A MATHEMATICAL ANALYSIS OF THE HUMAN OPERATOR IN A CLOSED-LOOP CONTROL SYSTEM

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

CLAUDE ELLSWORTH WALSTON, B.S., M.S.
The Ohio State University
1953

Approved by:

C. E. Warner
Adviser
ACKNOWLEDGEMENTS

The author is greatly indebted to Professor C. E. Warren of the Department of Electrical Engineering of The Ohio State University who suggested the problem investigated. His valuable advice and guidance contributed greatly to the development of this dissertation.

The author also wishes to express his sincere appreciation to Dr. Paul M. Fitts of the Department of Psychology for his valuable suggestions and his interest in this investigation.
FOREWORD

The investigation here reported was carried out as part of a program of research dealing with motor skill and skill learning. The objectives of this research program are a better understanding of the nature of human motor behavior in skill tasks, as related both to task and organismic variables, and the development of improved techniques for the precise measurement, analysis, and identification of components of skill.

This research program is supported by the Human Resources Research Center of the USAF Air Training Command and is monitored by the Perceptual and Motor Skills Research Laboratory. Investigations are conducted through the use of the facilities of the Psychology, Electrical Engineering, and other Departments of The Ohio State University. Paul M. Fitts, Director of the Aviation Psychology Laboratory, Department of Psychology, is project supervisor and Claude E. Warren, Department of Electrical Engineering is associate supervisor.
TABLE OF CONTENTS

FOREWORD ...................................... iii
LIST OF FIGURES ................................ vi
LIST OF TABLES ................................ x
I INTRODUCTION .................................. 1
II EXPERIMENTAL PROCEDURE ..................... 3
   A. Apparatus .................................. 3
   B. Experimental Variables .................... 7
   C. Design and Method .......................... 8
   D. Data Recorded .............................. 9
III MATHEMATICAL ANALYSIS ...................... 11
   A. General Case. ............................. 11
   B. The Compensatory Task .................... 16
      1. Human noise ............................. 39
      2. The simple harmonic tracking task .. 54
      3. The random tracking task .............. 57
   C. The Following Task ........................ 63
      1. The noise term ......................... 67
      2. The simple harmonic tracking task .. 70
      3. The random tracking task .............. 71
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>IV Evaluation of the Human Constants</td>
<td>73</td>
</tr>
<tr>
<td>A. The Compensatory Task</td>
<td>73</td>
</tr>
<tr>
<td>1. The simple harmonic tracking task</td>
<td>73</td>
</tr>
<tr>
<td>2. The random task</td>
<td>86</td>
</tr>
<tr>
<td>B. The Following Task</td>
<td>102</td>
</tr>
<tr>
<td>1. The simple harmonic tracking task</td>
<td>102</td>
</tr>
<tr>
<td>2. The random input</td>
<td>111</td>
</tr>
<tr>
<td>V Discussion and Conclusions</td>
<td>121</td>
</tr>
<tr>
<td>A. The Human Constants</td>
<td>121</td>
</tr>
<tr>
<td>B. Autocorrelation Functions</td>
<td>136</td>
</tr>
<tr>
<td>C. Learning</td>
<td>140</td>
</tr>
<tr>
<td>D. Summary</td>
<td>142</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>151</td>
</tr>
</tbody>
</table>
LIST OF FIGURES

Figure 1. Block diagram of the pursuit apparatus .................. 4

2. Typical records of the operator's response and error in a compensatory task. 15

3. A plot of H, H, N space showing one way of classifying the operators tracking behavior for different tracking conditions 22

4. Block diagram of the compensatory situation incorporating the assumed human transfer function. 24

5. A plot of the stability region for the compensatory tracking situation. 33

6. The loci of several of the roots of the characteristic equation of the compensatory tracking task 36

7. Comparison of the root loci for the exact and approximate characteristic equations of the compensatory tracking situation. 45

8. The effect of a decrease in L upon $G_{EN}^N(\omega)$. 51

9. The effect of a uniform increase in B upon $G_{EN}^N(\omega)$ 52

10. The effect of an increase in AL/B upon $G_{EN}^N(\omega)$ 52

11. A Rahm recording of a typical output of the random function generator. 59
Figure 12. The autocorrelation function of a typical sample of the output of the random function generator... 60

13. A block diagram of the following tracking situation incorporating the assumed human transfer function... 65

14. A plot of the mean-square error for the sinusoidal compensatory task showing the effects of the scale factors... 76

15. Absolute, angular mean-square error for the sinusoidal compensatory task 79

16. A plot of $E^2$ in the sinusoidal compensatory task for the case $K_d = 1$. 83

17. A plot of $E^2$ in the sinusoidal compensatory task for the case $K_d = 1$. 84

18. A plot of $E^2$ in the random compensatory task showing the effect of the scale factors... 87

19. The normalized autocorrelation functions of the error of Subject 1 in the random compensatory task... 89

20. Autocorrelation functions of the error of two subjects in the random compensatory task... 90

21. Autocorrelation functions of the error of Subject 2 in the random compensatory task... 93

22. Effects of changes in the control scale factors upon the steady-state error spectral density... 97
Figure 23. Root locus for the case $AL/B = 4$, showing the loci of roots for particular values of $B$. .......................... 97

24. A plot of $E^2$ for Subject 1 in the random compensatory task. ........................... 100

25. A comparison of the theoretical and experimental autocorrelation function of Subject 1 for the random compensatory task. ........................... 101

26. A plot of $E^2$ for the sinusoidal following task. ........................... 103

27. A plot of $E^2$ for the sinusoidal following task. ........................... 105

28. Mean-square error of three subjects in the sinusoidal following task. ........................... 106

29. Autocorrelation function of the error of a subject in the sinusoidal following task. ........................... 109

30. Autocorrelation function of the error of a subject in the sinusoidal following task. ........................... 110

31. Mean-square error scores in the random following task. ........................... 112

32. Autocorrelation functions of the error of two subjects in the random following task. ........................... 113

33. Autocorrelation functions of the error of a subject in the random following task. ........................... 114

34. Mean-square error of Subject 4 in the random following task. ........................... 119
Figure 35. A comparison of the theoretical and experimental autocorrelation functions of the error of Subject 4 ............................................................ 120

36. Determination of the maximum allowable value of $\omega$ ................. 139

37. Autocorrelation function of the error of Subject 6 showing the effects of learning .......................................................... 141

38. A suggested analog of the human transfer function ....................... 150
<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Human Parameters Computed for the Sinusoidal Compensatory Task</td>
<td>85</td>
</tr>
<tr>
<td>2.</td>
<td>Values of the Human Parameters of Subject 1 in the Random Compensatory Task</td>
<td>99</td>
</tr>
<tr>
<td>3.</td>
<td>Human Parameters Computed for the Sinusoidal Following Task</td>
<td>107</td>
</tr>
<tr>
<td>4.</td>
<td>Human Parameters Evaluated for Subject 4 in the Random Following Task</td>
<td>118</td>
</tr>
<tr>
<td>5.</td>
<td>Evaluation of the Performance Criteria Factors for the Sinusoidal Following Task</td>
<td>129</td>
</tr>
</tbody>
</table>
A MATHEMATICAL ANALYSIS OF THE HUMAN OPERATOR
IN A CLOSED-LOOP CONTROL SYSTEM

I INTRODUCTION

In the past fifty years great technological advances have been made in all the branches of science. As a result of these advances our civilization has become an increasingly complex mechanism and we have been forced to develop control systems capable of processing and handling large quantities of information for both industrial and military applications with much stress being placed upon the latter as a result of World War II and the cold war which has followed it. Consequently much time and effort has been applied to the design and development of the material portion of control systems. It has only been realized in the past few years that the human element enters somewhere in the operation of nearly every control system and human behavior may greatly influence the effectiveness of the system. In some situations efforts have been made to eliminate the human element altogether, but this is not always practicable. In those systems in which the human operator appears, a knowledge of the behavior of human beings...
relative to these systems would aid engineers to design equipment which could capitalize upon those characteristics of the human operator which would yield a more efficient system and which would compensate for his shortcomings. This knowledge is also essential for an adequate understanding of human learning and individual differences and for an effective solution to selection and training problems. As a result, research programs have been established in an effort to learn more about human behavior in control systems and especially to learn more about the ability of the human operator to act as an information-handling device.

One phase of the investigation of human behavior in these situations has been an attempt to determine the human transfer function, particularly the transfer function relating to a visual-input, manual-output operation. There exist two techniques for attempting to determine the human transfer function. One approach is to use the methods of electrical engineering analysis to measure the human transfer function experimentally. The disadvantage of this approach is that although it yields a numerical description of the human transfer function for a specific situation it does not provide any means for predicting what the behavior of the human will be in
another and different situation. The second approach is to use a priori reasoning to obtain an analytic expression for the human transfer function taking into account as many factors as possible and then to check the validity of this expression by seeing how closely it agrees with actual experimental data. The latter technique is the one that we have utilized in our efforts to develop a mathematical analysis of the human operator in closed-loop control systems.

II EXPERIMENTAL PROCEDURE

Before undertaking a mathematical analysis of the human operator, we shall describe briefly the apparatus used, the procedure followed and the data collected in the experimental portion of this analysis.

A. Apparatus

The apparatus used in this study was a pursuit apparatus designed and constructed in the Department of Electrical Engineering of the Ohio State University under the supervision of Professor C. E. Warren. A simplified block diagram of the apparatus is shown in Figure 1. This pursuit apparatus provides a means of
Figure 1. Block diagram of the pursuit apparatus showing arrangements for the following and the compensatory tasks.
simulating a tracking problem and of scoring the performance of the human operator in tracking the given problem. The simulated tracking problem, which may be one or two-dimensional, is displayed upon the face of a cathode-ray tube. In this study, however, only one-dimensional problems were considered. The output of the operator is manual and is effected by any one of several different types of tracking controls. In this study only one control was used, an arm control, which consisted of a lever 17 inches in length capable of motion in a horizontal plane. The pivot end of the lever, upon which the operator rested his elbow, was geared to a potentiometer which translated the angular movement of the operator's arm into a voltage. This arm control had practically no inertia, friction, or elasticity; hence the added effects of the arm control on the inherent feedback of proprioceptive information generated in the course of the arm's activity can be neglected.

Two different types of tracking tasks can be presented on the pursuit apparatus. The first task, which has been termed the following task, may be described with reference to Figure 1(A). A problem, generated by the problem generator, is fed through a time sharing switch into a cathode-ray oscilloscope where it causes a
motion of the target on the face of the tube. The target appears as a vertical straight line about 1/32 inch in width and 5/8 inch in length. The output of the human operator, that is, the voltage output of the arm control potentiometer, is also fed into the oscilloscope through the time sharing switch. The operator's output appears as a second vertical line, the cursor, on the face of the cathode-ray tube. The cursor is located below the target; its top portion slightly overlapping the bottom portion of the target. The switching speed of the time sharing switch is sufficiently high that the target and the cursor seem to the human operator to appear on the cathode-ray tube simultaneously. The operator's task is to keep the cursor superimposed upon the target.

The second type of tracking task is the compensatory task which can be described with reference to Figure 1(B). In this case the operator's output and the problem are subtracted and their difference (the operator's error) is presented to the oscilloscope through the time sharing switch which has one side grounded so that one of the vertical lines remains stationary at the center of the cathode-ray tube. The remaining line then moves in proportion to the tracking error. Again the operator's task is to keep the two lines superimposed.
B. **Experimental Variables**

In this study three variables were controlled by the experimenter. The first of these was the nature of the problem supplied to the operator. Two types of problems were considered:

(i) a simple harmonic motion of thirty cycles per minute, and

(ii) a random motion, Gaussian in amplitude distribution, but with a low-frequency spectrum.

The second variable was the amplitude of the target and cursor motion displayed on the face of the oscilloscope. The several amplitudes used were 1/4 inch, 1/2 inch, 1 inch, 2 inches and 4 inches. These values represent the maximum horizontal movement of the target. The relation existing in the pursuit apparatus between the display on the oscilloscope and the input to the display control is given by

\[ D(t) = 0.44K_dE(t) \] (1)

where \( D \), the distance between the target and cursor lines, is expressed in inches, \( E \), the error, is in volts, and \( K_d \), the display scale factor, is a numeric. For \( K_d = 1 \),
the maximum target motion on the oscilloscope is 4 inches; when \( K_d = 1/2 \), the maximum target motion is 2 inches, etc.

The third variable used in this study was the amount of angular displacement of the arm control required to track the target between its maximum right and left excursions. Five different values of maximum arm control angular displacements were used: 80 degrees, 40 degrees, 20 degrees, 10 degrees and 5 degrees. The relation existing between the angular arm control displacement and the voltage output of the arm control potentiometer is

\[
S_c(t) = 6.45K_m S_H(t)
\]

where \( S_c \), the output of the arm control potentiometer, is expressed in volts, \( S_H \), the angular motion of the arm control, is in radians and \( K_m \), the manual control scale factor, is a numeric. For a setting of \( K_m = 1 \), the total arm control movement required is 80 degrees; for \( K_m = 2 \), the total movement is 40 degrees; etc.

C. Design and Method

The experimental data used in the analysis of the following and compensatory tasks for the simple, sine-wave input were collected by Bryce Hartman in the process
of gathering data for his dissertation, and a description of the experimental methods he used can be found there.

The data for the random tracking tasks were collected by the author using as subjects graduate students at the Ohio State University. The subjects were given two and one-half hours of practice (approximately 140 one minute trials) and then tracked one and one-half hours on compensatory tasks and one and one-half hours on following tasks under various combinations of the display and manual control scale factors presented in a random fashion.

D. Data Recorded

The pursuit apparatus was so constructed that two different types of data could be collected, qualitative and quantitative. The qualitative data consisted of graphic records of the input or target signal, the operator's output signal, and the operator's error signal recorded on a Rahm graphic recorder and on film by means of a specially designed camera. The quantitative data recorded were the mean square error, the average error, time-on-target scores and target hit scores for several target bandwidths.
The Rahm recordings could be used as visual indicators of certain trends in the operators performance, but beyond that they did not provide much information. The photographically recorded data were valuable in that they could be used to obtain autocorrelation functions of the signals recorded. Unfortunately, the autocorrelator was not available when the study of simple harmonic tracking was being made. However, it was available during the investigation of the random tracking task and autocorrelation functions of the operator's error signal were obtained for a wide range of the experimental variables.

As far as the quantitative data are concerned, the most important information was given by the mean square error scores. These scores were used as the primary criterion of the goodness of the tracking performed. The time-on-target scores indicate the number of seconds during each trial that the operator was able to contain his error within certain predetermined tolerance bands. The hit scores counted the number of times he passed in and out of these tolerance bands. These two scores could also be used as a criterion of the goodness of the tracking performance, but do not lend themselves very readily to a mathematical analysis.
III MATHEMATICAL ANALYSIS

A. General Case

The electronic pursuit apparatus when arranged for either a following or a compensatory task forms a closed-loop control system as shown in Figure 1. In attempting to analyze these systems we find that the operator is the only unknown element in the system. If the transfer function of the operator were known exactly, an analysis of these systems would be a straightforward process. Unfortunately the human transfer function is not known, so the problem we posed was to deduce the human transfer function (or a decent approximation of it) from a priori reasoning. The validity of this deduced transfer function could then be ascertained by comparing the results of a theoretical analysis based on this deduction with the actual experimental data. The situations presented by the two tracking tasks under consideration are representative of the most common situations in which the human operator is likely to perform a tracking task; hence a knowledge of the characteristics of the operator in both of these cases would be extremely useful in engineering design.

In developing a model of the human operator in a
tracking task we certainly desire to be guided as much as possible by the psycho-physiological relations which have been learned about the human operators in situations similar to those under investigation. Hick and Bates have prepared reports which briefly describe some of the psycho-physiological processes that may be of importance in an analysis of the human operator in tracking tasks. Unfortunately, however, as we investigate the literature available on this subject, we find that there is very little information available in a form which could be utilized in making an engineering analysis.

For example, since a visual display is used in the tracking tasks under consideration, the perceptual ability of the operator must play an important part in his behavior. Unfortunately, the available data relating to the matter of human perception of length and velocity is based upon detached experiments which have only an indirect relationship to a tracking task in which there are continuous changes in displacements and velocities which are continuously under observation.

We have proceeded on the assumption that the visual information presented to the operator is processed in some manner and results in a motor response which is manifested by the muscular movement required to manipulate
the arm control. The molecular pattern of the eye-brain-motor response including the internal feedback loops, such as kinaesthetic feedback, is so complex and so little understood at the present, especially for continuous tasks such as we are analyzing here, that we cannot hope to incorporate all of its details into our model of the human. All we can hope to do is to obtain a model which simulates at a molar level the overall human behavior in the tracking task. Then as more specific knowledge is obtained about the detailed functioning of the human operator it can be utilized to modify our original model.

Another aspect of human behavior which must be incorporated into our model is the effect of the human lag time (reaction time). This lag time is the interval which elapses between the instant the information reaches the visual input of the operator and the instant at which he first begins a motor response to it. This lag time would appear to consist mainly of analysis time; that is, the operator utilizes this interval to determine what the nature of his response should be.

One other point which we shall consider briefly is the manner in which the operator performs; does he perform as a continuous system, as an intermittent system, or does
he perhaps alternate from one mode of performance to another. Judging from the inherent intermittency with which nerve impulses are transmitted through the nervous system, we might assume that the operator performs as an intermittent system. However, examination of graphic recordings of the operator's response in a typical tracking task as shown in Figure 2 sheds no real light on the answer to this question. It may be that the nervous system is capable of multiple sequence operations, in which case we could consider the operator to perform essentially as a continuous system. For simplicity in the present preliminary analysis we shall use a continuous model to represent the human operator.

In the tracking situation we are attempting to analyze, the human operators were placed in sound-shielded booths which contained only the oscilloscope upon which the display was presented and the manual arm control. The only direct information upon which the operator could base his responses was that which he could obtain from the oscilloscope display. We have assumed that there exists some relation between the oscilloscope display and the resulting action of the operator. Furthermore we have assumed that this relation has some degree of permanence and can be expressed mathematically. Finally, we have based our
Figure 2. Typical records of the operator's response and error (not to the same scale) in a compensatory task, for $K_m=1$, $K_d=1$. 
analysis upon the assumption that all the operators used in these studies were well trained and had reached a relatively stable level of performance.

With our general assumptions in mind we are now ready for an analysis of the compensatory and the following tasks. We shall consider the two cases separately, commencing with the compensatory task.

B. The Compensatory Task

As we have already mentioned, the only direct information concerning the tracking task available to the operator is that which is displayed upon the face of the oscilloscope. In order to construct a suitable model for the operator's behavior in this case we need to know what information he uses; that is, what aspects of the oscilloscope display the operator uses as an input. It would seem logical that the operator's response in the compensatory task is based upon the size of the displacement $D$ he observes on the screen as well as the time rates of change of this displacement. Expressing this mathematically,

$$\text{Human response} = F[D(t), \frac{dD(t)}{dt}, \frac{d^2}{dt} D(t)\ldots], \quad (3)$$
where $D$ is measured in inches and $t$ in seconds. We do not mean to imply in equation 3 that the operator consciously uses the first and second derivatives of the displacement to help determine his response. However, he is undoubtedly capable of discriminating incremental changes of the displacement with respect to time and possibly (although unlikely in many cases) of discriminating incremental changes in the velocity of the displacement. Most likely it is these incremental changes which influence the operator's response, but since we are using a continuous model, we express these incremental changes as the derivatives of the displacement.

It may be true that the operator utilizes other quantities as sources of direct information. For example, it has been suggested that he may attempt to split the displacement $D$ into its two components $S_p(t)$ and $S_H(t)$. From his own experience and from conversations with the subjects the author has learned that the displacement and its time rate of change are consistently used as major sources of direct information. In constructing our model of the human operator we want to keep it as exact, yet as simple, as possible. Incorporating these other possible information sources in equation 3, while greatly increasing
the complexity of the analysis, would not materially improve the accuracy of our model of the human operator.

The response of the operator is manifested as an angular velocity of the arm control. Thus, equation 3 can be rewritten as

\[
\frac{dS_H}{dt} = F[D(t), \frac{dD(t)}{dt}, \frac{d^2D(t)}{dt}, \ldots],
\]

where \( S_H \) is the angular displacement, expressed in radians, of the arm control about a reference line. As suggested above, it is assumed that the human operator probably bases his response only upon the displacement and its first time derivative, being relatively insensible to any higher time derivatives of the target motion. Thus, equation 4 becomes, as a first order, linear approximation,

\[
\frac{dS_H}{dt} = H_1D(t) + H_2\frac{dD(t)}{dt}
\]

The quantities \( H_1 \) and \( H_2 \), measured in radians per inch-seconds and radians per inch, respectively, are weighting functions which are characteristic of the human operator. In our model they are used in conjunction with two additional quantities, which will be discussed below, to
describe the operator's behavior in a tracking task.

Equation 5 requires further modifications to make it more analogous to the human response. The nature of the delay time of the human operator was briefly discussed in an earlier paragraph. To incorporate this factor into our expression for the human operator, equation 5 is rewritten as a mixed difference-differential equation. We shall assume in our analysis that the delay time $L$, expressed in seconds, of a skilled operator is constant. Equation 5 thus becomes

$$\frac{dS_H(t)}{dt} = H_1 D(t-L) + H_2 \frac{dD(t-L)}{dt}. \quad (5A)$$

Equation 5(A) expresses mathematically the fact that the operator's response at a given instant of time is a function of the displacement and its time rate of change which he observed at an instant of time $L$ seconds earlier.

Finally there is one other item to be included in our model. The response of the human in these tracking situations is not always a clearly defined function but appears to have a random component. This fact can be seen by examining the operator's error in tracking a simple sine-wave as illustrated in Figure 2. Furthermore, there are
unquestionably random variations in the operator's perception of errors as presented on the face of the oscilloscope and in his responses to them. To account for this stochastic portion of the human response, a random function of time $N(t)$ is added to the expression for the operator's response. It appears that we can safely assume as a first approximation that this random term, which we shall call the "human noise," is a purely random function of time. We further assume that no cross-correlation exists between $N(t)$ and the problem signal presented to the operator.

Incorporating the human delay time and the human noise into our model of the operator we obtain

$$\frac{dS_H(t)}{dt} + N(t) = H_1D(t-L)+H_2\frac{dD(t-L)}{dt} \quad (6)$$

Before proceeding with the analysis, there is one other detail which will be examined briefly. This is the manner in which the four quantities $H_1$, $H_2$, $N(t)$ and $L$ enter into a discussion of the operator's behavior in a tracking task. The delay time $L$ probably remains fairly constant (we shall consider that it does so in the remainder of our analysis), decreasing to some limiting
value as the operator approaches a highly trained condition. However, there may be daily and secular changes in the value of L. The effect of these changes, which are very likely random, can be incorporated into the N(t) term. The noise term N(t) is a function over which the operator has little or no control. The weighting factors $H_1$ and $H_2$, which shall henceforth be designated as the human constants, are the two quantities which, according to our assumptions, the operator has at his disposal to control the quality of his tracking performance under different experimental conditions. Thus one method of classifying the operator's behavior in terms of the model we have developed would be by a subset of points in the three dimensional $H_1 H_2 N$ space as suggested by Figure 3. Each point in the $H_1 H_2 N$ space would represent the state of the operator for a given set of operating conditions and inputs. Effectively, what is being suggested is that the human operator is not a linear system over the entire range of situations with which he might be confronted, but rather an attempt is being made to describe his behavior in different situations by breaking it into a series of subclasses in each of which he can be considered to behave effectively as a linear system.

Based upon the above assumptions and deductions, the
Figure 3. A plot of $H_1H_2N$ space showing one way of classifying the operator's tracking behavior for different tracking conditions.
expression for the human transfer function in the tracking task becomes, in operation form,

\[ KG(s) = \frac{R}{s + H_2} e^{-Ls}. \]  

(7)

where \( s = \sigma + j\omega \) is the complex, generalized angular frequency of Laplace transform analysis. However, in addition to the above transfer function, it has been shown necessary to consider that the operator has a noise generator which behaves essentially as a second input to the overall system. Thus, a block diagram for the compensatory tracking situation incorporating these assumptions about the human operator will appear as shown in Figure 4.

The error signal of the tracking task is defined to be:

\[ E(t) = S_p(t) - S_c(t), \]  

(8)

the difference between the problem signal and the output signal of the manual control. Also, as we presented earlier the equations defining the constants of the pursuit apparatus are:
Figure 4. The block diagram of the compensatory tracking situation incorporating the assumed human transfer function.
Using these three equations in conjunction with equation 6, we are then in a position to determine expressions for any of the functions in the system in which we may be interested. Since our method of evaluating the performance of the operator is based upon the error he makes while tracking the problem, the first step in our analysis is to solve equations 1, 2, 6 and 8 to obtain an expression for his error. Taking the Laplace transforms of these equations, combining them, and solving for \( E(s) \), the transform of the error signal, yields

\[
E(s) = \frac{s}{s + e^{-Ls}(A+Bs)} S_p(s) + \frac{6.45K_m}{s + e^{-Ls}(A+Bs)} N(s) \tag{9}
\]

where \( A = 2.83K_mK_dH_1 \) and \( B = 2.83K_mK_dH_2 \).

In equation 9 we designate

\[
Y_1(s) = \frac{s}{s + e^{-Ls}(A+Bs)} \tag{10}
\]

and

\[
D(t) = 0.44K_dE(t), \tag{1}
\]

\[
S_c(t) = 6.45K_mS_H(t). \tag{2}
\]
\[ Y_2(s) = \frac{6.45K_m}{s e^{-Ls}(A+Bs)} \]  

Equation 9 then can be written symbolically as

\[ E(s) = Y_1(s)S_P(s) + Y_2(s)N(s). \]  \(9A\)

Since \( N(t) \) is a stochastic or random function, the statistical properties of time-variable data must be used to continue the analysis. Furthermore, the criterion of goodness which is to be used in the analysis of the system is the mean square error \( \overline{E^2} \) which is defined to be

\[ \overline{E^2} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} E(t)^2 dt \]  \(12\)

Two other quantities which will be utilized in our analysis are the autocorrelation function and the spectral density of a time function. The autocorrelation function \( R(\tau) \) of a given function is defined by

\[ R(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(t)f(t+\tau) dt \]  \(13\)

The value of the autocorrelation function of the error \( R(\tau) \) at \( \tau = 0 \) equals the mean square error:
\[ E^2 = R(0), \]  

(14)

as can be seen from the definitions given in 12 and 13.

A third relation which will be of use in our analysis is the fact that the autocorrelation \( R(\tau) \) and the power density spectrum \( G_E(\omega) \), where the angular frequency \( \omega \) equals \( 2\pi f \), of the error function are uniquely related by their Fourier transforms; that is,

\[ G_E(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau) e^{-j\omega \tau} d\tau \]  

(15)

and conversely

\[ R(\tau) = \int_{-\infty}^{\infty} G_E(\omega) e^{j\omega \tau} d\omega \]  

(16)

From equations 14 and 16 we obtain the relation between the error spectral density and the mean-square error, namely:

\[ \overline{E^2} = \int_{-\infty}^{\infty} d\omega G_E(\omega). \]  

(17)

This relation will prove to be very useful in determining the mean-square error of the operator in these tracking tasks.
Now it can be proved\(^6\) that since in 9(A) \(Y_1(j\omega)\) and \(Y_2(j\omega)\) are the transfer functions of a linear time-invariant mechanism and \(G_P(\omega)\) and \(G_N(\omega)\) are the spectral densities of \(S_P(t)\) and \(N(t)\), respectively, the error spectral density is

\[
G_E(\omega) = |Y_1(j\omega)|^2G_P(\omega) + |Y_2(j\omega)|^2G_N(\omega) + Y_1^*(j\omega)G_{PN}(\omega)Y_2(j\omega) + Y_1(j\omega)G_{NP}(\omega)Y_2^*(j\omega)
\]  

(18)

where \(Y_1^*(j\omega)\) is the complex conjugate of \(Y_1(j\omega)\) and \(G_{NP}(\omega)\) is the cross-spectral density of \(S_P(t)\) and \(N(t)\). We shall assume that the noise term \(N(t)\) and the problem \(S_P(t)\) are uncorrelated; thus

\[
G_{NP}(\omega) = 0 = G_{PN}(\omega).
\]  

(19)

Equation 18 then simplifies to

\[
G_E(\omega) = G_{EP}(\omega) + G_{EN}(\omega),
\]  

(18A)

where \(G_{EP}(\omega) = |Y_1(j\omega)|^2G_P(\omega)\) and \(G_{EN}(\omega) = |Y_2(j\omega)|^2G_N(\omega)\).

Applying equation 17, the mean-square error for the compensatory system is seen to be
\[ E^2 = \int_{-\infty}^{\infty} d\omega G_{EP}(\omega) + \int_{-\infty}^{\infty} d\omega G_{EN}(\omega). \]

With the mathematical relations given above, an operational procedure for attacking the problem under consideration can be evolved. Thus, given a specific input \( S_p(t) \), its spectral density \( G_p(\omega) \) can be determined. Utilizing this information, in addition to the assumptions concerning the human transfer function, a theoretical expression for the mean-square error of the operator can be calculated. By comparing this expression with the experimental data obtained from the studies on the electronic pursuit apparatus, the validity of the theoretical expression can be determined.

Thanks to the construction of an autocorrelator, a further check can be made upon the validity of the assumed transfer function, for the mean-square error \( E^2 \), the autocorrelation function of the error \( R(\tau) \) and the error spectral density \( G_E(\omega) \) are all closely related, as we have already seen. Having utilized equation 18(A) to obtain the spectral density of the operator's error, we can then go one step further and use equation 16 to determine a theoretical expression for the autocorrelation function of the error. This provides a further check on the validity of our model for we can compare the
theoretical autocorrelation functions with those actually obtained from the autocorrelation of experimental data. Hence, if our assumptions about the human transfer function are valid, they will also explain the characteristics of the autocorrelation functions of the error in a satisfactory manner.

Before undertaking any extensive analysis of the compensatory system, one vital piece of information is needed: namely, the boundary of the region of stable operation of the overall system. According to the model which has been developed, this can be obtained from the characteristic equation of the system

\[ s + e^{-Ls}(A+Bs) = 0, \]  
\[ s = a+j\omega. \]
Substituting 22 into equation 21 and equating the real and imaginary parts to zero yields

\[ a + e^{-aL}[(Ba + A)\cos \omega L + B\omega \sin \omega L] = 0 \]  \hspace{1cm} (23)

and

\[ \omega + e^{-aL}[(B\omega \cos \omega L) - (Ba + A)\sin \omega L] = 0. \]  \hspace{1cm} (24)

These equations are linear in A and B, and solving for these quantities, we obtain:

\[ A = \frac{(a^2 + \omega^2)\sin \omega L}{\omega e^{-aL}} \]  \hspace{1cm} (25)

and

\[ B = \frac{\omega \cos \omega L - a \sin \omega L}{\omega e^{-aL}} \]  \hspace{1cm} (26)

Now the boundary of the region of stability is given by the condition that \( a = 0 \). For this condition, we find

\[ A = \omega \sin \omega L \]  \hspace{1cm} (27)

and
It is convenient to work in terms of non-dimensional variables; consequently letting \( u = \omega L \) and rewriting 27 and 28 we obtain

\[ AL = u \sin u \]  
\[ (27A) \]

and

\[ B = \cos u. \]  
\[ (28A) \]

These two relations determine the region of stability plotted in Figure 5. This plot gives the limitations which must exist on \( AL \) and \( B \) in order that the system remain stable. The use of Figure 5 can be illustrated by means of an example. Let us assume that \( B \) has the value 0.45, then we want to determine the limitations on \( AL \) in order that the system remain stable. Locating this value of \( B \) on the \( B \)-axis in Figure 5, we construct a vertical line which intersects the curve defining the boundary of the stability region. This point of intersection determines the upper limit on \( AL \). Thus if \( B = 0.45 \), the value of \( A \) must be less than 1.80 in order
Figure 5. A plot of the stability region for the compensatory tracking situation.
that the system remain stable.

This stability information is very useful, but it is not enough. To continue the analysis the roots of the characteristic equation must be found. The best method of obtaining these roots is through the application of the root locus method of W. R. Evans. Since \( S = 0 \) is not a solution of \( 21 \), we can rewrite \( 21 \) to obtain

\[
\frac{Be^{-Z(Z+AL/B)}}{A} = 1
\]

(29)

where \( Z = \text{Ls} \). Designating the real and imaginary components of \( Z \) as \( Z = v+jw \), we have \( u = Lw \) and \( v = La \). Since equation 29 is complex, it can be broken into two equations by equating the magnitude and the phase of both members of 29. Thus

\[
B \left| \frac{e^{-Z(Z+AL/B)}}{Z} \right| = 1
\]

(29A)

and

\[
\text{Ang}(e^{-Z}) + \text{Ang}(Z+AL/B) - \text{Ang}(Z) = \pm 180^\circ,
\]

(29B)

where \( \text{Ang}(Z) \) represents the argument of the complex
function $Z$. For different ratios of $AL/B$, equation 29 defines a series of curves which are the loci of the roots of 29. Figure 6 illustrates the nature of these loci for several prescribed ratios of $AL/B$. Each point on a given locus is determined by one particular value of $B$, and this value is given by the expression

$$B = \left| \frac{Z}{e^{-Z}(Z+AL/B)} \right|.$$  \hspace{1cm} (29C)

This value of $B$ can either be calculated or determined graphically. In Figure 6 only the loci in the third quadrant have been plotted since we are only interested in the roots corresponding to stable operation. It might also be noted that these loci are symmetrical with respect to the real axis. In Figure 6 the two groups of curves represent the loci of the first two roots of the characteristic equation given in equation 21. Inasmuch as this is transcendental, it has an infinite number of roots, so that we need an infinite number of groups of curves in Figure 6 to define the loci of all its roots. Usually, only the first few roots of the characteristic equation are of interest, so that Figure 6 is sufficient for our purposes.
Figure 6. The loci of several of the roots of the characteristic equation of the compensatory tracking task.
To illustrate the use of Figure 6 let us consider a specific example. Let us assume that $AL = 0.8$ and $B = 0.2$ in equation 21 and we want to determine the first few roots for this particular case. Since $AL/B = 4$, the roots will lie on the loci marked $AL/B = 4$ in Figure 6. Then by a trial and error procedure we select several points on these loci and calculate the value of $B$ at that particular point by the use of equation 29(C). In this manner we locate the points corresponding to $B = 0.2$. Then from Figure 6 we read the values of the first three roots; namely $LS_1 = -3.45$, $LS_2 = -0.9 + j1.2$ and $LS_3 = -1.6 + j9$.

Before discussing the root locus plot any further it would be well to consider the significance of the quantity $AL/B$. Since $A = 2.83K_mK_dH_1$ and $B = 2.83K_mK_dH_2$, the ratio $AL/B$ is, in fact, the ratio $H_1/L/H_2$, so that each particular locus is determined by the value of the human constants and the human delay time. Since each point on a given locus is determined by a particular value of $B$, the control scale factors $K_m$ and $K_d$ help determine the point on the locus corresponding to the state of the operator. This means that any change in $K_m$ and $K_d$ may cause a shift in the point on the locus describing the state of the system.
Furthermore, it should be noted that on a particular root locus (that is, for a particular value of $AL/B$) as the value of $B$ is increased, as would happen when $K_m$ and $K_d$ are increased, we move along the locus toward the imaginary axis. For a particular ratio of $AL/B$, there is an upper limit to the value which $B$ (and hence $AL$) can assume, yet allowing the system to remain stable. These values can be calculated from Figure 6 but can be more easily seen by referring to Figure 5 where the range of $B$ is indicated for several ratios of $AL/B$. Figure 5 indicates that as the ratio $AL/B$ is increased, the maximum value which $B$ can assume is decreased. Figure 6 will thus provide a useful service in indicating what effect any change in $A$, $B$, or $L$ should have upon the performance characteristics of the operator.

The evaluation of the roots of the characteristic equation 21 provides essential information for determining the two functions $R(\tau)$ and $E^2$. However, before pursuing the analysis any further, let us pause and consider the earlier statements concerning the nature of the error. Referring to equations 18(A) and 20, it is seen that the result of our approach has been to split the operator's mean-square error into two parts; one part, which will be
designated $\overline{E_P^2}$, is the steady-state, mean-square error associated with the input $S_P$, and the other $\overline{E_N^2}$ is associated with the operator's noise. From the assumptions made earlier about the human noise its effect should be the same for all input functions. Therefore, let us consider it first.

1. **Human noise.** It has been assumed that the human noise is purely random; that is, that its spectral density is flat ("white noise"). The spectral density of the human noise is then given by

$$G_N(\omega) = N,$$  \hspace{1cm} (30)

where $N$ is a constant independent of the angular frequency $\omega$. The first function which shall be considered is the autocorrelation function $R_N(\tau)$ of the "noise-error" term $E_N(t)$. This is obtained from

$$R_N(\tau) = \int_{-\infty}^{\infty} Y_2(j\omega)|^2 N e^{j\omega \tau} d\omega.$$ \hspace{1cm} (31)

For convenience in calculating, a change of variable is made in equation 31 by replacing $j\omega$ by the complex
variable \( s = \sigma + j \omega \),

\[
R_N(\tau) = \frac{1}{j} \int_{-\infty}^{\infty} |Y_2(s)|^2 N e^{\tau \sigma} ds. \tag{31A}
\]

Equation 31(A) is the contour integral equivalent of equation 31, with the integration being along the axis of imaginaries of the s-plane. It might also be noted that inasmuch as only stable modes of operation will be considered \( |Y_2(s)|^2 N \) will never have any singularities on the imaginary axis. Although 31(A) is not a closed contour, the path of integration may be artificially closed by a large semicircle, first in the left-half s-plane for positive values of \( \tau \) and then in the right-half s-plane for negative values of \( \tau \). However, since \( R_N(\tau) \) is an even function of \( \tau \), only \( \tau > 0 \) need be considered. The evaluation of 31(A) is then given by

\[
R_N(\tau) = \sum_n K_n \tag{31B}
\]

where the \( K_n \) are \( 2\pi \) times the residues of \( |Y_2(s)|^2 N e^{\tau \sigma} \) at its singularities inside the semicircle. Thus, to evaluate 31(A) we must know the singularities of the integrand. From equation 11 it is seen that \( |Y_2(s)|^2 \)
is given as

\[ |Y_2(s)|^2 = \frac{41.6K_m^2}{[s+e^{-Ls(A+Bs)}][-s+e^{-Ls(A-Bs)}]} \]  \hspace{1cm} (32)

Since we are considering only stable operation, the values of A, L and B must be such that they fall within the stability region of Figure 5. The roots of

\[ s+e^{-Ls(A+Bs)} = 0 \]  \hspace{1cm} (21)

will then be located in the left hand side of the s-plane and will be located symmetrically about the real axis. These roots can be obtained from the root locus plot of Figure 6 (remembering that \( s = \frac{1}{L}(\nu+j\omega) \)) and are symmetrically located about the imaginary axis with respect to the roots of

\[ -s+e^{Ls(A-Bs)} = 0 \]  \hspace{1cm} (33)

which are all located in the right hand s-plane for a stable system. Consequently, once we have determined the roots of equation 21, we have in effect located all the poles of \( |Y_2(s)|^2 \), and we can evaluate \( 31(A) \) by means of
residue theory. Furthermore the roots occur as complex conjugates in the left-half $s$-plane so that the residues need only be evaluated at the singularities occurring in the third quarter of the $s$-plane; that is, the portion of the plane represented in Figure 6. Although the integrand of equation 31 has an infinite number of poles inside the path of integration chosen, only the first two poles will be used to approximate $R_N(t)$. Designating the values of $s$ at which the first two poles occur as $s_1 = -a_1 + j\omega_1$ and $s_2 = -a_2 + j\omega_2$ with $(\omega_1 < \omega_2)$, the approximation for $R_N(t)$ can be symbolized as

$$R_N(t) \cong K_1(-a_1 + j\omega_1) e^{-(a_1 + j\omega_1)t} + K_2(-a_2 + j\omega_2) e^{-(a_2 + j\omega_2)t}$$

$$+ K_1^* e^{-(a_1 - j\omega_1)t} + K_2^* e^{-(a_2 - j\omega_2)t}$$

(34)

where $K_1^*$ and $K_2^*$ are the complex conjugates of $K_1$ and $K_2$ respectively. Equation 34 can be rewritten to yield

$$R_N(t) \cong 2K_1 e^{-a_1 t} \cos (\omega_1 t + A_1) + 2K_2 e^{-a_2 t} \cos (\omega_2 t + A_2)$$

where $A_1 = \text{argument of } K_1(-a_1 + j\omega_1)$ and

$$A_2 = \text{argument of } K_2(-a_2 + j\omega_2).$$

(34A)

The expression obtained for $R_N(t)$ will be a sum of
exponentially damped cosine terms. Occasionally reference will be made to the frequencies appearing in the autocorrelation function; the frequencies implied are the frequencies of the cosine factors of the exponentially damped cosine terms. Figure 6 will be useful in providing the magnitude of the damping factor of these terms and their frequencies but it does not provide any information concerning the relative magnitude of the $K_i$. This information must be obtained by actually calculating the residues of the roots in question.

Up to this point attention has been directed to the autocorrelation function of the noise $R_N(\tau)$ but nothing has been said about the mean-square value $E_N^2$ of the noise. It is true that it can be approximated by setting $\tau = 0$ in equation 34(A), but this does not show the effects of variations in $A$, $B$ and $L$ upon $E_N^2$. The mean-square noise error can be obtained from

$$E_N^2 = \int_{-\infty}^{\infty} G_E(\omega) d\omega = \int_{-\infty}^{\infty} |Y_2(j\omega)|^2 N d\omega.$$  \hspace{2cm} (35)

Unfortunately, the evaluation of $E_N^2$ is not a simple task. As we have already discovered, the denominator of $|Y_2(j\omega)|^2$ is a transcendental expression having an infinite number of zeros so that, conversely, the integrand
of $E_N^2$ has an infinite number of poles. Thus, an evaluation of $E_N^2$ in closed form is impossible. If specific values of $A$, $B$ and $L$ are known then the same approach as was used to obtain the autocorrelation function $R_N(\tau)$ could be used to evaluate $E_N^2$. However, as we have stated above, since we desire a general expression for the mean-square error in terms of $A$, $B$ and $L$, that method is not practicable. The only other alternative is to approximate the exponential term $e^{-j\omega L}$ in the integrand. An approximation which can be used is

$$e^{-j\omega L} = \frac{1}{1+j\omega L/n}^n \text{ for } n = 1, 2, 3 \ldots . \quad (36)$$

The next question which naturally arises is what is the smallest value of $n$ which yields a satisfactory approximation. One guide that can be used is to plot the root locus of the characteristic equation 21 in which the exponential term has been approximated by the relation in equation 36. The root loci obtained in this manner are plotted in Figure 7 for $n = 1, 2, 3$ and 4 for the particular case $AL/B = 2$. The exact locus for this case has also been plotted in Figure 7. It is seen that $n = 3$ is the smallest value of $n$ that could possibly be used.
Figure 7. Comparison of root loci for the exact and the approximate characteristic equations of the compensatory tracking situation for the case $AL/B=2$. 

**Exact value**

- $n=1$
- $n=2$
- $n=3$
- $n=4$
A larger value of \( n \) would be preferable but the labor involved in evaluating the integral becomes prohibitive. Substituting equation 36 with \( n = 3 \) into the expression for \( Y_2(j\omega) \), we obtain

\[
Y_2(j\omega) = \frac{6.45K_m}{j\omega+(A+Bj\omega)/(1+j\omega L/3)} = \frac{6.45K_m(1+j\omega L/3)^3}{j\omega(1+j\omega L/3)^3+A+Bj\omega} = \frac{g_1(j\omega)}{h_1(j\omega)}. \tag{37}
\]

Now

\[
|Y_2(j\omega)|^2 = Y_2(j\omega)Y_2^*(j\omega) = Y_2(j\omega)Y_2(-j\omega)
\]

\[
= 41.6K_m^2 \frac{g_1(j\omega)g_1(-j\omega)}{h_1(j\omega)h_1(-j\omega)} = 41.6K_m^2 \frac{g(j\omega)}{h_1(j\omega)h_1(-j\omega)},
\]

where

\[
g(j\omega) = g_1(j\omega)g_1(-j\omega) = (1+j\omega L/3)^3(1-j\omega L/3)^3
\]

\[
= \left(\frac{\omega L}{3}\right)^6 + 3\left(\frac{\omega L}{3}\right)^4 + 3\left(\frac{\omega L}{3}\right)^2 + 1
\]

and
Making a change of variable through $x = \frac{\omega L}{3}$, we obtain

$$
\begin{align*}
\frac{E_N^2}{A} &= \frac{L}{3} \int_{-\infty}^{\infty} \frac{41.6K_m^2 \text{Ng}(x)dx}{h_1(x)h_1(-x)} \\
\end{align*}
$$

where now

$$
g(x) = x^6 + 3x^4 + 3x^2 + 1
$$

and

$$
h_1(x) = x^4 - 3jx^3 - 3x^2 + j(B+1)x + AL/3
$$

Equation 35(A) is now in a form so that it can be evaluated by use of the appendix of reference (6) which contains a table of integrals prepared by utilizing the theory of equations and residue theory to evaluate integrals of this nature. This yields

$$
\begin{align*}
\frac{E_N^2}{A} &= \frac{43.6K_m^2N}{A} \left( \frac{(AL)^2 - 6ABL - 15AL + 3B - 24}{B^2 - 7B - 8 + 3AL} \right) \\
\end{align*}
$$

Replacing $A$ and $B$ by the quantities given in equation 9
yields

\[
E_N^2 = \frac{15.4K_N}{H_1d} \times \frac{(K_mK_HL)^2 - 6K_m^2K_d^2H_2L - 531K_mK_HL + 1.06K_mK_H - 3}{(K_mK_dL)^2 - 2.4K_mK_dH_2 + 1.06K_mK_dL - 1}
\] (38A)

Equation 38(A), while it certainly is not a simple expression to use for the evaluation of $E_N^2$, will still prove useful later in making quantitative evaluations of the effects of changes in $K_m$ and $K_d$ upon the contributions of the noise to the total mean-square error. As a matter of fact, we see at once that an increase in $K_m$ causes at least a proportional increase in $E_N^2$ while an increase in $K_d$ causes a decrease in $E_N^2$.

At this stage of the analysis nothing is known about the possible values which $A$, $B$ and $L$ will assume, since the values of the human constants are not known. This means that for the present only the qualitative effects of these three quantities can be examined. There are three conditions which would seem to merit examination:

(i) a decrease in the magnitude of $L$,

(ii) a uniform increase in the magnitudes of
A and B with L fixed (AL/B remaining constant), and

(iii) an increase in B occurring at a faster rate than an increase in A, for a fixed value of L (AL/B decreasing).

Since we have already assumed that the operator has reached a high level of skill and operates with a fixed lag, we should point out that our interest in the effect of a decrease in L stems from the fact that it appears that this is one of the multiple changes occurring during learning. If A and B remain fixed as L decreases, the effects as can be seen from examining Figure 6 are two-fold. First, the frequency of the autocorrelation function increases and secondly, the magnitude of the damping factor is increased. Furthermore, we note from equation 38 that the value of $E_N^2$ also decreases. To illustrate this effect consider a typical example in which $B = 0.2$, $A = 3.2$ seconds$^{-1}$ and L assumes the values 1/2, 1/3 and 1/4 seconds. Locating the points corresponding to these values on Figure 6, reveals that the first roots $s_1$ for each of these values are:

(i) for $AL/B = 8$, $L = 1/2$ and $B = 0.2$,

$$s_1 = -0.5 + j3.5,$$
(ii) for $AL/B = 5.3$, $L = 1/3$ and $B = 0.2$,
\[ s_1 = -1.6 + j4.5, \]

(iii) for $AL/B = 4$, $L = 1/4$ and $B = 0.2$,
\[ s_1 = -3.6 + j4.8. \]

Figure 8 is a plot of $G_{EN}(\omega)/41.6K_m^2N$ versus $\omega$ corresponding to this situation, illustrating graphically the points we have discussed; namely, the predominant frequency term in the noise-error spectral density function decreases as $L$ decreases, as does $\overline{E_N^2}$, which is proportional to the area under the noise-error spectral density curve. Thus a decrease in the human delay would cause an improvement in the tracking performance.

The second situation to be considered is the effect of a uniform increase in $A$ and $B$ such that $AL/B$ remains constant. Again two effects are seen from Figure 6; namely, the damping factor decreases and the frequency increases (at least the latter statement is true for $AL/B = 2$). Figure 9 shows a plot of $G_{EN}(\omega)/41.6K_m^2N$ versus $\omega$ for the case $AL/B = 8$, $L = 0.25$ seconds, $B = 0.05, 0.10$ and $0.15$. Under these conditions a pronounced frequency term is seen to appear in the spectral density as $B$ increases. If it is assumed that $K_m$ is held constant, we note an interesting fact about the behavior of
\[
\frac{G_{EN}(\omega)}{41.6K_m^2N}
\]

Figure 8. The effect upon \(G_{EN}(\omega)\) of a decrease in \(L\), plotted for the case \(A=3.2\) seconds\(^{-1}\) and \(B=0.2\).
Figure 9. The effect upon $G_{EN}(ω)$ of a uniform increase in $B$ for the case $AL/B=8$.

Figure 10. The effect upon $G_{EN}(ω)$ of an increase in $AL/B$, for the case $L=0.25$ seconds.
\[ E_N^2 \]; as \( B \) is increased, the magnitude of \( E_N^2 \) decreases at first, reaching a minimum, and then it begins to increase very rapidly until finally it becomes infinite when \( B \) reaches the value at which instability occurs.

Finally, let us consider the results of the situation in which \( B \) increases at a faster rate than \( A \) so that \( AL/B \) decreases. Figure 6 shows that the two effects are a decrease in the damping factor and a larger shift in frequency than occurred in the second situation. Also it is seen that this change results in a decrease in \( E^2 \) provided \( K_m \) remains constant. Figure 10 is a graphical illustration of these effects for the particular cases \( AL/B = 8, 4 \) and \( 2 \) for \( B = 0.1, 0.2 \) and \( 0.4 \), respectively, and \( L = 0.25 \) seconds.

The noise term and its contribution to the mean-square error score have been briefly considered above. However, the importance of the contribution of the noise term will be found to be involved with two other factors: (i) the nature of the input which is used, and (ii) the value of the manual scale factor \( K_m \) employed. For inputs which are periodic and not too complex, the noise term plays an important role in the error score and quite often outweighs the steady state error. Increasing the magnitude of \( K_m \) tends to magnify the effects of \( N(t) \).
Hence for settings of $K_m$ of 8 or 16, the noise term undoubtedly must be considered in any analysis that is undertaken.

The general analysis of the compensatory system and the noise term has been carried about as far as is practicable. Our attention will now be turned to a consideration of the operator's response to two specific inputs: (i) a simple harmonic tracking task (sine-wave input), and (ii) a random tracking task. The analysis of the sine-wave input will be undertaken first.

2. The simple harmonic tracking task. The quantity which interests us most in these tracking studies is the mean-square error. This means that we have to determine

$$E^2 = E_p^2 + E_n^2 = \int_{-\infty}^{\infty} |Y_1(j\omega)|^2 G_p(\omega) d\omega + \int_{-\infty}^{\infty} |Y_2(j\omega)|^2 N d\omega.$$  

In the preceding section we have already obtained an expression for $E_n^2$, so that all that remains is to evaluate $E_p^2$ for the case of a sinusoidal input. The input used in these studies was the sine-wave

$$S_p(t) = M \sin \omega_0 t$$  

(39)
where \( M \) is expressed in volts, \( \omega_0 \) is in radians per second, and \( t \) is in seconds. The spectral density of this input is

\[
G_p(\omega) = \frac{M^2}{4}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)].
\]

Substituting \( j\omega \) for \( s \) in equation 10, we obtain

\[
Y_1(j\omega) = \frac{j\omega}{j\omega+e^{-j\omega L(A+Bj\omega)}}.
\]

Accordingly,

\[
\frac{\overline{E_p^2}}{E_p^2} = \int_{-\infty}^{\infty} \frac{(M^2/4)\omega^2[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]}{[j\omega+e^{-j\omega L(A+Bj\omega)}][-j\omega+e^{j\omega L(A-Bj\omega)}]} \, d\omega,
\]

and this integral can be evaluated exactly, yielding

\[
\overline{E_p^2} = \frac{M^2\omega_0^2/2}{(B^2+1)\omega_0^2+A^2+2B\omega_0^2 \cos \omega_0L-2A\omega_0 \sin \omega_0L}.
\]

For the particular case which was investigated experimentally \( M \) and \( \omega_0 \) had the values 4.5 volts and \( \pi \) radians per second respectively. Substituting these values into equation 42 along with the relations for \( A \) and \( B \) given in equation 9 we obtain
\[
E_p^2 = \frac{10.1}{8(K_m K_d H_1)^2 - 1.8K_m K_d H_1 \sin \pi L + 5.66K_m K_d H_2 \cos \pi L + 8(K_m K_d H_2)^2 + 1}. \tag{42B}
\]

Combining equations 38(A) and 42(A) yields an expression for the mean-square error which would be made by the operator in tracking a sine-wave in a compensatory task. This expression for \( E^2 \), although complex and fairly unwieldy to handle, will enable us to hypothesize about the effects of the various parameters upon the mean-square error.

A special case which will prove to be of interest is that case in which the lag time is zero; i.e., \( L = 0 \). The expression for the mean-square error then simplifies to

\[
E^2 = \frac{M \omega_o^2 / 2}{8(K_m K_d H_1)^2 + \omega_o^2(2.83K_m K_d H_2 + 1)^2} + \frac{15.4K_m N}{K_d H_1(2.83K_m K_d H_2 + 1)}. \tag{42C}
\]

The autocorrelation function of the error can also be obtained in closed form for the case \( L = 0 \); it is,
Equation 43 implies, therefore, that if the operator is tracking the sine-wave with zero lag, the autocorrelation function of his error will be a cosine term whose frequency is that of the problem plus an exponential term whose damping is dependent upon the scale factors and upon $H_1$ and $H_2$.

This is as far as we intend to carry our analysis of the compensatory task with a sinusoidal input for the present. However, the expressions derived above will be utilized when the experimental data for the sine-wave input are analyzed in a later section.

3. The random tracking task. The sine-wave was chosen as the original type of input to be used in these tracking studies because of its simplicity. Among other things it was easy to analyze and easy to generate. However, sine-wave inputs leave much to be desired. They are simple periodic functions which are quite easily learned.
and once the operator learns them he ceases to act strictly as a follower and becomes a predictor. Thus, it was decided that more information could be obtained about the nature of the human operator if he were forced to track an input which was random in nature and whose future could not be so easily predicted.

A random function generator was consequently constructed under the supervision of Professor C. E. Warren of the Department of Electrical Engineering of the Ohio State University. Its purpose was to provide a random input with a restricted bandwidth suitable for use in a tracking study. Basically, this random generator utilizes the output of a 6D4 gas tube, which is a Gaussian random process with essentially a flat spectrum ("white noise"). The output of the gas tube is then subjected to a heterodyning process and to filtering to produce an output whose amplitude distribution is Gaussian and whose spectral density has a peak at 30 cycles per minute and half-power points at roughly 15 and 45 cycles per minute.

Figure 11 is a Rahm recording made of a typical sample of the output of the random function generator. This output was also autocorrelated and the normalized autocorrelation function shown in Figure 12 was obtained.
Figure 11. A Rahm recording of a typical output of the random function generator.
Figure 12. The autocorrelation function of a typical sample of the output of the random function generator.
This autocorrelation function can be approximated for all practical purposes by the expression

\[ R_p(\tau) = Ke^{-d\tau} \cos rt. \quad (44) \]

The spectral density of the output of the random function generator based upon the approximation given in equation 44 is found to be

\[ G_p(\omega) = \frac{Kd}{2\pi} \left[ \frac{1}{d^2 + (\omega + r)^2} + \frac{1}{d^2 + (\omega - r)^2} \right], \quad (45) \]

or combining fractions,

\[ G_p(\omega) = \frac{Kd}{\pi} \frac{d^2 + r^2 + \omega^2}{(d + jr + j\omega)(d - jr - j\omega)(d - jr + j\omega)(d + jr - j\omega)}. \quad (45A) \]

The values of the parameters in equation 44 were found by measurement to be \( d = 0.9 \) seconds\(^{-1} \), \( r = 2.9 \) radians per second, and \( K = 5.79 \) volts\(^2 \). Substituting these values into equation 45(A) yields

\[ G_p(\omega) = \frac{1.65(9.2 + \omega^2)}{(0.9 + 2.9j + j\omega)(0.9 - 2.9j - j\omega)(0.9 - 2.9j + j\omega)} \times \frac{1}{(0.9 + 2.9j - j\omega)} \quad (45B) \]
Our problem now is to derive an expression for the steady-state error $E_p^2$ arising from the random input. Utilizing the expression for the spectral density of the input given in equation 45(B), we have that the steady-state mean-square error is given by the integral

$$E_p^2 = \int_{-\infty}^{\infty} \frac{1.65\omega^2(9.2+\omega^2)}{\left[ j\omega+e^{-j\omega L(A+Bj\omega)}\right]\left[ -j\omega+e^{j\omega L(A-Bj\omega)}\right](0.9+2.9j+j\omega)} \cdot \frac{d\omega}{(0.9-2.9j+j\omega)(0.9-2.9j-j\omega)(0.9+2.9j-j\omega)} \quad (46)$$

Evaluation of this integral in closed form is impossible, so we are forced to use the same technique to obtain an approximation for $E_p^2$ as was used to evaluate $E_n^2$. Thus, we approximate $e^{-j\omega L}$ by $1/(1+j\omega L/3)^3$, rewrite the integrand in the form $g_n(\omega)/h_n(\omega)h_n(-\omega)$, and utilize the table of integrals in reference (6) to obtain $E_p^2$. This expression is quite complex and is given by

$$E_p^2 = 5.2 \frac{M}{D_{26}} \quad (47)$$

where
\[ M_\delta = 9.2\alpha^2 + 1.2\beta\gamma - 100\alpha - 13\beta + (L/3)(106A + 91.4A\beta + 59.8A\beta^2) + 88.1\beta - 9.9\gamma - 6.5A\beta\gamma + (L/3)^2(24\gamma^2 + 3.1\beta^2\gamma - 28.6A\alpha\beta - 28.3\gamma) + (L/3)^3(9.2A\alpha\gamma - 85.6A\beta - 200A\alpha - \beta\gamma^2) + (L/3)^4(811A^2), \]  

\[ (47A) \]

\[ D_\delta = 100A\alpha^2 - 10.8A\beta\gamma + A^2\beta\gamma - 9.2A^3\alpha + (L/3)(37.9\gamma^2 + 9.6\beta^2 - 88.4A\alpha\beta - 3.5A\gamma^2) + (L/3)^2(83.5A\alpha\gamma - 6.19A\tau - 6.2\beta\gamma^2) - (L/3)^3(2520A^2 + \gamma^3) \]  

\[ (47B) \]

and

\[ \alpha = B + 1 + 1.8L + 3.06L^2 \]

\[ \beta = A + 1.6B + 1.8 + 9.2L \]

\[ \gamma = 1.8A + 9.2B + 9.2 \]  

\[ (47C) \]

**G. The Following Task.**

In the preceding sections the operator's performance in a compensatory situation has been considered and equations describing his performance have been developed. In this section an attempt will be made to analyze the operator's performance in a following task.

Figure 1(B) illustrates the arrangement of the electronic pursuit apparatus for simulating a following
In this task, inasmuch as the operator now sees both the target and the cursor, the situation is quite distinct from that occurring in the compensatory task. The operator is presented with a different type of information display and consequently his behavior would be expected to be different. It would seem logical in approximating the operator's behavior to assume that he bases his response on two quantities: (i) the velocity of the target, and (ii) the displacement $D(t)$ between the target and the cursor. In this situation, as in the compensatory situation, the operator may also utilize other direct sources of information to help him determine the correct response to make. As a result of conversations with the subjects it was ascertained that these two quantities probably provide a major portion of the information used by the operator. A first order linear approximation of the human operator in the following task is thus given by:

$$\frac{dS_H(t)}{dt} + N(t) = H_1 D(t-L) + K_d H_2 \frac{dS_p(t-L)}{dt}$$  \(48\)

where $H_1$, expressed in radians per inch-second, and $H_2$, expressed in radians per inch, are weighting functions characteristic of the human operator. Figure 13 is a
Figure 13. A block diagram of the following tracking situation incorporating the assumed human transfer function.
block diagram of the following tracking situation incorporating the assumed human transfer function.

The relations given in equations 1, 2 and 8 are also valid for the following task. Taking the Laplace transforms of these equations, combining them and solving for \( E(s) \) yields

\[
E(s) = \frac{s(1-Be^{-Ls})}{s+Ae^{-Ls}} S_p(s) + \frac{6.45Km}{s+Ae^{-Ls}} N(s). \quad (49)
\]

Let

\[
Y_1(s) = \frac{s(1-Be^{-Ls})}{s+Ae^{-Ls}} \quad (50)
\]

and

\[
Y_2(s) = \frac{6.45Km}{s+Ae^{-Ls}}. \quad (50A)
\]

The mean-square error score for the following task is then given by

\[
\bar{E}^2 = \bar{E}_P^2 + \bar{E}_N^2 = \int_{-\infty}^{\infty} |Y_1(j\omega)|^2 C_P(\omega) d\omega + \int_{-\infty}^{\infty} |Y_2(j\omega)|^2 N d\omega \quad (51)
\]
Again we are confronted with an integrand whose denominator is transcendental. The roots of the characteristic equation of this system,

\[ s + Ae^{-Ls} = 0, \]  

(52)

can also be obtained from the root locus plot of Figure 6. They lie on the locus marked \( AL/B = \infty \). The conditions for stability in the following case are much simpler than those in the compensatory case; the only requirement necessary is that

\[ AL < 1.57 \]

or

\[ 2.83K_{KH}L < 1.57 \]  

(53)

1. **The noise term.** As we did in the analysis of the compensatory system, we shall first consider the contribution of the noise to the total mean-square error, since \( E_N^2 \) will have the same form in all following tasks regardless of the type of problem course. The assumptions about the human noise which were made earlier for the compensatory case still apply to the following case. Thus, again, we have that the noise spectral density is
Combining this with \( Y_2(j\omega) \), obtained by replacing \( s \) by \( j\omega \) in equation 50(A), we have that

\[
\begin{align*}
\frac{E_N^2}{A} &= \int_{-\infty}^{\infty} \frac{41.6K_m^2Nd\omega}{(j\omega + Ae^{-j\omega L})(-j\omega + Ae^{j\omega L})} \cdot (j\omega + A e^{j\omega L})(-j\omega + A e^{-j\omega L}) \tag{54}
\end{align*}
\]

Approximating \( e^{-j\omega L} \) by \( 1/(1+j\omega L/3)^3 \) and integrating, we obtain

\[
\begin{align*}
\frac{E_N^2}{A} &= \frac{43.6K_m^2N}{8-3AL} \cdot \frac{24+15AL-(AL)^2}{24+15AL-(AL)^2} \tag{54A}
\end{align*}
\]

or

\[
\begin{align*}
\frac{E_N^2}{Hd} &= \frac{15.4K_mN}{K_3H(L)} \cdot \frac{3+5.3K_mH_3L-(K_mH_3L)^2}{1-1.08K_mH_3L} \tag{54B}
\end{align*}
\]

Equation 54 enables us to determine quantitatively the effects of the various parameters on the noise error term.

If we compare equations 38 and 54, the expressions for the noise error term in the compensatory task and the following task, respectively, we see that they resemble one another quite closely. In fact, if we set \( B = 0 \) in
the compensatory case, the two expressions become identical. Therefore, it will not be necessary to go into as much detail in the following case when we discuss qualitatively the effects of variations in A and L upon the value of $E_N^2$.

If we assume for the moment that A is constant, then a decrease in the value of the lag L is seen to have the following consequences: (i) the magnitude of $E_N^2$ is decreased, and (ii) the magnitudes of the real and imaginary components (the damping and the frequency components) of the roots of the characteristic equation are increased. On the other hand, assuming that L is fixed, an increase in the size of A results in a decrease in the size of the damping factor but an increase in the frequency component. Furthermore, as A is increased (assuming $K_m$ is held constant), the value of $E_N^2$ decreases to a minimum, then begins to increase until finally it becomes infinite when $AL = 8/3$. This differs from the value 1.57 given in equation 53 because of the use of an approximation for $e^{-j\omega L}$.

This concludes our discussion of the effects of the human noise in the following task. We now want to investigate the nature of the steady-state mean-square error for the sine-wave input and the random input.
2. The simple harmonic tracking task. As in the compensatory task, the sine-wave input used was

\[ S_p(t) = M \sin \omega_o t, \]

and its spectral density is

\[ G_p(\omega) = \frac{M^2}{4} \left[ \delta(\omega - \omega_0) + \delta(\omega + \omega_0) \right]. \]

The steady state error is obtained from

\[ \bar{E}_p^2 = \int_{-\infty}^{\infty} \frac{\omega^2 (1-e^{-j\omega L})(1-e^{j\omega L})}{(j\omega + A e^{-j\omega L})(-j\omega + A e^{j\omega L})} \frac{M^2}{4} (\omega - \omega_0) + (\omega + \omega_0) \, d\omega. \]

Hence

\[ \bar{E}_p^2 = \frac{M^2}{2} \frac{\omega_0^2 (1+B^2-2B \cos \omega_0 L)}{\omega_0^2 + A^2 - 2A \omega_0 \sin \omega_0 L}, \]

or in terms of the parameters of the system,

\[ \bar{E}_p^2 = 10.1 \frac{(1+8(K_m K_d H_2)^2 - 5.66K_m K_d H_2 \cos \pi L)}{1+0.81(K_m K_d H_1)^2 - 1.8K_m K_d H_1 \sin \pi L}. \]

The total mean-square error is then given by the sum of the expressions in equations 54 and 55.

For the particular case in which \( L = 0 \) and
$S_p = 4.5 \sin \omega t$, the mean-square error becomes

$$\bar{E}^2 = \frac{10.1(1-2.83K_mK_H^2)}{1+0.81(K_mK_H^2)}^2 + \frac{46.2K_mN}{H_1K_d}. \quad (56)$$

The autocorrelation function of the error is

$$R(\tau) = \frac{10.1(1-2.83K_mK_H^2)}{1+0.81(K_mK_H^2)^2} \cos \omega \tau + \frac{46.2K_mN}{H_1K_d} e^{-2.83K_mK_H \tau}. \quad (57)$$

We see then that for the case in which the operator follows the sine-wave with zero lag the autocorrelation function of his error is the sum of two terms — a cosine term having the frequency of the input and an exponential whose damping is a function of $K_m, K_d$ and $H_1$.

3. **The random tracking task.** The final situation which will be analyzed is that in which the operator follows a random input. Utilizing the spectral density of the problem signal given in equation 45 and the expression for $Y_1(j\omega)$ obtained from equation 50, we obtain the expression for the steady-state mean-square error $E_p^2$; namely,
\[ \frac{E_P^2}{E_P^2} = \int_{-\infty}^{\infty} \frac{(j\omega - B - j\omega L)(-j\omega - B - j\omega L)(1.65)(9.2 + \omega^2)}{(j\omega + A - j\omega L)(-j\omega + A - j\omega L)(0.9 + 2.9j + j\omega)(0.9 - 2.9j - j\omega)} \]

\[ \times \frac{d\omega}{(0.9 - a.9j + j\omega)(0.9 + 2.9j - j\omega)} \cdot (58) \]

Once again we are forced to use our approximation technique for evaluating \( E_P^2 \). Replacing \( e^{-j\omega L} \) by its approximation \( 1/(1+j\omega L/3)^3 \), rearranging the integrand in the form \( g_n(\omega)/h_n(\omega)h_n(-\omega) \) and utilizing the table of integrals in reference \( (6) \), we obtain

\[ \frac{E_P^2}{E_P^2} = 5.2 \frac{M_6}{D_6} \]

where

\[ M_6 = -\lambda(113+9)+2(L/3)[33\lambda(\lambda+1)+\lambda+2](115\lambda-9\lambda^2) \]

\[ + (L/3)^2[3\beta^2(\sigma+7)+3\alpha^2-28\beta\lambda-39\lambda^2] \]

\[ - (L/3)^3(257\lambda^2+\alpha^2)+\lambda(1/3)^4(761\lambda^2), \]

\[ (59A) \]

\[ D_6 = 159\lambda-12\alpha\beta+(L/3)(9\lambda^2+35\beta^2-116\lambda^2) \]

\[ -(L/3)^2(448\lambda^2+6\alpha^2+5^2)-(L/3)^3(2280\lambda^2+\beta^3) \]

\[ (59B) \]

and
\[
\alpha = A + 1.8 + 9.2L
\]
\[
\beta = 1.8A + 9.2
\]
\[
\gamma = 2B + 1 + L^2
\]
\[
\lambda = (B-1)^2
\]  

IV. EVALUATION OF THE HUMAN CONSTANTS

In the preceding section theoretical expressions have been derived to describe the operator's behavior under two different tracking conditions and two different classes of inputs. The next step in this analysis is the evaluation of the human constants in these expressions and the determination of the accuracy of agreement between the theoretical and the experimental situations. The human constants will be evaluated in the same order in which the theoretical expressions were derived in the preceding section.

A. The Compensatory Task

1. The simple harmonic tracking task. Before evaluating the human constants let us make a few observations concerning the nature of the human response in compensatory tracking tasks of the type we have been considering.
First, let us rewrite equation 6, which describes the human response in compensatory tasks, by replacing \( D(t) \) by \( 0.44K_dE(t) \) as given in equation 1. This yields

\[
\frac{dS_H(t)}{dt} + N(t) = 0.44K_d H_1 E(t-L) + 0.44K_d H_2 \frac{dE(t-L)}{dt}. \tag{60}
\]

Now for a given value of the manual control scale factor (that is, for \( K_m = \) a constant) a given value of error requires the same magnitude of manual control response regardless of the size of the oscilloscope display or, in other words, regardless of the value of the display scale factor \( K_d \). That is to say, if this model of the human operator were a noiseless mechanism with perfect judgment, then changing the magnitude of \( K_d \) should have no effect on the response, or mathematically speaking,

\[
K_d H_1 = C_1, \text{ a constant, and} \tag{61A}
\]

\[
K_d H_2 = C_2, \text{ a constant.} \tag{61B}
\]

Now, if relations 61(A) and 61(B) were true for the human operator, an examination of the expression for the mean-square error in the sine-wave case shows that a
plot of $E^2$ versus $K_d$ for a constant $K_m$ would yield a horizontal straight-line. Referring to Figure 14, which shows the plots of $E^2$ versus $K_d$ obtained experimentally for two operators in the compensatory task, we note that to a certain extent the curves of $E^2$ versus $K_d$ are straight lines. For a given $K_m$, excluding the case $K_d = 1/16$, the mean-square error scores lie within a fairly narrow range of values. This phenomenon appears to be a matter of practice; the more skilled a subject becomes, the flatter his $E^2$ versus $K_d$ curves become.

For example, in Figure 13 subject G had nearly fifty percent more practice than had subject E.

In the above discussion we have excluded the case $K_d = 1/16$; let us now consider it briefly. An examination of the experimental mean-square error scores shows that the mean-square error is usually much larger for this value of $K_d$ than for any of the larger values of $K_d$. The main reason for this is that $K_d = 1/16$ corresponds to only a 1/4" display on the face of the oscilloscope. For such a limited display, amplitude errors in tracking which are so small as to be barely discernible by the operator may constitute a sizeable portion of the mean-square error score. For example, if the operator tracks so that his error when displayed upon the oscillo-
Figure 14. A plot of the mean-square error, in volts$^2$, of two subjects for the sinusoidal compensatory task showing the effects of the scale factors.
scope is a constant $1/65$ of an inch, his mean-square error score will be $0.316 \text{ volts}^2$ at $K_d = 1/16$ and only $0.00123 \text{ volts}^2$ at $K_d = 1$. Consequently, we are not surprised that the experimental data show that the operators made their largest mean-square error scores at the smallest display size. This indicates that the human is able to adjust his constants so that suitable performance is maintained down to about $1/2$ inch display. This result is comparable to a very general finding in psychology, that the Weber fraction (the ratio, a detectable increment in stimulus magnitude to the reference value) is constant beyond some minimum stimulus value. For example, Grether and Williams$^{16}$ report that relative accuracy in scale reading is constant for scale intervals beyond 0.5 inch.

In addition to illustrating the effects of small errors at small display scale factor settings, the example used above illustrates another important point; namely, the mean-square errors are relative to the display and control scale factors employed. The relation in equation 2 is used to convert relative mean-square errors into absolute, angular mean-square errors. Thus,

$$\frac{\Theta^2}{E^2} = \frac{E^2}{41.6K_m^2}$$

(62)
where $\Theta_E$, the absolute, angular mean-square error, is in radians$^2$; $\overline{E}$, the relative mean-square error, is in volts$^2$ and $K_m$ is the manual control scale factor. The absolute, angular mean-square error in degrees$^2$ is given by

$$\Theta_E^2 = 80.8E^2/K_m^2.$$  

(62A)

Figure 15 shows the absolute angular mean-square error of subject G in the sinusoidal compensatory task.

Next, let us attempt to evaluate the effects of $K_m$ on the human response. From equation 60 we obtain, neglecting the noise term $N(t)$,

$$\frac{dS_C}{dt} = 6.45K_m \frac{dS_H}{dt} = 2.83K_m K_d H_1 E(t-L) + 2.83K_m K_d H_2 \frac{dE(t-L)}{dt}.$$  

(63)

If the judgment of the noiseless operator whose response is represented in equation 63 is perfect, a change in the manual control scale factor $K_m$, for a fixed setting of $K_d$, should have no effect. For if $K_m$ were doubled, for example, the operator with perfect judgment would halve his response to compensate for the change; that is,
Figure 15. Absolute mean-square error in degrees squared of Subject G for the sinusoidal compensatory task.
he could be considered to be adjusting $H_1$ and $H_2$ so that the products $K_m H_1$ and $K_m H_2$ remained constant. This is a difficult feat and an examination of Figure 14 shows that the human operator does not accomplish it. However, this change in the mean-square error which occurs when $K_m$ is increased from 1 to 8 to 16 involves more than just an inability on the part of the operator to adjust $H_1$ and $H_2$. If we refer to our "imperfect" model whose response is given by

$$\frac{dS_C}{dt} + 6.45K_m N(t) = 2.83K_m d_1 E(t-L) + 2.83K_m d_2 \frac{dE(t-L)}{dt},$$

(63A)

we see that we can also account for this effect in a much better way; for increasing $K_m$ amplifies the effect of the human noise. Thus, we see from equation 38 that the contribution of the noise term to the mean-square error term varies nearly as $K_m^{3/2}$. In tracking a sine-wave input the noise term comprises a large percentage of the mean-square error, hence anything that increases the effect of the noise increases the operator's error. Furthermore for a small magnitude of control movements ($K_m > 4$), the noise term may become large enough to prevent the operator from maintaining $K_d H_1$ and $K_d H_2$ constant,
resulting in a more evident departure from constancy of the curves of $\bar{E}^2$ versus $K_d$ for this range of $K_m$.

For the sine-wave input we shall assume, therefore, that the operator is able to maintain $K_dH_1$ and $K_dH_2$ constant over the range $1<K_m<16$. Thus, in the expression for the mean-square error there are three parameters, namely, $K_dH_1$, $K_dH_2$ and $N$, to evaluate for each operator. It will be noted that no mention is made of the human lag time $L$. We have purposely avoided it since it appears that, in analyzing the sine-wave case, little error is introduced by considering that the lag time $L$ is effectively zero. This reduction of the lag to zero appears to result from the fact that for a sine-wave input the operator behaves as a predictor. Once he has learned the nature of this type of input he can predict its behavior far enough into the future to compensate for his own lag time. Utilizing this assumption, the theoretical expression for the mean-square error occurring in the compensatory tracking task with a sine-wave input $S_p(t) = 4.5 \sin \pi t$ is

$$\bar{E}^2 = \frac{10.1}{0.81(K_mK_dH_1)^2+(2.83K_mK_dH_2+1)^2} + \frac{15.4K_mN}{K_dH_1(2.83K_mK_dH_2+1)}.$$

(42B)
We now want to determine how closely this expression can be made to match the experimental data.

In Figures 16 and 17 the solid line curves are the graphical records of the experimental mean-square errors for four different subjects, each curve being the average over a series of trials. The subjects used in the sinusoidal tracking studies (both compensatory and following tasks) were well trained, each subject having had at least two hours of tracking practice (120 one-minute trials) before these mean-square errors were tabulated.

The three constants which we desire to evaluate can be obtained by making a three point fit between equation 42(B) and the experimental curves of each subject. This leads to three simultaneous equations in three unknowns which are quite difficult and tedious to solve without a computer. A trial and error graphical approximation of fitting equation 42(B) to the experimental curves was used and the values obtained for the three constants for each subject are listed in Table 1. A visual comparison of the accuracy with which equation 42(B) matches the experimental data can be made from Figures 16 and 17, in which the dashed-line curves are the plots of theoretical mean-square errors given by equation 42(B).
Figure 16. A plot of the mean-square error in volts$^2$ for two subjects in the sinusoidal compensatory task for the case $K_d=1$. 
Figure 17. A plot of the mean-square error in volts$^2$ for two subjects in the sinusoidal compensatory task for the case $K_d=1$. 
**TABLE 1**

Human Parameters Computed for the Sinusoidal Compensatory Task

<table>
<thead>
<tr>
<th>Subject</th>
<th>$H_1 K_d$ rad./in.-sec.</th>
<th>$H_2 K_d$ rad./in.</th>
<th>$N$ volts$^2$/rad.</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>4.09</td>
<td>0.071</td>
<td>43.0x10$^{-4}$</td>
</tr>
<tr>
<td>D</td>
<td>31.1</td>
<td>0.035</td>
<td>441.0x10$^{-4}$</td>
</tr>
<tr>
<td>E</td>
<td>7.01</td>
<td>0.018</td>
<td>39.0x10$^{-4}$</td>
</tr>
<tr>
<td>G</td>
<td>6.24</td>
<td>0.018</td>
<td>35.2x10$^{-4}$</td>
</tr>
</tbody>
</table>
2. The random task. Figure 18 shows the effects of the scale factor upon the mean-square error scores of four different operators for the compensatory task with a random input. The curves are quite interesting especially when compared with those in Figure 14 which were obtained for the sinusoidal input.

The first difference we note is in the level of the mean-square errors for the two inputs. Although the mean-square value of the sine-wave input signal is $10.1 \text{ volts}^2$ and the mean-square value of the random input signal is only $5.8 \text{ volts}^2$, the mean-square error for the sinusoidal case is less than $1.9 \text{ volts}^2$ in every instance, while for the random case it lies between $1.2$ and $4.5 \text{ volts}^2$. This result is not unexpected, however, when we compare the complexity of the two inputs. For both of these inputs, the contribution of the human noise to the total mean-square error is probably of the same order of magnitude, but the steady-state mean-square error resulting from the random input is much larger than the steady-state mean-square error in the sine-wave case.

The second pronounced difference between the curves in Figures 14 and 18 lies in the role which the manual control scale factor plays in these two tasks. Increasing
Figure 18. A plot of the mean-square error in volts$^2$ for four subjects in the random compensatory task showing the effects of the scale factors.
$K_m$ in the sine-wave case increases the mean-square error, while increasing $K_m$ in the random case decreases the mean-square error. The explanation for this difference appears to lie in the ratio $E_N^2/E_P^2$ in these two cases. For according to the expressions we have developed for $E_N^2$ and $E_P^2$ in both of these cases, increasing $K_m$ increases the values of $E_N^2$ but decreases $E_P^2$ (assuming the other parameters remain constant). Since $E_N^2$ is greater than $E_P^2$ in the sine-wave case, the resulting effect is to produce an increase in $E^2$ as $K_m$ increases. The situation is just reversed in the case of the random input, however, for $E_P^2$ is greater than $E_N^2$ and the overall effect is to decrease $E^2$ as $K_m$ increases. However, it should be noted that one subject out of the six tested on the random task did not behave this way. This is subject 2 whose mean-square error scores are included in those shown in Figure 18. It may be that in the case of this subject $E_P^2$ is not larger than $E_N^2$.

Figures 19, 20 and 21 are the normalized autocorrelation functions of the error functions of three different subjects performing in the compensatory task with a random input. The autocorrelation functions in Figures 19 and 20 were made to determine the effects of
Figure 19. The normalized autocorrelation functions of the error of Subject 1 in the random compensatory task for the case $K_d=1$. 
Figure 20. Autocorrelation functions of the error of two subjects in the random compensatory task for the case $K_d=1$. 
changes in $K_m$ upon the operator's response. For all three subjects we see that an increase in $K_m$ tends to cause an increase in the predominant frequency appearing in the autocorrelation function. In addition, we note that at the largest value of $K_m$, that is, at $K_m = 16$, a higher frequency term whose frequency lies in the range 1 to 2 c.p.s. begins to appear in the autocorrelation functions of all three subjects. The predominant frequency in all these autocorrelation functions is near the predominant frequency of the random input; namely, 1/2 c.p.s. (30 c.p.m.). This is to be expected, however, since the steady-state error constitutes the major part of the total mean-square error score. If we examine the autocorrelation function of subject 2 in Figure 20 for the case $K_m = 16$, however, we notice that the problem frequency is not present, suggesting that in this case $E_N^2$ may be greater than $E_p^2$. As a matter of fact, this is the subject whose mean-square error scores increased as $K_m$ increased, as we mentioned in the preceding paragraph. Thus, the autocorrelation function of this subject's error would tend to corroborate the statement we made earlier, namely, that in his case $E_p^2$ was not greater than $E_N^2$. 
In Figure 21 are shown the autocorrelation functions of the operator's error which were made in an effort to ascertain whether there were any detectable changes in the operator's performance as the display scale factor was changed. One interesting feature is now noticed. At $K_d = 1/16$ and $1/8$, the predominant frequency in the autocorrelation function is about $1/4$ c.p.s., while the second frequency appearing in the autocorrelation function has a value in the neighborhood of the predominant frequency of the random input signal. When $K_d$ was increased to its maximum value, however, the predominant frequency in the autocorrelation function increased until it equaled that of the input signal, and while there is a suggestion of the presence of higher frequency terms, their contribution is so small as to be negligible.

The question that now arises is whether our model of the operator can account for these changes, and if it can, how closely does it match them from a quantitative point of view. Since we have found that the mean-square error in the random case is composed mainly of the steady-state error $E_p^2$, the quantity we want to examine is $G_{EP}(m)$ which is the product of the two functions $|Y_1(j\omega)|^2$ and
Figure 21. Autocorrelation functions of the error of Subject 2 in the random compensatory task for the case $K_m = 1$. 
According to our model of the operator, this is given by

\[ G_p(o) = \frac{\omega^2}{j\omega + e^{j\omega L}(A+B\omega) - j\omega + e^{j\omega L}(A-B\omega)} \times \frac{1.65(\omega^2+9.2)}{(\omega^2-1.8j\omega-9.2)(\omega^2+1.8j\omega-9.2)} \]  

(64)

where \( A = 2.83K_mK_dH_1 \) and \( B = 2.83K_mK_dH_2 \). Before making a direct attempt at evaluating the human parameters to see how closely our model fits the experimental situation, we shall try to show graphically how our model can be made to explain the changes occurring in the autocorrelation function. Before doing this, we shall make one further assumption about the behavior of the human, and that is that the operator acts as a linear system over the given range of the manual control and display scale factors used; that is, the parameters \( H_1 \) and \( H_2 \) are constants. Since \( AL/B = H_1L/H_2 \), this means that the operator may be considered to have selected one locus in the root locus plot and his consequent behavior may be described in reference to it. Since each point on the locus corresponds to a specific value of \( B \) and since \( B = 2.83K_mK_dH_1 \), then any change in the scale factors will
cause a change in the position on the locus corresponding to the given state of the system. Furthermore, as B is increased we move to the right along the locus, and if we examine $E_p^2$, where

$$E_p^2 = \int_{-\infty}^{\infty} G_{EP}(\omega) d\omega,$$

as B increases, we find that it decreases as B increases until it reaches a minimum and then as B is further increased, $E_p^2$ begins to increase until it becomes infinite as the point of instability of the system is reached. (We might add that in our model for the specific input used, the preceding statement is true only if L is equal to or less than 0.25 seconds, approximately. For larger values of L, as B is increased, $E_p^2$ increases monotonically.)

To illustrate this we have plotted $G_{EP}(\omega)$ versus $\omega$ in Figure 22 for the case $AL/B = 4$ and $L = 0.25$ seconds. We see that as B increases from 0.1 to 0.28 the steady-state error $E_p^2$, proportional to the area under the spectral density curves, decreases. As B is increased to 0.35, $E_p^2$ begins to increase and it will become infinite when B reaches 0.44. To facilitate an understanding of what is happening, we have sketched the root locus for
Figure 22. Graphical illustration of the effect of changes in the control scale factors upon the steady-state error spectral density.

Figure 23. Root locus for the case $AL/B=4$, showing the locus of roots for particular values of $B$. 
this particular case in Figure 23 and the points corresponding to these particular values of $B$ have been indicated on it.

Now for nearly all the subjects tested, their minimum mean-square error scores were made at $K_m = 16$, $K_d = 1$. This would correspond to a selection of $H_2$ in the example under consideration so that $B = 0.28$, when $K_m = 16$, $K_d = 1$; this point lying near the region of the locus corresponding to the minimum value of $E_p^2$. Thus, the case $K_m = 8$, $K_d = 1$ would correspond to the point $B = 0.14$ on the root locus; the point $B = 0.018$ would correspond to $K_m = 1$, $K_d = 1$, etc. But, these points on the root locus also tell us what will be the frequency of the components appearing in the autocorrelation function. (We already know from the nature of $G_p(\omega)$ that one of the frequencies appearing in the autocorrelation will be the predominant frequency of the input; namely, 0.5 c.p.s.) Now examining the root locus plot, we see that we would expect a frequency 0.63 c.p.s. to appear in the autocorrelation function of the operator's error for the case $K_m = 1$, $K_d = 1$, although the damping factor is so high in this case that the term may be damped out before we can distinguish its presence. Similarly for $K_m = 8,$
$K_d = 1$ we would expect a frequency of 0.7 c.p.s. to be present; and for $K_m = 16$, $K_d = 1$, we would expect a frequency of 1.13 c.p.s. to be present. Now comparing this behavior with what actually happened in the experimental situations, we see our model matches the actual situation very nicely, at least from a qualitative point of view.

We have shown by the above example that our model corresponds to the actual situation in a qualitative manner. Unfortunately, when a numerical evaluation of the human constants for each operator is required, the task becomes hopelessly involved. No simplified procedure has been worked out whereby these constants can be evaluated; the only thing we can do is to use a trial and error procedure to attempt to find the ranges in which the constants must fall to yield results similar to those obtained experimentally. The determination of the operator's lag was the first problem, and unfortunately there was no way of determining it experimentally from the data which was taken. As we mentioned above, it was determined that if our model was to simulate the performance of the human, $L$ would have to be equal to or less than 0.25 seconds. Even after a value of $L$ has been assumed, the
determination of $H_1$, $H_2$ and $N$ is laborious. The technique used was to attempt to find the range of values of these constants which would fit $E^2$, as given by the sum of equations 38 and 47, to the experimental data.

The human parameters were determined for subject 1 and are listed below in Table 2. In Figure 24, the theoretical values of $E^2$, computed by using these parameters in equations 38 and 47 and adding the results, have been plotted as the dashed-line curve. The experimental data for the same conditions have been plotted as the solid line. Utilizing these values of the human parameters, the theoretical autocorrelation functions of the operator's error have been computed and plotted in Figure 25.

### TABLE 2

Values of the Human Parameters of Subject 1 in the Random Compensatory Task

<table>
<thead>
<tr>
<th></th>
<th>$H_1$</th>
<th>$H_2$</th>
<th>$L$</th>
<th>$N$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.102 rad./in.-sec.</td>
<td>0.006 rad./in.</td>
<td>0.25 seconds</td>
<td>$0.464 \times 10^{-4}$ volts^2/rad.</td>
</tr>
</tbody>
</table>
Figure 24. A plot of the mean-square error in volts$^2$ of Subject 1 for the case $K_d = 1$, in the random compensatory task.
Figure 25. A graphical comparison of the theoretical and experimental autocorrelation functions of the error of Subject 1 for random compensatory task in which $K_m=16$ and $K_d=1$. 
B. The Following Task

1. The simple harmonic tracking task. Figure 26 shows the effects of the control scale factors on the mean-square errors of two operators for a following task with a sinusoidal input. Comparing these curves with those in Figure 14 for the compensatory case, we see that they are quite similar in nearly every respect. The only difference between them is in the level of the scores, the operators' mean-square errors being lower in the following task. This is to be expected, however, since the operator now receives more information upon which to base his responses than he did in the compensatory task.

The discussion presented for the compensatory task in connection with the sinusoidal input also applies to the following task. We see that with the exception of the case $K_d = 1/16$, where errors so small as to be barely discernible may contribute a large percentage of the total score, the operator maintains $E^2$ at a constant level over a wide range of the display scale factors for a fixed value of $K_m$. Hence, in this case also it would be justifiable to consider that the operator maintains $K_dH_1$ and $K_dH_2$ constant. Here also it appears that
Figure 26. A plot of the mean-square error in volts$^2$ for two subjects in the sinusoidal following task.
the operator's noise term contributes quite heavily to the total mean-square error. As in the compensatory task, the increase in $E_N^2$ with an increase in $K_m$ also accounts for the fact that the operator's error increases as the arm control sensitivity is increased ($K_m$ made larger).

After following a sine-wave for a short period of time, the operator learns the nature of the problem and soon begins to behave as a predictor. Consequently his lag time is reduced to zero. Then his mean-square error score as given by equation 56 is

$$\overline{E^2} = \frac{10.1(1-2.83K_mK_dH_2)^2}{1+0.81(K_mK_dH_1)^2} + \frac{46.2K_mN}{H_1K_d}$$  \hspace{1cm} (56)

Our problem now becomes that of seeing how closely equation 56 matches the experimental data shown in Figures 27 and 28 for six different subjects following a sinusoidal problem. The solid line curves represent the experimental mean-square errors plotted against the values of the manual control scale factor for the case $K_d = 1$. The operators followed a sine-wave with a frequency of 30 c.p.m. The values of the constants determined for the six operators are listed in Table 3.
Figure 27. Mean-square scores in volts$^2$ for three subjects in the sinusoidal following task for the case $K_d=1$. 
Figure 28. Mean-square error in volts$^2$ for three subjects in the sinusoidal following task for the case $K_d=1$. 
**TABLE 3**

Human Parameters Computed for the Sinusoidal Following Task

<table>
<thead>
<tr>
<th>Subject</th>
<th>$H_1K_d$ rad./in.-sec.</th>
<th>$H_2K_d$ rad./in.</th>
<th>$N$ volts$^2$/rad.</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>28.4</td>
<td>1.07</td>
<td>$89.2 \times 10^{-4}$</td>
</tr>
<tr>
<td>B</td>
<td>5.11</td>
<td>0.19</td>
<td>$5.1 \times 10^{-4}$</td>
</tr>
<tr>
<td>C</td>
<td>2.76</td>
<td>0.035</td>
<td>$31.1 \times 10^{-4}$</td>
</tr>
<tr>
<td>D</td>
<td>51.5</td>
<td>2.72</td>
<td>$162.1 \times 10^{-4}$</td>
</tr>
<tr>
<td>E</td>
<td>34.2</td>
<td>1.90</td>
<td>$166.0 \times 10^{-4}$</td>
</tr>
<tr>
<td>F</td>
<td>55.6</td>
<td>2.86</td>
<td>$222.0 \times 10^{-4}$</td>
</tr>
</tbody>
</table>
Figures 27 and 28 afford a visual check of the accuracy with which the theoretical curves, which are represented by dashed-line curves, match the experimental data.

Although, as we have already mentioned, the autocorrelator was not available when the data for the sine-wave input were being obtained, we were able at a later date to obtain some autocorrelation functions of error made by two trained operators while following a sine-wave. These have been plotted in figures 29 and 30. Unfortunately these two operators were not members of the group whose constants are tabulated in Table 3. The correlation functions in Figure 29 were obtained to see what the effects of the display size might be upon the operator's performance. As far as this subject was concerned, the effects were negligible. It is interesting to note that these autocorrelations consist of a cosine term plus an exponential term. Referring to the autocorrelation function, equation 57, which was derived for our model of the operator with zero lag, we see that there is nice agreement. The autocorrelation functions in Figure 30 were made to determine the effects of the arm control sensitivity on the operator's performance. For $K_m = 1$ and $K_m = 8$, the operator's
Figure 29. Autocorrelation function of the error of a subject in the sinusoidal following task in which $K_d=1/16$.

Figure 29. Autocorrelation function of the error of a subject in the sinusoidal following task in which $K_d=1$. 
Figure 50. Autocorrelation function of the error of a subject in the sinusoidal following task for the case $K_d = 1$. 

Figure 30. Autocorrelation function of the error of a subject in the sinusoidal following task for the case $K_d = 1$. 
performance again can be nicely represented by an exponential term plus a cosine term whose frequency is that of the problem. However, at $X_{m} = 16$, we see that a higher frequency has appeared in the operator's response. Thus it would appear that in this case the operator no longer operates with zero lag as we have assumed.

2. The random input. Figure 31 shows the effect of the scale factors upon the mean-square error scores of four operators for the following task with a random input. Comparing these curves with those obtained in the compensatory task we see that they are similar, the major difference being that the effects of changes in the scale factors are more pronounced in the following case.

In Figures 32 and 33 are shown the normalized autocorrelation functions of the error functions of three different subjects performing in the following task with a random input. The autocorrelation functions in Figure 32 were made to determine the effects of changes in $X_{m}$ upon the operator's performance. In this case we notice that subject 4 shows very little change in his performance, the only change being a slight decrease in the predominant frequency appearing in his autocorrelation functions.
Figure 31. Mean-square error scores in volts$^2$ for four subjects in the random following task.
Figure 32. Autocorrelation function of the error of Subjects 3 and 4 in the random following task for the case $K_d=1$. 
Figure 33. Autocorrelation functions of the error of Subject 3 in the random following task for the case $K_m = 1$. 
as $K_m$ is increased. Furthermore we note the absence of a higher frequency terms, even at $K_m = 16$. Subject 3, on the other hand shows the presence of a higher frequency component as $K_m$ is increased. At $K_m = 1$, the predominant frequency is approximately that of the problem while at $K_m = 8$, the problem frequency does not appear to be present, but a frequency component of 1 c.p.s. is quite evident.

The correlation functions in Figure 33 were made in an effort to determine the effect of the display size upon the operator's performance. We see that surprisingly enough the results appear to be similar to those obtained in the compensatory case. At $K_d = 1/16$ and $1/8$, the predominant frequency is about $1/3$ c.p.s. As $K_d$ is increased the predominant frequency increases to a value of 0.6 c.p.s.

As in the compensatory case, the question arises as to whether our model can account for these changes. Since the mean-square error in this case also appears to be mainly due to the steady-state error $E_p^2$, we want to examine the quantity $G_{KP}(\omega)$ which in this case is given by
\begin{equation}
G_{KP}(\omega) = \frac{\omega^2(1-Be^{-j\omega L})(1-Be^{j\omega L})1.65(\omega^2+9.2)}{(j\omega+Ae^{-j\omega L})(-j\omega+Ae^{j\omega L})(\omega^2-1.8j\omega-9.2)}
\tag{65}
\end{equation}

where \( A = 2.83K_mK_H \) and \( B = 2.83K_mK_d \). In this case the root locus plot is the single locus, designated \( AL/B = \omega \) in Figure 6. The point upon this locus describing the state of the operator is determined only by \( A \) in this case. Although \( B \) does not play a part in determining the frequencies which will appear in the autocorrelation functions of the error, it does determine the rate at which the mean-square error will decrease as \( K_m \) or \( K_d \) is increased and hence it determines what the relative magnitudes of the frequency components in the autocorrelation function will be. Now assuming that \( H_1 \) and \( H_2 \) are constants, the position of each point on the locus is determined by the magnitude of \( A \), and hence, as \( K_m \) or \( K_d \) is increased, the point describing the state of the system moves to the right along the locus. Now these points on the root locus tell us what will be the frequency of the components appearing in the autocorrelation function. (We already know that one of the
frequencies appearing will be the predominant frequency of the random input, namely 0.5 c.p.s.) Hence as $K_m$ or $K_d$ is increased, we would expect an increase in the frequency of some of the components in the autocorrelation function. However, if $H_1$ and, consequently, if $A$ is small enough, the magnitudes of these components may be so small and their damping may be so large that they may be insignificant when compared to the contribution of $G_p(\omega)$, and any change in them is not noticeable. This is apparently true in the case of Subject 4.

As was true in the compensatory task, the actual evaluation of the human constants is a very laborious task. The constants were evaluated for Subject 4 and are listed in Table 4. Figure 34 gives a visual comparison of the theoretical values of $E^2$, computed from equations 54 and 59, with the experimental data. The autocorrelation function of the error was also computed utilizing these values and has been plotted in Figure 35.
### TABLE 4

Human Parameters Evaluated for Subject 4 in the Random Following Task

<table>
<thead>
<tr>
<th></th>
<th>( H_1 )</th>
<th>( H_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>0.104 rad./in.-sec.</td>
<td>0.018 rad./in.</td>
</tr>
<tr>
<td>N</td>
<td>0.2 sec.</td>
<td>0.35x10^{-4} volts^2/rad.</td>
</tr>
</tbody>
</table>
Figure 34. Mean-square error in volts$^2$ of Subject 4 in the random following task for the case $K_d=1$. 
Figure 35. A comparison of the theoretical and experimental autocorrelation functions of the error of Subject 4 for the case $K_m=8$, $K_d=1$. 
V DISCUSSION AND CONCLUSIONS

The results of the preceding section show that our model is in many respects a good first approximation to the human operator in the tracking task. In addition, our model has enabled us to describe the operator's behavior in the tracking task by means of the "human parameters" $H_1$, $H_2$, $N$ and $L$. Having established these parameters, we are now concerned with their significance in describing individual differences in tracking behavior.

A. The Human Constants

1. The sinusoidal tracking task. The sinusoidal compensatory task will be considered first. In this case it was assumed that the lag $L$ of the practiced operator was essentially zero. The values of the constants $H_1$, $H_2$ and $L$ were determined from four subjects and tabulated in Table 1. At first glance it may appear that there are such wide individual differences in these parameters that it is impossible to attach any significance to them. Fortunately, however, this is not the case. If the ratio of $H_1/H_2$ is calculated for all four
subjects, it is discovered that in every case \( \left( \frac{H_1}{H_2} \right) \) is much greater than one, varying from a minimum of 58 seconds\(^{-1}\) for Subject C to a maximum of 889 seconds\(^{-1}\) for Subject D. This indicates that every subject relied more heavily upon the displacement information than upon the velocity information in the display. Although the ratio \( \frac{H_1}{H_2} \) varies quite widely for the four subjects tested, if the subjects are classified according to the magnitude of their \( H_1K_d \) term, it is found that they fall into two distinct groups. Subject D having an \( H_1K_d \) term whose magnitude is 31.1 rad./in-sec. can be placed in one group while the remaining subjects having \( H_1K_d \) terms whose magnitudes are 7.01 rad./in-sec. or less can be placed in the second group. This is a strong indication that these particular subjects used two different tracking techniques. For example, Subject D when confronted with a given displacement would use a large response rate to correct it; the net result being an erratic tracking performance. The remaining three subjects, on the other hand, when confronted with the same displacement would use a smaller response rate to correct it, the results in this case being smoother tracking performances. This reasoning is further
justified by an examination of the noise figure \( N \) of each subject. It will be noted that the \( N \) of Subject D is roughly ten times greater than that of the other subjects, thereby indicating a much greater degree of randomness in his tracking performance.

An important problem which should be considered is the determination of a suitable criterion to use in selecting the better trackers from a given group of subjects. Since the noise term contributes the major portion of the total mean-square error in the case of a sinusoidal input, an examination of the \( E_N^2 \) term in equation 42(B) provides a basis for a suitable criterion. Since \( \xi \) is much greater than \( \xi' \) in the sinusoidal compensatory task, the quantity \( N / H_1 K_d \) is the predominant factor in this term. Hence, for the sinusoidal compensatory task, the performance criterion requires the selection of those subjects having the smallest ratio \( N / H_1 K_d \). Of the four subjects in Table 1, we see from Figures 16 and 17 that Subjects E and G, having \( N / H_1 K_d \) ratios of \( 5.56 \times 10^{-4} \) and \( 5.65 \times 10^{-4} \), respectively, had lower mean-square errors than Subjects C and D whose \( N / H_1 K_d \) ratios were \( 10.5 \times 10^{-4} \) and \( 14.2 \times 10^{-4} \), respectively.
One other question which should be considered is the relationship between the computed constants in Table 1 and the stability criteria developed in Section III. If the lag time $L$ is zero, then referring to equation 21, the characteristic equation of the compensatory tracking situation, it is seen that the system will always be stable. However, if $L$ is not zero, a reference to Figure 5 shows that the magnitude of $B$ must always be less than one if the system is to be stable. Now computing the magnitude of $B$ (from the relation $B = 2.83 K_m K_d H_2$) for Subjects E and G at the maximum value of $K_m$, that is, $K_m = 16$, we obtain $B = 0.81$. From Figure 5, we see that the stability criterion requires that, for this value of $B$, $AL$ must be less than 1.43. On the basis of this inequality the lag of Subject E must be less than 0.0045 seconds and that of Subject G must be less than 0.005 seconds. If the magnitudes of $B$ are calculated for Subjects C and D, they are 3.15 and 1.55, respectively, both of which lie outside the stability region if $L$ is not zero, that is, if the subject is not acting as a perfect predictor. One possible explanation of this anomaly, of course, is that both of these subjects performed this
task with zero lag. One other possibility, which may be true for all subjects in a sine-wave task, is that the lag \( L \), while extremely small at all times as a result of operator's prediction may vary in magnitude in a random fashion during a given trial. Thus, it may be that the lag is positive during one cycle of the trial and then is negative during the next. The net result of these small variations is that the lag can be considered effectively zero over the entire trial, and the effects of these random variations can be included in the noise term.

Since the question of the operator's stability has been mentioned, it should be pointed out that there are two kinds of instability relative to these tracking situations. The first kind is the inherent instability of the operator which might be caused, for example, by such an extremely small display scale setting that the operator is unable to perceive his error any longer or which might be caused by an unusually sensitive manual control system. Instability as affected by organismic variables such as fatigue, tension, hormone effects, etc. can also be included under the heading of the inherent stability of the operator. The effect of these
organismic variables is to cause the magnitudes of $H_1$ and $H_2$ to be increased to such a value that instability would occur. The second type of instability is that induced by the problem, as would occur, for example, if the frequency of the sine-wave input were increased to the point where the operator can no longer follow the problem.

The sinusoidal following task will now be considered. An examination of the human constants tabulated in Table 3 shows again a large variation in their magnitudes for the six subjects in this group. The ratio $H_1/H_2$ is much larger than one in this case, ranging from a minimum of 18.0 seconds for Subject E to a maximum of 78.9 seconds for Subject C. This means that in this situation the operators relied more heavily upon the information provided by the displacement between the target and the cursor in the display than upon the target velocity information.

Utilizing the magnitudes of $H_1K_d$ for the six operators in the sinusoidal following task, these subjects can be separated into two groups. The first group, whose members are subjects B and C, have $H_1K_d$ terms whose magnitudes are less than 5.11 rad./in.-sec. The second
group, whose members are Subjects A, D, E and F, have $H_1 K_d$ terms whose magnitudes are greater than 28.4 rad./in.-sec. This is a strong indication that these two groups also used two different tracking techniques. The members in the second group used large response rates, the result being erratic tracking performances. The members of the first group used smaller response rates, resulting in smoother tracking performances. This classification is further justified by examining the noise figures $N$ of the members of the two groups. The members of the first groups, Subjects B and C have smaller noise figures than the members of the second group thereby indicating a less erratic tracking behavior.

The next problem is the determination of a performance criterion to aid in the selection of the better trackers in the sinusoidal following task. The expression for the operator's mean-square error is given in equation 56. Since $E_N^2$ constitutes the major part of the total mean-square error, this term must be minimized. Then one of the conditions of our performance criterion is that the ratio $N/H_1 K_d$ be small.

An examination of the expression for $E_P^2$ in
equation 56 shows that it can be greatly reduced by a proper choice of $K_dH_2$. As a matter of fact, if $2.83K_mK_dH_2$ were equal to one, its contribution would be zero. Since $E_p^2$ is important only at the smaller values of $K_m$, we can consider specifically the case $K_m = 1$. Then the second condition of our performance criterion is that the quantity $(1-2.83H_2K_d)$ should be as small as possible.

As an example, consider Subjects A and D. Their $N/H_1K_d$ ratios are nearly equal, being $3.14 \times 10^{-4}$ for Subject A and $3.16 \times 10^{-4}$ for Subject D. However, the magnitude of $(1-2.83H_2K_d)$ is 2.03 for Subject A and 6.76 for Subject D. Accordingly Subject A should be the better tracker. This is verified by Figures 27 and 28.

The magnitudes of the quantities $N/H_1K_d$ and $(1-2.83H_2K_d)$ have been calculated for each subject and their values tabulated in Table 5. Application of the performance criterion indicates that Subjects A and B are the best trackers and that Subject C is the poorest. This is verified by comparing the curves in Figures 27 and 28.

The stability criterion for the following task is
### TABLE 5

Evaluation of the Performance Criteria Factors for the Following Task

<table>
<thead>
<tr>
<th>Subject</th>
<th>((N/H_1K_d)\times10^{-4})</th>
<th>((1-2.83H_2K_d))</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3.14 watts-in/rad.(^2)</td>
<td>2.03</td>
</tr>
<tr>
<td>B</td>
<td>1.00</td>
<td>0.46</td>
</tr>
<tr>
<td>C</td>
<td>11.2</td>
<td>0.90</td>
</tr>
<tr>
<td>D</td>
<td>3.16</td>
<td>6.76</td>
</tr>
<tr>
<td>E</td>
<td>4.86</td>
<td>4.36</td>
</tr>
<tr>
<td>F</td>
<td>4.00</td>
<td>7.10</td>
</tr>
</tbody>
</table>
that $AL 1.57$, or $2.83K_{m} dH_{l} 1.57$. If we consider the case $K_{m} = 16$ and utilize the values of $H_{l}K_{d}$ in Table 3 we find that the value of the lag $L$ ranges from 0.0006 for Subject F to 0.013 for Subject C. These subjects can be effectively considered to track with zero lag.

Although the compensatory and the following tasks represent two different tracking situations, it is interesting to compare the human constants obtained for both cases. First of all we note that the general level of the noise figure $N$ appears to be the same, although the range of values of $N$ is larger in the compensatory task. An examination of the behavior of Subjects C and E, whose constants have been evaluated for both tasks, shows that Subject C uses a smoother tracking technique in the following task than in the compensatory, while Subject E does just the opposite. Thus, it may not be wise to try to judge an operator's performance in the following task on the basis of his performance in the compensatory task, and vice versa.

2. The random tracking task. Before discussing the results of the study of the random input problem, we should point out certain limitations in the
experimental data used in this part of the analysis.

This study was the first one made in which the random function generator was used, and there were certain features about its output signal which were not discovered until the study was underway. It had been our practice on earlier tracking studies to have each operator track for a minute and fifteen seconds with only the last sixty seconds of his performance being scored. This procedure was also used in the studies of the random problem. However, it was discovered that the mean-square value of the output signal of the random function generator was extremely variable for one minute trials. Fortunately, the mean-square value of this problem signal had been recorded for each trial, thereby allowing us to adjust the mean-square errors of the subject's performance during each trial on a percentage basis to coincide with a theoretical input whose mean-square value was $5.8 \text{ volts}^2$. This procedure is not entirely correct in principle, but it is believed that the conclusions drawn from the adjusted data will not be affected.

The human parameters were evaluated for only one subject in both the random compensatory and the random
following tasks. However, it is interesting to compare them with those obtained for the sinusoidal input. Let us consider the compensatory task first. Comparing the data in Table 2 with that in Table 1, four differences in the data are noted; namely, (i) the magnitudes of $H_1$ and $H_2$ are smaller in the random case, (ii) the noise figure is smaller in the random case, (iii) the lag is larger in the random case, and (iv) the ratio $H_1/H_2$ is smaller in the random case. The first difference indicates that Subject 1 was using much smaller response rates in the random case than had been used by any of the subjects in the sine-wave case. This indicates that Subject 1 had a smoother tracking performance than did the subjects in the sine-wave case. This is reflected in the noise figure $N$ of Subject 1 which is smaller than that of the other subjects. The lag is larger in the random case since the operator no longer behaves as a perfect predictor due to the increased complexity of the problem. The smaller $H_1/H_2$ ratio in the random case indicates that Subject 1 was using proportionately more velocity information than did the subjects in the sine-wave case.
Let us now consider the following task. A comparison of the human constants in Table 3 for the sinusoidal case with those in Table 4 for the random case shows the same four differences that were discovered in the compensatory task: namely, (i) the magnitudes of $H_1$ and $H_2$ are smaller in the random case, (ii) the value of the noise figure $N$ is smaller in the random case, (iii) the value of the lag time $L$ is larger in the random case, and (iv) the ratio $H_1/H_2$ is smaller in the random case. One possible explanation of these differences in the human constants in these two cases is that the value of the lag time $L$ is increased in the random case since the complexity of the problem precludes perfect prediction. This necessitates a reduction in $H_1$ to prevent instability. This reduction in $H_1$ results in a smoother performance with a corresponding decrease in the operator's randomness, as reflected by a smaller value of $N$.

The magnitudes of $AL$ and $B$ for Subject 1 in the random compensatory task for the case $K_m = 16$ are $AL = 1.13$ and $B = 0.26$. Locating this point on Figure 5, we find that it is situated well within the stability region of the region. Similarly, in the random following task Subject 4 has an $AL$ term whose magnitude is 0.92
which also satisfies the stability criterion for this system; namely that \( AL < 1.57 \).

The establishment of a performance criterion for either the random compensatory or the random following tasks is a very difficult job. Since in both of these cases the steady-state mean-square error constitutes the major part of the total mean-square error, an examination of \( E_p^2 \) should provide a basis for establishing a performance criterion.

The expression for \( E_p^2 \) in the random compensatory task is given in equation 46. Certainly one important condition of a performance criterion for this case would be to select those operators having small values of \( L \). A second condition would be to select those operators having large values of \( H_1 \) and \( H_2 \).

In the case of the random following task the expression for \( E_p^2 \) is given in equation 58. Here again one important condition of our criterion would be to select those operators having small values of \( L \). A second criterion would be to select those operators having small values of the quantity \( (1-2.83KmKdH) \) and large values of \( H_1 \).
Since the values of the human constants are very difficult to determine in the random case, one possible method of selecting the better trackers for any tracking task might be to evaluate their performances for the sine-wave input, determine their human constants and apply the performance criteria developed for this case. Although at present there are no data available to substantiate this, it would appear logical that those subjects who are the better trackers in the case of the sine-wave input would also be the better trackers in the case of more complex inputs. This would assume of course a positive correlation between ability in the sine-wave situation and ability in the random situation. This matter would have to be investigated quite thoroughly.

The use of the human parameters $H_1$, $H_2$, $N$ and $L$ and the development of performance criteria involving these parameters represent the preliminary stages of a program for screening trackers. The present method of evaluating these parameters admittedly involves much time and labor. However, if we could devise simpler tests for quantifying these parameters, then the application of the performance criteria would provide a very
useful means of selecting the best trackers from a group.

We should mention that this study probably encompasses the two extremes of any tracking situation which the human operator is likely to encounter. The sine-wave task represents one of the simplest and most easily predictable inputs the operator will be called upon to track. In contrast, the random task, in which prediction of future target positions can be made only as a probability estimate, represents the most difficult input he will be required to track. Nearly every situation the operator may encounter in real life will lie somewhere between these extremes.

B. Autocorrelation Functions.

The use of autocorrelation functions proved to be an invaluable accessory in the analysis of the operator's behavior. However, before autocorrelation functions could be used in our study several important questions had to be answered. The most important question was whether the operator's output was a stationary time series. The autocorrelation functions of the operator's error are, in effect, computed from finite samples taken
from a process which has infinite duration. If we are
to be able to use these finite samples to describe the
original process from which they are taken, that original
process must be stationary. It would appear, based on
our experience, that the operator's output is stationary
over short intervals of time (about one to five minutes).
Hence, we are justified in using the autocorrelation
functions taken during our tracking studies to describe
the operator's behavior.

A second question which must be answered is what
is the maximum length of the autocorrelation record that
we can safely use in our analysis. Since we are using
finite lengths of records and are integrating over
finite period of time, we would certainly expect this to
limit the range of $\tau$ over which our autocorrelation
function are valid. One means of answering this question
is to select several samples of the record to be used
and autocorrelate each of these samples. When normalized
and plotted on the same set of axes, the maximum value
of $\tau$ which can be utilized is that value of $\tau$ at which
a phase shift between these several autocorrelation
functions becomes noticeable. This has been done for
the output signal of the random function generator, and
the autocorrelations of these samples have been plotted in Figure 36. These data suggest that the maximum value of $\gamma$ in this case should be no larger than two seconds. The final decision about the maximum value of $\gamma$ to use in any particular case thus depends upon the quality of the record involved.

A final question which should be answered concerns the confidence limits which can be placed upon the autocorrelation function which has been computed. In other words, the error tolerance at each value of $\gamma$ must be determined. This is a difficult piece of information to calculate and at the present has not been fully determined.

During the time at which these studies were made, only a one channel photographic recorder for the autocorrelator was available. This meant that only one function could be recorded for autocorrelation during any given trial. Since it was felt that the operator's error function yielded more information about his behavior than his response function, the error signal was the only function recorded for autocorrelating. A two channel recorder is now in preparation so that additional data
Figure 36. Determination of the maximum allowable value of $\tau$ by comparing the autocorrelation functions of samples of the same record.
can be recorded on each trial. With this additional information cross-correlation functions can also be obtained which may prove useful in expanding and enlarging our theoretical analysis. One valuable application of cross-correlation functions in an analysis of the tracking task will be to determine empirically the value of $L$, the operator's lag time. If the input function and the operator's response function are cross-correlated, that value of $\tau$ at which the cross-correlation function becomes a maximum may be interpreted as the operator's lag time.

C. Learning

Although this study was an investigation of the steady-state performances of skilled operators and the effects of learning were only of a secondary interest, several autocorrelation functions of the error of Subject 6 were made as he progressed through the initial stages of learning. These records which were made for the random compensatory task with scale factor settings of $K_m = 1$ and $K_d = 1$ are plotted in Figure 37. This subject started out as a completely naive subject, and trial one was his first attempt at a tracking task.
Figure 37. Autocorrelation function of the error of Subject 6 for the random compensatory task with $K_m=1$, $K_d=1$. 
The autocorrelation function of his error in this trial is practically indistinguishable from the autocorrelation function of the random problem signal. By his fifteenth trial a slight difference in his tracking behavior is noted, and by the sixtieth trial a complete change has occurred in his tracking behavior. By the time he has reached this stage of practice, his tracking behavior is relatively stable and will probably show only minor variations from trial to trial. This change in the operator's tracking behavior during the learning period can be attributed to two changes in terms of our analytic theory: (i) a decrease in L, (ii) an increase in H₁ and H₂.

D. Summary

In concluding our discussion of the human operator in a closed-loop system we shall briefly summarize some of the factors relating to the usefulness of our model in describing human tracking behavior. Our model:

a. Simulates the statistical characteristics of the human tracking behavior. In Section IV we demonstrated that our model matched
the mean-square errors of our subjects and to a reasonable extent accounted for the effects of changes of the scale factor settings upon the autocorrelation functions of the operator's error.

b. Defines the human tracking behavior in terms of the human constants $H_1$, $H_2$, $L$ and $N$. These constants are believed to remain stable during short periods of tracking by trained subjects, but to vary slowly with learning. These constants provide one means of classifying individual differences in tracking behavior and can be utilized to establish a performance criterion for screening trackers.

c. Shows the contribution of the human noise to the operator's performance. Our model enables us to split the operator's mean-square error into two components $E_P^2$ and $E_N^2$ and thereby to determine the contribution of the noise to the total mean-square error.
d. Defines the limits of the inherent stability of the operator. We can determine the boundary of the stability region of the operator and ascertain the effects of the scale factors upon the stability of his performance. It is possible that this analytical approach will also be useful in studies of muscular tension, stress, fatigue, etc.

e. Provides one means of accounting for changes in the operator's behavior during the learning period. Speaking in terms of our model, the learning period is the time during which the operator reduces his lag time to a minimum and determines the proper values of $H_1$ and $H_2$ to yield a satisfactory performance.

f. Yields values of constants useful in engineering design. Our model, by providing a mathematical expression for the human transfer function along with the values of the human constants, provides a useful
aid to anyone who has to design control systems in which the human operator must perform a tracking task, especially control systems containing computer or booster elements such as are commonly used in Air Force fire control or aircraft control systems.

There are several points of disagreement between our model and the actual operator. Summarizing and briefly discussing them, they are:

a. The human noise. In our model we assumed that the noise was perfectly random with a flat spectrum. Although this assumption may not be perfectly true, it is sufficiently accurate to yield valid results. There are indications, however, that the human noise may be band-limited. In the case of the sine-wave input, by filtering the problem frequency out of the operator's response, it should be possible to get a clear picture of the operator's noise.

An amplitude analyzer now under construction
would then enable us to gain some knowledge of the first probability distribution of this noise. In the case of the random input, however, no simple separation of the input problem and the operator's noise is possible. The second assumption which was made about the human noise and which may not be exactly true is that the operator's noise and the problem signal are uncorrelated. The actual verification of this assumption would be a difficult undertaking, especially in the case of the random input signal.

b. Nonlinearity. We have assumed that the operator performs as a linear mechanism over the range of scale factors used for a given input. It may be true that he is not linear over the entire range. However, the linear approximation yields acceptable results so that the extra work involved in attempting to determine the nature of the nonlinearity does not seem to be
justified at this stage of the analysis.

c. Discontinuity. As we pointed out earlier, there are many indications that the human operates as a discontinuous mechanism. Before analyzing the tracking situation as a discontinuous system, we will need to know more about the human operator as an information-handling device.

d. Prediction. The operator undoubtedly tends to behave as a predictor. The important question, it seems, is what information does the operator use in attempting to predict the future course of the problem. The answer to this question may necessitate a modification in our model of the operator. In its present form our model can account for prediction in one of two ways: (i) by a decrease in L, or (ii) by an increase in the use of velocity information.

e. Memory. In our model of the operator we have made no provision for a memory. This fact may account for some error in the
model, for the operator undoubtedly uses stored information to help determine his responses.

f. Goodness of performance. It is not known at the present what sort of figure of merit the operator uses in scoring his performance. There are several possibilities, for example he may attempt to reduce his mean-square error, or he may attempt to minimize his average error. Whatever performance criterion he establishes will affect his tracking behavior. The use of special instructions can alter the operator's selection of a performance criterion.

In concluding we might mention one further aspect of our model. At the present we do not know how the operator determines the values of $H_1$ and $H_2$ which he uses or how this learning process can be modified by special instructions, etc. It may be that he performs in some sense as an optimum prediction filter and may adjust $H_1$ and $H_2$ so that he resembles as nearly as possible
the optimum filter required for the particular input used. Once we have determined what statistical information the human extracts from the input, how he utilizes it to act as a predictor and to adjust $H_1$ and $H_2$, we may find that we will be able to modify our present model of the human to obtain a model similar to that shown in Figure 38.
Figure 38. A suggested analog of the human transfer function.
REFERENCES


I, Claude Ellsworth Walston, was born in Barberton, Ohio, June 15, 1926. I received my secondary school education in the public schools of the city of Barberton, Ohio. Enlisting in the U. S. Navy in 1943, I was qualified for the Naval V-12 program and attended Tulane University and the University of South Carolina, receiving the degree Bachelor of Science in Electrical Engineering from the latter university in 1946. Upon release from active duty in 1947 I was employed by the Allis-Chalmers Manufacturing Company. Entering graduate school, I received the degree Master of Science in Electrical Engineering from the University of Wisconsin in 1950. I received an appointment as Research Fellow in the Department of Electrical Engineering of The Ohio State University in 1950. In the succeeding two years I held appointments as Research Assistant and Research Fellow in The Ohio State University Research Foundation.