THE ESTIMATION OF PARAMETERS IN THE ORNSTEIN-UHLENBECK PROCESS

Dissertation

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By

Walter Wilson Hoy, B.Sc., M.A.
The Ohio State University

1953

Approved by:

[Signature]
Adviser

OHIO STATE UNIVERSITY
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INTRODUCTION

The study and mathematical theory of stochastic processes is a branch of mathematics that is progressing rapidly, with a great deal of work having been done during the past twenty years. The author is interested in the consideration of stochastic processes from the viewpoint of a mathematical statistician and in particular the theory as presented by Professor H.S. Mann [1]. It is hoped that this paper will aid those using stochastic processes in practical applications as well as provide further results useful in the theory. One specific stochastic process will be of primary interest, namely the Ornstein-Uhlenbeck Process (abbreviated O.U.P.).

The Ornstein-Uhlenbeck Process is determined by certain parameters and a mean value function. Methods for estimating the parameters, of the process, and the parameters involved in the mean value function, will be discussed. Three estimates of the parameters of the mean value function, and their comparison with each other, will be presented. The variance-covariance matrix for each of the three estimates will be determined for two particular examples of the mean value function, namely polynomials and trigonometric polynomials. The matrix for each example is given in as general and complete a form as possible in order to be readily applicable to a specific polynomial.

In our estimation procedures, we shall assume that we have
one sample curve available registering the value of the process over an interval \([0,T]\). Actually it is sufficient to know values of the process for any dense set of this interval. We will compute our estimates and their variances from these values.

The notation and basic definitions used in this paper will be those originally presented in reference [1]. Many of them will be redefined as they occur in this paper, but if not, proper references will be made.
CHAPTER I
BASIC CONCEPTS

Section 1: Stochastic Processes

It will be well to start with the definition of a stochastic process and some of its more basic properties. Following these the definition and properties of an O.U.P. will be given.

Definition 1: Consider a finite or infinite set of symbols \((x, y, ...)\) such that to every finite set of symbols \(x_1, ..., x_n\) there is defined a right continuous distribution function

\[
F_{x_1, x_2, ..., x_n}(a_1, a_2, ..., a_n) = P(x_1 \leq a_1, ..., x_n \leq a_n)
\]

called the probability of the event \(x_1 \leq a_1, ..., x_n \leq a_n\).

The distribution functions of the family satisfy the equations

\[
F_{x_{i_1}, ..., x_{i_n}}(a_{i_1}, ..., a_{i_n}) = F_{x_1, ..., x_n}(a_1, ..., a_n)
\]

where \(i_1, ..., i_n\) is any permutation of the numbers 1, 2, ..., \(n\);

\[
F_{x_1, ..., x_n}(a_1, ..., a_{n-1}, \infty) = F_{x_1, ..., x_n-1}(a_1, ..., a_{n-1})
\]

The symbol \(x_i\) is called a random variable. For every Borel set \(A\) in the \(n\)-dimensional Euclidean space we define the symbol

\[
P \left[ (x_1, ..., x_n) \subset A \right]
\]

called the probability that the "point" \((x_1, ..., x_n)\) lies in \(A\) by the equation

\[
P \left[ (x_1, ..., x_n) \subset A \right] = \int_A dx_{1}, ..., x_n(a_1, ..., a_n)
\]

Definition 2: A set of random variables \(x_t\), where \(t\) is chosen out of some set of real numbers, is called a stochastic process. If
the set of indices $t$ is an interval, then the stochastic process is said to depend on a continuous parameter.

The processes considered in this paper will depend on such a continuous parameter $t$ where $t$ will range over the interval $[0, T]$.

**Definition 3:** A sequence of random variables $\{x_n\}$ converges in probability to zero with increasing $n$, in symbols $\lim_{n \to \infty} x_n = 0$, if for every $\epsilon > 0$ we have $\lim_{n \to \infty} P(|x_n| \geq \epsilon) = 0$. Further $\lim_{n \to \infty} x_n = x$, $x$ a random variable, if $\lim_{n \to \infty} (x_n - x) = 0$.

**Definition 4:** A sequence of random variables $\{x_n\}$ converges in the mean to a random variable $x$, in symbols $\text{l.i.m. } x_n = x$, if $\lim_{n \to \infty} E(x_n - x)^2 = 0$.

**Definition 5:** A process, with continuous parameter, is called continuous in $[a, b]$ if for every sequence $\{h_i\}$, with $\lim_{i \to \infty} h_i = 0$, $\lim_{i \to \infty} x_{t+h_i} = x_t$ for $a \leq t \leq b$.

**Definition 6:** The process $x_t$ is called differentiable at the point $t$ if $\lim_{h \to 0} \frac{x_{t+h} - x_t}{h} = x'_t$ exists. The stochastic process $x'_t$ is called the derivative of $x_t$.

**Definition 7:** Let $x_t$ be a stochastic process defined for $a \leq t \leq b$. We subdivide the interval from $a$ to $b$ into $n$ parts by means of the points $a = t_0, t_1, \ldots, t_n = b$ and put $\max (t_{i+1} - t_i) = \delta$. The number $\delta$ is called the modulus of the subdivision. Within every interval $t_{i-1} \leq t \leq t_i$ we choose a value $t_i^*$ and form the sum $X_n = \sum_{i=1}^{n} x_{t_i^*} (t_i - t_{i-1})$, $X_n$ is a random variable. Now consider a sequence $\{S_n\}$ of subdivisions $S_n$ with moduli $\delta_n$ such that
\[ \lim_{n \to \infty} s_n = 0. \] Let \( X_n \) be the random variable corresponding to the subdivision \( S_n \) and some choice of the \( t^*_i \). If then \( \lim_{n \to \infty} X_n = X \) exists and is equal for all sequences \( \{S_n\} \) with moduli converging to zero and all choices of the \( t^*_i \) then \( X \) is called the integral of \( x_t \) and we write \( \int_a^b x_t \, dt = X \).

We shall say that
\[ x_t = \int_a^t x_r \, dr \] exists l.i.m.
if the Riemann sums \( \sum x_{t_i}^* (t_i - t_{i-1}) \) converge in the mean.

Let \( x_t, y_t \) be two stochastic processes, then to every subdivision \( S = (a = t_0, t_1, \ldots, t_n = b) \) we can form Riemann-Stieltjes sums
\[ X(S) = \sum_{i=1}^n x_{t_i}^* (y_{t_i} - y_{t_{i-1}}) \]

If, for every sequence \( \{S_n\} \) of subdivisions, with modulus converging to zero, \( \lim_{n \to \infty} X(S_n) \) exists and is independent of the particular sequence \( \{S_n\} \) and of the choice of points \( t^*_i, (t_{i-1} \leq t^*_i \leq t_i) \), then we shall write
\[ X = \lim_{n \to \infty} X(S_n) = \int_a^b x_t \, dy_t. \]
We shall call \( X \) the integral of \( x_t \) with respect to \( y_t \). If the random variables \( X(S_n) \) converge in the mean to \( X \) we shall say that \( \int_a^b x_t \, dy_t \) exists l.i.m.

A complete study of these basic concepts and their relations to each other is given in Chapter I of \([1]\).

Section 2: The Ornstein Uhlenbeck Process

As pointed out in the Introduction, the interest of the
author is to work with the Ornstein Uhlenbeck Process. Hence the use of the word process or abbreviation O.U.P. will refer to this particular process.

Consider the stochastic process $x_t$ satisfying the difference equation

$$x_{t+r} = a_r x_t + \epsilon_{t,r}; \quad a_r = e^{-\rho r}, \quad \rho > 0, \quad r \geq 0$$

where $\epsilon_{t,r}$ is normally distributed with mean zero and variance $\sigma^2(1-e^{2\rho})$ and is independent of $x_t$ and $\epsilon_{t',r'}$ if the intervals $(t,t+r)$, $(t',t'+r')$ do not overlap. We shall further assume that $x_t$ is normally distributed with mean zero and that the process is stationary. That is to say, that the joint distribution of $x_{t_1}, \ldots, x_{t_n}$ is, for every $h$, the same as that of $x_{t_1+h}, \ldots, x_{t_n+h}$. Under this assumption $\sigma^2_{x_{t+r}} = \sigma^2_{x_t} = \sigma^2$ and $\sigma^2_{x_t x_{t+r}} = \sigma^2 \epsilon^{-\rho r}$.

The process $x_t$, satisfying the assumptions above, is called an Ornstein Uhlenbeck Process. This process is a suitable model for many physical phenomena, one example being the Brownian Motion. The O.U.P. arises in systems which are acted upon by random disturbances and are subject to a restoring force proportional to the disturbances. For instance, a free particle in Brownian Motion is subject to a restoring force due to the fact that more impacts decelerate than accelerate the motion of a moving particle. Thus the velocity of a free particle, after a lapse of time $r$, whose velocity was $x_t$ at time $t$ will be given by (1.4) \(^1\).
CHAPTER II
ESTIMATION OF PARAMETERS

Consider now the process \( y_t \) given by

\[
(2.1) \quad y_t = x_t + f(t)
\]

where \( x_t \) is an O.U.P. and \( f(t) \) is the mean value function of the process. The process \( y_t \) is determined by \( f(t) \) and the parameters \( \beta \) and \( \sigma^2 \). In the following sections we shall consider the case that

\[
(2.2) \quad f(t) = k_1 \phi_1(t) + \ldots + k_s \phi_s(t)
\]

and give estimates for the parameters \( \beta, \sigma^2 \) and \( k_1, \ldots, k_s \).

We shall assume that the functions \( \phi_1(t), \ldots, \phi_s(t) \) are linearly independent in the interval \([0, T]\) and twice differentiable. We shall also assume that the process \( y_t \) has been observed in the interval \([0, T]\).

Section 1: The Estimation of \( \beta \sigma^2 \).

We have the following theorem:

**Theorem:** If \( y_t = x_t + f(t) \) where \( x_t \) is an O.U.P. determined by the two parameters \( \beta \) and \( \sigma^2 \) and \( f(t) \) a mean value function satisfying a Lipschitz condition with constant \( M \), and if the values of \( y_t \) are known in a dense set in any interval \( 0 \leq t \leq T \), then it is possible to determine \( \beta \sigma^2 \) with arbitrarily high precision independently of \( f(t) \).

**Proof:** Let \( \tau = \frac{T}{N} \) and form

\[
(2.3) \quad D = \frac{1}{N} \sum_{n=1}^{N} \left( \frac{y_{nr} - y_{nr-1}}{\tau} \right)^2
\]
\[
E(D) = \frac{b\sigma^2}{\tau} (1 - a_{\tau}) + \frac{1}{N\tau} \sum_{n} [f(n\tau) - f(n-1\tau)]^2
\]

We have

(2.6) \[E(D) = \frac{2\sigma^2}{\tau} (1 - a_{\tau}) + \frac{1}{N\tau} \sum_{n} [f(n\tau) - f(n-1\tau)]^2\]

which for \( \tau \to 0 \) converges to \( 2b\sigma^2 \).

To compute the variance of \( D \) we will need the following values:

\[
\begin{align*}
E[(x_{n\tau} - x_{n-1\tau})(x_{m\tau} - x_{m-1\tau})] &= -\sigma^2 (1 - a_{\tau})^2 e^{-\tau(m-n)}, \quad m > n \\
E(x_{n\tau} - x_{n-1\tau})^2 &= 2\sigma^2 (1 - a_{\tau}) \\
E[(x_{n\tau} - x_{n-1\tau})(x_{m\tau} - x_{m-1\tau})^2] &= 0 \\
E[(x_{n\tau} - x_{n-1\tau})^2] &= 2\sigma^2 (1 - a_{\tau})^2 \left[2 + (1 - a_{\tau})^2 e^{-\tau(m-n)}\right] \\
E(x_{n\tau} - x_{n-1\tau})^3 &= 0 \\
E(x_{n\tau} - x_{n-1\tau})^4 &= 12\sigma^4 (1 - a_{\tau})^2
\end{align*}
\]

From (2.5) we have

\[
E(D^2) = \frac{1}{N\tau^2} \left\{4N(N+2)\sigma^4(1-a_{\tau})^2 + 12\sigma^2(1-a_{\tau}) \sum_{n} [f(n\tau) - f(n-1\tau)]^2 + \sum_{n} [f(n\tau) - f(n-1\tau)]^4 + 4\sigma^4(1-a_{\tau})^4 \left[\frac{N}{a_{\tau}} - \frac{1 - a_{\tau}}{(1 - a_{\tau})^2}\right] + 4\sigma^2(1-a_{\tau}) \sum_{n} (n-n) [f(n\tau) - f(n-1\tau)]^2 - 8\sigma^2(1-a_{\tau})^2 \sum_{m=1}^{N} \sum_{n=1}^{N} \left|\frac{a_{n-m}}{a_{\tau}}\right|^2 f(n\tau) - f(n-1\tau)]^2 + 4\sigma^2(1-a_{\tau}) \sum_{n} (n-n) [f(n\tau) - f(n-1\tau)]^2 \sum_{m=1}^{N} \sum_{n=1}^{N} f(n\tau) - f(n-1\tau)\right\};
\]

and

\[
[E(D)]^2 = \frac{4\sigma^4}{\tau^2} (1 - a_{\tau})^2 + \frac{1}{\tau^2} \left\{\sum_{n} [f(n\tau) - f(n-1\tau)]^2 \right\}^2 + \frac{4\sigma^2}{N\tau^2} (1 - a_{\tau}) \sum_{n} [f(n\tau) - f(n-1\tau)]^2
\]
so that

\[
\begin{align*}
\sigma_D^2 &= \frac{8 \sigma^4 (1-\alpha)^2}{N^2 \gamma^2} + \frac{4 (\alpha-N) \sigma^2 (1-\alpha)^2}{N^2 \gamma^2} \sum_m \left[ f(m) - f(m-1) \right]^2 \\
&\quad + \frac{4 \sigma^2 (1-\alpha)^2}{N^2 \gamma^2} \left( \frac{2N}{1+N} \right) \sum_m (m+1) a_T \\
&\quad \cdot \left[ f(m) - f(m-1) \right] \left[ f(m) - f(m-1) \right] + \frac{8 \sigma^2 (1-\alpha)^2}{N^2 \gamma^2} \sum_m (m-N) \left[ f(m) - f(m-1) \right]^2 \\
&\quad + \frac{4 \sigma^2 (1-\alpha)^2}{N^2 \gamma^2} \sum_m (m-N) \left[ f(m) - f(m-1) \right]^2
\end{align*}
\]

Equation (2.6) shows that \( \sigma_D^2 \) can be made arbitrarily small by making \( N \) large enough. In fact \( \lim_{N \to \infty} \frac{\sigma_D^2}{\sqrt{N} D} = 8 \sigma^4 (3^2) \). Also

\[
E[\sqrt{N} (D-2 \theta \sigma^2)] = 2 \sigma^2 \sqrt{N} \left[ \frac{1-e^{-\alpha T}}{\alpha} \right] + \frac{\sqrt{N}}{\sqrt{T}} \sum_m \left[ f(m) - f(m-1) \right]^2
\]

and therefore since \( \tau = \frac{T}{N} \) and \( |f(m) - f(m-1)| \leq \alpha T \)

\[
(2.7) \quad \lim_{N \to \infty} E[\sqrt{N} (D-2 \theta \sigma^2)] = 0
\]

We proceed to prove that the limit distribution of \( \sqrt{N} (D-2 \theta \sigma^2) \)

is normal.

\[
\sqrt{N} D = \frac{1}{\sqrt{N} T} \sum_m \left( X_{mt} - x_{m-1,t} \right)^2 - \frac{2}{\sqrt{N} T} \sum_m \left( X_{mt} - x_{m-1,t} \right) \left[ f(m) - f(m-1) \right] \\
&\quad + \frac{1}{\sqrt{N} T} \sum_m \left[ f(m) - f(m-1) \right]^2 \\
&\quad = \frac{1}{\sqrt{N}} \left( \frac{\alpha-1)^2}{1+N} \right) \sum_m X_{mt}^2 + \frac{2 (\alpha-1)}{\sqrt{N} T} \sum_m X_{mt} E_{m-1,t} + \frac{1}{\sqrt{N} T} \sum_m \epsilon_{m-1,t}^2 \\
&\quad - \frac{2}{\sqrt{N} T} \sum_m (X_{mt} - X_{m-1,t}) \left[ f(m) - f(m-1) \right] + \frac{1}{\sqrt{N} T} \sum_m \left[ f(m) - f(m-1) \right]^2
\]

As shown in [1], the first two sums converge to zero, in the mean, and the third sum is a sum of independent random variables, all having the same distribution, which converges to the normal distribution. Thus the normality of the limit distribution will
be proved if the last two sums can be shown to converge to zero. Therefore we let
\[ \Sigma_3 = \frac{2}{N \tau} \sum_n (\epsilon_{n-1} - \epsilon_{n-2})[f(\epsilon_{n-1}) - f(\epsilon_{n-2})] \]
and
\[ \Sigma_4 = \frac{1}{N \tau} \sum_n [f(\epsilon_{n-1}) - f(\epsilon_{n-2})]^2 \]
and by use of (2.45)
\[ \mathbb{E}(\Sigma_3^2) = \frac{8 \sigma^2 (1 - \lambda^2)}{T} \sum \sum \left( f(\epsilon_{n-1}) - f(\epsilon_{n-2}) \right)^2 - \frac{8 \sigma^2 (1 - \lambda^2)}{T} \sum \sum \epsilon_n \epsilon_{n-1} \left( f(\epsilon_{n-1}) - f(\epsilon_{n-2}) \right) \]}
Thus \( \lim_{N \to \infty} \mathbb{E}(\Sigma_3^2) = 0 \) and since \( f(t) \) satisfies a Lipschitz condition, \( \lim_{N \to \infty} \mathbb{E}(\Sigma_4^2) = 0 \). Therefore \( \lim_{N \to \infty} \Sigma_3 = 0 \) and \( \lim_{N \to \infty} \Sigma_4 = 0 \) so that \( \lim_{N \to \infty} \left( \sqrt{N} D - \frac{1}{\sqrt{N}} \sum \sum \frac{\epsilon_n \epsilon_{n-1}}{T} \right) = 0 \). Thus since the distribution of \( \sqrt{N} \left( \frac{1}{\sqrt{N}} \sum \sum \frac{\epsilon_n \epsilon_{n-1}}{T} \right) \) converges to a normal distribution with mean 0 and variance \( 8 \beta^2 \sigma^4 \), it follows that \( \sqrt{N} \left( D - 2 \beta \sigma^2 \right) \) is in the limit normally distributed with mean zero and variance \( 8 \beta^2 \sigma^4 \).

Section 2: Estimation of \( \beta \) and \( \sigma^2 \) when \( \beta \) is not Known.

The limiting form of the maximum likelihood equations of the O.J.P. when \( \beta \) is unknown is rather complicated. However, the parameters \( \beta \) can be estimated by least squares. We then have to minimize \( \int_0^T (y_t - f(t))^2 \, dt \). These estimates, denoted by \( \bar{K}_1 \), and their covariance matrix take the following forms:
(2.8) \( \bar{K}_i = \sum_{j=1}^{s} \psi_{ij} \int_0^T y_t \varphi_j(t) \, dt \)

where \( \psi_{ij} = \int_0^T \varphi_i(t) \varphi_j(t) \, dt \) and \( (\psi_{ij})^{-1} \) the inverse of the matrix \( (\psi_{ij}) \). It can easily be shown that \( \bar{K}_i \) is an unbiased estimate of \( K_i \). The covariance matrix of the \( \bar{K}_i \) is given by

(2.9) \[ \Sigma_{\bar{K}_i \bar{K}_j} = \sigma^2 \sum_{\alpha=1}^{s} \sum_{\gamma=1}^{p} \psi_{\alpha \gamma} \int_0^T \Phi_{\alpha}(t) \Phi_{\gamma}(t') e^{-|t-t'|} \, dt \, dt' \]

and if the functions \( \varphi_i(t) \) are orthogonal to each other this reduces to

(2.10) \[ \Sigma_{\bar{K}_i \bar{K}_j} = \sigma^2 \psi_{ij} \int_0^T \varphi_i(t) \varphi_i(t') e^{-|t-t'|} \, dt \, dt' \]

We set \( \bar{f}(t) = \sum_{i=1}^{s} \bar{K}_i \varphi_i(t) \).

To estimate \( \sigma^2 \) we consider

(2.11) \[ \frac{1}{T} \int_0^T (y_t - f(t))^2 \, dt = \frac{1}{T} \int_0^T (x_t + f(t) - \bar{f}(t))^2 \, dt \]

\[ = \frac{1}{T} \int_0^T x_t^2 \, dt + \frac{2}{T} \int_0^T x_t(f(t) - \bar{f}(t)) \, dt \]

\[ + \frac{1}{T} \int_0^T (f(t) - \bar{f}(t))^2 \, dt \]

From the least square equations we have

\[ \int_0^T (y_t - \bar{f}(t)) \varphi_i(t) \, dt = \int_0^T (x_t + f(t) - \bar{f}(t)) \varphi_i(t) = 0, \quad i=1, \ldots, s \]

Therefore

(2.12) \[ \int_0^T x_t \varphi_i(t) \, dt = - \int_0^T (f(t) - \bar{f}(t)) \varphi_i(t) \, dt, \quad i=1, \ldots, s \]
Multiplying the $i$th equation (2.12) by $k_i$ and adding we get

$$\sum_{i=1}^{T} x_t f(t) \, dt = - \sum_{i=1}^{T} (f(t) - \bar{f}(t)) f(t) \, dt \quad (2.13)$$

Similarly

$$\sum_{i=1}^{T} x_t \bar{f}(t) \, dt = - \sum_{i=1}^{T} (f(t) - \bar{f}(t)) \bar{f}(t) \, dt \quad (2.14)$$

Subtracting (2.14) from (2.13) we get

$$\sum_{i=1}^{T} x_t (f(t) - \bar{f}(t)) \, dt = - \sum_{i=1}^{T} (f(t) - \bar{f}(t))^2 \, dt \quad (2.15)$$

Substituting this in (2.11) we have

$$\frac{1}{T} \sum_{t=0}^{T} (y_t - \bar{f}(t))^2 \, dt = \frac{1}{T} \sum_{t=0}^{T} x_t^2 \, dt - \frac{1}{T} \sum_{t=0}^{T} (f(t) - \bar{f}(t))^2 \, dt \quad (2.16)$$

and hence

$$E \left[ \frac{1}{T} \sum_{t=0}^{T} (y_t - \bar{f}(t))^2 \, dt \right] = \sigma^2 \frac{T - S(T)}{T}$$

where

$$S(T) = \sum_{i=1}^{s} \sum_{\alpha=1}^{s} \gamma^{i \alpha} \int_{0}^{T} \int_{0}^{T} \phi_i(t) \phi_\alpha(t*) \, e^{-\alpha|t-t*|} \, dt \, dt*.$$

It follows that

$$\frac{\sigma^2}{T - S(T)} \sum_{t=0}^{T} (y_t - \bar{f}(t))^2 \, dt \quad (2.17)$$

is an unbiased estimate of $\sigma^2$. It can be shown that for large $T$, $\frac{\sigma^2}{T}$ may be treated as a random variable with variance $\frac{2 \sigma^4}{T}$.

However, the variance of $\frac{\sigma^2}{T}$ may actually be computed.

Without loss of generality, we may assume that the functions $\phi_i(t)$ are orthogonal and normalized. From (2.16) and (2.17) we have

$$\frac{\sigma^2}{T - S(T)} \left\{ \sum_{t=0}^{T} x_t^2 \, dt - \sum_{i=1}^{s} (k_i - \bar{k}_i)^2 \right\}$$

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We then have

\[
(2.18) \quad \frac{\sigma^2}{2} = \frac{1}{(T-s(T))^2} \left\{ E\left[ \int_0^T x_t^2 \, dt \right]^2 - \int_0^T \frac{\dot{x}_t^2}{x_t^2} \, dt \right\} - \frac{\sigma^4}{T^4}
\]

Therefore evaluating the three terms, in the brackets, we have

\[
(2.19) \quad E\left\{ \left[ \int_0^T x_t^2 \, dt \right]^2 \right\} = \frac{2 \sigma^4 T}{\beta} + \frac{\sigma^4 (e^{-2 \beta^T - 1})}{\beta} + T^2 \sigma^4
\]

From (2.8), recalling that the \( \Phi_i(t) \) are orthogonal and normalized we have \( \int_0^T x_t \Phi_i(t) \, dt = \overline{x}_1 - K_i \) and hence

\[
(2.20) \quad E\left[ (\overline{x}_1 - K_i)^2 \int_0^T x_t^2 \, dt \right] = \int_0^T \int_0^T \int_0^T E(\Phi_i(t) \Phi_j(t) \Phi_k(t) \Phi_l(t)) \, dt \, dt \, dt
\]

and

\[
(2.21) \quad E\left[ (\overline{x}_1 - K_i)^2 (\overline{x}_j - K_j)^2 \right] = \int_0^T \int_0^T \int_0^T \int_0^T E(\Phi_i(t) \Phi_j(t) \Phi_k(t) \Phi_l(t)) \, dt \, dt \, dt \, dt
\]

To evaluate (2.20) and (2.21) we need the formula for \( E(x_t x_{t'}, x_{t''} x_{t'''} \Phi_i(t)) \). Refering to formula (3.13) of \cite{1} we have for \( t', t'', t''' \geq t \)

\[
E(\Phi_i(t) \Phi_j(t) \Phi_k(t) \Phi_l(t)) = \sigma^4 \left\{ e^{-\beta(t''-t'+t''-t')} + 2e^{-\beta(t''+t''-t'-t')} \right\}
\]

In the computation of \( \frac{\sigma^2}{2} \) in (2.18), we shall consider the case that the \( \Phi_i(t) \) are normalized Legendre Polynomials over the interval \([0, T]\). That is, we let \( \Phi_i(t) = \sqrt{\frac{2i+1}{\pi}} P_i(\frac{2t}{T}-1) \)

where \( P_i(x) \) is the \( i^{th} \) Legendre Polynomial on the interval \([-1, 1]\).

We shall write \( \Phi_i(t) = b_{n_1} x_t^{n_1} + b_{n_2} x_t^{n_2} + \ldots + b_{n_k} x_t^{n_k} \)

Let \( t', t'', t''' \geq t \) and consider
\[
\begin{align*}
(2.22) \sum \sum \sum \sum \sum \sum \mathbb{E} \left( t' \right) Q_{i} \left( t'' \right) Q_{j} \left( t' \right) Q_{k} \left( t' \right) Q_{l} \left( t \right) dt dt' dt'' dt'''
&= \sigma^{-4} \left\{ \sum \sum \sum \sum \sum \sum e^{-\rho(t''-t'''+t'-t)} Q_{i} \left( t'' \right) Q_{j} \left( t' \right) Q_{k} \left( t' \right) Q_{l} \left( t \right) dt dt' dt'' dt'''
+ 2 \sum \sum \sum \sum \sum \sum e^{-\beta(t''-t'''+t'-t)} Q_{i} \left( t'' \right) Q_{j} \left( t' \right) Q_{k} \left( t' \right) Q_{l} \left( t \right) dt dt' dt'' dt'''ight\}
&= \sigma^{-4} \left( A_{\alpha, \delta, \mu} + 2 B_{\alpha, \delta, \mu} \right)
\end{align*}
\]

where \( A_{\alpha, \delta, \mu} \) equals the first integral and \( B_{\alpha, \delta, \mu} \) equals the second integral on the right hand side of equation \((2.22)\).

To evaluate \((2.21)\) twenty-four integrals must be considered when \( i \neq j \). However, since there are only two different subscripts \( i \) and \( j \) to be considered, we need to compute only six of the integrals corresponding to the six different arrangements of \( i, i, j, j \).

To further simplify the calculations, once \( A_{\alpha, \delta, \mu} \) and \( B_{\alpha, \delta, \mu} \) are computed, we shall obtain the six integrals by setting \( \alpha, \gamma, \delta, \mu \) equal to \( i \) or \( j \) according to the six different arrangements of \( i, i, j, j \). For example,

\[
A_{i,i,i,i} = \sum \sum \sum \sum \sum \sum e^{-\rho(t''-t'''+t'-t)} Q_{i} \left( t'' \right) Q_{i} \left( t' \right) Q_{i} \left( t' \right) Q_{i} \left( t \right) dt dt' dt'' dt'''
\]

Computing \( A_{\alpha, \delta, \mu} \) we have:

\[
A_{\alpha, \delta, \mu} = \sum \sum \sum \sum \sum \sum e^{-\rho(t''-t'''+t'-t)} \left\{ \sum \int_{0}^{at} Q_{\alpha} \left( t \right) dt \right\} Q_{\alpha} \left( t'' \right) Q_{\delta} \left( t' \right) Q_{\delta} \left( t' \right) dt dt' dt'' dt'''
= \sum \sum \sum \sum \sum \sum e^{-\rho(t''-t'''+t'-t)} \left\{ \sum \sum \sum \sum \sum \sum \sum \sum \sum \sum e^{-\rho(t''-t'''+t'-t)} \right\} Q_{\alpha} \left( t'' \right) Q_{\delta} \left( t' \right) Q_{\delta} \left( t' \right) dt dt' dt'' dt'''
\]
Computing in a similar manner we have:

\[(2.24)\]
\[B_{\alpha \mu \beta \nu} = \sum_{h=0}^{\infty} \frac{(2h)!}{h!} \left[ \frac{\sum_{\ell=0}^{\infty} \frac{b_{h,\mu} b_{h,\nu} b_{h+1,\mu} b_{h+1,\nu}}{\ell+1} \sum_{m=0}^{\infty} \frac{m! (2m)! (2m+1)!}{(2m+1)!} \right] + \frac{\sum_{h=0}^{\infty} \frac{h!}{(2h+1)!} \sum_{\ell=0}^{\infty} \frac{b_{h,\mu} b_{h,\nu} b_{h+1,\mu} b_{h+1,\nu}}{\ell+1} \sum_{m=0}^{\infty} \frac{m! (2m)! (2m+1)!}{(2m+1)!}}{\sum_{h=0}^{\infty} \frac{h!}{(2h+1)!} \sum_{\ell=0}^{\infty} \frac{b_{h,\mu} b_{h,\nu} b_{h+1,\mu} b_{h+1,\nu}}{\ell+1} \sum_{m=0}^{\infty} \frac{m! (2m)! (2m+1)!}{(2m+1)!}} \]

Thus when \(i \neq j\) we have for \(2.25\)

\[(2.25)\]
\[= 4 \delta^{ij} \left[ (A_{ij} + A_{ji} + A_{ij} + A_{ji} + A_{ij} + A_{ji} + A_{ij} + A_{ji}) + 2(B_{jj} + B_{ij} + B_{ji}) ight]

and when \(i = j\) we have for \(2.26\)

\[(2.26)\]
\[= 2 \delta^{ij} \left[ (A_{ii} + A_{ii} + A_{ii} + A_{ii} + A_{ii} + A_{ii} + A_{ii} + A_{ii}) + 2(B_{iii} + B_{i} + B_{ii}) \right] \]

The term \(\sum_{i} \sum_{j} E \left[ (K_i - K_i)^2 | (K_i - K_i)^2 \right]\) in equation \(2.18\) can be evaluated using \(2.25\) and \(2.26\).
In order to evaluate \[ \sum_{0 \leq t \leq t'} \sum_{0 \leq t' \leq t''} \mathbb{E}(X_{t'}^2 X_t^2) Q_{t'}(t'') dt' dt'' \]
we must consider the six possible arrangements for \( t, t' \) and \( t'' \).
However, it will only be necessary to compute three different integrals. Multiplying each of these by two and adding gives the desired evaluation for (2.20).

Consider \( t'' \geq t \geq t' \), then \[ \mathbb{E}(X_{t'}^2 X_t^2) = 3 \sigma^4 \beta(t''-t') \]
and let \[ \Lambda_{t,t'}^{t''} = \int_0^{t''} \int_0^{t'} \mathbb{E}(t'') Q_{t'}(t'') dt' dt'' . \]

Then \[ 3 \sigma^4 \Lambda_{t,t'}^{t''} = 3 \sigma^4 \int_0^{t''} \int_0^{t'} e^{-pt''} \mathbb{E}(t'') \left[ \sum_{m=0}^{\infty} \frac{2 \cdot \psi}{\beta^{m+1}} \right] \mathbb{E}(t') \left[ \sum_{m=0}^{\infty} \frac{2 \cdot \psi}{\beta^{m+1}} \right] dt' dt'' . \]

\[ = 3 \sigma^4 \int_0^{t''} e^{-pt''} \mathbb{E}(t'') \left[ \sum_{m=0}^{\infty} \frac{2 \cdot \psi}{\beta^{m+1}} \right] \left[ \sum_{m=0}^{\infty} \frac{2 \cdot \psi}{\beta^{m+1}} \right] dt'' . \]

\[ = 3 \sigma^4 \left\{ \sum_{m=0}^{\infty} \frac{2 \cdot \psi}{\beta^{m+1}} \right\} \left[ \sum_{m=0}^{\infty} \frac{2 \cdot \psi}{\beta^{m+1}} \right] . \]

(2.27)

If \( t'' \geq t' \geq t \) then

\[ \int_0^{t''} \int_0^{t'} \mathbb{E}(X_{t'}^2 X_t^2) Q_{t'}(t'') dt' dt'' . \]

\[ = 3 \sigma^4 \left\{ \sum_{m=0}^{\infty} \frac{2 \cdot \psi}{\beta^{m+1}} \right\} \left[ \sum_{m=0}^{\infty} \frac{2 \cdot \psi}{\beta^{m+1}} \right] . \]

\[ = 3 \sigma^4 \left( \Lambda_{t,t'}^{t''} + 2 \beta^{t''-t'} \right) \]
where

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\[ A_{t''}^{i, t''} = \sum_{r_0} \left[ \sum_{r_1=0}^{r_0} \beta_t (m) \right] \frac{e^{-\beta_t (m) \alpha_t (t')}}{2^m (m)!} \]
(2.30) \[ A_{i_{1}t_{1}t_{1}} = \int_{0}^{t_{1}} e^{-\beta t} Q_{1}(0) Q_{1}(t_{1}) dt_{1} dt + \int_{0}^{t_{1}} e^{\beta t} Q_{1}(0) \left[ \sum_{n=0}^{\infty} \frac{(\beta t)^{n} m! (-1)^{n}}{n!} \right] dt \]

\[ = \int_{0}^{t_{1}} e^{-\beta t} Q_{1}(0) \left\{ \sum_{n=0}^{\infty} \frac{b_{n} e^{\beta t} (\beta t)^{n} m! (-1)^{n}}{n!} \right\} dt \]

\[ = \int_{0}^{t_{1}} \sum_{n=0}^{\infty} \frac{b_{n} e^{\beta t} (\beta t)^{n} m! (-1)^{n}}{n!} \int_{0}^{t_{1}} e^{\beta t} \left[ \sum_{u=0}^{\infty} \frac{(\beta t)^{u} n! (-1)^{u}}{u!} \right] dt \]

\[ = \sum_{n=0}^{\infty} \frac{b_{n} e^{\beta t} (\beta t)^{n} m! (-1)^{n}}{n!} \int_{0}^{t_{1}} \left[ \sum_{u=0}^{\infty} \frac{(\beta t)^{u} n! (-1)^{u}}{u!} \right] dt \]

\[ = \sum_{n=0}^{\infty} \frac{b_{n} e^{\beta t} (\beta t)^{n} m! (-1)^{n}}{n!} \left[ \sum_{u=0}^{\infty} \frac{(\beta t)^{u} n! (-1)^{u}}{u!} \right] \frac{1}{\beta} + \frac{n!}{\beta^{n+1}} \]

\[ = \sum_{n=0}^{\infty} \frac{b_{n} e^{\beta t} (\beta t)^{n} m! (-1)^{n}}{n!} \left[ \sum_{u=0}^{\infty} \frac{(\beta t)^{u} n! (-1)^{u}}{u!} \right] \frac{1}{\beta} + \frac{n!}{\beta^{n+1}} \]

\[ = \sum_{n=0}^{\infty} \frac{b_{n} e^{\beta t} (\beta t)^{n} m! (-1)^{n}}{n!} \left[ \sum_{u=0}^{\infty} \frac{(\beta t)^{u} n! (-1)^{u}}{u!} \right] \frac{1}{\beta} + \frac{n!}{\beta^{n+1}} \]

\[ = \sum_{n=0}^{\infty} \frac{b_{n} e^{\beta t} (\beta t)^{n} m! (-1)^{n}}{n!} \left[ \sum_{u=0}^{\infty} \frac{(\beta t)^{u} n! (-1)^{u}}{u!} \right] \frac{1}{\beta} + \frac{n!}{\beta^{n+1}} \]

\[ = \sum_{n=0}^{\infty} \frac{b_{n} e^{\beta t} (\beta t)^{n} m! (-1)^{n}}{n!} \left[ \sum_{u=0}^{\infty} \frac{(\beta t)^{u} n! (-1)^{u}}{u!} \right] \frac{1}{\beta} + \frac{n!}{\beta^{n+1}} \]

\[ = \sum_{n=0}^{\infty} \frac{b_{n} e^{\beta t} (\beta t)^{n} m! (-1)^{n}}{n!} \left[ \sum_{u=0}^{\infty} \frac{(\beta t)^{u} n! (-1)^{u}}{u!} \right] \frac{1}{\beta} + \frac{n!}{\beta^{n+1}} \]

\[ = \sum_{n=0}^{\infty} \frac{b_{n} e^{\beta t} (\beta t)^{n} m! (-1)^{n}}{n!} \left[ \sum_{u=0}^{\infty} \frac{(\beta t)^{u} n! (-1)^{u}}{u!} \right] \frac{1}{\beta} + \frac{n!}{\beta^{n+1}} \]

\[ = \sum_{n=0}^{\infty} \frac{b_{n} e^{\beta t} (\beta t)^{n} m! (-1)^{n}}{n!} \left[ \sum_{u=0}^{\infty} \frac{(\beta t)^{u} n! (-1)^{u}}{u!} \right] \frac{1}{\beta} + \frac{n!}{\beta^{n+1}} \]

\[ = \sum_{n=0}^{\infty} \frac{b_{n} e^{\beta t} (\beta t)^{n} m! (-1)^{n}}{n!} \left[ \sum_{u=0}^{\infty} \frac{(\beta t)^{u} n! (-1)^{u}}{u!} \right] \frac{1}{\beta} + \frac{n!}{\beta^{n+1}} \]

\[ = \sum_{n=0}^{\infty} \frac{b_{n} e^{\beta t} (\beta t)^{n} m! (-1)^{n}}{n!} \left[ \sum_{u=0}^{\infty} \frac{(\beta t)^{u} n! (-1)^{u}}{u!} \right] \frac{1}{\beta} + \frac{n!}{\beta^{n+1}} \]

\[ = \sum_{n=0}^{\infty} \frac{b_{n} e^{\beta t} (\beta t)^{n} m! (-1)^{n}}{n!} \left[ \sum_{u=0}^{\infty} \frac{(\beta t)^{u} n! (-1)^{u}}{u!} \right] \frac{1}{\beta} + \frac{n!}{\beta^{n+1}} \]

\[ = \sum_{n=0}^{\infty} \frac{b_{n} e^{\beta t} (\beta t)^{n} m! (-1)^{n}}{n!} \left[ \sum_{u=0}^{\infty} \frac{(\beta t)^{u} n! (-1)^{u}}{u!} \right] \frac{1}{\beta} + \frac{n!}{\beta^{n+1}} \]

\[ = \sum_{n=0}^{\infty} \frac{b_{n} e^{\beta t} (\beta t)^{n} m! (-1)^{n}}{n!} \left[ \sum_{u=0}^{\infty} \frac{(\beta t)^{u} n! (-1)^{u}}{u!} \right] \frac{1}{\beta} + \frac{n!}{\beta^{n+1}} \]

\[ = \sum_{n=0}^{\infty} \frac{b_{n} e^{\beta t} (\beta t)^{n} m! (-1)^{n}}{n!} \left[ \sum_{u=0}^{\infty} \frac{(\beta t)^{u} n! (-1)^{u}}{u!} \right] \frac{1}{\beta} + \frac{n!}{\beta^{n+1}} \]

\[ = \sum_{n=0}^{\infty} \frac{b_{n} e^{\beta t} (\beta t)^{n} m! (-1)^{n}}{n!} \left[ \sum_{u=0}^{\infty} \frac{(\beta t)^{u} n! (-1)^{u}}{u!} \right] \frac{1}{\beta} + \frac{n!}{\beta^{n+1}} \]

\[ = \sum_{n=0}^{\infty} \frac{b_{n} e^{\beta t} (\beta t)^{n} m! (-1)^{n}}{n!} \left[ \sum_{u=0}^{\infty} \frac{(\beta t)^{u} n! (-1)^{u}}{u!} \right] \frac{1}{\beta} + \frac{n!}{\beta^{n+1}} \]

\[ = \sum_{n=0}^{\infty} \frac{b_{n} e^{\beta t} (\beta t)^{n} m! (-1)^{n}}{n!} \left[ \sum_{u=0}^{\infty} \frac{(\beta t)^{u} n! (-1)^{u}}{u!} \right] \frac{1}{\beta} + \frac{n!}{\beta^{n+1}} \]

\[ = \sum_{n=0}^{\infty} \frac{b_{n} e^{\beta t} (\beta t)^{n} m! (-1)^{n}}{n!} \left[ \sum_{u=0}^{\infty} \frac{(\beta t)^{u} n! (-1)^{u}}{u!} \right] \frac{1}{\beta} + \frac{n!}{\beta^{n+1}} \]

Therefore evaluating (2.20) we have

(2.32) \[ E[\xi_{1} - \xi_{1}^{2}] \int_{0}^{t_{1}} dt = 2E_{1} \left( 3A_{1}^{2}t_{1}^{2} + A_{1}^{2}t_{1}^{2} + A_{1}^{2}t_{1}^{2} + 2B_{1}^{2}t_{1}^{2} + 2B_{1}^{2}t_{1}^{2} \right) \]
Thus the second term on the right hand side of equation (2.18) may be determined from equation (2.32). Hence from (2.19), (2.25) and (2.32)

\[
\frac{1}{\sigma^2} = \left[ T - S(T) \right]^2 \left\{ \frac{2\sigma^4 \gamma^n + \sigma^4 (e^{-2\gamma^2}) + T^2 \sigma^4 + 4\sigma^4 \sum \left[ 2A_{f,1} + A_{f,2} \right] + 2B_{f,1} + 2B_{f,2} \right\} + \left[ 2C_{f,1} + C_{f,2} \right] \}
\]

Section 3: Estimation of the \( K_1 \) When \( \Psi \) is Known

Since from Section 1 the quantity \( 2 \Psi \sigma^2 \) can be estimated with arbitrary accuracy and since \( \Psi \) is known, \( \sigma^2 \) can thus be estimated with arbitrarily high precision.

Considering a process of the form (2.1) with a mean value function of the form (2.2), we will present the maximum likelihood estimates of the \( K_1 \) of \( f(t) \) when \( \Psi \) is known.

The interval \([0, T]\) is divided into \( N \) equal parts with \( \gamma = \frac{T}{N} \). The random variables \( y_{nT} = a_f y_{(n-1)T} \); \( n = 1, \ldots, N \), are normally and independently distributed with means \( f(nT) = a_f f((n-1)T) \) hence their joint distribution is given by

\[
(2.33) c \exp -\frac{1}{2\sigma^2} \left[ (y - f_0)^2 + \sum \left\{ \frac{(y_{nT} - a_f f_{(n-1)T}) - (f_{(nT)} - f_{(n-1)T}))^2}{1 - a_f^2} \right\} \right]
\]

and the maximum likelihood equations for the \( K_1 \) are therefore

\[
(2.34) (y_{nT} - f_{nT})Q_{nT} + \frac{1}{1 - a_f} \left\{ \frac{(y_{nT} - a_f f_{(n-1)T}) - (f_{(nT)} - a_f f_{(n-1)T}))^2}{1 + a_f^2} \right\} \left( Q_{(nT)} - a_f Q_{((n-1)T)} \right) = 0
\]
for \( i = 1, \ldots, s \). Let \( \hat{f}(t) = \hat{k}_1 \mathcal{Q}_1(t) + \cdots + \hat{k}_s \mathcal{Q}_s(t) \) where the \( \hat{k}_i \) are the maximum likelihood estimates of the \( k_i \).

Before solving for \( \hat{k}_i \) and computing \( \mathcal{Q}_i \), it is necessary to consider a new kind of derivative and integral [3].

We introduce the following definitions with \( a \tau = e^{-\alpha t} \), \( \beta > 0 \)

**Definition 1:**
\[
\frac{d(a\tau)\mathcal{Q}(t)}{d(a\tau)(t)} = \lim_{\tau \to \infty} \frac{\mathcal{Q}(t+\tau) - a\tau\mathcal{Q}(t)}{1 - a\tau}
\]

**Definition 2:**
\[
\int_a^b f(t) \, d(a\tau)\mathcal{Q}(t) = \lim_{\tau \to \infty} \sum_i f(t_i^*) \left( g(t_i^*) - g(t_{i-1}) \right) + \lim_{\tau \to \infty} \sum_i f(t_i^*) q(t_i^*) q(t_{i-1}^*) k_i - g(t_{i-1}^*)
\]
where \( \delta = \max(t_{i-1} - t_{i-1}) \), \( t_{i-1} \leq t_i \leq t_i^* \), \( t_0 = a \), \( t_N = b \).

In terms of ordinary derivatives, if they exist, we have
\[
\frac{d(a\tau)\mathcal{Q}(t)}{d(a\tau)(t)} = \lim_{\tau \to \infty} \frac{\mathcal{Q}(t+\tau) - a\tau\mathcal{Q}(t)}{1 - a\tau} = \lim_{\tau \to \infty} \frac{\mathcal{Q}(t+\tau) - \mathcal{Q}(t)}{1 - a\tau} + \mathcal{Q}(t)
\]
\[
= \lim_{\tau \to \infty} \frac{\mathcal{Q}'(t^*) + \mathcal{Q}(t), t \leq t^* + \tau}
\]
\[
= \frac{1}{(3)} \mathcal{Q}'(t) + \mathcal{Q}(t)
\]

\[
\int_a^b f(t) \, d(a\tau)\mathcal{Q}(t) = \lim_{\tau \to \infty} \sum_i f(t_i^*) \left( g(t_i^*) - g(t_{i-1}) \right) + \lim_{\tau \to \infty} \sum_i f(t_i^*) q(t_i^*) q(t_{i-1}^*) k_i - g(t_{i-1}^*)
\]
\[
= \int_a^b f(t) \, dq(t) + \lim_{\tau \to \infty} \left( \sum_i f(t_i^*) q(t_i^*) q(t_{i-1}^*) (t_i - t_{i-1}) + \theta \right) \lim_{\tau \to \infty} \sum_i f(t_i^*) q(t_i^*) q(t_{i-1}^*) (t_i - t_{i-1})
\]

We have
\[
\left| \sum_i f(t_i^*) g(t_i^*) (t_i - t_{i-1}) \right| \leq \max f(t) g(t) \delta \sigma (b-a)
\]

\[\sigma \geq b\]

and since \( \delta \to 0 \) the third term vanishes and we get
\[
\int_a^b f(t) \, dq(t) = \int_a^b f(t) \, dq(t) + \left( \int_a^b f(t) \, g(t) \, dt \right)
\]
\[-21-\]
We next prove the formula

\[(2.37) \quad \int_{a}^{b} f(t) \, d(\omega)(t) = \int_{a}^{b} f(t) \, \omega(t) \, d(\omega)(t)\]

if \( g(\omega)(t) \) exists. The right side of (2.37) equals by (2.35)

\[\int_{a}^{b} f(t) \, g(\omega)(t) \, d(\omega)(t) + \frac{1}{2} \int_{a}^{b} f(t) \, g(\omega)(t) \, d(\omega)(t) = \int_{a}^{b} f(t) \, g(\omega)(t) \, d(\omega)(t) + \frac{1}{2} \int_{a}^{b} f(t) \, g(\omega)(t) \, d(\omega)(t) \]

which by (2.36) is the left side of (2.37).

Now from (2.34), if we let \( \gamma \) approach 0, we get in the limit

\[(2.38) \quad \lim_{\gamma \to 0} \phi_i^{(\gamma)}(\omega) = \frac{1}{2} \int_{0}^{T} \phi_i^{(\gamma)}(\omega(t)) \, dt + \frac{1}{2} \int_{0}^{T} \phi_j^{(\gamma)}(\omega(t)) \, dt + \frac{1}{2} \int_{0}^{T} \phi_i^{(\gamma)}(\omega(t)) \, dt \]

Let \( \phi_i^{(\gamma)} = \phi_i^{(\gamma)}(\omega(t)) + \frac{1}{2} \int_{0}^{T} \phi_i^{(\gamma)}(\omega(t)) \, dt \). Using (2.35) we have

\[(2.39) \quad \phi_i^{(\gamma)} = \frac{1}{2} \left[ \phi_i^{(\gamma)}(\omega(t)) + \frac{1}{2} \int_{0}^{T} \phi_i^{(\gamma)}(\omega(t)) \, dt \right] \]

Since the \( \phi_i(t) \) are linearly independent we have \( \phi_i^{(\gamma)} \neq 0 \) and if we put \( (\phi_i^{(\gamma)})^{-1} \) then

\[(2.40) \quad \hat{K}_i = \sum_{j} \phi_i^{(\gamma)}(\omega(t)) \left[ \phi_i^{(\gamma)}(\omega(t)) + \frac{1}{2} \int_{0}^{T} \phi_i^{(\gamma)}(\omega(t)) \, dt \right] \]

This estimate may be referred to as the \( \alpha_\gamma \) Estimate.

Taking the expected value of \( \hat{K}_i \) in (2.40) we find that \( \hat{K}_i \) is an unbiased estimate of \( K_i \). The covariance takes the form

\[(2.41) \quad \sigma^2 \phi_i^{(\gamma)}(\omega(t)) \]
Section 4: The Covariance Matrix of Fundamental Random Process

Estimates if the Process is an O.U.P.

It may happen that under the assumption that \( x_t \) is a Fundamental Random Process, \([2]\), the maximum likelihood estimates determined under this assumption are being used, although the process is actually an O.U.P. These estimates take the form

\[
(2.42) \quad \tilde{\kappa}_i = \frac{1}{T} \sum_{j=1}^{\infty} \alpha_i j \int_0^T \alpha_i j \, dt
\]

where \( \tilde{\alpha}_i j = \sum_{j=1}^{\infty} \alpha_i j \alpha_i j \, dt \) and \( \tilde{\alpha}_i j \) is an unbiased estimate of \( \alpha_i j \). The covariance matrix, if \( x_t \) is an O.U.P., becomes

\[
(2.43) \quad \sigma_{\tilde{\kappa}_i \tilde{\kappa}_j} = 2 \sigma^2 \int_0^T \left( \sum \alpha_i j \alpha_i j \right) \, dt
\]

It can be shown \([3]\) that the second term on the right hand side of (2.43) is greater than or equal to zero, hence the variances of the estimates, if \( x_t \) is an O.U.P., are not larger than the variances

A stochastic process \( x_t \) is called a Fundamental Random Process if it satisfies the following assumptions and for every \( \tau \) the difference equation

\[
x_{t+\tau} = x_t + \varepsilon_{t,\tau}
\]

where \( \varepsilon_{t,\tau} \) is normally distributed with mean 0 and variance \( \sigma^2 \tau \) and independent of \( x_t \) and \( \varepsilon_{t',\tau'} \) if the intervals \( (t, t + \tau) \), \( (t', t' + \tau') \) do not overlap. Assume also that \( x_0 = 0 \).
if $x_t$ is a F.R.P. with variance constant $2\sigma^2 [1]$. This gives us the result that if the estimates $\hat{K}_1$ are time consistent under the assumption that $x_t$ was a F.R.P., then they are also time consistent if $x_t$ is actually an O.U.P.

If $x_t$ is an O.U.P. then

$$C_{x_t}^2 \leq 2\sigma^2 \mathbb{E}^t.$$ 

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1 An estimate $\hat{K}_1$ of $K_1$ will be called time consistent if $\hat{K}_1$ converges stochastically to $K_1$ when $T$ converges to infinity.
CHAPTER III
ESTIMATION OF THE $K_i$ FOR EXAMPLES OF THE MEAN VALUE FUNCTION

Section 1: The Mean Value Function Is A Polynomial

In this section we shall consider the special case when
$f(t) = \sum_i K_i \phi_i(t)$ is a polynomial in $t$. The three different estimates of $K_i$, and the corresponding covariance matrices, will be considered. The evaluation and results of the matrices are given in as complete a form as possible in order that they may be easily computed for any particular polynomial. Since any polynomial, $\sum_{i=1}^n L_i t^i$, may be expressed in the form $\sum_{i=1}^n K_i \phi_i(t)$, where the $\phi_i(t)$ are Legendre Polynomials, we will consider our mean value function as one of this form. In particular, if $P_i(x)$ is the $i^{th}$ Legendre Polynomial,

$$P_i(x) = \sum_{m=\frac{1}{2}i}^{\infty} (-1)^m \frac{(2i-2m)!}{2^i i! (i-m)! (i-2m)!} x^{i-2m} \quad (3.1)$$

where $m = \frac{1}{2}i$ or $\frac{1}{2}(i-1)$ whichever is an integer, over the interval $[-1,1]$, we let $x = \frac{2t}{T} - 1$ and let $\phi_i(t) = P_i(\frac{2t}{T} - 1)$. Thus the $\phi_i(t)$ are orthogonal over the interval $[0, T]$ and the integral

$$\int_0^T \phi_i(t)^2 dt = \frac{T}{2i+1}.$$

For convenience, the first five orthogonal polynomials are listed:

$$\phi_0(t) = 1$$
$$\phi_1(t) = 2 \frac{t}{T} - 1$$
$$\phi_2(t) = 6 \left( \frac{t}{T} \right)^2 - 6 \left( \frac{t}{T} \right) + 1$$
$$\begin{align*}
\Phi_3(t) &= 20\left(\frac{t}{2}\right)^3 - 30\left(\frac{t}{2}\right)^2 + 12\left(\frac{t}{2}\right) - 1 \\
\Phi_4(t) &= 70\left(\frac{t}{2}\right)^4 - 140\left(\frac{t}{2}\right)^3 + 90\left(\frac{t}{2}\right)^2 - 20\left(\frac{t}{2}\right) + 1 \\
\Phi_5(t) &= 252\left(\frac{t}{2}\right)^5 - 630\left(\frac{t}{2}\right)^4 + 560\left(\frac{t}{2}\right)^3 - 210\left(\frac{t}{2}\right)^2 + 30\left(\frac{t}{2}\right) - 1
\end{align*}$$

Part I: Estimation of $K_1$ when $\beta$ is Not Known

The least square estimate is given by

$$\hat{K}_i = \sum_{j=1}^{\infty} \psi_{ij} \int_0^T \Phi_j(t) \, dt$$

where

$$\psi_{ij} = \int_0^T \Phi_i(t) \Phi_j(t) \, dt$$

and if the $\Phi_i(t)$ are orthogonal then

$$\sum_{j=1}^{n} \psi_{ij} = \sigma^{-2} \sum_{j=1}^{n} \int_0^T \Phi_i(t) \Phi_j(t) \, dt \quad \text{for all } i \neq j.$$  

Let $f(t) = \sum_{n=0}^{\infty} K_n \Phi_n(t)$ where $\Phi_n(t)$ are Legendre Polynomials of the form $P_n\left(\frac{2t}{T} - 1\right)$. Then calculating $\psi_{ij}$ we have:

$$\psi_{ij} = 0 \text{ if } i \neq j \quad \text{or} \quad \psi_{ij} = \frac{T}{2^{i+j+1}} \text{ if } i = j.$$  

To evaluate $\sum_{j=1}^{n} \psi_{ij}$ we consider the decomposition

$$\int_0^T \Phi_i(t) \Phi_j(t) \, dt = \int_0^T \Phi_i(t) \Phi_j(t) \, dt = \int_0^T \Phi_i(t) \Phi_j(t) \, dt'$$

where

$$A_{ij} = \int_0^T \Phi_i(t) \Phi_j(t) \, dt'$$

Let $\Phi_i(t) = a_0 t^0 + a_{-1} t^{-1} + \ldots + a_{k_i} t + b_{-k_i}$ and we get

$$\int_0^t \Phi_i(t') \, dt' = e^{\alpha t} \left\{ \sum_{m=0}^{\infty} \frac{b_{-m}}{m!} \left[ (\alpha t)^m - m (\alpha t)^{m-1} + \ldots + (-1)^m \right] + \frac{b_i}{\alpha} \right\}$$

Consider

$$\int_0^t \sum_{m=0}^{\infty} \frac{b_{-m}}{m!} \left[ (\alpha t)^m - m (\alpha t)^{m-1} + \ldots + (-1)^m \right] + \frac{b_i}{\alpha}$$

as a polynomial in $t$ of degree $j$, then we write it as
where the $Q_j(t)$ are Legendre Polynomials orthogonal over the interval $[0,T]$.

Let

$$A_{ij} = \sum_{m=0}^{\infty} \frac{b_{im} m! (-1)^m}{\beta^{m+1}}$$

then for $A_{ij}$ using (3.3) we have:

$$A_{ij} = \int_0^T \left\{ c_{jj} Q_j(t) + \cdots + c_{j0} Q_0(t) \right\} dt - R_j \int_0^T Q_j(t) e^{qt} dt$$

$$= \frac{c_{ji} T}{2i+1} + R_j \left\{ \frac{1}{\beta^{i+1}} \sum_{m=0}^{\infty} \frac{b_{im} m!}{\beta^{m+1}} \frac{(\beta T)^m}{m!} - \frac{1}{\beta^{i+1}} \right\}$$

$$= \frac{c_{ji} T}{2i+1} + R_j \frac{1}{\beta^{i+1}} \sum_{m=0}^{\infty} \frac{b_{im} m!}{\beta^{m+1}} \left\{ e^{-\beta T} \frac{1}{m!} \left( \frac{(\beta T)^m}{m!} - 1 \right) \right\}$$

where $c_{ji} = 0$, $i > j$. Therefore

$$\sigma_{pp}^{(p,p)} = \begin{cases} \frac{\sigma^2(2i+1)!}{m^2} \left\{ A_{ii} + A_{ji} \right\} & j \neq i \\ \frac{\sigma^2(2i+1)^2}{m^2} \left\{ 2 A_{ii} \right\} & j = i \end{cases}$$

**Part II: Estimation of $K_j$ when $Q$ is Known**

The $a_j$ estimates are given by (2.40), (2.41) and we have

$$\hat{K}_j = \sum_j \phi^{(a)}_{ij} \left[ \frac{1}{\beta} \int_0^T \phi_j(t)^{a_j} Q_j(t) Q_j(t) dt + \phi_j(t)^{a_j} Q_j(t) \right]$$

where

$$\phi_j^{(a_j)} = \frac{1}{\beta} \int_0^T \phi_j(t)^{(a_j)} Q_j(t) Q_j(t) dt + a_j \phi_j(t)^{(a_j)}$$

Also

$$Q_j^{(a_j)}(t) = \frac{1}{\beta} Q_j(t)^{(a_j)} + \phi_j(t)^{(a_j)}$$

and

$$d^{(a_j)}(t) = \beta dt$$

from (2.35).

Hence

$$\phi_j^{(a_j)} = \frac{1}{\beta} \left\{ \frac{1}{\beta} \int_0^T \phi_j(t) Q_j(t) dt + \phi_j(t) Q_j(t) dt + Q_j(t) Q_j(t) \right\}$$

$$= \frac{1}{\beta} \int_0^T \phi_j(t) Q_j(t) dt + \frac{1}{\beta} \int_0^T Q_j(t) Q_j(t) dt + \frac{1}{\beta} \left[ \phi_j(t) Q_j(t) + \phi_j(t) Q_j(t) \right]$$
Let \( f(t) = \sum_{n=0}^{h} K_n \Phi_n(t), \) where \( \Phi_n(t) = P_n(\frac{2t}{T} - 1) \) are Legendre Polynomials. To evaluate \( \Phi_{ij} \) it is necessary to determine
\[
\int_0^T \Phi_i(t) \Phi_j(t) \, dt,
\]
and we therefore use the relation for Legendre Polynomials from [5]
\[
(3.6) \quad \Phi_{n+1}(t) = (2n+1) \Phi_n(t) + \Phi_{n-1}(t)
\]
to obtain:
\[
\begin{align*}
\Phi_i'(t) &= (2i-1) \Phi_{i-1}(t) + \Phi_{i-2}(t) \\
\Phi_{i-2}'(t) &= (2i-5) \Phi_{i-3}(t) + \Phi_{i-4}(t) \\
\Phi_{i-4}'(t) &= (2i-9) \Phi_{i-5}(t) + \Phi_{i-6}(t) \\
&\quad \vdots \\
\Phi_{i-m}'(t) &= [2i-(2m+1)] \Phi_{i-(m+1)} + \Phi_{i-(m+2)},
\end{align*}
\]
where \( m = \begin{cases} \text{The first even number less than } i \text{ for } i > 2 \\ 0 \text{ if } i = 1 \text{ or } 2 \end{cases} \)

Thus
\[
(3.7) \quad \Phi_i'(t) = (2i-1) \Phi_{i-1}(t) + (2i-5) \Phi_{i-3} + \ldots + [2i-(2m+1)] \Phi_{i-(m+1)}
\]

Note: If \( i \) is even the last terms are \( 3\Phi_1(t) + \Phi_0(t) = 3\Phi_1(t) \)
If \( i \) is odd the last terms are \( \Phi_0(t) + \Phi_{-1}'(t) = \Phi_0(t) = 1 \)

Hence \( \int_0^T \Phi_i'(t) \Phi_j'(t) \, dt = 0 \) if either \( i \) or \( j \) is even and the other is odd.
(3.8) \[ T \sum_{n=0}^{i-j} [2i-(ln+1)] = \frac{T}{2} i(i+1) \] if both $i$ and $j$

are even and $i \leq j$.

\[ = T \sum_{n=0}^{i-j} [2i-(ln+1)] = \frac{T}{2} i(i+1) \] if both $i$ and $j$

are odd and $i \leq j$.

Thus $\phi_{ij}(a_r)$ may be determined and we have

\[ \phi_{ij}(a_r) = \frac{1}{2} \left[ \phi_{ij}(T) + \phi_{ij}(0) \right] \] if either $i$ or $j$

is even and the other is odd.

\[ = \frac{T}{4} i(i+1) + \frac{3}{2} \left[ \phi_{ij}(T) + \phi_{ij}(0) \right] \] if both $i$ and $j$

are even (or odd) and $i < j$.

\[ = \frac{T}{4} i(i+1) + \frac{3}{2} \left[ \phi_{ij}(T) + \phi_{ij}(0) \right] \] if both $i$ and $j$

are even (or odd) and $i = j$.

\[ = \frac{3}{2} \left[ \phi_{ij}(T) + \phi_{ij}(0) \right] \] if either $i$ or $j$ is 0.

\[ = \frac{3}{2} T + 1 \] if $i$ and $j$ are 0.

The matrix $\left( \phi_{ij}(a_r) \right)$ is a symmetric matrix which must be inverted
to evaluate $K_{ij}$ and $\sum_{i} P_{ij}$ for $i, j = 0, \ldots, h$.

The matrix $\left( \phi_{ij}(a_r) \right)$ takes the following forms with the
inverses $\left( \phi_{ij}(a_r) \right)$ also given. From (3.9) we have:

Second order: $\left( \phi_{ij}(a_r) \right) = \left( \begin{array}{cc} \frac{T+1}{T} & 0 \\ 0 & \frac{T+1}{T+1} \end{array} \right) \quad i, j = 0, 1$

\[ \left( \phi_{ij}(a_r) \right) = \left( \begin{array}{cc} \frac{2}{T+2} & 0 \\ 0 & \frac{6T}{3T+6T+6T} \end{array} \right) \]
Third order: \( \left( \phi_{ij}^{(a_r)} \right) = \begin{pmatrix} \frac{\partial^T}{\partial t^2} + 1 & 0 & 0 \\ 0 & \frac{T}{\alpha} + \frac{\partial^T}{\partial t} + 1 & 0 \\ 0 & 0 & \frac{\partial^T}{\partial t} + \frac{\partial T}{\alpha} + 1 \end{pmatrix} \)

\( i, j = 0, 1, 2 \)

Let \( \phi_{ij}^{(a_r)} \) denote the determinant of the matrix \( \left( \phi_{ij}^{(a_r)} \right) \) then we have

\[
\mid \phi_{ij}^{(a_r)} \mid = \left( \frac{\partial^T}{\partial t^2} + 1 \right) \left( \frac{T}{\alpha} + \frac{\partial^T}{\partial t} + 1 \right) \left( \frac{\partial^T}{\partial t} + \frac{\partial T}{\alpha} + 1 \right) - \left( \frac{T}{\alpha} + \frac{\partial T}{\alpha} + 1 \right).
\]

Hence

\[
\left( \phi_{ij}^{(a_r)} \right) = \frac{1}{\mid \phi_{ij}^{(a_r)} \mid} \begin{pmatrix} \left( \frac{T}{\alpha} + \frac{\partial^T}{\partial t} + 1 \right) \left( \frac{\partial^T}{\partial t} + \frac{\partial T}{\alpha} + 1 \right) & 0 & -\left( \frac{T}{\alpha} + \frac{\partial T}{\alpha} + 1 \right) \\ 0 & \left( \frac{T}{\alpha} + \frac{\partial^T}{\partial t} + 1 \right) \left( \frac{\partial^T}{\partial t} + \frac{\partial T}{\alpha} + 1 \right) - 1 & 0 \\ -\left( \frac{T}{\alpha} + \frac{\partial T}{\alpha} + 1 \right) & 0 & \left( \frac{T}{\alpha} + \frac{\partial^T}{\partial t} + 1 \right) \end{pmatrix}
\]


From (2.42)

\[
\hat{R}_1 = \sum_{j=1}^{s} \Phi_{ij} \int_0^T \mathcal{Q}_j(t) \, dy \text{ where } \Phi_{ij} = \int_0^T \mathcal{Q}_i(t) \mathcal{Q}_j(t) \, dt
\]

and from (2.43)

\[
\overline{\sigma}_{K_j} = 2 \beta \sigma^2 \alpha^2 \int_0^T \alpha \sum \sum \alpha \mathcal{Q}_j(t) \mathcal{Q}_j(t) \mathcal{Q}_i(t) e^{-\beta |t-t'|} \, dt \, dt
\]

Let \( f(t) = \sum_{n=0}^{s} \mathcal{Q}_n(t) \) where \( \mathcal{Q}_n(t) \) are Legendre Polynomials of the form \( \mathcal{Q}_n(t) = P_n \left( \frac{2t}{T} - 1 \right) \) where \( P_n \) is the \( n \)th Legendre Polynomial.
To calculate $\mathbf{M}_{ij}$ we use the expansion (3.7) for $Q^i_j(t)$ and $Q_j^i(t)$ and using the results from (3.8) we get

$$\int_0^T Q^i_j(t) Q_j^i(t) \, dt = 0 \text{ if either } i \text{ or } j \text{ is even and the other is odd.}$$

$$= \frac{T}{2} \delta(i+1) \text{ if both } i \text{ and } j \text{ are even (or odd)}$$

and $i \leq j$ also $i, j > 0$.

Therefore

$$\begin{pmatrix} \alpha_{ij} \end{pmatrix} = \begin{pmatrix} T & 0 & 0 & \cdots & 0 \\ 0 & 3T & 0 & 3T & \cdots & 3T \\ T & 0 & 6T & 0 & \cdots & 0 \\ 0 & 3T & 0 & 10T & 0 & \cdots & 10T \\ T & 0 & 6T & 0 & 15T & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 3T & 0 & 10T & 0 & \cdots & \frac{T}{2} s(i+1) \end{pmatrix}$$

Denoting the inverse of $\begin{pmatrix} \alpha_{ij} \end{pmatrix}$ by $\begin{pmatrix} \alpha^{ij} \end{pmatrix}$ we have:

Second Order

$$\begin{pmatrix} \alpha^{ij} \end{pmatrix} = \begin{pmatrix} \frac{1}{T} & 0 \\ 0 & \frac{1}{3T} \end{pmatrix}$$

(3.11)

Third Order

$$\begin{pmatrix} \alpha^{ij} \end{pmatrix} = \begin{pmatrix} \frac{1}{T} + \frac{1}{5T} & 0 & -\frac{1}{5T} \\ 0 & \frac{1}{5T} & 0 \\ -\frac{1}{5T} & 0 & \frac{1}{5T} \end{pmatrix}$$

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Fourth Order

\[
(\mathbf{M}^{ij}) = \begin{pmatrix}
\frac{1}{4} + \frac{i}{2\pi} & 0 & -\frac{1}{2\pi} & 0 \\
0 & \frac{1}{2\pi} + \frac{i}{2\pi} & 0 & -\frac{1}{2\pi} \\
-\frac{1}{2\pi} & 0 & \frac{1}{2\pi} & 0 \\
0 & -\frac{1}{2\pi} & 0 & \frac{1}{2\pi}
\end{pmatrix}
\]

Fifth Order

\[
(\mathbf{M}^{ij}) = \begin{pmatrix}
\frac{1}{4} + \frac{i}{2\pi} & 0 & -\frac{1}{2\pi} & 0 & 0 \\
0 & \frac{1}{2\pi} + \frac{i}{2\pi} & 0 & -\frac{1}{2\pi} & 0 \\
-\frac{1}{2\pi} & 0 & \frac{1}{2\pi} & 0 & 0 \\
0 & -\frac{1}{2\pi} & 0 & \frac{1}{2\pi} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2\pi}
\end{pmatrix}
\]

In order to evaluate \( \sum_{k_i} \mathbf{M}_{ij} \), as given above, we must calculate

\[
\iint T' T' \mathcal{Q}_{t'}(t) \mathcal{Q}_{t'}^*(t') e^{-\beta[T-t]} dt dt'
\]

Again using (3.7) we have

\[
(3.12) \int \int T \mathcal{T} \mathcal{Q}_{\alpha}(t) \mathcal{Q}_{\alpha}^*(t) e^{-\beta[T-t]} dt dt' = \sum \sum [2 \alpha - (4 \omega - 2)][2 \gamma - (4 h - 2)]
\]

\[
\int \int T \mathcal{T} \mathcal{Q}_{\alpha}(t) \mathcal{Q}_{\alpha}^*(t) e^{-\beta[T-t]} dt dt'
\]

where the limits of the sum over \( \alpha \) are 1 to \( \frac{\alpha}{2} \), if \( \alpha \) is even,

and 1 to \( \frac{\alpha + 1}{2} \) if \( \alpha \) is odd. Similarly for the sum over \( h \) in
terms of $\tau$. To simplify the notation in the calculations in (3.12) we let $i = \kappa - 2\omega + 1$ and $j = \kappa - 2\eta + 1$.

The calculation of $\sum_{i,j} \sum_{\sigma} \alpha_{\sigma}(i,j) e^{-\sigma|t-t'|} dt dt'$ is given by equations (3.3), (3.4) and (3.5).

Thus with the inverse of $(\mathbf{A}_{ii})$, in equations (3.11) and the evaluation of (3.12) we may determine $\overline{r_i e_i}$.

Section 2: The Mean Value Function Is A Trigonometric Polynomial.

We now consider a mean value function of the form

\[(3.13) \quad f(t) = a_0 + \sum_{n=1}^{s} (a_n \cos nt + b_n \sin nt)\]

The covariance matrix for each of the three estimates (Sections 2, 3 and 4 of Chapter II) is computed.

Part I: Estimation of $K_i$ When $\varphi$ is not Known

From (2.8), (2.9) and (2.10) we have for the Least Square Estimate

\[\overline{r_i} = \sum_{j=1}^{s} \psi_{ij} \int_{0}^{T} y_t \alpha_j(t) dt\]

where $\psi_{ij} = \int_{0}^{T} \alpha_i(t) \alpha_j(t) dt$. If the $\alpha_i(t)$ are orthogonal then

\[\overline{r_i^2 e_i^2} = \sigma^2 \psi_{ii} \int_{0}^{T} \int_{0}^{T} \alpha_i(t) \alpha_j(t') e^{-\sigma|t-t'|} dt dt'\]

Let $T = 2\pi$ and $f(t) = a_0 + \sum_{n=1}^{s} (a_n \cos nt + b_n \sin nt)$.

If $\alpha_i(t) = \sin it$ then $\alpha_i'(t) = i \cos it$ and
if $\varphi_i(t) = \cos it$ then $\varphi_i'(t) = -i \sin it$.

We have the following forms for $\psi_{nm}$, putting $a_{s+n} = b_n$

\[
\psi_{nn} = \int_0^T \sin^2 nt \, dt = \pi \lambda \quad \text{if } n \leq s
\]

\[
= \int_0^T \cos^2 nt \, dt = \pi \lambda \quad \text{if } n > s
\]

\[
= \int_0^T dt = 2\pi \lambda \quad \text{if } n = 0
\]

\[
\psi_{nm} = \int_0^T \sin nt \sin nt \, dt = 0 \quad \text{if } n, m \leq s
\]

\[
= \int_0^T \sin nt \cos mt \, dt = 0 \quad \text{if } n \leq s, \ m > s
\]

\[
= \int_0^T \cos nt \cos mt \, dt = 0 \quad \text{if } n, m > s
\]

\[
= \int_0^T \sin mt \, dt = 0 \quad \text{if } n > 0, \ m \leq s
\]

\[
= \int_0^T \cos mt \, dt = 0 \quad \text{if } n = 0, \ m > s
\]

Therefore

\[
\psi_{nn} = \frac{1}{\pi \lambda} \quad \text{if } 0 \leq n \leq 2s
\]

(3.14) \quad = \frac{1}{2\pi \lambda} \quad \text{if } n = 0

\[
\psi_{nm} = 0 \quad n \neq m
\]

Substituting into (2.10) and for $n > 0, m > 0$, we have:
\[ q_{kn} = \frac{1}{2\pi^2} \int_0^T \int_0^T \cos nt \cos mt \ e^{-\beta(t-t')} \ dt \ dt' \]

\[ q_{kn}^{*} = \frac{1}{2\pi^2} \int_0^T \int_0^T \cos nt \ sin mt \ e^{-\beta(t-t')} \ dt \ dt' \]

\[ a_{kn} = \frac{1}{2\pi^2} \int_0^T \int_0^T \sin nt \ sin mt \ e^{-\beta(t-t')} \ dt \ dt' \]

\[ a_{kn}^{*} = \frac{1}{2\pi^2} \int_0^T \int_0^T \cos nt \ e^{-\beta(t-t')} \ dt \ dt' \]

\[ a_{kn}^{**} = \frac{1}{2\pi^2} \int_0^T \int_0^T \sin nt \ e^{-\beta(t-t')} \ dt \ dt' \]

\[ a_{kn}^{***} = \frac{1}{2\pi^2} \int_0^T \int_0^T \ e^{-\beta(t-t')} \ dt \ dt' \]

Using the decomposition

\[ \int_0^T \int_0^T \sum_{n,m \geq 0} \left( q_{kn} \cos nt \cos mt + q_{kn}^{*} \cos nt \ sin mt + a_{kn} \ sin nt \ sin mt + a_{kn}^{*} \ cos nt \ e^{-\beta(t-t')} + a_{kn}^{**} \ sin nt \ e^{-\beta(t-t')} + a_{kn}^{***} \ e^{-\beta(t-t')} \right) \ dt \ dt' \]

where

\[ \int_0^T \int_0^T \cos nt \ cos mt \ e^{-\beta(t-t')} \ dt \ dt' = \delta_{nn} \ \frac{\pi}{\beta^2 + m^2} - \frac{\beta^2}{(\beta^2 + m^2)^2} \ (1 - e^{-2\pi\lambda}) \]

\[ \int_0^T \int_0^T \sin nt \ cos mt \ e^{-\beta(t-t')} \ dt \ dt' = \delta_{nn} \ \frac{\pi}{\beta^2 + m^2} - \frac{2\beta^2}{(\beta^2 + m^2)^2} \ (1 - e^{-2\pi\lambda}) \]

\[ \int_0^T \int_0^T \cos mt \ sin nt \ e^{-\beta(t-t')} \ dt \ dt' = \delta_{nn} \ \frac{\pi}{\beta^2 + m^2} + \frac{2\beta^2}{(\beta^2 + m^2)^2} \ (1 - e^{-2\pi\lambda}) \]

\[ \int_0^T \int_0^T \sin nt \ sin nt \ e^{-\beta(t-t')} \ dt \ dt' = \delta_{nn} \ \frac{\pi}{\beta^2 + m^2} + \frac{m n}{(\beta^2 + m^2)^2} \ (1 - e^{-2\pi\lambda}) \]

We get from (3.15) and (3.16) for \( n \geq 0, \ m \geq 0 \)
(3.17) \[ \sigma_{\tilde{a}_n\tilde{b}_m} = \frac{2\sigma^2}{\pi^2 \lambda^2} \left[ \delta_{nm} \frac{\pi \lambda \beta}{\beta^2 + \mu^2} - \frac{\beta^2}{(\beta^2 + \mu^2)(\beta^2 + \mu^2)} \left(1 - e^{-2\pi \lambda \beta} \right) \right] \]

(3.18) \[ \sigma_{\tilde{a}_n\tilde{b}_m} = 0 \]

(3.19) \[ \sigma_{\tilde{a}_n\tilde{b}_m} = 0 \]

(3.20) \[ \sigma_{\tilde{b}_n\tilde{b}_m} = \frac{2\sigma^2}{\pi^2 \lambda^2} \left[ \delta_{nm} \frac{\pi \lambda \beta}{\beta^2 + \mu^2} + \frac{\mu^2}{(\beta^2 + \mu^2)(\beta^2 + \mu^2)} \left(1 - e^{-2\pi \lambda \beta} \right) \right] \]

(3.21) \[ \sigma_{\tilde{a}_n\tilde{a}_n} = \frac{\sigma^2}{\pi^2 \lambda^2} \left(1 - e^{-2\pi \lambda \beta} \right) \]

(3.22) \[ \sigma_{\tilde{a}_n^2} = \frac{\sigma^2}{4\pi^2 \lambda^2} \left[ \frac{4\pi \lambda}{\beta} - \frac{2}{\beta^2} \left(1 - e^{-2\pi \lambda \beta} \right) \right] \]

**Part II: Estimation of \( K \) When \( \beta \) is Known**

A. If the time \( T \) during which the process is observed is large and if we do not use the value of \( y_0 \) then the matrix \( (\phi)^{a_0}_{(a_0)} \) simplifies and adjusting (2.39), (2.40) and (2.41) we have

\[ \phi_{ij}^{(a_0)} = \int_0^T \phi_i(t) \phi_j(t) \, d\theta(t) \]

(3.23) \[ = \frac{1}{\beta} \int_0^T \phi_i(t) \phi_j(t) \, dt + \beta \int_0^T \phi_i(t) \phi_j(t) \, dt + \phi_{tj} \phi_{ji}(T) - \phi_{ti} \phi_{j} \phi_{ij}(0) \]

Let \( f(t) = a_0 + \sum_{n=1}^s [a_n \cos nt + b_n \sin nt] \) and \( T = 2\pi \lambda \), then putting \( a_{s+n} = b_n \) and for \( i = j > 0 \) we have:

\[ \phi_{ij}^{(a_0)} = \frac{1}{\beta} \int_0^T \phi'_i(t)^2 \, dt + \beta \int_0^T \phi_i(t)^2 \, dt = \frac{T \lambda}{\beta} (i^2 + \beta^2) \]

and for \( i \neq j \) and \( i, j > 0 \),

\[ \phi_{ij}^{(a_0)} = \phi_i(T) \phi_j(T) - \phi_i(0) \phi_j(0) = 0 \]

and for \( i = j = 0 \), \( \phi_{00}^{(a_0)} = 2(\beta T \lambda) \)

and for \( j = 0, i > 0 \), \( \phi_{ij}^{(a_0)} = 0 \).
Since we are not using the value \( y_0 \) we have 
\[
\sigma_{k_i k_j} = 2 \sigma_i^2 \phi_i^{(a_i)}
\]
hence we have:

(3.24) \[
\sigma_{a_n}^2 = \sigma_{b_n}^2 = \frac{2 \beta \sigma_i^2}{\pi \lambda (\beta^2 + n^2)} \quad \text{if } n > 0
\]

(3.25) \[
\sigma_{a_n a_m} = \sigma_{b_n b_m} = \sigma_{a_n b_m} = 0 \quad \text{if } n \neq m \text{ and } n, m > 0
\]

(3.26) \[
\sigma_{a_n a_n} = \sigma_{b_n b_n} = 0 \quad \text{if } n > 0
\]

(3.27) \[
\sigma_{a_0}^2 = \frac{2 \sigma_i^2}{\beta \pi \lambda}
\]

B. If we consider the observation at \( y_0 \) and compute the estimates of the parameters we have from (2.39)

\[
\phi_i^{(a_i)} = \frac{1}{2} \int_0^\infty \left( \Psi_i(t) \Psi_i^{(a_i)}(t) d(t) + \Omega_i(t) \Omega_i^{(a_i)}(t) \right)
\]

\[
= \frac{1}{2 \beta} \left\{ \int_0^\infty \Psi_i(t) \Psi_i'(t) dt + \frac{1}{2} \int_0^\infty \Omega_i(t) \Omega_i'(t) dt + \frac{1}{2} \left\{ \Omega_i(t) \Omega_i(t) + \Omega_i(t) \Omega_i(t) \right\} \right\}
\]

When \( \Psi_i(t) = \sin \lambda t \) then \( \Psi_i'(t) = i \cos \lambda t \), or if \( \Omega_i(t) = \cos \lambda t \)

then \( \Omega_i'(t) = -j \sin \lambda t \) and we have putting \( a_{n+1} = b_n \)

\[
\phi_i^{(a_i)} = \frac{e^{-2 \pi \lambda / \beta}}{2 \pi} + \frac{e^{2 \pi \lambda / \beta}}{2 \pi} + 1 \quad \text{if } i > s
\]

(3.28) \[
= \frac{e^{-2 \pi \lambda / \beta}}{2 \pi} + \frac{e^{2 \pi \lambda / \beta}}{2 \pi} + 1 \quad \text{if } 0 < i \leq s
\]

\[
\phi_i^{(a_i)} = 1 \quad \text{if } i \neq j \text{ and } 0 < i, j \leq s
\]

\[
= 0 \quad \text{if } i \neq j \text{ and } i, j > s
\]

(3.29) \[
\phi_{i,j}^{(a_i)} = 1 \quad \text{if } i = 0 \text{ and } 0 < j \leq s
\]

\[
= 0 \quad \text{if } i = 0 \text{ and } j > s
\]
Hence the matrix \( \phi_{ij}^{(a_{ni})} \), given by (3.28) and (3.29), must be inverted to evaluate the estimates \( \hat{a}_i, \hat{b}_i \) and their covariances.

Suppose \( f(t) = a_0 + \sum_{n=1}^{2} (a_n \cos nt + b_n \sin nt) \) then we have

\[
\phi_{ij}^{(a_{ni})} = \begin{pmatrix}
\frac{\pi \lambda + \frac{\pi \lambda}{2} + \frac{\pi \lambda}{2}}{2} & 1 & 0 & 0 \\
1 & \frac{\pi \lambda + \frac{\pi \lambda}{2}}{2} + 1 & 0 & 0 \\
0 & 0 & \frac{\pi \lambda}{\beta} + \frac{\pi \lambda}{2} & 0 \\
0 & 0 & 0 & \frac{\pi \lambda}{\beta} + \frac{\pi \lambda}{2}
\end{pmatrix}
\]

or \( \phi_{ij}^{(a_{ni})} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \) where \( A \) is the upper left three by three block and \( B \) is the lower right two by two block. Then to find \( \phi_{ij}^{(a_{ni})} \), the inverse of \( \phi_{ij}^{(a_{ni})} \), we first compute \( A^{-1} \) and \( B^{-1} \) (that is, considering \( A \) and \( B \) as matrices).

\[
A^{-1} = \frac{1}{|A|} \begin{pmatrix}
\left(\frac{\pi \lambda}{2} + \frac{\pi \lambda}{\beta} \right) \left(\frac{\pi \lambda}{\beta} + \frac{\pi \lambda}{2} \right) & 1 & 0 & 0 \\
1 & \left(\frac{\pi \lambda}{\beta} + \frac{\pi \lambda}{2} \right) & 1 & 0 \\
0 & 0 & \left(\frac{\pi \lambda}{\beta} + \frac{\pi \lambda}{2} \right) & 1 \\
0 & 0 & 1 & \left(\frac{\pi \lambda}{\beta} + \frac{\pi \lambda}{2} \right)
\end{pmatrix}
\]
where \[ |A| = \left( \frac{\pi a}{2} + \frac{\pi a}{2} \right) \left( \frac{\pi A}{2} \right) + \left[ 1 - \left( \frac{\pi A}{2} + \frac{\pi A}{2} \right) \right] \]

and \[ B^{-1} = \begin{pmatrix} \frac{9 A^2}{B + \frac{\pi A}{2}} & 0 \\ 0 & \frac{\pi A}{B + \frac{\pi A}{2}} \end{pmatrix} \]

Therefore \[ \left( \phi_i(A, B) \right) = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} \]

Part III: F.R.P. Estimates if the Process is an O.U.P.

We have from (2.142) and (2.143) \[ \hat{R}_i = \sum_{j=1}^{s} \phi_{ij} \int_0^T \phi_j'(t) \, dt \]

where \[ \phi_{ij} = \int_0^T \phi_i(t) \phi_j(t) \, dt \]

and if the \( \phi_i(t) \) are orthogonal then

\[ C_{j_1 j_2} = \delta_{j_1 j_2} \left( \phi_1^2 + \phi_2^2 - \phi_1^2 \phi_2^2 \right) \int_0^T \phi_1'(t) \phi_2'(t) e^{-\left( \phi_1^2 + \phi_2^2 \right) t} \, dt \]

Let \( f(t) = a_0 + \sum_{n=1}^{s} \left( a_n \cos nt + b_n \sin nt \right) \) and \( T = 2 \pi \lambda \). Putting \( a_{s+n} = b_n \) we have for \( n, m > 0 \):

\[ \phi_{nm} = n^2 \pi \lambda \] if \( n = m \)

and

\[ \phi_{nm} = 0 \] if \( n \neq m \)

Therefore

\[ \phi_{nm} = \frac{1}{n^2 \pi \lambda} \] if \( n = m \)

and

\[ \phi_{nm} = 0 \] if \( n \neq m \).
Hence \( \Omega \) takes the following forms:

\[
\begin{align*}
\sigma_{\alpha}^2 &= \frac{2\pi^2}{n^2\pi^2} - \frac{2\pi^2}{n^2\pi^2} \int_0^\infty \int_0^\infty n^2 \sin nt \sin nt \ e^{-\beta|t-t'|} \ dt \ dt' \\
\sigma_{\beta}^2 &= \frac{2\pi^2}{n^2\pi^2} \int_0^\infty \int_0^\infty nm \sin nt \sin mt \ e^{-\beta|t-t'|} \ dt \ dt' \quad \text{if } n \neq m \\
\sigma_{\alpha \beta}^2 &= \frac{2\pi^2}{n^2\pi^2} \int_0^\infty \int_0^\infty -nm \sin nt \cos mt \ e^{-\beta|t-t'|} \ dt \ dt' \\
\sigma_{\alpha}^2 &= \frac{2\pi^2}{n^2\pi^2} \int_0^\infty \int_0^\infty n^2 \cos nt \cos nt \ e^{-\beta|t-t'|} \ dt \ dt' \\
\sigma_{\beta}^2 &= \frac{2\pi^2}{n^2\pi^2} \int_0^\infty \int_0^\infty -nm \cos nt \cos mt \ e^{-\beta|t-t'|} \ dt \ dt' \quad \text{if } n \neq m
\end{align*}
\]

Using the decomposition

\[
\int_0^T \int_0^T f(t) g(t') e^{-\beta|t-t'|} \ dt \ dt' = \int_0^T \int_0^T f(t) \exp(-\beta(t-t')) \ dt \ dt' + \int_0^T \int_0^T f(t') \exp(-\beta(t'-t)) \ dt \ dt'
\]

and the results of equations (3.16) we have for \( n, m > 0 \)

\[
\begin{align*}
(3.30) \quad \sigma_{\alpha n}^2 &= \frac{2\pi^2}{n^2\pi^2} - \frac{2\pi^2}{n^2\pi^2} \left\{ \frac{\pi n \beta}{\beta^2 + n^2} + \frac{n^2}{\beta^2 + n^2} \right\} \left(1 - e^{-2\pi n^2}\right) \\
(3.31) \quad \sigma_{\beta n}^2 &= \frac{2\pi^2}{n^2\pi^2} \left(1 - e^{-2\pi n^2}\right), \quad n \neq m \\
(3.32) \quad \sigma_{\alpha \beta}^2 &= 0 \\
(3.33) \quad \sigma_{\alpha n}^2 &= \frac{2\pi^2}{n^2\pi^2} - \frac{2\pi^2}{n^2\pi^2} \left\{ \frac{\pi n \beta}{\beta^2 + n^2} - \frac{n^2}{\beta^2 + n^2} \right\} \left(1 - e^{-2\pi n^2}\right) \\
(3.34) \quad \sigma_{\beta n}^2 &= \frac{2\pi^2}{n^2\pi^2} \left(1 - e^{-2\pi n^2}\right), \quad n \neq m
\end{align*}
\]

Part IV: The Relative Efficiency of the Least Square Estimates

When \( f(t) \) is a Trigonometric Polynomial.
We are also interested in the comparison of the Least Square Estimate and the $a_r$ Estimate. From \([4]\) we have the following definition.

**Definition:** Let $\overline{K}_1, \ldots, \overline{K}_s$ be any estimate of $K_1, \ldots, K_s$ we write 
$$\bar{f}(t) = \overline{K}_1 a_1(t) + \ldots + \overline{K}_s a_s(t),$$
then the ratio
$$e = \frac{E\left\{ \int_0^T [f(t) - \bar{f}(t)]^2 dt \right\}}{E\left\{ \int_0^T [f(t) - \bar{f}(t)]^2 dt \right\}}$$
will be called the relative efficiency of the set of estimates $\overline{K}_1, \ldots, \overline{K}_s$. Also for a sequence of estimates $\overline{K}_1, \ldots, \overline{K}_s$, depending on $T$, the expression \( \lim_{T \to \infty} e \) will be called the asymptotic efficiency of the estimates $\overline{K}_1, \ldots, \overline{K}_s$.

Consider the relative efficiency of the Least Square Estimates, of Part I, and the $a_r$ Estimates of Part II, Section 2. From the definition of $e$ we must compute 
$$E\left\{ \int_0^T [f(t) - \bar{f}(t)]^2 dt \right\}$$
and 
$$E\left\{ \int_0^T [f(t) - \bar{f}(t)]^2 dt \right\}.$$ 
Let \( T = 2 \pi \lambda \) and 
$$\hat{f}(t) = a_0 + \sum_n (\hat{a}_n \cos nt + \hat{b}_n \sin nt)$$
then from \((3.24)\) and \((3.27)\) we have
$$E\left\{ \int_0^T [f(t) - \hat{f}(t)]^2 dt \right\}$$
$$= E\left\{ \int_0^T \left( a_0 - \hat{a}_0 \right)^2 + \sum_n \sum_m (a_n - \hat{a}_n)(a_m - \hat{a}_m) \cos nt \cos mt \right\}$$
\[
+ \sum_{n} \sum_{m} (b_n \hat{b}_n)(b_m \hat{b}_m) \sin nt \sin mt + 2 \sum_{n} (a_0 \hat{a}_0)(a_n \hat{a}_n) \cos nt
\]

\[
+ 2 \sum_{n} (a_0 \hat{a}_0)(b_n \hat{b}_n) \sin nt + 2 \sum_{n} (a_n \hat{a}_n)(b_m \hat{b}_m) \cos nt \sin mt \] \ dt
\]

(3.35) \quad = \frac{25^2}{\theta} + \sum_{n=1}^{\infty} \frac{2 \theta \sigma^2}{\beta_n^2 + \sigma^2}

In a similar manner from (3.17) and (3.22), we have:

(3.36) \quad \mathbb{E} \left\{ \int_{0}^{T} [f(t)-\tilde{f}(t)]^2 \ dt \right\} = \frac{25^2}{\theta} + \sum_{n=1}^{\infty} \frac{2 \theta \sigma^2}{\beta_n^2 + \sigma^2} - \frac{\sigma^2}{\beta_n^2}(e^{-2\beta_n^2}) \left[ \frac{1}{\theta} + \frac{2 \sigma^2}{\beta_n^2} \right] - \frac{2 \sigma^2}{\beta_n^2} \frac{2 \sigma^2}{\beta_n^2}

Thus the relative efficiency \( \varepsilon \) will be close to one and the two estimates differ very little.

For example if we have the following values for \( \beta, \lambda \) and \( s \) we find the resulting values of \( \varepsilon \) using (3.35) and (3.36).

- If \( \beta = 1, \lambda = 2, s = 1 \) then \( \varepsilon = 1.0585 \)
- If \( \beta = 1, \lambda = 6, s = 2 \) then \( \varepsilon = 1.0187 \)
- If \( \beta = \frac{3}{2}, \lambda = 2, s = 1 \) then \( \varepsilon = 1.1394 \)

Using the same conditions for \( \beta, \lambda \) and \( s \) as in the previous example, consider the relative efficiency computed when \( y_0 \) is utilized. That is, the numerator of \( \varepsilon \) is evaluated using (3.28) and (3.29) and the denominator of \( \varepsilon \) is given by (3.36). We have the following results:

- If \( \beta = 1, \lambda = 2, s = 1 \) then \( \varepsilon = 0.9468 \)
- If \( \beta = 1, \lambda = 6, s = 2 \) then \( \varepsilon = 0.9780 \)
- If \( \beta = \frac{3}{2}, \lambda = 2, s = 1 \) then \( \varepsilon = 0.9257 \)
Section 3: The Efficiency of the Least Square Estimate

In the previous section, we computed the relative efficiency of the Least Square and $a_\tau$ Estimates when the observation $y_0$ was not used. This in many instances, with a large number of observations, can readily be done. However, we are also interested in the comparison of the Least Square and $a_\tau$ Estimates when $y_0$ is utilized. Again referring to [1] we have the following results for Trigonometric Polynomials and Polynomials.

Theorem 1: If $f(t) = a_0 + \sum_{n=1}^{S} [a_n \cos nt + b_n \sin nt]$ then the asymptotic efficiency of the Least Square Estimates of the $a_n$ and $b_n$ is unity.

Theorem 2: If $f(t)$ is a polynomial then the asymptotic efficiency of the Least Square Estimates $k_i$ is unity.

These results are important since $\theta$ will rarely be known, and the Least Square Estimates can be carried out without such knowledge.


I, Walter Wilson Hoy, was born in St. Louis, Missouri, June 3, 1924. I received my secondary school education in the public schools of the city of Houston, Texas. My undergraduate training was obtained at Hendrix College in Conway, Arkansas. This training was interrupted for three years while I served in the U.S. Navy during World War II. After the war, I returned to Hendrix and received the degree Bachelor of Science in 1947. From the Ohio State University, I received the degree Master of Arts in 1949. While in residence at the Ohio State University I had a teaching assistantship in the Department of Mathematics. After completing my work for the Masters degree, I remained in residence at the Ohio State University and specialized again in the Department of Mathematics. While completing the requirements for the degree Doctor of Philosophy I held several teaching assistantships and for one year held an Assistant Instructorship.