOR THE CASE OF A CYLINDRICAL CONTAINER WITH HEMISPHERICAL ENDS WHICH HAVE THE SAME THICKNESS AS THE CYLINDER - The desired energy-volume relation can be found by combining equations (6.22) and (6.31).

6.5.7 ENERGY-VOLUME RELATION AS A FUNCTION OF MAXIMUM ALLOWABLE STRESS FOR THE CASE OF A CYLINDRICAL CONTAINER WITH FLAT ENDS

6.5.7.1 Analysis - The desired energy-volume relation is obtained by combining equations (6.22) and (6.32). This gives

$$\frac{E_g}{V_c} = \left(\frac{p_0 c_v}{R}\right) \left[1 - \left(\frac{p_f}{p_0}\right)^{(\gamma - 1)/\gamma}\right] \left[1 - 2 \frac{p_0}{S_y}ight]. \quad (6.34)$$

Fig. 33 shows a plot of $E_g/V_g$ vs. $p_0$ for various pressure ratios. Air is used as the working medium and the allowable stress in the container has been taken as 100,000 psi.

A similar energy-volume relation can be derived for the case of a cylindrical container with hemispherical ends where the end thickness is related to the cylinder thickness in such a way that the maximum shearing stress in each section is the same.

6.5.7.2 Conclusions - The following conclusions can be arrived at on the basis of the analysis of section 6.5.7.1.
6.5.7.2.1 Inspection of Fig. 33 reveals that for a given working medium and for given values of expansion ratio and maximum allowable stress in the container, there is some value of initial pressure which maximizes the energy storage capacity per unit volume.

6.5.7.2.2 Inspection of Fig. 33 reveals that cylindrical/spherical containers have similar characteristics except at high pressures. In the region of high pressure, the spherical container is superior to the cylindrical container.

6.5.7.2.3 Examination of equation (6.34) reveals that for a given working medium and for given values of initial pressure and pressure ratio, the energy storage capacity per unit volume increases as the allowable stress in the container increases.

6.5.7.2.4 The analysis of section 6.5.7.1 has shown that the maximum energy storage capacity per unit volume for a cylindrical-container-compressed-gas system is in the vicinity of 1,000,000 psi. This is the value for a steel container with an allowable stress of 100,000 psi where air is used as the working medium.

6.6 RATIO OF ISOThERMAL WORK TO ISENTROPIC WORK - The previous sections were devoted to an evaluation of energy-weight relations and the energy-volume relations on the basis of the assumption that the energy is extracted from the gas by expanding it isentropically. This section is devoted to an analysis of the ratio of the energy which can be extracted by an isothermal expansion to the energy which has been evaluated in the previous sections.

6.6.1 ANALYSIS - The energy which can be extracted from a mass of gas which is expanded isothermally is given by
\[ E_{gi} = W_g T_1 R \ln\left(\frac{p_o}{p_f}\right). \quad (6.35) \]

Combination of equations (6.2) and (6.35) gives

\[ \frac{E_{gi}}{E_g} = \left( \frac{R \ln\left(\frac{p_o}{p_f}\right)}{c_v} \right) \left[ 1 - \left(\frac{p_f}{p_o}\right)^{\gamma-1} \right]. \quad (6.36) \]

Since

\[ \frac{R}{c_v} = \gamma - 1, \quad (6.37) \]

equation (6.36) can be written as

\[ \frac{E_{gi}}{E_g} = \left[ (\gamma-1) \ln\left(\frac{p_o}{p_f}\right) \right] \left[ 1 - \left(\frac{p_f}{p_o}\right)^{\gamma-1} \right]. \quad (6.38) \]

Fig. 3h is a plot of this ratio vs. \( \frac{p_o}{p_f} \) where air has been used as the working medium. From this plot it appears probable that \( \frac{E_{gi}}{E_g} \) increases monotonically with \( \frac{p_o}{p_f} \) and that its least value is \( \sqrt{\gamma} \). That the ratio has a least value of \( \sqrt{\gamma} \) can be shown as follows. \( \frac{E_{gi}}{E_g} \) approaches \( \sqrt{\gamma} \) as \( \frac{p_o}{p_f} \) decreases if

\[ \lim_{\Delta p \to 0} \left[ (\gamma-1) \ln\left(\frac{p_o-(p_o-\Delta p)}{p_o}\right) \right] \left[ 1 - \left(\frac{p_o-\Delta p}{p_o}\right)^{\gamma-1} \right] = \sqrt{\gamma} \quad (6.39) \]

This can be shown to be true by using L'Hospital's rule for evaluating the limit of an indeterminate form of the type

\[ \lim_{x \to c} \left[ \frac{f(x)}{g(x)} \right] \]

where

\[ f(c) = g(c) = 0. \quad (6.40) \]

6.6.2 CONCLUSIONS - The following conclusions can be arrived at on the basis of the analysis of section 6.6.1.

6.6.2.1 Inspection of Fig. 3h reveals that more work can be extracted from a gas by expending it isothermally than can be extracted by expanding it isentropically. This is true since if the gas is initially at ambient temperature, then when the gas expands isothermally, energy will
be extracted from the surroundings. If the initial temperature of the
gas is other than ambient temperature, then a heat source must be sup-
plied in order to maintain the isothermal condition.

6.6.2.2 It has been determined that the ratio $E_g / E_s$ increases mono-
tonically with $p_0 / p_f$ and has as its least value a value of $\gamma$.

6.7 CONSIDERATION OF WORK DONE AGAINST THE ATMOSPHERE - In the previ-
ous sections it was assumed that the energy which can be extracted from
a compressed gas is equal to the isothermal or isentropic work of ex-
pansion. If the expansion takes place in a piston-cylinder device, work
must be done on the atmosphere as the gas expands. If compression and
expansion take place successively, then the atmosphere will do an equal
amount of work on the piston when the gas is compressed. However, it
is felt that an analysis of the magnitude of the net energy which can
be extracted when doing work against the atmosphere is of value. Such
an analysis is carried out in this section.

6.7.1 ANALYSIS - The work done against the atmosphere when gas expands
in a piston-cylinder device is given by

$$E_a = p_a (V_2 - V_1)$$  \hspace{1cm} (6.41)

or

$$E_a = p_a V_1 (V_2 / V_1 - 1).$$  \hspace{1cm} (6.42)

Assuming that expansion takes place isentropically, equation (6.42) can
be written as

$$E_a = p_a V_1 [(p_0 / p_f)^{1/\gamma} - 1].$$  \hspace{1cm} (6.43)

The expression for the energy given up by the gas is given by equation
(6.2). This can also be written as

$$E_g = p_0 c_v V_1 [1 - (p_f / p_0)^{(\gamma - 1)/\gamma}] / R.$$  \hspace{1cm} (6.44)
The net energy extracted from the gas by using a piston-cylinder device is given by

\[ E_n = E_g - E_a \]  \hspace{1cm} (6.45)

or

\[ E_n = E_g (1 - \frac{E_a}{E_g}) \]  \hspace{1cm} (6.46)

Combination of equations (6.43), (6.44) and (6.46) gives

\[ E_n = E_g \left[ 1 - p_a R \left( \frac{1}{\gamma} - 1 \right) \right] \left[ \frac{1 - (P_f/P_o)^{1/\gamma}}{P_o} \right] c_v \left[ 1 - \left( \frac{P_f}{P_o} \right)^{\gamma/\gamma} \right] \]  \hspace{1cm} (6.47)

The second term in the braces can be considered as a correction factor which must be applied to the results of sections 6.2 through 6.5 in order to take into account the energy which is expended in doing work against the atmosphere. Fig. 35 is a plot of this correction factor vs. the initial pressure for various final pressures. For this it has been assumed that air is the working medium. For any final pressure, these plots indicate that the correction factor increases as the initial pressure decreases. However, the initial pressure must always be greater than or equal to the final pressure. It is possible, therefore, to draw the limiting curve shown on Fig. 35. This curve represents the case of zero pressure drop during expansion or the case of an infinitesimal expansion.

6.7.2 CONCLUSION - If the initial pressure of a gas is much higher than the final pressure, then when energy is extracted from the gas by expanding it, the energy expended in doing work against the atmosphere can be neglected. Otherwise, the magnitude of the deviation between the energy which is indicated as available in sections 6.2 through 6.5 and the available energy when work is done against the atmosphere can be found by using the plot of Fig. 35 and equation (6.46).
6.8 CONSIDERATION OF LOSSES - The suitability of using compressed gas as a means of storing energy is not entirely determined by its energy-weight or its energy-volume relations. One of the important factors to consider is the inherent losses involved in any device using a compressed gas for energy storage. These losses are discussed in this section.

6.8.1 CONSIDERATION OF HEAT LOSSES - When energy is stored in a compressed gas for long periods of time, the energy which can be extracted at any time will be a function of the ambient temperature. In this section the ratio $\frac{E_g}{E_{gc}}$ will be evaluated. The symbol $E_g$ represents the energy which can be extracted by expanding a gas immediately after compression, while $E_{gc}$ represents the energy which can be extracted by expanding a gas after it has been compressed and then cooled to ambient temperature.

6.8.1.1 Assumptions - The process of cooling of the contained gas is considered to be a constant volume process. There is a slight deviation from the constant volume process due to the elasticity of the container.

6.8.1.2 Analysis - The energy extracted when a gas is expanded immediately after compression can be considered the same as the energy put into the gas in compressing it. Thus, the energy expression can be written as

$$E_g = W_g c_v T_0 \left[ \left( \frac{p_2}{p_1} \right)^{\gamma - 1} - 1 \right]. \quad (6.48)$$

The energy which can be extracted from a gas which has been cooled to the temperature it had before compression is given by

$$E_{gc} = W_g c_v T_c \left[ 1 - \frac{T_{ce}}{T_o} \right]. \quad (6.49)$$
Since the energy is to be extracted by isentropic expansion, this can be written

\[ E_{gc} = W_g c_v T_c \left[ 1 - \left( \frac{p_1}{p_3} \right)^{\left( \frac{\gamma - 1}{\gamma} \right)} \right]. \]  \hspace{1cm} (6.50)

However, since cooling is to take place to ambient temperature

\[ T_c = T_0. \]  \hspace{1cm} (6.51)

Also, since expansion will take place to atmospheric pressure,

\[ P_4 = P_1. \]  \hspace{1cm} (6.52)

Using these conditions in equation (6.50) gives

\[ E_{gc} = W_g c_v T_0 \left[ 1 - \left( \frac{p_1}{p_3} \right)^{\left( \frac{\gamma - 1}{\gamma} \right)} \right]. \]  \hspace{1cm} (6.53)

Using assumption 6.8.1.1,

\[ P_3 = P_2 \frac{T_0}{T_f}. \]  \hspace{1cm} (6.54)

Using equation (6.51) in this gives

\[ P_3 = P_2 \frac{T_0}{T_f}. \]  \hspace{1cm} (6.55)

But since the compression is adiabatic

\[ \frac{T_0}{T_f} = \left( \frac{p_1}{p_2} \right)^{\left( \frac{\gamma - 1}{\gamma} \right)}. \]  \hspace{1cm} (6.56)

Combination of equations (6.53), (6.55) and (6.56) gives

\[ E_{gc} = W_g c_v T_0 \left[ 1 - \left( \frac{p_1}{p_2} \right)^{\left( \frac{\gamma - 1}{\gamma} \right)} \right]. \]  \hspace{1cm} (6.57)

Combination of equations (6.4.8) and (6.57) gives

\[ \frac{E_g}{E_{gc}} = \left[ \left( \frac{p_2}{p_1} \right)^{\left( \frac{\gamma - 1}{\gamma} \right)} - 1 \right] \left[ 1 - \left( \frac{p_1}{p_2} \right)^{\left( \frac{\gamma - 1}{\gamma} \right)} \right] \]. \hspace{1cm} (6.58)

This is the desired ratio. Fig. 36 is a plot of \( E_g/E_{gc} \) vs. \( p_2/p_1 \) using air as a working medium.

6.8.1.3 Conclusions - Inspection of Fig. 36 reveals that the energy loss due to cooling is very significant. The energy which can be extracted from a gas before cooling occurs is approximately 2.5 times

136
FIG. 35: RATIO OF ENERGY STORAGE CAPACITY
BEFORE COOLING TO ENERGY STORAGE CAPACITY
AFTER COOLING VS. PRESSURE RATIO

$\gamma \times L$
the energy which can be extracted after the gas has cooled when the
initial pressure ratio $p_2/p_1$ is 10.

6.9 SUMMARY - In this chapter, the suitability of using a gas as an
energy storage medium has been evaluated. This section is a summary
of the most significant equations and conclusions which have arisen
from the analyses of this chapter. Graphs which are a pictorial repre-
sentations of the analyses have been included in previous sections.

6.9.1 SUMMARY OF EQUATIONS

**Spherical Container**

$$\frac{E_g}{W_t} = \left( c_v \frac{p_0}{R} \right) \left[ 1 - \left( \frac{p_f}{p_0} \right)^{\gamma-1} / \gamma \right] \frac{p_0}{RT_1}$$

$$+ p_c \left[ 3 \frac{p_0}{S_{yp}} / (1 - 3 \frac{p_0}{S_{yp}}) \right]$$

$$\frac{E_g}{V_c} = \left( p_o \frac{c_v}{R} \right) \left[ 1 - \left( \frac{p_f}{p_0} \right)^{\gamma-1} / \gamma \right] \left( \frac{r_{1c}}{r_{2c}} \right)^3$$

**Cylindrical Container (Flat Ends)**

$$\frac{E_g}{W_t} = \left( p_o \frac{c_v}{R} \right) \left[ 1 - \left( \frac{p_f}{p_0} \right)^{\gamma-1} / \gamma \right] \frac{p_0}{RT_1}$$

$$+ \frac{p_c}{\gamma} \left[ (2 \frac{p_0}{S_{yp}}) / (1 - 2 \frac{p_0}{S_{yp}}) \right]$$

$$\frac{E_g}{V_c} = \left( p_o \frac{c_v}{R} \right) \left[ 1 - \left( \frac{p_f}{p_0} \right)^{\gamma-1} / \gamma \right] \left( \frac{r_{1c}}{r_{2c}} \right)^2$$

**Ratio of Isothermal to Isentropic Work**

$$\frac{E_{gi}}{E_g} = \left( \gamma - 1 \right) \ln \left( \frac{p_0}{p_f} \right) / \left[ 1 - \left( \frac{p_f}{p_0} \right)^{\gamma-1} / \gamma \right]$$

**Net Energy When Work Is Done Against Atmosphere**

$$E_n = E_g \left\{ 1 - P_a R \left[ \frac{p_0}{p_f} \right]^{1/\gamma} - 1 \right\} / p_o c_v \left[ 1 - \left( \frac{p_f}{p_0} \right)^{\gamma-1} / \gamma \right]$$

**Losses**

$$\frac{E_g}{E_{gc}} = \left[ \left( \frac{p_2}{p_1} \right)^{\gamma-1} / \gamma - 1 \right] / \left[ 1 - \left( \frac{p_1}{p_2} \right)^{\gamma-1} / \gamma^2 \right]$$

138
6.9.2 SUMMARY OF CONCLUSIONS - The following conclusions can be arrived at on the basis of the analyses of this chapter.

Case 1 (Not Including Container)

6.9.2.1 The energy storage capacity per unit weight is independent of the pressure level at which the gas is stored.
6.9.2.2 To obtain the maximum energy storage capacity per unit weight, the gas with the maximum value of $c_V$ should be used.
6.9.2.3 For any given $p_0$ the energy storage capacity per unit weight and the energy storage capacity per unit volume both increase as the pressure ratio $p_f/p_0$ decreases.
6.9.2.4 The energy storage capacity per unit volume is independent of the temperature at which the gas is stored.

Case 2 (Including Container)

6.9.2.5 When the energy storage capacity is evaluated by calculating the amount of energy which can be extracted by expanding the gas to atmospheric pressure, there is some value of initial pressure which maximizes the energy storage capacity per unit weight.
6.9.2.6 For any given pressure ratio through which the gas is expanded the energy storage capacity per unit weight decreases as the initial pressure increases.
6.9.2.7 The energy storage capacity per unit weight and the energy storage capacity per unit volume both increase as the allowable stress in the container increases.
6.9.2.8 On the basis of the energy per unit weight characteristic, and on the basis of the energy per unit volume characteristic, the spherical shaped container utilizes material more effectively than does the cylindrical shaped container.
6.9.2.9 For any given working medium and for given values of expansion ratio and maximum allowable stress in the container, there is some value of initial pressure which maximizes the energy storage capacity per unit volume.

6.9.2.10 The ratio of isothermal work of isentropic work increases monotonically as the pressure ratio $p_o/p_f$ increases and has as its least value a value of $\sqrt{2}$.

6.9.2.11 The maximum energy storage capacity per unit weight for a spherical-container-compressed-gas system is in the vicinity of 22,600 ft lb/lb. This is the value for a steel container with an allowable stress of 100,000 psi where air is the working medium.

6.9.2.12 The maximum energy storage capacity per unit weight for a cylindrical-container-compressed-gas system is in the vicinity of 18,400 ft lb/lb. This is the value for a steel container with an allowable stress of 100,000 psi where air is the working medium.

6.9.2.13 The maximum energy storage capacity per unit volume for a spherical-container-compressed-gas system is in the vicinity of 5,200,000 ft lb/ft$^3$. This is the value for a steel container with an allowable stress of 100,000 psi where air is the working medium.

6.9.2.14 The maximum energy storage capacity per unit volume for a cylindrical-container-compressed-gas system is in the vicinity of 4,000,000 ft lb/ft$^3$. This is the value for a steel container with an allowable stress of 100,000 psi where air is the working medium.
CHAPTER VII

ENERGY STORAGE CAPACITY OF AN ELECTROCHEMICAL CELL

7.1 INTRODUCTION - As was pointed out in section 1.1, the energy storage capacity of any system is a function of the process which causes the change in internal energy. The process which is considered in this section is one which involves a chemical reaction. The electrochemical cell is the well known energy storage system which converts chemical energy into work. This chapter is a summary of the energy storage capacities of various types of electrochemical cells.

7.2 LEAD-ACID AND EDISON CELLS - Table 3 is a summary of energy storage capacities of Edison nickel-iron alkaline cells.

<table>
<thead>
<tr>
<th>Cell Type</th>
<th>Capacity ft lb/lb x 10^-3</th>
<th>Capacity ft lb/ft^3 x 10^-6</th>
</tr>
</thead>
<tbody>
<tr>
<td>D12</td>
<td>34.0</td>
<td>4.56</td>
</tr>
<tr>
<td>D8</td>
<td>34.8</td>
<td>4.69</td>
</tr>
<tr>
<td>D4</td>
<td>31.0</td>
<td>4.70</td>
</tr>
<tr>
<td>C1</td>
<td>31.8</td>
<td>4.77</td>
</tr>
<tr>
<td>C8</td>
<td>35.0</td>
<td>4.80</td>
</tr>
<tr>
<td>C12</td>
<td>33.6</td>
<td>4.56</td>
</tr>
<tr>
<td>B2</td>
<td>22.1</td>
<td>3.05</td>
</tr>
<tr>
<td>B6</td>
<td>29.8</td>
<td>3.59</td>
</tr>
<tr>
<td>B12H</td>
<td>27.1</td>
<td>3.16</td>
</tr>
<tr>
<td>A1</td>
<td>31.8</td>
<td>4.48</td>
</tr>
<tr>
<td>A8</td>
<td>34.1</td>
<td>4.45</td>
</tr>
<tr>
<td>A16</td>
<td>34.3</td>
<td>4.15</td>
</tr>
</tbody>
</table>

Inspection of this table reveals that the energy storage capacity of an Edison cell falls in the range of 3.0 to 4.8 million ft lb/ft³ or 20 to 35 thousand ft lb/ft. A similar survey indicates that the energy storage capacity of a lead-acid cell falls in the range of 3.5 to 5 million ft lb/ft³ or 25,000 to 38,000 ft lb/ft.

7.3 MISCELLANEOUS CELLS - Considerable work has been done in recent years in an effort to improve the energy storage capacity of electrochemical cells. One of the recently developed types is the Silver-Zinc alkaline cell. The energy storage capacity of this particular cell is indicated by Table 4.

### TABLE 4

<table>
<thead>
<tr>
<th>Cell Type</th>
<th>Energy Storage Capacity ft lb/lb</th>
<th>ft lb/ft³ x 10⁻⁶</th>
</tr>
</thead>
<tbody>
<tr>
<td>HR 100</td>
<td>146,000</td>
<td>14.80</td>
</tr>
<tr>
<td></td>
<td>141,000</td>
<td>14.3</td>
</tr>
<tr>
<td></td>
<td>63,000</td>
<td>6.41</td>
</tr>
<tr>
<td>HR 60</td>
<td>119,000</td>
<td>13.98</td>
</tr>
<tr>
<td></td>
<td>92,500</td>
<td>10.87</td>
</tr>
<tr>
<td></td>
<td>77,700</td>
<td>5.59</td>
</tr>
<tr>
<td>HR 40</td>
<td>111,000</td>
<td>11.06</td>
</tr>
<tr>
<td></td>
<td>77,400</td>
<td>8.48</td>
</tr>
<tr>
<td></td>
<td>77,600</td>
<td>5.94</td>
</tr>
<tr>
<td>HR 20</td>
<td>100,000</td>
<td>8.93</td>
</tr>
<tr>
<td></td>
<td>70,200</td>
<td>6.25</td>
</tr>
<tr>
<td></td>
<td>60,100</td>
<td>5.35</td>
</tr>
</tbody>
</table>

Another of these recently developed cells is the Zinc-Potassium Hydroxide-Silver Peroxide cell. According to Fischbach ², cells


of this type have an energy storage capacity as high as 16,500 ft lb/lb.

The primary losses which are encountered in using an electrochemical cell as an energy storage device are due to the internal resistance of the cell. Lead-acid cells have an efficiency (ratio of watt hour output to watt hour input) of 75 to 80%. Nickel-Iron-Alkaline cells have an efficiency of 55 to 60%.
CHAPTER VIII

MISCELLANEOUS ENERGY STORAGE SYSTEMS

8.1 INTRODUCTION - This chapter is devoted to discussions of various energy storage systems which do not warrant sufficient attention to occupy an entire chapter by themselves.

8.2 ENERGY STORAGE CAPACITY OF A TRANSLATING MEDIUM - If a system involves a translating medium, work can be obtained from the system by changing the velocity of the medium. This process is commonly called a conversion of linear kinetic energy into work. For the ideal process where $Q = 0$, it is possible to write

$$U = \frac{1}{2}Mv^2$$

(8.1)

where $M$ is the mass of the medium and $v$ is its translational velocity. This system is impractical for storing energy for any appreciable period of time because of the necessary space required.

8.3 ENERGY STORAGE CAPACITY OF A RAISED MEDIUM - If the elevation of a medium is increased, the energy storage capacity of the medium is increased. The process of obtaining work from an elevated medium, is the process which is commonly referred to as the conversion of potential energy into work. The energy storage capacity of such a system can be expressed as

$$E = Wh.$$  (8.2)

8.4 ENERGY STORAGE CAPACITY OF AN ELECTROSTATIC CAPACITOR - The final energy storage system which will be considered is one which involves an electrostatic capacitor. Table 5 is a summary of the energy storage capacity of some commercially built electrostatic capacitors.
<table>
<thead>
<tr>
<th>Voltage</th>
<th>Capacitance mfd</th>
<th>Energy Storage Capacity ft lb/ft³</th>
</tr>
</thead>
<tbody>
<tr>
<td>100,000</td>
<td>.2</td>
<td>4.88</td>
</tr>
<tr>
<td>50,000</td>
<td>.5</td>
<td>305</td>
</tr>
<tr>
<td>20,000</td>
<td>1</td>
<td>273</td>
</tr>
<tr>
<td>15,000</td>
<td>1</td>
<td>230</td>
</tr>
<tr>
<td>6,000</td>
<td>1</td>
<td>168</td>
</tr>
<tr>
<td>5,000</td>
<td>1</td>
<td>155</td>
</tr>
<tr>
<td>4,000</td>
<td>4</td>
<td>265</td>
</tr>
<tr>
<td>3,000</td>
<td>2</td>
<td>274</td>
</tr>
<tr>
<td>1,000</td>
<td>10</td>
<td>190</td>
</tr>
</tbody>
</table>

From this table it can be seen that the energy storage capacity of commercially built electrostatic capacitors falls in the range from 150 to 500 ft lb/ft³.
I, Leo Virgil Kline, was born in West Salem, Ohio, February 22, 1929. I received my secondary school education in the centralized schools of Albion, Ohio and Homerville, Ohio. My undergraduate training was obtained at The Kent State University and at The University of Akron. I received the degree Bachelor of Mechanical Engineering from The University of Akron in 1950. I then received an appointment as a Sigma Tau Scholar and continued studies at The Ohio State University. In 1951 I received the degree Master of Science from The Ohio State University. I continued studies at The Ohio State University as the Stillman W. Robinson Fellow for two successive years. I then received an appointment as a Research Associate with The Ohio State University Research Foundation. While holding this position I completed the work for the degree Doctor of Philosophy.