GROWTH OF ELEMENTARY-SCHOOL TEACHERS IN ARITHMETICAL UNDERSTANDINGS THROUGH IN-SERVICE PROCEDURES

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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CHAPTER I

THE PROBLEM

Of all elementary school subjects, arithmetic appears to be the most hated, the most liked, the most difficult, and the easiest for pupils. Furthermore, this subject results in a large number of pupil failures, yet is the basis for the successful achievements of many students. There must be a reason for the paradoxical conditions which are produced by arithmetic.

Even though arithmetic is both liked and disliked, the unfortunate circumstance is that those persons who dislike arithmetic seem to outnumber those who like it.

Since a strong interdependence exists between successes or failures and attitudes, it is likely that a major source of the difficulties surrounding arithmetic as a school subject is that of attitudes toward it. The attitudes which students demonstrate toward a school subject are the result of a variety of experiences. Furthermore, many of the experiences are not the result of the school's educational program. However, it seems that much of the responsibility for the development of attitudes must be assumed by elementary-school teachers.

A study of the attitudes of 586 college students preparing for teaching at the elementary-school level
revealed the fact that 434 students disliked arithmetic while only 152 liked it. According to Dutton:

Lack of understanding, teaching dissociated from life, pages of word problems, boring drill, poor teaching, lack of interest and poor motivation, and fear of making mistakes were the seven most frequently mentioned reasons for unfavorable attitudes. These seven accounted for 65 per cent of all statements pertaining to students' unfavorable feelings toward arithmetic.¹

A scrutiny of the seven categories revealed by Dutton's study, tends to place the major responsibility for attitudes against arithmetic upon the classroom teachers, since each of the seven categories is indicative of a type of learning situation. Teacher responsibility for attitudes toward arithmetic was given additional emphasis by Dutton when he said:

A few quotations from students' papers will show the quality of feelings expressed.

When it comes to mathematics I have a storehouse of frustrations that few people can equal. . . . The "queen of the sciences" indeed was dressed as Satan himself for me. And who has dressed her thus? Who, but my teachers!

Ever since I can remember mathematics has always been my greatest weakness and my most unpleasant experience.²

²Ibid., pp. 85-87.
It would seem that classroom teachers are largely responsible for unfavorable attitudes toward arithmetic, but what about the favorable attitudes? A teacher who likes arithmetic would surely find it convenient to encourage her pupils to derive a pleasure from the subject, and to see the many interesting things about numbers and their uses in daily life. Furthermore, the teacher would be likely to furnish enrichment materials, thus kindling the enthusiasm of the pupils for the subject. Dutton shared this point of view, and suggested that elementary-school teachers are very much responsible for the favorable attitudes toward arithmetic when he stated:

A good teacher, a challenging experience, and numerous practical applications of arithmetic are highly significant factors in the development of favorable attitudes toward the subject.\(^3\)

The development of attitudes toward arithmetic is apparently in the hands of elementary classroom teachers, and those teachers determine almost entirely whether the attitudes will be favorable or unfavorable. In fact, the influence of a classroom teacher is so great that it is likely to offset an attitude which has been established by the home environment, even though the attitude in the home may be a desirable one. In reporting a study of factors determining attitudes toward arithmetic and

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\(^3\)Ibid., p. 87.\(^3\)
mathematics, Poffenberger and Norton placed strong emphasis upon the teacher factor when they stated:

One girl said "I liked arithmetic at first in grammar school, because my mother helped me and made it interesting. I lost interest about the ninth grade when I had a very poor teacher."

A boy indicated that he had had a series of favorable experiences with arithmetic. His teachers in the lower grades had been good and both his father and his mother had encouraged him in mathematics. He said that he had always liked mathematics and science up to his junior year in high school. He reported, "Mathematics would have been my major until I got into second year algebra. I disliked the teacher and the way he taught. I didn't want to work for him. . . ." He indicated also that both his parents and adviser had encouraged him to take trigonometry but that he had refused because the same teacher taught this subject. 4

Since one teacher may destroy favorable attitudes toward arithmetic and mathematics established over a period of years, it is important that all mathematics teachers perform their teaching duties in a highly satisfactory manner. The performance of the mathematics teacher takes on additional significance when it is realized that the mathematicians and scientists of the future are as modeling clay in the hands of today's teachers. The foundations for the scientific future of the United States are being determined in the classrooms of America.

Obviously, the effectiveness of the teacher's work is strongly dependent upon her understanding of arithmetic, since it is to be assumed that a teacher who does not understand arithmetic cannot hope to convey meanings and understandings to her students. Therefore, the importance of teacher understanding of the basic concepts of arithmetic should be emphasized in programs of teacher education. That the present conditions of teaching understandings in arithmetic leave much to be desired was pointed out by Orleans when he made the following statement:

There are apparently few processes, concepts, or relationships in arithmetic that appear to be understood by a large percentage of teachers.\(^5\)

As further evidence of the lack of understandings on the part of the teachers, and the resultant handicap which is consequently placed upon the improvement of the teaching of arithmetic, Mueller and Moser have suggested:

Some of the obstructions to progress are even now beginning to take shape. One of these is especially important. Reports from consultants working with the in-service training of

teachers indicate that the biggest single barrier to a more effective implementation of meaningful arithmetic is the inadequacy in the mathematical background of the teachers themselves. Teachers cannot do a creditable job teaching that which they neither practice nor understand.⁶

Elementary teachers in service must assume much of the responsibility for improving programs in arithmetic, yet many of these teachers do not understand arithmetic. Arithmetical understandings on the part of teachers are clearly in need of improvement.

Statement of the problem.- It was the purpose of this study: (1) to develop in-service procedures likely to promote growth in arithmetical understandings with groups of elementary teachers; (2) to determine, if possible, the extent to which procedures used were effective; and (3) to recommend procedures for a program of in-service education intended to improve arithmetic teaching in the elementary school.

The importance of the study.- This study is important in terms of existing conditions regarding the teaching of arithmetic, and the fact that the in-service education of

teachers in basic arithmetical understandings has been seriously neglected.

A brief examination of the historical development of numbers should serve as an explanation for some of the conditions found in the arithmetic program of many schools today.

Prior to the sixteenth century, the area of mathematics which is now known as arithmetic was divided into two separate parts. According to Smith:

The Ancient Greeks distinguished between arithmetic, which was the theory of numbers and was therefore even more abstract than geometry, and logistic, which was the art of calculating. These two branches of the study of numbers continued as generally separate subjects until the time of printing, although often with variations in their names; but about the beginning of the 16th century the more aristocratic name of "arithmetic" came to be applied to both disciplines.7

Calculating meant performing the operations with the calculi or pebbles, and the use of concrete objects was implicit in the term; consequently, wide use was made of mechanical devices. Furthermore, most number systems were not positional, and, therefore, did not lend themselves to operations in the abstract. Also, since writing materials had not been refined, each civilization utilized manipulative devices for calculating. These devices were

used extensively in trade and commerce. In most cases the device was an abacus, an instrument on which values were assigned to objects (usually stones) depending upon their relative positions. By use of this amazing device the basic operations of arithmetic were performed exclusively in a concrete form, with very little memorization necessary. Since work was at the concrete level, the operations performed were usually understood.

After the development of better writing materials and the invention of printing, the Hindu-Arabic system of numbers, in which the operations could be performed by the use of symbols, rapidly replaced the mechanical devices. The pressure for efficiency produced many short-cuts of the system, and in its standard form the understandings or basic principles were well hidden. The fact that the operations could be performed with the numbers was the most likely reason for combining arithmetic and logistic.

Refinement of the operations in the positional number system reached such a degree that answers could be derived by merely following a few simple rules. The fact that the rules were learned by rote and had little meaning for the participant was not a matter of great concern. The tendency was toward following the rules and getting the correct answers. This trend resulted in arithmetic becoming a tool subject to be acquired through a program of
drill on the fundamental operations. In spite of the many recent improvements in the teaching of arithmetic through the introduction of meaningful programs, similar conditions exist in many elementary classrooms even today.

In a great number of schools throughout the United States, the stress is upon speed of operations with almost complete disregard for either mathematical understandings or the social usefulness of the subject. Efficiency with a series of routine short-cuts which are completely void of understandings brings top recognition from both the teacher and the peer group, so that students strive for efficiency irrespective of whether or not they understand the why of arithmetic. An unwarranted premium is placed upon the how of the process. The natural consequence is that students resort to blind procedures which they hope will produce the correct answers. Often they do not know what a reasonable answer would be or how to check their own solutions. Children become confused and highly frustrated when teachers ask them to drill on meaningless exercises in order to develop proficiency in a task which seems to have very little usefulness for them. Obviously there is a basis for the fact that of all elementary school subjects, arithmetic has the dubious honor of being chosen most frequently as both the best liked and the most hated subject, and that basis is the uninteresting manner in which arithmetic is taught by teachers who are forced to rely upon
methods of rote learning because they do not possess an understanding of the basic concepts of arithmetic.

The current weakness in mathematics programs at the elementary level can be traced to the manner in which arithmetic is taught in the schools. However, the programs at the elementary level are directly dependent upon the teachers; consequently, the problem is one of teacher understandings in arithmetic. Orleans feels that the present approach to numbers precludes the possibility of meaningful teaching.

It would seem reasonable to conclude that the teaching of arithmetic has become so routinized that whatever understanding is introduced in the learning process is soon lost. Considering the lack of understanding possessed by most teachers, it is doubtful that they can introduce much meaning to the learning process for their pupils.  

A great deal of thought is being given to the basic curricula requirements for teacher education at the high school and college levels for there seems to be a lack of emphasis upon mathematics courses for prospective teachers at these levels. Layton pointed out that most colleges completely ignore mathematics as an entrance requirement for those persons entering upon a program of teacher training.

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8Orleans, op. cit., p. 8.
Only one-fourth of the catalogs studied specify mathematics as a requirement for college entrance for freshmen who enter with a teacher training objective. This seems to indicate a weak foundation for our potential teachers of mathematics in the elementary school as far as admission requirements are concerned. In the four-year curricula, the requirement in content art and also in geography is more than two and one-half times that in mathematics. The mean in English is approximately seven times the mean in mathematical content.⁹

Apparently this unbalanced requirement of high-school subjects for college entrance lacks justification. One likely reason for neglecting mathematics is that it is often taught in such a manner that neither the mathematical meanings nor the social significance is apparent. Consequently, the trend toward a more functional type curriculum has resulted in meager mathematical requirements. This situation presents a serious threat to the establishment of adequate programs of teacher education, and makes possible the unfortunate circumstance of having teachers who are attempting to teach a subject about which they have a small amount of understanding. It would seem that a major factor in the solution of the problem of college students with an inadequate background of mathematics is that of making certain that high schools offer students

a program of mathematics which meets high standards both as to quantity and to quality.

Not only are the programs at the high-school level considered to be inadequate for prospective teachers, but also many college programs fail to prepare their students for the task of teaching arithmetic meaningfully.

An investigation by Grossnickle of 129 teachers' colleges regarding the training of teachers of arithmetic revealed that in three-fourth of these colleges a student did not need credits in mathematics at the secondary level to be admitted. Furthermore, two-thirds of the colleges offering a curriculum for kindergarten and primary grades did not require a course in background mathematics, and more than half of the colleges neglected such a course for teachers preparing for the intermediate and advanced grades. And, in addition, at least one-third of the colleges did not require a course in the teaching of arithmetic.\footnote{Foster E. Grossnickle, "The Training of Teachers of Arithmetic," The Teaching of Arithmetic. Fiftieth Yearbook of the National Society for the Study of Education, Part II, Ch. XI. (Chicago: University of Chicago Press, 1951), pp. 210 and 228.}

The conditions reported by Grossnickle, permit prospective elementary school teachers to graduate from teachers' colleges without a course in mathematics beyond the eighth grade. This deplorable condition is matched only
by the fact that graduates from these colleges are in elementary classrooms, and are being entrusted with the teaching of arithmetic to children.

Unfortunately, a serious lack of understanding of the basic concepts of arithmetic is present not only among teachers, but also in groups at all levels of the educative process. According to Glennon:

Neither the pupils who have completed a study of the computational processes of the first six grades, nor high school students, nor persons preparing to teach arithmetic, nor the teachers who are teaching arithmetic at the present time seem to understand arithmetic.12

This condition indicates a weakness in the teaching of arithmetic in elementary schools, secondary schools, and colleges. However, the programs of instruction at the elementary level are almost completely responsible for the weakness since only a small percentage of the population studies arithmetic beyond the elementary school. In the words of Wilburn and Wingo, "It is a peculiar fact that people usually do not study arithmetic except in the elementary school."13 Therefore, the elementary schools are


responsible for existing conditions in the area of arithmetic, and it would seem that it is at this step of the educational ladder that corrective measures should be taken.

Although the current conditions in the teaching of arithmetic leave much to be desired, it must be assumed that attempts are being made to correct the deficiencies. In fact, the past two decades have witnessed a movement toward the meaningful approach to the teaching of arithmetic. Regarding the modern trends in instruction in arithmetic, Buckingham states:

The modern movement in arithmetic is characterized by such key words as discovery, meaning, understanding, significance, resourcefulness, seeing sense, concepts, relationships, etc. The implication is that children shall learn arithmetic not as an abstract science of numbers but as an avenue of understanding the quantitative in modern society.14

In recent years, many new materials devised to facilitate understandings have been made available. These include a wide assortment of manipulative materials and learning devices; a wealth of audio-visual materials, such as films, charts, and diagrams; colorful and illustrative books and materials which draw heavily upon the semi-concrete level of experiences, particularly at the primary level; well organized curriculum guides; and many free and

inexpensive materials which companies have produced to be placed in the hands of both teachers and children. Furthermore, there have been many desirable changes in the curricula of elementary schools. The programs have been broadened to include many phases of community life, explorations in natural science, and many other activities which would not have been included a few years ago.

Modern practices are based upon the psychological principle that learning is a process involving the whole child in his total surroundings. Consequently, the trend is toward a close integration of the many aspects of the educational program into a logical pattern. The result is a program which utilizes large blocks of time for a unified course of study. Arithmetic has a major role in this new type of curriculum which makes possible the functional use of numbers as a powerful and indispensable means of communication. The structural nature of the number system, though often taught in semi-isolation from other areas of the program, can be developed as a skill to be understood and practiced and, subsequently, to be used. This program of meaningful learning experiences has a tendency to make sense to the youngsters. In addition, never before have so many facilities been made available to assist in the teaching of arithmetic with both meaning and social significance. The manner in which arithmetic is taught rests, of course, with the classroom teachers.
In spite of the excellent facilities and desirable curriculum organization for teaching arithmetic, the subject seems to be poorly taught in many schools. The problem apparently centers around the fact that many teachers do not have a sufficient knowledge of the basic concepts of arithmetic to know how to use materials effectively. Many book companies are aware of this situation and publish comprehensive teachers' editions to accompany their textbooks. Although these supplements may be helpful, their usefulness depends upon their adaptation rather than their adoption. Teacher adaptation of materials presupposes teacher understandings. Materials for teaching arithmetic are as strong as the ability of the teacher to employ such materials effectively. A prerequisite to the process of teaching for understanding is teacher understanding, regardless of the materials used.

Other studies have dealt with the status of teacher understandings and with the effectiveness of specific classroom procedures. However, this study dealt directly with the problem of teacher growth in understandings. Some of the areas in which major contributions were made are these: incidence of teacher understandings, teachers' attitudes toward arithmetic and the measurement of understandings, effectiveness of an in-service course designed to promote growth in understandings, objective treatment
of resultant changes in teachers and the arithmetic programs in their classrooms, and recommended procedures for programs of in-service education in arithmetic.

The present status of the teaching of arithmetic leaves much to be desired. In general, the subject is poorly taught because the teachers are inadequately prepared. Teachers are ill prepared as a result of insufficient pre-service education and almost a complete lack of in-service education. In view of current conditions in the teaching of arithmetic this study seems to be significant.

**Approach to the problem.**—Since children cannot be expected to obtain the basic understandings of numbers from teachers who lack such understandings, a logical place to attack the problem seemed to be at the in-service level. Wilburn and Wingo reinforced this point of view when they made this statement:

One thing which is probably needed is for teacher-training institutions to pay more attention to providing prospective elementary-school teachers with a better understanding of arithmetic and the number system. At the present time very few of them appear to do this, so the responsibility must fall on those responsible for in-service training of teachers.15

An in-service training course with small groups of teachers, utilizing an experiential approach to the basic understandings of arithmetic and aimed at promoting teacher

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growth in arithmetical understandings emerged as a possible means for improving present conditions in the teaching of arithmetic. Such a course would provide not only an introduction to the basic concepts of arithmetic for those teachers with a meager preparation for teaching the subject, but also a review of the basic concepts for those teachers with a better background of pre-service training. This approach would afford an opportunity for working with teachers in service, most of whom would not be returning to teacher education institutions, and would be carried out close to elementary classrooms where children were making initial contacts with the basic foundations for mathematics.

It readily became apparent that procedures for determining a solution to the major problem of teacher growth in arithmetical understandings through in-service procedures presented a natural sub-division into four sequential steps: (1) the contacting of several groups with whom to develop arithmetical understandings; (2) the development of understandings with groups of teachers by means of an in-service course given at the local level; (3) the measurement of teacher growth in understandings and consequently the effectiveness of in-service procedures used with the teachers; and (4) the establishment of certain recommendations relative to desirable procedures to be utilized in a program of in-service education in arithmetic.
The procedures used in initiating and carrying out the in-service program are described in Chapter III. Chapter IV is devoted to a description of the in-service course. The instruments used in measuring the extent of teacher growth and a description of procedures for using them are described in Chapter V, while a list of recommendations for a program of in-service education in arithmetic is included in Chapter VII.

Definition of terms. — Certain terms which are used frequently in the study are defined as follows:

a. Understandings. Arithmetical understandings are defined as those basic insights into the language of numbers resulting from a knowledge of the structural nature of number systems and particularly the Hindu-Arabic number system, and the operations which are the outgrowth of this structure. This definition includes the operational relationships in arithmetic.

b. Meanings. The term meaning has been subdivided into meaning of and meaning for, with meaning for referring to functional use of arithmetic. When the term meaning is used in this study, it has reference to the meaning of arithmetic and is, therefore, synonymous with the term understanding, as both terms refer to basic insights into the language of numbers.
Brownell has defined the meanings of arithmetic as mathematical understandings and has placed them into four categories:

The meanings of arithmetic can be roughly grouped under a number of categories.
1. One group consists of a large list of basic concepts. Here, for example, are the meanings of whole numbers, of common fractions, of decimal fractions, of percent, and most persons would say, of ratio and proportion. . . . Here, too, are the technical terms of arithmetic--addend, divisor, common denominator, and the like. . . .

2. A second group of arithmetical meanings includes understanding of the fundamental operations. Children must know when to add, when to subtract, when to multiply, and when to divide. . . .

3. A third group of meanings is composed of the more important principles, relationships, and generalizations of arithmetic. . . .

4. A fourth group of meanings relates to the understanding of our decimal number system and its use in rationalizing our computational procedures and our algorisms.16

Each of the four groups of meanings listed by Brownell was given some attention in this study, with major emphasis being placed upon the fourth group.

c. "Why" of arithmetic. The why of arithmetic is defined as the structural nature of the system which permits the operations, or answers the question, "by what right are the operations performed?"

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d. "How" of arithmetic. The how of arithmetic has reference to the procedures to be followed in working with numbers or to the actual performing of the operations. It includes work with both objects and symbols.

e. Growth. Growth has reference to those measurable changes in the teachers involved in the study which indicate that their arithmetical understandings have shown some degree of improvement and that this improvement is further manifested in their classroom procedures. The extent of change was determined by the use of an objective test, conferences, observations, questionnaires, evaluative summaries, and check lists.

f. Pre-service. Pre-service education has reference to the programs of teacher education institutions engaged in the preparation of teachers for elementary schools.

g. In-service. The term in-service has reference to on-the-job training of teachers. This training is limited to the area of the development of arithmetical understandings and to the teaching of arithmetic, and is centered in a course designed to promote growth in understandings, and the resultant conferences and classroom visitations.
h. **Procedure.** As used in this study, the term procedure has two meanings: (1) that course of action utilized in developing the basic understandings of arithmetic, in which case reference is generally made to the in-service course in the basic concepts of arithmetic; and (2) the manner in which the study is conducted, which definition is more general and includes all of the four phases of the study. In each case the contextual use of the term is indicative of which meaning is intended.

i. **Elementary.** The term "elementary" has reference to grades one through six.

**Basic assumptions of the study.**—It is assumed that arithmetical understandings on the part of both teachers and children are desirable, and that they are a necessary part of an effective learning situation. The Second Report of the Commission on Post-War Plans of the National Council of Teachers of Mathematics stated the case for the mathematical aim of arithmetic as follows:

... Both skills and ideas should be made sensible to children through their mathematical relationships. This means that children must understand whole numbers, the number system, common fractions, decimal fractions, per cents, units of measure, etc.; that they must understand the functions of basic operations, and that they must understand the rationale of our
methods of computation. To teach well these understandings and the essential skills is to achieve the mathematical aim.\(^{17}\)

A vital part of a good program in arithmetic is that of emphasis upon understandings.

The study was based upon the assumption that arithmetical understandings exist in degrees and are subject to change. The possibility of measuring the understandings held by teachers is assumed. Furthermore, it is assumed that the means for measuring both growth and resultant classroom changes are both valid and reliable.

It is assumed that desirable in-service procedures for developing growth in arithmetical understandings involves an in-service course given at the local level with small groups of teachers.

An objective test, teacher conferences, questionnaires, summaries, classroom visitations, and a teacher opinion check list are assumed to be valid and reliable to promote growth in arithmetical understandings and to determine teacher attitudes and reactions toward arithmetic and the teaching of the subject.

Limitations of the study.- The study was limited in scope to the development of seventy-two arithmetical understandings with groups of elementary teachers, and to

the measurement of the effectiveness of procedures used in developing the understandings. Some attention was given to both materials and procedures for teaching arithmetic; however, major emphasis was placed upon the development of understandings with the expectation that teachers would adapt many of the materials and methods of the course for use in their classrooms. The means for measuring growth in understandings were somewhat limited. Many aspects of learning did not lend themselves completely to objective measurement, thus necessitating the employment of conferences, evaluative summaries, classroom visitations, questionnaires and a teacher opinion check list. A very significant indication of teacher growth seemed to be that of improved procedures in the classroom; yet, it was difficult to get an accurate measure of this aspect of teacher growth. The fact that a relatively short period of time was devoted to the collection of the data served to make even small gains on the inventory quite significant in terms of the long range approach to the problem. In referring to growth in understandings, Glennon remarked, "Mathematical understandings, like all other understandings, grow slowly."\textsuperscript{18}

Undoubtedly, a period of years would be necessary in order to measure some aspects of the study accurately. Some teachers stated that they had not had an opportunity to put to use some of the classroom practices gleaned from the course. Unfortunately, depending upon their grade level, some of the teachers will not have an opportunity to apply the learnings to their own classrooms; however, the learning situations were designed to give them an over-all perspective which each teacher should have regardless of the grade level she teaches.

For many years an apathetic attitude toward the teaching of arithmetic has been demonstrated by persons who are ordinarily concerned about other weaknesses in educational programs. Many parents are highly concerned about a slight reading difficulty on the part of their children, but when confronted with their child's difficulties in arithmetic, they are quick to remark, "So what? I had a hard time with arithmetic when I was in school." And many teachers are eager to remark that they teach a primary grade and should not be expected to know very much about arithmetic.

Fortunately, such apathy toward the teaching of arithmetic is passing from the scene, as suggested by Eads when she said:

Arithmetic is growing in importance at all school levels. This was quite apparent at the April, 1954 meeting of the National Council of
Teachers of Mathematics in Cincinnati. The theme of the meeting was "Mathematics on the March" but arithmetic was given serious consideration by speakers and panelists representing schools, colleges, and industry.\(^{19}\)

The attention being given to arithmetic is a necessity for additional progress in this area of the curriculum, but concern alone will not solve the problem. Research investigations must be made. Regarding such studies Eads added:

Types of research difficult to plan and interpret but very much needed are studies in: concept formation in mathematics, appreciation in mathematics, the place of mathematics in various types of curriculum programs, sequences in learning mathematics.\(^{20}\)

Studies must be made to determine effective means for promoting teacher growth in arithmetical understandings. These studies are essential because the most logical direction for teacher training in arithmetic is toward understanding. As Glennon stated; "Understanding--this is the frontier of needed redirection in the training of teachers of arithmetic."\(^{21}\)

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\(^{20}\)Ibid., p. 13.

\(^{21}\)Glennon, *op. cit.*, p. 396.
Thus, in spite of the limitations of this study, its procedures seem to be justified by its importance as frontier research in in-service education.

**Summary**—Arithmetic is an unusual school subject. It is unusual because of the wide variety of effects it has upon children. These effects are the product of a wide variation of attitudes toward arithmetic, and the attitudes are undoubtedly the result of the many degrees of understanding or in many cases the degrees of misunderstanding which are prevalent. The extent to which children understand arithmetic is in a large measure due to the extent of arithmetical understanding possessed by teachers. Consequently, the unusual characteristics of arithmetic as a school subject seem to be the product of the varying degrees of teacher understandings in arithmetic.

Attempts to teach arithmetic meaningfully are being made in many elementary schools; however, the lack of teacher understandings presents a major problem and is generally considered to be a serious hindrance to the progress of such programs. Seemingly, much could be done to solve the problem by means of pre-service educational programs which would insure the competence of prospective teachers in both understandings and teacher methods; however, the teacher training institutions are falling far short of this goal. The result is that many persons at all steps of the educational ladder are seriously lacking in arithmetical understandings.
A solution to the problem apparently is dependent upon in-service procedures for helping teachers to improve their understanding of the basic concepts of arithmetic.

This study dealt with the in-service education problem of improving teacher understandings in arithmetic, and attempted to find a desirable solution to the problem by utilizing the procedures of an in-service course in arithmetical understandings. The course was conducted at the local level with four groups of elementary teachers, and involved thirty teachers and three principals. These teachers and principals were selected from fourteen groups of teachers from six different school systems.

The in-service procedures were evaluated in terms of their effectiveness in producing teacher growth in arithmetical understandings as measured by an objective test, teacher conferences, a teacher questionnaire, a teacher summary, classroom visitations, and a teacher opinion check list. The results of the in-service course were used as a basis for formulating conclusions and making recommendations for a program of in-service education of teachers in the area of arithmetical understandings.
CHAPTER II

REVIEW OF THE LITERATURE

A review was made of the literature dealing with the teaching of arithmetic and with the in-service education of arithmetic teachers. The review of the literature on the teaching of arithmetic was divided into that prior to the twentieth century, and that during the twentieth century. This seemed to be a logical division since the turn of the century witnessed a revolution in the teaching of arithmetic brought about in part by the progressive movement in education.

The teaching of arithmetic prior to the twentieth century. - The teaching of arithmetic has undergone many changes since the early Egyptian era when the subject was taught by the priests. According to Karpinski:

The pedagogue in Greece was the slave who accompanied the child to and from school. Both in Greece and in Rome elementary instruction including numbers was frequently given by such a slave.¹

Medieval instruction in arithmetic was provided in the church schools, with the major objective of computing

Easter. Karpinski further summarized the early teaching of arithmetic when he said:

In Germany and Holland the Rechenmeister was appointed by the city to act as town clerk, and was given a practical monopoly of the business of instruction in arithmetic. . . . In English grammar schools arithmetic was rarely taught, but appeared in private schools for a separate fee. . . . In colonial America instruction including arithmetic received the early attention of the English and Dutch settlers, while the Spanish were more occupied with the conversion of the natives to Christianity and of gold and silver to Spain. Towards the end of the seventeenth century "to cipher and to cast accounts" appears in the schools of New England, and a little later in the schools of New York and Pennsylvania.²

The methods of instruction used in colonial times were primarily those of teacher-telling pupil-doing (solving examples), and copy books were used extensively. According to Buswell and Judd:

The chief method of instruction in early American schools was the dictation of practical problems by the teacher. These problems were neatly copied by the pupils in so-called "ciphering books."³

The teachers did not attempt to help the pupils to gain an understanding of the processes used. Pride was taken in neat copywork, and the work in arithmetic was in part an exercise in handwriting.

²Ibid., pp. 171 and 173.

The role of educational programs of the twentieth century in American society is likely to be misleading and to generate the conclusion that schools have always assumed a heavy responsibility in the American culture. Such is not the case. In fact, the "common school" in America is a little more than a hundred years old, and prior to the nineteenth century schools existed only in the towns and districts. Furthermore, the colonial schools devoted almost all of their attention to reading, spelling, and writing. In schools where arithmetic was taught, stress was given the perfunctory approach. In the words of Cajori:

And even where schools were kept, the study of mathematics was often not pursued at all, or consisted simply in learning to count and to perform the fundamental operations with numbers.\(^4\)

In the early schools routine drill as an initial approach to arithmetic was standard procedure. The dictatorial methods were, obviously, the natural outgrowth of the belief that education should be a telling, listening reciting process. Also, the scarcity of materials forced the use of oral methods, and much arithmetic became drill, memorization, puzzles, and even rhymes.\(^5\)


\(^5\)Buswell and Judd, *op. cit.*, p. 160.
The rhymes and puzzles were obviously aimed at promoting an interest in arithmetic, and at helping pupils to realize that numbers can be interesting; however, so much stress was placed upon mental exercises with seemingly nonsensical materials that most pupils were unable to grasp the social utility of the subject. For many students, even today, arithmetic consists of memorization, drill, and the juggling of numbers in a meaningless manner. A great deal of value may be derived from puzzles and rhymes when they are used as enrichment materials, and are based upon the interests of the pupils.

In regard to the scarcity of arithmetic books in the colonial period in America, Cajori stated: "Regular arithmetics were a great rarity in this country in the seventeenth century."\(^6\)

Cajori further pointed out that the only arithmetic contained on the famous "horn-book" was the Roman Numerals, and that the first arithmetic textbook to be written in America was that of Greenwood of Harvard College published in 1729. The next book in America devoted entirely to arithmetic was written by Pike in 1788. Then, after the turn of the nineteenth century many arithmetic textbooks

\(^6\)Cajori, *op. cit.*, p. 11.
were published; however, most of these books had very little impact upon the teaching of arithmetic and "enjoyed only a mushroom popularity."\(^7\)

In view of the extreme lack of teaching materials, the responsibility for a program of instruction was directly upon the teacher and the success of the program depended heavily upon her abilities. Nevertheless, the "normal school" had not been established and the standards for teaching were very low. Many teachers were poorly qualified, and unfortunately the profession was viewed as a position for those persons who had been unsuccessful in other occupations. In regard to the lack of qualifications possessed by teachers in the nineteenth century, Cajori remarked:

The representative school-masters of by-gone times were itinerant school-masters. They were mostly foreigners. Their qualifications seemed to be the inability to earn anything in any other way.\(^8\)

The idea that anyone can teach has been a serious drawback to teacher education programs. It is most fortunate that the attitude of the public toward teaching has improved in recent years.

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\(^7\)Ibid., p. 49.

\(^8\)Ibid., p. 52.
Not only were the meager materials used by poorly prepared teachers but the teaching methods were not designed to promote understandings. Cajori reinforced this point of view when he remarked that "The statement of rules took the place of explanations and reasoning."\(^9\)

Unfortunately, many teachers in modern elementary schools are following procedures which are as void of understanding as those used in colonial times. They, too, rely almost entirely upon a statement of the rules instead of helping pupils to understand the basic structure of the number system. Many of the teachers do not realize that the rules are only short-cuts which were invented to facilitate the efficient use of numbers. Consequently, they encourage their pupils to accept the rules instead of seeking an understanding of their mathematical basis.

Cajori described a classroom situation in which the teacher, having stated the rule for addition, viewed with displeasure the work of a pupil who was too confused to perform in the conventional manner. The pupil misinterpreted "add the figures in the first column" and proceeded to solve the example by working from left to

\(^9\)Ibid., p. 52.
right, all of which received strong disapproval from
the teacher. A teacher with a knowledge of the "scratch"
method of addition and an understanding of the base of
the number system could have provided a real learning
experience for the pupil by using the abacus to demon­
strate the meaning of "carrying," and the process of
addition by beginning from either the left or the right;
and by explaining the "scratch" method of addition and
relating the fact that this method was "current in
Europe in the sixteenth century."\(^\text{10}\)

The nineteenth century reforms in arithmetic were
brought about mostly by the work of Colburn who published
his first arithmetic textbooks in 1821. His books were
very popular and used widely in the schools for almost a
half century. In regard to the wide use of Colburn's
books, Buswell and Judd remarked, "It is reported that
Colburn's texts sold in this country in one year to the
extent of more than 100,000 copies. They also circulated
extensively in England."\(^\text{11}\)

Since Colburn had been influenced by Pestalozzi,
his approach to arithmetic was through the use of concrete

\(^{10}\)Francis J. Mueller, *Arithmetic Its Structure and

\(^{11}\)Buswell and Judd, *op. cit.*, p. 162.
objects. The objects were used to introduce the pupils to arithmetic, and work in the abstract always followed the work with manipulative materials. Cajori described Colburn's approach when he stated:

Colburn's First Lessons embodied what was then a new idea among us. . . . The idea was to begin with the concrete and known, instead of the abstract and unknown, and then to proceed gradually and by successive steps to subjects more difficult.\(^2\)

The concrete approach to arithmetic used by Colburn had many aspects of a sound approach to teaching arithmetic. Consequently, it is surprising that the teaching of arithmetic reverted to formalized drill in the second half of the century. Regarding this period, Buswell and Judd made the following statement:

Following the period of Colburn's strong influence there was what one of the historians has described as the "static" period in American arithmetic.\(^3\)

Buswell has suggested that the teaching of arithmetic during the period from 1800 to 1900 did not seek to develop rational understanding.

As revealed by the textbooks of the period, the subject was taught in a very formal manner, the instruction consisting primarily of the memorization of rules followed by their


\(^3\)Buswell and Judd, op. cit., pp. 162-163.
application. There was no attempt to develop a rational understanding on the part of pupils.14

The recommendations of the "Committee of Ten" of the National Council of Education in 1893 initiated several changes in curricular offerings. The committee favored an extension of the curriculum to include work in the practical arts, and emphasized the social usefulness of courses. Members of the committee felt a need for a change in the teaching of arithmetic and indicated the direction in which they felt changes should be made in their report in which they stated:

The conference recommends that the course in arithmetic be at the same time abridged and enriched; abridged by omitting entirely those subjects which perplex and exhaust the pupil without affording any really valuable mental discipline, and enriched by a greater number of exercises in simple calculation and in the solution of concrete problems.15

The report of the "Committee of Fifteen" of the National Council of Education in 1895 included some specific recommendations in regard to the teaching of arithmetic. The committee felt that too much time was being spent on arithmetic and that more time should be devoted to the

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other subjects. It sharply criticized the common practice of using two class periods each day for arithmetic (one for mental arithmetic and the other for written work), and recommended that:

> With the right methods, and a wise use of time in preparing the arithmetic lesson in and out of school, five years are sufficient for the study of mere arithmetic—the five years beginning with the second school year and ending with the close of the sixth year.¹⁶

A brief historical perspective of the teaching of arithmetic prior to the twentieth century pointed out the fact that the subject was taught by priests, slaves, city clerks, and other persons who were often forced to depend upon outside earnings for their financial support. Also, the subject was taught in a manner which was not conducive to an understanding of the structural nature of numbers and very meager materials were used. In the American colonial times the subject was either not taught at all or was taught in a perfunctory manner. Furthermore, the emphasis was upon formal drill and the skills of arithmetic with very little concern for either mathematical understandings or the social usefulness of the subject.

The changes initiated by Colburn seemed to be in the direction of much needed improvement in both procedures

and materials. It would seem that Colburn's work coupled with the works of Mann and Barnard in the area of teacher education should have produced more lasting results in the teaching of arithmetic. Nevertheless, the latter part of the century witnessed the return to the formal drill approach. A significant carry-over from Colburn's reformation, however, was an expansion of arithmetic as a school subject. According to Buswell, "Arithmetic expanded too greatly; in some schools as much as one-half of the total school time was devoted to the subject."17 The overemphasis produced a desirable result in that it set the stage for twentieth century investigative research in arithmetic.

The teaching of arithmetic during the twentieth century.- The widespread interest in the curriculum and particularly in arithmetic which was created during the last decade of the nineteenth century carried over into the twentieth century, and for the first time an attempt was made to measure and to evaluate the teaching of arithmetic. A pioneer in this testing movement was Rice whose tests were first given in 1902. According to Brown and Coffman, Rice was attempting to answer three questions:

What results should be accomplished whenever a subject is incorporated in the school program? How much time shall be devoted to the branch?

17Buswell, op. cit., p. 447.
Why do some schools succeed in securing satisfactory results with a reasonable appropriation of time, while others cannot get reasonable results with a satisfactory appropriation of time?\(^{18}\)

Rice gave his achievement tests to 6,000 pupils in the fourth to eighth grades, and reached the final conclusion that, "the supervisor is the controlling factor determining differences in achievement in arithmetic."\(^{19}\) Rice also concluded that there is no direct relation between time spent on arithmetic and the results achieved, and that teaching methods are not the controlling element in producing results.\(^{20}\)

The second contribution to the testing movement in arithmetic was that of Stone who developed a series of achievement tests for sixth grade pupils, and who was interested in the notion of setting up standards for comparing the achievement of groups of pupils. In regard to the purpose of Stone's tests, Buswell and Judd remarked:

The purpose of the Stone test, like that of the Rice tests, was to make possible a comparison of the achievements of different school systems.\(^{21}\)

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\(^{19}\)Ibid., p. 20.

\(^{20}\)Ibid., pp. 19-20.

\(^{21}\)Buswell and Judd, *op. cit.*, pp. 37-38.
The first attempt at standardizing arithmetic tests was made by Courtis in 1909. In addition to constructing a series of tests:

Courtis also collected results on a large scale from widely scattered school systems in the different states and consciously set about the task of establishing standards through the averaging of the scores thus secured.  

During the second decade of the twentieth century, work was done in the refinement of arithmetic tests, and in further standardization. These efforts pointed out some of the weaknesses of instructional programs. For example, one of Courtis' conclusions was that of the poor promotion practices used in the schools. However, it is likely that the emphasis upon test scores had the effect of directing instructional programs toward the objective of performing the fundamental operations, and tended to produce a serious neglect for the meanings of arithmetic.

The premium upon speed in performing the operations, regardless of understandings or functional use, which is found in many elementary schools today is, obviously, a carry-over from the instructional programs of the early part of the century. It is only within the past two decades that serious attempts have been made to develop tests which would measure the understandings of arithmetic.

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22 Ibid., p. 39.
The strong movement in curriculum organization and its functional approach to learning, during the early part of the century seemed to maintain a balance with the testing movement and undoubtedly prevented an overemphasis upon the testing of isolated skills. Brownell described the first quarter of the twentieth century as the revolution in arithmetic, and stated clearly that a functional program must make provision for meaningful learning:

It was demonstrated that teaching for transfer could yield large amounts of transfer and thus could reduce considerably the necessity of mastering isolated elements one by one. Too, it was shown that memorization, when introduced prematurely, blocked sound learning; or, stated differently, that children could be expected to profit greatly from instruction which emphasized understanding, prior to repetitive practice. From all this research and from experimentally oriented teaching emerged the notion that one ingredient in a functional program in arithmetic is provision for meaningful learning.23

Thorndike described the new methods in arithmetic in contrast with the old when he said:

The older methods taught arithmetic for arithmetic's sake, regardless of the needs of life. The newer methods emphasize the processes which life will require and the problems which life will offer.24

The functional approach to the teaching of arithmetic led to increased emphasis upon understandings and meaningful


learnings. However, this movement was gradual and was accompanied by work in the area of arithmetic readiness.

"The Committee of Seven" of the Northern Illinois Conference on Supervision did research to determine at what mental age a child was most likely to succeed in mastering certain topics in arithmetic. A procedure used by the committee was that of determining the success with which pupils were able to handle certain topics in arithmetic at the level at which the topic is usually taught, and at the levels immediately above and below that which it is usually taught. The committee arrived at the following conclusion:

There is a point in a child's mental growth before which it is not effective to teach a given process in arithmetic, and after which that process can be taught reasonably effectively.²⁵

Additional evidence resulting from the work of the committee suggested that for most effective learning, many topics of arithmetic should be placed one or two grade levels above the level at which it is usually taught.

The work in the area of pupil readiness for learning arithmetic seemed to create an interest in the preparation of teachers or teacher readiness for handling arithmetic. A study by Robinson contributed some pertinent information

regarding the professional preparation of elementary teachers in the area of arithmetic. One of the purposes of Robinson's study was that of determining to what extent teachers at the elementary-school level are handicapped by a lack of knowledge of the basic principles of arithmetic, and what effect the teacher's knowledge of arithmetic, or the lack of knowledge of the subject, has on the pupils she teaches. Furthermore, his study dealt with the extent to which teachers of arithmetic have a knowledge of teaching techniques and their application to classroom situations. Robinson was also interested in how well the courses offered at teacher training institutions are preparing prospective elementary-school teachers for teaching arithmetic. 26

The catalogues of teacher training institutions were studied by Robinson and questionnaires were used to determine both the training and experience of teachers of the professional courses. The content of the professional courses was studied by an examination of the descriptions of the courses and the classroom activities in the courses. Teacher examination scores in arithmetic were analyzed, and examinations were given to teachers in service. Furthermore, many visits were made to elementary classrooms, and

detailed accounts of these observations, together with the results of conferences with the teachers, provided information regarding teachers in service.

As stated by Robinson, a significant conclusion of the study was:

Elementary school teachers have at best only a mechanical knowledge of arithmetic. . . . Anything like a clear and versatile knowledge of the fundamental principles of arithmetic and their mathematical significance is all but totally lacking on the part of such teachers. Because of this fact, it is difficult to see how the arithmetic they teach or would be called upon to teach to elementary school children could be anything more than mechanical.27

In further summarizing his study, Robinson expressed this point of view:

After all, the ultimate individuals to suffer or profit in any scheme of professional education for elementary teachers are the children in the public schools. It is believed that too many professional courses in arithmetic now offered are merely completing a vicious circle by which arithmetic in the elementary school is doomed to a mechanical existence and is therefore stripped of its peculiar power as a factor in the fundamental education of public school children.28

Members of the National Council of Teachers of Mathematics gave a great deal of attention to arithmetic and its place in the school curriculum and reported their

27Ibid., p. 180.
28Ibid., p. 185.
work in two yearbooks, *The Teaching of Arithmetic* in 1935, and *Arithmetic in General Education* in 1941.

Immediately following World War II, the board of directors of the National Council of Teachers of Mathematics created a commission on post-war plans. In its second report, the commission made eight specific recommendations for teaching arithmetic in grades one through six. One of the recommendations was that more emphasis must be given to the development of meanings. By meanings was meant both the mathematical and the social meanings, and it was pointed out that neither of these had been receiving proper attention in the classrooms.

The commonly used procedure of "following the rules" was criticized, and members of the commission felt that more attention should be given to the development of those understandings for which the rules were short-cuts. The point was made that understandings are brought about through a series of guided experiences rather than by the telling-listening process.

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It is the function of the teacher to provide an abundance of relevant experiences and to assist the child to isolate the critical elements and to build them into the desired understandings.\textsuperscript{31}

In 1946 Sueltz developed a battery of tests for the intermediate level, grades 4, 5, and 6; and for the upper level, grades 7, 8, and 9. A specific test was designed for each of the three areas: quantitative understanding, problem solving, and basic computations. These tests were administered to selected groups of pupils. For each grade level, 100 students of normal ability were selected. Many of the children were interviewed both during and after the tests, and the conclusion was reached that the tests, coupled with the interviews and discussions that followed, were a valuable means of measuring understandings. Sueltz listed ten generalizations based on the results of his experiment. Four of these generalizations which have a direct bearing on teaching arithmetic meaningfully are:

1. Children need a rich background of experience in learning arithmetic. This experience should be of two types: (a) experience with objects and real things of life, as suggested by Pestalozzi and Colburn, and (b) experience with the mathematical interrelationships of numbers and processes.

2. Schoolrooms should have at least a minimum of mathematical laboratory equipment, such as a meter, standard measures, various containers, geometric forms, etc. This equipment should be handled as well as seen by the pupils.

3. A new kind of education for teachers is needed— not more courses in psychology or more college mathematics in the usual sense. The new course might well be called "mathematical background for teachers." It should provide a point of view strengthened by much information and understanding.

4. School programs too frequently set proficiency in computation as their major aim of instruction in arithmetic and give little attention to basic understanding, meanings, judgments, etc.32

Sueltz's generalizations specifically point out the need for arithmetical understandings on the part of both students and teachers and the need for a strong program of professional training at both the pre-service and in-service levels. Furthermore, emphasis was placed upon the need for working with manipulative materials which are within the experiences of the pupils, and which will have a carry-over into social situations.

In 1947, Glennon inventoried the arithmetical understandings of both students in teachers colleges and teachers in service. According to Glennon:

It was the purpose of the study to determine the extent of growth and mastery of certain basic mathematical understandings possessed by three groups of persons: teachers college freshmen, teachers college seniors, and teachers in service.33

Glennon had difficulty locating a suitable instrument for testing understandings and consequently developed, as a part of his study, an eighty item test of basic mathematical understandings. Glennon found that:

Within the scope of this study there is a strong tendency for the understandings that are difficult on any one level to be difficult for persons on all levels; and for the understandings that are easy on any one level to be easy for persons on all levels.34

It was a conclusion of Glennon's study that significant growth in arithmetical understandings was not being accomplished by persons at any step of the educational ladder from the seventh grade students to teachers with several years of teaching experience.

In 1951, Rodney conducted a study the purpose of which was: "To evaluate the pre-service preparation for

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teaching mathematics in elementary schools as offered at the State College for Teachers at Buffalo.\textsuperscript{35}

One aspect of Rodney's study was that of developing a forty-item test of basic arithmetical understandings and administering the test to 151 freshmen and to 162 seniors at the State College for Teachers at Buffalo. He found that:

The teachers college freshmen who cooperated in this program have a knowledge of about half the mathematical understandings tested. . . . The findings reveal that teachers college seniors have achieved a knowledge of slightly over fifty per cent of these understandings.\textsuperscript{36}

Thus Rodney's study revealed evidence of the shortcomings of one teacher training institution in the area of preparing teachers to teach arithmetic with understanding. In summarizing his study, Rodney stated:

The findings of the several phases of this investigation indicate that teachers college students have not attained a high degree of competence in any of the three aspects of mathematics. Therefore, in view of the function of teachers colleges, a program for the professional preparation of teachers should be organized to alleviate, to the greatest extent possible, these apparent deficiencies.\textsuperscript{37}


\textsuperscript{36}\textit{Ibid.}, p. 105.

\textsuperscript{37}\textit{Ibid.}, p. 108.
An endorsement for a program of in-service education was made when Rodney said, "An in-service program for the study of the meanings of the content of the elementary mathematics should contribute to an improvement of teaching elementary-school mathematics."\(^3^8\)

A study by Orleans in 1952 involved 722 subjects in five categories: (1) undergraduates who were completing a methods course in the teaching of arithmetic; (2) undergraduates who were doing their elementary school student teaching; (3) students in graduate courses in education in two colleges; (4) classroom teachers; and (5) other persons. The purpose of the study was:

To ascertain the extent to which teachers and prospective teachers of arithmetic understand the processes and concepts which are represented by the short cuts they teach.\(^3^9\)

Orleans developed two tests and some supplementary questions designed to measure arithmetical understandings. One test utilized free-answer responses; however, the inability of teachers to verbalize their answers resulted in the development of the second test which was of the multiple-choice type. One of the conclusions of his study was:

\(^{3^8}\)Ibid., p. 107.

\(^{3^9}\)Ibid., p. 38.
 Being prepared as a teacher of arithmetic, and even having experience in teaching the subject does not seem to go with an understanding of what is to be taught, or with a much greater understanding of it than is possessed by other than professionally trained teachers.\(^4\)

In 1953, Hartsell did a study to determine:

What program of mathematics and the teaching of arithmetic in the teacher-training institutions of Alabama can be made effective in producing competent teachers for the elementary schools of Alabama?\(^4\)

After doing an analysis of the teacher training programs, Hartsell recommended five changes which should be made to strengthen the programs. Two of these recommended changes were:

Prospective teachers for elementary schools should have a background course in arithmetic and arithmetical knowledge. The prospective elementary teachers should also have a course in the teaching of arithmetic to acquaint them with the meaningful approach in teaching arithmetic.\(^4\)

Hartsell's recommendations for both a background course in arithmetic and a methods course in the teaching of the subject focus attention upon the nationally recognized weakness of inadequate preparation for teaching

\(^4\)Ibid., p. 38.


\(^4\)Ibid., p. 83.
arithmetic of prospective elementary teachers by the educational institutions which propose to be preparing teachers for the elementary classrooms of the nation.

Strong emphasis was placed upon the place of college courses in the preparation of arithmetic teachers by Morton when he said, "The college mathematics experience of prospective teachers of arithmetic should be secured through the medium of courses especially organized for this purpose." 43

The responsibility of the teacher training institutions for preparing prospective teachers to teach arithmetic meaningfully was pointed out by Mayor who stated:

Through the year's sequence the prospective arithmetic teacher must obtain clear understanding of the basic mathematical concepts so that he will be able to lead children to think meaningfully and independently in situations involving numbers. 44

Points of view similar to those held by Hartsell and Mayor regarding the inadequate programs of teacher training institutions in the pre-service preparation of arithmetic teachers and the resulting conditions of arithmetic teaching in elementary classrooms have been expressed by


Grossnickle,45 Weaver,46 Layton,47 Newsom,48 Schaaf,49 Phillips,50 and Boyer.51 Undoubtedly, reference was being 
made to these reports by Dyer, Kalin, and Lord when they 
suggested that:

There have been numerous studies of teacher 
competence in arithmetic. If they can be believed, 
it seems pretty clear that many elementary school 
teachers have a hard time keeping even half a 
jump ahead of their pupils.52


46J. Fred Weaver, "Teacher Education in Arithmetic," 
Review of Educational Research, XXI (October, 1951), 
317-20.

47Layton, op. cit., pp. 551-556.

48C. V. Newsom, "Mathematical Background Needed by 
Teachers of Arithmetic," The Teaching of Arithmetic, 
Fiftieth Yearbook, Part II (Chicago: National Society for 

49W. L. Schaaf, "Arithmetic for Arithmetic Teachers," 
School Science and Mathematics, LIII (October, 1953), 
537-43.

50C. Phillips, "Background and Mathematical Achieve­
ment of Elementary Education Students in Arithmetic for 
Teachers," School Science and Mathematics, LIII (January, 
1953), 48-52.

51L. E. Boyer, "Preparation of Elementary Arithmetic 
Teachers," Emerging Practices in Mathematics Education, 

52Henry S. Dyer, Robert Kalin, and Frederic M. 
Lord, Problems in Mathematical Education (Princeton, 
In a recent article, Weaver revived the problem of teacher education and referred to it as a crucial problem. He pointed out that:

Two things stand out clearly from the conditions discussed thus far: (1) the general level of arithmetic understanding on the part of undergraduates in representative teacher-training institutions is inexcusably low; (2) all too few teacher-training programs provide appropriate required work in background mathematics that could be definitely helpful in raising the undergraduates' level of arithmetic scholarship.53

A review of the literature on the teaching of arithmetic points out that much verbal allegiance has been given to a more functional approach to the subject; however, very little action has been taken. Emphasis has been placed upon the development of tests to measure skill in performing the fundamental operations. Also, the work in the area of teaching for understandings and meanings has been composed mostly of talking about what should be done. Teachers have continued to teach for the performance of skills upon which their pupils will be tested. Nevertheless, the movement toward the meaningful approach to the teaching of arithmetic has gained momentum in the twelve years since 1945, and it is now the accepted approach to

the teaching of arithmetic. This fact was emphasized by Brownell and Moser when they concluded that:

Almost without exception students of arithmetic are now agreed that the subject must be understood, that learning which is not grounded in understanding is pseudo learning, not worth the time and effort required to achieve it.¹⁵

In-service education of arithmetic teachers.- The inadequacy of pre-service programs of teacher training institutions coupled with the need for on-the-job training would seemingly furnish the stimulus for studies in the in-service education of elementary teachers in the area of arithmetic. However, such is not the case, and unfortunately very few studies dealing with programs of in-service education have been conducted.

Glennon recognized the importance of both in-service and pre-service teacher education in the area of arithmetical understandings when he stated:

At the present time little attention is given by the teacher and the supervisor to the need for measuring the degree to which the learner is growing in those outcomes which are the controls of behavior in number situations—outcomes such as the understanding and meanings that are inherent in the number system, the attitude toward number situations as they appear in life and in school, and appreciation of the uses of arithmetic. The teacher and the supervisor will seek out and use more adequate methods and devices for measuring the broader outcomes of instruction in arithmetic only after they have accepted these.

broader outcomes, these controls of behavior, as worth-while aims in arithmetic. Bringing about this acceptance on the part of the teacher is an important in-service and pre-service problem in teacher education.55

Having conducted a study concerned with the problem of developing teacher concern for meaningful arithmetic, Boylan concluded that:

In-service education and supervision face a great responsibility. Before meaningful teaching of arithmetic can be achieved, the attitudes of teachers who have been primarily concerned with mechanics must undergo a radical change. Concepts of learning and a general understanding of children are basic to reorientation. Knowledge of fundamental number ideas must be developed.56

Regarding the place of in-service education, Barr pointed out that there is no substitute for programs of in-service education and for the contributions which such programs make to the growth and development of teachers when he stated:

Increasing standards of pre-service education do not necessarily lessen the need for continued in-service education. With changing conditions, only continued study and growth in service will provide teachers sufficiently up-to-date to cope with the task at hand.57


Despite the fact that there is a paucity of studies relating to the in-service education of teachers in arithmetic, some work has been done particularly in the area of cooperative studies which are commonly known as action research.

A four-year study was made in the Chicago public schools from 1945 to 1959. A major result was the publication *Arithmetic Teaching Techniques*. "This was action research in that more than 1300 teachers and administrators participated in the development of the publication."58 As suggested by the title of the publication, the purpose of the study was to define the difficulties encountered in teaching arithmetic, and to determine what techniques were being used by teachers to cope with the situations in the schools. "Proof of the need for this study is found in the fact that reports of difficulties were received from teachers in more than three hundred elementary schools."59 Obviously, many values both tangible and intangible are to be derived from a long range project involving such a large number of teachers.

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59 Ibid., p. 76.
Another in-service co-operative study was begun in 1945 in the New York City public schools. Regarding this study, Eads reports:

During the past 10 years meaningful arithmetic has become the generally accepted method of teaching and learning in the elementary schools of New York City. This means that some 20,000 supervisors and teachers are now making varying efforts to help children think mathematically and to understand the mathematics they learn. It means also that these supervisors and teachers are learning or have learned the meaning of arithmetic themselves.60

One of the techniques used in the study was to get the supervisors, teachers, pupils, and parents so thoroughly involved in the development of the program that all of them experienced growth in understandings as they learned together. Their learning experiences centered around the development, evaluation, and use of concrete materials and the publication of curriculum bulletins and courses of study. In describing the co-operative evaluations during the study Eads points out that:

As the research proceeded evaluations and curriculum adaptations were made co-operatively by teachers and supervisors. Sometimes these were "on-the-spot" adaptations by children and teachers in the classroom. Parents too were involved. Engineer fathers devised teaching aids. Mothers suggested experience situations and sent concrete materials—beads, buttons, felt, etc. They helped their children make

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60 Laura K. Eads, "Ten Years of Meaningful Arithmetic in New York City," The Arithmetic Teacher, II (December, 1955), 142.
purchases and measurements and keep records of these.61

The New York study emphasized the fact that growth in arithmetical understandings is a gradual process, requiring time, energy, and much patience. "It wasn't easy for teachers to learn how long it takes for understanding to be developed."62

A co-operative in-service study in arithmetic was conducted by the Central New York School Study Council during the school year 1949-50. The objectives of the study, as reported by Lonsdale, were:

The study had definite and limited goals: (1) to explore the place of mathematical meanings in the teaching and learning of arithmetic and (2) to develop some teaching techniques through which pupils could be led to gain essential mathematical understandings and some test items through which pupils' mastery of these understandings could be evaluated.63

A technique of the study was to involve directly teachers, principals, supervisors, and both general and special consultants in a program which utilized committees

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61 Ibid., p. 144.
62 Ibid., p. 144.
at the local and district levels. A feature of the program which added professional status and undoubtedly contributed to the success of the study involved the scheduling of meetings during school hours, thus helping teachers to feel that the work of the Council was a part of their schoolwork and that they were not carrying an extra burden in order to do the work of the study. A second desirable feature was that expenses were paid out of Council funds, which included the travel expenses incurred by those persons who participated in the study. 64

Of the several results of the study, one which is pertinent to the growth of teachers in arithmetical understandings was suggested by Norem when she made the following statement:

"Teachers and supervisors themselves are understanding arithmetic processes for the first time and consequently are more interested in teaching them." 65

Other values derived from the study included; a more flexible program of arithmetic instruction; increased

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64 Ibid., p. ix.

cooperation between supervisors, principals, and teachers; teachers were able to discuss their common problems; and an awareness on the part of the teachers of the need for individualized instruction in arithmetic.

The Central New York School Study Council's Committee on Flexibility had been working for the three years, 1946-49. Consequently, the organizational machinery was already in operation, thus permitting more rapid progress in one year than otherwise might have been possible. Nevertheless, it was the feeling that "obviously we have just begun to scratch the surface."⁶⁶

It seems that the development of a meaningful program in arithmetic requires a long range program of planning, participation, and evaluation. Furthermore, the resultant learnings tend to follow a spiral pattern.

**Summary.** A review of the literature reveals many variations in the teaching of arithmetic from programs of education which neglect arithmetic to those programs which seem to place too much emphasis upon the subject; from programs which use few materials to programs which utilize a wealth of materials; from programs of rote learning to

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programs placing emphasis upon the meaningful approach; and from programs dependent upon slaves, town clerks and teachers without professional preparation to programs guided by professionally trained teachers. Points of emphasis have shifted from a mechanical approach to teaching to a meaningful approach which takes into consideration the organismic theory of learning.

A major conclusion which may be drawn is that present day teachers of arithmetic are not adequately prepared to teach the subject and as a group display a high degree of incompetence for teaching the basic understandings of arithmetic to elementary school children.

While other studies, particularly those of the past decade gave some attention to arithmetical understandings, much of the emphasis was upon the status of teacher understandings. A basic assumption of the action research studies seems to be that teachers will naturally grow in understandings through the process of developing these understandings with pupils. Although much can be said in favor of this "learning together" approach, it would seem to be in the best interests of both effective and economical learnings to have the teacher bring to the learning situation a better background of understandings. This point of view is clearly reinforced by Lonsdale when he said:
It became clear as the year progressed that the work of the Committee was having the greatest impact in those schools where definite provision was made for an organized in-service education program focused upon the need for a more meaningful approach to the teaching of arithmetic.\footnote{Lonsdale, op. cit., pp. ix-x.}

Obviously, more attention needs to be devoted to programs of in-service education for teachers, with very strong emphasis upon growth in arithmetical understandings. This study attempted to find a solution to the problem of teacher growth in arithmetical understandings by means of in-service procedures which are not as long-range as those of the co-operative research studies.
CHAPTER III

PROCEDURES USED IN THE STUDY

The procedures used in this study were: (1) contacting the schools, (2) selecting the schools, (3) contacting groups of teachers, (4) selecting groups of teachers for the administration of a test of basic mathematical understandings, (5) administering the test, (6) enrolling teachers and principals in an in-service course, (7) conducting the course with a pilot group and with three other groups, and (8) evaluating the effectiveness of the in-service course.

The desire to participate in a program of in-service education should emanate from the teachers themselves; however, in the area of arithmetic it seems that many teachers are either unaware of their in-service needs or prefer to overlook their needs for fear of exposing their inadequacies. Therefore, consideration was given to the problem of arousing an interest in the basic understandings of arithmetic. It was decided that a technique for motivating the teachers should involve the use of a test of basic mathematical understandings, and that a part of the experiment would be determination of the value of using such a test for the purpose of motivation.
The Glennon test of basic mathematical understandings was selected for the initial phase of the study. This test was developed by Glennon in 1947. It is an eighty-item multiple-choice test which was designed to inventory five aspects of basic mathematical understandings: (1) the decimal system of notation, (2) integers and processes, (3) fractions and processes, (4) decimals and processes, and (5) the rationale of computation. Since the test has been copyrighted but has not been published, it was necessary to obtain Glennon's written permission to duplicate and to use the test as an instrument in the study. The test was selected for three reasons: (1) it is non-computational and designed to measure understandings rather than performance; (2) its contents are highly appropriate for the elementary teachers; and (3) the time required to administer the test is within reasonable limits.

**Contacting the schools.**- Initial contact with the schools was made by a conference with the superintendent of each school system. During the conference, the experiment was described, and both phases of the projected program were explained. A request was made to discuss the study with the principals and eventually with the teachers, and in each case the superintendent pleasantly gave his approval. The superintendents of three school systems were consulted, and the result was the administration of the test to four groups of teachers in April of 1956. In September
of 1956, three other school systems were approached in a similar manner, and the result was the administration of the test to ten groups of teachers during the fall of 1956.

Selecting the schools. - A basic assumption of the study is that desirable in-service procedures involve working with the teachers in small groups at the local level. In order to carry out the work with small groups, it was necessary to select several schools for the study. A sampling technique was used to increase the probability that the teachers in the study would be representative of elementary teachers as a group.

Four considerations in selecting the schools for the study were: (1) the level of the educational program of the school system, (2) the in-service educational practices in the school, as evidenced by the emphasis upon consultant services, (3) the size of the school, and (4) the type of community in which the school was located, with reference to the socio-economic level. Table I gives information pertaining to each of the six school systems in terms of these four criteria, and also indicates which school systems were a part of the second phase of the study. Table I also shows that the four schools eventually selected for the in-service course constituted a sampling of the schools in terms of the established criteria.
<table>
<thead>
<tr>
<th>School System</th>
<th>Level of Educational Program</th>
<th>Type of Community</th>
<th>Consultant in Arithmetic</th>
<th>Consultant in Education</th>
<th>In-Service Education Programs</th>
<th>Number of Elementary Teachers</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Participated in second phase of study</td>
<td>Low socio-economic level</td>
<td>No</td>
<td>No</td>
<td>67</td>
<td>6</td>
</tr>
<tr>
<td>B</td>
<td>Participated in second phase of study</td>
<td>Low socio-economic level</td>
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<td>No</td>
<td>91</td>
<td>6</td>
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<td>C</td>
<td>Low socio-economic level</td>
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<td>No</td>
<td>63</td>
<td>10</td>
<td></td>
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<tr>
<td>D</td>
<td>Low socio-economic level</td>
<td>Yes</td>
<td>No</td>
<td>111</td>
<td>12</td>
<td></td>
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<tr>
<td>E</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>26</td>
<td>26</td>
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<tr>
<td>F</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>17</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

Table I
Contacting the teachers.- In every case, contacts with the teachers were made through the principal. Two of the seventeen principals interviewed refused permission for the discussion of the study with their teachers; in one of these schools the teachers were involved in another study which was taking a great deal of their time, and the other had an almost completely new staff with problems of human relations yet to be solved. The other fifteen principals either discussed the first phase of the study (the test) with teachers or made arrangements for a group conference in which the experiment was explained and the teachers were invited to participate in the test. The point was made clear that participation in the first phase of the study would not obligate the teachers to participate in the second phase. Several groups of teachers were reluctant to cooperate until it was explained that the test results would be kept strictly confidential. This promise of confidential treatment of scores put most teachers at ease.

Participation in either the Glennon test or the in-service course was placed on a completely voluntary basis, and the percentage of teachers in each school who took the Glennon test varied from a high of ninety-three per cent in one school to a low of forty-four per cent in two schools. The average for all of the schools was sixty-six per cent teacher participation. Only one of the fifteen groups of
teachers refused to take the Glennon test, and gave the reason that they were just too busy with other things.

Selecting the teachers.—The study was not limited to a particular grade level so teachers at all levels, one through six inclusive, were invited to take part. Kindergarten teachers were not discouraged from participation, and eight kindergarten teachers were included in the first phase of the study.

Since participation was placed on a voluntary basis, it was impossible to insure that all grade levels would be represented, or that there would be a desirable distribution of grade levels; however, it seemed reasonable to assume that each grade would be represented. This assumption was borne out by the fact that volunteers included eight kindergarten teachers, twenty-two first grade teachers, thirteen second grade teachers, twenty-two third grade teachers, eighteen fourth grade teachers, fourteen fifth grade teachers, and twenty-two sixth grade teachers. Table II describes the teachers who participated in the first phase of the study.

Administering the test.—The Glennon test was given to 116 teachers and to four principals at fourteen different sittings, in order to maintain the small group approach from the beginning, and also to retain proximity to the elementary classrooms where many of the values to be
<table>
<thead>
<tr>
<th>Grade</th>
<th>Number of Teachers</th>
<th>Average years of teaching experience</th>
<th>Average score on mental arithmetic tests</th>
<th>Average number of college mathematics courses</th>
<th>Average number of high school mathematics courses</th>
<th>Other</th>
<th>Masters Plus</th>
<th>Masters</th>
<th>Bachelors</th>
<th>No degree</th>
<th>Higher Education</th>
<th>Degrees</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kindergarten</td>
<td>3</td>
<td>4.13</td>
<td>12.13</td>
<td>2.13</td>
<td>3.13</td>
<td>0.03</td>
<td>0.25</td>
<td>0.38</td>
<td>3.17</td>
<td>1.00</td>
<td>University school of education</td>
<td>Liberal Arts</td>
</tr>
<tr>
<td>First Grade</td>
<td>5</td>
<td>4.07</td>
<td>12.07</td>
<td>2.07</td>
<td>3.07</td>
<td>0.07</td>
<td>0.24</td>
<td>0.36</td>
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<td>1.05</td>
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<td>Masters</td>
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<td>Second Grade</td>
<td>12</td>
<td>4.96</td>
<td>11.96</td>
<td>2.96</td>
<td>3.96</td>
<td>0.06</td>
<td>0.24</td>
<td>0.34</td>
<td>2.94</td>
<td>1.00</td>
<td>College of education</td>
<td>Masters Plus</td>
</tr>
<tr>
<td>Third Grade</td>
<td>19</td>
<td>4.88</td>
<td>11.88</td>
<td>2.88</td>
<td>3.88</td>
<td>0.05</td>
<td>0.22</td>
<td>0.32</td>
<td>2.86</td>
<td>1.00</td>
<td>High school of education</td>
<td>Teachers</td>
</tr>
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<td>19</td>
<td>4.79</td>
<td>11.79</td>
<td>2.79</td>
<td>3.79</td>
<td>0.04</td>
<td>0.21</td>
<td>0.30</td>
<td>2.77</td>
<td>1.00</td>
<td>High school of education</td>
<td>Teachers</td>
</tr>
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<td>Fifth Grade</td>
<td>15</td>
<td>4.70</td>
<td>11.70</td>
<td>2.70</td>
<td>3.70</td>
<td>0.03</td>
<td>0.20</td>
<td>0.29</td>
<td>2.70</td>
<td>1.00</td>
<td>High school of education</td>
<td>Teachers</td>
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<td>Sixth Grade</td>
<td>15</td>
<td>4.61</td>
<td>11.61</td>
<td>2.61</td>
<td>3.61</td>
<td>0.02</td>
<td>0.19</td>
<td>0.28</td>
<td>2.60</td>
<td>1.00</td>
<td>High school of education</td>
<td>Teachers</td>
</tr>
<tr>
<td>TOTAL</td>
<td>76</td>
<td>4.60</td>
<td>11.60</td>
<td>2.60</td>
<td>3.60</td>
<td>0.02</td>
<td>0.19</td>
<td>0.28</td>
<td>2.60</td>
<td>1.00</td>
<td>High school of education</td>
<td>Teachers</td>
</tr>
</tbody>
</table>

**Table II**

DATA ON TEACHERS WHO PARTICIPATED IN THE STUDY
derived from the group work could be applied. The test was administered by the same person each time, in order to maintain standard testing conditions. Each teacher was allowed sixty-three minutes for the test, and this period proved adequate for most teachers. In twelve of the fourteen groups, the teachers were from the same school within the system and the test was given at their school. The two remaining groups, one of which participated in the in-service course, were composed of teachers from more than one school within the system, and the test was given at a centrally located building.

The administration of the test resulted in some informative data both about the attitudes of teachers toward an inventory of mathematical understandings and about the incidence of understandings among teachers at the various grade levels. The test served the purpose not only of determining to what extent such a test would motivate teachers, but also of taking an inventory of teacher understandings of arithmetic.

Many teachers commented that they found the test interesting and stimulating, and at the same time it gave them the feeling of self-evaluation. The test of basic mathematical understandings was considered to be highly satisfactory for the purposes for which it was used, and provided an excellent introduction to the study.
Enrollment of teachers in the course. - It was decided that from the fourteen groups of teachers who participated in the first phase of the study four groups would be selected for the in-service course, and that one of the four groups would be used for a pilot course to be completed prior to the beginning of the work with the three remaining groups.

Four groups were utilized in order to provide a sampling of groups of teachers from different school systems, helping to determine to some extent the reliability of the in-service procedures. Work with the pilot group served as a guide for the standardization of procedures. However, the variations between the pilot group and the three other groups were so small that all four groups were considered together in analyzing the results of the study.

As a preliminary step to the second phase of the study, groups of teachers were invited to participate in the in-service course to be held at their school. In one of the school systems, the assistant superintendent suggested that instead of working with the teachers from only one school building, he would prefer to form a group representing more than one individual school. This plan was approved as it would provide some variation in in-service procedures and thereby form a basis for comparing this procedure with that used in the other three school systems.
where each group was made up of the teachers from only one building. The result was a large group, seventeen teachers and two principals, which made an excellent basis for comparison.

The seventeen teachers and two principals represented seven different elementary schools as follows: school A, one teacher; school B, one teacher; school C, one teacher and the principal; school F, three teachers; and school G, five teachers.

The number of teachers from the other three school systems were: school One, six teachers and their principal; school Two, four teachers; and school Three, three teachers. Table III describes the teachers who participated in the in-service course.

The selection of groups of teachers for the course depended upon several factors such as the teachers' schedules, in-service policies of the school department, the degree of interest shown by both administrators and teachers, and appropriateness of times for meetings. It became evident that the selection would have to be made from the teachers who could and would participate, instead of from those who could benefit most from the study. Small groups of teachers were then invited to participate in the small group work until the total of thirty teachers and three principals had volunteered.
<table>
<thead>
<tr>
<th>Grade</th>
<th>Average Number of Teachers</th>
<th>Average Number of Teachers with Bachelor's Degree</th>
<th>Average Number of Teachers with Master's Degree</th>
<th>Average Number of Teachers with Masters Plus Degree</th>
<th>Average Number of Teachers with Liberal Arts Liberal Arts College Background</th>
<th>Average Number of Teachers with University School of Education Background</th>
<th>Average Number of Teachers with Teachers College Background</th>
<th>Average Number of Teachers with No Degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sixth</td>
<td>71.00</td>
<td>20.35</td>
<td>19.33</td>
<td>13.72</td>
<td>11.37</td>
<td>9.71</td>
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<td>7.00</td>
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<tr>
<td>Fifth</td>
<td>42.00</td>
<td>10.75</td>
<td>9.57</td>
<td>6.33</td>
<td>5.70</td>
<td>4.73</td>
<td>3.20</td>
<td>3.33</td>
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<tr>
<td>Fourth</td>
<td>3.20</td>
<td>1.25</td>
<td>1.00</td>
<td>0.67</td>
<td>0.67</td>
<td>0.67</td>
<td>0.67</td>
<td>0.67</td>
</tr>
<tr>
<td>Third</td>
<td>5.20</td>
<td>1.25</td>
<td>1.00</td>
<td>0.67</td>
<td>0.67</td>
<td>0.67</td>
<td>0.67</td>
<td>0.67</td>
</tr>
<tr>
<td>Second</td>
<td>1.25</td>
<td>1.25</td>
<td>1.00</td>
<td>0.67</td>
<td>0.67</td>
<td>0.67</td>
<td>0.67</td>
<td>0.67</td>
</tr>
<tr>
<td>First</td>
<td>3.33</td>
<td>3.33</td>
<td>3.33</td>
<td>3.33</td>
<td>3.33</td>
<td>3.33</td>
<td>3.33</td>
<td>3.33</td>
</tr>
</tbody>
</table>

**Data on Teachers Who Participated in the Early Education Course**

**Table III**
Since the desired distribution was one group of teachers from each of four different school systems, as soon as one group volunteered from a system other groups in the same system were not invited. In three school systems the first group to be invited enrolled in the course. In a fourth system, two groups of teachers refused to participate in the in-service course, and gave as reasons a lack of time and other commitments. It was felt that the in-service policies of the school system may have had a bearing on the teachers' decisions. In the fifth school system, the first group to be invited enrolled in the course. In the sixth school system, no one was invited to participate since the quota of four groups had been reached.

Conducting the course. - During the in-service course, eight meetings were held in each of the four schools. In two schools weekly meetings were arranged, and in the other two the meetings were held semi-weekly. In every case the meetings were devoted to comprehensive treatment of the structural nature of arithmetic. In two schools, arrangements were made for in-service credit for those teachers who participated in the small group sessions; in a third school, in-service credit was eventually awarded though no definite decision was reached before the course was completed; and in the fourth school, in-service credit was not
given. During the eight-session course attention was given to seventy-two arithmetical understandings. Each of the eight sessions is described in Chapter IV by a statement of the major objective of the session, a list of the understandings developed, and paragraph descriptions of the procedures used with the teachers. A list of the seventy-two understandings is included in the Appendix.

**Evaluative procedures.** The procedures for evaluating the effectiveness of the in-service course are described in Chapter V which is devoted to the evaluative instruments and their uses.

**Summary.** The procedures used in the study involved making the initial contacts with the schools, meeting with groups of teachers to acquaint them with the study and to solicit their participation, administering the test of basic mathematical understandings, conducting the in-service course with groups of teachers, and evaluating the effectiveness of the procedures used in the course. The basic procedure of working with small groups at the local level was maintained throughout the study.
CHAPTER IV

DESCRIPTION OF IN-SERVICE COURSE

I. INTRODUCTION

A major assumption of the study is that desirable in-service procedures for correcting teacher deficiencies in arithmetic should have as their nucleus a course designed to promote arithmetical understandings. Basic criteria for such a course are as follows:

1. It should be conducted at the local level, that is, within the school system.

2. The course should devote at least eight one-hour sessions to the development of the understandings of the structural nature of number systems, and particularly to that of the Hindu-Arabic number system. Consideration should be given to number relationships and to the rationalization of the four fundamental operations with whole numbers and with both common and decimal fractions.

3. It should make use of a variety of materials helpful in teaching the understandings of arithmetic. The materials should include manipulative objects, diagrams, charts, guide sheets, the chalkboard, sheets of scratch paper, and books and pamphlets.

4. The course should provide an interesting approach to arithmetical understandings, and should operate at a level which is well within the range of comprehension of the teachers for whom the subject is new, yet should not lag for the teachers for whom it is a review.
5. It should utilize the procedures of demonstration, explanation, participation, and discussion, at a professional level.

6. The general pattern of the course should be established prior to work with the pilot group. However, the work with the pilot group should provide a means for evaluating the course, and the changes which would seem to be improvements should be included in the work with the other three groups.

7. The course should cut across the grade levels from one through six, and should place emphasis upon arithmetical understandings instead of grade placement.

8. The course should present a standard approach to arithmetical understandings for each group of teachers.

9. The course should be limited in scope to arithmetical understandings, and very little emphasis should be given the many aspects of methods of teaching arithmetic.

10. No formal home-work assignments should be made. However, the teachers should be encouraged to pursue further their interests in the work of the course by discussions with other teachers and by application in their work with children in the classrooms.

The ten basic criteria for the general organization of the course, as well as the arithmetical understandings developed in the course reflect, to a large extent, experiences with an undergraduate methods course in arithmetic, with college extension workshops in arithmetic, with summer arithmetic workshops designed to help teachers in service, and with an in-service arithmetic seminar with the staff of a small elementary school.
The course followed the same pattern for each of the four groups with the greatest amount of variation being used with the pilot group. The time for each session with the pilot group was one and one-half hours in order to meet a class-time requirement for in-service credit. The increased length of the session was not feasible with the other three groups either because they had to meet after school or because the lack of in-service credit for the course served to discourage participation in a session of more than one hour in length. The additional time with the pilot group was spent in discussion of methods for teaching arithmetic and placed more emphasis on arithmetic games and other enrichment materials. It was clearly evident from the work with the pilot group that the one-hour session would not be enough time to include more than a treatment of arithmetical understandings. Consequently, the one-hour sessions were devoted almost exclusively to the development of understandings.

Each of the eight sessions is treated separately in three steps: (1) a statement of the major objective of the session; (2) a listing of the specific understandings which were developed; and (3) paragraph descriptions of procedures used to develop understandings. The descriptions are supplemented with appropriate illustrations. Samples of the guide sheets used in the sessions are a part of the
appendix. Duplicate guide sheets for the first and second sessions are included in the descriptions of the two sessions, in order that they may be more illustrative of procedures to be used with teachers. During the eight sessions, seventy-two understandings of arithmetic were developed.

Table IV lists the materials used in the course, according to the session or sessions in which the materials were used. Most of the materials are illustrated with the charts and guide sheets. Brief descriptions of other materials are as follows:

1. The wooden beads were of two sizes; one-fourth inch in diameter, and one-half inch in diameter.

2. The cardboard discs were one-half inch in diameter.

3. The cards were three-inch squares cut from large sheets of plain white cardboard.

4. Each flannel board was a large piece of flannel stapled to a two-foot square piece of three-fourths inch plywood.

5. The Winston fractional parts were made by the John C. Winston Company, and are sold under the name of "enlarged fractional parts with cohere-o-graph."

6. The abaci and the ten-tens counting frame were made from strips of lumber for the frames, eleven-inch bicycle spokes for the wires, and one-fourth inch wooden beads.

7. The box of place value cans was a wooden box made of light materials with discarded orange-juice cans screwed to the bottom, and with aluminum plates to hold the cards.
<table>
<thead>
<tr>
<th>Materials</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<tr>
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<td></td>
<td></td>
<td>x</td>
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<td></td>
<td></td>
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<tr>
<td>Cardboard discs</td>
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<td>x</td>
<td>x</td>
<td></td>
<td>x</td>
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<td></td>
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<td>Guide sheets</td>
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<td>x</td>
<td></td>
<td>x</td>
<td>x</td>
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<tr>
<td>Chalkboard</td>
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<td>x</td>
<td></td>
<td>x</td>
<td>x</td>
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<td>Hogben's book &quot;The Wonderful World of Mathematics&quot;</td>
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<td></td>
<td></td>
<td></td>
<td></td>
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<td>Booklet &quot;Story of Figures&quot;</td>
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<td></td>
<td></td>
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<tr>
<td>Booklet &quot;Abacus to Monroe&quot;</td>
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<td></td>
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<td>Booklet &quot;The Amazing Story of Measurement&quot;</td>
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<td></td>
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<td>Tongue depressors</td>
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<tr>
<td>Box of place value cans</td>
<td>x</td>
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<tr>
<td>Five bead abacus</td>
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<tr>
<td>Nine bead abacus</td>
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<tr>
<td>Ten bead abacus</td>
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<td></td>
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<tr>
<td>Twenty bead abacus</td>
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<td>x</td>
<td>x</td>
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<tr>
<td>Bundles of cards</td>
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<td>Chart showing different bases</td>
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<td>Chart for Number Game</td>
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<tr>
<td>Ten-tens counting frame</td>
<td>x</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Paper and pencil abacus</td>
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<td></td>
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<td></td>
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<tr>
<td>Chart of addition-subtraction facts</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Chart of multiplication-division facts</td>
<td>x</td>
<td></td>
<td></td>
<td>x</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Set of Napier's bones</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Flannel board</td>
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<td>x</td>
<td>x</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Winston's fractional parts</td>
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<td>x</td>
<td>x</td>
<td></td>
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<td></td>
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<tr>
<td>Decimal chart</td>
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</table>
II. RECORD KEEPING AND NUMBERS

Major objective.- The major objective of the first session was to develop understanding about the early methods of record keeping and the early number systems which man invented.

Understandings.- The following understandings were given consideration in the first session.

1. Early methods of record keeping involved the use of marks on the ground, notched trees, notched sticks, knotted cords, and other objects, thus utilizing the concept of one-to-one correspondence.

2. Systems for record keeping were the product of man's inventive genius.

3. Counting is a process of ordering a group in such a manner as to determine both "how many" and "which one."

4. The "cardinal" and "ordinal" characteristics of number were early refinements.

5. Many different number systems have been used in the past 5,000 years.

6. Most early number systems used five, ten, or twenty as a base, and invented a new symbol to represent this number.

7. Number names, number symbols, and compounding plans were the products of man's inventive genius, and were arbitrarily determined.

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The term "compounding" refers to a grouping plan whereby previously used names, objects, or symbols may be joined or compounded to represent a number without the use of completely new names, objects, or symbols.
8. Early systems of notation utilized the basic principles of repetition of symbols, and the addition of assigned values.

9. The principles of multiplication and subtraction were used in early number systems.

10. The ability to use a number system effectively is dependent upon a knowledge of both the symbols and the compounding principles used.

Description of procedures.—The teachers were asked to imagine a civilization in which man knew neither the process of counting nor a system of numbers, yet had a desire to keep a record of his belongings. The story was then told of the sheep herders in ancient times who kept a bag of pebbles to record, not the number of their sheep, but whether or not they had all of them. Marbles instead of pebbles were used to dramatize the story. Then, illustrations from Hogben's book,2 The Wonderful World of Mathematics, were used to focus attention upon the early use of notched trees, notched sticks, and knotted cords as devices for keeping records.

Wooden beads and cardboard discs were distributed to members of the group, and the teachers were asked to use the objects to set up a one-to-one correspondence with each person in the room. Then the question was asked, "How many objects are in your group, and therefore how many persons in class?" Of course, the teachers knew the

answer to "How many?" They were then reminded that the reason they knew the answer was because they had advanced much beyond the concept of one-to-one correspondence, and that primitive man most likely could not have answered the question other than by saying, "This many," with reference to his model group of objects. In this way the use of model groups was pointed out as the next stage in record keeping. The size of the model group apparently created an interest in the "how many" and from this interest man most likely invented words to describe the model groups. Such words as few, many, flock, hive, crowd, and school (of fish) surely resulted from this interest. Words such as pair, head, hands, and feet most likely resulted from this same source. The use of the names of parts of the human body and of animals for number names was emphasized. The teachers were asked to use the beads and discs in forming other model groups for objects in the room such as chairs, tables, and erasers. It was pointed out that there are many situations today where the one-to-one correspondence concept is used, as demonstrated by the examples of taking the class roll, and keeping the score of a ball game on the playground.

Emphasis was placed upon the fact that man invented a name for each of the model groups, and that this led to counting, or ordering a group of objects into a sequence of model groups. By this comparatively advanced system,
men could communicate and answer the question "how many?" by stating the name for their model group. The word cardinal is used today to indicate a reference to the group or to the total value of the group, and man has found it convenient to use a slight variation of the group names when he desires to communicate about only one object of the group. Thus in a group of fifteen chairs, it is possible to talk about groups of chairs or about one of the chairs. When one chair is the center of attention, words such as third (the three-chair), seventh (the seven-chair), or tenth (the ten-chair) are used. Stress was placed upon the social uses of both cardinal (how many) numbers and ordinal (which one) numbers in numbering streets, houses, rooms, and many other objects.

The teachers were reminded that counting does not depend upon a system of notation, but upon a system of naming groups, and that number names obviously were invented long before the advent of notational systems. However, it was pointed out that man usually dealt with only small groups, and was able to name or to count them; otherwise he used a large group word such as "many." Note was taken of the fact that in some civilizations today the people have invented a surprisingly small amount of number names, and that there seems to be a direct relationship between a civilization's cultural advancement and its
number system. Each teacher was presented with a copy of the booklet, "The Story of Figures," along with the booklets, "From Abacus to Monroe" and "The Amazing Story of Measurement." The teachers were encouraged to read the booklets at their convenience.

The teachers were then asked to practice counting different sized groups of beads and cardboard discs. Teachers were asked one at a time to count small groups, and when the counting passed the number twelve the question was raised as to what relationship exists between number names beyond twelve and those below twelve. One of the teachers pointed out that a specific system was used for naming numbers above twelve, that is, thirteen (three-ten), fourteen (four-ten), fifteen (five-ten) and sixteen (six-ten). This procedure served to re-emphasize the fact that the counting of large groups followed and was dependent upon a compounding system.

At this point some attention was given to the many inconsistencies in the naming of the number groups. It was suggested that the teens not only have unusual spelling, but

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also are in reversed form from the naming of the numbers in the higher decades. It was also noticed that other spelling inconsistencies exist, such as twenty (two-ty), thirty (three-ty), forty (four-ty), and fifty (five-ty), and that with the use of a pattern consistent with positional notation, ten itself would have been named onety (one-ty), thus re-using one combined with ty (the letters symbolizing base) instead of inventing a completely new name. One of the teachers pointed out that the spelling changes were most likely in the interest of easier pronunciation, rather than the result of the haphazard nature of the system.

The teachers were then asked to use the beads and discs to invent a compounding plan for dealing with model groups. They were interested in this procedure, and some struggled with the notion of grouping, while others immediately used ten as a base and represented the ten by substituting a different sized bead or by mixing the use of beads and discs. Emphasis was placed upon the fact that the selection of a base was highly arbitrary, and the teachers were asked to invent a means for representing a model group by using a base other than ten. Systems using five, twelve, and twenty were invented by the teachers. They saw that these systems did not use place value, but object value because a different object was used for the grouping plan.
Attention was then directed to the Egyptian number system, and a guide sheet was distributed showing the Egyptian symbols and offering some practice in using the symbols for notation. Emphasis was placed upon the fact that both the Greeks and the Egyptians used ten as a compounding point, and that whereas the Greeks used only the principle of addition, the Egyptians used the principles of both addition and repetition. This point alone suggests a wide time difference in the development of the two systems, since a system using the principle of repetition has a great deal in common with tallying and one-to-one correspondence. To avoid this simple principle undoubtedly was considered as a great cultural advancement in symbolic notation, even though it was at the expense of many additional symbols.

Attention was then turned to the Roman system of notation which used not only the principles of repetition and addition but also a limited use of positional value and the principle of multiplication. A guide sheet was distributed which illustrated the Roman symbols and also gave the teachers some practice in both writing and reading Roman numerals.

The number system of the Mayas of Central America was described, which primarily used twenty as a compounding point and which used only three symbols: a dot for one,
Egyptian Number Symbols:

one - I  
ten thousand - \[\]  
ten -  
hundred thousand - \[\]  
hundred - \(\)  
million - \[\]  
thousand - \(\)  

Write Egyptian Symbols for:

two -  
seven -  
twelve -  
eighteen -  
twenty-one -  
twenty-three -  
one hundred twenty-five -  
four thousand -  
one hundred thousand -  
one million four thousand three hundred sixty-four -  
Roman Numerals System of Notation:

- one - I
- five - V
- ten - X
- fifty - L
- one hundred - C
- five hundred - D
- one thousand - M

Write the following Hindu-Arabic numbers in Roman Numerals:

- 3 - I
- 14 - XIV
- 26 - XXVI
- 48 - XLVIII
- 91 - XCI
- 106 - CVI
- 520 - DCCCLXX
- 839 - DCCCXXXIX
- 949 - CMXLIX
- 3,487 - MMMCDLXXXVII
- 8,785 - MMMMMMMDCCLXXV

Write the following Roman Numerals in Hindu-Arabic Numbers:

- XXI - 21
- CCCLXII - 362
- XLIV - 44
- MMDLXXXII - 2588
- XCVI - 96
a horizontal mark for five, and an egg-shaped symbol for twenty. This system used the principles of repetition, addition, and multiplication.

**Summary.**- The first session placed emphasis upon the long struggle of man in inventing and further developing methods of record keeping, counting, and systems of notation. Also, the fact that certain basic principles were in operation in each number system, and that the effective use of a system pre-supposes a knowledge of both the symbols and the compounding principles. It was suggested that even in tallying, a compounding principle is generally used, and stress was placed upon the futility of a system which would attempt to create the necessary number of new symbols in lieu of a compounding plan.

As a closing remark, the teachers were asked to give some consideration to the special characteristics of the Hindu-Arabic number system, and to decide for themselves what gives the system superiority over the other systems.

Thus, session one placed strong emphasis upon the development of early methods of record keeping, and the evolution of systems of notation.
III. POSITIONAL NOTATION AND NUMBER SYSTEMS

Major objective.—The major objective of the second session was to stress positional notation, the arbitrary assignment of a base for a number system, and the special characteristics of the Hindu-Arabic system of notation.

Understandings.—The following understandings were given consideration in the second session.

1. Size value is in contrast to place value.

2. The advantages of positional notation or place value extend a number system beyond the base without the invention of new symbols.

3. The abacus is a calculating device which utilizes place value or positional notation, and it is possible to represent consecutive numbers on the abacus, the limit being determined by the number of rods or places.

4. Any of several different bases may be used for a number system utilizing place value.

5. Most bases for a number system have both advantages and disadvantages.

6. The duodecimal system of numbers has many advantages.

7. The Hindu-Arabic system does not use the principles of repetition, but utilizes the principles of addition and place value, which can be demonstrated on the abacus.

8. The Hindu-Arabic number system includes these desirable characteristics:
   a. symbols, including the zero.
   b. a base.
   c. place value, or positional notation.
   d. the principle of addition.
Description of procedures.- Tongue depressors were used to demonstrate separate units, bundles of tens, and a bundle of ten-tens. The size value was emphasized by holding the bundles in different positions in relation to each other and to the units, and by asking the question, "How many are there now?" The answer was obvious in each case, since value was determined by the size and not by the position of the bundles. Then, each teacher was given several tongue depressors and asked to show certain numbers by grouping the tongue depressors. Strong emphasis was placed upon the fact that as long as the bundles are retained as tools the determining factor is size value and not place value, or if a different object is used to represent a bundle the determining factor is the value assigned to the object, and place value is still lacking. It was suggested that the concept of place value or positional notation was obviously beyond the comprehension of the best minds for hundreds of years.

The concept of place value was likely first introduced in the form of a calculator known as an abacus. An early form of this device involved parallel grooves in the sand with pebbles placed in the grooves to represent model groups. In order for the teachers to experience a type of abacus, guide sheets with lines having both unassigned and assigned values on them were given the teachers along with
Fig. 3. - DEMONSTRATION OF SIZE VALUE

How many?  How many?  How many?  How many?
the small cardboard discs which were used in the first session. The teachers were asked to show consecutive numbers by placing the discs on the sheets of paper. When ten was reached, it became necessary to move over one line from the units line to the tens line. One sheet had four lines of unassigned value, and the other sheet had four sets of lines and demonstrated that the direction of place value from right to left for whole numbers was arbitrarily decided, and that the direction could just as easily have been from left to right.

The teachers saw that place value was being utilized because the same disc might be used on any of the lines, and the sole determinant of the value of the disc was its position and the value assigned to the line on which it was located. Therefore, a number could be represented by placing discs in such positions that the values when added together would give the desired number. Teachers worked in teams, with one person asking another to demonstrate a number by placing discs on the lines.

A brief period of practice with the discs served to illustrate the basic concept of positional notation, and the advantages of this scheme over that of size value or shape value. It was suggested that the evolution of the abacus likely involved a shift from grooves in the open sand to grooves in a sand box, and eventually beads or other
Fig. 4

Number 3

GUIDE SHEET - SECOND SESSION
<table>
<thead>
<tr>
<th>Number 4</th>
<th>GUIDE SHEET - SECOND SESSION</th>
</tr>
</thead>
<tbody>
<tr>
<td>ones</td>
<td>tens</td>
</tr>
<tr>
<td>ones</td>
<td>fives</td>
</tr>
<tr>
<td>hundreds</td>
<td>hundreds</td>
</tr>
<tr>
<td>five-hundreds</td>
<td>five-hundreds</td>
</tr>
</tbody>
</table>
objects on wires or strings. Also, abaci were commonly used as counting and calculating devices by merchants who frequently had a flat surface in their stores with a "built-in" abacus, and this area became known as the counter. Hence the counter in stores today is the flat surface over which the business transactions are completed.

Abaci were distributed to the teachers in order that each teacher could practice representing numbers on an abacus. Five-bead abaci, nine-bead abaci, ten-bead abaci, and twenty-bead abaci were given to the teachers. The abaci were collected at the end of the session, but later in the course a ten-tens counting frame was presented as a gift to each teacher to be retained as a part of her classroom materials for teaching arithmetic. The abaci were used by the teachers to illustrate the representation of numbers by the placement of objects or groups of objects. Emphasis was placed upon the fact that consecutive numbers could be shown on the abacus.

Each teacher was then given a bundle of small cards with a Hindu-Arabic symbol on each card. The symbols 1, 2, 3, 4, 5, 6, 7, 8, and 9 were on the cards. The "0" card was purposely withdrawn from the bundle. The teachers were asked to use the cards to represent certain numbers. Most of the teachers saw that the zero card was needed to complete the system and to make possible the representation
Fig. 6. - FIVE-BEAD ABACUS
FIG. 7. - NINE-BEAD ABACUS
of any number. The point was made that such limitations did not occur on the abacus because "zero" was represented by a vacant rod. Consequently, the extensive use of the abacus prior to the introduction of writing materials, coupled with the fact that the early number systems invented a new symbol when the compounding point was reached, undoubtedly accounted for the fact that the first actual recorded use of the zero was not until 876 A.D. The teachers were given the "0" card to finish the set, then guide sheets were passed out, each sheet with four parallel lines which extended almost the length of the paper.

In order to initiate the transition from place value with objects to place value with symbols, the teachers were asked to represent numbers by placing the cards with the symbols on a line of designated value. This procedure emphasized the convenience of symbolic notation, because instead of placing a group of objects in a position, one symbol was used to represent the group. Then, instead of using the cards, the teachers were asked to represent numbers by writing the symbols on the lines. Thus the extreme convenience of positional notation was demonstrated. Also, it was obvious that each symbol including "0" is used as a place holder in the Hindu-Arabic system, and that the major function of each symbol is to designate "how many" for each denominate position.
The question was raised as to why most of the early number systems used a base of five, ten, or twenty. Many of the teachers obviously did not know the reason for the selection of these bases; however, some of the teachers gave the answer that it was because of five fingers on a hand, ten fingers on both hands, and twenty fingers and toes. Several teachers seemed surprised to learn of such a practical basis for the early number systems.

A box of place value cans was used, with positions for the placement of a card in front of each can. Numbers of objects were first represented by the placement of tongue depressors (not in bundles) in the value-designated cans. Then cards with symbols on them were placed in front of each can to represent symbolically the number of objects which had been represented with the tongue depressors. The cards with symbols were replaced with other cards which indicated base, base times base, and base times base times base, demonstrating the fact that the value symbolized by each place may be expressed in terms of powers of the base.

Then the teachers participated in oral practice in which a number was represented by the tongue depressors, and a base was designated for the system. The teachers were asked to determine the number of objects being represented by the tongue depressors.
Fig. 10. - BOX OF PLACE VALUE CANS
A chart demonstrating different bases was the subject of a discussion to establish further the fact that any of several bases could be used for a number system. The chart illustrated the fact that a small base has the advantage of requiring only a few different symbols, but has the disadvantage of requiring the more frequent use of these symbols. However, the opposite is true when a large base is used. That is, a large base has the advantage of only a few symbols being needed to represent a number of objects, but has the disadvantage of requiring additional symbols. The conclusion was reached that the ability to use number symbols to communicate by means of positional notation is dependent upon the assigned base for the system and upon the operation of the principle of addition. It was also concluded that the number of different symbols required for a strictly positional system of notation is the same as the size of the base. A system using base five would require five symbols (including zero), a system using twelve as a base would require twelve symbols, and a system based on twenty would require twenty symbols.

Attention was called to the duodecimal system, and to its advantage over the decimal system of having the additional factors of three, four, and six, which greatly increase facility in dealing with common fractions. The binary system was described and emphasis was placed upon
Fig. 11.- CHART SHOWING SYMBOLS USED IN DIFFERENT BASES

FOR THIRTY-SIX OBJECTS:

<table>
<thead>
<tr>
<th>Base Ten</th>
<th>Base Five</th>
<th>Base Eight</th>
</tr>
</thead>
<tbody>
<tr>
<td>36</td>
<td>121</td>
<td>44</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Base Twelve</th>
<th>Base Fifteen</th>
<th>Base Twenty</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>26</td>
<td>12</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Base Twenty-five</th>
<th>Base Thirty</th>
<th>Base Thirty-six</th>
</tr>
</thead>
<tbody>
<tr>
<td>1x</td>
<td>16</td>
<td>10</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Base Two</th>
<th>Base Three</th>
<th>Base Four</th>
</tr>
</thead>
<tbody>
<tr>
<td>100,100</td>
<td>1,100</td>
<td>210</td>
</tr>
</tbody>
</table>
its advantage of only two symbols and its disadvantage of requiring many places to represent a relatively small number of objects. The use of the binary system as a basis for modern calculating machines was pointed out along with the fact that the extensive re-use of the two symbols is not a disadvantage since the machines are electrically controlled.

A discussion centered around the necessary characteristics of a good system of number notation. It was decided that the first requirement is a set of arbitrarily determined, but commonly agreed upon, symbols which include a symbol for zero. After the selection of symbols, a base must be selected. Then, positional notation must be utilized to eliminate the need for additional new symbols and to make it possible to write any number regardless of size. The fourth desirable characteristic is the use of the principle of addition; that is, the total value of a number symbol is equal to the sum of its place values. It was agreed that the Hindu-Arabic number system alone has all of the four desirable characteristics, and that for this reason it is far superior to many other systems of number notation.

The teachers were urged to give further consideration to the basic understandings of arithmetic treated in session two, and to use them where appropriate in their classroom teaching.
Summary. - In session two consideration was given to size value as contrasted to place value, the importance of the abacus as a counting and calculating device utilizing place value, the significance of different bases for number systems, and those special characteristics which enhance the value of a number system and which are a part of the Hindu-Arabic number system.

IV. WHOLE NUMBERS AND THE GROUPING PROCESS

Major objective. - The major objective in session three was to give further consideration to place value, whole numbers, and the processes of addition and subtraction.

Understandings. - The following understandings were developed in the third session.

1. Numbers may be read in different ways depending upon the amount of transformation.  

2. Addition is a process of combining two groups which may or may not be the same size to form a larger group. However, the groups to be added must have the same "last name," or the quality of likeness.

3. Subtraction is a separation process in which one group is separated into two smaller groups, one of the smaller groups is known, and the second group is to be determined. Groups to be subtracted must have the same "last name," or the quality of likeness.

The term "transformation" refers to a change in the units in which a number is expressed. For example, the number thirty-four (three tens and four ones) may be transformed to thirty-four ones.
4. The related facts of addition and subtraction may be easily demonstrated on the ten-tens counting frame.

5. The related facts of addition and subtraction may be easily demonstrated with a chart.

6. The operations of addition and subtraction of whole numbers may be rationalized on the abacus.

7. Whole numbers may be added in any order. This is commonly known as the commutative law for addition.

8. Whole numbers to be added may be grouped in several different ways without changing the final answer. This is commonly known as the associative law for addition.

Description of procedures.- The session was introduced with a number game which demonstrated some of the "magic" nature of numbers. A chart with a six-square arrangement of numbers from one to sixty-three was shown and the teachers were asked to select a number between one and sixty-three. Then certain teachers were selected to play the game one at a time by answering the question, "Is the number you have selected in this square?" for each of the six squares of numbers. After answering each question by either yes or no, the teacher was told the number she had selected. After a brief period of practice with the game, the explanation of its "magic" nature was given. Its basis is the same as that used for the duplation method of multiplication; namely, that any number may be expressed as a summation of powers of the base two, and that six places (hence the six squares) are required to represent all the numbers one
Fig. 12.- CHART FOR NUMBER GAME

<table>
<thead>
<tr>
<th>1 3 5 7 9 11 13</th>
<th>2 3 6 7 10 11 14</th>
</tr>
</thead>
<tbody>
<tr>
<td>15 17 19 21 23 25 27</td>
<td>15 18 19 22 23 26 27</td>
</tr>
<tr>
<td>29 31 33 35 37 39 41</td>
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<td>43 46 47 50 51 54 55</td>
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<tr>
<td>57 59 61 63</td>
<td>58 59 62 63</td>
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</table>

<table>
<thead>
<tr>
<th>4 5 6 7 12 13 14</th>
<th>8 9 10 11 12 13 14</th>
</tr>
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<tbody>
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</tr>
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<td>30 31 36 37 38 39 44</td>
<td>30 31 40 41 42 43 44</td>
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<td>45 46 47 56 57 58 59</td>
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<tr>
<td>60 61 62 63</td>
<td>60 61 62 63</td>
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</tbody>
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<table>
<thead>
<tr>
<th>16 17 18 19 20 21 22</th>
<th>32 33 34 35 36 37 38</th>
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<tbody>
<tr>
<td>23 24 25 26 27 28 29</td>
<td>39 40 41 42 43 44 45</td>
</tr>
<tr>
<td>30 31 48 49 50 51 52</td>
<td>46 47 48 49 50 51 52</td>
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<tr>
<td>53 54 55 56 57 58 59</td>
<td>53 54 55 56 57 58 59</td>
</tr>
<tr>
<td>60 61 62 63</td>
<td>60 61 62 63</td>
</tr>
</tbody>
</table>
through sixty-three in the binary system. Consequently, even though the numbers on the chart were in base ten, they were arranged in accordance with the base two plan of positional notation and could be isolated by the "yes" or "no" answers. The puzzle game was used as an introduction to the session for two reasons: (1) to demonstrate the use of an arithmetic game or puzzle for motivation; and (2) to illustrate the use of the concept of different bases in an enrichment situation.

The fact that multi-place numbers may be read in more than one way was demonstrated by use of the chalkboard. The number twenty-three was represented on the chalkboard, and then the question was asked, "How do you read the number?" The answer was given "Two tens and three ones." Then the question was asked, "Are there other correct ways of reading the number?" The answer, "Two bases and three ones," emphasized the work done on bases in the second session. The question was asked, "Is there still another way to read the number?" The answer of "twenty-three ones" was given. It was suggested that although the place value interpretation of a number is in terms of the base, it is permissible to read a number as ones since any whole number is symbolic of a group of ones. The number fifty-eight was then placed on the chalkboard, and the teachers were asked
to read it two different ways. The answers were; "five tens and eight ones," and "fifty-eight ones."

The teachers were asked to practice representing numbers on their scratch sheets of paper, and then to read them in as many correct ways as possible. The teachers discovered that a three-place number may be read in at least three different ways. That is, the number 236 may be read as: two hundreds, three tens, six ones; twenty-three tens, six ones; or two hundred thirty-six ones.

An understanding of transformation of numbers was stressed as a very important pre-requisite to an understanding of the four fundamental processes of whole numbers, and particularly the process of division. Also, the point was made that children should be guided in the process of breaking numbers into their component parts.

Then it was stated that after man learned to count a group of objects, he was faced with the problem of combining groups of objects to determine "how many" in all. It would seem that even primitive man probably realized that he must deal with objects having a common name. Surely, having killed two bears and three dear, he would not have stated that he killed either five bears or five deer, but five objects or animals. Undoubtedly one of the reasons why it is so easy to form the habit of operating blindly and without understanding in arithmetic is the fact that
place value naturally forces the application of the principle of likeness without focusing attention upon the principle itself.

The teachers used the loose beads and discs to practice combining or adding groups of objects, and also in separating or subtracting groups of objects. Then, they were asked to use the ten-tens counting frames to practice combining two small groups of beads to form a new group, thereby demonstrating the process of addition on the counting frame. Then the teachers were guided in performing the operation of subtraction by starting with a group of beads on the counting frame, and separating the group into two smaller groups. Thus, the opposite relationship between addition and subtraction was pointed out.

A demonstration was given of the addition facts whose sums did not exceed ten, by use of the ten-tens counting frame. Along with the demonstration of the addition facts, the subtraction facts with a minuend of ten or less were also shown. Then the teachers were given a brief period of time to show these facts on their ten-tens counting frames.

Following the work with the counting frames, a chart showing the one hundred basic addition and subtraction facts was demonstrated. The use of the chart further emphasized the opposite nature of the two processes. It was pointed out that whereas many children seem to be able to grasp
Fig. 13.- TEN-TENS COUNTING FRAME
Fig. 14.- CHART OF ADDITION-SUBTRACTION FACTS

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the facts of addition and subtraction from having experienced them by the use of objects, other children find such a crutch as the chart of basic facts a valuable aid, and that seemingly it should be the responsibility of the teacher to help the children in gaining the ability to operate without the crutch instead of completely avoiding it.

The question was raised as to when the addition and subtraction facts should be presented; that is, should all the addition facts be presented and then all the subtraction facts, or should the order of presentation to children be first an addition fact and then the corresponding subtraction fact? A variety of opinions existed among the teachers as to the best order of presentation. It was suggested that the procedure of presenting first an addition fact and then the corresponding subtraction fact serves to point out the relationships between the two, and therefore seems to be a more meaningful approach to teaching arithmetic.

The demonstrations of the addition facts on the counting frame and by use of the chart served also to point out the fact that with the exception of the doubles, there are pairs of very similar facts; for example, 2 plus 4 equals 6, but 4 plus 2 also equals six. That is, the order of the addends does not affect the sum. This fact is known as the commutative law for addition. The chalkboard was
used to show that when more than two addends are involved, the addends may be grouped in different ways without changing the result; this fact is known as the law of association for addition.

Examples of both addition with "carrying" and subtraction with "borrowing" were demonstrated on the twenty bead abacus. The teachers saw that "borrowing" and "carrying" involve the process of transforming a number into a different unit or units. A teacher who had a twenty bead abacus volunteered to do an addition example involving "carrying". Emphasis was placed upon the fact that the process was really one of transforming instead of carrying, and that "carrying" and "borrowing" are really poor terms to use. Another teacher with a twenty bead abacus volunteered to demonstrate a subtraction example involving "borrowing."

The teachers were then given guide sheets with ruled lines representing an abacus frame, and were guided in performing addition and subtraction operations on the paper and pencil abaci. After performing the operations by using marks to represent objects, the teachers were asked to place the answers in Hindu-Arabic symbols in their appropriate positions above the lines. Thus the processes were demonstrated at the semi-concrete level, with a symbolic representation of the results.
The process of addition using symbols was shown on the chalkboard. The distributive forms of two and three-place numbers were used to demonstrate the operation of the principle of likeness and to show the close similarity between addition on the abacus and addition with the symbols in their distributive form.

The scratch method of addition was demonstrated as a plan for solving an example by moving from left to right. Thus it was pointed out that addition examples may be solved by working from either direction, but that "carrying" may be included in the column addition by moving from right to left, thus lending efficiency to the right-to-left method.

The teachers were urged to utilize the understandings treated in the third session in their classrooms whenever possible, and to have the courage to experiment with various procedures, determining those which seem to be effective for them.

Summary.—In the third session strong emphasis was placed upon the process of transforming numbers, rounding off numbers, and the fundamental operations of addition and subtraction.
V. THE SEQUENCE OF ESTIMATE-SOLVE-CHECK
FUNDAMENTAL OPERATIONS WITH WHOLE NUMBERS

Major objective.- The major objective in the fourth session was to give consideration to: (1) the sequence of estimate, solve, and check as a desirable approach to arithmetic examples; (2) the excess of nines check; (3) the process of subtraction; and (4) the processes of multiplication and division.

Understandings.- The following understandings were developed in the fourth session.

1. Estimation is a convenient means for determining an approximate answer, and utilizes rounding off numbers.

2. The sequence of estimate, solve, and check is a procedure which is likely to generate confidence in the person doing the operation.

3. Two ways of checking addition are: Adding in a different order, and the excess of nines.

4. The subtraction principle of compensation means that if both the minuend and the subtrahend are increased by the same amount or decreased by the same amount the answer will remain unchanged.

5. There are at least six possible methods of subtraction.

6. A commonly used check for subtraction is addition; however, it is possible to use the excess of nines check.

7. Multiplication is the process of adding or combining two or more groups of equal size.
8. Division is the process of repeated subtraction, and is opposite to the process of multiplication.

9. The multiplication and division facts may be easily demonstrated on the ten-tens counting frame.

10. The multiplication and division facts may be easily demonstrated with a chart.

Description of procedures. - The sequence of estimate, solve, and check was demonstrated by adding the three numbers 37, 22, and 58. The estimate involved rounding off each number to tens and it was obvious that the answer would be very close to 40, plus 20, plus 60, or to twelve tens. Consequently, the estimate for the example was 120. The question arose as to whether it is desirable to encourage children to deal with approximate answers and whether this procedure would result in a feeling on the part of the children that the answer does not have to be exact. It was suggested that the answer does not have to be exact. It was suggested that a great value to be derived from estimating the answer to an example is the confidence which the child gains from knowing a reasonable answer. Furthermore, it was re-emphasized that in many social uses of arithmetic the ability to determine readily an approximate answer would prove a valuable aid in deciding whether the transaction was within reasonable limits, and would undoubtedly serve to correct some of the many errors that are made in everyday transactions. The teachers agreed that the
approximate answer is not intended to be a substitute for the correct answer, and that children should be encouraged to take the next step of solving the example. The addition of the three numbers gave an answer of 117, which was observed to be very close to the estimate of 120.

The two checking procedures of adding in the opposite direction and the excess of nines were discussed. Emphasis was placed upon the fact that a checking procedure provides the child with self-confidence, and the sequence of estimate, solve, and check was highly recommended. Most of the teachers were unfamiliar with the excess of nines check, yet showed a strong interest in it. The possibility of the excess of nines check being invalid because of the fact that it would not detect a reversal of digits error was pointed out; however, it is an exceptional case when an error in computation produces a reversal of digits. In one group, a teacher was familiar with the commonly used bookkeeping check for a reversal of digits error by determining whether or not the extent of the error is evenly divisible by nine, and the relationship between this procedure and the excess of nines check was pointed out.

A subtraction example was placed on the chalkboard, and the principle of compensation was demonstrated by solving the example and then increasing both the minuend and the subtrahend by the same amount and solving the new
example to show that the difference was the same in each case. Then, another example was used. The minuend and the subtrahend were decreased by the same amount and the solution demonstrated the fact that the difference remained unchanged. Therefore, the conclusion was made that if both the minuend and the subtrahend are increased by the same amount or decreased by the same amount, the answer remains unchanged.

Emphasis was placed upon the fact that the process of subtraction as performed on the abacus involved two processes: (1) taking a small group away from a larger group; and (2) borrowing or transforming the larger group when necessary to make possible the "take away" process. Consequently, this very commonly used method for the subtraction process is generally known as the take-away borrow method. Also, this method is known as the decomposition method, since the larger group is transformed or decomposed.

The teachers were asked if they knew another method, and the addition-borrow method was suggested by one of the teachers as a plan whereby instead of taking away, the process involved the decision as to how large a group would need to be added to the small group or subtrahend to produce the larger group or minuend. This method is often considered as having the advantage of using already known
addition facts for the process of subtraction. However, most social situations involving subtraction seem to be a "how much is left" situation, rather than "how much more is needed."

Still another method for subtraction was proposed whereby borrowing is not used, and the process involves increasing the minuend by ten, and then increasing the subtrahend by the same amount transformed to the next higher unit. In other words, if the minuend is increased by ten ones, the subtrahend is increased by one ten, and if the minuend is increased by ten tens, the subtrahend is increased by one hundred. A teacher suggested that this process merely involved the principle of compensation. Some of the teachers stated that they had been taught this method of subtraction, and a few of them said that they have been teaching it in this manner. Many of the teachers seemed to disapprove of this method for performing the subtraction operation on the ground that it depended too much on "following the rule." The take-away-borrow method certainly seemed more consistent with the meaningful approach to the teaching of arithmetic. Since the process involves adding the same amount to the minuend and to the subtrahend, it is known as the equal additions method; however, it is also commonly known as the "carry" method. It was further pointed out that the "carry" method may be
sub-divided into two methods in a manner similar to that in which the "borrow" method was divided into either the take-away-borrow or the addition-borrow methods. Thus, a third method known as the take-away-carry, and a fourth method known as the addition-carry were designated.

Brief attention was given to a method which utilizes complements of ten; that is, seven subtracted from fifteen is considered as seven and three are ten, plus five are fifteen, and three and five are eight, therefore, seven from fifteen leaves eight. As in the other two general methods, the complementary method may be sub-divided into the complementary method with borrowing, and the complementary method with carrying. The complementary method does have the one advantage of placing strong emphasis upon ten as a base for the Hindu-Arabic number system.

The sequence of estimate, solve, and check was suggested as a very desirable approach to the subtraction process. The example of 89 minus 28 served to illustrate the sequence. An estimate utilizing "rounding off" would lead the student to think nine tens minus three tens or about six tens (60) should be the answer. The take-away-borrow method of subtraction was performed and the result was 61. The addition check was applied to verify the answer. The excess of nines check was used as an additional check of the answer. The excess of nines in 89 is 9.
From 9, the excess of nines in 28 (namely 1) was subtracted and the 7 was in agreement with the excess of nines in 61. Although the excess of nines check may be used for subtraction, the addition check was recommended as a more desirable method for most subtraction examples. However, when the addition check is used care must be taken to insure that the children are discouraged from keeping the minuend in sight, and thereby working for a known answer while doing the check.

Emphasis was placed upon multiplication as a process of adding or combining groups when the groups or addends are of equal size. The teachers were asked to manipulate the beads and discs, demonstrating the process of grouping groups or multiplication. They were asked also to use the beads and discs to demonstrate group subtraction or division. Then the ten-tens counting frames were used to demonstrate both the multiplication facts and the division facts. The teachers were guided in performing the processes of multiplication and division on both the bead abaci and the pencil and paper abaci. The use of the abaci served to emphasize further that multiplication is the process of repeated addition of equal-sized groups, and that division is repeated subtraction of equal-sized groups.

A chart of multiplication and division facts was also used to point out the opposite relationship between
Fig. 15. - CHART OF MULTIPLICATION-DIVISION FACTS

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the two processes, and that the chart may be used to an advantage as a crutch in the classroom.

The teachers were urged to give further consideration to the understandings developed in session four, and to utilize them in their classrooms whenever the occasion presented itself.

Summary. - In the fourth session emphasis was placed upon estimate, solve, and check as a desirable approach to performing the fundamental operations of arithmetic, and particular emphasis was placed upon the excess of nines check for the operations of addition and subtraction. The subtraction process was analyzed, and six different methods of subtraction were discussed with special endorsement being given to the take-away-borrow method. The processes of multiplication and division were introduced, and the multiplication and division facts were demonstrated both on the ten-tens counting frame and with a chart.

VI. A CONTINUATION OF MULTIPLICATION AND DIVISION WITH WHOLE NUMBERS

Major objective. - The major objective for the fifth session was to give consideration to: (1) multiplication and division by a power of the base; (2) the early multiplication algorisms; (3) the basic concepts of the multiplication operation; (4) the use of the estimate, solve,
and check sequence for multiplication; and (5) the basic concepts of division and the algorithms for performing the division and the algorithms for performing the division operation.

**Understandings.** The following understandings were developed in the fifth session.

1. The process of multiplication or division by a power of the base may be accomplished by moving the digits the appropriate number of places to the left for multiplication and to the right for division.

2. Multiplication algorithms were developed and used prior to the standard form which is used extensively today:
   a. Lattice method
   b. Napier's bones
   c. Finger multiplication
   d. Duplation or doubling
   e. Lightning method.

3. In multiplication a change in the order in which the numbers are multiplied does not change the answer, and this is known as the commutative law for multiplication.

4. The numbers to be multiplied may be regrouped without changing the answer, and this is known as the associative law for multiplication.

5. Multiplication of multi-place numbers results in distribution into partial products which may be added in any order. This is the distributive law.

6. The sequence of estimate, solve, and check is a procedure which is likely to generate confidence in the person doing the process of multiplication.

7. Either the division check or the excess of nines check may be used for multiplication.
8. There are two different aspects of division: measurement, and partition.

9. Division of multi-place numbers results in distribution into partial quotients which may be added in any order. This is the distributive law.

10. The principle of compensation applies to division; that is, if both the dividend and the divisor are multiplied or divided by the same number, not zero, the quotient will remain the same.

Description of procedures. - The abaci were used to demonstrate that multiplication by a power of the base is made easy by the fact that positional notation is a system whereby each place indicates a power of the base. The example of multiplying 7 by 10 was performed on the abacus by showing 7 beads in the tens (base to the first power) position. Then the teachers were asked to do the example of multiplying 7 by 100. This example was followed by practice at multiplying by a power of the base by use of the place value guide sheets. The demonstration of multiplication was followed by a demonstration of division by a power of the base in a manner opposite to that of multiplication.

The abacus was used to demonstrate the fact that division by a multiple of the base may be accomplished by moving the objects the appropriate number of places to the right, thus determining the position with the proper value. The example of forty divided by ten was used to show first
that groups of ten may be subtracted from forty four times;
then, that the short cut for this operation would be to
move the four in forty one place to the right, thus getting
the answer of four. The example of six hundred divided by
one hundred was used. Then the work was shifted to the
place value guide sheets where symbols were properly
placed to represent objects, and the teachers used the short
cut for dividing by a multiple of the base. Again, the
opposite relationship between multiplication and division
was clearly shown.

A procedure was demonstrated which has the same
result as moving the multiplicand for multiplication or the
dividend for division to the correct position relative to a
power of the base. It is that of moving the decimal point
to the right for multiplication and to the left for division.
Consequently, the standard algorisms for multiplication or
division by a power of the base need not be used, but the
short cut of moving the decimal point the appropriate number
of places accomplishes the objective and serves to empha­
size a basic characteristic of a system of positional nota­
tion.

Consideration was given to the early forms of algor­
isms which were used for multiplication. A guide sheet was
used to help the teachers in developing an understanding of
the Lattice method for multiplication, and in guiding their
attention to the place value similarities between the Lattice method and the standard multiplication algorithm. It was pointed out that the Lattice method of multiplication served the purpose of providing a place value pattern for recording the numbers, including the partial products.

A set of Napier's bones was demonstrated to the teachers, and a guide sheet was given each teacher to provide a practice multiplication example using this type of procedure. The Lattice pattern was combined with the multiplication facts to provide this ingenious plan to avoid committing those facts to memory.

Emphasis was placed upon another plan for performing the multiplication operations with a minimum amount of memorization of the basic facts. It was that of the finger method of multiplication. This method was demonstrated and teachers were given a brief period of time to practice using the method.

Duplication or doubling was then pointed out as another scheme for multiplication, and this method was briefly explained as being based on the premise that any number can be expressed as a power of two, and that multiplication is a process which involves cumulative addition of partial products.

The lightning method of multiplication was demonstrated as a plan in which the final answer may be obtained without writing the partial products. This method of
performing multiplication is dependent upon the ability of the person doing the operation to work mentally with the partial products.

The distributive form for multiplication was then considered and its close similarity to multiplication on the abacus was noted. A multiplication example having both a two-place multiplicand and a two-place multiplier was demonstrated on the abacus, and teachers were encouraged to do a similar example of two-place multiplication with objects, gaining a better understanding of the part played by partial products in multiplication.

The chalkboard was used to emphasize again the grouping characteristic that is inherent in positional notation and that an understanding of the fundamental operations is dependent upon a recognition of the presence of the grouping plan.

Guide sheets were distributed to the teachers showing both the distributive and the standard forms of the multiplication algorism. Emphasis was also placed upon the fact that the answer to a multiplication example remains unchanged even though the order for multiplying the numbers is changed, and that this is commonly known as the commutative law. The numbers to be multiplied may be re-grouped without changing the answer, and this is known as the associative law. The fact that the partial products in a multiplication example
may be added in any order was pointed out by the use of both the abacus and the guide sheets.

The sequence of estimate, solve, and check which had been stressed for both addition and subtraction was given special emphasis in multiplication. It was suggested that in checking a multiplication example, the procedure of performing the process a second time is a poor check as the person doing the operation is very likely to repeat any error that may have been made the first time. The division check was suggested as being appropriate for multiplication. Also, the excess of nines check was recommended as having a great deal of value in checking a multiplication example, and as having the additional advantage of consuming much less time than other methods.

The teachers were asked to use the cardboard discs to form a group of eight objects. Then they were asked to determine how many groups of two objects were contained in the larger group of eight objects. They determined the answer to this problem by repeatedly separating a group of two until there were none left in the big group. Thus the answer was four groups of size two, or there are four groups of two contained in a group of eight, or it is possible to subtract two from eight a total of four times. In this manner two aspects of division were pointed out: (1) it can be considered as repeated subtraction; and (2)
it has an opposite relationship to multiplication in that multiplication involves the combining of a specified number of small groups to determine the size of the resulting large group while division involves determining the number of known size small groups contained in a known size large group.

The teachers were asked to work again with a group of eight discs. This time instead of knowing the size of the small groups, the number of groups was known and the size of each group was to be determined. They were asked to separate the group of eight objects into two groups and to determine the size of each group. It was obvious that in practical re-grouping problems either type of division example might well present itself. Therefore, the teacher should be aware of the two approaches to division. Some of the teachers were aware of the two approaches and knew them by the names of measurement (how many divisions?), and partition (what is the size of the parts?).

In a situation which demands re-grouping, the two approaches are obviously different; however, in operations with symbolic notation, not only is the answer the same but also one division operation would be adequate for either situation. Consequently, when the operation is with symbols it would appear superfluous and perhaps confusing to children to expect them to keep in mind the two concepts
of division, when the measurement concept is entirely adequate and is very much in harmony with the understanding of division as repeated subtraction.

Attention was again focused upon the guide sheet which was used for multiplication, and which included both the distributive and the standard forms for division. Emphasis was placed upon the fact that division of multi-place numbers results in partial quotients, that these partial quotients may be added in any order, and that this is known as the distributive law for division.

The similarity between the subtraction principle of compensation and the operation of the same principle in regard to division was pointed out. However, it was suggested that whereas in subtraction the process was one of dealing with single groups and the compensation was achieved by the individual group processes of addition or subtraction, in division the process is one of dealing with multiple groups and the multiple group processes of multiplication and division are required to achieve compensation.

The teachers were urged to give further consideration to the basic understandings of multiplication and division, and to use these understandings in their classroom teaching whenever possible. It was also suggested that the teachers make use of the distributive forms for multiplication and division in their classrooms, and that
they experiment with the estimate, solve, and check approach to the fundamental operations and determine for themselves what its values are.

**Summary.** - In the fifth session emphasis was placed upon several understandings of the fundamental operations of multiplication and division with whole numbers and the forms or algorisms for each of these two processes.

**VII. DIVISION WITH WHOLE NUMBERS CONTINUED AND INTRODUCTION TO FRACTIONS**

**Major objective.** - The major objective for the sixth session was to give consideration to: (1) the sequence of estimate, solve, and check in performing division, (2) the importance of estimation as a means for determining the trial quotient in division, and (3) the basic introductory concepts of fractions.

**Understandings.** - The following understandings were developed in the sixth session.

1. The sequence of estimate, solve, and check is a desirable procedure for performing the operation of division.

2. Either the multiplication check or the excess of nines check may be used for division.

3. Estimation is significant in determining a trial quotient.

4. A fraction may be conceived as a symbol used to represent a part of a whole object, or a part of a group of objects.
5. A fraction may be conceived as a symbol to show a division example, or to express the comparative relationship of two numbers.

6. Early civilizations represented fractions by using vertical place value, and by keeping one of the two numbers constant.

7. The Egyptians and the Greeks retained a numerator of one for most of their fractions, while both the Babylonians and the Romans retained constant denominators.

8. The principle of compensation may be applied to fractions. That is, the numerator and the denominator of a fraction may be multiplied or divided by the same number, not zero, without changing the value of the fraction.

Description of procedures.- The sequence of estimate, solve, and check was advocated as a highly desirable approach to the division process. The importance of estimation in division was emphasized by demonstration on the chalkboard and by the teachers working on sheets of scratch paper. Particular importance was given two-place divisor examples, and the two methods of "apparent" and "increase-by-one" were explained as applications of the estimation process. The use of the two names have a tendency to confuse youngsters, and it seems preferable simply to call the process estimation, and thus to avoid the use of the two names which in reality represent the application of the one process.

Just as the addition process provides a proof for subtraction, the multiplication process is a valid check for division, and with most examples, is as efficient as the excess of nines check. It also avoids the weakness of
repeating the same basic facts. Consequently, the excess of nines check was not advocated as strongly for division as for multiplication. However, the excess of nines check was explained, and the teachers demonstrated a great deal of interest in it.

Statements introducing the basic concepts of fractions placed emphasis upon the original concept of the fraction as a part of a whole. Felt pieces of circles were used to demonstrate the concept on the flannel board. A guide sheet with a large circle on it was given to each teacher, and they were asked to divide the circle into fractional parts by drawing lines through it. The sharp contrast between whole numbers and fractions was suggested by the fact that in fractions, the greater the number of pieces into which the whole is divided, the smaller the size of each piece; and the smaller the number of pieces, the larger the size of each piece; whereas in dealing with whole numbers, the greater the number of objects, the greater the size of the group, and the smaller the number of objects the smaller the size of the group. The teachers were cautioned that failure to recognize this difference is undoubtedly responsible for much of the difficulty encountered by both children and teachers in dealing with fractions. The distinction between work with whole objects as whole numbers, and with pieces of an object as fractions.
was proposed as a prerequisite understanding for dealing with the symbolic representation of fractions.

Some discussion was devoted to the problem of representing a fraction by means of a symbol. The problem was reduced to two alternatives: either to invent a new set of symbols, or to make a new use of the whole number symbols. The teachers realized that fractions are represented using the Hindu-Arabic symbols by means of vertical place value; however, a great deal of interest was shown when it was pointed out that the early civilizations kept constant either the numerator or the denominator, but the Hindu-Arabic system permits both to vary. The Egyptians permitted the denominator to vary and thus dealt exclusively with unit fractions; but both the Babylonians and the Romans maintained constant denominators (sixty for the Babylonians and twelve for the Romans), and changed the numerator to fit the size of the fraction.

Emphasis was placed upon the numeration technique which is consistent with the symbolic representation; that is, the symbol \( \frac{2}{3} \) is read two-thirds and means two of the three equal parts into which a whole object has been divided. The consistent pattern for numeration makes for easy reading of fractions.

The flannel board was used to demonstrate the fact that a fraction symbol may be used to show a part of a group.
Again the example two-thirds was used. Two whole circles were placed on the flannel board, and the question was asked to determine one-third of the two circles. The suggestion was made that each of the two circles be divided into thirds and then a third of one circle plus a third of the second circle would be a third of both circles or two-thirds in all. Thus it was demonstrated that the symbol $\frac{2}{3}$ can also be used to signify one-third of a group of two.

The cardboard discs were distributed and the teachers practiced finding a fraction of a group by partitive division. The teachers saw that some examples could be solved by a separation of the group into smaller groups, but that other examples required breaking whole objects into pieces.

The use of the fraction symbol to represent division was explained by a restatement of the concept of a fraction as a part of a group; namely, if a group of two is separated into three equal parts what will be the size of each part? It was suggested that the partition concept of the division example two divided by three asks precisely this same question. Therefore, another use of the fraction symbol is to indicate division.

The fourth use of the fraction symbol was in comparing or expressing the ratio of two numbers. The example of two-thirds was used again, and instead of the three previous
uses of the symbol, it was used to answer the question, what part of three is two? Or what is the ratio or comparison of two and three?

In summarizing the discussion on the four different uses of the fraction symbol $\frac{2}{3}$, strong emphasis was placed upon the fact that the symbol was the same in all four cases; therefore, the same computational procedures would be used in each case. The question was asked, "Why should emphasis be placed upon the four cases since there is no computational difference?" The question was answered by placing stress upon the existence of the four cases in problem situations, and by stating that an appreciation for the four cases seemingly would tend to facilitate arithmetical understandings, and should aid the reasoning process, thus being a great help in problem solving and mental arithmetic.

A fraction symbol used to indicate a division example was the basis for a discussion of the application of the principle of compensation to a fraction; that is, the numerator and the denominator of a fraction may be multiplied or divided by the same number, not zero, without changing the value of the fraction.

The teachers were urged to give further consideration to the use of objects in developing the basic concepts of fractions, and to use this approach in their own classrooms.
In order to encourage the teachers to use the objective approach in their classrooms, each teacher was presented with a flannel board, and with some additional flannel to be used on the back of those objects which the teacher decided to use. Therefore each teacher, regardless of grade level, could make use of this equipment by using those objects appropriate for her grade.

The gifts of the flannel boards and the abaci to the teachers later served as a basis for determining to some extent what uses the teachers would make of these manipulative materials if they were provided with them.

**Summary.**—In the sixth session further treatment was given to the basic understandings of division of whole numbers, and the basic concepts of fractions were introduced. Strong emphasis was placed upon the use of objects in developing the basic concepts of fractions.

**VII. FRACTIONS AND THE FUNDAMENTAL OPERATIONS**

**Major objective.**—The major objective for the seventh session was to give further consideration to the basic concepts of fractions, and to the four fundamental operations with fractions.

**Understandings.**—The following understandings were developed in the seventh session.

1. The reduction of a fraction is based upon the principle of compensation.
2. Reduction of fractions is not a reduction in the strict sense of the word, but is a transformation with no change in value.

3. Addition and subtraction of fractions is dependent upon the quality of likeness. That is, the fractions must have the same "last name," or a common denominator. This is very similar to the principle of likeness used with whole numbers.

4. Multiplication of fractions may be rationalized in each of three cases: a fraction by a whole number; a whole number by a fraction; and a fraction by a fraction.

5. Multiplying by a proper fraction results in a product that is a smaller number than the multiplicand; whereas, dividing by a proper fraction results in a quotient which is larger than the dividend.

6. The inverse relationship of multiplication and division is demonstrated in multiplication by a unit fraction.

7. The commutative and associative principles apply to fractions in a manner very similar to their application to whole numbers.

8. The process of cancellation may be rationalized as a reduction prior to multiplication.

Description of procedures.- The fraction four-sixths was placed on the chalkboard, and then the fraction was "reduced" to two-thirds. The question was asked, "By what right can four-sixths be "reduced" to two-thirds?" Most of the teachers seemed surprised by the question, and obviously had given very little thought to the why of the procedure. However, some of the teachers realized that it was simply the application of the principle of compensation.
The question was then asked, "Is the process really one of reduction?" Most of the teachers agreed that a change of value does not occur, that the only reduction is in the cardinal value of the symbols, and that "reduction" is a poor word for the process.

The teachers were reminded of a basic understanding treated in session three; namely, that the processes of addition and subtraction with whole numbers are dependent upon the quality of likeness. It seemed reasonable to assume that the processes of addition and subtraction with fractions could be performed with the fractions which have the quality of likeness and therefore a common denominator. An attempt to verify this assumption was made by overlapping the flannel parts on the flannel board to show that only fractions having a common denominator could be added or subtracted. It was shown that once the fractions have common denominators, the processes of addition and subtraction may be performed by adding or subtracting numerators. Therefore, a major concern in adding or subtracting fractions is to find a common denominator. This problem does not present itself in the operations with whole numbers because place value forces the quality of likeness. The conclusion was reached that the process of "reduction" provides a most convenient means for finding the lowest common denominator of fractions, and involves
"reducing" the fractions to either lower or higher terms until the desirable denominators are reached.

A guide sheet was given to each teacher to facilitate the rationalization of basic understandings regarding multiplication with fractions. Consideration was given first to the example of a fraction multiplied by a whole number, and the attention of the teachers was turned to the example of \( \frac{1}{2} \times 4 \), by the suggestion that this example really means, "What is the sum of four one-halves?" The diagram on the guide sheet showed that the sum is two wholes. The flannel board was used to show that the sum of four one-halves is two. Other examples were demonstrated on the flannel board, and the teachers were given an opportunity to practice with their sheets of scratch paper.

The multiplication of a whole number by a fraction was next considered, and reference was made to both the flannel board and to the guide sheet. The example, \( 3 \times \frac{1}{2} \), was demonstrated on the flannel board by using three circles, taking one-half of each circle and assembling the three halves for an answer of \( 1 \frac{1}{2} \). This was the first case where the process of multiplication had produced an answer which was smaller than the multiplicand, and this contrast with whole number operations was seen to be the source of much of the difficulty encountered in working with fractions. The flannel board was used to
demonstrate the sequence of first multiplying by a whole number greater than one and getting an answer which was greater than the multiplicand, then multiplying by the whole number, one, and getting an answer which was equal to the multiplicand, and then multiplying by a proper fraction, less than one, and arriving at an answer which was smaller than the multiplicand. The whole number, one, seemed to be the dividing line between multiplication which produced a larger result than the multiplicand and multiplication which produced a result which was smaller than the multiplicand. Consequently, when a proper fraction is used as the multiplier it would seem reasonable to expect the answer to be smaller than the multiplicand, and this is just the opposite to the result which may be expected when multiplying by a whole number.

The question was then raised as to what result could be expected when performing the operation of division with a proper fraction as a divisor. It was reasoned that since the division process is the opposite or inverse of the multiplication process, then the opposite result should be expected; namely, that division by a proper fraction should produce a quotient which is larger than the dividend. The flannel board was used to demonstrate the validity of the assumption. Much of the confusion resulting from the apparent impossibility of multiplying and getting less and
of dividing and getting more could seemingly be avoided by remembering that work with fractions involves parts instead of wholes. Consequently, when a fraction is used as the multiplier, the answer will be a part of (hence smaller than) the multiplicand. Likewise, when a fraction is used as the divisor, the answer will be indicative of the number of parts (hence a larger number) contained in the dividend.

Consideration was then given to the multiplication of a fraction by a fraction, and the guide sheet was used to demonstrate the fact that multiplying a fraction by a fraction involves a part of a part. Consequently, the procedures of multiplication of denominators for a new denominator and of multiplication of numerators for a new numerator produce the correct answer for multiplication of fractions. The chalkboard was used to demonstrate the algorism of multiplying a fraction by a fraction.

The question was raised as to whether multiplication by a unit fraction is really multiplication or division, since division by the denominator of the fraction determines the correct answer. This provided an excellent opportunity to demonstrate further the inverse relationship that exists between multiplication and division. This was done by emphasizing that the inverse of a unit fraction is the denominator of the fraction used as a whole number (for example, the inverse of \( \frac{1}{3} \) is \( \frac{3}{1} \) or three wholes), and
by using examples to show that multiplication by a number and division by the inverse of the number produces the same result. Therefore the conclusion was reached that the example is really one of multiplication, but that the inverse relationship between a unit fraction and a whole number brings about the use of division in solving the example.

The fact that the numerator of a fraction is a whole number symbol, and that only the numerators are added is obvious proof that the commutative and associative laws apply not only to the process of addition with whole numbers but also to addition with fractions. Furthermore, since the process of multiplication of fractions is performed by multiplying the numerators which are the whole number symbols and the denominators which are also whole number symbols, the commutative and associative laws are valid for multiplication of fractions.

The example, $\frac{2}{3}$ multiplied by $\frac{1}{2}$, was used to demonstrate that the procedure known as reduction of a fraction may be used either before the application of the multiplication process or after the multiplication has been performed. Other examples demonstrated the fact that the use of reduction before performing the multiplication was the preferred method because it usually resulted in a shorter process using smaller numbers. This reduction
before multiplying is commonly known as the process of cancellation.

The teachers were urged to give further consideration to the basic concepts of fractions and to the rationalization of the fundamental operations with fractions, and to experiment with the use of objects in teaching fractions in their classrooms.

**Summary.** - In the seventh session consideration was given to some of the understandings of fractions, and to the use of the four fundamental processes of addition, subtraction, multiplication, and division in working with fractions.

**IX. DIVISION WITH FRACTIONS. DECIMALS AND THE FUNDAMENTAL OPERATIONS**

**Major objective.** - The major objective for the eighth session was to give consideration to the rationalization of the process of division of fractions, the basic concepts of decimal fractions, and the four fundamental operations with decimals.

**Understandings.** - The following understandings were developed in the eighth session.

1. Division of fractions by the common denominator method may be rationalized in each of three cases: a whole number by a fraction; a fraction by a whole number; and a fraction by a fraction.
2. Division of fractions by the inversion method may be rationalized in each of three cases: a whole number by a fraction; a fraction by a whole number; and a fraction by a fraction.

3. Decimal fractions are the result of an extension of the Hindu-Arabic number system to include fractions, and therefore use place value.

4. Facility in reading and writing decimals is dependent upon an understanding of the base of the number system.

5. Decimals may be transformed to common fractions and common fractions may be transformed to decimals.

6. The processes of addition and subtraction of decimals may be performed in a manner very similar to that used with whole numbers.

7. The distinguishing feature between operations with whole numbers and operations with decimals is the placement of the decimal point.

8. Multiplication with decimals involves multiplying with whole numbers and locating the decimal point in the product.

9. Division with decimals involves dividing with whole numbers and locating the decimal point in the quotient.

10. Since decimals are an extension of the Hindu-Arabic system, the processes of multiplying by a power of the base or dividing by a power of the base may be performed in a manner similar to that used with whole numbers.

Description of procedures. - The flannel board and the flannel parts were used to demonstrate the use of objects in developing an understanding of division of fractions. A demonstration was given of each of the three cases: (1) division of a whole number by a fraction; (2)
division of a fraction by a whole number; and (3) division of a fraction by a fraction. Two circles were placed on the flannel board and the example was 2 divided by $\frac{1}{3}$. The example was considered as involving the measurement concept of division and therefore as asking the question, how many one-thirds in two. The division of the two circles into thirds proved that there were six one-thirds in two.

The division of a fraction by a whole number was demonstrated by the example, $\frac{1}{2}$ divided by 2. One-half of a felt circle was placed on the flannel board and the question was asked, "If $\frac{1}{2}$ is divided into two equal parts what will be the size of each part?" This was recognized as the partition concept of division. When the one-half was divided into two equal parts, the answer of $\frac{1}{4}$ was obvious.

The division of a fraction by a fraction was considered as measurement division and very similar to the division of a whole number by a fraction; that is, the question "how many?" was asked. The example of $\frac{1}{2}$ divided by $\frac{1}{4}$ served to demonstrate the solution. The one-half piece of flannel was placed on the board, then the one-fourth pieces were superimposed to determine that there were two one-fourths in one-half. These examples served to illustrate the use of objects in showing the three cases of division involving fractions.
Attention was then turned to the rationalization of division of fractions by the use of symbols. Each teacher was given guide sheets on which each of the cases of division with fractions was illustrated. Consideration was given to the first example of 3 divided by 1/2. It was shown on the flannel board that there are six halves in three; however, the problem was to arrive at this solution by a procedure with the symbols. It was suggested that since the three circles had been divided into halves, a comparable procedure would be to transform the 3 into halves, namely 6/2. Then the example would become 6/2 divided by 1/2. Although the answer was obvious, the problem of symbolic procedure had not been solved. The next step was to press into service the principle of compensation for division, and when both the numerator (6/2) and the denominator (1/2) were multiplied by two, the answer was 6 divided by 1 or 6. The same procedure of finding a common denominator was used for the other two cases, and the generalization was made that one method for division of fractions is to find the common denominator, and then apply the principle of compensation. Many of the teachers had never heard of the common denominator method for division of fractions, and some of them felt it was a longer method than the one they knew, that of invert and multiply.

The teachers requested a rationalization of the inversion method for division of fractions, and some of them
remarked that they had always felt that the inversion method was simply a rule to follow and could not be rationalized. Two explanations were given for the inversion method.

The first explanation involved consideration of the division example, $3$ divided by $1/2$, as a fraction in which $3$ was the numerator and $1/2$ was the denominator. Since by the rule of compensation the value of the fraction remains unchanged when the numerator and the denominator are multiplied by the same number, not zero, then both numerator and denominator were multiplied by the $2$ which was pointed out as the inverse of $1/2$. The result was $6$ divided by $1$ or $6$. The teachers were given other examples and were asked to show by multiplying both the numerator and the denominator by the inverse of the denominator that the inversion method is valid and is based upon the principle of compensation.

The second explanation for the inversion method of division was based upon the inverse relationship between multiplication and division of whole numbers. The teachers were reminded that this relationship was emphasized in session seven with the discussion of the question whether the example of multiplying by a unit fraction was really a multiplication or a division example. It was suggested that in division of fractions the same inverse relationship
exists but more generally, not only with a unit fraction but also with any common fraction.

The rationalization of the inversion method as shown by the division of a fraction by a whole number and a fraction by a fraction obviously follows the same general pattern as that of dividing a whole number by a fraction; and the only emphasis given to these cases was the existence of this similarity.

The concept of decimal fractions as a special case in which the denominator of the fraction is ten or a multiple of ten was emphasized. Furthermore, it was pointed out that the scheme for decimal fractions is an extension of the Hindu-Arabic system of positional notation. The placement of the decimal point between the ones place and the tenths place is the only means for distinguishing between a whole number and a decimal fraction since both use the same symbols and also a similar scheme of positional notation. These common characteristics greatly facilitate the four fundamental operations with decimals.

A place value chart was used to demonstrate not only the scheme for the notation of decimal fractions, but also to plan for naming the decimals. Stress was placed upon the three important aspects of decimals: (1) writing decimals; (2) reading decimals; and (3) maintaining the correct placement of the decimal point when each of the four fundamental operations is performed.
The distributive form for the operations was demonstrated in a manner very similar to that used for whole numbers, and the standard algorism was that of performing the whole number operations and then properly placing the decimal point in the answer. The effectiveness of using the process of estimation in determining the correct placement of the decimal point was illustrated.

The change of a decimal into a common fraction and a common fraction into a decimal was demonstrated on the chalkboard; then the teachers were asked to practice some examples on their scratch paper.

Multiplication with decimals was demonstrated on the chalkboard with stress upon estimating, performing the operation as with whole numbers, determining the position of the decimal point in accordance with the estimate, and checking the example. Two examples were used: (1) the multiplication of a decimal by a whole number; and (2) the multiplication of a decimal by a decimal.

Division with decimals was demonstrated by the estimate-solve-check procedure. The operation was performed as with whole numbers, then the decimal point was placed in accordance with the estimate. Two examples were used: (1) the division of a decimal by a whole number; and (2) the division of a decimal by a decimal.
It was pointed out that since decimals are an extension of the Hindu-Arabic system, multiplication by a power of the base and division by a power of the base may be performed by moving the decimal point to the right for multiplication, and to the left for division.

The teachers were urged to give further consideration to the rationalization of the process of division of fractions, to the basic concepts and the operations with decimals, and to use these understandings in their classrooms when possible.

**Summary.** - In the eighth session consideration was given to the rationalization of the process of division of fractions by the common denominator method and by the inversion method. Also, the basic concepts of decimals together with the four fundamental operations with decimals were considered.

**Summary of chapter.** - The in-service course attempted to develop seventy-two basic understandings of arithmetic with the teachers in accordance with the ten basic criteria established for the course. Extensive use was made of objects and the means whereby the teachers could see and experience the relationships and operations of the basic concepts of arithmetic.

Although the brevity of each session made mandatory an efficient procedure, time was allotted for the teachers
to participate in working with the manipulative materials and in group discussions. The teachers seemed to benefit from the variety of procedures used, and, in many cases, made excellent contributions to the class sessions. Each group of teachers participated with interest and enthusiasm.

Time was a factor in the course and prevented the development of understandings in other areas of arithmetic and in a more extensive consideration for methods of teaching arithmetic; however, it seems that a longer period of time would have tended to discourage participation, especially in view of the heavy work loads of teachers in service.

In Chapter IV, each session of the in-service course was described in three steps: (1) a statement of the major objective of the session; (2) a statement of those specific understandings which were developed; and (3) paragraph descriptions of procedures used to develop understandings.
CHAPTER V

EVALUATIVE INSTRUMENTS AND THEIR USES

The instruments used to evaluate the effectiveness of the in-service procedures were: (1) the Glennon test of basic mathematical understandings, (2) teacher conferences utilizing a list of guide questions, (3) a teacher questionnaire, (4) a teacher summary, (5) classroom visitation and an observation check list, and (6) a teacher opinion check list. Samples of the six evaluative instruments are included in the Appendix.

The Glennon test.- The Glennon test served two purposes in the study: (1) as described in Chapter III, it was used in the initial phase of the study as both an instrument to motivate the teachers and an inventory of teacher understandings; (2) it was used to measure objectively the growth in understandings of the teachers and principals who participated in the in-service course. The test was administered a second time during a special session, after the conclusion of the course. Following a similar time lapse, twelve teachers who were not enrolled in the course took the test a second time for the purpose of determining
to some extent what gain in test score could be expected simply from taking the test a second time.

**Conferences with teachers.** At the conclusion of the second phase of the study, thirty-minute individual conferences were held in the classroom of each of the six teachers in the pilot group. These conferences were designed to determine the reactions of the teachers to the in-service course, the arithmetic materials and procedures used in their classrooms, the general attitudes of both children and parents toward arithmetic, and some of the teachers' feelings toward the teaching of arithmetic.

A list of ten guide questions was used as a basis for discussion during the conference. The conferences had a great deal of value but a small group conference coupled with a questionnaire and a summary to be completed by each teacher would have better served the purpose. The questionnaire and the summary would have given the teachers more time to organize their statements, thus providing a more careful and comprehensive evaluation.

Either individual or small group conferences were then held with the remaining twenty-four teachers at their schools. These included three individual conferences, four three-teacher conferences, one four-teacher conference, and one five-teacher conference. The teacher questionnaire and the teacher summary were used as the basis for
discussion, and at the conclusion of each conference the two forms were left with the teachers to be completed at their convenience.

The group conferences appeared to be highly satisfactory as teachers freely discussed the questionnaire and the summary and in this way prepared for the task of completing them with a clarity of purpose. Many of the teachers obviously spent two hours or more filling out the two forms.

A teacher questionnaire.—An eleven-item questionnaire was designed to assist the twenty-four teachers in recording virtually the same information as that recorded from the thirty-minute individual conferences with the six teachers in the pilot group. The questionnaire was completed by each teacher without a time limitation.

A teacher summary.—The summary which was written by each of the twenty-four teachers was based on a sheet of nine guide questions. The questions were designed to assist each teacher in writing a descriptive evaluation of the course, and in stating some reactions to the course as a procedure for in-service education in arithmetic. Also, the guide questions provided an outline for organization of the summary.

The summary placed emphasis upon the extent to which the teachers had found the course practical and helpful in
their classroom teaching. Most of the teachers wrote a very comprehensive summary, and the instrument proved of great value for measuring teacher growth in understandings as estimated by the teachers themselves.

**Classroom visitation.**—It was felt that a follow-up visitation with each of the thirty teachers would yield evidence of that teacher's philosophy for teaching arithmetic as evidenced by classroom practices, the materials which were being used for teaching arithmetic, the procedures which were being utilized, and to some extent the amount of carry-over from the course in arithmetical understandings. A classroom visit of approximately one hour was then arranged with the administrative personnel and with each teacher. As requested, each teacher spent part of the time on some phase of arithmetic, and in most cases from forty-five minutes to one hour was devoted to the subject. In some classrooms, the program for the day had been modified to provide a convenient time for the observer; however, the change in schedule did not appear to create a hardship, and in only one classroom was there any strong evidence of superficiality. In some cases, the shift in schedule seemed to produce restlessness, especially at the primary level where desirable work habits are strongly dependent upon a regular schedule which gives consideration to the short attention span of the children.
The follow-up visitation was made in each case at least two months after the conclusion of the course in arithmetical understandings, and the teachers were not forewarned that there would be follow-up visits until the week before the visits were made. This was done in order that valid observations could be made concerning the naturalness of the teaching situation and the extent of carry-over from the in-service course. In an attempt to objectify the results of the visits, a twenty-three item observation check list was completed after each classroom visit.

Teacher opinion check list. - A check list, designed to register teacher opinions in regard to the teaching of arithmetic and to the in-service education of teachers, was developed. In its original form the check list contained more than seventy items, but refinement of the items and a reduction in number to keep the check list within reasonable time limits resulted in the list of forty items. Each teacher was asked to check a single category for each item. The categories were "disagree strongly," "disagree," "agree," "agree strongly," and "of little significance." In addition to checking the forty items each teacher was asked to write a brief paragraph describing what she considered to be the greatest on-the-job need of elementary teachers in the area of arithmetic.

The check list served two purposes: (1) it secured some of the opinions of those teachers who had participated
in the in-service course; and (2) it determined what differENCES of opinion exist between those teachers who were enrolled in the course and other teachers in the same school system who did not take part in either the Glennon test or the in-service course.

The check lists were filled out by thirty teachers and three principals who participated in the course, and by a total of 100 teachers who did not take the course. It was then assumed that a reasonable difference of opinion between the two groups of teachers represented a change of opinion resulting from the in-service procedures utilized in this study.

**Summary.**—The instruments used in the study included a test of basic mathematical understandings, teacher conferences based on a list of guide questions, a teacher questionnaire, a teacher summary, classroom visitation and an observation check sheet, and a teacher opinion check list. Each of the instruments has been described and its uses have been indicated. By serving a distinct purpose in the study, each instrument played a vital role in accomplishing the over-all objective of evaluating a developmental approach to the in-service education of elementary teachers in arithmetical understandings.
CHAPTER VI

RESULTS OF THE STUDY

The results of the study revealed evidence of the extent to which the teachers who participated in the study had an understanding of the basic concepts of arithmetic, and the extent to which the in-service procedures were effective in producing growth in arithmetical understandings.

The incidence of teacher understandings was determined by an analysis of the data secured through the use of the Glennon Inventory, while the effectiveness of the in-service procedures was measured by six evaluative instruments used in the study: (1) the re-use of the Glennon Inventory, (2) teacher conferences, (3) teacher questionnaire, (4) teacher summary, (5) classroom visitations, and (6) teacher opinion check list.

Incidence of teacher understandings. - It would seem reasonable to expect that the incidence of understandings would vary from one grade level to another because of the fact that certain basic concepts are given more emphasis at some grade levels than at others. Furthermore, many teachers tend to be interested only in the work of their
grade level. Table V shows the incidence of understandings according to grade level of the 116 teachers and four principals who participated in the Glennon Inventory. With the exception of the second grade teachers, there was a small gain in score from one grade to the next higher grade level.

Table VI shows the incidence of understandings, by grade level, of the thirty teachers and three principals who participated in the in-service course. It is to be noted that as a group the teachers and principals who took part in the course had slightly higher scores on the test than the large group of 120 teachers and principals. However, when it is realized that the low scoring kindergarten teachers were not included in the in-service course, the conclusion may be made that the group participating in the course was representative of the larger group.

An examination of the items on the Glennon Test and the scores shown by Tables V and VI indicate that the arithmetical understanding of the 120 teachers and principals leaves much to be desired.

Teacher reaction to a test of arithmetical understandings. - The reactions of the teachers to the test of understandings was observed with interest since it was expected that some teachers would have reservations about a test of their own understandings. The major concern of the
Table V

INCIDENCE OF UNDERSTANDINGS ACCORDING TO GRADE LEVEL FOR ALL TEACHERS IN THE STUDY.

BASED ON SCORES ON THE GLENNON TEST

<table>
<thead>
<tr>
<th>Grade level</th>
<th>Number of teachers</th>
<th>15 possible</th>
<th>15 possible</th>
<th>15 possible</th>
<th>20 possible</th>
<th>15 possible</th>
<th>80 possible</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>The decimal system of notation</td>
<td>Integers and processes</td>
<td>Fractions and processes</td>
<td>Decimals and processes</td>
<td>Rationale of computation</td>
<td>TOTAL TEST</td>
</tr>
<tr>
<td>Kindergarten</td>
<td>8</td>
<td>11.63</td>
<td>11.13</td>
<td>5.86</td>
<td>7.25</td>
<td>6.13</td>
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<tr>
<td>First Grade</td>
<td>18</td>
<td>11.17</td>
<td>11.22</td>
<td>8.28</td>
<td>10.06</td>
<td>7.39</td>
<td>48.12</td>
</tr>
<tr>
<td>Second Grade</td>
<td>13</td>
<td>10.77</td>
<td>10.62</td>
<td>6.92</td>
<td>9.08</td>
<td>8.38</td>
<td>45.77</td>
</tr>
<tr>
<td>Third Grade</td>
<td>21</td>
<td>12.05</td>
<td>11.86</td>
<td>8.52</td>
<td>8.57</td>
<td>8.36</td>
<td>49.86</td>
</tr>
<tr>
<td>Fourth Grade</td>
<td>19</td>
<td>12.58</td>
<td>12.21</td>
<td>9.16</td>
<td>10.84</td>
<td>9.00</td>
<td>53.79</td>
</tr>
<tr>
<td>Fifth Grade</td>
<td>15</td>
<td>13.13</td>
<td>12.13</td>
<td>10.73</td>
<td>12.47</td>
<td>9.47</td>
<td>57.93</td>
</tr>
<tr>
<td>Sixth Grade</td>
<td>22</td>
<td>12.23</td>
<td>12.15</td>
<td>10.68</td>
<td>14.00</td>
<td>10.18</td>
<td>59.54</td>
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<td>Principal</td>
<td>4</td>
<td>13.24</td>
<td>13.50</td>
<td>10.75</td>
<td>15.00</td>
<td>11.25</td>
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<td></td>
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<td>11.83</td>
<td>9.32</td>
<td>9.98</td>
<td>8.83</td>
<td>52.01</td>
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Table VI

INCIDENCE OF UNDERSTANDINGS ACCORDING TO GRADE LEVEL FOR THE TEACHERS WHO PARTICIPATED IN THE IN-SERVICE COURSE. BASED ON SCORES ON THE GLENNON TEST

<table>
<thead>
<tr>
<th>Grade level</th>
<th>Number of teachers</th>
<th>The decimal system of notation</th>
<th>15 possible</th>
<th>15 possible</th>
<th>15 possible</th>
<th>20 possible</th>
<th>15 possible</th>
<th>TOTAL TEST</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Grade</td>
<td>3</td>
<td>13.00</td>
<td>12.33</td>
<td>9.67</td>
<td>11.67</td>
<td>8.33</td>
<td>55.00</td>
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<tr>
<td>Second Grade</td>
<td>4</td>
<td>10.50</td>
<td>10.75</td>
<td>8.50</td>
<td>9.50</td>
<td>9.00</td>
<td>48.25</td>
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<td></td>
</tr>
<tr>
<td>Fifth Grade</td>
<td>3</td>
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<td>12.33</td>
<td>11.67</td>
<td>15.67</td>
<td>11.33</td>
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<tr>
<td>Sixth Grade</td>
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<td>12.13</td>
<td>12.75</td>
<td>10.88</td>
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<td>10.38</td>
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<tr>
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<td>12.67</td>
<td>13.33</td>
<td>10.00</td>
<td>15.33</td>
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<td></td>
</tr>
<tr>
<td>TOTAL</td>
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<td></td>
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<td>12.27</td>
<td>9.67</td>
<td>12.06</td>
<td>9.88</td>
<td>56.61</td>
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</tbody>
</table>

170
teachers was with the purpose of the test and with the expected use of the test scores. Most of the teachers were interested in taking the test after its purpose was explained to them and they were assured that their scores would remain confidential. However, as shown in Table VII four groups of teachers who took the test showed some disinterest in it. One group of teachers not used in the study refused to take the test even after its purpose was explained to them.

**Effectiveness of the in-service procedures.** The effectiveness of the in-service procedures and consequently of the in-service course was measured with particular emphasis upon teacher growth in understanding by using the six previously mentioned evaluative instruments.

**Re-use of the Glennon Test.** - The second use of the Glennon Inventory revealed the extent to which there had been a change in test scores of the thirty teachers and two principals who took part in the course. A control group of eleven teachers and one principal, who were included in the initial inventory but who did not participate in the course, took the test a second time.

Table VIII compares the growth in understanding of the experimental group and the control group. In each of the five parts of the test, the net gain of the experimental group exceeded that of the control group, and the
Table VII

GENERAL REACTIONS OF THE FOURTEEN GROUPS OF TEACHERS TO TAKING
THE GLENNON TEST OF BASIC MATHEMATICAL UNDERSTANDINGS

<table>
<thead>
<tr>
<th>Code letter of School</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
<th>I</th>
<th>J</th>
<th>K</th>
<th>L</th>
<th>M</th>
<th>N</th>
<th>Total Types of Reaction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strong interest in taking the test</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>Some interest in taking the test</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>Lack of interest in taking the test</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>Strong lack of interest in taking the test</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>x</td>
<td>1</td>
</tr>
</tbody>
</table>
Table VIII

COMPARISON OF INCIDENCE OF GROWTH OF UNDERSTANDINGS BETWEEN THE EXPERIMENTAL GROUP AND THE CONTROL GROUP. BASED ON SCORES ON THE GLENNON TEST

<table>
<thead>
<tr>
<th></th>
<th>Number of teachers</th>
<th>15 possible</th>
<th>15 possible</th>
<th>15 possible</th>
<th>20 possible</th>
<th>15 possible</th>
<th>30 possible</th>
<th>TOTAL TEST</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>The decimal system of notation</td>
<td>Integers and processes</td>
<td>Fractions and processes</td>
<td>Decimals and processes</td>
<td>Rationale of computation</td>
<td>TEST</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Experimental Group</td>
<td>33</td>
<td>0.94</td>
<td>0.88</td>
<td>0.91</td>
<td>0.91</td>
<td>1.09</td>
<td>4.73</td>
<td></td>
</tr>
<tr>
<td>Control Group</td>
<td>12</td>
<td>-0.08</td>
<td>-0.16</td>
<td>0.33</td>
<td>0.59</td>
<td>0.25</td>
<td>0.93</td>
<td></td>
</tr>
</tbody>
</table>

Figures represent average gain in score from first to second testing.
average difference in net gain was almost four points. For each teacher of the experimental group the average gain was one point or more for each of the five parts of the test, which indicates a balanced growth in all areas. Also, in the experimental group a consistent net gain was present, but the control group showed cases of net loss and other cases in which the difference was very slight. The greatest gain by any one person in the experimental group was nineteen points and the greatest single loss was three points, as compared to the control group where the greatest gain was six points and the greatest loss was four points. Although a gain of only 4.73 points on an eighty-item test is relatively small, this gain when associated with the other evidence is an important part of the total picture of teacher growth in arithmetical understandings.

Teacher conferences.—The ten questions which were used as a basis for individual conferences with the pilot group and both the kind and frequency of responses are shown in Table IX. Most of the teachers gave more than one answer to each question, which accounts for the number of replies exceeding the number of teachers. The fact that a teacher did not mention the same response that other teachers emphasized does not necessarily indicate a negative attitude toward that answer, but simply that other factors took precedence in her thinking. The table contains only those responses which were given three or more times, as
<table>
<thead>
<tr>
<th>Question</th>
<th>Sample responses from the teachers</th>
<th>Frequency of response</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Have you found the in-service course helpful?</td>
<td>Yes</td>
<td>6</td>
</tr>
<tr>
<td>If so, how?</td>
<td>No</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>It has helped improve my understanding of numbers.</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>The course made use of helpful materials.</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>I feel much more confident now.</td>
<td>3</td>
</tr>
<tr>
<td>2. What are your criticisms of the course?</td>
<td>Not enough time to discuss teaching of arithmetic.</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>Twice a week is too often to meet.</td>
<td>3</td>
</tr>
<tr>
<td>3. What materials do you find helpful in teaching arithmetic?</td>
<td>Guide sheets</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>Abacus</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>Flannel board</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>Textbooks</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>Objects</td>
<td>3</td>
</tr>
<tr>
<td>4. What additional materials do you need for teaching arithmetic?</td>
<td>More manipulative materials.</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>A simpler teacher's guide.</td>
<td>3</td>
</tr>
<tr>
<td>5. How many groups do you have, in arithmetic?</td>
<td>Two</td>
<td>3</td>
</tr>
<tr>
<td>6. What is the general attitude of your students toward arithmetic?</td>
<td>Most of them like it.</td>
<td>5</td>
</tr>
<tr>
<td>7. What interest in arithmetic is shown by parents of your pupils?</td>
<td>They are interested.</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>Not much reaction.</td>
<td>3</td>
</tr>
<tr>
<td>8. What difficulties do you find in teaching arithmetic?</td>
<td>It is hard to get the children to learn the facts.</td>
<td>5</td>
</tr>
<tr>
<td>9. What part of arithmetic do you find easy to teach?</td>
<td>Using manipulative materials to perform the fundamental operations.</td>
<td>3</td>
</tr>
<tr>
<td>10. What are your general comments about the course?</td>
<td>The course has been helpful.</td>
<td>3</td>
</tr>
</tbody>
</table>
responses mentioned less frequently did not seem to be sufficiently significant.

It is significant that five of the six teachers felt the course had helped improve their understanding of numbers, and that three teachers mentioned both the materials used and the feeling of confidence which they had gained from the course.

Although more time was spent with the pilot group in considering the teaching of arithmetic than with the other groups, the most frequent criticism of the course was that more time should have been spent on methods of teaching. Obviously the teachers became so concerned with teaching procedures that they forgot the limitation of the course to the basic arithmetical understandings, and the basic assumption that many of the methods used in teaching the in-service course were intended to serve as a demonstration of the general procedures to be used in teaching arithmetic to children.

In response to the question regarding materials which the teachers found useful in teaching arithmetic, eight different materials were mentioned. A question arises whether the teachers mentioned materials which they used in their classrooms or the materials which were used in the in-service course. However, subsequent classroom visitations revealed evidence of teacher use of the variety of basic materials demonstrated in the course.
The teachers felt a need for additional manipulative materials, and of particular significance is the fact that three of the six teachers mentioned a desire for a more simple teachers' guide which would provide them with a ready reference for teaching arithmetic.

Five of the six teachers utilized homogeneous grouping for teaching arithmetic, and two of the teachers had as many as three arithmetic groups in their classrooms. This organizational plan was obviously in effect prior to the in-service course, and was not a result of the course. However, the grouping plans indicated the extent to which the teachers were attempting to provide for the individual differences of children in their classrooms.

The fact that the teachers felt that their pupils liked arithmetic was a reflection of the teachers' own attitudes toward the subject.

The greatest single difficulty encountered by the teachers in teaching arithmetic was in helping the children to learn the basic facts.

It was realized that the use of teacher conferences to measure growth in understanding has limitations. Some of the teachers responded more freely and spoke more frankly than others; however, it was felt that the teachers were both sincere and honest in their reactions and that a degree of validity was present in the conferences. A major
limitation of the conferences as a means of measuring growth was the absence of factors of control, and the fact that the questions used in the conferences were largely concerned with current conditions. However, the fact that eighty-three per cent of the teachers stated that the course had helped them to broaden their arithmetical understandings and that they were making use of the demonstrated materials indicates a growth in their basic understanding of arithmetic.

The teacher questionnaire. - The limitations of the teacher questionnaire were in many ways similar to the limitations of the teacher conferences. The questionnaire had the advantage of giving the teachers more time to prepare their answers and of making it possible for them to write their responses. Since the results of the teacher questionnaire did not lend themselves to tabular form they are included in the Appendix.

It is significant that the teachers placed a great deal of stress upon the fact that they enjoy teaching arithmetic because the children enjoy arithmetic. This close relationship of teacher-pupil attitudes is to be expected in arithmetic as in other teaching fields.

In addition to the difficulty of teaching children the basic facts of the whole number operations, which was mentioned by the six teachers in the pilot group, the other groups of teachers made frequent mention of the
difficulties of teaching problem solving and of helping the children develop habits of care and accuracy.

It was thought that teachers would be reluctant to admit that their pupils did not like arithmetic, because such a dislike would reflect upon their teaching methods, therefore the validity of teacher responses to the question regarding the attitudes of their pupils toward arithmetic might be questioned. The two principals replied that "students' liking for the subject diminishes as they reach the upper grades," and also, "Pupils like arithmetic if the teacher likes it, and when the manipulative materials are sufficient to establish real concepts—not the blind feeling that comes with some teaching."

The frequent mention of the wide variety of materials for teaching arithmetic compared favorably with the responses of the pilot group.

The response to the question of whether or not children should understand arithmetic reveals to some extent the emphasis which the teachers place upon understandings, and whether they are able to state clearly the values to be derived from an understanding of the basic concepts of arithmetic.

Only three teachers and the two principals responded to the "additional comments" part of the questionnaire. One of the principals considered the primary grades as the
place in the educational program where the foundations for later success in mathematics are established, and the other principal made reference to the need for teacher education in the area of arithmetic.

The teacher questionnaire did not include a question regarding evaluation of the in-service course, since a separate nine-item evaluative summary was designed for that purpose.

The teacher summary. - Twenty-four teachers and two principals were asked to write an evaluation of the in-service course, and all except one teacher completed such a summary. Seventeen of the twenty-four teachers and principals adhered to the nine-item guide sheet, facilitating a summary of their responses; eight teachers either omitted certain questions or wrote a general summary which did not follow the guide questions.

The evaluative summary, though not lending itself to the usual tabular analysis of data, was undoubtedly one of the most valuable means of measuring growth in understanding, in terms of the changes resulting in the teachers and their work. It is unlikely that teachers would be able to write a summary of an experience which was as comprehensive as were many of the teacher summaries, unless the experience had been a meaningful one for them. The extent to which the responses of the teachers to the evaluative summary favorably compared with the initial objectives of
the course served to evaluate further the effectiveness of the procedures used in the course. The responses to the first three items on the summary sheet varied from teacher to teacher, so it seemed advisable to list the responses instead of attempting to report them in tabular form. It was decided that ten of the most clearly stated responses of the teachers to each of the first three items would be listed.

First item.- The following quotations are the responses of ten teachers to the first item on the summary sheet, which requested that they write a brief but quite specific description of the course giving attention to the major objectives, procedures, specific arithmetical understandings covered, materials used, and the extent to which the course has been practical.

1. This course dealt with the development of understandings underlying the structure of our number system and the four fundamental processes applied to whole numbers, fractions, and decimals. Emphasis was put on place value and the 'tenness' of the Hindu-Arabic system. Introductory material on other systems pointed out the ideas of the symbolism of number and the possibility of a base other than ten. Materials used included wooden sticks, wooden beads of varying sizes, tin cans, the ten-tens counter. The course was a practical value in that it not only developed basic concepts but did so by methods which the teacher can use in her own classroom.

2. The course was planned, I believe, to present the basic understandings of our number system. By comprehending these we will be better able to present meaningful arithmetical concepts and processes to our pupils.
The history of number and computation from primitive methods to our present use of Hindu-Arabic system was developed. From man's need to keep track of his belongings to our processes of computation were many steps of progress. The Hindu-Arabic number system is now universally accepted and its success is based on its symbols, base of ten, place value, and the use of zero as a place holder.

From these latter were developed the processes of computation which we use. Also developed was the relationship of these processes to one another.

Specific arithmetical understandings gained are: (1) addition is counting like things to find the total number. Since our system's base is ten, place value plays an important part in arriving at results. Each number has a definite place and each place has a definite value from ones to tens, hundreds, thousands to billions, or more. (2) Multiplication is a quick way of adding. Again place value is important as 486 is multiplied by 48. The product is eight 486's and forty 486's added together. By moving one place to the left we take care of place value and so mechanically follow the process and sometimes teach it only as such. (3) Subtraction is an opposite process to addition by separating a certain number of objects from another number of same. (4) Division is subtraction performed as many times as possible in order to find out how many times one number is contained in another. In our division process we again learn certain routine steps to follow and do not always consider their meaning.

Estimating and looking for reasonable or sensible answers can increase understandings and can be a practical application of arithmetic concepts.

The abacus was used to bring out the development of the need for place value. A flannel board, beads, sticks or counters of many kinds can be used for concrete illustrations of number.

3. The purpose of the course was to present to teachers a meaningful understanding of the basic concepts in arithmetic. The course was enriched by using concrete, practical methods for the teaching of these basic understandings in the classroom. The course maintained that number and number operations must be taught in such a way that they are meaningful to children. Children must be given experiences which will make clear to them how number functions in the affairs of daily life.
The procedures used were lecture, use of illustrations, discussion, and pre-testing and final testing. The arithmetic understandings covered in the course included the development of our number system, the history of numbers, one-to-one correspondence, grouping, place value, the importance of zero, and the special characteristics of the Hindu-Arabic system which give it simplicity yet unlimited power to represent objects.

The skills developed were: reading and writing numbers; the processes of addition, subtraction, multiplication, and division with whole numbers and also with common fractions and with decimal fractions; estimation in problem solving; and checking.

Some of the materials used were counters, beads, sticks, circles, abacus, pocket chart, and the flannel board.

This course was practical because the materials and methods used, talked about, and presented could be used effectively in the classroom.

4. During this course we were placed in the artificial position of being primitive man faced with the necessity of keeping track of our earthly goods. Armed with such elementary tools as discs, beads, and sticks, we eventually discovered the need for a base on which to group our numbers and all the contingent problems evolving from this need.

In facing and solving these problems, we came up with the same system that thousands and thousands of years of mankind's labor came up with namely the system of arithmetic as we used to know it but now understand it.

In experiencing this evolution ourselves many of the concepts of arithmetic became more apparent.

Other tools used in this experiment were the abacus, flannel boards and number graphs.

5. This course has been most worthwhile and interesting in my opinion. The purpose seemed to be to determine the teachers' concept of number; to develop these concepts; and to encourage a true attempt to give the school child real number understanding.

The time devoted to the history of number in itself gave me a new outlook and a deeper understanding of our number system. We covered basic arithmetic processes and the actual manipulation of materials, such as beads, counters, and counting frames, to explain these processes helped to develop concepts
in the teachers. I found that this handling of materials encouraged me to think of methods to do the same with my class.

6. This course has shown me how closely the study of figures has been related to the history of civilization.

To insure lasting learning process it has been demonstrated time and again throughout the course that understanding is the major objective to mastery, rather than the old fashioned drill method which produces a mere temporary mastery. The understanding of place value is basic to understanding of many important arithmetical concepts, hence, by means of the tens relationship the four fundamental processes plus fractions and decimals are problems conceived with lasting knowledge rather than to satisfy the teacher for the time being, and then forgotten. The abacus and flannel board which I had never used will always be a part of my necessary equipment for classroom procedure.

7. This course as its chief goal was designed to help us (teachers) gain understandings in the arithmetical processes, to show us how manipulative materials need to be utilized to illustrate meanings, to make us aware of how we are teaching presently, and finally to make us teach in the future with emphasis on understanding.

The course began with an inventory of our own understandings. Next, we were taken back to the origin of our number system. In this way the basic principles of our number system were developed before our eyes. These include: symbols; base or group; place value; zero; and enumeration.

Each of these was illustrated by means of such materials as beads, abaci, place holders, and diagrams. Worksheets were passed out and examples were cited.

Following this, we worked with fractions, using the flannel board. An understanding of division of fractions was explained by two types of diagrams: (a) partition; and (b) measurement.

The importance of estimating and use of devices as casting out nines were stressed.

The meaning of addition, subtraction, multiplication, and division and how they are related to one another were explained and each defined.
We were made aware of free materials available for classroom use, and the course finished up with a retake of the previously-taken inventory.

The course was given for practical use. Its suggestions are very helpful. There are, of course, a few devices that I do not intend to try—namely; because I don’t feel that my fourth graders are ready and would be confused, and because of the time element.

8. I felt that the course was a concentrated compilation of some of the basic understandings children need in order to understand their number system. It was apparent that displaying available materials designed for understanding was part of the plan of the course. Because it was concentrated and because it covered a wide range of material, some of which applies to your particular level, it was reasonably practical. As usual, I would prefer emphasis on just what I can use and some pupil demonstration.

9. The course began with a brief survey of counting and the number systems of early civilizations and continued to a close examination of our number system, and computation with it. Advantages of our system were displayed. Materials such as number frames, abacus, games for drill and materials for checking understanding of the number system were used. The course was confined to necessary knowledge for elementary teaching.

10. The course dealt with: (1) the importance of understanding basic concepts in arithmetic; and (2) methods by which a teacher could teach children to develop these basic concepts.

We began with a passing glance at the history of numbers and the development of the basic processes. Next, we stressed the make up of our own number system, fundamental operations, relationships, and generalizations.

The chalkboard, abacus, counting boards, flannel boards, and various type discs and wooden beads were used. We were given helpful printed matter in the form of booklets, pamphlets, and mimeographed materials.

I think the course was very useful. It was certainly a spur to quantitative thinking in arithmetic.
The responses to the first item on the evaluative summary indicate that most of the teachers who participated in the in-service course experienced growth in their understanding of the basic concepts of arithmetic.

**Second item.**—The second item on the summary was, "Has your understanding of the basic concepts of arithmetic changed during this course? How? Give some specific examples." The responses of ten teachers to the second item are as follows:

1. I believe that I have acquired a better understanding of place value and the base ten as it applies to decimals. I can see how moving the point one place to the right multiplies by ten while moving it one place to the left divides by ten. In this course, for the first time, I understood the idea of 'casting out nines.' I have been impressed with the usefulness of estimating as a vital part of computation.

2. My understanding of the basic concepts of arithmetic was clarified by the specific illustrations used by the instructor. Moving from simple counting and symbols to more complicated tallying and then to the use of place value aided by zero as a place holder has increased my background knowledge of my use of numbers. Specifically, why we carry, borrow, move one place to the left in multiplication, etc. has some meaning now; whereas, it used to be purely a mechanical process.

3. Some of my basic concepts of arithmetic have changed since taking this course. In the first steps in arithmetic, it is not desirable to work for speed and skill, but for definite insight and understandings. Children can gain understandings through a wide variety of concrete materials which are inexpensive and at times free.
4. Although our present arithmetic system deals with basic concepts and a former course was an introduction to their value, I feel that the instructor greatly enhanced my knowledge. The use of the abacus to explain our numerical system proved interesting and more understandable.

5. As a student I always enjoyed mathematics of most any sort, but during this course found that although I thought I understood some processes I really did not. My concepts of division of numbers and estimation of the answers after decimal point change have been strengthened. Also, although I knew about the 'tenness' of our number system, I had not understood how very important it is.

6. Being of the 'old school' I have had little to do with the manipulative aids. This course has proved the benefits of such aids in the understanding of basic concepts.

7. I don't believe my understanding of basic concepts changed greatly. I had been acquainted with the base of five and had participated in demonstrations with manipulative materials in college. However, it made me aware that I hadn't been using it to advantage in class. The diagrams of multiplication of fractions were new to me.

8. Yes. The base times base, etc. was new to me. Also, I did not realize how to teach multiplication and division of fractions with concrete materials.

9. Yes, my understanding of the basic concepts of arithmetic have changed. I had never taken a course similar to this before. I secured insights and noted relationships which were new to me. I never stopped to realize why I indented when multiplying by the second digit in the multiplier. I did not know that overlining a Roman numeral designated a different value. I never thought about rationalizing the division algorism, either.

10. I feel my understanding of the basic concepts of arithmetic have definitely improved. I understand the need for a base such as our use of the ten base now which I never did before. The possibility of a base of five or twelve is also clear to me now.
The responses to the second item of the summary expressed a feeling of growth in the teachers themselves. It is significant that teachers with many years of experience in classrooms were not aware of the base of the number system, the place value characteristic of numbers, or the use of manipulative materials for developing an understanding of the basic concepts of arithmetic. Furthermore, even though one teacher felt that she had experienced very little growth in understandings, she felt that the course had helped a great deal by reminding her that she was making little use of the understandings which she "knew". It seems appropriate to challenge the validity of an understanding which is not sufficiently genuine to produce resultant action in the classroom. It is quite likely that the teacher's understanding, in this particular case, was superficial.

Third item. - The third item on the summary was, "Your homework assignments for this course have been that you apply the work of the course in your classroom situations. Carefully describe those parts of the course for which this has been possible, stating what uses you have already made. Also state what uses you expect to make in the future." Ten teachers wrote the following responses to the third item.
1. Since I teach the sixth grade there was much that I could utilize. I reviewed portions of each session with my youngsters. The history of arithmetic tied in nicely with their study of ancient history. They enjoyed the Lattice method of multiplying—the general comment being: 'That's a neat trick.' Many expressed surprise at the fact that the division example with 880 as the dividend, 8 as the divisor, and 110 as the quotient meant that there were 110 eights in 880 and that 110 eights could be taken out of 880. I feel that stressing the latter will help with the enormities of three-figure division. I used the counting board to review place value in decimals. The children liked this—possibly because it was a tangible visible aid. Finding a common denominator in multiplication of fractions was not at all popular—probably because they had already been taught another method. In the future, I intend to spend much more time on explaining ten as the base of our number system.

2. The abacus which I gladly accepted and introduced to my class has already proven itself. Place value is an important aspect in arithmetic in the third grade. In the short time we have used the abacus some of my children have made discoveries of their own. They clearly see and understand the use of zero as a place holder. Before introducing the abacus, I had used a place value pocket board, but still some of the children could not see why and where a zero was used and what purpose it had. To me this proves the use of a variety of manipulative materials that are necessary for understanding meanings and relationships.

3. Many of the things we discussed in class, such as fractions (except halves, thirds, and fourths), decimals, division, multiplication (for the most part), carrying, and casting out nines, though helping to strengthen my understanding, could not be used with my second grade. However, the flannel board and the ten-tens counting frame were greatly appreciated by the children. We have used the counting frame to help understand the place value of ones, tens, and hundreds, and the children enjoyed it even more than I had anticipated. My slower pupils are using the counting frame to check their addition and subtraction facts on daily papers. The flannel board we will use as we start the facts larger than ten. I have also found the counting frame useful in reviewing counting by two's, five's and ten's.
4. I like to use the flannel board with milkbottle tops covered to represent dimes and pennies, to teach carrying in addition and borrowing in subtraction. Thus far I have used the ten-tens counter only with individuals for counting and for help in addition and subtraction, but I expect to use it and the flannel board for teaching multiplication and division. It will be useful earlier in the year in teaching place value and addition and subtraction combinations. I also plan to develop a pocket device for showing place value.

5. Since this course began, I have used the flannel board as an aid in building fraction concepts. I have used both the collections of equal parts of things and fraction scales such as those on rulers and yardsticks.

6. In the classroom I have tried to get the children to 'thick with numbers' and to 'think through' all arithmetic situations. We have tried to do more estimating and stressing the 'reasonableness of the answer.' I have used the flannel board for fractions, and the children use it independently. Also, I have used the abacus with slow youngsters for help in subtraction and multiplication facts. Too, we have done more of this kind of thing to develop understanding and meaning:

$$\begin{align*}
263 \\ \times 317 & \quad \text{equals the sum of} \\
& \quad 7 \times 263 \\
& \quad 300 \times 263
\end{align*}$$

7. I have used a flannel board for some teaching of fractions. As we are just commencing this study, I will use this aid more frequently. What I have seen in the classroom has been very good because the children can 'see' the parts and even play with them, feel them and work them out themselves. A great game for them at this time is to test one another by taking away certain parts of wholes and making simple problems for one another to solve. This is all done in their spare time so I feel it shows some interest and understanding.

As to the future, I am looking forward to using many more materials next year. I would particularly like to take two weeks at the beginning of the year to give the children the same experience I had, of evolving an arithmetic system from beads, et cetera. The first four sessions left a tremendous impression on me, and I would imagine that this same would be true of the children.
8. While working on the course I have become more conscious of the value of the flannel board especially for fractions at the introductory level. Emphasis has been placed on place value of our number system and more time has been spent on concepts pertaining to decimals.

9. Because of the course I have been more conscious of the need for estimation and mental arithmetic, and have done more with my group. I am also planning on splitting the class into three groups; one for continued independent work, one for more drill, and one for re-teaching of number concepts.

10. I was able to apply the use of the flannel board and the abacus in my teaching of arithmetic in the third grade. I tried this out, that is the using of the abacus in teaching units, tens, hundreds—also teaching carrying in addition and borrowing in subtraction. The flannel board was helpful in teaching addition and subtraction facts.

The responses to the third item indicate that the teachers found many aspects of the course practical and adaptable to their classroom teaching, that many of the teachers were already utilizing parts of the course, and that the teachers had been able to project their thinking into the future sufficiently far to determine other practical uses of the experiences they had gained from the course. One teacher's comment upon the resentment of her pupils to the common denominator method for division of fractions, which was new to them, is expressive of the teacher's willingness to experiment with different methods for the teaching of arithmetic. A value of the course for at least one teacher was that of becoming aware of the need to provide for the individual differences of the children in her classroom.
The responses to items four through eight were tabulated and are reported as a part of the Appendix. Both the variety and frequency of responses revealed the diversity of teacher reactions to the in-service course. At one extreme, several teachers were very enthusiastic about the course and seemed to have derived many benefits from it, while at the other extreme a few teachers obviously had gained very little from the course and saw little value in it for other teachers. Many teachers placed strong emphasis on two significant values derived from the course: (1) the opportunity for discussion of mutual problems with teachers in the same school system; and (2) the opportunity, provided by proximity to the classroom, to put to early use what they learned. Also, the inspiration to help children gain a better understanding of arithmetic was mentioned nine times.

The ninth item requested other comments which the teachers desired to make concerning the course, but only nine teachers and the two principals responded. Since replies were repetitious of those to the first eight items, they were neither listed nor tabulated.

The evaluative summaries prepared by the teachers indicated several degrees of teacher growth in arithmetical understandings. It was obvious that some of the teachers' growth had already manifested itself, both in their basic
philosophy of arithmetic education and in their classroom practices. Undoubtedly the extent of growth experienced by each teacher was in a large measure proportional to the teacher's participation in the course, and in her willingness to make application to classroom practices.

**Classroom observations.** It is realized that a one-hour visit to an elementary classroom has limited use in a valid evaluation of the educational program of that classroom; however, some evidence regarding the type of educational program was obtained during the one-hour visit.

A one-sheet classroom observation check list was prepared in an attempt to objectify the results of the observations. After each classroom visit a check list was completed by the observer. Table X is a tabulation of the check list items showing the results of the observations. It seems significant that in twenty-six of the thirty classrooms there was positive evidence of a carry-over from the in-service course, and in fourteen classrooms the carry-over was highly evident.

While verbal comments do not constitute objective evidence, some unsolicited remarks about the in-service course are worthy of mention. The assistant superintendent in one of the four schools involved in the study remarked that he had followed the experiment with a great deal of interest and that he would like to have a similar course
### Table X

<table>
<thead>
<tr>
<th>Check List Items</th>
<th>Highly Evident</th>
<th>Some Evidence</th>
<th>Not Evident</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. To develop an understanding of the basic concepts of arithmetic.</td>
<td>14</td>
<td>11</td>
<td>5</td>
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<tr>
<td>2. To develop desirable attitude toward arithmetic.</td>
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<td>11</td>
<td>5</td>
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<td>3. To teach arithmetic as mechanical drill.</td>
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<td>9</td>
<td>16</td>
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<td>4. To teach the social usefulness of arithmetic.</td>
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<tr>
<td>5. To use the objects-pictures-symbols approach to teaching arithmetic.</td>
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<td>6. To provide for the individual differences of children.</td>
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<td>7. Textbooks, workbooks, chalkboard, paper and pencils.</td>
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<td>8. The abacus, counting frame, flannel board and other manipulative materials.</td>
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<td>9. Guide sheets and diagrams.</td>
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<td>10. Games and enrichment materials.</td>
<td>9</td>
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<td>11. Teacher-made problems.</td>
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<td>13</td>
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<td>12. Emphasis on pupil discovery.</td>
<td>10</td>
<td>12</td>
<td>8</td>
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<tr>
<td>13. Emphasis on estimating, solving, and checking.</td>
<td>14</td>
<td>12</td>
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<tr>
<td>14. A variety of procedures.</td>
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<td>15. Emphasis on speed.</td>
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<tr>
<td>16. Emphasis on accuracy.</td>
<td>23</td>
<td>7</td>
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<tr>
<td>17. Demonstrations and explanations.</td>
<td>16</td>
<td>12</td>
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<tr>
<td>18. Evaluation in terms of textbook.</td>
<td>3</td>
<td>14</td>
<td>13</td>
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<tr>
<td>19. Evaluation in terms of social application.</td>
<td>5</td>
<td>16</td>
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<tr>
<td>20. Evaluation in terms of reasonableness of answer.</td>
<td>14</td>
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<tr>
<td>21. Evaluation by both teacher and pupils.</td>
<td>17</td>
<td>8</td>
<td>5</td>
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<tr>
<td>22. The transfer of learnings from the in-service course.</td>
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<tr>
<td>23. The presence of superficiality in the classroom situation.</td>
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given for other teachers in the school system during the coming year. A principal in one of the schools commented that she was amazed at the favorable attitudes of the teachers toward the course, that the usual complaints about in-service courses had been completely lacking and that the teachers had even demonstrated a high degree of enthusiasm for the course. Similar remarks were volunteered by four other principals during the follow-up visits to the schools. One principal felt that a major factor contributing to the success of the course had been the friendly, helpful manner in which the course was conducted.

The teacher opinion check list.—The forty-item teacher opinion check list was filled out by the thirty teachers and three principals who participated in the course and by one hundred other teachers in the same four schools systems. Table XI gives a comparison of the opinions of the two groups who filled out the check list. The listings are in terms of percentages of responses so that the comparison can readily be seen. A sample of the check list is included in the Appendix.

The fallacy of drawing conclusions based upon specific items of a non-standardized check list is obvious; however, some general patterns of response deserve attention. In general, the opinions of the two groups of teachers were not in sharp contrast, yet the teachers who
Table XI

RESPONSES TO TEACHER OPINION CHECK LIST OF THE TEACHERS IN THE IN-SERVICE COURSE AND ONE HUNDRED OTHER TEACHERS*

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<th>Number of Item</th>
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<th>Disagree</th>
<th>Agree Strongly</th>
<th>Agree</th>
<th>Of Little Significance</th>
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*Reported in percentages of teacher responses.

E = Teachers in in-service course.  C = Teachers in control group.
participated in the course tended to feel more strongly about a majority of the items than members of the control group. At least twenty-three of the forty items showed the experimental group in stronger agreement than the other group, and they demonstrated stronger disagreement on at least two items.

A careful scrutiny of the items upon which the experimental group tended to differ from the control group revealed that many of the items were given attention in the course. The items included: teacher understanding of the basic concepts of arithmetic; the counting frame, abacus, flannel board, and other manipulative materials; in-service education in arithmetic; the importance of estimating and checking; teacher groups to discuss the problems of teaching arithmetic; and classroom teaching procedures.

In addition to the forty-item check list, the teachers were asked to "write a brief paragraph describing what you consider to be the greatest on-the-job need of elementary teachers in the area of arithmetic." Eighty-two per cent of the experimental group and sixty-six per cent of the control group responded to this last item. The teachers in both groups mentioned a wide variety of needs. The five needs most frequently mentioned by the experimental group, in descending order, were: (1) classroom materials, (2) a better foundation of understandings, (3) knowledge of
grade placement, (4) an arithmetic consultant, and (5) an in-service course. The five needs most frequently mentioned by the control group, in descending order, were: (1) basic, concrete materials, (2) better means for effective drill, (3) basic teacher understandings, (4) enrichment materials, and (5) means for developing pupil understanding. A point of contrast is that the control group placed emphasis upon the need for a better program of drill work, but this item was not mentioned by the experimental group.

Summary.- Teacher scores on the Glennon Test of Basic Mathematical Understandings revealed that the elementary teachers participating in the study displayed a knowledge of about fifty-three of the eighty items on the test. An examination of the nature of the items on the test seems to justify the conclusion that a person who understands only fifty-three of the basic concepts tested has a weak foundation for the meaningful teaching of arithmetic.

An analysis of the initial inventory of understandings showed that in ascending order the levels of understanding were: kindergarten, second grade, first grade, third grade, fourth grade, fifth grade, sixth grade, and principals. Also, in each case, the difference from each level to the next higher level was about three or four points. The low scores of the teachers of the first five
grades on "Decimals and process" is an indication of a weakness in the basic understanding of the place value characteristic of the number system. The modest gain by the teachers on the re-test seems to indicate that teacher growth in understandings was not large enough to produce a wide margin of improvement in test scores. Since responses to the test items were either correct or incorrect, thus failing to measure partial understandings, it appears that an objective type test is not the most valid means for measuring the effectiveness of an in-service course.

The results of each of the six methods for determining the effectiveness of procedures for developing arithmetical understandings indicated that most of the teachers who participated in the study experienced growth in the basic understandings, and that for some teachers the extent of the growth was large. Perhaps the most conclusive evidence of teacher growth was found in the results of the teacher summary, in which the teachers evaluated in their own words the effectiveness of the procedures used in the in-service course.
CHAPTER VII

OBSERVATIONS, CONCLUSIONS, AND RECOMMENDATIONS

A basic factor in a learning situation is the willingness of the persons involved to take part in the learning process. Therefore, a desirable program of education should place emphasis upon voluntary learning instead of attempting to force learning. The potential learner should be provided with a set of conditions conducive to learning, together with a motivation to participate in meaningful experiences.

A desirable approach to the in-service education of teachers appears to be that of providing a program of experiences from which the teachers are likely to benefit. An equally important phase of such a program, that of motivating the teachers to participate, involves the problem of self-evaluation, a diagnosis of one's own abilities and inabilities in such a manner as to create a feeling of need for participation in the program provided. This study investigated both phases of an in-service program; however, the order was reversed and motivation was given first consideration. The study utilized a technique intended to create self-evaluation on the part of the
teachers, before making provision for a program of in-service education. It was assumed that a person is more likely to volunteer for that for which he already has a felt need. A problem of in-service education centers around the ability of the persons in charge of the educational program to create a desire within the teachers for the type of experience which will be beneficial for them.

The technique for self-evaluation was that of administering to the teachers a test of basic mathematical understandings. The test was used during the introductory phase of the study, and was followed by an in-service course which provided for the teachers an opportunity to participate in experiences likely to be meaningful for them.

Observations.- The following observations, having a bearing on the procedures to be used for a program of in-service education in arithmetic, were made during the course of the study. Although the observations are limited to this study and to the teachers who were involved, it is felt that a more extensive sampling would yield similar results and that a general application would have some validity.

1. School administrators value arithmetic, and are willing to have their teachers tested in the area of arithmetical understandings.
2. School administrators welcome a program of in-service education, and are agreeable to arranging a meeting with the teachers to discuss the initiation of such a program.

3. School administrators prefer that the final decision regarding an in-service program in arithmetic be left to the teachers.

4. School administrators are interested in the scores of the teachers on a test of basic mathematical understandings, but are willing for the scores to remain a confidential matter between the teachers and the person administering the test.

5. Teachers demonstrate a wide variety of reactions to a test of their basic mathematical understandings. Many are anxious to know their scores, but some are reluctant to sign their name to such a test. Many teachers working at the primary level make excuses for their lack of understanding of arithmetic.

6. A value to be gained by teachers from taking a test of understandings is the opportunity for self-evaluation.

7. The group of teachers in the first six grades of the elementary school having the least knowledge of the basic concepts of arithmetic are the teachers at the second grade level. The group having the best knowledge of arithmetical understandings are teachers working at
the sixth grade level. With the exception of the second grade level, the teachers at each successively higher level have a better knowledge of arithmetical understandings than the teachers at the preceding level.

8. Principals have a better understanding of arithmetic than their teachers.

9. Teachers at all grade levels have a poor understanding of the basic concepts of fractions.

10. Primary teachers have an extremely poor understanding of the basic concepts of decimals.

11. All elementary-school teachers, and particularly those at the primary level, have a poor understanding of the rationale of computation.

12. Most teachers are anxious to participate in an in-service course, and enjoy working with manipulative materials in arithmetic.

13. There appears to be a need for an in-service course in arithmetic which gives consideration to the problems of teaching arithmetic.

14. Even teachers with many years of experience are likely to volunteer for an in-service course in arithmetic.

15. Most teachers will make effective use of manipulative materials in teaching arithmetic if they have experienced the use of the materials, and if they are provided with the materials.
Conclusions.- In order to measure the effectiveness of in-service procedures used to develop growth in arithmetical understandings, six evaluative instruments were used. After using the six instruments to evaluate the course the following conclusions were reached.

1. Teacher growth in arithmetical understandings may be produced by means of the procedures used in an in-service course of this kind.

2. It is advisable to use several means for measuring teacher growth in understandings, as such growth is a slow process which does not lend itself readily to objective measurement. Although improved test scores tend to indicate growth, the re-use of an objective test has definite limitations and should be considered in conjunction with other measurements and in particular with observation of the changes which teachers have made in their classroom methods.

3. Teacher conferences are a valid means for measuring teacher growth in understandings, especially when they are followed by a written questionnaire and summary.

4. A very effective means for evaluating an in-service course is a written summary of the course by each teacher. Teachers show a strong willingness to evaluate the results of such a course, and express stronger feelings about arithmetic in general than does the
broad trend of teacher opinion as measured by a check list.

5. Classroom visitations furnish evidence of the extent of carry over from an in-service course to the elementary classrooms.

6. It is probable that at no level of pre-service or in-service education is the content of a course in the basic concepts of arithmetic more effectively dealt with than at the local level in close proximity to the elementary classrooms.

Recommendations for an in-service program. - The third purpose of the study was to recommend procedures for a program of in-service education intended to improve arithmetic teaching in the elementary school. The following recommendations are based upon the results of the study.

1. An inventory of the basic arithmetical understandings of teachers is strongly recommended as an initial step in an in-service program. Teachers should be approached in a group session, at which time the purpose of the inventory should be explained and discussed with the teachers.

2. Participation of teachers in an in-service course in arithmetic should be a voluntary matter for the teachers, and they should be permitted to attend the first session before feeling obligated to take the course.
3. An in-service course in arithmetic should focus attention upon teaching the basic concepts through an experiential approach designed to promote teacher confidence in working with numbers.

4. In-service credit should be given the teachers for participating in the course.

5. The course should utilize, as much as time will permit, the classroom procedures of discussion and teacher participation, in an atmosphere of patient friendly help for individual difficulties.

6. Manipulative materials such as an abacus, a ten-tens counting frame, and a flannel board should be given to each teacher who participates in the course. The teachers should be encouraged to furnish other inexpensive materials for their classrooms.

7. The course should utilize basic classroom materials for teaching arithmetic, and strong emphasis should be placed upon the adaptation and application of the materials and procedures of the course to the elementary classrooms.

8. It is recommended that the course meet once a week for a period of ten weeks, but that the time and place for the course be determined by mutual consent of the teachers.
9. It is recommended that each class session be not less than one hour nor more than one and one-half hours in length.

10. The class size for the in-service course should be not less than five and not more than twenty teachers.

11. The in-service course should cut across elementary grade levels one through six, and should include teachers at all grade levels.

12. Teachers should be encouraged to share their experiences in an in-service course with other teachers in the same school system.

13. The in-service course should be given in the early part of the school year, so that the teachers will be able to make maximum application of the understandings developed in the course.

14. Both during the course and after its completion, conferences and classroom visitations should be made available to the teachers in order that they may receive additional help in the area of arithmetic.

Recommendations for further study.- An area of in-service education which seems to warrant additional research is that of desirable means for motivating teachers for in-service work in arithmetic. A conclusion of this study is that an inventory test is one desirable means; however, the validity of other means should be investigated.
These might include classroom visitations, individual teacher conferences, seminar sessions, and classroom demonstrations.

A second area for research is that of determining better instruments for evaluating teacher growth in arithmetical understandings. One means not used in this study is that of solicited statements from principals and other administrative and supervisory personnel who have occasion to observe teacher growth. A more comprehensive teacher opinion check list would be a desirable development, not only for measuring teacher growth, but also for use as an instrument for motivation of teachers for in-service work.

Apparently there is need for an in-service course dealing almost exclusively with teaching methods instead of with basic arithmetical understandings. The values of such a course should be determined by means of research.

Summary.- The observations made during this study were based upon procedures for initiating a program of in-service education in arithmetic, and the implementation of in-service procedures for developing arithmetical understandings. The conclusions place emphasis upon the use of six instruments in evaluating the in-service course, and the values which the teachers apparently gained from participating in the course.
In relation to the initiation and implementation of a program of in-service education, fifteen observations were made. A consideration of the use of six evaluative instruments in the study resulted in six conclusions. Fourteen recommendations were made for a program of in-service education designed to promote growth in arithmetical understandings.

Recommendations for further study included the problem areas of: (1) testing the validity of a variety of means for motivating teachers for in-service work in arithmetic; (2) determining better instruments for evaluating teacher growth in arithmetical understandings; and (3) determining the values to be gained from an in-service course dealing almost exclusively with methods for teaching arithmetic.

Growth in arithmetical understandings is considered to be a slow process; yet it is possible to produce a measurable amount of growth within a relatively short period of time. One means for producing teacher growth in arithmetical understandings is that of the in-service procedures used in conducting a course within the school system.
APPENDIX
1. Early methods of record keeping involved the use of marks on the ground, notched trees, notched sticks, knotted cords, and other objects, thus utilizing the concept of one-to-one correspondence.

2. Systems for record keeping were the product of man's inventive genius.

3. Counting is a process of ordering a group in such a manner as to determine both "how many" and "which one."

4. The "cardinal" and "ordinal" characteristics of number were early refinements.

5. Many different number systems have been used in the past 5,000 years.

6. Most early number systems used five, ten, or twenty as a base, and invented a new symbol to represent this number.

7. Number names, number symbols, and compounding plans were the products of man's inventive genius, and were arbitrarily determined.

8. Early systems of notation utilized the basic principles of repetition of symbols, and the addition of assigned values.

9. The principles of multiplication and subtraction were used in early number systems.

10. The ability to use a number system effectively is dependent upon a knowledge of both the symbols and the compounding principles used.

11. Size value is in contrast to place value.

12. The advantages of positional notation or place value extend a number system beyond the base without the invention of new symbols.
13. The abacus is a calculating device which utilizes place value or positional notation, and it is possible to represent consecutive numbers on the abacus, the limit being determined by the number of rods or places.

14. Any of several different bases may be used for a number system utilizing place value.

15. Most bases for a number system have both advantages and disadvantages.

16. The duodecimal system of numbers has many advantages.

17. The Hindu-Arabic system does not use the principles of repetition, but utilizes the principles of addition and place value, which can be demonstrated on the abacus.

18. The Hindu-Arabic number system includes these desirable characteristics:
   a. symbols, including the zero.
   b. a base.
   c. place value, or positional notation.
   d. the principle of addition.

19. Numbers may be read in different ways depending upon the amount of transformation.

20. Addition is a process of combining two groups which may or may not be the same size to form a larger group. However, the groups to be added must have the same "last name," or the quality of likeness.

21. Subtraction is a separation process in which one group is separated into two smaller groups, one of the smaller groups is known, and the second group is to be determined. Groups to be subtracted must have the same "last name," or the quality of likeness.

22. The related facts of addition and subtraction may be easily demonstrated on the ten-tens counting frame.

23. The related facts of addition and subtraction may be easily demonstrated with a chart.
24. The operations of addition and subtraction of whole numbers may be rationalized on the abacus.

25. Whole numbers may be added in any order. This is commonly known as the commutative law for addition.

26. Whole numbers to be added may be grouped in several different ways without changing the final answer. This is commonly known as the associative law for addition.

27. Estimation is a convenient means for determining an approximate answer, and utilizes rounding off numbers.

28. The sequence of estimate, solve, and check is a procedure which is likely to generate confidence in the person doing the operation.

29. Two ways of checking addition are: Adding in a different order, and the excess of nines.

30. The subtraction principle of compensation means that if both the minuend and the subtrahend are increased by the same amount or decreased by the same amount the answer will remain unchanged.

31. There are at least six possible methods of subtraction.

32. A commonly used check for subtraction is addition; however, it is possible to use the excess of nines check.

33. Multiplication is the process of adding or combining two or more groups of equal size.

34. Division is the process of repeated subtraction, and is opposite to the process of multiplication.

35. The multiplication and division facts may be easily demonstrated on the ten-tens counting frame.

36. The multiplication and division facts may be easily demonstrated with a chart.

37. The process of multiplication or division by a power of the base may be accomplished by moving the digits the appropriate number of places to the left for multiplication and to the right for division.
38. Multiplication algorithms were developed and used prior to the standard form which is used extensively today:
   a. Lattice method.
   b. Napier's bones.
   c. Finger multiplication.
   d. Duplation or doubling.
   e. Lightning method.

39. In multiplication a change in the order in which the numbers are multiplied does not change the answer, and this is known as the commutative law for multiplication.

40. The numbers to be multiplied may be re-grouped without changing the answer, and this is known as the associative law for multiplication.

41. Multiplication of multi-place numbers results in distribution into partial products which may be added in any order. This is the distributive law.

42. The sequence of estimate, solve, and check is a procedure which is likely to generate confidence in the person doing the process of multiplication.

43. Either the division check or the excess of nines check may be used for multiplication.

44. There are two different aspects of division: measurement, and partition.

45. Division of multi-place numbers results in distribution into partial quotients which may be added in any order. This is the distributive law.

46. The principle of compensation applies to division; that is, if both the dividend and the divisor are multiplied or divided by the same number, not zero, the quotient will remain the same.

47. The sequence of estimate, solve, and check is a desirable procedure for performing the operation of division.

48. Either the multiplication check or the excess of nines check may be used for division.
49. Estimation is significant in determining a trial quotient.

50. A fraction may be conceived as a symbol used to represent a part of a whole object, or a part of a group of objects.

51. A fraction may be conceived as a symbol to show a division example, or to express the comparative relationship of two numbers.

52. Early civilizations represented fractions by using vertical place value, and by keeping one of the two numbers constant.

53. The Egyptians and the Greeks retained a numerator of one for most of their fractions, while both the Babylonians and the Romans retained constant denominators.

54. The principle of compensation may be applied to fractions. That is, the numerator and the denominator of a fraction may be multiplied or divided by the same number, not zero, without changing the value of the fraction.

55. The reduction of a fraction is based upon the principle of compensation.

56. Reduction of fractions is not a reduction in the strict sense of the word, but is a transformation with no change in value.

57. Addition and subtraction of fractions is dependent upon the quality of likeness. That is, the fractions must have the same "last name," or a common denominator. This is very similar to the principle of likeness used with whole numbers.

58. Multiplication of fractions may be rationalized in each of three cases: a fraction by a whole number; a whole number by a fraction; and a fraction by a fraction.

59. Multiplying by a proper fraction results in a product that is a smaller number than the multiplicand; whereas, dividing by a proper fraction results in a quotient which is larger than the dividend.
60. The inverse relationship of multiplication and division is demonstrated in multiplication by a unit fraction.

61. The commutative and associative principles apply to fractions in a manner very similar to their application to whole numbers.

62. The process of cancellation may be rationalized as a reduction prior to multiplication.

63. Division of fractions by the common denominator method may be rationalized in each of three cases: a whole number by a fraction; a fraction by a whole number; and a fraction by a fraction.

64. Division of fractions by the inversion method may be rationalized in each of three cases: a whole number by a fraction; a fraction by a whole number; and a fraction by a fraction.

65. Decimal fractions are the result of an extension of the Hindu-Arabic number system to include fractions, and therefore use place value.

66. Facility in reading and writing decimals is dependent upon an understanding of the base of the number system.

67. Decimals may be transformed to common fractions and common fractions may be transformed to decimals.

68. The processes of addition and subtraction of decimals may be performed in a manner very similar to that used with whole numbers.

69. The distinguishing feature between operations with whole numbers and operations with decimals is the placement of the decimal point.

70. Multiplication with decimals involves multiplying with whole numbers and locating the decimal point in the product.

71. Division with decimals involves dividing with whole numbers and locating the decimal point in the quotient.

72. Since decimals are an extension of the Hindu-Arabic system, the processes of multiplying by a power of the base or dividing by a power of the base may be performed in a manner similar to that used with whole numbers.
A TEST OF BASIC MATHEMATICAL UNDERSTANDINGS

Directions

NOTE: This test has been copyrighted (1947) by Dr. Vincent J. Glennon, School of Education, Syracuse University. The test has been reproduced, and is being used, by permission of the author.

This is a test to see how well you understand arithmetic. You do not have to do any written work to find the answers. In fact, you will not be permitted to work out any written computations whatsoever.

The test is divided into five parts:

I. The Decimal System of Notation.
II. Basic Understandings of Integers and Processes.
III. Basic Understandings of Fractions and Processes.
IV. Basic Understandings of Decimals and Processes.
V. Basic Understandings of the Rationale of Computation.

Read each statement or question carefully and decide which of the suggested answers is the correct one. Then write the letter for this answer on the proper line on the answer sheet. All answers are to be recorded in this way on the separate answer sheet. MAKE NO WRITTEN MARKS WHATSOEVER ON ANY OF THE TEST SHEETS.

Sample Item

Which of the following numbers has the largest value?

A. 23  B. 9  C. 35  D. 45  E. 11

Since 45 is the correct answer, you would write the letter D on the proper line on the answer sheet.

Try each example but do not stay too long on any one statement or question. If you cannot find the answer you may go on to the next example and come back to the one which you omitted if time permits.

You may go all the way through the test without stopping. When you finish the examples in one section, go right on to the next section.
In Section III you will find shaded diagrams similar to the one at the right of this page. This diagram should be read as $\frac{3}{4}$ (i.e., three-fourths). Read all diagrams in this way. Remember: The value of the fraction is indicated by the white or unshaded part of the diagram.

When you are told to do so, begin at the top of the next page and proceed thru the test in the manner which has been indicated.

Remember: DO NO WRITTEN WORK TO FIND THE ANSWERS. Make no written marks on any of the test sheets. Record the letter of your choice for each correct answer on the proper line on the answer sheet.

Section I. The decimal system of notation.

1. If you rearranged the figures in the number 43,125 which of the following arrangements would give the smallest number?
A. 54,321  
B. 21,345  
C. 12,345  
D. 14,532  
E. 13,245

2. If you rearranged the figures in the number 53,429 which of the following arrangements would give the largest number?
A. 95,324  
B. 95,432  
C. 59,432  
D. 95,234  
E. 95,243

3. Which of the following has a 3 in the hundreds' place?
A. 23,069  
B. 86,231  
C. 49,563  
D. 39,043  
E. 42,304

4. In the number 2,222 the 2 on the left represents a value how many times as large as the 2 on the right?
A. 1 (same value)  
B. 10  
C. 100  
D. 200  
E. 1,000

5. About how many tens are there in 6542?
A. 65  
B. 65 1/2  
C. 654  
D. 6,540  
E. 65,000

6. If the figures in 23,469 were rearranged, which of the following would place the smallest figure in the tens' place?
A. 46,932  
B. 96,432  
C. 69,234  
D. 34,629  
E. 92,346
7. In the number 7,255 the 5 on the left represents a value how many times as large as the 5 on the right?
   A. 1 (same value)  B. 2  C. 5  D. 10  E. 100

8. Which of the following statements best tells why we write a zero in the number 4,039 when we want it to say "four thousand thirty-nine?"
   A. Because the number would say 'four hundred thirty-nine' if we wrote a zero in some other place.
   B. Writing a zero helps us to read the number.
   C. Writing a zero tells us to read the hundreds' figure carefully.
   D. Because the number would be wrong if we left out a zero some place.
   E. Because we use zero as a place-holder to show that there is no amount to record in that place.

9. Which of the following has a 4 in the ten thousands' place?
   A. 423,102  B. 643,142  C. 433,116  D. 374,942  E. 763,420

10. If the figures in 86,473 were arranged differently, which of the following would place the largest figure in the thousands' place?
   A. 73,648  B. 38,467  C. 76,483  D. 87,643  E. 86,734

11. In the number 3,944 the 4 on the right represents a value how many times as large as the 4 on the left?
    A. 1/10  B. 1/2  C. 5  D. 1 (same value)  E. 10

12. In the number 5,492 the 4 represents a value how many times as large as the 2?
    A. 2  B. 10  C. 20  D. 100  E. 200

13. About how many hundreds are there in 34,820?
    A. 3 1/2  B. 35  C. 350  D. 3,500  E. 35,000

14. Which of the following methods is the best for determining the value of a figure in a number? For example, the value of the 7 in 3748.
    A. Its position in the number.
    B. Its value when compared with other figures in the number.
    C. Its value in the order from 1 to 9.
D. Its value when compared with the whole of the number.
E. Its position in the number and its value.

15. In the number 7,843 the 4 represents a value how many times as large as the 8?
A. \(\frac{1}{10}\)  B. \(\frac{1}{20}\)  C. \(\frac{1}{2}\)
D. 2  E. 20

(Go right on to Section II)

Section II. Basic understandings of integers and processes.

1. If you had a bag of 365 marbles to be shared equally by 5 boys, which would be the quickest way to determine each boy's share?
A. counting  B. adding  C. subtracting
D. multiplying  E. dividing

2. When a whole number is multiplied by a whole number other than 1, how does the answer compare with the whole number multiplied?
A. larger  B. smaller  C. same
D. 10 times as large  E. can't tell

3. When a whole number is divided by a whole number other than 1, how does the answer compare with the whole number divided?
A. larger  B. smaller  C. same
D. one-half as large  E. can't tell

4. Which of the following is the quickest way to find the sum of several numbers of the same size?
A. by counting  B. by adding  C. by subtracting
D. by multiplying  E. by dividing

5. If the zeros in the two numbers in this example were left off, how would the answer be charged?
A. The answer would be ten times as large.  \(\frac{60}{3720}\)
B. The answer would be one hundred times as large.
C. The answer would be one-tenth as large.
D. The answer would be one-hundredth as large.

6. Here is an example in subtraction in which letters have been used instead of figures. Which statement is true?
A. \(AFGB + CXU\) added together equal \(TWMY\).
B. \(CXU + TWMY\) added together equal \(AFGB\).  \(AFGB\)
C. \(AFGB + TWMY\) added together equal \(CXU\).  \(-CXU\)
D. \(TWMY\) subtracted from \(CXU\) equals \(AFGB\).  \(TWMY\)
E. \(CXU\) subtracted from \(TWMY\) equals \(AFGB\).
7. How would the answer to this example be changed, if a zero were added (annexed) to the right of each number?
   A. The answer would be ten times as large.  
   B. The answer would be one hundred times as large. 
   C. The answer would not change. 
   D. Cannot tell until you add both ways. 
   E. The answer would be one thousand times as large.

8. Adding (annexing) two zeros to the right of a whole number has the same effect as:
   A. Adding ten to the number. 
   B. Adding one hundred to the number. 
   C. Multiplying the number by ten. 
   D. Multiplying the number by one hundred. 
   E. Dividing the number by one hundred.

9. What would be the effect on the answer if you added (annexed) two zeros to 439 and took away the zero from 450?
   A. The answer would be ten times as large. 
   B. The answer would be one hundred times as large. 
   C. The answer would remain the same. 
   D. The answer would be one-tenth as large. 
   E. The answer would be one-hundredth as large.

10. Crossing off a zero from the right side of a number has the same effect as:
    A. Subtracting ten 
    B. Subtracting one hundred 
    C. Multiplying by ten 
    D. Multiplying by one 
    E. Dividing by ten

11. What would be the effect on the answer if you added (annexed) two zeros to 92 and changed 4500 to 450?
    A. The answer would be ten times as large. 
    B. The answer would be one-tenth as large. 
    C. The answer would be one hundred times as large. 
    D. The answer would be one-hundredth as large. 
    E. The answer would be one-thousandth as large.

12. Which one of the following methods could be used to find the answer to this example?
    A. Multiply 17 by the quotient. 
    B. Add 17 six hundred twelve times. Answer would be the sum. 
    C. Subtract 17 from 612 as many times as possible. Answer would be number of times you were able to subtract.
D. Add 612 seventeen times. Answer would be the sum.
E. Multiply 17 by 612. Answer would be the product.

13. If the numbers in a large addition example were changed so that the top number was placed at the bottom and the bottom number was placed at the top, how would the answer be affected?
A. Answer would be larger.  B. Answer would be smaller.  
C. Answer would not change.  D. Could not do the example.  
E. Cannot tell until you add both ways and compare.

14. How would the example be affected if you put the 29 above 4306?
A. The answer would be larger.  
B. The answer would be smaller.  
C. The answer would be the same.  
D. Cannot tell until you multiply both ways.  
E. You cannot do the example when the large number is on the bottom and the small number on top.

15. What would be the effect on the answer if you added (annexed) two zeros to 39?
A. The answer would be one hundred times as large.  
B. The answer would be one-hundredth as large.  
C. The answer would be one-thousandth as large.  
D. The answer would not change.  
E. You could not do the example.

(Go right on to Section III)

Section III. Basic understandings of fractions and processes.

1. Which of the following fractions is the largest?
A. 1/7  B. 5/7  C. 3/7  D. 11/7  E. 6/7

2. Which of these statements best tells why we cannot say that the unshaded parts of this picture represent 5 "eights"?
A. Because more than 5/8 of it is unshaded.  
B. Because the unshaded parts are not together.  
C. Because all the unshaded parts are not the same size.  
D. Because less than 5/8 of it is unshaded.  
E. Because the parts are not the same shape.
3. Which of the following fractions is the smallest?
   A. $\frac{1}{9}$  B. $\frac{1}{5}$  C. $\frac{1}{2}$  D. $\frac{1}{7}$  E. $\frac{1}{3}$

4. Which picture shows how the result would look if you divided the numerator and denominator of $\frac{10}{8}$ by 2?
   A. 
   B. 
   C. 
   D. 
   E. 

5. When a whole number is multiplied by a common (proper) fraction, how does the answer compare with the whole number?
   A. larger  B. smaller  C. same  D. cannot tell  E. half as large

6. Which picture shows how the result would look if you divided the numerator of this fraction by 2?
   A. 
   B. 
   D. 
   E. 

7. Which picture best shows the example, $4 \times \frac{2}{3}$?

A. 
B. 
C. 
D. 
E. 

8. When a common (proper) fraction is divided by a common fraction, how does the answer compare with the fraction divided?
A. larger  B. smaller  C. same  D. cannot tell  
   E. twice as large

9. Which picture shows how the result would look if you multiplied the numerator and denominator of $\frac{3}{5}$ by 2?

A. 
B. 
C. 
D. 
E. 

10. Which picture shows how the result would look if you multiplied the denominator of this fraction by 2?

A.  

B.  

C.  

D.  

E.  

11. When a whole number is divided by a common (proper) fraction, how does the answer compare with the whole number?

A. larger  

B. smaller  

C. same  

D. cannot tell  

E. varies  

12. Which picture looks like this example: $3 \div 1/2$?

A.  

B.  

C.  

D.  

E.  

13. Which sentence best tells why the answer is larger than the 5?

$5 \div 3/4 = 6 \frac{2}{3}$

A. Because inverting the divisor turned the $3/4$ upside down.  

B. Because multiplying always makes the answer larger.  

C. Because the divisor $3/4$ is less than 1.  

D. Because dividing by proper and improper fractions makes the answer larger than the number divided.  

E. Inverting a fraction puts the larger number on top.
14. Which sentence is shown by this picture?

A. Fractions with common denominators may be added.
B. The value of a fraction is changed if a number is subtracted.
C. Dividing the numerator and denominator of a fraction by the same number does not change the value of the fraction.
D. Fractions with the same denominators are equal.
E. Fractions with the same numerators are equal.

15. When a common (proper) fraction is multiplied by a common fraction, how does the answer compare with the fraction multiplied?
A. larger  B. smaller  C. same  D. cannot tell  E. varies

Section IV. Basic understandings of decimals and processes.

1. How should you write the decimal, "eighty and eight hundredths"?
(A) .8008 (B) 80.800 (C) 80.08 (D) 80.008 (E) 8008.08

2. How should you read this decimal: .0309?
A. Three and nine hundredths.
B. Three hundred nine thousandths.
C. Three hundred nine ten-thousandths.
D. Thirty-nine thousands.
E. Three hundred nine hundredths.

3. Which decimal tells how long line Y is when compared with line X?
line X  __________
line Y  __________
(A) .5  (B) .625  (C) 1.25  (D) 76  (E) 33

4. About how many tenths are there in 1.25?
(A) .13  (B) 1.3  (C) 13  (D) 125  (E) 1250

5. About how many hundredths are there in .635?
(A) 1/2  (B) 6.35  (C) 63.5  (D) 635  (E) 6350

6. What would be the effect of the answer if you dropped the zero from 23.90?
(A) The answer would have the same value 23.90
(B) The answer would be one-tenth as large. x 2.75
(C) The answer would be ten times as large.
(D) You would point off three places.
(E) It would be the same as subtracting zero from the answer.

7. How would the answer be changed if you changed 6.5 to .65 and 84.5 to 845?
A. The answer would be the same.  
B. The answer would be ten times as large.  
C. The answer would be one hundred times as large.  
D. The answer would be one-tenth as large.  
E. The answer would be one-hundredth as large.

8. Which seems to be the correct answer to this example:  
   ten divided by five-tenths.  
(A) 1/2  (B) 2  (C) 10  (D) 20  (E) 50

9. Which decimal tells how long line Y is when compared with X?
   Line X ____________  
   Line Y ________________  
(a) 1.25  (B) 1.50  (C) 2  (D) 2.40  (E) 2.50

10. Which of the following decimals has the largest value?
    (A) 30.3  (B) 30.03  (C) 20.0333  (D) 30.303  
    (E) 30.003

11. What would be the effect on the answer if you changed 368 to 3680 and 24 to 2.4?
    A. The answer would be smaller.  
    B. It would not change the answer  
    C. It would be the same as adding a zero to the answer.  
    D. The answer would be one-tenth as large.  
    E. Cannot tell until you do the example both ways.

12. Which decimal has the smallest value?
    (A) .3  (B) .09  (C) .048  (D) .693  (E) .0901

13. How would the answer be affected if you moved the point one place to the left in both numbers?
    (A) The answer would be one-tenth as large.  
    (B) The answer would be one-hundredth as large.  
    (C) The answer would be one hundred times as large.  
    (D) It would be the same as subtracting 100 from the answer.  
    (E) The answer would have the same value.
14. How would the answer be changed if you moved the point two places to the right in both numbers?
   A. The answer would have the same value
   B. The answer would be one thousand times as large.
   C. You would point off differently.
   D. You cannot move the point in the top number two places.
   E. The answer would be 10,000 times as large.

15. How would the answer be affected if you moved the point in 485.3 one place to the right?
   A. The answer would be ten times as large.
   B. The answer would be 10 larger.
   C. The answer would be one-tenth as large.
   D. The answer would have a zero at the right.
   E. The value of the answer would be the same.

16. How would the answer be affected if you changed 7.3 to 73 and 1390 to 13.90?
   A. The answer would be one hundred times as large.
   B. The answer would be one-tenth as large.
   C. The answer would be one thousand times as large.
   D. The answer would be one-hundredth as large.
   E. The answer would be one-thousandth as large.

17. About how many tenths are there in .055?
   (A) 0  (B) 1/2  (C) 5  (D) 10  (E) 50

18. About how many thousandths are there in 16.5?
   (A) 1.7  (B) 18  (C) 170  (D) 1,700  (E) 17,000

19. Why is the answer smaller than the top number?
   A. Because 8 is more than .5
   B. Because you are finding how many .5's in 8. 8 x .5
   C. Because .5 is less than 8.
   D. When you multiply by a decimal the answer is always smaller than the top number.
   E. Because multiplying by .5 is the same as finding half of the number.

20. How would the answer be changed if you changed 1.47 to 147?
   A. You would get the same answer.
   B. The answer would be ten times as large.
   C. The answer would be one hundred times as large.
   D. The answer would be one-tenth as large.
   E. The answer would be one-hundredth as large.
Section V. Basic understandings of the rationale of computation.

1. Why do we find a common denominator when adding fractions with unlike denominators?
   A. You cannot add together things that are different.
   B. It is easier to add fractions when they have a common denominator.
   C. The denominators have to be the same in order to add.
   D. We learned to add unlike fractions that way.
   E. So that all the fractions will have the same value.

2. When dividing by a decimal, why do we move the point to the right?
   A. Multiplying by a multiple of ten is a quick way of changing a decimal to a whole number.
   B. It places the decimal point in the quotient correctly.
   C. You can only divide by a whole number.
   D. To make the divisor equal to the dividend.
   E. It is easier to divide by a whole number than a decimal.

3. Which one of the following would give the correct answer to this example? 2.1 x 21
   A. The sum of 1 x 2.1 and 21 x 2.1
   B. The sum of 10 x 2.1 and 2 x 2.1
   C. The sum of 10 x 2.1 and 20 x 2.1
   D. The sum of 1 x 2.1 and 20 x 2.1
   E. The sum of 1 x 2.1 and 2 x 2.1

4. Which statement best tells why we "invert the divisor and multiply" when dividing a fraction by a fraction?
   A. It is an easy method of finding a common denominator and arranging the numerators in multiplication form.
   B. It is an easy method for dividing the denominators and multiplying the numerators of the 2 fractions.
   C. It is a quick, easy and accurate method of arranging two fractions in multiplication form.
   D. Dividing by a fraction is the same as multiplying by the reciprocal of the fraction.
   E. It is a quick method of finding the reciprocals of both fractions and reducing to lowest terms (cancelling).

5. Why do we move the second partial product one place to the left when we multiply by the 6?
   A. Because the answer has to be larger than 729.
   B. Because the six means six tens.
   C. Because 6 is the second figure in 68.
   D. Because we learned to multiply that way.
E. Because the 6 represents a greater value than the 8 represents.

6. Which statement best tells why we arrange numbers in addition the way that we do?
A. It is an easy way to keep the numbers in straight columns.
B. It helps us to add correctly.
C. It helps us add only those numbers in the same position.
D. It helps us to carry correctly from one column to another.
E. It would be harder to add if the numbers were mixed.

7. When you multiply by the 4 in 48 you will get a number that is how large compared with the final answer?
A. One-twelfth as large.
B. One-tenth as large.
C. One-half as large.
D. Five-sixth as large.
E. Twice as large.

8. The answer to this example will be how large when compared with the 69?
A. Twice as large.
B. Sixty-nine times as large.
C. One sixty-ninth as large.
D. Eight hundred twenty-seven times as large.
E. 1 as large.

9. Which statement best tells why it is necessary to "borrow" in this example?
A. Because the top number is smaller than the bottom number.
B. You cannot subtract 92 from 67.
C. You cannot subtract 9 tens from 6 tens.
D. You cannot subtract 39 tens from 56 tens.
E. You cannot subtract 9 from 6.

10. Which statement best tells why we carry 2 from the second column?
A. The sum of the second column is 23 which has two figures in it. We have room for the 3 only, so we put the 2 in the next column.
B. The sum of the second column is more than 20, so we put the 2 in the next column.
C. Because we learned to add that way.
D. The value represented by the figures in the second column is more than 9 tens, so we put the hundreds in the next column.
E. If we do not carry the 2, the answer will be 20 less than the correct answer.

11. In this example you multiply by the 6, then by the 3. How do the two results (partial products) compare?
A. The second represents a number one-half as large as the first. 749
B. The second represents a number twice as large as the first. 36
C. The second represents a number five times as large as the first.
D. The second represents a number ten times as large as the first.
E. The second represents a number twenty times as large as the first.

12. Which would give the correct answer to 439 x 563?
A. Multiply 439 x 3; 439 x 6; 439 x 5 then add answers.
B. Multiply 439 x 3; 439 x 63; 439 x 563; then add answers.
C. Multiply 563 x 9; 563 x 3; 563 x 4 then add answers.
D. Multiply 563 x 9; 563 x 39; 563 x 439 then add answers.
E. Multiply 439 x 3; 439 x 60; 439 x 500 then add answers.

13. Which statement best explains the 4 in the answer?
A. The 4 means that there are forty-eight 26's in 1248.
B. The 4 in the answer means that there are four 26's in 1248.
C. The 4 means that 2 goes into 12 four times, and 5 would be too large.
D. The 4 means that there are at least forty 26's in 1248.
E. The 4 means that the answer will come out even.

14. Here is an example in subtraction of mixed numbers in which it is necessary to "borrow." Which statement best explains the borrowing.
A. you cannot subtract 5/8 from 3/8, so you take 1 from the 5 and put in front of the 3 making 13.
B. You cannot subtract 5/8 from 3/8 so you add the 3 and the 8 making 11/8.
C. You cannot subtract 5/8 from 3/8, so you turn them around and subtract 3/8 from 5/8.
D. You cannot subtract 5/8 from 3/8, so you take 1 from the 5 and add it to 3/8 making it 4/8.
E. You cannot subtract $\frac{5}{8}$ from $\frac{3}{8}$, so you take 1 from the 5 and change it to $\frac{8}{8}$; then add the $\frac{8}{8}$ to $\frac{3}{8}$ making $\frac{11}{8}$.

15. Which statement best explains what happens when you reduce a fraction to lowest terms?
A. The size of the terms and the value of the fraction become smaller.
B. The value of the fraction does not change. The size of the part represented by the new denominator is smaller, and the number of parts represented by the new numerator is less.
C. The value of the fraction does not change. The terms are smaller, but they represent more parts of larger size.
D. The value of the fraction does not change, but the parts of the fraction represented by the new numbers become fewer in number and larger in size.
E. The value of the fraction changes because the new numbers are smaller.


## ANSWER SHEET FOR A TEST OF BASIC MATHEMATICAL UNDERSTANDINGS

<table>
<thead>
<tr>
<th>Section I</th>
<th>Section II</th>
<th>Section III</th>
<th>Section IV</th>
<th>Section V</th>
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<tbody>
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</tr>
</tbody>
</table>

**TOTAL SCORE:** Wrong _____ Right _____
TEACHER DATA SHEET

1. Degrees:
   ____ None       ____ Master's       ____ Doctor's
   ____ Bachelor's ____ Master's plus thirty hours

2. College Training:
   ____ Teachers College       ____ University - School of
       ___ Liberal Arts.       ___ Other.

3. Courses in Arithmetic Methods:
   ____ Undergraduate. More than one course? ____ When? ____
   ____ Graduate or in in-service. When? ____

4. Courses in Mathematics:
   High School:      ____ Arithmetic       ____ Algebra I       ____ Algebra II
                      ____ Gen. Math.       ____ Plane Geometry       ____ Solid
                      ___ Geometry
   College:            ____ Gen. Math.       ____ Algebra       ____ Geometry
                         ____ Calculus       ____ Other courses

5. Number of years teaching experience: ____ years.

6. Levels of teaching experience:
   Kdgn. ____ yrs.      4th ____ yrs.
   1st ____ yrs.       5th ____ yrs.
   2nd ____ yrs.       6th ____ yrs.
   3rd ____ yrs.  Other teaching ____ yrs.
GUIDE QUESTIONS FOR TEACHER CONFERENCES

1. Have you found the course helpful? If so, how?

2. What are your criticisms of the course?

3. What materials do you find helpful in teaching arithmetic?

4. What additional materials do you feel a need for in teaching arithmetic?

5. How many groups do you have in arithmetic?

6. What is the general attitude of your students toward arithmetic?

7. What interest in arithmetic is shown by the parents of your pupils?

8. What difficulties do you find in teaching arithmetic?

9. What part of arithmetic do you find easy to teach?

10. What are your general comments about the course?
TEACHER QUESTIONNAIRE

Name ______________________  Grade now teaching ______

1. How would you describe your understanding of the basic concepts of arithmetic?
   ___ well below ___ below ___ average ___ above ___ above average average average

2. Do you enjoy teaching arithmetic?
   Why, or why not?

3. Do you find it easy to teach arithmetic?
   Why, or why not?

4. What do you consider as your three greatest difficulties in teaching arithmetic?

5. Do you feel a need for an arithmetic consultant in your school?

6. How many children in your classroom? ______

7. In a sentence, describe the attitude of your pupils toward arithmetic.

8. Do you have more than one arithmetic group in your classroom? If yes, how many groups?

9. List the materials which you find most helpful in teaching arithmetic.
TEACHER QUESTIONNAIRE CONCLUDED

10. If the school department should approve an unlimited amount of money to be spent for arithmetic materials, what would you order?

11. Do you feel that youngsters should understand what they are doing in arithmetic? Why, or why not?

12. Other comments which you may wish to make regarding the teaching of arithmetic.
**TEACHER RESPONSES TO ITEMS ONE THROUGH ELEVEN OF THE TEACHER QUESTIONNAIRE**

<table>
<thead>
<tr>
<th>Items</th>
<th>Responses of Teachers</th>
<th>Frequency of Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. How would you describe your understanding of arithmetic?</td>
<td>Average</td>
<td>17</td>
</tr>
<tr>
<td></td>
<td>Above average</td>
<td>9</td>
</tr>
<tr>
<td>2. Do you enjoy teaching arithmetic? Yes</td>
<td>It is so easy to see a child make improvement in arithmetic</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>I have always enjoyed doing arithmetic myself</td>
<td>8</td>
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<tr>
<td></td>
<td>It is a concrete subject</td>
<td>7</td>
</tr>
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<td></td>
<td>Because the children are so anxious to learn arithmetic</td>
<td>6</td>
</tr>
<tr>
<td>3. Do you find it easy to teach arithmetic? Yes</td>
<td>Children enjoy it</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>I have always enjoyed numbers</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>It is so definite and so easy to evaluate</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>Children with difficulties in arithmetic present quite a problem</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>No</td>
<td>4</td>
</tr>
<tr>
<td>Items</td>
<td>Responses of teachers</td>
<td>Frequency of Response</td>
</tr>
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<tr>
<td>4. What do you consider as your three greatest difficulties in teaching arithmetic?</td>
<td>Teaching problem solving</td>
<td>13</td>
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<td></td>
<td>Helping children to learn the facts</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>Helping children to overcome the habit of making careless mistakes</td>
<td>8</td>
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<td></td>
<td>Lack of pupil understanding of number concepts</td>
<td>7</td>
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<td></td>
<td>Provision for individual differences</td>
<td>7</td>
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<td></td>
<td>Teaching dollars and cents</td>
<td>4</td>
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<td></td>
<td>Teaching division of whole numbers</td>
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<td></td>
<td>Maintaining an interest in arithmetic</td>
<td>3</td>
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<td></td>
<td>Teaching the operations with fractions</td>
<td>2</td>
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<td></td>
<td>Justifying drill</td>
<td></td>
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<tr>
<td>5. Do you feel the need for an arithmetic consultant in your school?</td>
<td>No</td>
<td>18</td>
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<td></td>
<td>Yes</td>
<td>6</td>
</tr>
<tr>
<td>6. How many children in your classroom?</td>
<td>18</td>
<td>1</td>
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<td></td>
<td>19</td>
<td>1</td>
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<td>21</td>
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<td>32</td>
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<td>36</td>
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<td>40</td>
<td>1</td>
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<tr>
<td>Items</td>
<td>Responses of teachers</td>
<td>Frequency of Response</td>
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<tr>
<td>-------</td>
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<tr>
<td>7. In a sentence, describe the attitudes of your pupils toward arithmetic.</td>
<td>Most of them like it</td>
<td>10</td>
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<tr>
<td></td>
<td>They enjoy it</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>Most of them like to do the operations, but are baffled by problems.</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>A most favorable attitude.</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>They are willing to work at it.</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>The above statements are in essence the responses of the teachers.</td>
<td></td>
</tr>
<tr>
<td>8. Do you have more than one arithmetic group in your classroom?</td>
<td>Yes</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td>No</td>
<td>13</td>
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<tr>
<td></td>
<td>Two</td>
<td>8</td>
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<tr>
<td></td>
<td>Three</td>
<td>3</td>
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<tr>
<td>If yes, how many groups?</td>
<td>Some individual help</td>
<td>8</td>
</tr>
<tr>
<td>9. List the materials which you find most helpful in teaching arithmetic.</td>
<td>Flannel board</td>
<td>14</td>
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<tr>
<td></td>
<td>Textbook</td>
<td>12</td>
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<tr>
<td></td>
<td>Abacus</td>
<td>10</td>
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<td></td>
<td>Place value chart</td>
<td>9</td>
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<tr>
<td></td>
<td>Workbooks</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>Chalkboard</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>Objects</td>
<td>8</td>
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<td></td>
<td>Measuring devices</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>Counting frame</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>Flashcards</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>Charts</td>
<td>5</td>
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<tr>
<td></td>
<td>Clock face</td>
<td>4</td>
</tr>
<tr>
<td>10. If the school department should approve an unlimited amount of</td>
<td>Flannel board cut-outs</td>
<td>11</td>
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<td></td>
<td>Games to drill facts</td>
<td>9</td>
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<td></td>
<td>Enrichment materials</td>
<td>8</td>
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<td></td>
<td>Workbooks</td>
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<td></td>
<td>Measurement devices</td>
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<td></td>
<td>Counting frames</td>
<td>5</td>
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<tr>
<td></td>
<td>Textbooks</td>
<td>5</td>
</tr>
<tr>
<td>Items</td>
<td>Responses of teachers</td>
<td>Frequency of Response</td>
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<tr>
<td>money to be spent for arithmetic materials, what would you order?</td>
<td>One or more abaci, Play money, Flannel board, Commercial charts, Flashcards, Individual clocks, Reference books, Arithmetic films, Self-help cards</td>
<td>5, 4, 3, 3, 2, 2, 2, 1</td>
</tr>
</tbody>
</table>

11. Do you feel that children should understand what they are doing in arithmetic? Why, or why not? Yes

- Understanding helps to build confidence and independence: 7
- Understanding leads to greater interest and accuracy: 5
- Understanding helps to improve problem solving ability: 4
- Understanding aids learning: 3
- Understanding leads to social application: 1
- Understanding helps pupils to enjoy arithmetic: 1
- Understanding should begin in the primary grades, and be built up gradually, grade by grade: 1
- The above statements are in essence the responses of the teachers.

I have mixed feelings on this. My brighter students can understand why they do certain things, whereas some cannot see the why, but can do the process very well.
EVALUATIVE SUMMARY OF A COURSE IN THE BASIC UNDERSTANDINGS OF ARITHMETIC.

The following questions are meant to be used as a guide in preparing your summary of this course. I am hopeful that you will have some additional comments for question 9.

1. Write a brief but quite specific description of this course as you see it, giving attention to our major objectives, procedures, specific arithmetical understandings covered, materials used, and the extent to which the course has been practical.

2. Has your understanding of the basic concepts of arithmetic changed during this course? How? Give some specific examples.

3. Your "homework" assignments for this course have been that you apply the work of the course in your classroom situations. Carefully describe those parts of the course for which this has been possible, stating what uses you have already made. Also state what uses you expect to make in the future.

4. What values have you gained from this course that you feel could not have been gained by taking a similar course on a college campus?

5. What inspiration for teaching arithmetic have you derived from this course?

6. Which parts of the course were new to you?

7. If you have discussed this course with other teachers who have not taken it, what has been their interest?

8. Would you recommend that a course such as this be an in-service requirement for all elementary teachers? Why, or why not?

9. Other comments which you desire to make concerning this course.
## RESPONSES OF TEACHERS TO ITEMS FOUR THROUGH EIGHT OF THE SUMMARY OF THE COURSE

<table>
<thead>
<tr>
<th>Items</th>
<th>Responses of teachers</th>
<th>Frequency of Response</th>
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</thead>
<tbody>
<tr>
<td>4. What values have you gained from this course that you feel could not have been gained by taking a similar course on a college campus?</td>
<td>Opportunities for discussion of mutual problems with teachers in same school.</td>
<td>18</td>
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<td></td>
<td>Early use of learnings, due to proximity of the classroom.</td>
<td>7</td>
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<td>A homogeneous group.</td>
<td>6</td>
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<td></td>
<td>Convenient as to time and place</td>
<td>4</td>
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<tr>
<td></td>
<td>Materials geared to classroom use</td>
<td>3</td>
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<td></td>
<td>It was good not to have undergraduates and laymen in the group, thus avoiding impertinent discussions.</td>
<td>1</td>
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<td></td>
<td>None</td>
<td>1</td>
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</table>

Nineteen teachers responded to the fourth item. Some teachers mentioned more than one item.

<table>
<thead>
<tr>
<th>Items</th>
<th>Responses of teachers</th>
<th>Frequency of Response</th>
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<tbody>
<tr>
<td>5. What inspiration for teaching arithmetic have you derived from this course?</td>
<td>To help children to gain a better understanding of arithmetic.</td>
<td>10</td>
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<tr>
<td></td>
<td>An incentive to make wider use of visual aids, games, and devices.</td>
<td>9</td>
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<td></td>
<td>A better understanding of the basic concepts of arithmetic.</td>
<td>6</td>
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<td></td>
<td>To make certain that understanding precedes drill.</td>
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<td></td>
<td>Arithmetic can be fun</td>
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</table>

Twenty-two teachers responded to the fifth item. Some teachers mentioned more than one item.
Items | Responses of teachers | Frequency of Response
--- | --- | ---
6. Which parts of the course were new to you? | Historical development of numbers | 9
 | The many excellent uses of the abacus | 8
 | The importance of manipulative materials | 6
 | An understanding of place value | 5
 | Different bases | 3
 | Rationalization of the fundamental operations. | 3
 | Casting out nines | 3
 | Practically the whole course | 2
 | None | |

Twenty teachers responded to the sixth item. Some teachers mentioned more than one item.

7. If you have discussed the course, they were interested in the extent to which it was helpful in teaching arithmetic. Five teachers have not discussed it with others. Fourteen teachers responded to the seventh item. Some teachers mentioned more than one item.
<table>
<thead>
<tr>
<th>Items</th>
<th>Responses of teachers</th>
<th>Frequency of Response</th>
</tr>
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<tbody>
<tr>
<td>8. Would you recommend that a course such as this be an in-service requirement for all elementary teachers?</td>
<td>A good review of the basic concepts of arithmetic.</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>To help the teachers to understand the &quot;why&quot; of arithmetic.</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>I got a lot out of the course.</td>
<td>5</td>
</tr>
<tr>
<td>8. Would you recommend that a course such as this be an in-service requirement for all elementary teachers?</td>
<td>Not all teachers need the course.</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>Even though it is worthwhile, the fact that a course is required causes it to lose appeal.</td>
<td>7</td>
</tr>
<tr>
<td>Why, or why not?</td>
<td>Twenty-two teachers responded to the eighth item.</td>
<td></td>
</tr>
</tbody>
</table>
CLASSROOM OBSERVATION CHECK LIST

Teacher __________________ School __________________ Grade ______
Number of pupils ______

I. A philosophy for teaching arithmetic.
   1. To develop an understanding of the basic concepts of arithmetic.
   2. To develop a desirable attitude toward arithmetic.
   3. To teach arithmetic as mechanical drill.
   4. To teach the social usefulness of arithmetic.
   5. To use the objects-pictures-symbols approach to teaching arithmetic.
   6. To provide for the individual differences of children.

II. The use of materials.
   7. Textbooks, workbooks, chalkboard, paper, and pencils.
   8. The abacus, counting frame, flannel board and other manipulative materials.
   10. Games and enrichment materials.
   11. Teacher-made problems.

III. Procedures used.
   15. Emphasis on speed.
   17. Demonstrations and explanations.

IV. Evaluation.
   18. In terms of textbook.
   19. In terms of reasonableness of solution.
   20. In terms of social application.

V. The transfer of learnings from the in-service course.

VI. The presence of superficiality in the classroom situation.
Check List of Teacher Opinions Regarding
the Teaching of Arithmetic

In the following check list are statements which are in some way connected with the teaching of arithmetic in the elementary school. The purpose of the check list is to determine the importance which you as an elementary teacher assign to each statement. Please remember that there is no "correct" response. Furthermore, you are not asked to sign your name and your check list will remain completely anonymous.

Read each statement carefully, then check the category which you feel is most appropriate for that item.

1. Elementary teachers should be expected to understand the arithmetic they teach.
   __Disagree strongly __Disagree __Agree __Agree strongly
   __Of little significance

2. Most teachers find that arithmetic is easy to teach.
   __Disagree strongly __Disagree __Agree __Agree strongly
   __Of little significance

3. Teachers need help in learning to use arithmetic materials effectively.
   __Disagree strongly __Disagree __Agree __Agree strongly
   __Of little significance

4. Each teacher should know what arithmetic is taught at each level of the elementary school.
   __Disagree strongly __Disagree __Agree __Agree strongly
   __Of little significance

5. Most teachers seem to understand arithmetic.
   __Disagree strongly __Disagree __Agree __Agree strongly
   __Of little significance

6. Most teachers enjoy teaching arithmetic.
   __Disagree strongly __Disagree __Agree __Agree strongly
   __Of little significance

7. Teachers need someone to whom they can go for help in teaching arithmetic.
   __Disagree strongly __Disagree __Agree __Agree strongly
   __Of little significance
8. A very desirable way to teach arithmetic is to follow the rules.

Disagree strongly  Disagree  Agree  Agree strongly
Of little significance

9. Many teachers do not understand arithmetic and are afraid to admit it.

Disagree strongly  Disagree  Agree  Agree strongly
Of little significance

10. We may as well face up to the fact that many teachers have a fear of arithmetic, and very little can be done about it.

Disagree strongly  Disagree  Agree  Agree strongly
Of little significance

11. Teachers need to understand that the Hindu-Arabic number system is based on ten.

Disagree strongly  Disagree  Agree  Agree strongly
Of little significance

12. Most teachers would welcome a course in the basic understandings of numbers if it were given in their school with in-service credit.

Disagree strongly  Disagree  Agree  Agree strongly
Of little significance

13. Most teachers do not have adequate materials to teach arithmetic effectively.

Disagree strongly  Disagree  Agree  Agree strongly
Of little significance

14. A desirable in-service program should acquaint teachers with available enrichment materials for teaching arithmetic.

Disagree strongly  Disagree  Agree  Agree strongly
Of little significance

15. Groups of teachers should be expected to develop an arithmetic course of study for their school system.

Disagree strongly  Disagree  Agree  Agree strongly
Of little significance

16. Teachers should make extensive use of the abacus in teaching arithmetic.

Disagree strongly  Disagree  Agree  Agree strongly
Of little significance
17. A curriculum guide in arithmetic is an excellent teacher aid.  
|                       | Disagree strongly | Disagree | Agree | Agree strongly | Of little significance |

18. Arithmetic games should be used in the classroom.  
|                       | Disagree strongly | Disagree | Agree | Agree strongly | Of little significance |

19. The flannel board has many excellent uses in teaching arithmetic.  
|                       | Disagree strongly | Disagree | Agree | Agree strongly | Of little significance |

20. Teachers need better achievement tests for use in teaching arithmetic.  
|                       | Disagree strongly | Disagree | Agree | Agree strongly | Of little significance |

21. There appears to be a great need for in-service workshops in arithmetic.  
|                       | Disagree strongly | Disagree | Agree | Agree strongly | Of little significance |

22. Teachers need outside speakers to talk to them about teaching arithmetic.  
|                       | Disagree strongly | Disagree | Agree | Agree strongly | Of little significance |

23. A special consultant in arithmetic could be a great deal of help to teachers.  
|                       | Disagree strongly | Disagree | Agree | Agree strongly | Of little significance |

24. Teachers need someone to show them how to teach arithmetic.  
|                       | Disagree strongly | Disagree | Agree | Agree strongly | Of little significance |

25. Teachers should do a better job of helping each other teach arithmetic.  
|                       | Disagree strongly | Disagree | Agree | Agree strongly | Of little significance |

26. Teachers should solicit the help of the parents in teaching arithmetic.  
|                       | Disagree strongly | Disagree | Agree | Agree strongly | Of little significance |
27. A good in-service program in arithmetic should give teachers an opportunity to discuss mutual problems.  
   __Disagree strongly __Disagree ___Agree ___Agree strongly 
   ___Of little significance

28. Teachers should be given tests to determine how well they understand arithmetic.  
   __Disagree strongly __Disagree ___Agree ___Agree strongly 
   ___Of little significance

29. Teachers need better diagnostic tests for use in teaching arithmetic.  
   __Disagree strongly __Disagree ___Agree ___Agree strongly 
   ___Of little significance

30. Textbooks are extremely helpful in teaching arithmetic.  
   __Disagree strongly __Disagree ___Agree ___Agree strongly 
   ___Of little significance

31. Many of the arithmetic materials used in classrooms should be made by the teachers.  
   __Disagree strongly __Disagree ___Agree ___Agree strongly 
   ___Of little significance

32. Arithmetic games and puzzles are interesting, but of little value in teaching arithmetic.  
   __Disagree strongly __Disagree ___Agree ___Agree strongly 
   ___Of little significance

33. In-service education in the teaching of arithmetic should help teachers learn to use audio-visual materials.  
   __Disagree strongly __Disagree ___Agree ___Agree strongly 
   ___Of little significance

34. Children should not be permitted to use their fingers in learning arithmetic.  
   __Disagree strongly __Disagree ___Agree ___Agree strongly 
   ___Of little significance

35. In helping children to learn arithmetic it is advisable to use many objective materials such as buttons, beads, sticks, etc.  
   __Disagree strongly, Disagree ___Agree ___Agree strongly 
   ___Of little significance

36. The place value box is a very valuable device for teaching arithmetic.  
   __Disagree strongly __Disagree ___Agree ___Agree strongly 
   ___Of little significance
37. All teachers should know how to use the counting frame in teaching arithmetic.
   Disagree strongly  Disagree  Agree  Agree strongly
   Of little significance

38. Teachers need a great variety of materials for teaching arithmetic.
   Disagree strongly  Disagree  Agree  Agree strongly
   Of little significance

39. In teaching arithmetic teachers should place strong emphasis upon estimation and checking.
   Disagree strongly  Disagree  Agree  Agree strongly
   Of little significance

40. Teachers should do a better job of sharing arithmetic materials.
   Disagree strongly  Disagree  Agree  Agree strongly
   Of little significance

Write a brief paragraph describing what you consider to be the greatest on-the-job need of elementary teachers in the area of arithmetic.
Egyptian Number Symbols:

- one - /
- ten - \( \cap \)
- hundred - \( \odot \)
- thousand - \( \Delta \)
- ten thousand - \( \mathcal{V} \)
- hundred thousand - \( \mathcal{N} \)
- million - \( \mathcal{H} \)

Write Egyptian Symbols for:

- two -
- seven -
- twelve -
- eighteen -
- twenty-one -
- twenty-three -
- one hundred twenty-five -
- four thousand -
- one hundred thousand -
- one million four thousand three hundred sixty-four -
Roman Numerals System of Notation:

one  -  I  one hundred  -  C
five -  V  five hundred  -  D
ten -  X  one thousand  -  M

Write the following Hindu-Arabic numbers in Roman Numerals:

3  -  520  -
14 -  839  -
26 -  949  -
48 -  3,487  -
91 -  8,785  -
106 -

Write the following Roman Numerals in Hindu-Arabic Numbers:

XXI -  CCCLXII -
XLIV -  MMDLXXXII -
XCVI -
<table>
<thead>
<tr>
<th>Number 4</th>
<th>GUIDE SHEET - SECOND SESSION</th>
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<tbody>
<tr>
<td>ones</td>
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<td></td>
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<tr>
<td>ones</td>
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The Lattice Method of Multiplication.

34
x 28
618
x 42

206
x 38

618
x 435

8 5 3 1 0 7
x 4 3
Number 7. GUIDE SHEET - FIFTH SESSION

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<td>24</td>
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<td>1608</td>
<td>18224</td>
</tr>
<tr>
<td>120</td>
<td>8</td>
</tr>
<tr>
<td>2000</td>
<td>8</td>
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<td>15000</td>
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<td>528</td>
<td>6</td>
</tr>
<tr>
<td>480</td>
<td>6</td>
</tr>
<tr>
<td>48</td>
<td>6</td>
</tr>
<tr>
<td>48</td>
<td>6</td>
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</tbody>
</table>
Number 8. GUIDE SHEET - FIFTH SESSION

Demonstration of "Napier's Bones."

\[
\begin{array}{c|c|c|c}
4 & 1 & 2 & 1 \\
8 & 4 & 8 & 4 \\
1 & 8 & 8 & 8 \\
6 & 2 & 2 & 2 \\
2 & 3 & 3 & 3 \\
0 & 5 & 5 & 5 \\
4 & 1 & 1 & 1 \\
8 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 \\
6 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 \\
3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 \\
4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & 36 \\
5 & 10 & 15 & 20 & 25 & 30 & 35 & 40 & 45 \\
6 & 12 & 18 & 24 & 30 & 36 & 42 & 48 & 54 \\
7 & 14 & 21 & 28 & 35 & 42 & 49 & 56 & 63 \\
8 & 16 & 24 & 32 & 40 & 48 & 56 & 64 & 72 \\
9 & 18 & 27 & 36 & 45 & 54 & 63 & 72 & 81 \\
\end{array}
\]

\[
\begin{array}{c}
\frac{472}{3776} \times \frac{28}{944} \frac{13216}{3776} = 944 \\
\frac{3776}{944} = 3776
\end{array}
\]
Number 9. GUIDE SHEET - SIXTH SESSION
Number 11. GUIDE SHEET - SEVENTH SESSION

$\frac{1}{2} \times 4$

\[ \begin{array}{cccc}
\text{Half-circles} & \text{Half-circles} & \text{Half-circles} & \text{Half-circles} \\
\end{array} \]

\[ \begin{array}{cccc}
\text{Full circles} & \text{Full circles} & \text{Full circles} & \text{Full circles} \\
\end{array} \]
Number 12. GUIDE SHEET - SEVENTH SESSION

3 \times \frac{1}{2}
Number 13.  

$\frac{1}{2} \times \frac{1}{3}$
Number 14. GUIDE SHEET - EIGHTH SESSION

\[
\frac{1}{2} = 2
\]

Measurement:

Partition:
Number 15.  

GUIDE SHEET - EIGHTH SESSION

3 ÷ 1/2

Measurement:
Number 16.  

GUIDE SHEET - EIGHTH SESSION

\[ \frac{1}{2} + \frac{1}{2} \]

Measurement:

\[ \frac{1}{2} + \frac{1}{4} \]

Measurement:
A. BOOKS


Buswell, Guy T. and Judd, Charles H. *Summary of Educational Investigations Relating to Arithmetic*. Chicago: The University of Chicago, 1925.


E. PUBLICATIONS OF LEARNED SOCIETIES, AND OTHER ORGANIZATIONS


C. PERIODICALS


D. UNPUBLISHED MATERIALS


I, Lonie Edgar Rudd, was born in Marshall County, Kentucky, September 13, 1920. I received my elementary and secondary education in the public schools of Marshall County and the town of Benton, Kentucky. My undergraduate training was obtained at Murray State Teachers College, Murray, Kentucky, from which I received the degree of Bachelor of Science in June, 1943, having interrupted my program of undergraduate work between the Junior and Senior years to teach for one year in an eighth-grade one-room elementary school. My undergraduate majors were mathematics and commerce. In June of 1943, I was called to active duty in the United States Naval Reserve, receiving my commission as Ensign, USNR, in October of 1943. I served as a Line Officer until September 27, 1946, at which time I was released to inactive duty. I held a position as teacher of commercial subjects at Stryker High School, Stryker, Ohio, from 1946 to 1948. I began my graduate study for the degree Master of Arts at The Ohio State University in the summer of 1947, and my major area was mathematics education. In the fall of 1948, I received an appointment as Instructor in the Department of Naval Science, The Ohio State University, which position I held until July, 1951. I
received the Master of Arts Degree from The Ohio State University in June of 1949. I began my graduate study for the degree Doctor of Philosophy in the fall of 1949, and selected the major area of Elementary Education. From September, 1951, until March, 1952, I was graduate assistant to Professor Ruth Streitz in the Department of Education. For the spring quarter of 1952, I received an appointment as Instructor in the Department of Education, The Ohio State University. In August of 1952, I accepted a position as sixth grade teacher in the Newton Public Schools, Newton, Massachusetts, which position I held until June, 1955. From February, 1953, until September, 1955, I also held the position of Lecturer in Education at Tufts University, Medford, Massachusetts. From September, 1955, until the present time I have held the position of Assistant Professor of Education, Tufts University.