A STUDY OF SPACE-CHARGE-LIMITED
POTENTIAL DISTRIBUTION IN
ELLIPSOIDAL AND PARABOLOIDAL DIODES

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

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Approved by:

E. M. Boone
Adviser
Department of Electrical Engineering
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Chapter I

Introduction

The purpose of this investigation is to study space-charge-limited potential distributions in certain new geometries, with a view to developing better high-density electron beams. The problems studied include confocal and nonconfocal ellipsoidal and paraboloidal geometries.

The growth of electron optics began with the invention of the electron microscope and the cathode ray tube. These devices use sharp, well-defined, electron beams. However, currents required are usually small, and the familiar techniques in geometrical optics and other empirical procedures are quite adequate for most of the problems.

During the last fifteen years, progress in electron optics has been greatly accelerated with the evolution of modern microwave tubes, klystrons, magnetrons and travelling wave tubes. The requirements placed on electron beams necessary in these devices are very stringent. Although it is difficult to define the characteristics of an ideal beam, the following are some of the common desirable features in most types of beams:

(a) The beam has a very small diameter and a high current density.

(b) The profile of the beam changes very little over longitudinal distances many times larger than its transverse dimension.
(c) The beam perveance is very high. This parameter effectively expresses the performance of a beam in a given device, and is defined as \( \frac{I}{V^{3/2}} \), where \( I \) is the total current in the beam, and \( V \) is the accelerating voltage.

(d) All the electrons in any transverse section of the beam have the same velocity, and electron flow is laminar.

Major limitations associated with such beams are:

(a) Emission densities available at the cathodes;

(b) Space charge spread;

(c) Spread due to thermal velocities.

It is proposed to begin with a review of important developments in design and analysis of dense electron beams.

**Cathode Materials**

An ideal electron emitting material should be capable of giving large current density at moderate temperature, be insensitive to ion bombardment, be easily fabricated, and have long life. The conventional oxide cathode discovered some thirty years ago is still largely relied upon by the industry. It has probably reached the peak of its development at about 2 amps/cm\(^2\) of constant current density around 800\(^0\)C. During the last few years, many new types of cathodes have been reported. The most promising types are probably (1) the L-cathode developed at Philips Laboratories in 1950 and (2) the nickel matrix cathode first reported by McNair et al in 1953. The latter type has
found many enthusiastic supporters in this Laboratory. Clifton\(^3\) has described the technique of fabricating such cathodes and has given an evaluation of their performance. A current density of 10 amps/cm\(^2\) can be obtained at 1000\(°\) C. An important feature of this cathode is its mechanical ruggedness and remarkable resistance to poisoning from ion bombardment.

**Theory of Space Charge**

Since the current density available from present-day cathodes is severely limited, the usual procedure to get a large current density is to have a large emitting surface and to converge the emitted current to a small diameter by suitable electrostatic focussing. The limiting factors in converging a beam to an arbitrarily small diameter are the mutual repulsive forces of electrons and the initial transverse velocities with which electrons are emitted.

The mathematics involved in determining the beam profile and current available from a given electrode arrangement consists of the solution of Maxwell's divergence equation under static conditions:

\[
(1) \quad \text{Div } D = \rho
\]
\[
(2) \quad \text{or Div } E = \rho/\varepsilon
\]

Here, $D$ is the electric flux density, $E$ the electric field, $\rho$ the volume space-charge density which is a function of space coordinates, and the free space permittivity is $\varepsilon$.

A complete analytical solution of this equation was obtained by Langmuir, Blodgett and Child\textsuperscript{4, 5, 6} in the cases of infinitely long parallel planes, coaxial cylinders and concentric spheres. In deriving this solution, complete space charge condition is assumed, i.e., the field at the cathode surface is zero, and the effect of emission velocities is disregarded. In all three cases, the electron paths are rectilinear and laminar.

In the case of parallel planes, current density is the same in any cross-section of the beam, and is given by the Langmuir-Child equation or "three-halves power law".

\begin{equation}
J = \frac{2.330 \times 10^{-6} V_a^{3/2}}{d^2}
\end{equation}

where $V_a$ is the potential difference between cathode and anode in volts.

d the distance between them in cm and \( J \) is the current density in amps/cm\(^2\). The equipotential surfaces are parallel planes, the potential varying as the 4/3 power of the distance from the cathode.

For coaxial cylinders and concentric spheres the solutions are more complicated. In the case of coaxial cylinders, the current density at any radius \( r \) is given by:

\[
J_r = \frac{2.330 \cdot 10^{-6} \cdot V_a^{3/2}}{\beta^2 \frac{r_a}{r}}
\]

(4)

Here, \( \beta \) depends on the ratio of anode to cathode radius \( (r_a/r_c) \) and is given by the solution of the following non-linear differential equation.

\[
3\beta \frac{d^2\beta}{dy^2} + \left( \frac{d\beta}{dy} \right)^2 + 4\beta \frac{d\beta}{dy} = 1 - \beta^2
\]

(5)

Here, \( y = \ln \frac{r}{r_c} \).

The corresponding expression for concentric spheres is as follows:

\[
J_r = \frac{2.330 \cdot 10^{-6} \cdot V_a^{3/2}}{\alpha^2 \frac{r}{r_c}}
\]

(6)

Here, \( \alpha \) is the solution of the following non-linear differential equation:

\[
3\alpha \frac{d^2\alpha}{dy^2} + \left( \frac{d\alpha}{dy} \right)^2 + 3\alpha \left( \frac{d\alpha}{dy} \right) - 1 = 0
\]

(7)

Here again \( y = \ln \frac{r}{r_c} \). Power series solutions of equations (5) and (7).
have been obtained by Langmuir and Blodgett and are:

\( \alpha = y - 0.3 y^2 + 0.075 y^3 - 0.014318 y^4 + 0.002161 y^5 + \)

\( \beta = y - 0.4 y^2 + 0.01967 y^3 - 0.01424 y^4 + 0.001679 y^5 + \)

Langmuir and Blodgett's work described above appeared in 1924. Until recently very few successful attempts have been made to extend this work to more complicated geometries. In problems where electron flow is no longer rectilinear, the mathematics becomes very involved. However, in the past ten years many new approaches have been made. The contributions of Ivey, Matricon and Trouve, and Meltzer are noteworthy in the theoretical area. Liebmann, Musson-Genon and others have developed new experimental techniques.

Ivey\(^7\) in 1952 made extensive calculations for the potential distribution between inclined planes. The method, first suggested by Spangenberg\(^8\), is based on the Hamiltonian action function. Ivey's


\(^8\)Spangenberg, K., "Use of the Action Function to Obtain the General Differential Equations of Space-charge Flow in more than one Dimension", J. of Franklin Institute 232, pp. 365-371 (1941).
results appear in polar co-ordinates: if $\theta = 0$ and $\theta = \lambda$ are cathode and anode surfaces, the potential distribution and space charge limited current density are expressed as:

$$\frac{V}{V_a} = \left( \frac{\theta}{\lambda} \right)^{4/3} \frac{f(\theta)}{f(\lambda)} ,$$

$$J(r, \theta) = \frac{2.330 \cdot 10^{-6} \cdot V_a^{3/2}}{r^2} \cdot F(\theta) .$$

Here

$$f(\theta) = 1 + 0.0267 \theta^2 = 0.00234 \theta^4 + 0.0002166 \theta^6$$

$$F(\theta) = \frac{1}{\theta^2 \left[ f(\theta) \right]^{3/2}}$$

Equipotential surfaces are inclined planes defined by $\theta = \text{constant}$. Current density at any point is a function of $\theta$ and radial distance $r$ measured from the midpoint of the line of intersection of cathode and anode planes.

Matricon and Trouve\textsuperscript{9} in 1950 developed a method applicable to a diode formed by a plane, cylindrical or spherical cathode and an anode of

any arbitrary shape. The anode is replaced by an equivalent anode of the same geometry as the cathode and located symmetrically with respect to it. The current in the equivalent diode may be calculated from the Langmuir and Blodgett results. The separation between the cathode and equivalent anode is adjusted until the electrostatic capacitance of the equivalent diode is equal to that of the original diode. Under these conditions the beam permeance in the two geometries remains unchanged. This method has been applied to eccentric cylinders with their axes parallel or inclined. Many other interesting cases have been worked out by Ivey.

Meltzer in 1949 proposed a procedure powerful enough in principle to handle any space charge problem involving curved electron flow. In his method the solution for the potential distribution in a given electrode configuration cannot be directly determined. The nature of electron flow is assumed and the solution for an electrode configuration corresponding to this particular flow is determined. First, a choice is made of velocity vector as a function of space co-ordinates with arbitrary parameters to be adjusted later to satisfy properties of electron dynamics.

Let $u_x, u_y, u_z$ be the velocity components in cartesian coordinates as definite functions of $x, y, z$. Then

---

du_x = \frac{ds}{ds} \cdot \text{grad} u_x

\begin{align*}
du_y &= \frac{ds}{ds} \cdot \text{grad} u_y \\
du_z &= \frac{ds}{ds} \cdot \text{grad} u_z
\end{align*}

(14)

where \( ds \) is a differential length along the electron path. The acceleration vector \( a \) can now be expressed as:

\[ a = \text{grad} u \cdot i + \text{grad} u \cdot j + \text{grad} u \cdot k \]

(15)

where \( i, j, k \) are unit vectors along \( x, y, \) and \( z \) axes respectively.

Next, \( a \) is substituted in the equation of motion,

\[ a = -\frac{e}{m} E \]

(16)

Since \( E \) is an irrotational vector, \( \text{curl} E = 0 \),

(17) \[ \text{curl} \ a = 0. \]

From the equations of continuity and Poisson, another condition on the form of \( a \) is established as follows:

\[ \text{div} \ D = \rho \text{ and div} (\rho u) = 0, \text{ therefore} \]

(18) \[ \text{div} (u \ \text{div} \ a) = 0. \]

Arbitrary constants in \( u \) and \( a \) are now adjusted so that equations (17) and (18) are simultaneously satisfied. Once the correct form of \( a \) is estab-
lished, $E$, $V$, $\rho$, and $J$ can be calculated from the following equations:

$$E = -\frac{m}{e} \cdot a$$
$$V = E \cdot \frac{ds}{a} = -\frac{m}{e} \cdot \frac{a}{ds}$$
$$\rho = \text{div} \ D = \mathcal{E} \text{div} \ E = -\mathcal{E} \frac{m}{e} \text{div} \ a$$
$$J = \rho \ u = -(\mathcal{E} \frac{m}{e}) \ u \text{div} \ a$$

Meltzer has solved a simple two dimensional problem. Several more cases need to be worked out to gain experience in selecting proper forms of velocity functions. Thurston (private discussion) at this laboratory has attempted to extend this method to three dimensional problems of circular symmetry. The mathematics in such cases becomes quite involved.

**Experimental Techniques**

The available analytical methods are not adequate for solving problems of practical interest. In recent years, several interesting experimental approaches have been developed and in the following some of the promising developments are reviewed.

**The Electrolytic Tank**

The paths of electric current and equipotential lines between conductors placed in a medium of uniform conductivity are identical to the field lines and equipotential lines when the same conductors are placed in a dielectric medium, provided there is no space charge. This analogy is the basis of the electrolytic tank, where solution of the LaPlace equation can be readily obtained by making large scale models.
of the given boundaries with thin conducting strips and immersing them in a tank with an insulating bottom. The tank is filled with a weak electrolyte, usually tap water. An alternating voltage is applied between the electrodes. A conducting probe mounted on a carriage dips into the electrolyte. Equipotential lines may be drawn by moving the probe in the electrolyte until its potential balances a reference potential.

In two dimensional problems, the bottom of the tank is flat so that depth of water is uniform. Three dimensional problems with axial symmetry can be investigated by a method suggested by Bowman-Manifold and Nicol. The bottom of the tank is tilted by a small angle to form a wedge of water. The line of intersection of the bottom and top surface of water represents the axis of symmetry.

Musson-Genon has extended electrolytic tank technique to include space charge problems. The Poisson equation in a dielectric medium is:

\[ \text{div } \mathbf{E} = \mathbf{K} \frac{J}{V^{1/2}} \]

where \( \mathbf{E} \) is the electric field intensity, \( J \) the current density and \( V \) the potential.

The analogous equation in the electrolytic tank is:

\[ \text{div } \mathbf{j} = \text{div } (\sigma \mathbf{E}) = 0 \]

where \( \mathbf{j} \) is the current density, \( \mathbf{E} \) the electric field intensity, and \( \sigma \) the conductivity of the medium. Let \( \sigma \) instead of being uniformly constant as in the representation of the LaPlace equation, be intentionally made a variable, in which case:

\[
\text{div } \mathbf{E} + \frac{\mathbf{E} \text{ grad } \sigma}{\sigma} = 0.
\]

This permits us to identify the space charge term in Poisson's equation with the second term in equation (19). Therefore,

\[
\frac{\mathbf{E} \text{ grad } \sigma}{\sigma} = - \mathbf{K} \frac{J}{V^{1/2}}
\]

A convenient manner to vary \( \sigma \) is to make the depth of the electrolyte a function of space variables. If \( h \) represents the depth of the electrolyte, the above equation may be rewritten as:

\[
\frac{\mathbf{E} \text{ grad } h}{h} = - \mathbf{K} \frac{J}{V^{1/2}}
\]

or, \( \mathbf{E} \text{ grad } \ln h = -\mathbf{K} \frac{J}{V^{1/2}} \).

To calculate \( h \), it is necessary to know current density, potential and
field distribution. A method of successive approximations can be used. In the first approximation, \( h \) is kept constant and a solution of the Laplace equation is obtained. The values of \( J_0, V_0, \) and \( E_0 \) thus obtained are used to calculate a first order solution for \( h \) by numerical integration of Eq. (20). A wax model of the bottom of the tank is constructed to give the calculated depth of the electrolyte. The new field distribution at the surface of the electrolyte is obtained from which an improved shape of the bottom can be determined. The procedure is repeated until successive approximations converge closely.

Kuwabara\(^{14}\) has applied this technique to analyze the Heil gun.

The electrolytic tank is a very versatile instrument, useful in many other fields like hydrodynamics and aerodynamics. However, the accuracy possible from it is limited by a number of factors. Surface tension of the electrolyte causes distortion of the field near the electrode surface and the probe. Polarization error cannot be entirely eliminated even when an alternating current source is used. The frequency of the source should not be very high to avoid excessive stray capacitance currents. In addition, a highly sensitive and well shielded detector is necessary. When best precautions are taken the accuracy attainable is between 1 and 2\%.

Rubber Membrane

The electron motion in two dimensional electrostatic fields may be simulated by a rubber membrane technique developed at Philips Laboratory. It is based on the principle that vertical deformation of a stretched membrane that is slightly distorted satisfies LaPlace's equation. If \( h \) is the height of the membrane from a horizontal plane, then,

\[
\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0
\]

A rubber sheet, 0.5 mm thick, is stretched uniformly over a horizontal frame by pulling its edge with clips. The membrane is slightly distorted by pressing it against a scale model of the electrode system. The positive electrodes are placed above and the negative electrodes below the membrane. The height of each electrode is set proportional to the voltage it represents, so that necessary boundary conditions are satisfied.

Small steel balls are released from the points along the cathode. The horizontal projection of their motion corresponds to electron trajectories. The movement of steel balls is photographed with a pulsed light source and the track appears as a series of dashes of varying lengths. The length of individual dashes gives an indication of relative particle speed.
The technique has been extended by Alma\textsuperscript{15}, et al to include space charge problems. Let the membrane be stretched by a uniform tension, $\sigma$, in the horizontal xy plane. In addition, let a vertical pressure, $p$, which is a function of $x$ and $y$, be applied to the membrane. The equation for $h$ is now:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = -\frac{p}{\sigma}$$

This is equivalent to Poisson's equation, $p$ corresponding to space charge density. Since $p$ is an unknown function, a method of successive approximations is used. The first order solution is obtained by assuming $p = 0$, or better still, corresponding to a plausible space charge density distribution.

The vertical pressure is applied by means of a number of sponge rubber pads connected by means of pulleys and strings to standard weights. The pressure on individual pads can be varied by adjusting the weights.

The method has been checked to solve certain problems for which exact solutions are known. The accuracy is limited by friction and sagging of the membrane. The method is applicable only to two-dimensional problems. The effect of emission velocities may be studied by

rolling steel balls with initial velocities proportional to an assumed emission velocity distribution.

**Resistance Network**

It is recalled from the discussion about the electrolytic tank that potential in a uniformly conducting medium satisfies the LaPlace equation. In the resistance network method, the continuously varying conductance in the tank is replaced by a large number of lumped conductances arranged in a mesh. The required boundary conditions can be simulated by shorting appropriate mesh points. The solution of Poisson's equation can be obtained by injecting currents, proportional to space-charge density, at various mesh points. Since this method has been adopted in the present investigation, a more detailed discussion follows in the next chapter.

**Pierce's Technique**

J. R. Pierce\(^\text{16}\) in 1940 proposed a very elegant design procedure that permits the user to take account of space charge. The method uses the theoretical results of Langmuir for rectilinear electron flow. A convergent electron beam may be formed by cutting a conical section from a diode formed by two concentric spheres, the outer sphere being the cathode and the inner sphere the anode. The radial character of electron motion in this conical section can be maintained only if the effect of the

Conical Sector cut from a spherical diode

Fig. 1. Pierce's Technique of Electron Gun Design for Rectilinear Beams.
'missing' electrons is simulated. Pierce was able to achieve this by introducing an additional electrode, named the focussing electrode. The purpose of this electrode is to create the following conditions along the proposed boundary of the beam:

a) The field normal to the beam edge is zero.

b) The potential distribution along the beam edge follows the Langmuir distribution.

The focussing electrode can be designed in the electrolytic tank. A large scale model of the cathode and the anode is set up in the tank. An insulating strip is placed along the beam edge, as illustrated in Fig. 1, which establishes the first condition. A number of conducting probes are mounted on the insulating strip and these permit measurement of potential distribution along the beam edge. The shape and location of the focussing electrode is determined by trial and error until the potential distribution along the beam edge follows the 4/3 power law of Langmuir and Blodgett. It has been found in practice that no unique shape of focussing electrode exists. This fact has been theoretically verified by Pardee.\(^{17}\)

The guns designed by the Pierce method perform quite well as long as micropervance is less than unity; however, deviation between pre-

dicted and actual performance is observed for higher perveance. Anode aperture and emission velocity effects, disregarded in Pierce's method, assume greater significance in higher perveance beams.

**Anode Aperture Effect**

An aperture in the anode causes various aberrations in the beam, the extent of which depends on the size of the aperture in relation to anode-cathode distance. In the first place, the field near the cathode surface is distorted. For a spherical gun, the field along the axis is weaker than along the beam edge, causing nonuniform emission. This would tend to give a non-uniform current density in the focussed beam. Secondly, a radial field is set up in and around the aperture causing the beam to diverge from its designed path. The extent of divergence may be predicted approximately by the well known Davisson-Calbick formula, if the aperture is regarded as a thin electrostatic lens. The focal length, \( F \), is given by

\[
F = -\frac{4 V}{E},
\]

where \( V \) is the potential of the anode and \( E \) is the electric field when the aperture is removed. The negative sign implies that it is a divergent lens. In the derivation of this formula, the field beyond the aperture is assumed zero.

In high perveance guns, the ratio of size of the anode aperture to anode-cathode spacing is quite large, and as a result, the thin lens
approximation is no longer valid. In the region beyond the anode aperture, the beam spreads also due to its own space charge. The contribution of the latter effect is not included in the Davisson-Calbick formula.

Brown and Suskind\textsuperscript{18} in 1954 suggested a method to calculate the effect of the anode aperture in high perveance Pierce guns. If the space-charge-limited and space-charge-free potential distributions in an ideal Pierce gun (no anode aperture) are compared, it is noted that the deviation is maximum near the cathode and reduces to zero toward the anode. Beyond a certain distance from cathode, the difference in the two values is only a small percentage of anode potential. In this region, the effect of space charge is insignificant. Therefore, the actual potential distribution may be approximated by the LaPlace solution. The latter solution is still valid when the anode aperture is introduced, because the charge distribution will not alter appreciably with the aperture. The region between cathode and anode in an apertured Pierce gun is divided into two parts: the first between actual cathode and "virtual cathode" in which potential varies in accordance with the Langmuir distribution, and the second between the virtual cathode and the anode in which the potential can be approximated by the LaPlace solution. The latter can be obtained

in the electrolytic tank.

A complete analysis of anode aperture effects for Pierce guns in micropervance range .1 -. 8 has been made by Danielson, Rosenfield and Saloom\(^\text{19}\). The results are presented in the form of design charts. Two independent approaches have been used to arrive at these results. The first is more or less an extension of the Davisson and Calbick formula taking account of space charge. This method is rather tedious, and is used to justify the second method, which is easier though less sophisticated. The latter method is similar to that of Brown and Suskind.

Brewer\(^\text{20}\) regards the effect of anode aperture as causing the potential distribution to depart from the Langmuir value. The departure is greatest along the axis of the beam. As a result, the equipotential lines, instead of being parallel to the cathode, are distorted; the inner edge of the equipotential lines are bent toward the anode. The potential distribution in the central part of the beam can be described more accurately by a Langmuir function for a reduced value of \(r_a/r_c\)


than by that for which the gun was designed. In other words, for this part of the beam, an equivalent anode with smaller radius of curvature can be assumed. A partial compensation for the anode aperture aberration can be effected by redesigning the focusing electrode, so that potential distribution along the beam edge corresponds to a reduced value of $r_a/r_c$. The outer edges of the equipotentials are shifted by the proper amount to make them parallel to the cathode. The beam perveance is reduced because of the general weakening of potential gradient near the cathode. Although no direct method of calculating the radius of the equivalent anode is available, a value equal to $2/3$ actual anode radius is suggested.

A second method suggested by Brewer is to have an auxiliary anode closer to the axis and at a higher potential than the main anode. This would tend to neutralize the diverging field due to the aperture.

**Effect of Thermal Velocities**

The effect of transverse velocities with which many electrons are emitted is to cause further defocussing or broadening of the beam. This constitutes a basic limitation to the design of narrow high density beams. Cutler and Hines\textsuperscript{21} have made a theoretical analysis of this effect. They

show that if the initial velocity distribution at the cathode is assumed Maxwellian, the nature of the distribution remains Maxwellian at any cross-section along the path of the beam. However, magnitudes of transverse velocities are increased as the square of beam compression. In other words, at any cross-section, the distribution corresponds to a different temperature. It must be emphasized that this theory applies only to well-designed Pierce-type guns, where electron flow is laminar and rectilinear and where current density is uniform.

The theory has been verified experimentally by Cutler and Saloom\textsuperscript{22} with the help of a pin-hole-type beam analyzer. Two diaphragms separated by a certain longitudinal distance are placed in the path of the outcoming beam. Each diaphragm has a small aperture, and can be moved transverely across the beam. The first aperture selects a small portion of the beam. The transverse velocity distribution in this sample can be obtained by moving the second aperture normal to the beam and measuring the current passing through it by placing a collector behind it. The device can also be used to measure current density distribution in the beam with the help of the first diaphragm alone.

In the original analysis of Cutler and Hines, the increase in beam

diameter due to thermal velocities was assumed very small in comparison to the actual diameter of the beam. In many actual situations, it is found that the effect of transverse velocities may be as much as 100 percent. Danielson et al have extended the Cutler and Hines analysis to include cases where the effect of transverse velocities is more severe.

It should be mentioned that the pin-hole camera of Cutler and Saloom is very useful in evaluating performance of guns for which no theoretical results are available. Campbell and Mueller have built such a device and have used it successfully to study the beam formed by the Heil gun.

Heil Gun

Heil gave the design of this gun in which a microperveance of four is attainable. This gun, developed by 'cut and try' experimental procedure, consists of an ellipsoidal cathode, a Pierce type focussing


Fig. 2. Electron trajectories and equipotential lines in Heil Gun.
electrode and a conical shaped anode (see Fig. 2). Most of the current in this gun comes from the edge of the cathode, which is deliberately given more curvature in an effort to compensate for diverging action of the anode aperture. Analyses of this gun made by theoretical and experimental methods indicate that electron flow is not laminar and current density in the focused beam is nonuniform.

Scope of the Present Investigation

The purpose of the present study is to investigate potential distribution and electron flow in new geometries with a view to developing better high-density beams. It is recalled that Pierce's method is applicable only for rectilinear electron flow and becomes approximate when the desired perveance is greater than one. In view of the rather singular success of Heil in obtaining a remarkable figure for perveance, a study of ellipsoidal geometry suggested itself. A confocal ellipsoidal diode with axis of symmetry along the minor axis was first examined. An analysis of this problem revealed that although enough current was available from the cathode, it was not possible to focus it into a small enough area. However, if the anode ellipsoid is moved further along the axis of symmetry so that the major axes are no longer coincident but parallel, and a third electrode (focusing electrode) is added, it is possible to get a good convergence. A nonconfocal ellipsoidal gun was designed and analyzed. It gives a beam of perveance 2, and a concentration ratio of about 200. Though perveance is lower than the figure given
by Heil, our study indicates that this gun has much less aberration. It may be possible to increase perveance by more extensive examination of the shape of the focusing cathode and the position of the anode.

A study of space-charge-limited potential distribution and electron paths in certain confocal paraboloidal diodes has also been made. Although enough current can be obtained from the cathode, convergence is poor. The current density at the cathode is nonuniform, highest along the axis and gradually decreasing toward the edge. This may have some significance; the effect of anode aperture in this case may be to make cathode current density more uniform. A preliminary investigation, without including space charge, reveals that it is possible to get good convergence in paraboloidal geometry if a focusing electrode is added and if the anode is moved farther away from the cathode to form a nonconfocal system.

A paraboloidal-ellipsoidal gun, consisting of paraboloidal cathode, oblate spheroidal anode and a conical focusing electrode of circular cross-section has been designed and analyzed on the resistor network.
Rigorous solution of LaPlace and Poisson equations is possible only in a few cases of practical interest. An approximate numerical solution can be found with the help of Liebmann's iterative procedure or Southwell's relaxation technique. In both these methods, the region between given boundaries is divided into a rectangular net. From the theory of finite differences, the value of the function at each netpoint can be related to its values at four neighboring points by a linear algebraic equation. A set of algebraic equations, equal in number to the number of netpoints within the given boundaries can be set up, and a simultaneous solution of these equations gives the values of the desired function at each netpoint. Southwell's method is quicker, while the iteration procedure is simpler and can be adapted for programming on a digital computer.

---

Hogan\textsuperscript{27} was the first to realize that solution of finite difference equations can be obtained automatically and instantly with a network of resistors. The first device was built by Redshaw\textsuperscript{28} and used to obtain solutions of the LaPlace equation in two-dimensional rectangular coordinates. De-Packh\textsuperscript{29} extended this method to the solution of the LaPlace equation in cylindrical co-ordinates. By injecting currents into the network, and by the use of an iteration process, Liebmann\textsuperscript{30, 31} was able to solve Poisson's equation.

Since the theory of the resistance network is intimately related

\begin{thebibliography}{9}
\end{thebibliography}
to the theory of finite difference equations, we shall first establish the finite difference approximation of Poisson's equation in rectangular and in cylindrical coordinates.

**Finite Difference Equation-Rectangular Coordinates**

Poisson's equation in two dimensional rectangular coordinates is expressed as:

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\frac{\rho}{\varepsilon_0} \]

\[ (21) \]

Let the coordinates of point 0 be \((x_0, y_0)\). Then coordinates of neighboring points 1, 2, 3 and 4, located unsymmetrically with respect to 0, are \((x_0 + h_1, y_0)\), \((x_0, y_0 + h_2)\), \((x_0 - h_3, y_0)\), \((x_0, y_0 - h_4)\) respectively.

Let the value of the function at 0 be \(u_0\). Then within the mesh 1, 2, 3, 4, the function \(u\) may be approximated by a quadratic function in \(x\) and \(y\), namely,

\[ u(x, y) = u_0 + a_1 (x - x_0) + a_2 (y - y_0) + a_3 (x - x_0)^2 + a_4 (y - y_0)^2 + a_5 (x - x_0) (y - y_0) \]

\[ (22) \]
Therefore

(23) \[ \left( \frac{\partial^2 u}{\partial x^2} \right)_o \approx 2a_3 \]

and

(24) \[ \left( \frac{\partial^2 u}{\partial y^2} \right)_o \approx 2a_4 \]

If \( h_1, h_2, h_3, h_4 \) are small enough, equation (21) may be replaced by

\[ \nabla^2 u \approx 2a_3 + 2a_4 = \frac{\rho}{\varepsilon_o} \]

The values of \( a_3 \) and \( a_4 \) can be obtained by substituting the coordinates of net points 1, 2, 3, 4 in equation (22)

(25) \[ u_1 = u_o + a_1h_1 + a_3h_1^2 \]

(26) \[ u_3 = u_o - a_1h_3 + a_3h_3^2 \]

(27) \[ u_2 = u_o + a_2h_2 + a_4h_2^2 \]

(28) \[ u_4 = u_o - a_2h_4 + a_4h_4^2 \]

From (25), (26), (27), and (28)

\[ a_3 = \frac{u_1 - u_o}{h_1(h_1 + h_3)} + \frac{u_3 - u_o}{h_3(h_1 + h_3)} \]
Multiply both sides by \( \frac{(h_2 + h_4)(h_1 + h_3)}{4} \);

Then

\[
(u_1 - u_0) \frac{(h_2 + h_4)}{2h_1} + \frac{(u_2 - u_0)(h_1 + h_3)}{2h_2} + \frac{(u_3 - u_0)(h_2 + h_4)}{2h_3} + \frac{(u_4 - u_0)(h_1 + h_3)}{2h_4} = -\frac{(h_1 + h_3)(h_2 + h_4)}{4} \frac{\rho}{E_0}
\]

Equation (29) is the most general form of the finite difference approximation corresponding to Equation (21).
Consider a resistance mesh formed by $R_1$, $R_2$, $R_3$ and $R_4$. Let $V_1$, $V_2$, $V_3$, and $V_4$ be the potentials of points 1, 2, 3 and 4 respectively and $V_o$ be the potential of mesh junction 0. A current $I_o$ is injected into mesh junction 0.

![An x-y resistor network.](image)

Application of Kirchhoff’s current law provides

$$
(30) \quad \frac{V_1 - V_o}{R_1} + \frac{V_2 - V_o}{R_2} + \frac{V_4 - V_o}{R_4} + \frac{V_3 - V_o}{R_3} - I_o = 0
$$

The function $u$ in Eq. (29) can be identified with $V$ in Eq. (30) if $R_1$, $R_2$, $R_3$, $R_4$ and $I_o$ are properly chosen, namely:

$$
(31) \quad R_1 = \frac{2 h_1}{h_2 + h_4} \quad R_N
$$

$$
(32) \quad R_2 = \frac{2 h_2}{h_1 + h_3} \quad R_N
$$
The quantity $R_N$ is an arbitrary constant of the resistance network.

Equations (31) through (34) are generalized design equations for an irregular mesh and can be specialized for particular cases:

Square Mesh: If the mesh size is uniformly $h$ in $x$ and $y$ directions, then $R_1 = R_2 = R_3 = R_4 = R_N$ and 

$$I_o = -\frac{\rho}{\varepsilon_0 R_N}.$$ 

Rectangular Mesh: If the mesh size in the $x$-direction is $h$, in the $y$-direction $h'$, then 

$$R_1 = R_3 = \frac{h}{h'} R_N , \quad R_2 = R_4 = \frac{h'}{h} R_N , \quad I_o = -\frac{h h' \rho}{\varepsilon_0 R_N}.$$ 

Finite Difference Equation-Cylindrical Coordinates

Poisson's equation in cylindrical coordinates in axially symmetrical problems is of the form:
\[
(36) \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = -\frac{p}{\varepsilon}
\]

Let \( u_0 \) be the value of the function at 0, and \( u_1, u_2, u_3, u_4 \) its value at four neighboring points. Coordinates of points 0, 1, 2, 3, 4 are respectively:

\[(z_0, r_0), \quad (z_0 + h_1, r_0), \quad (z_0, r_0 + h_2), \quad (z_0 - h_3, r_3), \quad (z_0, r_0 - h_4). \]

Let function \( u \) in the neighborhood of point 0 be defined as:

\[
(37) \quad u(r, z_0) = u_0 + a_1 (z - z_0) + a_2 (r - r_0) + a_3 (z - z_0)^2 + a_4 (r - r_0)^2 + a_5(z - z_0)(r - r_0)
\]

\[
\left(\frac{\partial u}{\partial r}\right)_0 = a_2
\]

\[
\left(\frac{\partial^2 u}{\partial r^2}\right)_0 = 2a_4
\]
\[
\left( \frac{\partial^2 u}{\partial z^2} \right) = 2 a_3
\]

Equation (36) may now be written as

\[
(38) \quad \nabla^2 u = \frac{a_2}{r_o} + 2 a_4 + 2 a_3 = - \frac{p}{\varepsilon_o}
\]

The values of constants \(a_2, a_3,\) and \(a_4\) can be obtained by substituting coordinates of points 1, 2, 3, 4 one at a time in equation (37), and solving the resulting equations simultaneously. Thus,

\[
a_2 = \frac{(u_2 - u_o) h_4}{h_2(h_2 + h_4)} - \frac{(u_4 - u_o) h_2}{h_4(h_2 + h_4)}
\]

\[
a_3 = \frac{u_1 - u_o}{h_1(h_1 + h_3)} + \frac{u_3 - u_o}{h_3(h_1 + h_3)}
\]

\[
a_4 = \frac{u_2 - u_o}{h_2(h_2 + h_4)} + \frac{u_4 - u_o}{h_4(h_2 + h_4)}
\]

substituting \(a_2, a_3, a_4\) in Equation (38), one obtains

\[
(39) \quad \nabla^2 u = \frac{2(u_1 - u_o)}{h_1(h_1 + h_3)} + \frac{2(u_3 - u_o)}{h_3(h_1 + h_3)} + \frac{2(u_2 - u_o)}{h_2(h_2 + h_4)} (1 + \frac{h_4}{2 r_o})
\]

\[
+ \frac{2(u_4 - u_o)}{h_4(h_2 + h_4)} (1 - \frac{h_2}{2 r_o}) = - \frac{p}{\varepsilon_o}
\]
Equation (39) is the most general form of the finite difference equivalent of Poisson's equation in cylindrical coordinates from which a generalized resistor network can be immediately developed.

In this analysis, \( \frac{\partial u}{\partial r} \) is assumed constant within a given mesh. This approximation may be in error in regions very close to the axis. Liebmann\(^{32}\) suggests that a better approximation can be made by assuming logarithmic variation of \( u \) with \( r \). His generalized design formulae are:

\[
R_1 = \frac{8h_1 R_N}{(h_2 + h_4)(4r_0 + h_2 - h_4)}
\]

\[
R_2 = 2R_N \ln \frac{r_0 + h_2}{r_0} \frac{(h_1 + h_3)}{(h_1 + h_3)}
\]

\[
R_3 = \frac{8h_3 R_N}{(h_2 + h_4)(4r_0 + h_2 - h_4)}
\]

\[
R_4 = 2R_N \ln \frac{r_0}{r_0 - h_4} \frac{(h_1 + h_3)}{(h_1 + h_3)}
\]

Construction of the Resistor Network

An axially symmetrical network with 30 meshes in the radial direction and 45 meshes in the axial direction was constructed. The design formulae for a square net $h_1 = h_2 = h_3 = h_4 = h$ may be derived from equation (40) and are:

\[
\begin{align*}
I_0 &= -\frac{(h_1 + h_3)(h_2 + h_4)(4r_0 + h_2 - h_4)}{16 R_N} \frac{\rho}{\varepsilon_0} \\
R_1 &= R_3 = \frac{R_N}{r_0} \\
R_2 &= R_N \ln \frac{r_0 + h}{r_0} \\
R_4 &= R_N \ln \frac{r_0}{r_0 - h} \\
I_0 &= -\frac{r_0^2 h^2 \rho}{R_N \varepsilon_0}
\end{align*}
\]

For meshes along the axis, $r = 0$, the following formulae can be derived:

\[
\begin{align*}
R_1 &= R_3 = 8 R_N \\
R_2 &= 2 R_N \\
I_0 &= -\frac{h^3 \rho}{8 R_N \varepsilon_0}
\end{align*}
\]
Quantity $R_N$ is an arbitrary constant of the resistor network. It is desirable to choose a high value of $R_N$ because the injected current, $I_0$, for a given space-charge density and mesh size, depends inversely on it. On the other hand, the values of resistors required are directly proportional to $R_N$. In view of these considerations a value of $R_N = 6000$ ohms was chosen. The resistor values in the network used vary from 48 K along the axis to 200 ohms along the edge of the network. The resistors are all 0.5 watt deposited carbon with specified tolerance of 0.2 to 0.4 per cent and were supplied by 'Constanta' Kircheim, West Germany, on special order. Metallic jacks are mounted in a one inch square array on a transparent lucite panel. This panel is hinged to a wooden frame resting on the wall. The resistors are assembled permanently to the jacks on the back side. The front ends of the jacks are left free for setting up boundary conditions and for injecting currents.

An independent current injection unit comprising two hundred current sources is provided. It consists of two hundred, 10 turn, helical potentiometers arranged in five groups. All the potentiometers in a group are wired in parallel and can be switched to a 100-volt d-c supply. The variable terminal of each potentiometer is connected through a high series resistance to a corresponding jack mounted on a separate panel. The potentiometers and the terminating jacks are suitably marked for easy identification. The connections between these jacks and the network are made with patch cords. Each potentiometer is provided with a calibrated dial which enables the user to set the values of injected currents directly
without necessity of measurement. The purpose of high series resistances is to avoid shunting of the network through current sources. In the present equipment there are 120 current sources with current range 0-100 microamperes, and 80 current sources having current range 0-1 milliampere. A photograph of the resistor network and the current injection unit appears in Fig. 6.

Operation of the Resistor Network

The shapes of electrodes in a given problem are reproduced on the resistor network by short circuiting corresponding net points with banana plugs. In case of curved electrodes, some of the boundary points will not fall on regular net points. This difficulty may be overcome by replacing the actual shape of the boundary with an approximate zig-zagged shape so that it will fall on the nearest netpoints. A curved boundary may be more accurately represented on the network by inserting suitable shunting resistances across netpoints between which the given boundary lies. This has the effect of reducing the mesh size locally by the correct amount so that the boundary point in question will lie along the subdivided netpoint.

The value of shunting resistor required in a given situation may be easily calculated from the actual value of network resistance between two points and the ratio of subdivided mesh size to actual mesh size. A set of resistors connected to pairs of banana plugs is provided for boundary correction.

The solution is generally required in the form of equipotential curves for use in graphical trajectory plotting. In order to trace the course of
Fig. 6a. A view of the resistor network and current injection unit.
Fig. 6b. Circuit diagram of resistor network and current injection sources.
an equipotential, it is often necessary to interpolate the potential values between two neighboring netpoints between which the equipotential lies. This is accomplished quickly by plugging in a high resistance (500 K in our case) calibrated interpolating potentiometer between the two net points in question. A reference potentiometer on which the value of desired equipotential can be set is provided. A highly sensitive galvanometer is connected between the interpolating and reference potentiometers. This device speeds up equipotential plotting considerably.

**Scale Factors in the Resistor Network Analogy**

In calculation of values of injected currents for simulating space charge, certain scale factors have to be considered. Poisson's equation in planar systems is

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\frac{\rho}{\varepsilon_0}
\]

where \( u \) is the potential function and \( \rho \) the space-charge-density function in any given problem. Let the corresponding potential values and distances on the resistor model be related by the following equations:

\[
V = S_1 u
\]

\[
X = S_2 x
\]

\[
Y = S_2 y
\]

where \( S_1 \) and \( S_2 \) are voltage and dimensional scaling factors. After
substituting these relations in Poisson's equation, one finds

\[ \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -\frac{\rho}{\varepsilon_0} \frac{S_1}{S_2^2} \]

The space charge density, \( \rho_m \) in the scaled model is related to actual space charge density, \( \rho \), by

\[ \rho_m = \frac{\rho S_1}{S_2^2} \]

The corresponding current to be injected follows from Equation (41) and is

\[ I_0 = -\frac{r}{R_N} \frac{h^2}{S_2} \left( \frac{\rho S_1}{S_2^2} \right) \]

**Accuracy of Resistor Network**

The accuracy of solution obtained with the network is limited by the mesh size, the tolerance of resistors and the accuracy of measuring equipment.

The mesh size should be small enough so that replacement of the original partial differential equation with a finite difference approximation is permissible. It is known from the theory of the relaxation method that the error in the solution is proportional to the square of the mesh size, so that reducing mesh size by half reduces the error by a factor of four.
In practice, the number of mesh points in a problem must be limited, because the labor of obtaining the solution becomes excessive. It is rather difficult to find a general expression for finite difference error for a given mesh size as it would depend on the nature of the problem. The error may be large in regions where the potential changes rapidly and may be small in other regions where the field is weak. The network described here has 45 meshes in the axial direction and 30 in the radial direction, and when the greater part of the network is used, the mesh size is small enough to permit accurate representation of most types of problems.

Liebmann has pointed out that if the deviations in the values of resistors are randomly distributed about the designed values, the network possesses certain averaging properties which make the results obtainable more accurate than the tolerance of resistors used. When these considerations are applied, the error in the solution due to resistor tolerance is less than 1 part in 1000 in this device.

The injected currents can be set up within 1 per cent of desired values. The interaction among various current injection sources, and between current injection sources and the network is hardly noticeable. The potential measurements on the network are made with a high precision potentiometer with a specified linearity of one tenth of one per cent.
The network was tested by setting up several problems for which the exact theoretical solution is available. The three standard problems are parallel plane, cylindrical and spherical diodes.

The arrangement of electrodes in the first problem is shown in Fig. 7. The cathode is located at \( z = 0 \) and the anode at \( z = 40 \). A direct potential of one volt is applied between them. The measured and theoretical potential distributions are compared in Table 1. With the same geometry, the solution of Poisson's equation was obtained. The space charge density distribution and corresponding injected currents were calculated from Langmuir and Blodgett's theoretical results discussed in Chapter 1. The array of current injection points used in this particular problem is shown in Fig. 7. The experimental and theoretical results are compared in Table 2.

The arrangement of electrodes in the second test problem is indicated in Fig. 8. It consists of a cylindrical diode with cathode radius equal to one and anode radius equal to 20 mesh units. This problem is more difficult than the parallel plane problem because of high field intensity at the inner conductor. The results for space charge free solution at several mesh points are given in Table 3; agreement with theoretical values is within 0.5 per cent. The array of current injection points used is illustrated in Fig. 8. Near the cathode, a finer mesh was deliberately selected because of large space-charge
Fig. 7. Arrangement of electrodes and current injection points in parallel plane diode test problem.


<table>
<thead>
<tr>
<th>Z</th>
<th>r = 1</th>
<th>r = 15</th>
<th>r = 27</th>
<th>Theoretical</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0.1265</td>
<td>0.127</td>
<td>0.1265</td>
<td>0.125</td>
</tr>
<tr>
<td>10</td>
<td>0.250</td>
<td>0.251</td>
<td>0.252</td>
<td>0.250</td>
</tr>
<tr>
<td>15</td>
<td>0.375</td>
<td>0.3747</td>
<td>0.3765</td>
<td>0.375</td>
</tr>
<tr>
<td>20</td>
<td>0.500</td>
<td>0.4995</td>
<td>0.502</td>
<td>0.500</td>
</tr>
<tr>
<td>25</td>
<td>0.625</td>
<td>0.624</td>
<td>0.6255</td>
<td>0.627</td>
</tr>
<tr>
<td>30</td>
<td>0.750</td>
<td>0.749</td>
<td>0.752</td>
<td>0.750</td>
</tr>
<tr>
<td>35</td>
<td>0.875</td>
<td>0.8735</td>
<td>0.8765</td>
<td>0.8765</td>
</tr>
<tr>
<td>40</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>
TABLE II

Theoretical and Experimental Solution of Poisson Equation in Parallel Plane Diode

<table>
<thead>
<tr>
<th>Z</th>
<th>r = 1</th>
<th>r = 15</th>
<th>r = 27</th>
<th>Theoretical</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0.0625</td>
<td>0.0595</td>
<td>0.066</td>
<td>0.0625</td>
</tr>
<tr>
<td>10</td>
<td>0.156</td>
<td>0.155</td>
<td>0.1585</td>
<td>0.1575</td>
</tr>
<tr>
<td>15</td>
<td>0.269</td>
<td>0.268</td>
<td>0.270</td>
<td>0.2704</td>
</tr>
<tr>
<td>20</td>
<td>0.394</td>
<td>0.394</td>
<td>0.396</td>
<td>0.3968</td>
</tr>
<tr>
<td>25</td>
<td>0.531</td>
<td>0.531</td>
<td>0.533</td>
<td>0.5344</td>
</tr>
<tr>
<td>30</td>
<td>0.679</td>
<td>0.680</td>
<td>0.682</td>
<td>0.6814</td>
</tr>
<tr>
<td>35</td>
<td>0.834</td>
<td>0.836</td>
<td>0.837</td>
<td>0.8369</td>
</tr>
<tr>
<td>40</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>
Fig. 8. Configuration of electrodes and current injection points in cylindrical diode test problem.
# TABLE III

Theoretical and Experimental Solution of Laplace Equation in Cylindrical Diode

<table>
<thead>
<tr>
<th>r</th>
<th>Z = 0</th>
<th>Z = 10</th>
<th>Z = 20</th>
<th>Z = 30</th>
<th>Z = 40</th>
<th>Theoretical Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.2325</td>
<td>0.2325</td>
<td>0.2317</td>
<td>0.2321</td>
<td>0.2321</td>
<td>0.2314</td>
</tr>
<tr>
<td>3</td>
<td>0.3687</td>
<td>0.3687</td>
<td>0.3687</td>
<td>0.3687</td>
<td>0.3687</td>
<td>0.3667</td>
</tr>
<tr>
<td>4</td>
<td>0.4650</td>
<td>0.4650</td>
<td>0.4650</td>
<td>0.4650</td>
<td>0.4650</td>
<td>0.4628</td>
</tr>
<tr>
<td>5</td>
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<td>0.5396</td>
<td>0.5396</td>
<td>0.5396</td>
<td>0.5396</td>
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</tr>
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<td>6</td>
<td>0.600</td>
<td>0.600</td>
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<tr>
<td>7</td>
<td>0.653</td>
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<td>0.653</td>
<td>0.650</td>
</tr>
<tr>
<td>9</td>
<td>0.737</td>
<td>0.737</td>
<td>0.737</td>
<td>0.737</td>
<td>0.737</td>
<td>0.733</td>
</tr>
<tr>
<td>11</td>
<td>0.803</td>
<td>0.803</td>
<td>0.803</td>
<td>0.803</td>
<td>0.803</td>
<td>0.800</td>
</tr>
<tr>
<td>13</td>
<td>0.858</td>
<td>0.858</td>
<td>0.858</td>
<td>0.858</td>
<td>0.858</td>
<td>0.856</td>
</tr>
<tr>
<td>15</td>
<td>0.905</td>
<td>0.905</td>
<td>0.905</td>
<td>0.905</td>
<td>0.905</td>
<td>0.904</td>
</tr>
<tr>
<td>17</td>
<td>0.947</td>
<td>0.947</td>
<td>0.947</td>
<td>0.947</td>
<td>0.947</td>
<td>0.946</td>
</tr>
<tr>
<td>19</td>
<td>0.984</td>
<td>0.984</td>
<td>0.984</td>
<td>0.984</td>
<td>0.984</td>
<td>0.983</td>
</tr>
</tbody>
</table>
## TABLE IV

Theoretical and Experimental Solution of Poisson Equation in Cylindrical Diode

<table>
<thead>
<tr>
<th>r</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>9</th>
<th>11</th>
<th>13</th>
<th>15</th>
<th>19</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z=5</td>
<td>0.089</td>
<td>0.167</td>
<td>0.255</td>
<td>0.328</td>
<td>0.388</td>
<td>0.442</td>
<td>0.552</td>
<td>0.645</td>
<td>0.736</td>
<td>0.814</td>
<td>0.964</td>
</tr>
<tr>
<td>Z=10</td>
<td>0.068</td>
<td>0.138</td>
<td>0.243</td>
<td>0.318</td>
<td>0.376</td>
<td>0.417</td>
<td>0.546</td>
<td>0.633</td>
<td>0.732</td>
<td>0.807</td>
<td>0.964</td>
</tr>
<tr>
<td>Z=15</td>
<td>0.082</td>
<td>0.156</td>
<td>0.243</td>
<td>0.317</td>
<td>0.377</td>
<td>0.431</td>
<td>0.545</td>
<td>0.640</td>
<td>0.731</td>
<td>0.811</td>
<td>0.964</td>
</tr>
<tr>
<td>Z=20</td>
<td>0.082</td>
<td>0.156</td>
<td>0.243</td>
<td>0.317</td>
<td>0.377</td>
<td>0.431</td>
<td>0.545</td>
<td>0.639</td>
<td>0.731</td>
<td>0.811</td>
<td>0.964</td>
</tr>
<tr>
<td>Z=25</td>
<td>0.083</td>
<td>0.157</td>
<td>0.244</td>
<td>0.317</td>
<td>0.378</td>
<td>0.432</td>
<td>0.546</td>
<td>0.640</td>
<td>0.732</td>
<td>0.812</td>
<td>0.964</td>
</tr>
<tr>
<td>Z=30</td>
<td>0.084</td>
<td>0.158</td>
<td>0.246</td>
<td>0.319</td>
<td>0.380</td>
<td>0.433</td>
<td>0.548</td>
<td>0.642</td>
<td>0.735</td>
<td>0.814</td>
<td>0.965</td>
</tr>
<tr>
<td>Z=35</td>
<td>0.085</td>
<td>0.161</td>
<td>0.250</td>
<td>0.324</td>
<td>0.384</td>
<td>0.439</td>
<td>0.553</td>
<td>0.650</td>
<td>0.741</td>
<td>0.822</td>
<td>0.966</td>
</tr>
<tr>
<td>Z=40</td>
<td>0.088</td>
<td>0.166</td>
<td>0.256</td>
<td>0.330</td>
<td>0.391</td>
<td>0.446</td>
<td>0.559</td>
<td>0.652</td>
<td>0.743</td>
<td>0.821</td>
<td>0.966</td>
</tr>
</tbody>
</table>

Theoretical
Fig. 9. Percent deviation in cylindrical diode test problem.
Fig. 10. Configuration of electrodes and current injection points in spherical diode test problem.
curves represent theoretical solution
points represent experimental solution

Fig. 11. Solution of Laplace equation in spherical diode -
  \( r_c = 20, \ r_a = 4 \).
curves represent theoretical solution
points represent experimental solution

Fig. 12. Solution of Poisson's equation in spherical diode.
density in this region. In the rows \( r = 2 \) and \( r = 3 \), current is injected at every other mesh point. The potential at these points is 10 to 15 percent lower than the theoretical value. However, this perturbation in the solution reduces very rapidly, and at \( r = 4 \) the maximum deviation is within 1 percent. The percent deviation as a function of \( z \) is plotted for \( r = 2, 3 \) and 4 in Fig. 9.

Finally, a spherical diode problem was solved. Since in this case the electrodes are curved, boundary adjusting resistors were necessary at several netpoints. The cathode sphere has a radius of 20 and the anode sphere has a radius of 4 units. The theoretical and experimental results for Laplace and Poisson potential distributions are given in Figs. 11 and 12. Electrode configuration and the array of current injection points used are shown in Fig. 10.
Potential Distribution in Certain Ellipsoidal Geometries

It has been mentioned earlier that Heil was able to achieve a microperveance of four in a gun with an ellipsoidal cathode. This fact suggested a general examination of the problem of space-charge-limited potential distribution in ellipsoidal diodes. However, in view of the rather large number of parameters involved, it was decided to confine the problem to the case of diodes formed by confocal oblate spheroidal electrodes. In this chapter, the solution of Poisson's equation in a typical confocal oblate spheroidal diode is presented. The method used is one of successive approximation and requires the space-charge-free solution as the starting point. The analytical solution of the Laplace equation in this geometry is known and has been used to check the solution obtained on the resistor network. Some general conclusions with regard to the limitations of this geometry from the standpoint of applications in high-density electron beam design may be derived from this study. The possibilities with nonconfocal oblate spheroidal geometry are indicated. An electron gun consisting of a cathode and an anode of nonconfocal spheroidal geometry is designed and the solution of Poisson's equation is obtained on the resistor network.
Analytical Solution of LaPlace Equation

In order to express the LaPlace equation in a simple form, it is advantageous to use oblate spheroidal coordinates. When the ellipses in Fig. 13 with common focal points are rotated about their semi-minor axis (Z-axis in this case), the resulting surfaces are confocal oblate spheroids.

The equation of these surfaces is

\[
\frac{x^2}{c^2 \cosh^2 \gamma} + \frac{y^2}{c^2 \cosh^2 \gamma} + \frac{z^2}{c^2 \sinh^2 \gamma} = 1
\]

where \( \gamma \) is a parameter associated with a particular oblate spheroid and is defined as \( \cosh \gamma = \frac{r_1 + r_2}{2c} \). Here \( r_1, r_2 \) are the distances measured from the two focal points to any point on the surface and \( 2c \) is the distance between focal points. The other set of surfaces, which are orthogonal to the spheroids, are generated by rotation of the hyperbolas in Fig. 13 about the Z-axis, and are called hyperboloids (of one sheet) of revolution. The equation of these surfaces is

\[
\frac{x^2}{c^2 \sin^2 \theta} - \frac{y^2}{c^2 \sin^2 \theta} - \frac{z^2}{c^2 \cos^2 \theta} = 1
\]

where \( \theta \) is a parameter associated with a particular surface of this family and is defined by the relation
Fig. 13. A system of confocal spheroidal coordinates.
\[
\cos \theta = \frac{r_1 - r_2}{2c}
\]

The coordinate surfaces are thus: (1) oblate spheroids \((\eta = \text{constant})\); (2) hyperboloids (of one sheet) of revolution \((\theta = \text{constant})\); (3) planes through z-axis \((\phi = \text{constant})\). The relationships between the rectangular and the oblate spheroidal coordinates may be derived from Equations (42) and (43) and are

\[
\begin{align*}
x &= c \cosh \eta \sin \theta \cos \phi \\
y &= c \cosh \eta \sin \theta \sin \phi \\
z &= c \sinh \eta \cos \theta
\end{align*}
\]

The metrical coefficients associated with this coordinate system are

\[
\begin{align*}
h_1^2 &= c^2 (\sinh^2 \eta + \cos^2 \theta), \\
h_2^2 &= c^2 (\sinh^2 \eta + \cos^2 \theta), \\
h_3^2 &= c^2 \cosh^2 \eta \sin^2 \theta.
\end{align*}
\]

The general form of the LaPlace equation in curvilinear coordinates is

\[
\sum_{i=1}^{3} \frac{\partial}{\partial x_i} \frac{\sqrt{g}}{h_i^2} \frac{\partial u}{\partial x_i} = 0
\]

(44)
where $\sqrt{g} = h_1 h_2 h_3$ and $x$ are the curvilinear coordinates. In order to express this equation in spheroidal coordinates, one may substitute the values of metrical coefficients in Equation 44 and replace $x_1, x_2, x_3$ by $\gamma, \theta, \phi$. It is further assumed that $u$ may be expressed as a product of three functions, $H(\gamma), \Theta (\theta), \Phi (\phi)$. With these substitutions, Equation (44) breaks up into the following three equations:

\begin{align}
(45) \quad & \frac{d^2 H}{d \gamma^2} + \tanh \frac{dH}{d \gamma} - \frac{m^2}{\cosh^2 \gamma} H = n(n+1) H \\
(46) \quad & \frac{d^2 \Theta}{d \theta^2} + \cot \theta \frac{d \Theta}{d \theta} - \frac{m^2}{\sin^2 \theta} \Theta = -n(n+1) \Theta \\
(47) \quad & \frac{d^2 \Phi}{d \phi^2} = -m^2 \Phi
\end{align}

The Equations (45 and (46) are modified Legendre equations and are satisfied by associated Legendre functions. The Equation (47) has a simple harmonic solution. The general solution of the Laplace equation is

\begin{align}
u &= K_1 P_n^m (i \sinh \gamma) P_n^m (\cos \theta) \cos m\phi + \\
&+ K_2 Q_n^m (i \sinh \gamma) P_n^m (\cos \theta) \cos m\phi
\end{align}
When the boundaries are chosen along one of the coordinate surfaces, the solution simplifies considerably. If the cathode and the anode are confocal oblate spheroids defined by constant $\eta$ surfaces, the solution of $u$ will be independent of $\theta$ and $\phi$. When these symmetry conditions are imposed in Eqns. (45), (46) and (47), one finds that $m = n = 0$. The only necessary solution is that for $H$, and the Equation (45) simplifies to

\[(48) \quad \frac{d^2 H}{d\eta^2} + \tanh \eta \frac{dH}{d\eta} = 0,\]

and its solution is

\[H = Q_o (i \sinh \eta)\]

The solution for $u$ is

\[u = K_1 Q_o (i \sinh \eta) + K_2,\]

where $K_1$ and $K_2$ are arbitrary constants to be fixed by boundary conditions. If $\eta = \eta_c$ and $\eta = \eta_a$ define the cathode and the anode oblate spheroids, then the solution is

\[(49) \quad \frac{u}{u_a} = \frac{Q_o (i \sinh \eta) - Q_o (i \sinh \eta_c)}{Q_o (i \sinh \eta_a) - Q_o (i \sinh \eta_c)}\]

The above solution will be referred to again in the next section.

Solution of LaPlace Equation with the Resistor Network

The solution of the LaPlace equation for any three-dimensional
problem with rotational symmetry may be readily obtained with the help of the resistor network discussed in the last chapter. It should be mentioned that no general solution is possible with this device, but each case has to be worked out as a separate boundary value problem.

The first problem solved with the network is that of a confocal oblate spheroidal diode. In terms of spheroidal coordinates already defined, the cathode surface has \( \xi_c = 1.02 \) and the anode surface has \( \xi_a = 0.1 \). The cathode and the anode may also be defined in terms of the ratios of their major and minor axes. In this example, the cathode has an axial ratio of 1.3 and the anode has an axial ratio of 10. The shape of the cathode was chosen to correspond to the Heil cathode. The anode was chosen such that it is located as far away from the cathode as possible within the same confocal system in an attempt to get a high beam concentration. The electrode arrangement and resulting equipotential lines are shown in Fig. 14. The equipotential surfaces can be identified as confocal oblate spheroids of the same family as the cathode and the anode. The potential variation as a function of \( \xi \) is shown in Fig. 15, and is compared with the theoretical solution given by Equation (49). In Fig. 14, electron trajectories have been plotted from different parts of the cathode surface by means of well known electron ray tracing methods. In regions where an electron path makes a large angle with an equipotential line the 'circle method' of Salinger is used, and in cases where this angle is small
Fig. 14. Space-charge free equipotentials and trajectories in confocal spheroidal diode - $\eta_c = 1.02$, $\eta_a = 0.1$. 

f is focal point
Fig. 15. Space-charge-free potential as a function of $\eta$. 

$u/u_a$ vs. $\eta$.
the 'parabola method' developed by Spangenberg and Field is used. In both these methods, the electrons are assumed to be emitted normal to the cathode surface with no initial velocity.

**Solution of Poisson's Equation**

The general form of Poisson's equation is

\[ \nabla^2 u = -\frac{\rho}{\varepsilon_0} \]

where \( \rho \) is the space charge density, and \( \varepsilon_0 \) the permitivity of free space. If the distribution of \( \rho \) is known, the solution for \( u \) can be readily obtained on the resistor network when currents proportional to \( \rho \) are injected at various network junctions. However, in our problem \( \rho \) is not known and depends on the distribution of the desired function \( u \) itself. A simultaneous solution for \( \rho \) and \( u \) can be obtained by a method of successive approximation. To start this process, we assume \( \rho = 0 \) and obtain the solution of the Laplace equation. This space charge free potential distribution is used to compute an approximate first order space charge density distribution function. The space charge density \( \rho = J/v \), where \( J \) and \( v \) are the current density and velocity functions respectively. The velocity \( v \) is known from the assumed potential distribution; however, \( J \) has to be computed. In general the cathode current density may not be constant if the potential gradient near the cathode surface is non-uniform. The cathode surface is divided into a large number
of regions so that current density in any such region may be approximately assumed constant. The electron paths starting from the end of each region are plotted. The current density for each portion of the cathode is approximately calculated. Each portion of the cathode and the corresponding portion of the equipotential of value 1/20th (or lower) of the anode potential are approximated as parallel planes, and the current density calculated from Child's formula. For laminar flow the total current between any two trajectories is constant. The current density anywhere in the zone between two trajectories may be calculated from the corresponding current density at the cathode and the ratio of sectional area of the zone at the particular point to the corresponding area of the cathode. From the current density distribution, space charge density is easily computed.

The first order solution of space charge density is used to calculate currents to be injected at various network junctions. An approximate solution of Poisson's equation corresponding to the above space charge density is determined, and this solution is used to obtain improved space charge density distribution by repeating the procedure outlined in the preceding paragraph. This process is repeated until two consecutive solutions agree fairly closely.

In practice it is found that if the injected currents for the first approximation are calculated from the Laplace solution, the resulting potential gradient at the cathode surface is negative. This implies that
the assumed space charge density distribution is greater than the space-
charge-limited condition. In order that the solution may converge within
a reasonable number of approximations it is necessary to assume a more
plausible space charge density distribution. It has been found advanta-
geous to set the injected currents corresponding to the Laplace solution
at the first instance. The potential gradient at a number of points on the
cathode surface is measured on the network and the injected currents at
various junctions of the network are reduced by the same proportion until
the potential gradient at all points along the cathode surface changes from
negative value to a slightly positive value. This adjustment can be made
very easily because all the current sources are supplied from the same
voltage source, so the input voltage to the current sources is varied.
It should be mentioned that the potential gradient at different points on
the cathode surface does not change sign simultaneously at all points, and
this necessitates simultaneous measurement of potential gradient at
various points along the cathode surface as injected currents are ad-
justed. In our experience if this method is adopted, the solution con-
verges very rapidly and usually two to three approximations are suffi-
cient.

The solution of Poisson's equation between confocal oblate spheroids
\( \eta_c = 1.02, \eta_a = 0.1 \), for which the Laplace solution has already been
discussed, was obtained. The equipotential lines and trajectories for the
Fig. 16. Space-charge-limited equipotentials and trajectories first approximation.
Fig. 17. Space-charge-limited equipotentials and trajectories - final approximation.
Fig. 18. Comparison of axial potential distribution for various approximations.
Fig. 19. Current density distribution at the cathode surface.
first and the second approximations are shown in Figs. (16) and (17). The solutions of these two approximations are compared in Fig. 18 to show how closely they agree. The results of final approximation may be summed up as follows:

(i) The electron trajectories are not rectilinear. The electron paths are laminar.

(ii) The cathode current density is nonuniform. It is minimum along the axis of symmetry and increases with radial distance from it. In Fig. 19 the variation of cathode current density with radial distance is shown. The current density in this diagram is normalized with respect to the axial current density at the cathode.

(iii) The average current density at the anode is only four times greater than at the cathode.

(iv) The space charge limited equipotential surfaces are oblate spheroids of the same confocal system as the cathode and the anode.

In so far as the applications of these results to the design of high density electron beams are concerned, it is observed that the space-charge-limited cathode current is quite high. It is approximately four times greater than in a Heil gun with identical cathode and the same applied voltage. However, the beam concentration is very poor. Although it would be of some theoretical interest to extend this calculation to obtain a general solution of Poisson's equation in confocal spheroidal geometry,
the chances of finding a pair of confocal spheroids which would give
good beam convergence are not promising. The beam convergence may
be improved by increasing the cathode-to-anode distance. The anode
in the above analysis is already near its optimum position being quite
close to the focal point. By pushing the cathode away from the anode,
that is to say by studying cathodes with lower values of axial ratio than
1.3, some improvement in beam convergence is possible at the expense
of reduced cathode emission. However, the improvement in resulting
anode beam current would not be significant.

Nonconfocal Oblate Spheroidal Gun

Preliminary experiments with the aid of an electrolytic tank and
automatic trajectory plotter were made to explore qualitatively the
possibilities with nonconfocal geometry. These experiments indicated
that sufficient convergence can be achieved if the anode spheroid is moved
farther along the axis of symmetry (which is also the minor axis in oblate
spheroids). The major axes of the cathode and anode are no longer coin-
cident but are parallel to each other. The cathode was chosen identical
to that in the Heil gun with \( \eta_c = 1.02 \) or an axial ratio of 1.3. The anode
has \( \eta_a = 0.1 \) corresponding to an axial ratio of 10. The cathode and
the anode are separated by a certain distance such that the distance
between their major axes is equal to half the semiminor axis of the
cathode. A suitable aperture is made in the anode and it is terminated
into a nozzle. A focusing electrode is placed close to the cathode, and is a part of a circular cone making an angle of 26° to the axis of symmetry. Its position and size were determined in the electrolytic tank by trial and error. The electron trajectories for various positions and sizes of the focusing electrode were drawn from different parts of the cathode until the beam diameter was half the diameter of the assumed anode aperture. It was arbitrarily assumed that when space charge effects are considered the beam diameter might increase by a factor of two so that all the cathode current would still be able to go through the designed aperture. With regard to the focusing electrode, it should be observed that its position is quite critical in relationship to the beam convergence and the potential gradient near the edge of the cathode. Since most of the current in this geometry comes from the edge of the cathode, the effect of a slight change in position of the focusing electrode on the cathode current and beam convergence will be quite appreciable. An increase in the angle of the focusing electrode reduces the cathode current while the beam convergence improves. A compromise has to be made between these two conflicting factors. The position of the focusing electrode has thus to be carefully determined to get optimum current density in the focused beam.

An analysis of this gun was made on the resistor network. In Fig. 20, the electrode arrangement and the space charge free potential distribution
Fig. 20. Solution of LaPlace equation in non-confocal oblate spheroidal gun.
Fig. 21. Solution of Poisson's equation-first approximation.
Fig. 22. Solution of Poisson's equation, final approximation.
Fig. 23. Current density distribution at the cathode surface.
and electron trajectories are plotted. The solution for Poisson's equation was determined in two approximations. The corresponding potential distributions and electron trajectories are plotted in Figs. 21 and 22. The results of this analysis may be summed up as follows:

(1) The focused beam microperveance is 1.9 while that of the cathode is 2.1.

(2) The transmitted current is 90 percent of cathode emission current.

(3) Electron trajectories between the cathode and the anode are laminar. However, the electrons coming from the edge of the cathode make larger angles with the axis of symmetry and as a result some cross over may be expected as the beam leaves the anode aperture. In this region the field is very weak and it is difficult to plot the electron trajectories very precisely.

(4) The maximum beam concentration ratio is 200:1.

(5) The current density at the cathode surface is non-uniform. It increases from the axis toward the edge of the cathode in the ratio of 2:1. In Fig. 23, the cathode current density variation as a function of radial position from the axis is shown.

Comparison With Heil Gun

The above analysis is approximate to the extent that the effect of thermal velocities of emission have been neglected. Depending on the characteristics of the cathode emitting material used, the above figures
for beam perveance and beam transmission may be 25 to 50 per cent higher than the values practically possible. The value of microperveance given by Heil for his gun is four. In practice, however, the maximum perveance achieved with the Heil Gun is about two. A theoretical analysis of the Heil Gun by the present author and independently by Kuwabara indicate that electron paths are not laminar. This conclusion is supported by experimental results obtained by Campbell with a pin hole type beam analyzer. In comparing the Heil gun with the nonconfocal oblate spheroidal gun, it may be stated that the latter has much less aberration although the beam perveance is not quite so high.

From the foregoing conclusion, it would seem desirable to undertake a general solution of Poisson's equation in nonconfocal spheroidal surfaces in order to find better electron focussing systems.
Chapter IV

Potential Distribution In Paraboloidal Diodes

In this chapter, the solutions of the Laplace and the Poisson equations in confocal paraboloidal diodes obtained with the resistor network are presented. Since the analytical solution of the Laplace equation in this geometry is known, it is used to compare the results obtained with the network. The characteristics of the confocal paraboloidal diode in application to high density electron beam design are discussed. A preliminary examination of properties of nonconfocal paraboloidal geometry is made. An electron gun consisting of paraboloidal cathode, a conical focusing electrode and an oblate spheroidal anode is designed with the help of the electrolytic tank and the solution of Poisson's equation obtained with the network.

Analytical Solution of the Laplace Equation

The solution of the Laplace equation may be obtained in a convenient form if parabolic coordinates are used. In Fig. 24, two mutually orthogonal sets of parabolas with common focus at the origin, are shown. These parabolas may be represented by equations
Fig. 24. A system of parabolic coordinates.
\[ x^2 = 2 \xi^2 \left( z^2 + \frac{\eta^2}{2} \right) \]

\[ y^2 = -2 \eta^2 \left( z^2 + \frac{\eta^2}{2} \right) \]

The vertices of all the parabolas lie on the z-axis at distances \( \xi \) and \( \eta \) respectively. If we rotate these parabolas about the z-axis, the resulting surfaces are paraboloids of revolution defined by the above equations if one replaces \( x^2 \) by \( r^2 = x^2 + y^2 \), \( x = r \cos \phi \), \( y = r \sin \phi \). We thus obtain

\[ (51) \quad x = \xi \eta \cos \phi \]

\[ y = \xi \eta \sin \phi \]

\[ z = \frac{(\eta^2 - \xi^2)}{2} \]

\[ h_1^2 = h_2^2 = (\xi^2 + \eta^2) \]

\[ h_3^2 = \xi^2 \eta^2 \]

The parabolic coordinate surfaces are: (1) paraboloids of revolution extending in the direction of positive z-axis (\( \xi \) = constant); (2) paraboloids
of revolution extending toward negative z-axis ($\gamma = \text{constant}$); (3) planes through z-axis ($\phi = \text{constant}$).

The general form of the LaPlace equation in curvilinear coordinates is

$$\sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left( \frac{\sqrt{g}}{h_i^2} \frac{\partial u}{\partial x_i} \right) = 0, \quad \sqrt{g} = h_1 \cdot h_2 \cdot h_3$$

After substituting parabolic coordinates $\xi$, $\eta$, $\phi$ and metrical coefficients and separating the variables, one obtains the LaPlace equation in the form

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi \frac{dF_1}{d\xi} \right) + \frac{1}{\eta^2} \frac{d}{d\eta} \left( \eta \frac{dF_2}{d\eta} \right) + \frac{\xi^2 + \eta^2}{\xi \eta} \frac{d^2F_3}{d\eta^2} = 0$$

where $u = F_1(\xi) \cdot F_2(\eta) \cdot F_3(\phi)$

When the boundaries are set up along either of the $\xi$ or $\eta$ coordinate surfaces, the above equation simplifies considerably. For instance, if the anode and cathode are confocal paraboloids of revolution defined by coordinate surfaces $\eta = \eta_c$ and $\eta = \eta_a$, the solution of $u$ would be independent of $\xi$ and $\phi$ and would be given by the solution of the equation.
Substituting the boundary conditions \( u(\eta_c) = 0 \) and \( u(\eta_a) = u_a \), one finds

\[
\frac{1}{\eta u} \frac{d}{d \eta} \left( \eta \frac{du}{d \eta} \right) = 0
\]

This result will be used in next section.

**Solution of Laplace Equation with Resistor Network**

As has been mentioned earlier, the Laplace solution is required as a starting point for developing space charge limited potential distributions. It may be obtained either from Equation (55) or determined experimentally on the resistor network. Three problems with confocal paraboloidal geometry were studied. In all the three problems, the anode has a shape defined by \( \xi_a = 1 \). Three different cathodes with shapes \( \xi_c = 4, 5, 6 \) were selected. The results of analysis on the network in the form of equipotential lines and electron trajectories are presented in Figs. 25 to 27. The equipotential surfaces can be identified as paraboloids with the same focal point as the boundaries. In Fig. 28 to 30, variation of potential with parabolic coordinate \( \eta \) is given. The theoretical solution obtained from Equation (55) is also indicated in these figures for the three cases. The agreement between theoretical
Fig. 25. Solution of Laplace equation in confocal paraboloidal diode; $\eta_c=4, \eta_a=1$. 
Fig. 26. Solution of Laplace equation in confocal paraboloidal diode; \( \eta_c = 5, \eta_a = 1.0 \).
Fig. 27. Solution of Laplace equation in confocal paraboloidal diode; $\eta_c = 6$, $\eta_a = 1$. 
Fig. 28. Variation of potential as a function of $\eta$ in confocal paraboloidal diode; $\eta_c = 4$, $\eta_a = 1.$
Fig. 29. Variation of potential with $\eta$ in confocal paraboloidal diode; $\eta_c = 5, \eta_a = 1$. 

- Theoretical space-charge-free distribution
- Space-charge-free experimental
- Space-charge-limited experimental
Fig. 30. Variation of potential with $\eta$ in confocal paraboloidal diode; $\eta_c = 6$, $\eta_a = 1$. 
and experimental results is quite good.

Solution of Poisson's equation in Confocal Paraboloids

The solution of Poisson's equation for the three confocal paraboloidal diodes described above was obtained with the resistor network using the method of successive approximation described in the last chapter. Only two to three approximations were necessary in each case. The solutions corresponding to first and final approximations are presented in form of equipotential lines and electron trajectories in Figs. 31 to 36. The space-charge-limited equipotential surfaces may be identified as paraboloids of revolution with the same focal point as the cathode and the anode. The space charge limited and space charge free potential variation with parabolic coordinate $\eta$ is given Figs. 28 to 30.

The potential gradient at the cathode surface is not uniform and as a result the cathode current density is not constant. The variation of space charge limited current density at the cathode surface as a function of $r$ is plotted in Fig. 37 for the three cathodes. In each case the current density has been normalized with respect to axial current density corresponding to a cathode with $\eta_c = 6$. It is noticed that cathode current density decreases with increasing $\eta_c$, which is understandable because by increasing $\eta_c$, the cathode paraboloid is moved farther away from the anode paraboloid. For a given cathode, the current density is maximum along the axis of symmetry, $r = 0$, and decreases with increasing $r$. This variation is opposite to that
Fig. 31. First approximation solution of Poisson's equation in confocal paraboloidal diode; \( \gamma_c = 4, \gamma_a = 1 \).
Fig. 32. Final solution of Poisson's equation.
Fig. 33. First approx. solution of Poisson's equation in confocal paraboloidal diode; $\eta_c = 5, \eta_a = 1$. 
Fig. 34. Final solution of Poisson’s equation.
Fig. 35. First approx. solution of Poisson's equation in confocal paraboloidal diode; $\eta_c = 6; \eta_a = 1.$
Fig. 36. Final solution of Poisson's equation.
Fig. 37. Cathode current density distribution in confocal paraboloidal diodes.
observed in the case of spheroids where the cathode current density is minimum along the axis and increases with r. In an electron gun, it is often desirable to have uniform cathode emission because with laminar electron flow the focussed beam may be expected to have uniform current density. This is difficult to realize even in a spherical Pierce type gun due to distortion in the field near the cathode caused by anode aperture. It is conceivable that in paraboloidal diode, the effect of anode aperture could be used to an advantage in making the cathode current density more uniform.

The electron paths are laminar. The increase in current density at the anode is again very small as in confocal oblate spheroids. It improves slightly with increasing value of \( \gamma_c \), but the increase is not significant for any application in producing a high density electron beam, and the increase in current density is offset by reduced emission from the cathode.

**Nonconfocal Paraboloids**

A preliminary and qualitative study of electron focussing properties of a nonconfocal paraboloidal diode was made with the help of the electrolytic tank and electron trajectory plotter. The cathodes and anodes selected have shapes given by \( \gamma_c = 4, 5, 6 \) and \( \gamma_a = 1, 2, 3 \) respectively. The electron trajectories for different combinations of these electrodes were obtained. The effect of distance between anode and cathode for different combinations was studied. The effect of addition of a focussing
electrode was studied. The results of these experiments may be summarized as follows:

(1) For a given pair of paraboloids, when the distance between the cathode and the anode is increased so that the focal points of the two are no longer coincident, the field at the cathode surface decreases with increased distance between the focal points of the two electrodes. The current density at the cathode surface falls off. Because of the nature of paraboloidal geometry the decrease in current density at the cathode surface is greater for greater values of \(r\) or for larger distances from the axis. As a result, the cathode emission current falls off very rapidly in noncon­focal paraboloids with increased distance between their focal points.

(2) For a given pair of paraboloids, the focussed beam diameter can be made as small as desired by moving the anode farther away from the cathode paraboloid and introducing a focussing electrode which is a part of a circular cone and is placed close to the cathode. The con­tribution of the focussing electrode in producing additional radially converging field is significant and its role is thus different from that in the Pierce gun where it is intended to compensate for space charge repulsion. The presence of the focussing electrode causes further weakening of the potential gradient near the cathode; this effect is appreciable at the edge of the cathode; as a result the total cathode current drops off.

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When the results of (1) and (2) are combined, it appears that neither confocal nor nonconfocal paraboloidal diodes are promising for high perveance beam design.

**Paraboloidal-Spheroidal Gun**

In an effort to increase the beam perveance and beam convergence, an electron gun consisting of a paraboloidal cathode, oblate spheroidal anode and a part of a circular cone as focusing electrode was designed with the help of the electrolytic tank and automatic trajectory plotter. The cathode selected has $\eta_c = 4$ and the anode oblate spheroid has an axial ratio of 10. The distance between the cathode and the anode and the position of the focusing electrode was determined by trial and error in the electrolytic tank. The beam convergence and the potential gradient at the cathode were simultaneously observed and the best positions of various electrodes determined to give desirable beam convergence and beam perveance. The analysis of this gun was obtained on the resistor network. The equipotential lines and electron trajectories corresponding to the Laplace solution are shown in Fig. 38. The solution of Poisson's equation is obtained by using the iteration procedure described in Chapter III. The equipotential lines and electron trajectories corresponding to first and second order approximations are presented in Fig. 39 and 40 respectively. The results of this analysis may be summed up as follows:

1. The current density at the cathode surface is non-uniform, increasing
Fig. 38. Solution of Laplace equation in paraboloidal-spheroidal gun.
Fig. 39. First approx. solution of Poisson's equation in paraboloidal-spheroidal gun.
Fig. 40. Final solution of Poisson's equation in paraboloidal-spheroidal gun.
Fig. 41. Cathode current density distribution in paraboloidal spheroidal gun.
Fig. 42. Cathode current distribution in paraboloidal-spheroidal gun.
from the axis to the edge of the cathode in the ratio 8:1. The current
density drops off slightly at the edge of the cathode after reaching the
maximum due to the influence of focussing electrode. In Fig. 41, the
cathode current density variation as a function of radial position from
the axis is shown. The percent current contribution of various regions
of the cathode to the total cathode current is plotted in Fig. 42.

(2) The cathode microperveance is four. The microperveance of the
focussed beam is also four, implying 100 percent transmission.
However, since the effect of thermal velocities has been neglected in
this analysis, the practical value of beam microperveance may be about
25 percent lower.

(3) Electron trajectories between the cathode and the anode are laminar.
The electron paths at the exit of the anode nozzle have a negative slope
\( \frac{dr}{dz} \) implying that beam minimum would occur outside the beam nozzle
in the field free region. The electron trajectories in this region may be
obtained from Pierce\(^{33}\) solution of paraxial ray equation. The position of
the beam minimum and the corresponding radius may then be obtained.
In the present case the beam edge has a radius of 6.5 mesh units at
the anode exit and it makes an angle of 22 degrees with the axis. The

\(^{33}\) Pierce, J.R., "Theory and Design of Electron Beams", D. Van
minimum radius of the focussed beam is 5.5 mesh units. The position of beam minimum would depend on the total current in the beam and the anode potential.

(4) The area compression ratio at the beam minimum is 30.

(5) If the mesh size is chosen equal to .0724 mm., the area of the resulting cathode will be nearly equal to the 1/5th size Heil cathode. The corresponding beam current at anode potential of 800 volts is 90 ma. The minimum radius of the focussed beam is .4 mm. and the position of the beam minimum is approximately .8 mm. from the anode exit.

Peck obtained the following data for the 1/5 size Heil gun: beam current 80 ma at anode potential of 1000 volts, minimum beam radius equal to .24 mm.

In comparing the paraboloidal-ellipsoidal gun to the Heil gun, it may be stated that the former has a beam microperveance of four and an area compression ratio of 30, while the corresponding values for the latter are 2.5 and 90. Higher perveance in the paraboloidal-ellipsoidal gun has been obtained by deliberately distorting the field at the cathode surface resulting in higher emission current density from the cathode edge. The beam aberation in this gun may be expected to be much less and the current density more uniform. It may be possible to improve the

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beam compression by better design of the anode and focussing electrode.

**Accuracy of the Results**

In the present analysis, the effects of thermal velocity distribution of the emitted electrons on the size and shape of the beam have been disregarded. The experimental and theoretical work of Cutler, Hines and Saloom indicates that broadening of the focussed beam due to transverse velocities of the emitted electrons depends on the cathode temperature and the anode potential. In extreme cases, the percentage increase in beam size may be as large as 100 to 200 percent. In view of this consideration, the values of beam convergence calculated for the two guns may not be practically achieved. The other errors are inherent in the iterative procedure used in determining the space-charge-density distribution function. It may be that the final solution may not have exactly converged to the actual solution. The accuracy of graphical trajectory plotting technique is not better than 5 per cent even when the best precautions are taken. However, the error due to the latter factor is not significant in comparison to the effects of thermal velocities. In conclusion, it may be stated that the results for the cathode and the beam perveance, the cathode current density distribution and the space-charge-limited potential distribution may be in error from 5 to 10 percent. The values of beam convergence derived in the present case do not include the effect of thermal velocities and, consequently, may be too high to be practically possible.
Chapter V

Summary and Conclusions

Pierce's method, which has become a cornerstone in the design of dense electron beams, applies only to cases where electron paths are rectilinear. In addition, the emission velocity and anode aperture effects are neglected; these factors are quite significant for beams with micro-perveance of one and higher. In recent years, some success has been achieved in developing guns with perveance greater than attainable with conventional Pierce-type guns. One such example is the Heil gun in which a perveance of four is reported. This gun, developed by 'cut and try' experimental procedure, consists of an ellipsoidal cathode, a Pierce-type focussing electrode, and a conical shaped anode. The electron paths in this gun are curved. In view of the success achieved by Heil, need has arisen to extend Pierce's method of gun design to cases where electron paths may be curvilinear. The first step in this study is to establish a solution of Poisson's equation in geometries of interest.

Two distinctly different approaches have been used. In the first by Meltzer, the nature of electron flow (vector velocity function) is assumed, and the solution for the configuration of corresponding electrodes is attempted. So far only a few problems having little practical interest have been worked out. The other method is to assume the boundaries and solve for
the potential distribution and electron motion. Analytical solution of Poisson's equation, except for a few simple cases worked out by Langmuir and Blodgett, is very difficult due to limitations in the function theory. The choice is then limited to purely numerical procedures or to analogue techniques. In the present study, the latter approach has been adopted. A two-dimensional resistance network, in which resistance values are suitably graded to permit representation of three-dimensional problems with circular symmetry, has been used. The following problems have been investigated:

1. Confocal Ellipsoids

The cathode and the anode were chosen as confocal ellipsoids with axial ratios of 1.3 and 10 respectively. This particular choice was guided by its closeness to the Heil ellipsoidal gun previously studied at this laboratory. The results may be summarized as follows:

(a) The cathode current density is higher (about 100 per cent) than the corresponding ellipsoidal gun.

(b) The electron flow is laminar.

(c) The concentration ratio is poor.

(d) The cathode current density is non-uniform. It increases from the axis to the edge of the cathode.

Since (c) and (d) would become worse when the effect of anode aperture is included, there seems little advantage of further study of confocal
ellipsoids from the standpoint of practical applications.

2. Nonconfocal Ellipsoids

It is possible to achieve good convergence if a focussing electrode is added and the anode ellipsoid moved further along the axis of symmetry (which is also the minor axis of the ellipsoids). The major axes of cathode and anode are no longer coincident but are parallel. A gun with the following particulars was designed:

- Cathode - ellipsoid, axial ratio 1.3
- Anode - ellipsoid, axial ratio 10

Distance between major axes of cathode and anode is equal to half the semi-minor axis of the cathode.

A suitable aperture was made in the anode and a focussing electrode designed with the help of the electrolytic tank. The analysis of this gun on the resistance network gave the following information:

(a) Beam microperveance is 2.

(b) Electron trajectories do not cross each other or the axis.

(c) Concentration ratio is about 200 to 1.

(d) Current density at the cathode surface is nonuniform increasing in the ratio 1:2 from the axis to the edge of the beam.

3. Confocal Paraboloids

A preliminary investigation of confocal paraboloids has been made.

It is convenient to describe various paraboloids with a common focal point
at the origin and the axis of symmetry along the \( z \)-axis in terms of parameter, \( \eta \), defined by

\[
\eta^2 = \sqrt{z^2 + x^2 + z}
\]

The anode chosen has \( \eta_a = 1 \). Three different cathode shapes with \( \eta_c = 4, 5, \) and \( 6 \) were tried. The results may be summarized as follows:

(a) The cathode current density decreases with \( \eta_c \).

(b) Concentration ratio increases with \( \eta_c \), but is still poor for any design.

(c) Electron flow is laminar.

(d) For a given cathode, current density is highest at the axis of symmetry and gradually decreases toward the edge.

The situation of (d) above is exactly the reverse of the one in confocal ellipsoids. The effect of the anode aperture in this case could be to make the cathode current density more uniform.

4. Nonconfocal Paraboloids

By moving the anode electrode farther away from the cathode so that their focal points no longer coincide, and adding a focussing electrode, it is possible to get good convergence. An approximate study of electron paths between various nonconfocal paraboloids has been made with the help of the electrolytic tank and automatic trajectory plotter. The
present results do not include space charge. The role of focussing electrode is rather 'baffling'. The Pierce focussing electrode was intended to match the fields on the two sides of the beam boundary and thereby provide compensation for space charge repulsion. However, experience with nonconfocal paraboloids and ellipsoids indicates that the focussing electrode makes a definite contribution in providing an additional radially converging field, for, without the focussing electrode it is not possible to get the necessary convergence.

5. Paraboloidal-Ellipsoidal Gun

In an attempt to increase the beam perveance and compression, an electron gun consisting of a paraboloidal cathode, oblate spheroidal anode and a part of a circular cone as focussing electrode was designed with the help of the electrolytic tank. The axes of symmetry of the cathode and the anode are coincident. A suitable aperture is made in the anode and it is terminated into a conical nozzle. The analysis of this gun on the resistor network gives a beam microperveance of four and an area compression ratio of 30.

Results obtained up to this time indicate definite possibilities with nonconfocal paraboloidal and ellipsoidal guns. However, the method used has an inherent weakness that each specific problem has to be analyzed separately, and therefore, no generalized solution can be deduced. Furthermore, the role of focussing electrode is not quite understood in these geometries. In view of these difficulties, it is difficult to formulate a general design procedure.
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33. See footnotes.

34. See footnotes.
AUTOBIOGRAPHY

I, Ram Prakash Anand, was born at Lahore, India, on December 20, 1930. I had my secondary school education from D.A.V. School, Lahore, and obtained the Bachelor and Master's degrees in Physics from the Punjab University in 1949 and 1950 respectively. From 1950 to 1953 I studied at the Indian Institute of Science and graduated with a Diploma in Electrical Technology. In October 1953, I joined the Graduate School of the Ohio State University. In 1955, I obtained the Master's degree in Electrical Engineering. While a graduate student at the Ohio State University, I held the positions of Research Assistant, Research Associate and Instructor in the Department of Electrical Engineering.