A FUNCTIONAL ASSOCIATED WITH A
CONTINUOUS TRANSFORMATION

DISSERTATION

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1. In this preface, we shall introduce notations and definitions used throughout this dissertation.

The bibliography is at the end of the dissertation. Throughout the text of the dissertation, numbers in square brackets will be used to refer to the references listed in the bibliography. For example, [4; pp. 599 - 600] refers to pages 599 and 600 of the fourth reference in the bibliography.

Square brackets will also be used to refer to other portions of this dissertation. For example, [cf. 1.3] will refer the reader to the third section in Chapter I.

2. Let \( \mathcal{N} \) be any set of elements. Let \( A \) and \( B \) denote subsets of the set \( \mathcal{N} \), and let \( w \) be an element of the set \( \mathcal{N} \).

If the element \( w \) belongs to the set \( A \), we shall write \( w \in A \). Otherwise, we shall write \( w \notin A \).

If the set \( A \) is contained in the set \( B \), we shall write \( A \subseteq B \). Otherwise, we shall write \( A \nsubseteq B \).

\( A + B \) will denote the set of elements which belong either to the set \( A \), or the set \( B \), or both of these sets \( A \) and \( B \). \( A \cdot B \) will denote the set of elements which belong to both of the sets \( A \) and \( B \).

The empty subset of the set \( \mathcal{N} \) will be denoted by \( \emptyset \).

3. Let \( A \) be any subset of the Euclidean plane \( \Pi \).
We shall denote the complement, closure, frontier, and interior of \( A \), each relative to \( \mathcal{P} \), by \( \complement(A) \), \( c(A) \), \( fr(A) \), and \( i(A) \), respectively.

We shall say that \( A \) is open (or closed) if \( A \) is open (or closed) relative to \( \mathcal{P} \).

If \( A = B \cdot \complement(D) \), where \( B \) and \( D \) are subsets of \( \mathcal{P} \), we shall write \( A = B - D \), and we shall say that \( A \) is the complement of \( D \) relative to \( B \).

If \( A = B \cdot D \), where \( D \) is an open (or closed) set, we shall say that \( A \) is open (or closed) relative to \( B \).

If \( A \) is open and connected, we shall say that \( A \) is a domain.

If \( A \) is a bounded domain, and if \( fr(A) \) consists of just one component, we shall say that \( A \) is a simply connected domain.

If \( A \) is a bounded domain, if \( fr(A) \) consists of a finite number of disjoint simple closed curves, and if \( B = c(A) \), then we shall say that \( B \) is a Jordan region.

If \( A \) is a bounded domain, if \( fr(A) \) consists of exactly one simple closed curve, and if \( B = c(A) \), then we shall say that \( B \) is a simply connected Jordan region.

If \( A \) is a bounded domain, if \( fr(A) \) consists of a finite number of disjoint simple closed polygons, and if \( B = c(A) \), then we shall say that \( B \) is a polygonal region.

If \( A \) is a bounded domain, and if \( fr(A) \) consists of exactly one simple closed polygon, and if \( B = c(A) \), then
we shall say that $B$ is a simply connected polygonal region.

4. A set will be said to be admissible if it is either a Jordan region or a bounded domain.

Note that every polygonal region is a Jordan region, and every simply connected polygonal region is a simply connected Jordan region.

Observe that if $B$ is a polygonal region, then $\mathcal{L}(B)$ and $\text{fr}(B)$ have the same number of components. For each component $C$ of $\text{fr}(B)$, there is exactly one component $D$ of $\mathcal{L}(B)$ such that $\text{fr}(D) = C$. The frontiers of the remaining components of $\mathcal{L}(B)$ do not intersect $C$. Further, for each component $E$ of $\mathcal{L}(B)$, there is exactly one component $F$ of $\text{fr}(B)$ such that $F = \text{fr}(E)$. The remaining components of $\text{fr}(B)$ do not intersect $\text{fr}(E)$. \[3; \text{p. } 118\]
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CHAPTER I

Discussion of the Lower Area of Radó and Related Concepts

1.1 Let $T: z = t(w), w \in R_0$, be a continuous transformation from a simply connected polygonal region $R_0$ in the Euclidean plane $\mathbb{R}$ into Euclidean three-space. Then the transformation $T$ is a representation for an $F$-surface of the type of the 2-cell in Euclidean three-space, which we shall call, in brief, a surface $S$. [5; II. 3.7, II. 3.44]

In connection with the transformation $T$, Radó defines a non-negative (possibly infinite) quantity $a(T)$, which he shows is independent of the representation $T$ for the surface $S$. [5; V. 1.6] Radó calls this quantity $a(T)$ the lower area of the surface $S$, and it plays an important part in the study of surface area. [5; V. 1.3]

1.2 In defining the lower area $a(T)$ of the surface $S$, Radó uses finite collections of disjoint domains. [5; V. 1.3] This suggests the possibility of using other collections of sets in this definition rather than collections of disjoint domains.

We shall consider the following: [cf. Preface]

(a) Finite collections of disjoint simply connected polygonal regions, each of which is contained in $R_0$. Collections of this kind will be called collections of the first class.
(b) Finite collections of disjoint polygonal regions, each of which is contained in \( R_0 \). Collections of this kind will be called collections of the second class.

(c) Finite collections of simply connected Jordan regions, with disjoint interiors, each of which is contained in \( R_0 \). Collections of this kind will be called collections of the third class.

(d) Finite collections of Jordan regions, with disjoint interiors, each of which is contained in \( R_0 \). Collections of this kind will be called collections of the fourth class.

(e) Finite collections of disjoint simply connected domains, each of which is contained in \( R_0 \). Collections of this kind will be called collections of the fifth class.

(f) Finite collections of disjoint domains, each of which is contained in \( R_0 \). Collections of this kind will be called collections of the sixth class.

Using collections of the \( j \)-th class in place of finite collections of disjoint domains in defining the lower area \( a(T) \) gives rise to a non-negative (possibly infinite) quantity \( a_j(T) \), for each \( j, j = 1, \ldots, 6 \). It will be noted later that the quantity \( a_6(T) \) is precisely the lower area \( a(T) \).

The definitions of these quantities will be stated in 1.4, after a discussion of certain topological concepts in 1.3.
The purpose of this dissertation is to show that \( a_j(T) = a(T) \) for each \( j, j = 1, \ldots, 6 \). This will establish the fact that any collections of the kind described above may be used in defining, or in determining, the lower area of a surface.

1.3 Let \( F: \mathbb{Z} = f(w), w \in R_0 \), be a continuous transformation from a simply connected polygonal region \( R_0 \) in the Euclidean plane \( \mathbb{F} \) into the Euclidean plane \( \mathbb{F} \).

If \( P \) is an admissible set [cf. Preface] contained in \( R_0 \), define \( F|P: \mathbb{Z} = f(w), w \in P \). The transformation \( F|P \) may be considered as the transformation \( F \) restricted to the set \( P \). Note that if \( E \subseteq P \), then \( F|P(E) = F(E) \).

If \( C \) is an oriented simple closed curve contained in \( R_0 \), let \( \overline{C} = F(C) \). Then \( \overline{C} \) is an oriented closed curve contained in \( \mathbb{F} \). We define a function \( \mu(\overline{Z},F,C), \overline{Z} \in \mathbb{F} \), as follows. If \( \overline{Z} \in \overline{C} \), let \( \mu(\overline{Z},F,C) = 0 \). If \( \overline{Z} \notin \overline{C} \), let \( \mu(\overline{Z},F,C) \) equal the topological index of the point \( \overline{Z} \) with respect to the oriented closed curve \( \overline{C} \). Then, if \( \overline{Z} \in \mathbb{F} \), the value of \( \mu(\overline{Z},F,C) \) is a positive or negative integer or zero. [5; II. 4.34]

If \( R \) is a Jordan region contained in \( R_0 \), let \( C_1, \ldots, C_{n-1} \), denote the interior boundary curves of \( R \), oriented in the clockwise sense, and let \( C_n \) denote the exterior boundary curve of \( R \), oriented in the counterclockwise sense. We define a function \( \mu(\overline{Z},F,R), \overline{Z} \in \mathbb{F} \), as follows. If \( \overline{Z} \in F(C_1 + \cdots + C_n) \), let \( \mu(\overline{Z},F,R) = 0 \). If
If \( P \) is an admissible set contained in \( R_0 \), and if \( \bar{z} \in \overline{\Pi} \), let \( (P|P)^{-1} \bar{z} \) denote the set of points \( w \) such that \( w \in P \) and \( F(w) = \bar{z} \). (Note that \( (P|P)^{-1} \bar{z} \subset F^{-1} \bar{z} \), but generally \( F^{-1} \bar{z} \notin (P|P)^{-1} \bar{z} \).) If \( (P|P)^{-1} \bar{z} \) is not empty, then the components of \( (P|P)^{-1} \bar{z} \) will be called maximal model components for \( \bar{z} \) under \( P|P \). If a maximal model component for \( \bar{z} \) under \( P|P \) is closed, then it is a continuum, and it will be called a maximal model continuum for \( \bar{z} \) under \( P|P \). (Note that if \( P \) is closed, then a maximal model component for \( \bar{z} \) under \( P|P \) is closed, and is therefore a maximal model continuum for \( \bar{z} \) under \( P|P \).)

If \( \gamma \) is a maximal model component for \( \bar{z} \) under \( P|P \), and if, for every open set \( G \) containing \( \gamma \), there is a Jordan region \( R \) such that \( \gamma = i(R) \subset R \subset G \cdot i(P) \), (note that this implies that \( \gamma \) is closed), and such that \( \mu(\bar{z}, F, R) \neq 0 \), then we shall say that \( \gamma \) is an essential maximal model continuum for \( \bar{z} \) under \( P|P \). [5; IV. 1.27]

Note that if \( \gamma \) is an essential maximal model continuum for \( \bar{z} \) under \( P|P \), then necessarily, \( \gamma \subset i(P) \).

If \( P \) and \( Q \) are admissible sets, and \( Q \subset P \subset R_0 \), and if \( \bar{z} \in \overline{\Pi} \), then \( \kappa(\bar{z}, F|P, Q) \) will denote the number (possibly infinite) of essential maximal model continua for \( \bar{z} \) under \( F|P \) which are contained in \( i(Q) \). Note that if \( \bar{z} \in \overline{\Pi} \), then \( \kappa(\bar{z}, F|P, Q) \), if finite, is a non-negative
integer.

Also, if \( \bar{z} \in \bar{\Omega} \), then

\[
\kappa(\bar{z}, F|Q, Q) = \kappa(\bar{z}, F|P, Q) = \kappa(\bar{z}, F, Q).
\]

Further, if \( P_1, \cdots, P_n \) is a collection of the \( j \)-th class, \( j = 1, \cdots, 6 \), [cf. 1.2], such that \( P_k \subset Q \), for each \( k \), \( k = 1, \cdots, n \), then

\[
\sum_{k=1}^{n} \kappa(\bar{z}, F, P_k) \leq \kappa(\bar{z}, F, Q). \quad [5; IV. 1.21]
\]

The set \( F(R_0) \) is bounded, since \( R_0 \) is compact, and since \( F \) is a continuous transformation. Clearly, if \( \bar{z} \notin F(R_0) \), then there are no maximal model continua for \( \bar{z} \) under \( F \), and hence \( \kappa(\bar{z}, F, R_0) = 0 \).

If \( P \) is an admissible set contained in \( R_0 \), then

\[
\kappa(\bar{z}, F, P), \bar{z} \in \bar{\Omega},
\]

is a lower semi-continuous function, and hence, a measurable function. [5; IV. 1.11]

If \( R \) is a Jordan region contained in \( R_0 \), then

\[
|\mu(\bar{z}, F, R)|, \bar{z} \in \bar{\Omega},
\]

is a lower semi-continuous function, and hence, a measurable function. [5; IV. 1.25]

If \( \kappa(\bar{z}, F, R_0), \bar{z} \in F(R_0) \), is summable, then we shall say that the transformation \( F: \bar{z} = f(w), w \in R_0 \), is of essential bounded variation. [5; IV. 4.1] Note that if \( P \) is an admissible set contained in \( R_0 \), and if

\[
F: \bar{z} = f(w), w \in R_0, \]

is of essential bounded variation, then

\[
\kappa(\bar{z}, F, P), \bar{z} \in F(P),
\]

is summable, and hence the set of points \( \bar{z} \) for which \( \kappa(\bar{z}, F, P) \) is infinite is a set of measure zero.

1.4 We shall return to the transformation

\[
T: z = t(w), w \in R_0,
\]

described in 1.1. Let us write
\( T: z = t(w) = (x_1(w), x_2(w), x_3(w)), w \in R_0, \) where \( x_1(w), x_2(w), x_3(w) \) are the rectangular coordinates of \( t(w) \). We shall define three plane transformations as follows.

\[ T_1: z_1 = t_1(w) = (x_2(w), x_3(w)), w \in R_0 \]
\[ T_2: z_2 = t_2(w) = (x_3(w), x_1(w)), w \in R_0 \]
\[ T_3: z_3 = t_3(w) = (x_1(w), x_2(w)), w \in R_0. \]

Then, for \( i = 1, 2, 3 \), \( T_i: z_1 = t_i(w), w \in R_0, \) is a continuous transformation from the simply connected polygonal region \( R_0 \) in the Euclidean plane \( \Pi \) into the Euclidean plane \( \Pi_i \).

If \( P \) is an admissible set contained in \( R_0 \), let
\[ g(T_i, P) = \int K(z_i, T_i, P), \text{ for } i = 1, 2, 3, \text{ and let} \]
\[ G(T, P) = \left( \int_{i=1}^{3} (g(T_i, P))^2 \right)^{\frac{1}{2}}. \]
These quantities are non-negative, and possibly infinite.

If \( \mathcal{I} \) is a collection of admissible sets contained in \( R_0 \), let
\[ g(T_i, \mathcal{I}) = \sum_{P \in \mathcal{I}} g(T_i, P), \text{ for } i = 1, 2, 3, \text{ and let} \]
\[ G(T, \mathcal{I}) = \sum_{P \in \mathcal{I}} G(T, P). \]

For \( j = 1, \cdots, 6 \), let \( a_j(T) = \text{l.u.b.} G(T, \mathcal{I}), \) where \( \mathcal{I} \) is a collection of the \( j \)-th class, and the least upper bound is taken with respect to all collections of the \( j \)-th class. These quantities are non-negative, and possibly infinite, and represent the quantities referred to in 1.2.

Observe that \( a_6(T) \) is precisely the lower area \( a(T) \).

[5; v. 1.3]

1.5 The definitions of the quantities \( a_j(T) \), \( j = 1, \cdots, 6 \), suggest the possibility of using denumerable
collections of sets, rather than finite collections of sets. In this connection, we shall define, for \( j = 1, \ldots, 6 \),
\[
a_j'(T) = \text{l.u.b.} G(T, \mathfrak{I}),
\]
where \( \mathfrak{I} \) is a denumerable collection of sets of the type used in defining \( a_j(T) \), and the least upper bound is taken with respect to all such collections of this kind. Clearly, \( a_j(T) \leq a_j'(T) \), \( j = 1, \ldots, 6 \).

On the other hand, if \( \mathfrak{I} \) is any denumerable collection of admissible sets contained in \( \mathbb{R}_0 \), and if \( k \) is a real number such that \( G(T, \mathfrak{I}) > k \), then there exists a finite subcollection \( \mathfrak{V} \) of \( \mathfrak{I} \) such that \( g(T, \mathfrak{V}) > k \). From this remark, it is clear that for \( j = 1, \ldots, 6 \), we have
\[
a_j(T) \geq a_j'(T),
\]
and hence \( a_j(T) = a_j'(T) \).

Consequently, we conclude that using denumerable collections of sets in defining the quantities \( a_j(T) \), for \( j = 1, \ldots, 6 \), will yield the same value as do finite collections of sets.

1.6 It is quite evident from the definitions set forth in 1.4, that \( a_1(T) \leq a_2(T) \), \( a_1(T) \leq a_3(T) \),
\[
a_2(T) \leq a_4(T) \), \( a_3(T) \leq a_4(T) \), and \( a_5(T) \leq a_6(T) \).
\]

Further, if \( R_1, \ldots, R_n \) is a collection of the third class, then \( i(R_1), \ldots, i(R_n) \), is a collection of the fifth class, while, for \( k = 1, \ldots, n \), and for \( i = 1, 2, 3 \), we have
\[
k(z_k, T_i, R_k) = \kappa(z_k, T_i, i(R_k)).
\]
From this, it follows readily that \( a_3(T) \leq a_5(T) \).

The same type of reasoning may be used to show that
\[
a_4(T) \leq a_6(T).
\]
1.7 In Chapter II, it will be shown that 
\[ a_1(T) = a_3(T) = a_5(T), \text{ and } a_2(T) = a_4(T) = a_6(T). \]

Having this information, an indirect proof may be used to show that \( a_j(T) = a(T) \) for \( j = 1, \ldots, 6 \).

Cesari defines a quantity \( a'(T) \) as follows. First, if \( R_1, \ldots, R_n \) is any collection of the third class such that 
\[ \sum_{k=1}^{n} R_k = R_0, \]
we shall say that \( R_1, \ldots, R_n \) is a subdivision of \( R_0 \).

If \( R \) is a Jordan region contained in \( R_0 \), let
\[
\begin{align*}
&h(T_i, R) = \left\| \mu(z_i, T_i, R) \right\|, \text{ for } i = 1, 2, 3, \text{ and let} \\
&H(T, R) = \left( \sum_{i=1}^{3} (h(T_i, R))^2 \right)^{1/2}.
\end{align*}
\]

If \( \bar{\mathcal{R}} \) is any collection of Jordan regions, each of which is contained in \( R_0 \), let
\[ H(T, \bar{\mathcal{R}}) = \sum_{R \in \bar{\mathcal{R}}} H(T, R). \]

Finally, let \( a'(T) = \text{l.u.b. } H(T, \bar{\mathcal{R}}) \), where \( \bar{\mathcal{R}} \) is a subdivision of \( R_0 \), and the least upper bound is taken with respect to all subdivisions of \( R_0 \).

It has been shown that \( a'(T) = a(T) \). \([5; \text{V. 2.65}]\)

Now, Radó has shown that, for \( i = 1, 2, 3 \), there exists a countable set \( B_i \) such that \( B_i \subseteq \mathbb{N}_i \), and such that if \( z_i \notin B_i \), then for every Jordan region \( R \) contained in \( R_0 \), we have
\[ |\mu(z_i, T_i, R)| \leq K(z_i, T_i, R). \] \([4; \text{pp. 599-600}]\)

Hence if \( R \) is a Jordan region contained in \( R_0 \), we have
\[ h(T_i, R) \leq g(T_i, R), \text{ for } i = 1, 2, 3, \text{ and} \]
\[ H(T, R) \leq G(T, R). \]
This implies that if \( \bar{\mathcal{R}} \) is any subdivision of \( R_0 \), then
\[ H(T, \bar{\mathcal{R}}) \leq G(T, \bar{\mathcal{R}}) \]
It follows readily from this that \( a'(T) \leq a_3(T) \).
But in 1.6, it was noted that $a_3(T) \leq a_5(T) \leq a_6(T) = a(T)$, so $a'(T) \leq a_5(T) \leq a_6(T)$. As noted above $a'(T) = a(T)$, so $a'(T) = a_3(T) = a_6(T) = a(T)$.

Since, as noted at the beginning of this section, we can show that $a_1(T) = a_3(T) = a_5(T)$, and $a_2(T) = a_4(T) = a_6(T)$, it follows that $a'(T) = a_j(T) = a(T)$, for $j = 1, \ldots, 6$.

1.8 Our purpose is to establish a direct proof of the equalities $a_j(T) = a(T)$, $j = 1, \ldots, 6$.

As remarked previously, in Chapter II, we shall show that $a_1(T) = a_3(T) = a_5(T)$ and $a_2(T) = a_4(T) = a_6(T)$.

In Chapter III, we shall develop some lemmas of a topological nature to be used in the succeeding chapter.

In Chapter IV, it will be shown that $a_1(T) = a_2(T)$, and this, combined with the information obtained in Chapter II, will yield $a_j(T) = a(T)$, for $j = 1, \ldots, 6$.

1.9 We shall say that the transformation $T: z = t(w)$, $w \in R_0$, is of essential bounded variation, if the transformations $T_i: z_i = t_i(w)$, $w \in R_0$, are of essential bounded variation, for $i = 1, 2, 3$.

Note that if $T$ is not of essential bounded variation, then $a_1(T) \geq G(T,R_0) = +\infty$, and, using the facts noted in 1.6, we have $a_j(T) = a(T) = +\infty$, for $j = 1, \ldots, 6$.

Consequently, in later chapters, in proving that $a_j(T) = a(T)$, $j = 1, \ldots, 6$, we may assume without loss of generality, that $T$ is of essential bounded variation.

-9-
1.10 Let us note that if $P$ is an admissible set contained in $R_0$, then $G(T,P) \leq g(T_1,P) + g(T_2,P) + g(T_3,P)$.

Hence, if $\mathcal{T}$ is a collection of the $j$-th class, we have $G(T,\mathcal{T}) \leq g(T_1,\mathcal{T}) + g(T_2,\mathcal{T}) + g(T_3,\mathcal{T})$

$\leq g(T_1,R_0) + g(T_2,R_0) + g(T_3,R_0)$, and so $a_j(T) \leq g(T_1,R_0) + g(T_2,R_0) + g(T_3,R_0)$, for $j = 1, \ldots, 6$.

Thus, if $T: z = t(w)$, $w \in R_0$, is of essential bounded variation, then, since $g(T_i,R_0)$ is finite for $i = 1,2,3$, it follows that $a_j(T)$ is finite, for $j = 1, \ldots, 6$. 

-10-
CHAPTER II

Elementary Results

2.1 Let $T: z = t(w), w \in \mathbb{R}_0$, be defined as in 1.1.

In this chapter, we shall show that

$$a_1(T) = a_3(T) = a_5(T), \text{ and } a_2(T) = a_4(T) = a_6(T). \quad [\text{cf. 1.4}]$$

We begin with two elementary lemmas.

2.2 Lemma. If $a_1, \ldots, a_n$, are non-negative real numbers, if $b_1, \ldots, b_n$, are non-negative real numbers, and if $c_1, \ldots, c_n$, are non-negative real numbers, then

$$\sum_{j=1}^{n}(a_j + b_j + c_j)^{\frac{1}{4}} \geq \left(\sum_{j=1}^{n}a_j\right)^{\frac{1}{4}} + \left(\sum_{j=1}^{n}b_j\right)^{\frac{1}{4}} + \left(\sum_{j=1}^{n}c_j\right)^{\frac{1}{4}}.$$

Proof: See [2; p. 31].

2.3 Lemma. If $a_1, a_2, a_3$ are non-negative real numbers, and if $\varepsilon$ is an arbitrary positive number, then there exist non-negative real numbers $b_1, b_2, b_3$, such that, for $i = 1, 2, 3$, if $a_i = 0$, then $b_i = 0$, and if $a_i > 0$, then $b_i < a_i$, and such that $(b_1^2 + b_2^2 + b_3^2)^{\frac{1}{4}} > (a_1^2 + a_2^2 + a_3^2)^{\frac{1}{4}} - \varepsilon$.

Proof: This is an immediate consequence of the fact that $(x^2 + y^2 + z^2)^{\frac{1}{2}}$ is a continuous function of the variables $x, y, \text{ and } z$.

2.4 Throughout the following lemmas in this chapter we shall assume that the transformation $T: z = t(w), w \in \mathbb{R}_0$, is of essential bounded variation. [cf. 1.10]

2.5 Lemma. If $D$ is a domain contained in $\mathbb{R}_0$, and if $\varepsilon$ is an arbitrary positive number, then there exists a polygonal region $R$ contained in $D$, such that no component of $\mathcal{L}(R)$
is contained in \( D \), and such that \( g(T_i, R) > g(T_i, D) - \varepsilon \), [cf. 1.4], for each \( i, i = 1,2,3 \).

**Proof:** There exists a sequence \( R_1, \ldots, R_n, \ldots \), of polygonal regions, such that \( R_n \subset (R_{n+1}) \) for each \( n \), and \( \sum_{n=1}^{\infty} R_n = D \). [5; I. 2.48] Then, if \( z_i \in \Pi_i \), we have
\[
\lim_{n \to \infty} \kappa(z_i, T_i, R_n) = \kappa(z_i, T_i, D), \ i = 1,2,3. \quad [5; IV. 1.43]
\]
Since \( \kappa(z_i, T_i, R_n) \leq \kappa(z_i, T_i, R_{n+1}) \) for \( i = 1,2,3 \), and for each \( n \), and since \( \kappa(z_i, T_i, D) \) is summable for \( i = 1,2,3 \), [cf. 2.4], we have \( \lim_{n \to \infty} g(T_i, R_n) = g(T_i, D) \), \( i = 1,2,3 \).

Also, \( g(T_i, D) \) is finite, for \( i = 1,2,3 \), so it is clear that there exists a polygonal region \( R' \) contained in \( D \) such that \( g(T_i, R') > g(T_i, D) - \varepsilon \), \( i = 1,2,3 \).

If no component of \( \mathcal{L}(R') \) is contained in \( D \), then \( R' \) satisfies the requirements of the lemma.

Otherwise, let \( A_1, \ldots, A_r \) denote the bounded components of \( \mathcal{L}(R') \) which are contained in \( D \), and let \( R = R' + \sum_{j=1}^{r} A_j \). Then \( R \) is a polygonal region, \( R' \subset R \subset D \), and clearly \( g(T_i, R) > g(T_i, D) - \varepsilon \), \( i = 1,2,3 \).

Also, a bounded component (if any) of \( \mathcal{L}(R) \) is also a bounded component of \( \mathcal{L}(R') \) which is not contained in \( D \). Hence no bounded component of \( \mathcal{L}(R) \) is contained in \( D \).

Further, the unbounded component of \( \mathcal{L}(R) \) is obviously not contained in \( D \).

Thus the polygonal region \( R \) satisfies the requirements of the lemma.
2.6 Notice that if $D$ is a simply connected domain, then the polygonal region $R$, in the lemma of the preceding section, must be simply connected, because a bounded component of $\mathcal{C}(R)$ would necessarily be contained in $D$ if $D$ is simply connected. From this observation and the preceding lemma, we obtain the following lemma.

**Lemma.** If $D$ is a simply connected domain contained in $R_0$, and if $\epsilon$ is an arbitrary positive number, then there exists a simply connected polygonal region $R$ contained in $D$, such that $g(T_i, R) > g(T_i, D) - \epsilon$, $i = 1, 2, 3$.

2.7 The following lemmas are immediate consequences of the lemmas in 2.5 and 2.6.

**Lemma.** If $D_1, \ldots, D_n$ is a collection of disjoint domains, each of which is contained in $R_0$, and if $\epsilon$ is an arbitrary positive number, then there exists a collection of disjoint polygonal regions $R_1, \ldots, R_n$, such that for $j = 1, \ldots, n$, we have $R_j \subset D_j$ and no component of $\mathcal{C}(R_j)$ is contained in $D_j$, and such that

$$\sum_{j \neq i} g(T_i, R_j) > \sum_{j \neq i} g(T_i, D_j) - \epsilon, \quad i = 1, 2, 3.$$

2.8 Let us note that if $P$ and $Q$ are admissible sets
contained in \( R_0 \), and if \( P \subset Q \subset R_0 \), and if, for some \( i, 1 \leq i \leq 3 \), we have \( g(T_i, Q) = 0 \), then \( g(T_i, P) = 0 \).

**Lemma.** If \( D \) is a domain contained in \( R_0 \), and if \( \varepsilon \) is an arbitrary positive number, then there exists a polygonal region \( R \) contained in \( D \), such that \( G(T, R) > G(T, D) - \varepsilon \). [cf. 1.4]

**Proof:** By the lemma in 2.3, there are non-negative numbers \( b_1, b_2, b_3 \), such that, for \( i = 1, 2, 3 \), if \( g(T_i, D) = 0 \), then \( b_i = 0 \), and if \( g(T_i, D) > 0 \), then \( b_i < g(T_i, D) \), and such that \( (b_1^2 + b_2^2 + b_3^2)^{\frac{1}{2}} > G(T, D) - \varepsilon \).

By the remark above, and by the lemma in 2.5, there exists a polygonal region \( R \) contained in \( D \) such that \( g(T_i, R) \geq b_i, i = 1, 2, 3 \).

Hence \( G(T, R) \geq (b_1^2 + b_2^2 + b_3^2)^{\frac{1}{2}} > G(T, D) - \varepsilon \).

2.9 The following lemma is proved in the same manner as the lemma in 2.8.

**Lemma.** If \( D \) is a simply connected domain contained in \( R_0 \), and if \( \varepsilon \) is an arbitrary number (positive), then there exists a simply connected polygonal region \( R \) contained in \( D \), such that \( G(T, R) > G(T, D) - \varepsilon \).

2.10 The following lemmas are immediate consequences of the lemmas in 2.8 and 2.9.

**Lemma.** If \( \Psi \) is a collection of the sixth class [cf. 1.2], and if \( \varepsilon \) is an arbitrary positive number, then there exists a collection \( \mathcal{F} \) of the second class such that \( G(T, \mathcal{F}) > G(T, \Psi) - \varepsilon \). [cf. 1.4]
Lemma. If \( \Psi \) is a collection of the fifth class [cf. 1.2], and if \( \epsilon \) is an arbitrary positive number, then there exists a collection \( \Phi \) of the first class such that \( G(T, \Phi) > G(T, \Psi) - \epsilon \). [cf. 1.4]

2.11 Recall that the lemmas in this chapter have been derived under the assumption that \( T \) is of essential bounded variation.

Consequently, if \( T \) is of essential bounded variation, then the lemmas in 2.10 imply that \( a_5(T) \leq a_1(T) \) and \( a_6(T) \leq a_2(T) \) and hence \( a_1(T) = a_3(T) = a_5(T) \) and \( a_2(T) = a_4(T) = a_6(T) \). [cf. 1.6]

If \( T \) is not of essential bounded variation, then \( a_j(T) = a(T) = +\infty \). [cf. 1.10]

Thus, in every case, \( a_1(T) = a_3(T) = a_5(T) \), and \( a_2(T) = a_4(T) = a_6(T) \).
CHAPTER III

Some Lemmas Concerning Upper Semi-Continuous Collections of Continua

3.1. In this chapter, we shall establish topological lemmas to be used in Chapter IV. These lemmas will be concerned with the concept of upper semi-continuous collections of continua, and with connected sets contained in the complement of a closed set.

3.2 Lemma. If A is a polygonal region, and if $C = B \cdot \text{fr}(A)$, where B is a connected set open relative to A, then C is open and connected, and $B \cdot \text{fr}(A) \subseteq \text{fr}(C)$.

Proof: Since B is open relative to A, we have $B = G \cdot A$, where G is open. Then $C = G \cdot A \cdot \text{fr}(A) = G \cdot \text{fr}(A)$, so C is open.

Now suppose that C is disconnected and $C = C_1 \cup C_2$ is a separation of C. If $w \notin B - C$, then there is an open circle H and an open circular sector G, both of which have their center at w, such that $C \subseteq C$ and $C \cdot H \cdot \mathcal{L}(G) = 0$. Now G is connected, so either $G \subseteq C_1$ or $G \subseteq C_2$.

If $G \subseteq C_1$, then $w \in c(C_1)$, and $H \cdot C_2 = 0$, so $w \notin c(C_2)$.
If $G \subseteq C_2$, then $w \in c(C_2)$, and $H \cdot C_1 = 0$, so $w \notin c(C_1)$.

Consequently, $B - C = B \cdot \text{fr}(A)$

$$(B - C) \cdot [c(C_1) + c(C_2)] = B - C$$

$$(B - C) \cdot c(C_1) \cdot c(C_2) = 0,$$
and so $B = C_1 + (B - C) \cdot c(C_1) \mid C_2 + (B - C) \cdot c(C_2)$ is a
separation of B. But this is a contradiction since B is connected, and so C is connected.

The above reasoning shows that \( B - C \subseteq c(C) \). Now clearly \( C \subseteq c(C) \), so \( B \subseteq c(C) \). Hence

\[
B \cdot fr(A) = B - C = B \cdot \zeta(C) \subseteq c(C) \cdot c(\zeta(C)) = fr(C).
\]

3.3 Suppose that \( A \) is a polygonal region, and that \( B \) is a closed set contained in \( A \). Then \( A - B \) is open relative to \( A \). The components of \( A - B \) are connected and open relative to \( A \), and form at most a denumerable collection. Denote these components by \( C_1, \ldots, C_n, \ldots \), and let \( D_n = C_n \cdot i(A) \) for each \( n \).

**Lemma.** Under the assumptions in this section, we have \( fr(D_n) \subseteq B + fr(A) \) and \( fr(D_n) \cdot i(A) \subseteq B \) for each \( n \).

**Proof:** By the lemma in 3.2, \( D_n \) is open and connected for each \( n \). Let \( D = \sum D_n \). Then \( D \) is open, and \( D \) is a component of \( D \) for each \( n \), and

\[
D = i(A) \cdot \sum C_n = i(A) \cdot (A - B).
\]

Hence, for each \( n \),

\[
fr(D_n) \subseteq fr(D) \subseteq \zeta(D) = \zeta(i(A)) + \zeta(A \cdot \zeta(B)).
\]

Also, \( fr(D_n) \subseteq A \), so

\[
fr(D_n) \subseteq A \cdot \zeta(i(A)) + A \cdot [\zeta(A) + \zeta(\zeta(B))].
\]

But \( A \cdot \zeta(i(A)) = fr(A) \) and

\[
A \cdot [\zeta(A) + \zeta(\zeta(B))] = B, \text{ so } fr(D_n) \subseteq B + fr(A), \text{ and}
\]

\[
fr(D_n) \cdot i(A) \subseteq B, \text{ completing the proof.}
\]

3.4 Suppose that \( A \) is a polygonal region, and \( B_1, \ldots, B_n \) are open connected sets contained in \( A \). Let

\[
B = \sum_{k=1}^n c(B_k) = c(\sum_{k=1}^n B_k).
\]

\( B \) is closed, and \( B \subseteq A \), so \( A - B \)
is open relative to A. The components $C_1, \ldots, C_n, \ldots$ of $A - B$ are connected and open relative to A and form at most a denumerable collection. [cf. 3.3] Let $D_k = C_k \cdot i (A)$ for each k.

**Lemma.** Under the assumptions in this section, we have

$$\sum_{k=1}^n B_k + \sum_{k=1}^n \text{fr}(B_k) + \sum_{k=1}^n D_k + \sum_{k=1}^n \text{fr}(D_k) = A,$$

and

$$\sum_{k=1}^n B_k + \sum_{k=1}^n \text{fr}(B_k) + \sum_{k=1}^n D_k + \text{fr}(A) = A.$$

**Proof:** By the lemma in 3.3, $\text{fr}(D_k) \subset B + \text{fr}(A)$ for each k, so

$$\sum_{k=1}^n \text{fr}(D_k) \subset B + \text{fr}(A) = \sum_{k=1}^n B_k + \sum_{k=1}^n \text{fr}(B_k) + \text{fr}(A).$$

Also, $\text{fr}(D_k) \subset c(D_k) \subset c(\sum_{k=1}^n B_k) = \sum_{k=1}^n B_k$ for each k.

Hence

$$\sum_{k=1}^n \text{fr}(D_k) \subset \sum_{k=1}^n B_k + \text{fr}(A).$$

Now, $B = \sum_{k=1}^n B_k + \sum_{k=1}^n \text{fr}(B_k)$.

Also, since $C_k \cdot \text{fr}(A) \subset \text{fr}(D_k)$, by the lemma in 3.2, we have

$$A - B = \sum_{k=1}^n C_k = \sum_{k=1}^n C_k \cdot i(A) + \sum_{k=1}^n C_k \cdot \text{fr}(A)$$

$$= \sum_{k=1}^n D_k + \sum_{k=1}^n C_k \cdot \text{fr}(A)$$

$$\subset \sum_{k=1}^n D_k + \sum_{k=1}^n \text{fr}(D_k).$$

Consequently,

$$\sum_{k=1}^n B_k + \sum_{k=1}^n \text{fr}(B_k) + \sum_{k=1}^n D_k + \sum_{k=1}^n \text{fr}(D_k) = A,$$

and

$$\sum_{k=1}^n B_k + \sum_{k=1}^n \text{fr}(B_k) + \sum_{k=1}^n D_k + \text{fr}(A) = A,$$

completing the proof.

3.5 In this section we shall introduce two important definitions, which will be used often in this paper.

**Definition.** We shall say that a set $\gamma$ separates two sets A and B in the Euclidean plane $\mathcal{E}$, if $\gamma$ is closed, if A and B are connected sets, and if $A \subset U$ and $B \subset V$, where $U$ and $V$ are distinct components of $\zeta(\gamma)$.

**Definition.** We shall say that a set $\gamma$ does not
separate two sets A and B in the Euclidean plane \( \Pi \), if \( \gamma \) is closed, if A and B are connected sets, and if \( A \subset U \) and \( B \subset U \), where \( U \) is a component of \( \zeta(\gamma) \).

3.6 The following lemmas are elementary, but useful. The proofs will be omitted.

**Lemma.** If \( \gamma \) separates A and B, and if \( \gamma \) does not separate B and C, then \( \gamma \) separates A and C.

**Lemma.** If \( \gamma \) does not separate A and B, and if \( \gamma \) does not separate B and C, then \( \gamma \) does not separate A and C.

**Lemma.** If \( A_1 \) and \( B_1 \) are connected sets, if \( A_1 \subset A \), and \( B_1 \subset B \), and if \( \gamma \) separates A and B, then \( \gamma \) separates \( A_1 \) and \( B_1 \).

**Lemma.** If \( \gamma_i \) separates A and B, if \( \gamma_i \) is contained in the closed set \( \gamma_i \), and if \( A \subset \zeta(\gamma_i) \) and \( B \subset \zeta(\gamma_i) \), then \( \gamma_i \) separates A and B.

**Lemma.** If P is a simple closed polygon, if \( D_1 \) and \( D_e \) are the interior and exterior components, respectively, of \( \zeta(P) \), if \( \gamma \) separates A and B, if \( \gamma \subset D_1 \), and if \( A \cdot D_e \neq \emptyset \), then \( B \subset D_1 \).

3.7 The following lemmas are useful.

**Lemma.** If F is a bounded closed set in the Euclidean plane \( \Pi \), and if F separates A and B, then there exists a component C of F such that C separates A and B.

**Proof:** See [3; p. 117].

**Lemma.** Suppose \( F_1 \) and \( F_2 \) are closed bounded sets in the plane \( \Pi \). Suppose \( F_1 \) does not separate A and B, and \( F_2 \)
does not separate $A$ and $B$. If, for every component $\gamma$ of $F_1$, $\gamma \cdot F_2$ is connected, then $F_1 + F_2$ does not separate $A$ and $B$.

**Proof:** See [3; p. 120]

**Lemma.** If $D$ is a bounded domain in the plane $\Pi$, then each component of $\mathcal{E}(D)$ contains exactly one component of $\partial(D)$.

**Proof:** See [3; p. 118].

3.8 Throughout the remainder of Chapter III, $R$ will denote a fixed but arbitrary polygonal region in the plane $\Pi$. $Q_1, \ldots, Q_q$ will denote the bounded components, if any, of $\mathcal{E}(R)$, and $Q_q$ will denote the unbounded component of $\mathcal{E}(R)$. $r_1, \ldots, r_q$ will denote the disjoint simple closed polygons which constitute the frontier of $R$, in such a way $r_k = \partial(Q_k)$, $k = 1, \ldots, q$.

**Definition.** We shall say that $\gamma$ separates in $R$, if $\gamma \subset R$, and if $\gamma$ separates the unbounded component $Q_q$ of $\mathcal{E}(R)$ and a bounded component $Q_k$ of $\mathcal{E}(R)$, $1 \leq k \leq q - 1$. [cf. 3.5]

**Definition.** We shall say that $\gamma$ does not separate in $R$, if $\gamma \subset R$, and if, for every bounded component $Q_k$ of $\mathcal{E}(R)$, $k = 1, \ldots, q - 1$, $\gamma$ does not separate $Q_k$ and the unbounded component $Q_q$ of $\mathcal{E}(R)$. [cf. 3.5]

3.9 Notice that if $R$ is a simply connected polygonal region, then there are no bounded components of $\mathcal{E}(R)$. Hence, in this case, if $\gamma \subset R$, and if $\gamma$ is closed, then
\[ Y \text{ does not separate in } R. \ [\text{cf. 3.5}] \]

Also, if \( Y \) separates in \( R \), there may possibly be two (or more) bounded components of \( \mathcal{Z}(R) \), \( \mathcal{Q}_j \) and \( \mathcal{Q}_k \), \( 1 \leq j < k \leq q - 1 \), such that \( Y \) separates \( \mathcal{Q}_q \) and \( \mathcal{Q}_j \), and \( Y \) separates \( \mathcal{Q}_q \) and \( \mathcal{Q}_k \).

3.10 Suppose \( A \) and \( B \) are connected sets in the plane \( \mathbb{R} \). Let \( Y_1 \) and \( Y_2 \) be disjoint, closed, connected sets contained in \( R \), both of which separate \( A \) and \( B \).

**Lemma.** Under the conditions above, either \( Y_1 \) is contained in the component \( A_2 \) of \( \mathcal{Z}(Y_1) \) which contains \( A \), and \( Y_2 \) is contained in the component \( B_1 \) of \( \mathcal{Z}(Y_1) \) which contains \( B \), or else \( Y_2 \) is contained in the component \( A_1 \) of \( \mathcal{Z}(Y_1) \) which contains \( A \), and \( Y_1 \) is contained in the component \( B_2 \) of \( \mathcal{Z}(Y_1) \) which contains \( B \).

**Proof:** Suppose \( Y_1 \not\subset A_2 \). Then \( Y_1 \subset C_2 \), where \( C_2 \) is a component of \( \mathcal{Z}(Y_2) \) and \( C_2 \cdot A_2 = 0 \). \( A \subset A_1 \), \( A \subset A_2 \), \( A_2 \subset \mathcal{Z}(Y_2) \), and \( A_2 \) is connected, so \( A_2 \subset A_1 \). Also, \( A_2 \subset c(A_2) \), \( c(A_2) = A_2 + \text{fr}(A_2) \), \( c(A_2) \) is connected, and \( c(A_2) \subset c(\mathcal{Z}(C_2)) = \mathcal{Z}(C_2) \subset \mathcal{Z}(Y_1) \). Hence \( c(A_2) \subset A_1 \) and \( \text{fr}(A_2) \subset A_1 \). However, \( 0 \not\subset \text{fr}(A_2) \subset \text{fr}(\mathcal{Z}(Y_1)) \subset Y_1 \), \( Y_2 \) is connected, and \( Y_2 \subset \mathcal{Z}(Y_1) \), so \( Y_2 \subset A_1 \).

Hence, if \( Y_1 \not\subset A_2 \), then \( Y_2 \subset A_1 \). Reasoning in exactly the same manner, we find that if \( Y_2 \not\subset A_1 \), then \( Y_1 \subset A_2 \); if \( Y_1 \not\subset B_2 \), then \( Y_2 \subset B_1 \); if \( Y_2 \not\subset B_1 \), then \( Y_1 \subset B_2 \).

Then, if \( Y_1 \subset A_2 \), it follows that \( Y_1 \not\subset B_2 \), and \( Y_2 \subset B_1 \).

On the other hand, if \( Y_1 \not\subset A_2 \), it follows that
\[ x_2 \subseteq A_1, \; y_2 \notin B_1, \text{ and } x_i \subseteq B_2, \text{ completing the proof of the lemma.} \]

Quite often, in the sequel, the sets \( A \) and \( B \) will be taken to be a bounded component \( Q_k \) of \( \mathcal{L}(R) \), \( 1 \leq k \leq q - 1 \) and the unbounded component \( Q_q \) of \( \mathcal{L}(R) \).

3.11 Throughout the remainder of Chapter III, let \( F^* \) be an upper semi-continuous collection of disjoint continua. We shall suppose that each continuum of \( F^* \) is contained in \( R \), and that the collection \( F^* \) covers \( R \).

[5; II. 1.10]

We shall let \( E^* \) denote the subcollection of continua of \( F^* \) consisting of those continua of \( F^* \) which separate in \( R \), and we shall let \( E \) denote the set of points which belong to continua of \( E^* \). Note that \( E \subseteq R \).

**Lemma.** Under the assumptions above, \( E \) is closed.

**Proof:** If \( E \) is empty, then clearly \( E \) is closed.

If \( E \) is not empty, let \( \{b_n\} \) be an infinite sequence of points such that \( b_n \in E \) for each \( n \), and \( \lim_{n \to \infty} b_n = b \). Since \( R \) is closed, and since \( E \subseteq R \), we have \( b \in R \).

We shall suppose that for each \( n \), \( b_n \in \gamma_n \), where \( \gamma_n \) is a continuum of \( E^* \), and \( b \in \gamma \), where \( \gamma \) is a continuum of \( F^* \). Since \( F^* \) is an upper semi-continuous collection, we have \( \limsup n \gamma_n \subseteq \gamma \). [5; I. 2.13, II. 1.10]

Each \( \gamma_n \) separates in \( R \), that is, each \( \gamma_n \) separates \( Q_q \) and some \( Q_k, 1 \leq k \leq q - 1 \), (\( k \) depends on \( n \), of course). Hence there is a fixed \( k \) such that infinitely many \( \gamma_n \)

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separate $Q_q$ and $Q_k$, $1 \leq k \leq q - 1$.

Suppose $b \notin E$. Then $\gamma \notin E^*$, so $\gamma$ does not separate $Q_q$ and $Q_k$. Consequently, there is a simple polygonal arc $P$ such that $P \subset \mathcal{L}(\gamma)$, $P \cdot Q_q \neq 0$, and $P \cdot Q_k \neq 0$. Let $\delta$ be the distance between $P$ and $\gamma$. Since $P$ and $\gamma$ are closed, bounded, and disjoint, we have $\delta > 0$.

Let $G$ be the set of points in $\mathbb{R}$ whose distance from $\gamma$ is less than $\frac{\delta}{2}$. $G$ is open relative to $\mathbb{R}$, and $\gamma \subset G \subset \mathbb{R}$. $c(G)$ is closed, and $c(G) \subset \mathbb{R}$. Also $c(G) \cdot P = 0$, and since $Q_k \subset \mathcal{L}(\mathbb{R})$ and $Q_q \subset \mathcal{L}(\mathbb{R})$, we have $c(G) \cdot Q_k = 0$ and $c(G) \cdot Q_q = 0$, so $c(G)$ does not separate $Q_q$ and $Q_k$.

For all but a finite number of $n$, we have $\gamma_n \subset G$. [5; I. 2.37] Hence, there is an $m$ such that $\gamma_m$ separates $Q_q$ and $Q_k$, and $\gamma_m \subset G \subset c(G)$. Then $\gamma_m \cdot P = 0$, $\gamma_m \cdot Q_q = 0$, and $\gamma_m \cdot Q_k = 0$, so, by the fourth lemma in 3.6, $\gamma_m$ does not separate $Q_k$ and $Q_q$, a contradiction.

Therefore, $E$ is closed.

3.12 Since $E$ is closed and $E \subset \mathbb{R}$, it follows that $\mathbb{R} - E$ is open relative to $\mathbb{R}$. The components of $\mathbb{R} - E$ are open relative to $\mathbb{R}$ and form at most a denumerable collection. We shall assume that $\mathbb{R} - E$ is not empty.

Throughout the remainder of Chapter III, we shall let $M$ denote a fixed but arbitrary component of $\mathbb{R} - E$, and we shall let $N = M \cdot i(\mathbb{R})$. By the lemma in 3.2, $N$ is connected and open.
We shall let \( A_q \) designate the subcollection of continua of \( E^* \) consisting of those continua of \( E^* \) which separate \( Q_q \) and \( N \). Let \( A_q \) be the set of points that belong to continua of \( A_q^* \). Note that \( A_q \subset E \).

**Lemma.** Under the assumptions in 3.11 and 3.12, \( A_q \) is closed.

**Proof:** If \( A_q \) is empty, then of course \( A_q \) is closed. If \( A_q \) is not empty, let \( \{ b_n \} \) be an infinite sequence of points such that \( b_n \in A_q \) for each \( n \), and \( \lim b_n = b \). Since \( E \) is closed and \( A_q \subset E \), we have \( b \in E \).

For each \( n \), let \( b_n \in \gamma_n \), where \( \gamma_n \) is a continuum of \( A_q^* \), and let \( b \in \gamma \), where \( \gamma \) is a continuum of \( E^* \).

The remainder of the proof is similar to the proof of the lemma in 3.11, except that in this case, the set \( G \) is taken to be the set of points in \( R \cdot \mathcal{C}(N) \) whose distance from \( \gamma \) is less than \( \frac{1}{2} \delta \). Note then that \( \gamma \subset G \subset R \cdot \mathcal{C}(N) \), \( R \cdot \mathcal{C}(N) \) is closed, and \( G \) is open relative to \( R \cdot \mathcal{C}(N) \).

**3.13 Lemma.** Under the assumptions in 3.11 and 3.12, if \( A_q \) is not empty, then there exists a continuum \( \gamma_q \) of the collection \( A_q^* \) such that every other continuum of the collection \( A_q^* \) is contained in the component of \( \mathcal{C}(\gamma_q) \) which contains \( Q_q \).

**Proof:** By the lemma in 3.12, \( A_q \) is closed. Let \( B \) be a simple polygonal arc, with \( w_1 \) and \( w_2 \) as end points, where \( w_1 \in N \) and \( w_2 \in Q_q \). Let \( B' \) be the component of \( B - A_q \) which contains \( w_1 \). Then \( w_1 \in B' \), \( B' \neq B \), and \( B' \) is open.
relative to B. Then \( c(B') = B' + w_0 \), where \( w_0 \in B \cdot A_q \). Let \( w_0 \in \bar{Y} \), where \( \bar{Y} \) is a continuum of \( A_q^* \).

Let \( Y \) be any other continuum of \( A_q^* \). Then \( c(B') \subseteq \zeta(Y) \), and since \( c(B') \) is connected, it follows that \( c(B') \subseteq U \), where \( U \) is a component of \( \zeta(Y) \). But \( w_1 \in N \cdot c(B') \), and \( w_0 \in \bar{Y} \cdot c(B') \), so \( N \cdot U \neq 0 \) and \( \bar{Y} \cdot U \neq 0 \). Therefore, \( \bar{Y} \) and \( N \) are contained in the same component of \( \zeta(Y) \). By the lemma in 3.10, \( Y \) is contained in the component of \( \zeta(\bar{Y}) \) which contains \( Q_q \). Hence \( \bar{Y} \) will serve as \( Y \).

3.14 Throughout the remainder of Chapter III, if \( A_q \) is empty, we shall let \( A_k^* \) denote the subcollection of continua of \( E^* \) consisting of those continua of \( E^* \) which separate \( N \) and \( Q_k \), \( k = 1, \ldots, q - 1 \).

Throughout the remainder of Chapter III, if \( A_q \) is not empty, let \( A_k^* = 0 \) if \( Q_k \) is not contained in the component of \( \zeta(Y_k) \) which contains \( N \), \( k = 1, \ldots, q - 1 \). If \( Q_k \) is contained in the component of \( \zeta(Y_k) \) which contains \( N \), let \( A_k^* \) denote the subcollection of \( E^* \) consisting of those continua of \( E^* \) which separate \( N \) and \( Q_k \), \( k = 1, \ldots, q - 1 \).

In each case, let \( A_k \) denote the set of points which belong to continua of \( A_k^* \), \( k = 1, \ldots, q - 1 \). Note that \( A_k \subseteq E \) for each \( k \).

**Lemma.** Under the assumptions in 3.11 - 3.14, \( A_k \) is closed, \( k = 1, \ldots, q - 1 \).
Proof: The proof is similar to the proof of the lemma in 3.12.

3.15 Lemma. Under the assumptions in 3.11 - 3.14, if \( y \) is a continuum of \( A_k^* \), \( 1 \leq k \leq q - 1 \), then \( y \) separates \( Q_k \) and \( Q_q \).

Proof: a. If \( A_q \) is empty, then \( y \) does not separate \( Q_q \) and \( N \). By assumption, \( y \) separates \( Q_k \) and \( N \), so by the first lemma in 3.6, \( y \) separates \( Q_k \) and \( Q_q \).

b. If \( A_q \) is not empty, then \( N \) and \( Q_k \) are contained in the same component of \( \mathcal{L}(y_q) \), that is, \( y_q \) does not separate \( N \) and \( Q_k \). [cf. 3.14]

Suppose \( y \) does not separate \( Q_q \) and \( Q_k \). Then, since \( y \) separates \( Q_k \) and \( N \) by assumption, it follows from the first lemma in 3.6 that \( y \) separates \( N \) and \( Q_q \), so \( y \in A_q^* \).

Now \( y \neq y_q \), since \( y_q \) does not separate \( N \) and \( Q_k \). [cf. 3.14] Then, by the lemma in 3.13, we find that \( y \) is contained in the component of \( \mathcal{L}(y_q) \) which contains \( Q_q \).

The component of \( \mathcal{L}(y_q) \) which contains \( Q_k \) and \( N \) is thus contained in \( \mathcal{L}(y) \), and hence is contained in a component of \( \mathcal{L}(y) \). \( Q_k \) and \( N \) are then contained in the same component of \( \mathcal{L}(y) \), a contradiction, since \( y \) separates \( Q_k \) and \( N \) by assumption. Consequently, if \( A_q \) is not empty, then \( y \) separates \( Q_k \) and \( Q_q \).

3.16 Lemma. Under the assumptions in 3.11 - 3.15, if \( A_k \) is not empty, \( 1 \leq k \leq q - 1 \), then there exists a
continuum $\gamma_k$ of $A_k^*$ such that every other continuum of $A_k^*$ is contained in the component of $\mathcal{C}(\gamma_k)$ which contains $Q_k$.

**Proof:** Let $C$ be a simple polygonal arc, with $w_1$ and $w_2$ as end points, where $w_1 \in N$ and $w_2 \in Q_k$. Reasoning as in the lemma in 3.13, we find that there is a continuum $\overline{\gamma}$ of $A_k^*$ such that, if $\gamma$ is any other continuum of $A_k^*$, then $\overline{\gamma}$ and $N$ are contained in the same component of $\mathcal{C}(\gamma)$. Since $\gamma$ separates $N$ and $Q_k$, it follows that $\overline{\gamma}$ and $Q_k$ are contained in different components of $\mathcal{C}(\gamma)$.

By the lemma in 3.15, it follows that $\gamma$ and $\overline{\gamma}$ both separate $Q_k$ and $Q_q$. By the lemma in 3.10, it follows that $\gamma$ is contained in the component of $\mathcal{C}(\overline{\gamma})$ which contains $Q_k$. Hence $\overline{\gamma}$ will serve as $\gamma_k$.

3.17 If $A_k$ is not empty, $1 \leq k \leq q$, then $\gamma_k$ is defined. [cf. 3.13, 3.16] If $A_k$ is empty, $1 \leq k \leq q$, let $\gamma_k = 0$.

Recall now that $E$ is closed, [cf. 3.11], $M$ is a component of $R - E$, and the components of $R - E$ form at most a denumerable collection. [cf. 3.12] Also $N = M \cdot i(R)$, and $N$ is open and connected.

**Lemma.** Under the assumptions in 3.11 - 3.17, we have $fr(N) \cdot i(R) \subset \sum_{k=1}^{q} \gamma_k$.

**Proof:** By the lemma in 3.3, we have $fr(N) = E + fr(R)$ and $fr(N) \cdot i(R) \subset E$. Let $\omega \in fr(N) \cdot i(R)$. Then $\omega \in E$, so $\omega \in \gamma$, where $\gamma$ is a continuum of $E^*$. Now $N + \omega$ is connected, since $\omega \in c(N)$. Also,
\[ N + \gamma = (N + \omega) + \gamma, \text{ and } \omega \in (N + \omega) \cdot \gamma, \] so \( N + \gamma \) is connected.

a. Suppose \( A_q \) is empty. Then \( \gamma \) does not separate \( N \) and \( Q_q \), so there is a \( k \) such that \( \gamma \) separates \( N \) and \( Q_k \), \( 1 \leq k \leq q - 1 \). Thus, \( \gamma \in A_k^* \), and \( A_k \) is not empty.

Suppose \( \gamma \not\in A_k^* \). Then \( N + \gamma \subset C(\gamma_k) \), so \( \gamma \) and \( N \) are contained in the same component of \( C(\gamma_k) \).

By the lemma in 3.16, \( \gamma \) is contained in the component of \( C(\gamma_k) \) which contains \( Q_k \). Therefore, by the second lemma in 3.6, \( N \) and \( Q_k \) are contained in the same component of \( C(\gamma_k) \), a contradiction, since \( \gamma_k \) separates \( N \) and \( Q_k \).

Hence \( \gamma = \gamma_k \), \( \omega \in \gamma_k \), \( \omega \in \frac{1}{k} \gamma_k \), so \( fr(N) \cup (R) \subset \sum_{k=1}^{q} \gamma_k \) if \( A_q \) is empty.

b. Suppose \( A_q \) is not empty, and suppose that \( \gamma \not\in \gamma_q \). Then \( N + \gamma \subset C(\gamma_q) \), so \( N \) and \( \gamma \) are contained in the same component of \( C(\gamma_q) \), and so \( \gamma \) is not contained in the component of \( C(\gamma_q) \) which contains \( Q_q \).

Since \( \gamma \in E^* \), it follows that there is a \( k \) such that \( \gamma \) separates \( Q_q \) and \( Q_k \), \( 1 \leq k \leq q - 1 \). \( \gamma \) does not separate \( N \) and \( Q_q \), for otherwise, by the lemma in 3.13, \( \gamma \) would be contained in the component of \( C(\gamma_q) \) which contains \( Q_q \), which, as we have seen above, is impossible. Consequently, \( \gamma \) separates \( Q_k \) and \( N \).

\( Q_k \) is not contained in the component of \( C(\gamma_q) \) which contains \( Q_q \), because this component is contained in
a component of \( \mathcal{C}(Y) \), (since \( Y \) is contained in the component of \( \mathcal{C}(Y_i) \) which contains \( N \)), and \( Q_q \) and \( Q_k \) would be contained in the same component of \( \mathcal{C}(Y) \), which is impossible.

Thus \( \gamma \) separates \( Q_q \) and \( Q_k \). Also, \( Y \) separates \( Q_q \) and \( Q_k \). By the lemma in 3.10, either \( Y \) is contained in the component of \( \mathcal{C}(Y_i) \) which contains \( Q_q \) (which is impossible), or else \( Y \) is contained in the component of \( \mathcal{C}(Y_i) \) which contains \( Q_k \). Therefore, \( N \) and \( Q_k \) are contained in the same component of \( \mathcal{C}(Y_i) \), and \( Y \) separates \( Q_k \) and \( N \), so \( Y \in A_k^* \), and \( A_k \) is not empty.

Suppose \( Y \neq Y_k \). Then the lemma in 3.15 implies that \( Y \) is contained in the component of \( \mathcal{C}(Y_k) \) which contains \( Q_k \). Also, \( N + Y \subset \mathcal{C}(Y_k) \), so \( Y \) and \( N \) are contained in the same component of \( \mathcal{C}(Y_k) \). This means that \( N \) and \( Q_k \) are contained in the same component of \( \mathcal{C}(Y_k) \), which is impossible, since \( Y_k \) separates \( N \) and \( Q_k \).

Hence, \( Y = Y_k \), \( \omega \in \gamma_k \), \( \omega \in \sum_{k=1}^p \gamma_k \), so \( \text{fr}(N) \cdot \text{int}(R) \subset \sum_{k=1}^p \gamma_k \) if \( A_q \) is not empty, completing the proof of the lemma.

Note that as a consequence of this lemma, we have \( \text{fr}(N) \subset \sum_{k=1}^p \gamma_k + \sum_{k=1}^p r_k \). (Recall that \( \sum_{k=1}^p r_k = \text{fr}(R) \).)

3.18 Throughout the remainder of Chapter III, let \( R' \) be a polygonal region such that \( R' \subset N \). Let \( Q_1', \ldots, Q_{p-1}' \) denote the bounded components of \( \mathcal{C}(R') \), if any, and let \( Q_t' \) denote the unbounded component of \( \mathcal{C}(R') \).
Let \( r_1', \ldots, r_t' \) denote the disjoint simple closed polygons which constitute the frontier of \( R' \), in such a way that
\[
r_k' = \text{fr}(Q'_k), \quad k = 1, \ldots, t.
\]

Suppose further that for each \( k \), \( Q'_k \not\subset N, \quad k = 1, \ldots, t. \)

Lemma. Let \( G^* \) be an upper semi-continuous collection of disjoint continua such that each continuum of \( G^* \) is contained in \( R' \), and such that the collection \( G^* \) covers \( R' \). Also, we shall suppose that each continuum of \( G^* \) is contained in some continuum of \( F^* \). \text{[cf. 3.11]} Then, under the assumptions in \( 3.11 - 3.18 \), no continuum of \( G^* \) separates in \( R' \).

Proof: Suppose that \( \gamma' \) is a continuum of \( G^* \) which separates in \( R' \). Then there is an \( h \) such that \( \gamma' \) separates \( Q'_h \) and \( Q'_t, \quad 1 \leq h \leq t - 1. \)

Since \( Q'_h \not\subset N \), by assumption, and since \( \text{fr}(Q'_h) = r_h' \) and \( r_h' \subset R' \subset N \), we have \( Q'_h \cdot \text{fr}(N) \neq 0. \) Recall that
\[
\text{fr}(N) = \sum_{h=1}^{t} \gamma_k + \sum_{k=1}^{t} r_k'. \quad \text{[cf. 3.17]}
\]

a. Suppose that there is a \( k \) such that \( Q'_h \cdot \gamma_k' \neq 0, \)
\( 1 \leq k \leq q. \) \( \gamma_k' \) is connected, and \( \gamma_k' \cdot r_k' = 0, \) since \( \gamma_k' \subset E \).

If \( \gamma_k' \subset Q'_h \), then, by the last lemma in \( 3.6 \), \( Q_q \subset Q'_h \), and \( \gamma_k' \subset Q'_h \).

If \( \gamma_k' \subset Q'_h \), then, by the last lemma in \( 3.6 \), \( Q_q \subset Q'_h \), which is impossible. Hence we have \( \gamma_k' \subset Q'_h, \quad 1 \leq k \leq q - 1. \)

By the lemma in \( 3.15, \) \( \gamma_k' \) separates \( Q_h' \) and \( Q_q. \)

Also, \( Q_q \subset \mathcal{C}(R'), \) \( Q_q \) is connected and unbounded, so \( Q_q \subset Q'_t. \) Since \( \gamma_k' \) separates \( Q_q \) and \( Q'_h \), since \( \gamma_k' \subset Q'_h, \) and since \( Q_q \subset \mathcal{C}(c(Q'_h)) \), it follows from the last lemma in \( 3.6 \).
that \( Q_k \subset Q_t' \). By the third lemma in 3.6, \( \Gamma \) separates \( Q_k \) and \( Q_q \).

Let \( \gamma \) be the continuum of \( F^* \) which contains \( \Gamma' \). By the fourth lemma in 3.6, \( \gamma \) separates \( Q_k \) and \( Q_q \), so \( \gamma \in E^* \) and \( \gamma \subset E \).

But, on the other hand, \( \Gamma \subset R' \), so \( 0 \neq \Gamma' \cdot R' \subset \gamma \cdot N \), and \( \gamma \cdot N \neq 0 \), a contradiction, since \( \gamma \subset E \) and since \( N \subset R - E \).

b. Suppose that there is a \( k \) such that \( Q_h \cdot r_k \neq 0 \), \( 1 \leq k \leq q \). \( Q_q + r_q \) is connected, \( Q_q + r_q \subset \zeta(R') \), and since \( Q_q \subset Q_t' \), we have \( r_q \subset Q_t' \). Hence \( Q_h \cdot r_k \neq 0 \), and \( 1 \leq k \leq q - 1 \). Further, \( r_k \subset Q_h' \), since \( r_k \) is connected and since \( r_k \cdot r_h' = 0 \). Since \( Q_k + r_k \) is connected, and since \( Q_k + r_k \subset \zeta(R') \), it follows that \( Q_k \subset Q_h' \). Also, \( Q_q \subset Q_t' \). By the third lemma in 3.6, \( \Gamma \) separates \( Q_k \) and \( Q_q \).

Reasoning as above, we again obtain a contradiction.

We may conclude, therefore, that no continuum of \( G^* \) separates in \( R' \).

3.19 Notice that the reasoning used in the proof of the lemma in 3.18 shows also, under the assumptions in 3.11 – 3.18, that if \( E^* \) is empty, that is, if no continuum of \( F^* \) separates in \( R \), then no continuum of \( G^* \) separates in \( R' \).
CHAPTER IV
Principal Results

4.1 Let \( T: z = t(w), w \in R_0 \), be defined as in 1.1, and let \( T_i: z_i = t_i(w), w \in R_0 \), be defined as in 1.4.

If \( P \) is an admissible set [cf. Preface] contained in \( R_0 \), let \( T|P; z = t(w), w \in P \), and let \( T_i|P: z_i = t_i(w), w \in P, i = 1,2,3 \).

Recall that if \( P \) and \( Q \) are admissible sets, \( P \subseteq Q \subseteq R_0 \), then, if \( z_i \in \mathcal{M}_i \),
\[
\kappa(z_i,T_i,P) = \kappa(z_i,T_i|Q,P) = \kappa(z_i,T_i|P,P) \quad \text{for} \quad i = 1,2,3.
\]
[cf. 1.3]

Throughout the proofs of the theorems in Chapter IV, we shall use the abbreviation m.m.c. for maximal model continuum, and the abbreviation e.m.m.c. for essential maximal model continuum. [cf. 1.3]

4.2 Notice that if \( T \) is not of essential bounded variation, then [cf. 1.9] \( a_j(T) = a(T) = +\infty \) for \( j = 1,\ldots,6 \). Therefore, we shall assume throughout the remainder of Chapter IV that \( T: z = t(w), w \in R_0 \), is of essential bounded variation, implying that the quantities \( a_j(T), j = 1,\ldots,6 \), are finite. [cf. 1.10]

4.3 Lemma. If \( R \) is a polygonal region contained in \( R_0 \), then, for \( i = 1,2,3 \), there exists a set \( K_i \subseteq T_i(R) \subseteq \mathcal{M}_i \), such that the measure of \( K_i \) is zero, and such that, if \( z_i \notin K_i \), then every maximal model continuum \( \not\in \) under \( T_i|R \),

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for which $T_i(\gamma) = z_i$, is also a maximal model continuum under $T|R$.

**Proof:** See [1; vol. 10, p. 287].

**4.4 Lemma.** If $R$ is a polygonal region contained in $R_0$, then, for $i = 1, 2, 3$, there is a set $B_i$, such that $B_i \subseteq T_i(R) \subseteq \pi_i$, the measure of $B_i$ is zero, and such that $\sum T_i(\gamma) \subseteq B_i$, where the summation is extended over every essential maximal model continuum $\gamma$ under $T_i|R$ such that $\zeta(\gamma)$ has more than one component.

**Proof:** Let $A_i$ be the set of points $z_i$ such that $z_i \in \pi_i$, $K(z_i, T_i, R)$ is finite, and for which there is an e.m.m.c. $\gamma$ under $T_i|R$ such that $\zeta(\gamma)$ has more than one component, and such that $T_i(\gamma) = z_i$. Then $A_i$ is at most a countably infinite set. [4; pp. 593 - 6]

Since $K(z_i, T_i, R)$, $z_i \in T_i(R)$, is summable, [cf. 4.2] the set $C_i$ of points $z_i$ such that $z_i \in \pi_i$ and $K(z_i, T_i, R)$ is not finite, is a set of measure zero.

Let $B_i = A_i + C_i$. Then the set $B_i$ satisfies the requirements of the lemma.

**4.5 Lemma.** Suppose that $R$ is a polygonal region contained in $R_0$. Suppose that, for $i = 1, 2, 3$, $F_i$ is a bounded measurable set contained in $\pi_i$. Let $\epsilon$ be an arbitrary positive number. Then, for $i = 1, 2, 3$, there exists a closed, totally disconnected set $E_i$, such that $E_i \subseteq F_i$, and such that $\sum_{E_i} K(z_i, T_i, R) > \sum_{F_i} K(z_i, T_i, R) - \epsilon$.

**Proof:** Since $\sum_{\pi_i} K(z_i, T_i, R)$, $X \subseteq \pi_i$, $X$ measurable,
is an absolutely continuous set function, there is a positive number \( \delta_i \) such that if \( X \) is measurable and contained in \( \mathcal{U}_i \), having measure less than \( \delta_i \), then

\[
\int_X \| K(z_i, T_i, R) \| < \varepsilon.
\]

Let \( M_1 \) be the set of points \( z_1 \) in the Euclidean plane \( \mathbb{R}^2 \) such that \( z_1 = (\alpha, \beta) \), where both coordinates \( \alpha \) and \( \beta \) are irrational numbers. Then \( \mathcal{C}(M_1) \) is a set of measure zero, so \( F_1 \cdot \mathcal{C}(M_1) \) is a set of measure zero. Hence \( F_1 \cdot M_1 \) is measurable, and the measure of \( F_1 \cdot M_1 \) is equal to the measure of \( F_1 \). Also, \( M_1 \) is totally disconnected, so \( F_1 \cdot M_1 \) is totally disconnected.

Since \( F_1 \cdot M_1 \) is measurable, we can find a closed set \( E_1 \) such that \( E_1 \subseteq F_1 \cdot M_1 \), and such that the measure of \( E_1 \) differs from the measure of \( F_1 \cdot M_1 \) by less than \( \delta_i \). Then the measure of \( F_1 \cdot M_1 \cdot \mathcal{C}(E_1) \) is less than \( \delta_i \), so, since

\[
\int_{E_1} \| K(z_i, T_i, R) \| + \int_{F_1 \cdot M_1 \cdot \mathcal{C}(E_1)} \| K(z_i, T_i, R) \| = \int_{F_1 \cdot M_1} \| K(z_i, T_i, R) \|,
\]

we have

\[
\int_{E_1} \| K(z_i, T_i, R) \| > \int_{F_1} \| K(z_i, T_i, R) \| - \varepsilon.
\]

Also, since \( E_1 \subseteq M_1 \cdot F_1 \), and \( M_1 \cdot F_1 \) is totally disconnected, it follows that \( E_1 \) is totally disconnected, and \( E_1 \subseteq F_1 \).

4.6 **Theorem.** If \( R \) is a polygonal region contained in \( R_0 \), and if \( \varepsilon \) is an arbitrary positive number, then there exists a collection \( F_i \) of the second class [cf. 1.2], and
a subcollection $\Phi_i$ of $\Phi$, such that

a) Every polygonal region of the collection $\Phi_i$ is contained in $R$.

b) $g(T_i, \Phi_i) > g(T_i, R) - \epsilon$, $i = 1, 2, 3$. [cf. 1.3]

c) $g(T_1, \Psi) > g(T_1, R) - \epsilon$.

d) If $\overline{R} \in \Psi$, then no maximal model continuum under $T_1|\overline{R}$ separates in $\overline{R}$.

e) If $\overline{R} \in \Psi$, and if, for some $i$, $1 \leq i \leq 3$, no maximal model continuum under $T_i|\overline{R}$ separates in $R$, then no maximal model continuum under $T_i|\overline{R}$ separates in $\overline{R}$.

(There exists similar collections $\Phi_2$, $\Psi_2$, and $\Phi_3$, $\Psi_3$, having similar properties with regard to the transformations $T_2$ and $T_3$, respectively.)

**Proof:** (1) If $R$ is simply connected, then the collections $\Phi_i$ and $\Psi_i$ may both be chosen to be the collection consisting of the polygonal region $R$ alone.

(2) If $R$ is not simply connected, let $Q_1, \ldots, Q_q$ denote the bounded components of $\mathcal{L}(R)$, and let $Q_q$ denote the unbounded component of $\mathcal{L}(R)$. Let $r_1, \ldots, r_q$ denote the disjoint simple closed polygons which constitute the frontier of $R$ in such a way that $r_k = \text{fr}(Q_k)$, $k = 1, \ldots, q$.

Consider $T_1|R$: $z_1 = t_1(w)$, $w \in R$. Let $F^*$ denote the collection of m.m.c.s under $T_1|R$. Then $F^*$ is an upper semi-continuous collection of disjoint continua.

[5; II. 1.13] Also, each m.m.c. under $T_1|R$ of $F^*$ is contained in $R$, and the collection $F^*$ covers $R$. Consequently,
all of the results stated in 3.11 - 3.19 are valid in this situation.

Let $E^*$ be the subcollection of $F^*$ consisting of those m.m.c.s under $T_1|R$ which separate in $R$. [cf. 3.8] Let $E$ be the set of points which belong to m.m.c.s under $T_1|R$ of the collection $E^*$. By the lemma in 3.11, $E$ is closed.

(3) If $E$ is empty, then the collections $\mathcal{F}_i$ and $\mathcal{U}_i$ may both be taken to be the collection consisting of the polygonal region $R$ alone.

If $i(R) \subseteq E$, then $E = R$, since $E$ is closed. In this case, every m.m.c. $\gamma$ under $T_1|R$ is such that $\mathcal{F}(\gamma)$ has more than one component. Consequently, by the lemma in 4.4, there is a set $B_1$, such that $B_1 \subseteq T_1(R) \subseteq \pi_1$, the measure of $B_1$ is zero, and such that $\sum T_1(\gamma) \subseteq B_1$, where the summation is extended over every e.m.m.c. $\gamma$ under $T_1|R$. Hence, if $z_1 \notin B_1$, we have $\mathcal{K}(z_1, T_1, R) = 0$, and so $g(T_1, R) = 0$. In this case we may choose the collection $\mathcal{F}_i$ to be the collection consisting of the polygonal region $R$ alone, and we may choose the collection $\mathcal{U}_i$ to be the empty collection.

(4) From (3) we may assume that $E \neq 0$. $R - E$ is open relative to $R$, and the components of $R - E$ are open relative to $R$. These components form at most a countably infinite collection, and we shall denote them by $C_1, \ldots, C_n, \ldots$. Let $D_j = C_j \cdot i(R)$ for each $j$. From (3), $D_j$ is not empty, and by the lemma in 3.2, $D_j$ is open and connected for each $j$. 

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(5) Suppose $\gamma$ is an e.m.m.c. under $T_1|R$. Then
$\gamma = i(R)$. [cf. 1.3] Hence either $\gamma \subseteq E$, or else
$\gamma \subseteq i(R) \cdot (R - E)$, that is, $\gamma \subseteq \sum D_j$.

In the first case, since $\gamma$ is an m.m.c. under $T_1|R$
of $E^*$, it follows that $\gamma$ separates in $R$, and so $\gamma$ is an
e.m.m.c. under $T_1|R$ such that $\zeta(\gamma)$ has more than one com­
ponent. By the lemma in 4.4, there is a set $B_1$ such that
$B_1 \subseteq T_1(R) \subseteq \prod$, the measure of $B_1$ is zero, and
$
\sum_{T_1(\gamma)} \subseteq B_1$, where the summation is extended over every
e.m.m.c. $\gamma$ under $T_1|R$ such that $\zeta(\gamma)$ has more than one
component.

In the second case, since $D_j$ is a component of $\sum_{j=1} D_j$
for each $j$, there is a $j$ such that $\gamma \subseteq D_j$. Hence $\gamma$ is an
e.m.m.c. under $T_1|D_j$.

Consequently, if $z_1 \notin B_1$, then

$$
\sum_{j=1}^{\infty} \kappa(z_1, T_1, D_j) = \kappa(z_1, T_1, R), \text{ and since the measure of } B_1
$$
is zero we have $\sum_{j=1}^{\infty} \kappa(T_1, D_j) = \kappa(T_1, R)$. [5; I. 3.11]

Therefore, there is an $n$ such that

$$
\sum_{j=1}^{n} \kappa(T_1, D_j) > \kappa(T_1, R) - \frac{\epsilon}{2}.
$$

(6) For each $j$, $j = 1, \ldots, n$, and for each $k$,
k = 1, \ldots, q, we have, by the lemma in 3.16, a set $\gamma_{jk}$,
such that either $\gamma_{jk}$ is empty or else $\gamma_{jk}$ is an m.m.c.
der $T_1|R$ of $E^*$, and such that

$$
\text{fr}(D_j) \cdot i(R) = \sum_{k=1}^{q} \gamma_{jk}, \text{ for each } j, j = 1, \ldots, n.
$$

Therefore, $\sum_{j=1}^{\infty} \text{fr}(D_j) \cdot i(R) = \sum_{j=1}^{\infty} \sum_{k=1}^{q} \gamma_{jk}$, and so

$$
T_1 \left( \sum_{j=1}^{\infty} \text{fr}(D_j) \cdot i(R) \right) \text{ is a finite set. Also, we have}
$$
\[
\sum_{j=1}^{n} \text{fr}(D_j) + \frac{\beta}{2} \sum_{k=1}^{n} r_k = \sum_{j=1}^{n} \text{fr}(D_j) + \frac{\beta}{2} \sum_{k=1}^{n} r_k.
\]

(7) Let \( F = \sum_{j=1}^{n} c(D_j) = c(\sum_{j=1}^{n} D_j) \). \( F \) is closed, \( F \subseteq R \), and \( R - F \) is open relative to \( R \). Let \( C_{n+1}, C_{n+2}, \ldots \) denote the collection (at most countably infinite) of components of \( R - F \). These components are open relative to \( R \). For each \( j \), let \( D_{n+j} = C_{n+j} \cdot i(R) \). Then, by the lemma in 3.2, \( D_{n+j} \) is open and connected.

By the lemmas in 3.2, 3.3, and 3.4, we find that

\[
\sum_{j=1}^{n} \text{fr}(D_{n+j}) \subseteq \sum_{j=1}^{n} \text{fr}(D_j) + \frac{\beta}{2} \sum_{k=1}^{n} r_k,
\]

\[
\sum_{j=1}^{n} D_j + \sum_{j=1}^{n} D'_{n+j} + \sum_{j=1}^{n} \text{fr}(D_{n+j}) + \sum_{j=1}^{n} \text{fr}(D_j) + \frac{\beta}{2} \sum_{k=1}^{n} r_k = R.
\]

(8) Consider the transformation \( T_2|_R: z_2 = t_2(w) \), \( w \in R \). Let \( \gamma \) be an e.m.m.c. under \( T_2|_R \). Then either \( \gamma \) intersects \( \sum_{j=1}^{n} \text{fr}(D_j) + \sum_{j=1}^{n} \text{fr}(D_{n+j}) \), or not.

In the first case, we have \( \gamma \) intersects \( \sum_{j=1}^{n} \text{fr}(D_j) \cdot i(R) \), using (7) and the fact that \( \gamma \subseteq i(R) \). In (6), we have seen that \( T_1(\sum_{j=1}^{n} \text{fr}(D_j) \cdot i(R)) \) is finite, so \( T_2(\sum_{j=1}^{n} \text{fr}(D_j) \cdot i(R)) \) is a set of measure zero.

Now \( T_2(\gamma) \subseteq T_2(\sum_{j=1}^{n} \text{fr}(D_j) \cdot i(R)) \), and so

\( \Sigma T_2(\gamma) \subseteq T_2(\sum_{j=1}^{n} \text{fr}(D_j) \cdot i(R)) \), where the summation is extended over every e.m.m.c. \( \gamma \) under \( T_2|_R \) such that

\( \gamma \cdot [\sum_{j=1}^{n} \text{fr}(D_j) + \sum_{j=1}^{n} \text{fr}(D_{n+j})] \neq 0. \)

In the second case, \( \gamma \subseteq \sum_{j=1}^{n} D_j + \sum_{j=1}^{n} D'_{n+j} \), using (7).

If for some \( j, 1 \leq j \leq n \), we have \( \gamma \cdot D_j \neq 0 \), then, since \( \gamma \) is connected, and since \( \gamma \cdot \text{fr}(D_j) = 0 \), it follows that \( \gamma \subseteq D_j \), and \( \gamma \) is an e.m.m.c. under \( T_2|_D_j \).
If there is a \( j \) such that \( \gamma \cdot D'_{n,j} \neq 0 \), then since \( \gamma \) is connected, and since \( \gamma \cdot \text{fr}(D'_{n,j}) = 0 \), it follows that \( \gamma \subseteq D'_{n,j} \) and \( \gamma \) is an e.m.m.c. under \( T_2|D'_{n,j} \).

Therefore, if \( z_2 \notin T_2(\sum_{j=1}^{n} \text{fr}(D_j) \cdot i(R)) \), we have

\[
\sum_{j=1}^{n} \kappa(z_2, T_2, D_j) + \sum_{j=1}^{n} \kappa(z_2, T_2, D'_{n,j}) = \kappa(z_2, T_2, R).
\]

Since \( T_2(\sum_{j=1}^{n} \text{fr}(D_j) \cdot i(R)) \) is a set of measure zero, we have

\[
\sum_{j=1}^{n} g(T_2, D_j) + \sum_{j=1}^{n} g(T_2, D'_{n,j}) = g(T_2, R).
\]

In similar fashion, we find that

\[
\sum_{j=1}^{n} g(T_3, D_j) + \sum_{j=1}^{n} g(T_3, D'_{n,j}) = g(T_3, R).
\]

(9) We can determine an integer \( n' \) such that

\[
\sum_{j=1}^{n'} g(T_1, D_j) + \sum_{j=1}^{n'} g(T_1, D'_{n,j}) > g(T_1, R) - \frac{\epsilon}{2}, \quad i = 1, 2, 3.
\]

Recall that \( \sum_{j=1}^{n} g(T_1, D_j) > g(T_1, R) - \frac{\epsilon}{2} \), by (5).

From the first lemma in 2.7, we can determine polygonal regions \( R_j, j = 1, \ldots, n + n' \), so that

\[
\sum_{j=1}^{n} g(T_1, R_j) > \sum_{j=1}^{n} g(T_1, D_j) - \frac{\epsilon}{2}, \quad \text{for } i = 1, 2, 3,
\]

\[
\sum_{j=1}^{n} g(T_1, R_{n,j}) > \sum_{j=1}^{n} g(T_1, D'_{n,j}) - \frac{\epsilon}{2}, \quad \text{so that } R_j \subseteq D_j \quad \text{and no component of } \zeta(R_j) \text{ is contained in } D_j \text{, for } j = 1, \ldots, n,
\]

and so that \( R_{n,j} \subseteq D'_{n,j} \) and no component of \( \zeta(R_{n,j}) \) is contained in \( D'_{n,j} \).

Let \( m = n + n' \). Then we have

\[
\sum_{j=1}^{m} g(T_1, R_j) > g(T_1, R) - \epsilon, \quad \text{for } i = 1, 2, 3, \text{ and}
\]

\[
\sum_{j=1}^{m} g(T_1, R_j) > g(T_1, R) - \epsilon.
\]

(10) For each \( j, j = 1, \ldots, n \), consider the transformation \( T_1|_R_j: z_1 = t_1(w), w \in R_j \). Let \( F^*_j \) denote the collection of m.m.c.s under \( T_1|_R_j \). Then \( F^*_j \) is an upper semi-continuous collection of disjoint continua such that each
m.m.c. under $T_1|R_j$ of $F_j^*$ is contained in $R_j$, and the collection $F_j^*$ covers $R_j$. Further, each m.m.c. under $T_1|R_j$ of $F_j^*$ is contained in some m.m.c. under $T_1|R$. In addition, no component of $\zeta(R_j)$ is contained in $D_j$. Applying the lemma in 3.18, we find that no m.m.c. under $T_1|R_j$ separates in $R_j$, $j = 1, \cdots, n$.

Also, by the remark in 3.19, we find that if, for some $i$, $1 \leq i \leq 3$, no m.m.c. under $T_1|R$ separates in $R$, then no m.m.c. under $T_1|R_j$ separates in $R_j$, $j = 1, \cdots, n$.

(11) Let $\mathcal{P}_i$ be the collection of disjoint polygonal regions $R_1, \cdots, R_m$, and let $\mathcal{U}_i$ be the subcollection of $\mathcal{P}_i$ consisting of the polygonal regions $R_1, \cdots, R_n$.

We assert that these collections satisfy the requirements of the theorem.

a) is satisfied because $R_j \subset D_j \subset R$, $j = 1, \cdots, n$, and $R_j \subset D_j^*$, $j = n + 1, \cdots, m$.

b) and c) are demonstrated in (9), and d) and e) are demonstrated in (10).

Let us notice that if $R \neq E$, then $\mathcal{U}_i$ can be chosen to be a non-empty collection. Hence, if $\mathcal{U}_i$ is the empty collection, we must have $R = E$, and, by (3), $g(T_1, R) = 0$.

4.7 Corollary. If $\Lambda$ is a finite collection of disjoint polygonal regions, each of which is contained in $R_0$, and if $\epsilon$ is an arbitrary positive number, then there exists a finite collection $\mathcal{P}_1$ of disjoint polygonal regions, each of which is contained in some polygonal region of the
collection \( \Lambda \), and there exists a subcollection \( \Psi_i \) of \( \Xi_i \) such that

a) \( g(T_i, \Xi_i) > g(T_i, \Lambda) - \epsilon \), for \( i = 1, 2, 3 \).

b) \( g(T_1, \Psi_1) > g(T_1, \Lambda) - \epsilon \).

c) If \( \overline{R} \in \Psi_i \), then no maximal model continuum under \( T_1|\overline{R} \) separates in \( \overline{R} \).

d) If \( \overline{R} \in \Psi_i \), if \( \overline{R} \subset \overline{R} \), where \( \overline{R} \in \Lambda \), and if, for some \( i, 1 \leq i \leq 3 \), no maximal model continuum under \( T_1|\overline{R} \) separates in \( \overline{R} \), then no maximal model continuum under \( T_1|\overline{R} \) separates in \( \overline{R} \).

(Similarly collections \( \Xi_2, \Psi_2 \) and \( \Xi_3, \Psi_3 \), may be obtained having similar properties with regard to the transformations \( T_2 \) and \( T_3 \), respectively.)

4.8 Theorem. Let \( R \) be a polygonal region contained in \( R_0 \), and let \( \epsilon \) be an arbitrary positive number. Let \( i_1, \ldots, i_h, 0 \leq h \leq 3 \), denote those subscripts such that no maximal model continuum under \( T_j|\overline{R} \) separates in \( R \), \( j = 1, \ldots, h \). Then there exists a finite collection \( \Xi \) of disjoint simply connected polygonal regions, each of which is contained in \( R \), such that

\[
g(T_{ij}, \Xi) > g(T_{ij}, R) - \epsilon, \ j = 1, \ldots, h.
\]

Proof: We shall prove the theorem in case no m.m.c. under \( T_1|\overline{R} \) separates in \( R \), no m.m.c. under \( T_2|\overline{R} \) separates in \( R \), and no m.m.c. under \( T_3|\overline{R} \) separates in \( R \). The proofs in the remaining cases are similar, and somewhat simpler.

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(1) If $R$ is simply connected, then the collection $\mathcal{I}$ may be taken to be the collection consisting of $R$ alone.

(2) If $R$ is not simply connected, let $Q_1, \ldots, Q_q$ denote the bounded components of $\mathcal{C}(R)$, and let $Q_q$ denote the unbounded component of $\mathcal{C}(R)$. Let $r_1, \ldots, r_q$, denote the disjoint simple closed polygons which constitute the frontier of $R$ in such a way that $r_k = fr(Q_k)$, $k = 1, \ldots, q$.

By the lemma in 4.3, there is, for $i = 1, 2, 3$, a set $K_i$ of measure zero, such that $K_i \subset T_i(R) \subset T_i$, and such that, if $\gamma$ is an m.m.c. under $T_i|_R$, and $T_i(\gamma) \notin K_i$, then $\gamma$ is an m.m.c. under $T|_R$.

By the lemma in 4.5, there is, for $i = 1, 2, 3$, a set $E_i \subset \mathcal{L}(K_i) \cdot T_i(R)$, such that $E_i$ is closed and totally disconnected, and such that

$$\sum_{z_i} \mathcal{K}(z_i, T_i|_R) > \frac{\varepsilon}{2}.$$  

Since $\sum_{z_i} \mathcal{K}(z_i, T_i|_R) = 0$, we have $\sum_{z_i} \mathcal{K}(z_i, T_i|_R) > 0$.

Let $\bar{E}_i = (T_i|_R)^{-1} E_i$, for each $i$, $i = 1, 2, 3$. Then $\bar{E}_i$ is closed, and also, the components of $\bar{E}_i$ are m.m.c.s under $T_i|_R$. No component of $\bar{E}_i$ separates in $R$, so, by the first lemma in 3.7, $\bar{E}_i$ does not separate in $R$, for $i = 1, 2, 3$.

(3) Let $\gamma_i$ be a component of $\bar{E}_i$. Suppose $\gamma_i \cdot \bar{E}_2 \neq 0$. Then there is a component $\gamma_2$ of $\bar{E}_2$ such that $\gamma_i \cdot \gamma_2 \neq 0$. Now $\gamma_i$ is an m.m.c. under $T_i|_R$, $T_i(\gamma_i) \notin K_i$, and $\gamma_2$ is an m.m.c. under $T_2|_R$, $T_2(\gamma_2) \notin K_2$. Consequently,
\( \gamma_1 \) and \( \gamma_2 \) are both m.m.c.s under \( T|R \), so \( \gamma_1 = \gamma_2 \).

Therefore, if \( \gamma_1 \) is a component of \( E_1 \), then either
\[ \gamma_1 \cdot E_2 = 0, \] or else \( \gamma_1 \) is a component of \( E_2 \), so \( \gamma_1 \cdot E_2 \) is connected. By the second lemma in 3.7, we conclude that
\[ E_1 + E_2 \] does not separate in \( R \).

Let \( \gamma_3 \) be a component of \( E_3 \). Then, as above, either
\[ \gamma_3 \cdot E_1 = 0, \] or \( \gamma_3 \cdot E_1 = \gamma_3 \), and either \( \gamma_3 \cdot E_2 = 0, \) or \( \gamma_3 \cdot E_2 = \gamma_3 \). Hence, either \( \gamma_3 \cdot (E_1 + E_2) = 0, \) or else \( \gamma_3 \cdot (E_1 + E_2) = \gamma_3 \), so \( \gamma_3 \cdot (E_1 + E_2) \) is connected. Again using the second lemma in 3.7, we find that \( E_1 + E_2 + E_3 \) does not separate in \( R \).

(4) Let \( E = E_1 + E_2 + E_3 \). \( E \) is closed, so \( \zeta(E) \) is open. Since \( E \) does not separate in \( R \), it follows that the sets \( Q_k, k = 1, \cdots, q, \) defined in (2), all are contained in the same component \( D \) of \( \zeta(E) \). \( D \) is open and connected, so there exist polygonal arcs \( p_k, k = 1, \cdots, q-1, \) so that, for each \( k \), we have \( p_k \cdot E = 0 \) and so that \( p_k + S_k + S_q \) is connected, (where \( S_k = Q_k + r_k, \) for each \( k, k = 1, \cdots, q. \))

Let \( G = \alpha(R) - \sum_{k=1}^{q-1} p_k \). Then \( G \) is open, and \( G \subset R \).

Let \( D_1, \cdots, D_j, \cdots \), be the components of \( G \). For each \( j \), we have \( D_j \subset R \), and
\[
fr(D_j) \subset fr(G) \subset \zeta(G) = \sum_{k=1}^{q} S_k + \sum_{k=1}^{q-1} p_k
\]
\[
= \sum_{k=1}^{q-1} (p_k + S_k + S_q) + S_q.
\]

Consequently, \( \zeta(G) \) is connected, and so \( \zeta(G) \) is contained in a single component of \( \zeta(D_j) \). But, by the
third lemma in 3.7, each component of \( \mathcal{D}_j \) contains just one component of \( \text{fr}(D_j) \), so \( \text{fr}(D_j) \) has only one component, and \( D_j \) is a simply connected domain.

(5) If, for some \( i, 1 \leq i \leq 3, \mathcal{Y} \) is an e.m.m.c. under \( T_i | \mathcal{R} \), then \( \mathcal{Y} \in \mathcal{I}( \mathcal{R} ) \). Either \( \mathcal{Y} \cdot \sum_{k=i}^{q-1} p_k \neq 0 \), or else \( \mathcal{Y} \in \mathcal{G} \).

If \( \mathcal{Y} \cdot \sum_{k=i}^{q-1} p_k \neq 0 \), then \( T_i(\mathcal{Y}) \notin \mathcal{E}_1 \), for otherwise, \( \mathcal{Y} \subset (T_i | \mathcal{R})^{-1} T_i(\mathcal{Y}) \subset (T_i | \mathcal{R})^{-1} \mathcal{E}_1 = \overline{\mathcal{E}_1} \), while \( \overline{\mathcal{E}_1} \cdot \sum_{k=i}^{q-1} p_k = 0 \).

Hence \( \mathcal{Y} \notin \mathcal{G} \) implies \( T_i(\mathcal{Y}) \notin \mathcal{E}_1 \).

If \( \mathcal{Y} \in \mathcal{G} \), then, since \( \mathcal{Y} \) is connected, it follows that \( \mathcal{Y} \) is contained in a component \( D_j \) of \( \mathcal{G} \), and \( \mathcal{Y} \) is an e.m.m.c. under \( T_i | D_j \).

Thus we see that if \( z_i \in \mathcal{E}_1 \), then each e.m.m.c. \( \mathcal{Y} \) under \( T_i | \mathcal{R} \), for which \( T_i(\mathcal{Y}) = z_i \), is also an e.m.m.c. under \( T_i | D_j \), for some \( j \).

Consequently, \( \sum_{j=1}^{\mathcal{G}} \int_{E_i} K(z_i, T_i, R) = \sum_{j=1}^{\mathcal{G}} \int_{E_i} K(z_i, T_i, D_j) \), and \( \sum_{j=1}^{\mathcal{G}} g(T_i, D_j) \geq \sum_{j=1}^{\mathcal{G}} \int_{E_i} K(z_i, T_i, D_j) \)

\[ = \int_{E_i} K(z_i, T_i, R) > g(T_i, R) - \frac{\mathcal{G}}{2}, \]

for \( i = 1, 2, 3 \).

(6) We can find an integer \( n \), such that, for \( i = 1, 2, 3 \), we have \( \sum_{j=1}^{n} g(T_i, D_j) > g(T_i, R) - \frac{\mathcal{G}}{2} \).

Recall, from (4), that \( D_j, j = 1, \ldots, n \), is a simply connected domain contained in \( R \). Then by the second lemma in 3.7, we have a finite collection \( \mathcal{F} \) of disjoint simply connected polygonal regions, each of which is
contained in $D_j$ for some $j$, $j = 1, \cdots, n$, such that, for $i = 1, 2, 3$, we have $g(T_i, \mathcal{F}) > \sum_{j=1}^{n} g(T_i, D_j) - \frac{\varepsilon}{3}$.

Therefore, $g(T_i, \mathcal{F}) > g(T_i, R) - \varepsilon$, for $i = 1, 2, 3$, and so the collection $\mathcal{F}$ satisfies the requirements of the theorem.

4.9 Corollary. Suppose $\Lambda$ is a finite collection of disjoint polygonal regions, each of which is contained in $R_0$, and suppose $\varepsilon$ is an arbitrary positive number. Let $i_1, \cdots, i_h$, $0 \leq h \leq 3$, denote those subscripts such that if $R \in \Lambda$, then no maximal model continuum under $T_{i_j} | R$ separates in $R$, $j = 1, \cdots, h$. Then there exists a finite collection $\mathcal{F}$ of disjoint simply connected polygonal regions, each of which is contained in some polygonal region of the collection $\Lambda$, such that $g(T_{i_j}, \mathcal{F}) > g(T_{i_j}, \Lambda) - \varepsilon$, for $j = 1, \cdots, h$.

4.10 Theorem. If $R$ is a polygonal region contained in $R_0$, and if $\varepsilon$ is an arbitrary positive number, then there exists a finite collection $\mathcal{F}_1$ of disjoint simply connected polygonal regions, each of which is contained in $R$, such that $g(T_1, \mathcal{F}_1) > g(T_1, R) - \varepsilon$.

(Similar collections $\mathcal{F}_2$ and $\mathcal{F}_3$ exist having similar properties with regard to the transformations $T_2$ and $T_3$ respectively.)

Proof: From the theorem in 4.6, there is a finite collection $\mathcal{U}_j$ of disjoint polygonal regions, each of which is contained in $R$, such that $g(T_1, \mathcal{U}_j) > g(T_1, R) - \frac{\varepsilon}{2}$, and
such that, if $R \in \Psi_i$, then no m.m.c. under $T_1|R$ separates in $R$.

If $\Psi_i$ is the empty collection, then $g(T_1,R) = 0$, [cf. 4.6, (11)], and the required collection $\mathcal{F}_i$ may be taken to be any collection of disjoint simply connected polygonal regions, each of which is contained in $R_0$. (A similar remark will hold in the proofs of theorems occurring later in this chapter.)

We may assume then that the collection $\Psi_i$ is non-empty. By the corollary in 4.9, there is a finite collection $\mathcal{F}$ of disjoint simply connected polygonal regions, each of which is contained in some polygonal region of the collection $\Psi_i$, such that $g(T_1,\mathcal{F}) > g(T_1,\Psi_i) - \frac{\epsilon}{2}$.

Therefore, $g(T_1,\mathcal{F}) > g(T_1,R) - \epsilon$, so the collection $\mathcal{F}$ will serve as the collection $\mathcal{F}_i$.

4.11 The following corollary is an immediate consequence of the theorem in the preceding section, and the proof will be omitted.

**Corollary.** If $\Psi$ is a finite collection of disjoint polygonal regions, each of which is contained in $R_0$, and if $\epsilon$ is an arbitrary positive number, then there exists a finite collection $\mathcal{F}_i$ of disjoint simply connected polygonal regions, each of which is contained in some polygonal region of the collection $\Psi$, such that $g(T_1,\mathcal{F}_i) > g(T_1,\Psi) - \epsilon$.

(Also, similar collections $\mathcal{F}_2$ and $\mathcal{F}_3$ exist having
similar properties with regard to the transformations \( T_2 \) and \( T_3 \), respectively.)

4.12 In the proof of the following theorem, it will be assumed that all finite collections of polygonal regions which occur in the proof are non-empty. In case one or more of these collections is empty, then the proof of the theorem is similar and somewhat simpler than the proof when the collections are all non-empty. [cf. 4.6, (11); 4.10]

Theorem. If \( R \) is a polygonal region contained in \( R_0 \), and if \( \epsilon \) is an arbitrary positive number, then there exists a finite collection \( \mathcal{E}_3 \) of disjoint simply connected polygonal regions, each of which is contained in \( R \), such that

\[
\gamma(T_1, \mathcal{E}_3) > \gamma(T_1, R) - \epsilon, \quad \text{and} \quad \gamma(T_2, \mathcal{E}_3) > \gamma(T_2, R) - \epsilon.
\]

(There exist similar collections \( \mathcal{E}_1 \) and \( \mathcal{E}_1 \) having similar properties with regard to the transformations \( T_1 \) and \( T_3 \), and \( T_2 \) and \( T_3 \), respectively.)

Proof: From the theorem in 4.6, there is a finite collection \( \Lambda_1 \) of disjoint polygonal regions, each of which is contained in \( R \), and there is a subcollection \( \Lambda_2 \) of \( \Lambda_1 \), such that

\[
\gamma(T_2, \Lambda_1) > \gamma(T_2, R) - \frac{\epsilon}{4}, \quad \text{and} \quad \gamma(T_1, \Lambda_1) > \gamma(T_1, R) - \frac{\epsilon}{4},
\]

and such that if \( \bar{R} \in \Lambda_2 \), then no m.m.c. under \( T_1 | \bar{R} \) separates in \( \bar{R} \).

From the corollary in 4.7, we find that there is a finite collection \( \Lambda_3 \) of disjoint polygonal regions, each of which is contained in some polygonal region of the collection \( \Lambda_2 \), and there is a subcollection \( \Lambda_4 \) of \( \Lambda_3 \), such that
\[ g(T_1, \Lambda_3) > g(T_1, \Lambda_4) - \frac{\xi}{q}, \text{ and } g(T_2, \Lambda_4) > g(T_2, \Lambda_4) - \frac{\xi}{q}, \]
and such that if \( \bar{R} \in \Lambda_4 \), then no m.m.c. under \( T_1|\bar{R} \) separates in \( \bar{R} \) and no m.m.c. under \( T_2|\bar{R} \) separates in \( \bar{R} \).

Now let \( \Lambda_5 \) denote the subcollection of polygonal regions of the collection \( \Lambda_1 \) which do not belong to the subcollection \( \Lambda_2 \), and let \( \Lambda_6 \) denote the subcollection of polygonal regions of the collection \( \Lambda_3 \) which do not belong to the subcollection \( \Lambda_4 \).

Then
\[ g(T_1, \Lambda_4) + g(T_1, \Lambda_6) > g(T_1, \Lambda_4) - \frac{\xi}{q}, \text{ and } g(T_2, \Lambda_4) + g(T_2, \Lambda_5) > g(T_2, \Lambda_4) + g(T_2, \Lambda_5) - \frac{\xi}{q}, \]
and
\[ g(T_2, \Lambda_4) > g(T_2, \Lambda_4) - \frac{\xi}{q}. \]

By the corollary in 4.11, we can obtain a finite collection \( \Lambda_7 \) of disjoint simply connected polygonal regions, each of which is contained in some polygonal region of the collection \( \Lambda_4 \), such that \( g(T_1, \Lambda_7) > g(T_1, \Lambda_4) - \frac{\xi}{q} \).

By the corollary in 4.11, we can obtain a finite collection \( \Lambda_8 \) of disjoint simply connected polygonal regions, each of which is contained in some polygonal region of the collection \( \Lambda_5 \), such that \( g(T_2, \Lambda_8) > g(T_2, \Lambda_5) - \frac{\xi}{q} \).

By the corollary in 4.9, we can obtain a finite collection \( \Lambda_9 \) of disjoint simply connected polygonal regions, each of which is contained in some polygonal region of the collection \( \Lambda_4 \), such that \( g(T_1, \Lambda_9) > g(T_1, \Lambda_4) - \frac{\xi}{q} \), and \( g(T_2, \Lambda_9) > g(T_2, \Lambda_4) - \frac{\xi}{q} \).

Consequently, it follows that
\[ g(T_1, \Lambda_7) + g(T_1, \Lambda_8) + g(T_1, \Lambda_9) > g(T_1, \Lambda_4) + g(T_1, \Lambda_5) - \frac{\epsilon}{2} > g(T_1, R) - \epsilon , \text{ and} \]
\[ g(T_2, \Lambda_7) + g(T_2, \Lambda_8) + g(T_2, \Lambda_9) > g(T_2, \Lambda_4) + g(T_2, \Lambda_5) - \frac{\epsilon}{2} > g(T_2, R) - \epsilon . \]

Let \( \mathcal{F}_3 \) denote the collection of disjoint simply connected polygonal regions which belong to the collections \( \Lambda_7, \Lambda_8, \text{ and } \Lambda_9 \). Then the collection \( \mathcal{F}_3 \) satisfies the requirements of the theorem, that is,
\[ g(T_1, \mathcal{F}_3) > g(T_1, R) - \epsilon , \text{ and } g(T_2, \mathcal{F}_3) > g(T_2, R) - \epsilon . \]

4.13 The following corollary is an immediate consequence of the theorem in the preceding section, and the proof will be omitted.

**Corollary.** If \( \Psi \) is a finite collection of disjoint polygonal regions, each of which is contained in \( R_0 \), and \( \epsilon \) is an arbitrary positive number, then there exists a finite collection \( \mathcal{F}_3 \) of disjoint simply connected polygonal regions, each of which is contained in some polygonal region of the collection \( \Psi \), such that \( g(T_1, \mathcal{F}_3) > g(T_1, \Psi) - \epsilon \), and \( g(T_2, \mathcal{F}_3) > g(T_2, \Psi) - \epsilon \).

(There exist similar collections \( \mathcal{F}_2 \) and \( \mathcal{F}_1 \) having similar properties with regard to the transformations \( T_1 \) and \( T_3 \), and \( T_2 \) and \( T_3 \), respectively.)

4.14 In the proof of the following theorem, it will be assumed that all finite collections of polygonal regions which occur in the proof are non-empty. In case one or more of these collections is empty, then the proof of the
Theorem is similar and somewhat simpler than the proof when the collections are all non-empty. [cf. 4.6, (11); 4.10]

**Theorem.** If $R$ is a polygonal region contained in $R_0$, and if $\epsilon$ is an arbitrary positive number, then there exists a finite collection $\mathcal{F}$ of disjoint simply connected polygonal regions, each of which is contained in $R$, such that, for $i = 1, 2, 3$, $g(T_i, \mathcal{F}) > g(T_i, R) - \epsilon$.

**Proof:** By the theorem in 4.6, there is a finite collection $\mathcal{A}_1$ of disjoint polygonal regions, each of which is contained in $R$, and there is a subcollection $\mathcal{A}_2$ of $\mathcal{A}_1$, such that $g(T_1, \mathcal{A}_1) > g(T_1, R) - \frac{\epsilon}{2}$,

$g(T_2, \mathcal{A}_1) > g(T_2, R) - \frac{\epsilon}{2}$,

$g(T_3, \mathcal{A}_1) > g(T_3, R) - \frac{\epsilon}{2}$,

and such that, if $R \in \mathcal{A}_1$, then no m.m.c. under $T_1|\overline{R}$ separates in $\overline{R}$.

By the corollary in 4.7, there is a finite collection $\mathcal{A}_3$ of disjoint polygonal regions, each of which is contained in some polygonal region of the collection $\mathcal{A}_2$, and there is a subcollection $\mathcal{A}_4$ of $\mathcal{A}_3$, such that $g(T_2, \mathcal{A}_3) > g(T_2, \mathcal{A}_2) - \frac{\epsilon}{2}$,

$g(T_1, \mathcal{A}_3) > g(T_1, \mathcal{A}_2) - \frac{\epsilon}{2}$,

$g(T_3, \mathcal{A}_3) > g(T_3, \mathcal{A}_2) - \frac{\epsilon}{2}$,

and such that, if $R \in \mathcal{A}_4$, then no m.m.c. under $T_1|\overline{R}$ separates in $\overline{R}$, and no m.m.c. under $T_2|\overline{R}$ separates in $\overline{R}$.

By the corollary in 4.7, there is a finite collection $\mathcal{A}_5$ of disjoint polygonal regions, each of which is contained in $R$. 

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tained in some polygonal region of the collection \( \Lambda_4 \), and there is a subcollection \( \Lambda_6 \) of \( \Lambda_5 \), such that

\[
\begin{align*}
\varepsilon(T_1, \Lambda_6) &> \varepsilon(T_1, \Lambda_4) - \frac{\varepsilon}{3}, \\
\varepsilon(T_2, \Lambda_5) &> \varepsilon(T_1, \Lambda_4) - \frac{\varepsilon}{3}, \\
\varepsilon(T_2, \Lambda_6) &> \varepsilon(T_2, \Lambda_4) - \frac{\varepsilon}{3},
\end{align*}
\]

and if \( R \in \Lambda_6 \), then no m.m.c. under \( T_1 | R \) separates in \( R \), no m.m.c. under \( T_2 | R \) separates in \( R \), and no m.m.c. under \( T_3 | R \) separates in \( R \).

Now let \( \Lambda_7 \) denote the subcollection of polygonal regions of the collection \( \Lambda_1 \) which do not belong to the subcollection \( \Lambda_2 \). Let \( \Lambda_8 \) denote the subcollection of polygonal regions of the collection \( \Lambda_3 \) which do not belong to the subcollection \( \Lambda_4 \). Let \( \Lambda_9 \) denote the subcollection of polygonal regions of the collection \( \Lambda_5 \) which do not belong to the subcollection \( \Lambda_6 \).

Then

\[
\begin{align*}
\varepsilon(T_1, \Lambda_6) + \varepsilon(T_1, \Lambda_7) + \varepsilon(T_1, \Lambda_7) &> \varepsilon(T_1, \Lambda_4) + \varepsilon(T_1, \Lambda_8) - \frac{\varepsilon}{6} \\
&> \varepsilon(T_1, \Lambda_4) - \frac{\varepsilon}{3} \\
&> \varepsilon(T_1, R) - \frac{\varepsilon}{2},
\end{align*}
\]

\[
\begin{align*}
\varepsilon(T_2, \Lambda_6) + \varepsilon(T_2, \Lambda_7) + \varepsilon(T_2, \Lambda_7) &> \varepsilon(T_2, \Lambda_4) + \varepsilon(T_2, \Lambda_8) - \frac{\varepsilon}{6} \\
&> \varepsilon(T_2, \Lambda_4) + \varepsilon(T_2, \Lambda_7) - \frac{\varepsilon}{3} \\
&> \varepsilon(T_2, R) - \frac{\varepsilon}{2}, \text{ and}
\end{align*}
\]

\[
\begin{align*}
\varepsilon(T_3, \Lambda_6) + \varepsilon(T_3, \Lambda_7) + \varepsilon(T_3, \Lambda_7) &> \varepsilon(T_3, \Lambda_4) + \varepsilon(T_3, \Lambda_7) - \frac{\varepsilon}{6} \\
&> \varepsilon(T_3, \Lambda_4) - \frac{\varepsilon}{3} \\
&> \varepsilon(T_3, R) - \frac{\varepsilon}{2}.
\end{align*}
\]

By the corollary in 4.13, we can obtain a finite
collection $\Lambda_{10}$ of disjoint simply connected polygonal regions, each of which is contained in some polygonal region of the collection $\Lambda_1$, such that
\[ g(T_1, \Lambda_{10}) > g(T_1, \Lambda_1) - \frac{\epsilon}{2} \text{, and } \]
\[ g(T_2, \Lambda_{10}) > g(T_2, \Lambda_1) - \frac{\epsilon}{2} \text{.} \]

By the corollary in 4.13, we can obtain a finite collection $\Lambda_{11}$ of disjoint simply connected polygonal regions, each of which is contained in some polygonal region of the collection $\Lambda_1$, such that
\[ g(T_1, \Lambda_{11}) > g(T_1, \Lambda_1) - \frac{\epsilon}{2} \text{, and } \]
\[ g(T_2, \Lambda_{11}) > g(T_2, \Lambda_1) - \frac{\epsilon}{2} \text{.} \]

By the corollary in 4.13, we can obtain a finite collection $\Lambda_{12}$ of disjoint simply connected polygonal regions, each of which is contained in some polygonal region of the collection $\Lambda_7$, such that
\[ g(T_2, \Lambda_{12}) > g(T_2, \Lambda_7) - \frac{\epsilon}{2} \text{, and } \]
\[ g(T_3, \Lambda_{12}) > g(T_3, \Lambda_7) - \frac{\epsilon}{2} \text{.} \]

By the corollary in 4.9, we can obtain a finite collection $\Lambda_{13}$ of disjoint simply connected polygonal regions, each of which is contained in some polygonal region of the collection $\Lambda_4$, such that
\[ g(T_1, \Lambda_{13}) > g(T_1, \Lambda_4) - \frac{\epsilon}{2} \text{, and } \]
\[ g(T_2, \Lambda_{13}) > g(T_2, \Lambda_4) - \frac{\epsilon}{2} \text{, and } \]
\[ g(T_3, \Lambda_{13}) > g(T_3, \Lambda_4) - \frac{\epsilon}{2} \text{.} \]

Consequently, we have

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\[ g(T_1, \Lambda_{10}) + g(T_1, \Lambda_{11}) + g(T_1, \Lambda_{12}) + g(T_1, \Lambda_{13}) \]
\[ > g(T_1, \Lambda_4) + g(T_1, \Lambda_8) + g(T_1, \Lambda_9) - \frac{\epsilon}{2} \]
\[ > g(T_1, R) - \epsilon \], and

\[ g(T_2, \Lambda_{10}) + g(T_2, \Lambda_{11}) + g(T_2, \Lambda_{12}) + g(T_2, \Lambda_{13}) \]
\[ > g(T_2, \Lambda_4) + g(T_2, \Lambda_7) + g(T_2, \Lambda_9) - \frac{\epsilon}{2} \]
\[ > g(T_2, R) - \epsilon \], and

\[ g(T_3, \Lambda_{10}) + g(T_3, \Lambda_{11}) + g(T_3, \Lambda_{12}) + g(T_3, \Lambda_{13}) \]
\[ > g(T_3, \Lambda_4) + g(T_3, \Lambda_7) + g(T_3, \Lambda_8) - \frac{\epsilon}{2} \]
\[ > g(T_3, R) - \epsilon \].

Let \( \Phi \) denote the collection of disjoint simply connected polygonal regions which belong to the collections \( \Lambda_{10}, \Lambda_{11}, \Lambda_{12}, \) and \( \Lambda_{13} \). Then the collection \( \Phi \) satisfies the requirements of the theorem, that is, for \( i = 1, 2, 3 \), we have \( g(T_i, \Phi) > g(T_i, R) - \epsilon \).

4.15 The following corollary is an immediate consequence of the theorem in the preceding section, and the proof will be omitted.

**Corollary.** If \( \Psi \) is a finite collection of disjoint polygonal regions, each of which is contained in \( R_0 \), and \( \epsilon \) is an arbitrary positive number, then there exists a finite collection \( \Phi \) of disjoint simply connected polygonal regions, each of which is contained in some polygonal region of the collection \( \Psi \), such that, for \( i = 1, 2, 3 \),

\[ g(T_i, \Phi) > g(T_i, \Psi) - \epsilon \].

4.16 **Theorem.** If \( R \) is a polygonal region contained in \( R_0 \), and if \( \epsilon \) is an arbitrary positive number, then
there exists a finite collection \( \mathcal{I} \) of disjoint simply connected polygonal regions, each of which is contained in \( R \), such that \( G(T, \mathcal{I}) > G(T, R) - \epsilon \).

**Proof:** By the lemma in 2.3, there are non-negative numbers \( b_1, b_2, \) and \( b_3 \), such that, for \( i = 1, 2, 3 \), if \( g(T_i, R) = 0 \), then \( b_i = 0 \), and if \( g(T_i, R) > 0 \), then \( b_i < g(T_i, R) \), and such that \( (b_1^2 + b_2^2 + b_3^2)^{\frac{1}{2}} > G(T, R) - \epsilon \).

By the theorem in 4.14, there is a finite collection \( \mathcal{I} \) of disjoint simply connected polygonal regions, each of which is contained in \( R \), such that, for \( i = 1, 2, 3 \), we have \( g(T_i, \mathcal{I}) \leq b_i \).

Consequently, it follows from the lemma in 2.2 that

\[
G(T, \mathcal{I}) \geq \sum_{R \in \mathcal{I}} G(T, R)
\]

\[
= \sum_{R \in \mathcal{I}} \left( (g(T_1, R))^2 + (g(T_2, R))^2 + (g(T_3, R))^2 \right)^{\frac{1}{2}}
\]

\[
\geq \left( \sum_{R \in \mathcal{I}} g(T_1, R)^2 + \left( \sum_{R \in \mathcal{I}} g(T_2, R)^2 + \left( \sum_{R \in \mathcal{I}} g(T_3, R)^2 \right) \right)^{\frac{1}{2}}
\]

\[
= \left( (g(T_1, \mathcal{I}))^2 + (g(T_2, \mathcal{I}))^2 + (g(T_3, \mathcal{I}))^2 \right)^{\frac{1}{2}}
\]

\[
\geq (b_1^2 + b_2^2 + b_3^2)^{\frac{1}{2}} \geq G(T, R) - \epsilon .
\]

4.17. The following corollary is an immediate consequence of the theorem in the preceding section, and the proof will be omitted.

**Corollary.** If \( \Psi \) is a collection of the second class, and if \( \epsilon \) is an arbitrary positive number, then there exists a collection \( \mathcal{I} \) of the first class, such that \( G(T, \mathcal{I}) > G(T, \Psi) - \epsilon \).

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4.18 We have assumed throughout Chapter IV that 
T: \( z = t(w), \ w \in R_0 \), is of essential bounded variation. 
[cf. 4.2] 
Consequently, if T: \( z = t(w), \ w \in R_0 \), is of essential 
bounded variation, it follows from the corollary in 4.17 
that \( a_1(T) \leq a_2(T) \). [cf. 1.4] We have noted in 1.6 that 
a_1(T) \leq a_2(T). Therefore, \( a_1(T) = a_2(T) \). Also, in 2.11, 
we have shown that \( a_1(T) = a_3(T) = a_5(T) \) and 
a_2(T) = a_4(T) = a_6(T). Hence we may conclude that if 
T: \( z = t(w), \ w \in R_0 \), is of essential bounded variation, 
then \( a_j(T) = a(T) \), for \( j = 1, \ldots, 6 \). [cf. 1.4] 

If T: \( z = t(w), \ w \in R_0 \), is not of essential bounded 
variation, then \( a_j(T) = a(T) = +\infty \), for \( j = 1, \ldots, 6 \). 
[cf. 1.9] 
Therefore, we have shown that, in every case, 
a_j(T) = a(T), for \( j = 1, \ldots, 6 \), which was the purpose of 
this dissertation.
BIBLIOGRAPHY


AUTOBIOGRAPHY

I, William M. Myers, Jr., was born in Dayton, Ohio, April 23, 1925. I received my secondary school education in the public schools of Dayton, Ohio. My undergraduate training was obtained at Denison University, from which I received the degree Bachelor of Arts in 1946. From the Ohio State University, I received the degree Master of Arts in 1948. While in residence at Ohio State University, I was a graduate assistant in the mathematics department 1947 - 1949, an instructor in 1950, and an assistant instructor in 1951.