THE IMPLICATIONS OF THE THEORY OF OPERATIONALISM AND OF SOME STUDIES IN PSYCHOLOGY AND ANTHROPOLOGY FOR THE TEACHING OF ARITHMETIC

DISSERTATION

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By

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Chapter I

INTRODUCTION

Origin of the Problem

The rather intense study of the field of arithmetic over the past half century has resulted in noteworthy changes in the content and methods of teaching arithmetic in the elementary school. The social aim of arithmetic instruction today has produced a content more significant to the learner in terms of his interests and in terms of its applications for more effective living. The mathematical aim emphasizes meaningful learning. Arithmetic is to be learned as a system of related ideas rather than as an accumulation of unrelated facts and computational skills. The child study movement has emphasized the close relationship between mental growth and other kinds of growth, such as emotional, social, and physical. Modern arithmetic programs reflect these changes in varying degrees.

It would be erroneous to think that the changes in the teaching of arithmetic have come about solely through investigations and writings in the field of arithmetic. The educator and the textbook writer have been influenced, albeit indirectly, by the philosopher, the physiologist, the psychologist, the sociologist, the

anthropologist, and by the wealth of data from other investigators in education. Often investigations in arithmetic have proceeded from the testing of hypotheses growing out of findings in other fields.

There is reason to believe, however, that certain areas in philosophy, psychology, and anthropology have had little effect on the teaching of arithmetic. With few exceptions summarizations of investigations in arithmetic have included only studies made in the field of the teaching of arithmetic by teachers and educators directly interested in these problems. Attempts to include investigations made in other fields which have bearings on the problems in arithmetic have been sporadic and incomplete. A preliminary review of modern arithmetic programs and of summarizations of investigations in arithmetic has revealed few instances in which the philosophy of operationism and certain studies in psychology, particularly those dealing with the perception and the discrimination of number, have been influential.

**Importance of the Problem**

One of the recurring themes in the literature on the teaching of arithmetic over the past quarter of a century has been the attempt to make explicit what is entailed in the meaningful teaching and learning of arithmetic.

Authorities differ noticeably in the materials and methods they recommend for initiating meaningful number ideas for the child when he first encounters systematic instruction in arithmetic. In view
of the emphasis on meaning in the teaching of arithmetic, a study of the findings in other areas which have implications for the meaningful teaching of arithmetic should help to clarify issues regarding the child's introduction to systematic arithmetic.

The Problem

The problem of this study may be stated in two parts as follows:

1. To report on the findings of certain psychological studies of the perception of the number of objects in a group by adults, children, and animals and the findings of certain anthropological studies which are indicative of the concrete basis of number ideas, as well as the literature of the theory of operationism; and to point out the implications for the meaningful teaching of arithmetic in the primary grades.

2. To analyze critically, in the light of the findings in the first part of the study, the ways in which representative current arithmetic programs deal with early number experiences.

Scope and Limitations of the Study

The findings of psychological studies in the perception of discrete number made with human adults, children and animals will be reported in Chapters II, III, and IV, respectively. The major investigations in number perception with adults have been made in America and the reports will be limited to these studies. Number perception studies with children are mainly English, although one
major French source will be included. Some number perception studies were made with children in Germany around the turn of the century which were discussed in detail by Howell; hence, the information from these early German studies necessary to add continuity to the picture will be taken from him. The chief number studies made with animals are found in English and German, with somewhat fewer studies in French. All three sources will be included in this study.

The findings from anthropological studies will be reported in Chapter V. The main anthropological sources used are English and French and will be limited to studies of the number vocabularies and the uses of the number ideas of primitive peoples.

The operational point of view of meaning is accepted for the purposes of this study. In Chapter VI the major points of the operational school of thought will be highlighted by a review of the literature. The review will be limited to American sources and special references will be made to the writings of the American physicist, P. W. Bridgman, who has been one of the foremost proponents for meaning in the operational sense.

The implications of operationism for the teaching of arithmetic concepts, symbols, and algorithmic processes with whole numbers will be analysed in Chapters VII and VIII. Particular attention will be given to showing how the meanings of arithmetical concepts are given by sets of physical operations.

The analysis of the arithmetic programs reported in Chapters

IX and X will be specifically limited to those modern programs which are used nationwide, and which make definite provisions through workbooks, textbooks, or other written materials for number experiences in the first through the fourth grades. The materials will be those which are designed for use in developing early concepts of whole numbers and of the four so-called fundamental operations with whole numbers—addition, subtraction, multiplication, and division.

**Definitions of Terms**

Since much of this study concerns investigations and writings in fields other than education, terms relatively unfamiliar to the reader may occur from time to time. Some of these will be commonly understood terms in philosophy, psychology, and anthropology, but terms strange to the person unfamiliar with these fields. In other cases, terms coined and defined by specific investigators will be used. In both of these cases such terms will be defined as they arise in the body of the report of this study.

The analyses made of the investigations reported in this study will have pertinence to the materials and methods used when the child first encounters a systematic or formal study of arithmetic in school. A systematic study of arithmetic will mean that there are provisions made in both time and materials for the study of number. This approach is in contrast to the incidental learning of number in which the child considers number when and if it arises in connection with other work he is doing. For the most part, pre-school experiences with number are incidental. Most authorities
today prefer the term *systematic instruction* to the term *formal instruction*. They feel that the term *formal instruction* carries connotations iminical to the spirit and purposes of a modern arithmetic program.

Some writers in the theory of operationism make a distinction between *physical operations* and *mental or symbolic operations*. A brief discussion of various points of view on this issue is contained in Chapter VI. In this study the term *operation* will be used, however, to mean *physical operations performed with concrete objects*.

The term *grouping* will be used extensively throughout this study. In contrast to *counting by ones*, *grouping* will mean any operation by which a subject perceives more than one object at a time and uses this operation, either alone or with various other physical operations or symbolic processes, to determine the number of objects in a group.

**A Review of Similar Studies**

**General Studies**

Howell.— In 1914, Howell\(^3\) reported studies he had made of certain arithmetical abilities of school children in the first eight grades. His investigations in number discrimination among school children will be reviewed in this study. Howell's reviews of early studies, a series of investigations made by German educators around the turn of the century concerning the apprehension

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\(^3\) *Ibid.*
of number among school children, are of considerable value. The fact that his is the most recent work, now forty-three years old, covering number perception studies makes it important to bring such reviews up to date.

Pratt.-- In 1948, K. C. Pratt wrote a theoretical introduction to a series of experiments he later carried out with others regarding the indeterminate number concepts of school children. Pratt reported some of the experimental literature on number from the field of animal psychology, from etymology having to do with the derivation of number words, from anthropology on the number concept, and from genetic psychology relating to the development of number concepts among children. His discussions were brief, mainly presenting results. The number of studies he reported on was quite limited; the studies for the most part were those with which mathematics educators are familiar.

Reviews of Animal Studies

Bierens de Haan, Honigmann, Moore.-- Investigations relating to animals and the number concept have been reviewed by Bierens de Haan, Honigmann, and, somewhat less thoroughly, by Moore. Of


the three, Horngmann's article is both the best and the latest. Although these articles are excellent as far as they go, further discussion of number investigations with animals seems to be in order for the following reasons:

1. The above three reviewers tend to emphasize questions concerning conceptual behavior in animals. For example, can the animal form some kind of a concept of number and can it learn to count in some way? In general, the answer is in the negative. The main concern in this study is with the question: can the animal learn to make discriminations on the basis of number?

2. References to animals and number have been sporadic and incomplete in educational literature. These references are often to anecdotal rather than to scientific sources. The members of a species of birds which always lay four and only four eggs do not do so because they can count to four or because they have an idea of four in any sense of the word. K. von Friesch's bees8–9 did not convey the information concerning the location of the nectar by any use of number. Such studies belong to the realm of biology and will not be considered here where the interest is in investigations in

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which attempts have been made to determine whether the animal can learn to make discriminations on the basis of number.

Anthropological Studies

Conant. — Conant's\(^{10}\) work, now over fifty years old, reports number words and counting systems extant among primitive peoples. His major emphasis is on reporting the number vocabularies of primitive peoples. His work will be referred to when it is appropriate in this study.

Operationism and Arithmetic

Van Engen. — In four articles written several years ago, Van Engen\(^{11-14}\) pleaded for the adoption of the operational approach in the teaching of elementary school mathematics. His discussions of operationism and his illustrations of the implications of the operational approach for the teaching of arithmetic were brief. His arguments were limited to a discussion of the importance of the child's becoming aware of a correspondence between a set of symbols


\(^{11}\) H. Van Engen, "An Analysis of Meaning in Arithmetic: I,"

Elementary School Journal, XLIX (March, 1949), 395-400.


and a set of operations, and the necessity for using manipulative materials that would highlight the operations involved. His illustrations were limited to examples of addition and subtraction, with a brief mention of division.

In a more thorough discussion of operationism as it relates to the formation of mathematical concepts, Van Engen considered three dimensions of meaning: (1) the syntactic dimension, in which words and symbols have meaning by virtue of their relation to other words and symbols in sentences or formulas; (2) the pragmatic dimension, in which the emotional overtones of words and symbols influence their meanings; and (3) the semantic dimension, in which the meanings of words or symbols are obtained through acquaintance with their referents. Van Engen then discussed at some length the semantic dimension of meaning and its application in the development of concepts. This article is an elaboration of the main points made briefly in his other articles. Van Engen is one of the co-authors of a series of arithmetic textbooks in which his ideas concerning the operational approach are put into practice. These materials will be among those analyzed in Chapters IX and X.  

Although the writings of Van Engen concerning operationism and arithmetic are of considerable value, he has failed to make

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16. See Scott, Foresman and Company series listed in Bibliography VII.
explicit certain aspects of the operational theory of meaning of
utmost importance in the development of early number concepts. A
review of the operational approach to meaning should be of value
to the reader, as should be a detailed analysis of the implications
of the operational approach for early number experiences.

Method of the Study

Chapter II will summarize psychological experiments dealing with
the perception of number with adults as subjects. Such experiments
have been limited largely to those concerning visual discrimination
of simultaneously presented stimuli. There have been some experiments,
however, concerned with successively presented visual stimuli, a few
others concerned with auditory discrimination of number, and fewer
still with tactile discrimination.

Chapter III will review, interpret, and determine the impli-
cations for the teaching of arithmetic of number perception studies
made with children.

Chapter IV will consider number discrimination studies made
with animals and the relevance of the findings of those studies
to the teaching of arithmetic.

In Chapter V, anthropological studies having to do with the
number concept among primitive tribes will be reviewed. Particular
attention will be given in that chapter to the concrete basis of
number ideas.

The literature pertaining to operationism will be reviewed in
Chapter VI. The purpose will be to point out the essential features
of the operational approach to meaning as it has crystallized in
the thinking of those men who have given of their time and effort
to working out the details of operationism.

In Chapters VII and VIII, implications of operational theory
for teaching arithmetic in the primary grades will be presented.

Chapters IX and X will consist of a report of an analysis of
representative current arithmetic programs in view of the findings
reported in Chapters II to VIII.

Chapter XI will include a summary of the study and recommenda-
tions for changes in arithmetic programs in the light of the
findings of the study.
Chapter II

PERCEIVING THE NUMBER OF OBJECTS IN A GROUP: EXPERIMENTS WITH ADULTS

Counting and Grouping in Arithmetic

The authors of various modern arithmetic programs do not agree on the relative importance of counting and grouping. This disagreement is centered around three questions: (1) which process is the better foundation upon which to build for the later concepts and skills of arithmetic?; (2) what is the relation between counting and grouping; that is, is one prerequisite to the other?; and (3) what are the limitations in the ability to group?

Is Counting or Is Grouping the Better Foundation for Learning Arithmetic?

Which process is the better foundation upon which to build for the later concepts and skills of arithmetic? Spitzer advocates the necessity of counting as follows:

Counting to find the answer is considered the foundation of all methods of solution advocated in this book. This assumption will be made especially evident when fundamental operations are studied. It is further assumed that all the problems used in introducing each of the fundamental processes can be solved by counting. The child who can count is thus assured of having at least one solution. Besides being a starting-point and a last resort, counting is also a definite aid to understanding, primarily because it is a process in which everyone has confidence.1

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Wheat is of the opinion that counting is basic to learning about the different-sized groups. He says the child uses counting to find how many in a group, to compare groups, to determine the number left when some are taken away, to find out how many there are in one group of a larger group, and to find out how many altogether. He contends that "good training in addition starts with and grows out of good training in counting."²

Spencer and Brydegaard feel that "counting is the one basic and fundamental operation with numbers."³ Hollister and Gunderson express their feeling about the importance of counting as follows:

The child frequently meets the question "How many?" It is one of the fundamental questions of all mathematics. Counting is the most direct approach to finding the answer. It is the avenue through which the child gains an insight into more complicated arithmetic problems or processes. As he matures and learns more mathematics, counting will be replaced by more efficient methods, but the basic process is enumeration. The ability to count is an important step in primary numbers and it should be mastered by every pupil before success can be expected in other operations.⁴

In support of the value of grouping procedures, Brueckner and Grossnickle write that "since grouping seems to be closely related to success in abstract number, the teacher should see to

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it that her pupils learn how to group and identify the number in a
group of any size through at least 5.\textsuperscript{5}

Many advantages for emphasizing grouping in early number work
have been given by other authorities. Several of those advantages
will be discussed in Chapter III during the review of studies with
children. At this point, however, before reference is made to those
studies, it seems reasonable to suppose that if a child had some
facility in perceiving small groups of objects, he could free himself
somewhat from the operation of counting and give his major attention
to the physical operations which give meanings to his other arith-
metic concepts. The grouping method, which involves perceiving
the number of objects in a group without counting, is accepted here
as better adapted to building meaningful arithmetic concepts through
the operational approach than is the counting method.

Is Counting Prerequisite to Grouping?

What is the relation between counting and grouping; is one
prerequisite to the other? Many authorities feel it is not possible
for a child to learn to recognize the number of objects in a small
group instantaneously until after he has become familiar with the
number of objects in that group through several experiences in

\textsuperscript{5} Leo J. Brueckner and Foster E. Grossnickle, \textit{Making Arithmetic
counting the group. Hickerson, for example, states his position as follows:

After a child has learned to determine the number of objects in a group or a collection by counting, he is now ready for the next step: learning to determine the number of objects in a group at a glance without counting. 6

Brueckner and Grossnickle list a sequence of counting experiences which terminate in grouping procedures. The authors believe that "there are six stages in the complete process of counting: (1) rote counting, (2) enumeration, (3) identification, (4) reproduction, (5) comparison, and (6) grouping." 7

On the other side of the issue, Risdan feels that the child can easily learn to recognize the number of objects in a small group. She feels that an emphasis on counting tends to fix the habit of seeing ones and thus discourages the ability to see groups. She maintains that counting should not be used as the first experience with numbers but that the emphasis should be placed on the recognition of groups. 8

Clark and Eads point out that children learn on their own to recognize the number of objects in a small group without counting:

Before young children are able to count, they are usually able to see groups of two or three or four things as groups rather than as collections of ones, and are generally


able to differentiate the groups from each other. Most children have also at some time learned the number names for such groups . . . Unfortunately, however, many children seem to lose this valuable ability to see the group as a whole when they are encouraged to see things as single ones, as they count by ones, starting with "one."  

What Are the Limitations of the Ability to Perceive the Number of Objects in a Group?  

Morton is of the opinion this ability is rather limited for first-grade children. He writes that "as the year goes by and there are many opportunities to do so, probably all pupils will learn to identify at once a group of 2, most pupils a group of 3, many pupils a group of 4, and a few pupils a group of 5."  

On the other hand, Hickerson feels that some six-year-olds soon learn to recognize up to six objects without counting with 100 per cent accuracy and a few may even go as high as seven, while still others have difficulty beyond three. He points out that this ability may be extended by grouping procedures to determine the number of objects in larger groups without counting: 

For example, the total group of nine objects is not recognized at a glance, but the groups of five and four are. The child, then, after discovering by counting that there are nine objects altogether, learns through repeated contacts that five blocks and four blocks are nine blocks.  

Risden is of the opinion that the ability to recognize at a glance the number of objects in a small group and to use grouping procedures to determine the number of objects in larger groups is capable of considerable extension. For example, she points out that the limit of the perception of numbers may be only 4 or 5 objects, but it is then possible to see 4 groups of 5 each or 5 groups of 4 each for a total of twenty.\footnote{12} (Throughout the remainder of this study numerals rather than number-words will be used in all references to numbers of objects perceived.)

Brueckner and Grossnickle who are also of the opinion that this ability can be developed to a greater extent than Morton suggests, feel that "with practice in group recognition, all pupils in the lower primary grades should be able to recognize groups as large as five."\footnote{13}

Preview of the Chapter

This chapter is a review of studies on the abilities of adults to perceive the number of objects in a group. The studies reviewed here should give some information concerning the upper limits of human ability to perceive the number of objects in a group or to discriminate between the number of objects in two groups on the basis of number. In these experiments, efforts have been made to control other factors so that the subject will have no chance to use any number abilities they may possess other than those of direct perception and knowledge of number names. More specifically, this chapter will

\footnote{12}{Risden, \textit{op. cit.}, p. 131.}

\footnote{13}{Brueckner and Grossnickle, \textit{op. cit.}, p. 182.}
knowledge of number names. More specifically, this chapter will review the following kinds of studies: (1) studies utilizing simultaneously presented visual stimuli, (2) studies using successively presented auditory stimuli, and (3) studies using successively presented visual stimuli. In addition, one study of the way in which the ability to perceive the number of objects in a group is developed will be included. Two recent studies regarding the factors which influence the limits of visual perception will also be reviewed.

**Experiments with Adults in Perceiving the Number of Objects in a Group**

**Terminology Pertinent to Studies in Perception and in Discrimination**

In their concern with the general problem of discrimination and perception, psychologists have made contributions specifically toward the understanding of the perception and the discrimination of number.

*Perception* is generally defined as the process of becoming immediately aware of something. The *perception of number*, then, is the process of becoming immediately aware of the number of objects in a group. The *problem of the perception of number* of interest here may be stated thus: how many presented stimuli can a subject perceive? One way of testing this is to expose fields of dots for very short periods of time by means of a tachistoscope. The subject reports the number of dots he perceives in each exposure. His limit of the perception of number is the *maximum* number of objects which he can consistently perceive correctly.
Discrimination is the perception of a difference. The discrimination of numerosness is the act of making a judgment about differences between two groups simply on the basis of which group has more (or fewer) objects in it. No attempt is made to determine how many objects there are in either group or how many more objects one group has than the other. One way of testing the discrimination of numerosness is to present two groups of objects to the observer for a very short period of time. The observer reports which group appears to have more objects in it. It is permissible, of course, to expose two groups of equal size; and it is possible that the observer will perceive two unequal groups as being equal.

In the discrimination of number the judgment is made on the basis of the number of objects. The problem of the discrimination of number which is of interest here is generally presented in the following form: what is the largest pair of number stimuli, \(x\) and \(x + 1\) \((x = 1, 2, 3, \ldots)\), between which the subject can distinguish, on the basis of number only, without counting or subgrouping; that is, with no aid from secondary factors such as areas occupied by the two groups, sizes of the objects, arrangement of the objects, etc.?

In studies dealing with either the discrimination of number or with the perception of number, the stimuli may be presented visually, audibly, or tactiley, although there appears to have been little work done in this area with tactile stimuli. Visual and tactile stimuli may be presented either simultaneously or successively. Most authorities are of the opinion that audible stimuli must be presented successively in such studies.
Since the ability to perceive a difference is the condition for discrimination, some writers make little distinction, if any, between the two terms perception and discrimination. Apparently other terms, such as the apprehension of number, the cognition of number, the judgment of number, and the estimation of number, have been used interchangeably or in lieu of either discrimination or perception or inclusive of both.

In view of the lack of unanimity regarding the use of these terms among the investigators, no attempt has been made in the summaries to distinguish among them. In fact, an effort has been made to preserve the terminology used by each author or authors. In the more important studies, sufficient information is given about the procedure so that the particular ability being tested is indicated.

Early Experiments with Adults: Simultaneous Visual Presentation

Jevons (1871).—One of the earliest recorded experiments in the perception of number was performed by W. Stanley Jevons¹¹ in 1871. Jevons wrote that others, especially Sir William Hamilton, had discussed the problem. Hamilton was of the opinion that the upper limit of perceiving simultaneously presented visual stimuli was six or, at most, seven.

Jevons lined a shallow circular box, four and one-half inches in diameter, with white paper. He would then toss a quantity of

uniform black beans toward the box so that an uncertain number fell into it. He immediately estimated the number of beans in the box and then counted to ascertain the actual number. There was a total of 1027 trials. He presented his data, actual number versus estimated number, in table form. Jevons then found the average errors of estimation for each number and the direction of these average errors. His data clearly show that he had a tendency to overestimate small numbers, up to 8, and to underestimate larger numbers, from 8 up to and including 15 objects.

Since Jevons made no errors in estimating 3 and 4 beans, but erred in approximately 5 per cent of the cases in estimating 5, he concluded that the limit of complete accuracy would be neither at 4 nor at 5, but halfway between them. He felt that such a conclusion in dealing with discrete data was puzzling, but he submitted it for what it was worth. Since Jevons was correct in 95 per cent of the cases for 5 objects, however, that number would appear to be the best value to take as his limit for the perception of the number of objects in a group.

Warren (1897).— In 1897 Warren considered two problems: (1) what is the largest number that can be directly apprehended, and (2) what is the part played by habitual acts of counting? Warren felt that the mental process concerned in number apprehension

was not the same thing as that generally called discrimination by psychologists. He used the term counting regardless of the method by which the number in the group was determined, but he provided for two kinds of counting. The name perceptive counting was given to those cases in which his subjects could determine the number in a group composed of several objects as quickly as they could perceive one object. Warren's use of the word counting here is questionable. Kaufman later called this operation subitizing. The name progressive counting was given to all cases where the subjects needed more time to determine the number of objects in the group than to perceive one object.

Warren exposed black circles arranged in a circle for 0.131 seconds. He concluded that adults estimate correctly 3 stimuli presented simultaneously; 3 was the limit for perceptive counting of simultaneously presented visual stimuli. Warren's conclusion here is surprising in view of the fact that his pattern of circles arranged in a circle gives rise to the square when the number of stimuli is 4 and arranged symmetrically. Although Warren claims he thwarted this by making sure that the respective stimulus dots were not placed on the vertical or horizontal diameters of the circles, nevertheless the group of 4 is one of the most easily recognized groups.

**Messenger (1903).**—Messenger, in 1903, reported a study of the perception of objects in groups where the presentations were

tactile and where they were visual. He was not attempting to determine how many stimuli one could perceive, but rather the mechanism of perception, how it takes place, and some of its characteristics. In the tactile experiments, Messenger used two qualitatively different contacts and concluded that perception arises with the association of number with perceived qualitative differences.

From his experiments with visually presented objects, Messenger concluded that the perception of objects in groups is a result of analysis and not of synthesis. Messenger maintained that the number of things in a group is not perceived directly; rather, one perceives the group as a unit and remembers that as one of its characteristics it is made up of so many separate parts. When a subject is presented a number of objects in a definite arrangement, if he determines the number by count and then drills on this arrangement, he will eventually be able to associate form with number and will come to think he perceives number directly. Messenger's study is reported here because of the influence such ideas have had on the teaching of number. Some present-day authors of arithmetic textbooks maintain that it is impossible to expect a child to learn to perceive number directly without the intermediate stage in which he learns, by counting, to determine the number associated with a particular group.

Certain findings reported by Messenger, however, have been generally substantiated by later studies. He found that judgment of number in regular and familiar arrangements is more accurate than in irregular arrangements, that an increase in the distance between the objects presented or an increase in the size of the
objects presented makes the group appear more numerous, and that practice improves the ability to judge number.

Burnett (1906).—The results of Burnett's study in 1906 generally agreed with Messenger's findings. Burnett found that, in general, the numerosness of the dots in a stimulus field was increased when the size, the vividness, or the total area covered by the dot pattern was increased. Better judgment was used when simple patterns were presented, as form and complexity affected the numerosness but varied in the degree to which they influenced the observer.

Whipple (1910).—Whipple, using college students and instructors as subjects, reported an investigation with visually presented stimuli. He exposed dots for three seconds, and found that his subjects differed widely in their ability to estimate number. He concluded that a process of analysis of the total group of objects into recognizable subgroups was of more value in the estimation of number than a process which involved staring at the group and attempting to estimate the total number directly. In his experiments, particularly in one involving the perception of letters of the alphabet, Whipple found that practice had little effect on the range of attention.


Fernberger (1921).— Fernberger employed the idea of the statistical limen to test visual apprehension. A limen is that stimulus value for which correct judgments are given in 50 per cent or more of the cases. Fernberger determined the individual limens for simultaneous visual apprehension to range from 6 to over 11 dots when he used exposure times of 0.100 and 0.060 seconds with adult subjects.

Oberly (1924).— Oberly exposed black dots on white cards for 0.0375 seconds to six psychologically sophisticated subjects and used the method of introspection in order to determine the ranges for three different "conscious patterns" operating in the perception of number. These ranges were as follows:

1. The range of attention. In this range the dots were perceived immediately and with an equal and high degree of clearness. The limits for the statistical limens of his subjects in this type of perception varied from 3.5 to 5.1 dots.

2. The range of cognition. Within this range the perception was immediate, but because of the different degrees of clearness of the dots, the subjects used a grouping process. Oberly found the upper limits for this type of perception to vary from 5.2 (lowest limen of a subject) to 9.4 (highest).


3. The range of apprehension. Within this range the subjects reimaged the stimulus-field and counted the objects. The highest limen in this case was 9.5 dots.

**Gill and Dallenbach (1926).** Gill and Dallenbach\(^2\) exposed various simultaneously presented geometric figures for 0.060 seconds. They found the individual limens for three psychologically sophisticated adults to be 16.6, 19.3, and 12.5, respectively.

**Glanville and Dallenbach (1929).** Glanville and Dallenbach\(^2\) reported an experiment on the immediate perception of the number of simultaneously presented visual stimuli with three subjects. One subject reported correctly on the number of objects up to and including 6, with a gradual falling off to 58 per cent correct responses on 12 objects, succeeded by a sharp drop to five per cent correct responses on 13 objects.

The second subject reported correctly up to and including 3 objects. Four was correctly discriminated by this subject 98 per cent of the time. There was a gradual falling off to 38 per cent accuracy on 7 objects, followed by a sharp drop to three per cent accuracy on 8 objects. The third subject reported correctly up to and including 5 objects and attained 98 per cent accuracy on both 6 and 7 presented stimuli. This subject was correct more than half the

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time on 9 objects but dropped abruptly to 16 per cent accuracy on 10 objects. Introspections showed that the subjects' reports were based upon two different types of apprehension: (1) an immediate type, and (2) a mediated type which was dependent on re-imaging, counting, and adding. Thus, the limits for the direct perception of the number of objects in a group for Glanville and Dallenbach's subjects ranged from 4 to 7 with not less than 98 per cent accuracy.

Recent Experiments with Adults: Simultaneous Visual Presentation

More recent investigations in the perception of number with adults have been characterized by their mathematical treatment and their operational approach.

Hunter and Sigler (1940).— Hunter and Sigler investigated the span of visual discrimination as a function of time and intensity of stimulation. They found that even irregular groups of from 1 to 7 dots could be discriminated as a single event; but for more dots than 7, counting was present in some form. When these investigators graphed presented number versus estimated number, they found that the curves for 8 dots and over had initially negative slopes. These slopes became positive only when the exposure time was increased sufficiently to permit some form of counting, such as combining subgroups to obtain the total. Hunter and Sigler concluded that time is a more important factor in determining span than is intensity. The authors write as follows:

Since we are certain that the discrimination of a large number of dots, let us say 12, involves counting and therefore that time is an important factor, we are led to infer that even the span of 8 dots has taken us over the boundary of a single discriminatory event and into the counting area because its curve also has a negative slope.24

Hunter and Sigler, then, found that their adult subjects judged from 1 to 7 dots as a single discriminatory event, but for 8 or more dots some form of counting and subgrouping was necessary.

Taves (1941).— Taves25 reported seven experiments having to do with the perception of visual numerosness. Three of these are of special significance here. The problem of the first experiment was this: how does visual numerosness vary as a function of the number of dots in the stimulus? Fields of light dots in haphazard order on dark backgrounds were projected tachistoscopically upon a screen. Two fields of dots were exposed on the screen simultaneously. The standard field was presented on the left half of the screen for 0.2 seconds. Five psychologically sophisticated subjects seated eight feet from the screen were instructed to vary, by the use of a mechanism constructed for the purpose, the right field until there appeared to be half as many dots in it as in the standard field. When visual numerosness as a function of the actual number of dots in the stimulus was studied, Taves found a discontinuity in the function at a point between 6 and 8 stimulus-dots.

24. Ibid., p. 177.

Taves concluded that this discontinuity indicated that there are two mechanisms for the perception of visual numerosness. The first mechanism operates when the number of dots is fewer than 6 or 8, and is apparently a direct, rapid recognition of number, without the use of any counting technique. The observers, themselves, reported that in this range they knew how many dots were in the standard field without counting. The second mechanism operates when the number of dots is so large as to prohibit this direct recognition of number. Taves felt that the perception of number in the latter cases was on the basis of area covered by the dots or the density of the dots or both, although he felt further investigation was necessary along these lines.

The problem of Taves' second experiment was the same as that of the first. Here, however, the subjects were shown three fields of dots. Two of these fields, the one on the left and the one on the right, were the standard fields. The observer was to vary the center field of dots until he thought it was equidistant, with respect to numerosness, from the two standard fields. The results here confirmed the conclusions of the first experiment.

In his fifth experiment, Taves used 133 general psychology students. The time of exposure of the dots was 0.2 seconds. The subjects were instructed to estimate the number of dots appearing in each field as it was projected on the screen and also to indicate the approximate degree of confidence in each estimate on a five-point scale, where five meant complete certainty. Taves found
that the subjects were just able to recognize 7 dots and that their
certainty in their estimates decreased abruptly beyond a point
between 6 and 8 dots.

Some of the findings in Taves' other experiments should be
mentioned briefly. In his third experiment, he used the same materials
and apparatus as in his first experiment. When his subjects were
instructed to make every effort to count, the point of discontinu-
ity on the curve increased to almost 10 dots in the standard field.
When instructions were to make every effort not to count, the point
of discontinuity decreased slightly.

Taves found his subjects tended to overestimate numbers beyond
8 objects. This finding is, of course, just the reverse of Jevon's
conclusion regarding overestimation and underestimation. Taves
also found that when the dots were arranged in some definite con-
figuration in accordance with Gestalt theory the result was to
reduce the numerosness of the group; that is, the observer thought
there were fewer dots in the stimulus field.

Taves' findings relative to the limits of the direct visual
perception of the number of objects in a group may be summarized
as follows:

1. For adults a direct, rapid recognition of the number of
objects in a group takes place when the group contains no more
than 6 or 8 objects. This takes place in the absence of any count-
ing techniques.

2. The limit for the direct perception of number appears to
be about 7 objects.
The upper limit of the range within which adults have confidence in their ability to perceive the correct number of objects in a group lies between 6 and 8 objects.

Saltzman and Garner (1948).—Saltzman and Garner26 investigated the span of attention with regard to number in two different ways. The first was by tachistoscopic presentation in which the investigators measured the subject's accuracy in estimating number. Saltzman and Garner used concentric rings as the stimulus-objects. These rings, from 2 to 10 in number and presented in a predetermined random order, were exposed for 0.5 seconds with six seconds given between exposures. The diameter of the largest ring was always three feet, and the distance between rings was always equal to the radius of the smallest ring. The subjects, five male college students, were instructed to guess if they were uncertain of the exact number.

The investigators found that no more than three rings were correctly identified 100 per cent of the time in the first ten trials. The average of trials thirty-one to forty, however, showed no errors in estimating 4 rings, and considerable improvement in estimating 5 rings. When the subjects were advised of the range of number of rings used, there was immediate improvement in their reports; practice produced little further improvement.

The second method of investigation used by Saltzman and Garner was a reaction time presentation in which the subjects were told

to be absolutely certain of their answers as to the number of rings presented, but to make their oral reports as rapidly as possible. There was a constant exposure duration of the rings, and the time it took each subject to give his answer in each case was recorded.

In this second experiment, Saltzman and Garner found that the reaction time steadily increased with increases in the number of rings, even at the low end of the range. For example, it took the subjects longer to identify 3 rings than 2 rings. The authors then ran an experiment with the number names and determined that the increased reaction time was not due to increased time needed to verbalize the various numbers. They also found, principally in the early trials, that practice decreased the reaction time. When this experiment was repeated with dots as the stimulus-objects, essentially the same results were obtained.

Saltzman and Garner reached four main conclusions: (1) there is no minimum number of objects below which the reaction time remains constant; (2) practice increases the span of attention; (3) a knowledge of the range of the number of objects to be presented increases the span of attention; and (4) "there is no such event as an immediate awareness of number." 27

Saltzman and Garner seem to rely heavily on the results of their second experiment in reaching conclusion (4) above. The very fact that the reaction time increased consistently as the number of stimulus objects was increased indicates that the subjects counted.

It obviously takes longer to count 3 objects than it does 2, and moreover the increase in the time to count would be approximately proportional to the increase in the number of objects. This relationship is approximate since, even if all other factors could be held constant, the time needed to verbalize the various number words is not constant. Saltzman and Garner found the number taking the longest time to verbalize is "six," while "eight" takes the shortest time.

Saltzman and Garner's findings that 4 was the most number of rings correctly perceived, even after practice, is not surprising. The configuration of concentric rings is a very difficult one in which to judge number. These investigators' conclusion that there is no such thing as an immediate awareness of number is not in agreement with the conclusions of other investigators. A criticism of their work and this particular conclusion will be given after other relevant material has been reviewed.

Kaufman et al. (1949)—An important investigation concerning the perception of visual number was reported by Kaufman et al. in 1949. The stimulus objects were white dots, one and one-fourth inches in diameter, projected in random arrangements on a dark screen. Thirty-five fields of dots were used—a field each of 1 to 15, and 17, 19, 22, 25, 28, 32, 37, 42, 57, 66, 77, 89, 102, 112, 134, 152, 170, 191, and 210. The subjects were nine college students, none of whom had served on experiments of this kind before, seated

thirty-two feet from the screen. Each subject made about twenty-one reports of each of the thirty-five fields for a total of about seven hundred reports per subject. Five of the subjects were instructed for speed; the other four, for accuracy. The subjects were not told whether their reports were correct or not. The presentation of dots was tachistoscopic with an exposure time of 0.2 seconds.

The investigators used a mechanical arrangement which measured the time between the exposure of the dots and the subject's vocal response. After each report of the number of dots seen, the subject indicated his confidence on a five-point scale, with five representing absolute certainty and one representing absolute uncertainty. In this way, the investigators collected three kinds of data to treat and evaluate: (1) the number of dots reported, (2) reaction time, and (3) confidence.

1. Number of dots reported. The median number of dots reported by the subjects for each stimulus-number presented was computed; and after treatment of these medians, certain results were apparent. Both groups, those instructed for speed and those instructed for accuracy, usually reported 1 to 5 presented dots correctly. For 6 to 9 dots the median reports were a little too high. For 10 dots and more the median reports were a little too low, with reports on 50 or more dots still lower.

2. Reaction time. The data here were treated in two different ways: (a) median report-time as a function of the dots presented, and (b) a reciprocal function of median report-time plotted against number of dots presented. For both groups (speed and accuracy),
both sets of graphs were discontinuous with respect to slope at a point near or at 6 dots. Graphs for individual subjects showed similar discontinuities in the same vicinity.

3. Confidence. Each median confidence was subtracted from 5.00 (maximum confidence possible), and the result was plotted on a logarithmic scale as a function of the presented number. Again there was a discontinuity in slope in the vicinity of 6 stimulus-dots.

These investigators pointed out that in psychology a discontinuity in slope when two related variables are graphed is indicative of a change in mechanism at the point of discontinuity. According to the findings, such a change in mechanism in the discrimination of visual number occurs in the vicinity of 6 stimulus-objects. Therefore, a distinction was made between these two mechanisms, and each was given a name. It was suggested that the term estimating, which is commonly used to report numerosness, be reserved for the discrimination of stimulus-numbers greater than 6, and that the term subitizing be applied to the discrimination of stimulus-numbers six and below. The word subitize was adopted from the classical Latin adjective, subitus, meaning "sudden" and the medieval Latin verb, subitare, meaning "to arrive suddenly."

The authors devised operational definitions for their terms when these terms are applied to the visual discrimination of numerosness. Under these conditions subitizing is what occurs when the stimulus-number is less than 6, and estimating is what occurs when the stimulus-number is greater than that. On the average, these
authors maintained, subitizing is both more accurate and more rapid, and is done with more confidence than is estimating. The authors were not sure about the stimulus-number 6, but they suggested that it may be interpreted as that number where subitizing and estimating are equally rapid, accurate, and confident.

Kaufman and the others pointed out that subitizing and estimating differ in one other important aspect. When a subject is shown an auxiliary field of dots (an anchoring stimulus), with which he is familiar, it can be used to improve the accuracy, speed, and confidence of estimating; but it will not affect subitizing. In one respect, estimating and subitizing are alike. The subject gives only one response for the entire group of objects. In this respect, they both differ from counting. Counting, of course, requires that the stimulus field remain in view long enough for the operation to be completed. Counting (by ones) is the operation of setting up a one-to-one correspondence between the objects in the stimulus field and the number names taken in order. These authors pointed out that counting by twos, threes, and so on, or even by unequal increments, is a combination of subitizing and counting.

The major findings of Kaufman and his co-workers may be listed as follows:

1. Adults subitize (perceive directly) the number of objects in a stimulus field for groups containing up to 6 objects. They estimate the number of objects in groups containing more than 6 objects.

2. Adults are confident of their judgments of the number of objects in groups containing no more than 6 objects.
Jensen et al. (1950)—Jensen and others sought by experiment the answer to a question Kaufman had left unanswered: when the stimulus fields are projected long enough so that the subjects may count if they wish to when they are instructed for accuracy, will they count the stimulus-numbers 6 and below or will they subitize? Jensen used much the same experimental procedure as had Kaufman, except that the exposure time was as long as the subject needed and the subject was instructed for accuracy. Jensen's mathematical discussion and treatment of his data are excellent, being even more elaborately presented and explained than Kaufman's. After fitting a curve to his data, Jensen found a discontinuity in the vicinity of 6 dots. Jensen concluded that his subjects subitized up to 5 or 6 dots, even though they had been instructed for accuracy and had been given enough time to count, and that they counted in some form the stimulus fields of more than 6 dots.

Recent Experiments with Adults: Successive Auditory Presentation

Up to now the studies have utilized simultaneously presented visual stimuli. This chapter will terminate with short reports on two recent experiments in the judgment of auditory number and two experiments in the judgment of visual number where the stimuli were presented successively.

Taubman (1950).—Taubman's study in the judgment of auditory number grew out of certain problems relating to the discrimination of the number of short tones in various signals in the International Morse Code. For example, overestimation and underestimation were known to occur rather frequently in such signals as 5 (.....), H (....), and S (...). Taubman's problem was to determine the functional relation between judgment of auditory number and (1) the actual number presented and (2) the time interval between successive short tones. He used five different time intervals between the successive tones. The longest such interval, 0.125 seconds, corresponds roughly to the speed of ten groups per minute (GPM) in Morse Code reception. The shortest, 0.062 seconds, corresponds roughly to twenty GPM. Taubman's data indicate that the difference between judged number and presented number for all time intervals is very small up to and including 4 tones. The subjects tended to underestimate beyond 2, and this tendency increased as the time interval decreased. The fact that these subjects were able to "judge" all numbers 1 through 10 correctly when the time interval was 0.125 seconds would seem to indicate that his subjects could count short tones when the time interval between them was one-eighth of a second or greater.

Garner (1951).—Garner reported a study in audible number. His problem, like Taubman's, was to determine the accuracy with


which his subjects could "count" successive short clicks or tones as a function of (1) the repetition rate and (2) the number of clicks or tones. He concluded: (1) that counting accuracy is a function of both repetition rate and number of tones; (2) that small numbers can be counted at repetition rates of 12 per second; and (3) that there is a consistent tendency to underestimate when the subject can no longer count. His most interesting finding here, however, was that at all numbers above 5 or 6 counting accuracy decreased noticeably. Garner consistently used the word counting, but his findings appear to indicate that his subjects were able to perceive audible numbers up to 5 or 6 without counting.

Recent Experiments with Adults: Successive Visual Presentation

Taubman (1950).— Taubman also investigated the judgment of visual number where the stimuli were presented successively. His problem was to investigate the same functional relations as in his experiment with audible number. He found that the mean stimulus-number which can be correctly identified was 10 at 0.500 seconds. This dropped to 3 at 0.333 seconds, and to less than 2 at intervals of 0.250, 0.200, and 0.143 seconds. His time intervals here were longer than those he used between successive audible stimuli, and yet the judgment of number was poorer. Moreover, the mean difference

between judged number and presented number was greater even for small numbers with visual presentations than it was with audible presentations. The most obvious conclusion from these facts is that the ear is more acute than the eye in perceiving number where the presentations are successive.

Cheatham and White (1952).— In 1952, Cheatham and White tested perceived number as a function of flash rates higher than 5 per second. Light flashes corresponding to 30, 22.5, 15, and 10 flashes per second were used. These investigators found that the highest perceived rate with their subjects was 6 and concluded that the subjective rate would probably not exceed 6 to 8 per second regardless of the objective flash rate.

When curves were plotted for each of these four flash rates, perceived number versus presented number, they were found to be somewhat stepwise, with plateaus. This pattern indicated that certain numbers of objective flashes were grouped into very stable perceptual units. Thus, 3, 9, or 10 flashes at a rate of 30 per second were perceived by all subjects on all trials as 3 flashes.

The authors felt they were dealing with some basic property of vision itself and not with a judgment process. In a later experiment, the authors tested to see whether the perceived number

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of flashes was limited by temporal characteristics of the retina, and they found that the light-adapted retina reacted to each separate flash in a series of flashes up to and including 45 flashes per second. The investigators concluded that the retina is not responsible for the temporal patterning of perceived number, but that this patterning must be ascribed to some higher process.

Summary of Experiments with Adults

Table I is a summary of all experiments reported in this chapter in which randomly arranged visual stimuli were presented simultaneously. The more recent experiments definitely indicate that the average adult can perceive up to 6 objects in the field of vision with a high degree of reliability; moreover, he has confidence in his ability to do so. If the stimuli are arranged in regular or familiar patterns, his judgment is more accurate; and, in such cases, he can perceive more than 6 objects. Other factors, such as the size of objects, the distance between objects, the kind of objects, and knowledge of the range of the numbers presented, affect the adult's perception of number. Some experimenters have found that practice increases the ability in number perception. However, this improvement appears to take place during the first few trials, and its effect is most pronounced in difficult configurations such as concentric circles.

Hunter and Sigler, Taves, and Kaufman all found two different mechanisms operating in adults' judgments of number. One of these mechanisms operates when fewer than 6 stimulus objects are presented,
Table I
THE PERCEPTION OF VISUAL STIMULI WHEN RANDOMLY ARRANGED OBJECTS ARE PRESENTED SIMULTANEOUSLY

<table>
<thead>
<tr>
<th>Investigator</th>
<th>Stimulus objects</th>
<th>Time exposed</th>
<th>Accuracy criterion</th>
<th>Upper limit of number perceived</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jevons (1871)</td>
<td>Beans</td>
<td>Immediate response</td>
<td>100%</td>
<td>4.2</td>
</tr>
<tr>
<td>Warren* (1897)</td>
<td>Circles</td>
<td>0.131 sec.</td>
<td>100%</td>
<td>3</td>
</tr>
<tr>
<td>Fernberger (1921)</td>
<td>Dots</td>
<td>0.1 and 0.06 sec.</td>
<td>50%</td>
<td>6 to 11</td>
</tr>
<tr>
<td>Oberly (1924)</td>
<td>Dots</td>
<td>0.37 sec.</td>
<td>50%</td>
<td>5.2 to 9.4**</td>
</tr>
<tr>
<td>Gill and Dallenbach (1926)</td>
<td>Various geometric forms</td>
<td>0.06 sec.</td>
<td>50%</td>
<td>17, 20.42***</td>
</tr>
<tr>
<td>Glanville and Dallenbach (1929)</td>
<td>Various geometric forms</td>
<td>0.07 sec.</td>
<td>100%</td>
<td>5 to 6</td>
</tr>
<tr>
<td>Hunter and Sigler (1940)</td>
<td>Dots</td>
<td>Change**** of slope</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>Taves (1941)</td>
<td>Dots</td>
<td>0.2 sec.</td>
<td>Change**** of slope</td>
<td>Between 6 and 3</td>
</tr>
<tr>
<td>Saltzman and Garner (1948)</td>
<td>Concentric rings, dots</td>
<td>0.5 sec.</td>
<td>100%</td>
<td>3 with improvement to 4</td>
</tr>
<tr>
<td>Kaufman et al. (1949)</td>
<td>Dots</td>
<td>0.2 sec.</td>
<td>Change**** of slope</td>
<td>6</td>
</tr>
</tbody>
</table>

* The patterns were arranged in a circle.
** These data include all non-counting mechanisms.
*** Experimental data were obtained for three subjects only.
**** A change in mechanism is indicated at that point on the graph of two related variables where the graph changes slope.
and it is apparently an immediate awareness of the number of objects presented. Kaufman gave this mechanism the name **subitizing**, meaning "to arrive at suddenly." The second mechanism operates when more than 6 stimulus objects are presented. Kaufman found that this mechanism is not accurate, nor is it done with confidence. For this mechanism he reserved the term **estimating**. To obtain accuracy when the number of stimulus objects is greater than 6 requires the use, in some form, of sub-grouping and combining groups, with or without counting; consequently, a longer time is needed in such cases.

Saltzman and Garner's findings and conclusions regarding the subitizing of small numbers do not agree with those of Jensen. The former investigators found that for the subjects they used there was no minimum number of objects below which the reaction time remained constant, and they concluded that there is no such event as the immediate awareness of number. Jensen found that his subjects subitized groups numbering up to 6 items.

The difference in the results of these two investigations may be explained in terms of the difference in the stimulus materials for that part of the study in which Saltzman and Garner used concentric rings. This does not explain the situation in their second experiment, however, when their stimulus objects were dots, the same as Jensen's. In both experiments the subjects were male college students. In each case the subjects were instructed for accuracy; they could count if they wished to, but they were not specifically instructed to count. There is a possibility that differences in
the wording of the instructions in the two investigations account for these variations. In Saltzman and Garner's study, "the observers were repeatedly cautioned to be absolutely certain that the reports which they made were correct . . ."35 Jensen and his co-workers instructed their subjects to report the number of dots "as accurately as possible."36

When auditory stimuli are presented successively, the evidence for the range of the perception of number is not so clear. Taubman found the judgment of auditory number up to and including 4 successively presented stimuli to be highly reliable in all the time intervals he used, and Garner observed that the "counting" accuracy of numbers above 5 or 6 decreased noticeably.

Taubman's investigation indicated that the sense of sight is not so acute as the sense of hearing in the perception of successive stimuli. Cheatham and White concluded that the highest perceived rate of successively presented visual stimuli would probably not exceed 6 or 8 regardless of the presented rate.

Further research needs to be done on the problem of the tactile perception of number. Most experiments involving tactiley presented stimuli have had to do with form and position rather than with number. The writer has been unable to find any experiments in the perception of number in which the sense to be used was either taste or smell.

Implications for the Teaching of Arithmetic

The studies with adults reviewed in this chapter provide a definite answer to the question: what is the limit of the human ability to subitize—to perceive the number of objects in a group? If grouping procedures are to be taught in arithmetic, attention must be given to developing the ability to subitize, since the subitizing of small groups is needed for effective grouping procedures. It must be recognized, however, that the ability to subitize is limited, that the limit is in the neighborhood of 6 objects, and that it would be educationally wrong to expend time and energy in attempting to develop the ability to subitize beyond this limit for all students.

There is some indication in the studies with adults that it is more difficult to perceive number when the stimuli are presented successively than it is when they are presented simultaneously. It would appear, then, that children would tend to use subitizing to determine the number of objects in a group to a greater extent if number stimuli were presented to them simultaneously rather than successively.
Chapter III

EXPERIMENTS WITH CHILDREN IN THE PERCEPTION OF THE NUMBER OF OBJECTS IN A GROUP

Introduction

A discussion of questions associated with counting and grouping in arithmetic was used to introduce Chapter II. Those questions may be stated again as follows: (a) Is counting or is grouping the better foundation for learning arithmetic? (b) Is counting prerequisite to grouping? (c) What are the limitations of the ability to group?

The purpose of this chapter is to report on relevant studies made with children. The information obtained may be of value in the following ways: (a) to determine the abilities of children of various ages to perceive the number of objects in a group; (b) to make an estimate of the kinds of materials which encourage children to use grouping procedures in their work; and (c) to determine the relationship between counting and grouping in the development of early number ideas.

Perceiving the Number of Objects in a Group

In the studies reviewed in Chapter II, an upper limit in the neighborhood of 6 was established for adults in the visual perception of groups of irregularly arranged objects. Adults, particularly
those used in most of the experiments reviewed in Chapter II, have considerable number knowledge. In particular, they are able to count well beyond the number of objects they can immediately perceive; they have had many experiences with various representations of different sized groups; and undoubtedly many of them have developed individual techniques in grouping and subgrouping objects to aid them in their perception.

Certain questions arise in regard to the perception of number by children which do not arise in the studies with adults. Some of these are the following:

1. How do the abilities of children of various ages differ from those of adults in the power to perceive the number of objects in a group?

2. Is it possible for children to perceive the number of objects in a group and to discriminate between two groups on the basis of number before they can actually count the number of objects in a group?

3. To what extent and under what conditions will children use procedures which involve perceiving the number of objects in a group in preference to using counting techniques?

This section of Chapter III discusses studies which pertain to the perception of number by children and which relate to these questions.
Binet: The Discrimination of Visual Number by Children Who Could
Not Yet Count

In 1890 Alfred Binet reported a study made with two young girls, aged fifty-two and thirty-two months. Neither could count beyond three. Binet's problem was to determine how well the child who does not know how to count can perform in situations involving the perception of number and length.

In order to determine whether the four-year-old girl could recognize the number of objects in a group, without counting, Binet placed two groups of objects or counters on a table and asked her which group had the larger number of objects. By varying the differences between the two groups, Binet hoped to arrive finally at the smallest difference perceptible to the child. On this test Binet found that this child, who could count only to three, was nevertheless capable of comparing two groups of 17 and 18 objects as to size.

Binet wondered whether the girl were making the comparison on the basis of the relative areas occupied by the groups. To test this, he used 18 green counters, each two and one-half cm. in diameter, and 16 white counters, each four cm. in diameter. The larger group now occupied the smaller area. The child erred consistently. Evidently her judgment was based on the relative areas occupied by the two groups. Through a series of trials, however,

in which Binet varied the sizes of the groups of small green counters and large white counters, he eventually found that his subject could determine which group had the greater number of objects up to and including a comparison of 5 of the smaller green counters with 4 of the larger white counters. She was not able to make the correct comparison of 6 small green counters with 5 large white counters, nor of any larger groups, with any consistency.

In a second test Binet would place 2, 3, 4, etc., counters on the table. As soon as the two children had looked at the group, he would cover the counters with his hand and remove all but one of them. Then he would replace one counter at a time, in each case asking the girls if there were then as many counters on the table as there had been in the original group. On this test, the limit of the perception of number for the older child (fifty-two months) was 4 objects. After some practice, however, she was able to perceive a group of 5 with high reliability. The two-and-one-half-year-old child could perceive only 3 objects with exactness.

Binet's main finding may be stated as follows: the child who cannot count beyond three may still be able to perceive—subitize—the number of objects in a group up to and including 5 and to discriminate between 5 and 4 objects. Binet's findings are in contrast to the belief held by many authorities that counting is basic to other number ideas and that one must first count to determine the number of objects in a group. For example, Taylor and Mills state that "children get their first notions of number by counting. Hence,
for purposes of teaching elementary arithmetic, number may be regarded as a property of a group that is first found by counting." Other authorities who agree with Taylor and Mills will be referred to later. Binet's findings show that counting ability is not prerequisite for the perception of the number of objects in small groups. The implication here for early number work is that children do not need to depend solely on counting for the development of number concepts.

Howell: The Visual Perception of the Number of Objects in a Group, First- to Eighth-Grade Pupils

Prior to the report of his own study, Howell reviewed earlier investigations, made by German educators, which had to do mainly with perceptual materials. One of the German educators, W. A. Lay, had performed a considerable number of experiments with children. Since Howell gives a detailed account of Lay's experiments and based his own study on their results, only a brief summary of these findings will be given here.

1. First-grade children in Lay's experiments encountered more difficulty in apprehending successive auditory stimuli, even

4. Ibid., pp. 154-97.
5. Ibid., pp. 193-97.
when a rhythm was used to aid them, than they did in apprehending objects presented either visually or tactiley. The limit for the perception of audible stimuli presented successively for first-grade children was about 3 sounds. Lay concluded that early number experiences should be based largely on spatial things, using the senses of sight and touch, rather than on successive auditory stimuli using the sense of hearing.

2. To test the theory that number concepts arise only through counting, Lay experimented with kindergarten children. He exposed number pictures of the quadratic form for one-half to one second and asked three and one-half to six-year-old children to reproduce them on their slates. For example, if a child saw or thought he saw a number picture of three, he was to draw it on his slate. The quadratic number pictures used by Lay were of the following form:

- 6

The children were able to reproduce various number pictures of from 2 to 10 dots. Lay concluded that distinct and vivid number concepts can arise, exist, and be reproduced without counting.

3. Lay also found that when objects were presented in rows, 3 was the limit of visual apprehension for first-grade children. In the perception of number by touch he found that first-grade children could perceive up to 12 buttons if they were arranged in quadratic form, but no more than 3 if they were arranged in a row.

6. Ibid., p. 154.
In his own study, Howell presented quadratic number pictures to school children under school conditions.\textsuperscript{7} The time of exposure was five seconds. Howell was of the opinion that such an exposure time was necessary in order to be reasonably sure that the children did not use counting or, at least, that they could use it only sparingly.

Howell presented a table and a graph\textsuperscript{8} showing the per cents of errors in the apprehension of the number of objects, 5 through 12, by half years from the first grade to the eighth grade inclusive. His data clearly indicate that only the first-grade children who had just entered school had any appreciable trouble, and they made only six per cent errors.

Howell did not attempt to determine how his subjects apprehended the number of objects in his number pictures. In view of the length of his exposure time, five seconds (which was sufficiently long to permit all or most of his subjects to count the number of objects), the only reasonable conclusion which can be drawn is that he did not measure the limits of the perception of number for children except to the extent that many of his subjects may have grouped by twos and fours.

\textsuperscript{7} Ibid., pp. 203-51.

\textsuperscript{8} Ibid., pp. 212-13.
Freeman: The Visual Perception of the Number of Objects in a Group, Six- to Eight-Year-Old Children

In 1912, Freeman reported an investigation of number perception with children and adults. Freeman felt it was necessary in his experiments to present number pictures in which the objects portrayed neither had any regular pattern of grouping nor any form by which the individual could mentally assign them a regular-pattern grouping. Freeman arranged equally spaced, light circles in a horizontal line on dark backgrounds. The time of exposure was 0.05 seconds. Freeman's main findings follow:

1. Children six to eight years of age gave such incomplete and varying answers that Freeman felt no positive conclusions could be made regarding their perception of number. He believed this was probably the result of their inability to grasp the stimuli in the short period of time he allowed.

2. Less difference existed between adults and children in the ability to perceive the number of objects in a group than might be expected. Freeman makes this statement: "If we place the scope of attention of adults and older children at about five, that of younger children would not be less than four."  

3. As the number of objects increased beyond the scope of attention—that is, beyond the limits of subitizing—the correctness

9. Frank N. Freeman, "Grouped Objects as a Concrete Basis for the Number Idea," Elementary School Teacher, XII (March, 1912), 306-14.

10. Ibid., p. 309.
of judgment of the children fell off much more rapidly than it did with adults. Freeman found that even in groups lacking any pattern the adults tended to organize their perceptions into regular groupings; that is, they either tended to see a pattern in the whole group or to see it as made up of patterned subgroups. When the whole group was patterned, adults and older children nearly always saw the pattern. On the other hand, young children either were much slower to recognize the patterns or else saw them as far-fetched arrangements.

Douglas: The Visual Perception of the Number of Objects in a Group, Four and One-Half- to Six-Year-Old Children

Douglas, 11 in 1925, reported a study of the development of number concepts in pre-school children. His purpose in the study was to find out to what degree children from age four and one-half to age six have concepts of the numbers from one to ten. Douglas decided to base his estimate of these children's concepts of numbers on the results of three visual perception tests in which the stimuli were presented simultaneously. The three tests were these: (1) a dot recognition test, (2) a dot selection test, and (3) a marble test.

In the dot recognition test, the children were asked to estimate, without counting, the number of blue dots on a 4 x 6-inch white

card. Of forty pupils taking the test, none of them made errors in perceiving 1 dot and 2 dots. Thirty-four children, or 85 per cent of them, perceived 3 dots correctly. Seventy per cent perceived 4 dots correctly; 25 per cent 5 and 6 dots; 17 per cent 7 and 8 dots; 15 per cent 9 dots; and 12\(\frac{1}{2}\) per cent 10 dots.

The same cards were used in the dot selection test. After the cards had been arranged in a predetermined order on a table, the children were asked to designate, without counting, the card bearing the number of dots asked for. On this test all forty pupils were correct for 1 and 2 dots; 78 per cent of them were correct for 3; 48 per cent for 4; 35 per cent for 5; 22 per cent for 6; 20 per cent for 7; 25 per cent for 8; 12\(\frac{1}{2}\) per cent for 9; and 25 per cent for 10 dots.

For the third test the experimenter displayed marbles just as they happened to group themselves when the hand was opened. Of twenty-six six-year-old children taking this test, all of them were correct for 1 and 2 marbles; 81 per cent of them were correct for 3 marbles; 50 per cent for 4; 73 per cent for 5; 50 per cent for 6 and 7; 61 per cent for 8 and 9; and 54 per cent for 10. Fifty per cent or more of the children were correct in each case for the numbers 1 to 10 marbles.

Although Douglas' sampling was small, his findings give some indication that preschool children, who have had no formal instruction in number and who have probably had little or no formal instruction in the direct perception of the number of objects in a group,
nevertheless can perceive up to and including 4 objects with a rather high degree of reliability.

Brownell: The Visual Perception of the Number of Objects in a Group, First- to Seventh-Grade Pupils

Brownell\textsuperscript{12} used group tests to determine the ability of children in grades I to VII to apprehend number. He used groups of dots (3 to 12) arranged in quadratic, diamond, triangular, domino, linear, and odd patterns. These patterns were built up as follows:

- Quadratic number pictures.

```
3  4  5  6  7  8  ...
```

- Diamond number pictures.

```
3  4  5  6  7  8  ...
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- Domino number pictures.

```
3  4  5  6  7  8  ...
```

- Triangular number pictures.

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3  4  5  6  7  8  ...
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\textsuperscript{12} William A. Brownell, \textit{The Development of Children's Number Ideas in the Primary Grades, Supplementary Educational Monographs, No. 55} (Chicago: University of Chicago Press, August, 1928), 3-117.

\textsuperscript{13} Ibid., p. 11.
Brownell found that in irregular patterns the ease of apprehension decreased as the number increased from 3 to 12 objects. A large per cent of the children in the first grade (approximately 80 per cent) and in the second grade (approximately 65 per cent) used the counting method for numbers 3 through 12, even when they were arranged in the quadratic patterns. Brownell concluded that children in the first two or three grades use counting methods, which he called immature methods, for determining the number of objects in a group; they do not readily discriminate visual concrete numbers.

Studies which indicate that children receive rather extensive home instruction in counting will be reported. Undoubtedly this emphasis on counting continues in most first- and second-grade classrooms today. Brownell's findings indicate that an emphasis on counting arrests rather than promotes more mature grouping procedures.

Long and Welch: The Number Discrimination, Number Matching, and Group Matching Abilities of Young Children

In 1941, Long and Welch\(^\text{11}\) reported a study in which they sought to determine the number discrimination, number matching, and group matching abilities of 135 children who ranged in age from thirty months to eighty-three months. In a number discrimination test, these investigators found that for six-year-old children the average number of marbles which could be discriminated from a group of 10 marbles was 7.87. In a number matching test, the average number

\(^{11}\) Louis Long and Livingston Welch, "The Development of the Ability to Discriminate and Match Numbers," *Journal of Genetic Psychology*, LIX (December, 1941), 377-87.
of marbles which could be correctly matched by these six-year-old children was 6.87. In a group matching test, these same children could discriminate two equal groups with an average of 6.93 marbles in each from two other equal groups containing an average of 5.93 marbles.

No very definite conclusions can be drawn from Long and Welch's investigation regarding the limit of the discrimination of number by young children, however, since the mechanics of the experiment permitted the children to discriminate on the basis of area covered by the two groups as well as on the basis of the numbers of objects in the two groups.

Carper: The Extent of the Use of Grouping Procedures by First-Grade Children

In 1941, Doris Carper reported a study in which she gave careful attention to the methods and procedures used by first-grade children in dealing with various types of number activity. In the first part of her study, Carper attempted to find out not only how accurate the children's responses were to the various items on her test, but also the methods they used in obtaining their answers. She states that she determined two aspects of each response except for those involving counting and enumeration: (1) the quantitative aspect, the numerical answer given; and (2) the qualitative aspect, the method the child used.

15. Doris V. Carper, "A Study of Some Aspects of Children's Number Knowledge Prior to Instruction" (Unpublished Ph. D. dissertation, Graduate School of Arts and Sciences, Duke University, 1941).

16. Ibid., p. 34.
Miss Carper used the word enumeration in her study to mean rational counting by ones, that is, the operation of assigning to each individual unit in the group a number word, using the number words in order as the objects are counted. Any procedure used by the children which was not enumeration Carper classified as some form of grouping. Thus, she called an operation such as counting by twos and threes a form of grouping. She also differentiated part counting from enumeration. Part counting consists of perceiving the number of objects in part of a group directly and then counting the remaining objects. Thus, a person may perceive 3 of the objects in a group of 7 objects directly and think as follows: "three, four, five, six, seven."

The results of Carper's research which have to do with the perception of the number of objects in a group will be treated in two parts: (a) recognizing the number of objects in a group, and (b) reproducing groups.

Recognizing the number of objects in a group.— In her first investigation, Carper used 270 children who had been in the first grade less than three weeks. The arithmetic test she gave them was given individually. One part of this test concerned the recognition of groups. Black dots, one-half inch in diameter, were used on 5 x 7-inch white cards.

In the use of regular patterns, which were essentially the same as some of those used by Brownell, Carper found that a high per cent of the pupils used grouping for the numbers 2 and 3, approximately
90 per cent for 2 dots and approximately 80 per cent for 3 dots. Approximately 66 per cent were able to recognize groups of 4; and 52 per cent, groups of 5 dots. For 6 dots, approximately 10 per cent more of the pupils used counting than used grouping.

In another test of recognition of groups, Carper asked the pupils to identify the card in a set of four cards which had a designated number of pictures of children on it. Her first-grade children used the grouping process even more in this test than in the first one, with no regular decrease in use as the numbers increased and with only a slight decrease in accuracy. Although Carper does not say so, these results would seem to indicate that when children are faced with the alternatives of excessively clumsy counting procedures—counting each card in turn until the correct one is found—and the easier grouping method, they will group and not lose too much in accuracy by so doing.

Reproducing groups.— Carper used two different tests which she called tests in reproducing groups. In the first of these two tests, rows of pictures of animals were presented to the pupils and they were asked to draw boxes around a designated number (given verbally) of the animals throughout the row. Since the children readily grasped the 2- and 3-groups in this test, Carper felt this was a significant indication that these two groups are functionally meaningful concepts for these first-grade children. The per cent of pupils using grouping for 4, however, dropped abruptly to less than 50 per cent, and the number using grouping for 5 in this test was only about five per
cent of the total. This indicated to Carper that the children did not have clear concepts for the latter two numbers.

In her second test of reproducing numbers, Carper asked the children to draw as many windows in one house as there were already shown in another house. In this test Carper found that approximately 60 per cent of the children used a pure grouping method to reproduce 3 objects. Carper used the term pure grouping to mean immediate perception of the total number in the group. As the number of objects in a group increased, the per cent of children using pure grouping decreased in the following manner: 34 per cent for 4 objects; 16 per cent for 5 objects; and one per cent for 6 objects.

Carper's findings regarding the extent of the use made of grouping procedures by first-grade children may be summarized as follows:

1. Better than 50 per cent of her first-grade children could perceive the number of objects up to and including groups of 5 regularly arranged objects, and more than a third of them could recognize 6 dots immediately on her number recognition test. Moreover, they used this method of direct perception even though they were permitted to use other methods, such as counting.

2. In the tests of reproducing groups, most of the children in Carper's study used pure grouping for groups up to and including 3 objects. For larger groups, however, the per cents of pupils using grouping procedures fell off abruptly.
Dawson: The Limit of the Apprehension of Number by First-Grade Children

In one part of a study reported in 1953, Dawson used four sets of number pictures, each set containing four cards, one each for 4, 5, 6, and 7 stimulus-objects. The patterns and sizes of the number groups in the four sets were the same; only the picture units varied. Dawson found that 4 or 5 was the limit of number apprehension under his experimental conditions for first-grade children. When dealing with 6 or 7, the children decomposed the pictures into subgroups or counted.

Summary: The Ability of Children to Perceive the Number of Objects in a Group

The information concerning the ability of children to perceive the number of objects in a group, which in part answers the questions proposed at the beginning of this section, may be summarized as follows:

1. The number of objects in a group which children can perceive varies widely, and appears to depend on a number of factors which include age, previous experience with numbers, and the type of perceptual materials being used at the time. It would appear that a majority of children from four and one-half to six or seven years of age can perceive visually at least 4 objects when they are presented simultaneously in various arrangements. Thus, children of these ages, although they have admittedly had fewer experiences with

numbers and groups than have adults, nevertheless do not differ in
this ability by more than 2 objects from the average of adults, which
was found to be approximately 6 objects.

2. Children can perceive the number of objects in a small
group and can discriminate between two small groups on the basis of
number even when the numbers involved are beyond their counting
ability. Binet found that his four-year-old subject, who could only
count to three, could nevertheless discriminate between 4 and 5
objects when the two groups were presented simultaneously and after
some practice could determine when all of the objects of a previously
presented group of 5 had been replaced.

3. (a) Carper found that beginning first-grade children tended
to use grouping in small groups up to and including 5 objects,
although they were allowed to use any means they desired, to determine
the number of objects in the group. In number tasks where counting
procedures were clumsy, even more of Carper’s subjects used grouping
methods.

(b) Brownell found that children in first and second grades
tended to use counting methods in determining the number of objects
in a group. Brownell’s subjects had been exposed longer to the
emphasis on counting in the number work of the primary grades than
had Carper’s. His findings appear to indicate that the emphasis
on counting arrests rather than promotes the ability to perceive
the number of objects in a group, or at least arrests the use of
grouping procedures by young children.
Implications for the Teaching of Arithmetic

The results of the studies reported in this section hold the following implications for the teaching of arithmetic in the primary grades:

1. Young children do not need to depend solely on counting in order to develop concepts of numbers. The assumption that arithmetic programs must begin with lessons in counting is not supported by the results of the studies reported. Probably a considerable amount of quantitative learning can be attained before counting is needed.

2. Pre-school children from approximately four years to six years of age apparently develop the ability to subitize the number of objects in groups up to 4, without the aid of any systematic instruction along these lines. Yet, after they are introduced to the usual emphasis on counting in formal arithmetic, they tend to count even in situations in which they could subitize. Arithmetic programs in the primary grades should provide early for experiences which will keep alive and develop further the efficient method of subitizing the number of objects in small groups.

Studies Concerning the Qualities of Good Perceptual Materials

Some investigators have been interested in the kinds of materials which facilitate the perception of number. The main question involved in these studies is this: in what kinds of materials do children most readily apprehend number? More specifically, since these studies have been almost universally concerned with number pictures, the question may be stated thus: in what forms of number
pictures do children most readily apprehend the number of objects portrayed? A number picture is generally in the form of a card, a paper, or a screen upon which several figures, or units, such as dots, circles, tally marks, or pictures of objects, are portrayed.

Howell: Report on Investigations by German Educators

Howell discussed several kinds of number pictures which had been used and tested prior to 1914, particularly by German educators.18 Some of these were as follows:

1. Vertical strokes arranged in linear patterns. These had been used by Pestalozzi.

```
1 11 111 1111 11111 ...
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2. The Russian calculating machine form (horizontal abacus).

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o oo ooo oooo oooooo ...
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3. The normal picture form in which the stimuli were arranged in vertical pairs with a constant distance between pairs.

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o o oo oo ooo ooo ooo...
o o oo oo ooo ooo ooo...
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4. A form similar to that of the normal picture where there was a greater distance after every two pairs. This had the effect of grouping by fours and was called the quadratic number picture.

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o o oo oo oo o oo o oo o...
o o oo oo oo o oo o oo o...
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5. Another form similar to the normal picture, except that from 6 on the extensions were made vertically.

\[
\begin{array}{cccccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

6. Certain arbitrary groupings. The following is attributed to a German educator named Beetz.\(^19\)

\[
\begin{array}{cccccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

These German educators' experiments with children tended to verify the following hypotheses: (1) the apprehension of things in rows is more difficult than the apprehension of things in other regular arrangements; (2) ball and circle forms are noticeably easier to apprehend than rod-shaped objects; (3) horizontal arrangements of objects are superior to similar arrangements built up vertically; (4) the normal number picture and the quadratic number picture are superior to other number pictures.

McNamara: The Effect of Pattern and Units Used on Number Apprehension

Knight\(^20\) reported a study by J. J. McNamara on the relative perceptual value of various configurations of the numbers 6 to 12 inclusive. McNamara used photographs (4 x 6-inch) of dots, splints,

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and dolls arranged variously in linear, diamond, and domino patterns. These were randomly exposed in a Whipple tachistoscope for one second each. One hundred second-, third-, and fourth-grade pupils took part in the tests.

McNamara found that there was a wide variation in the per cents of error for the different numbers, the variation apparently depending on the size of the number, the pattern, and the stimulus picture used. The number 11 was especially difficult in all configurations and produced as high as 82 per cent error when presented as a linear arrangement of dolls. On the other hand, there was only 7 per cent error for the number 10 when presented as a domino pattern of dots. Knight believes McNamara's study supports the following practical suggestion:

In presenting number patterns to children: (1) use large dots in domino configurations; (2) enrich the number 8 by using the diamond configuration; and (3) give special attention to the number 11.

Russell: The Effect of Size, Color, and Pattern in Number Pictures

Russell experimented with fifty-four children in the kindergarten, the first grade, and the second grade. His major finding regarding perceptual materials was that children could and did use subitizing oftener and with greater accuracy in instances when the

21. Ibid., p. 329.

number pictures portrayed were composed of congruent units, were
colored the same, and were arranged in regular patterns.

Carper: How Variations in the Pattern, the Units Used, and the
Context of Number Pictures Affect the Perception of Number

In the second part of her study, Carper\textsuperscript{23} was interested in
determining how the various factors in presented number pictures
influence methods of solution and accuracy of responses. Her sub-
jects for this test, which was administered individually, were forty-
eight kindergarten children between five and six years of age. The
test items included recognition of groups and reproduction of groups.
These items were so constructed that only one of the following con-
ditions varied at a time: (1) nature of patterns (regular or
irregular); (2) nature of pictured units (geometrical, pictorial, real
object); (3) nature of context (none, simple, complex). Carper
discussed a number of interesting findings from this study:

1. When the test materials involved simple geometric figures
arranged in regular patterns on plain backgrounds, a large propor-
tion of the pupils recognized the numbers immediately.

2. When the units were pictured in irregular patterns and
imbedded in complex picture contexts, the children used counting to
determine the number of objects presented.

\textsuperscript{23} Carper, op. cit., pp. 150-55. See also Doris V. Carper, "Seeing
Numbers as Groups in Primary-Grade Arithmetic," Elementary
School Journal, XLIII (November, 1942), 167-70.
3. When the units were pictures of semi-abstract geometric designs, the children did as well as when the units were pictures of real objects.

Carper concluded that when materials are used to develop understandings of mathematical relationships and concepts in situations where grouping is the most mature method, these materials should be selected on the basis of good units, regular patterns, and simple backgrounds.24

Dawson: The Effect of the Complexity of the Pictured Units on the Perception of Number

The hypothesis for Dawson's study grew out of a 1935 article in which DeMay presented four levels of representation of numbers to be used by the teacher in helping the child to progress from the concrete to the abstract. Her four levels were these: (1) three-dimensional concrete materials; (2) two-dimensional pictures of concrete objects; (3) two-dimensional number pictures or number patterns of dots, rings, lines, etc.; (4) the number symbols.26

While textbook writers have generally supported DeMay's order of representation, to Dawson her article is suggestive of what she felt should be done and is not a report of research. He questioned


her hierarchy of representations of number and proposed the hypothesis that "when the pattern and size of the perceptual field are held constant, the critical factor in the apprehension of number as a group is the degree of complexity of the perceptual field."\(^2\)

Dawson achieved differences in complexity by varying the kinds of units he pictured. The units used in his number pictures varied from geometric figures, the least complex, which he used in his Type I number pictures, to complex pictorial units, which he used in his Type IV number pictures. Dawson felt his Type IV number pictures were the most complex because of the variety of pictures on any one card, the lack of symmetry in the irregular outlines of the pictures, and the variety of colors used. His subjects were seventy-six first-grade children.

Dawson reported the following findings:

1. The greater the complexity of the number picture, the greater was the amount of counting; the simpler the presentation, the less was the amount of counting.

2. The size and complexity of the group affected the speed of response. Complexity affected counting time more than it did grouping time. The fast responders were found to be those who used grouping, and the slow responders those who applied counting.

In his recommendations for teaching, Dawson maintained that it makes little difference in the perception of the number of objects in a group whether the number pictures portray things or geometric

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\(^2\) Dawson, \textit{op. cit.}, p. 37.
forms, so long as they meet the criterion of simplicity. Dawson concluded that DeMay's hierarchy was in the wrong order.

Summary and Implications for Arithmetic: Studies Concerning the Qualities of Good Perceptual Materials

Several of the investigations with children reported here have stressed the need for simplicity in the number pictures used in early number work. Number pictures which operate for ease of apprehension and thus encourage the use of grouping techniques are composed of simple geometric figures in regular patterns imbedded in simple contexts.

One conclusion reached by most of these investigators is that the units in number pictures should be depicted in regular patterns, since it tends to make number perception easier. In fact, it appears that it is the pattern that is perceived and not the number of objects in the pattern. After a person becomes familiar with a given pattern and knows the number of objects in it, he can report accurately on the number of objects in that pattern each time he sees it.

Familiarity with the number of objects in various regular patterns, such as the square, the triangle, and the domino arrangements, is of considerable value, since definite pattern arrangements occur in a number of real situations. The more of these regular patterns the person knows, the more command he will have of numbers. Perhaps more often than not, though, groups of objects occur in irregular patterns. It would seem that any program which limits its perceptual materials to the use of regular patterns would be
restricting rather than developing the children's abilities to perceive the number of objects in a group.

There appears to have been little or no work done in determining the criteria for the characteristics of concrete objects which make them the most suitable for early number work. However, there are implications for judging concrete objects from the findings of the studies in which number pictures were used. Some of these which would appear to be applicable to three-dimensional representations of numbers for early number work follow:

1. Three-dimensional materials should be simple in design. Blocks, checkers, counters, etc., fit this criterion.

2. Small objects which the child can manipulate within the area of one visual span are preferable to larger objects. If the child has to shift his sight often or over a considerable distance, he tends to count rather than to use grouping procedures.

3. At first, it is probably better if all the objects of a group with which the child is working are congruent and of the same color. However, as the child progresses, variations in the qualities of the objects portrayed should be made so that he will have experiences in attending only to the number quality of the group while he disregards all other qualities.

The Relationship of Counting to Grouping

A number of writers and the results of some investigations in the development of number concepts have tended to support Messenger's
analysis of the mechanism of the perception of number (Chapter II). Messenger's analysis may be repeated here in three steps: (1) one perceives the group as a unit; (2) one recognizes that the group is made up of so many units which may be determined by counting; (3) after sufficient practice in counting a given group, one comes to associate a number with that group and thinks he perceives number directly.

McLaughlin's study led her to a similar analysis of the development of the number concept and the ability to perceive number. The steps in her analysis follow:

1. Perception of simple spatial forms as aggregates.
2. Analysis by counting single units.
3. Analysis by recognition of small numbers, such as 2 and 3, and counting the other units of the group.
4. Small groups combined by counting 2's and by combining "doubles."
5. The mature stage, characterized by prompt recognition and naming of aggregates as groups of cardinal numbers.28

Similar analyses have been made by Judd,29 Delacroix,30 and Riess,31 among others. The work and opinions of Russell and Carper,

both of whom have considered the relationship of counting to grouping, and have reached conclusions different from the previous authors, will be reported in this section.

**Russell: The Relationship of Counting to Grouping**

The conclusion Russell reached is that the idea of counting as the basis for the acquisition of number ideas is erroneous. He felt his data supported the following hypotheses:

1. The cardinal (quantity) and ordinal (series) ideas of number differentiate or develop together.
2. Counting accompanies this process of differentiation, but is not a cause of the development.\(^{32}\)

Russell felt his data supported the hypothesis that counting by ones is a difficult method for differentiating groups, after having discovered that counting is not accurate above five for the young child. Moreover, he found that at first the young child forms subgroups which have equal value mathematically; that is, the child performs an operation which involves matching small equal subgroups of the two total groups until a differentiation between the two total groups can be made.

Russell's conclusion here is not very clear. However, if he is being interpreted correctly, it appears that he concluded that abilities in subitizing and counting develop together and are, or

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\(^{32}\) Russell, *op. cit.*, p. 656.
possibly can be made to be, mutually interdependent, but that one
cannot be called the cause of the other.

Carper: The Relationship of Counting to Grouping

In one part of her test, Carper\textsuperscript{33} presented the children with
some simple addition and subtraction combinations, given in three
different forms: (1) concrete (pennies were used), (2) verbal problems
(without the use of any concrete objects), (3) abstract (using the
Hindu-Arabic number symbols only). From the information gained on
this part of the test Carper arrived at the following conclusion:
grouping is not a counting process and it "cannot be said to have
developed from the counting process.\textsuperscript{34} She believes that the group-
ing process develops from dealing with numbers as groups rather than
as aggregates of individual objects the total of which is determined
by counting, and that it is a mistake to make counting basic to the
other operations.

Evidently Carper's conclusion may be interpreted in two different
ways. If Carper means that counting is of no aid in developing and
using grouping operations, then it would appear that her statement
does not agree with the facts. Adults who very probably have pro-
gressed through the usual school conditions in which counting was
stressed, nevertheless develop the ability to use grouping operations.
The adult studies reviewed in Chapter II, as well as Freeman's

\textsuperscript{33} Carper, "A Study of Some Aspects of Children's Number Knowledge
Prior to Instruction," \textit{op. cit.}, p. 32-34.

\textsuperscript{34} \textit{Ibid.}, p. 67.
findings about the comparative frequency of the use of grouping procedures by adults and children, are proof of this point.

If by her statement Carper means that counting is not the only, or even the best, basis for learning grouping operations, then she is very probably correct. Throughout her study, it appears Carper wishes to point out that the usual school and home practices of emphasizing counting actually deter rather than abet the development of more mature grouping operations. She is of the opinion that most of the early number work required of children "could actually be reduced to enumeration tasks—the most immature form of number solution." Brownell's findings too indicate that such is the case.

Young children want to please their parents and teachers. If these adults emphasize counting, the child will use his counting abilities, sometimes even in places where he knows how to use more mature methods. The findings of several of the studies reviewed in this chapter support Carper's belief that grouping processes develop better from experiences in which the child deals with numbers as groups than from experiences in which he deals with numbers as aggregates of individual objects.

A Short Review of Studies in Counting

There is little necessity to stress counting in early number work, since most children can count before they enter school. A

35. Ibid., p. 2.
host of investigators have reported on the counting abilities of children at various ages. Baldwin and Stecher found the average six-year-old child could do rote counting to 25.3 and rational counting to 27.6.36 Buckingham and McLatchy found that 90 per cent of first-grade children could repeat the number names in order to ten.37 McLatchy reported that the median child in the lowest quarter intellectually of Cincinnati children in the first grade could count by rote to twelve.38 Mott observed that 96 per cent of beginning school children could count beyond ten and that more than half the five-year-olds tested could count objects beyond their rote counting ability.39 Grigsby determined that the average six-year-old could count 26.50 objects.40 Thirty-eight per cent of the first-grade children in Woody's study could count by rote to 100 by ones, and 79 per cent of them could count 20 circles.41


subjects who were just beginning school could count ten objects, and 92 per cent could count at least eight objects. In a recent study regarding the abilities of seventy beginning school children in certain arithmetic tasks, Priore ascertained that the average rote-counting ability by 1's was 29.69 and that the average rational-counting ability by 1's was slightly higher at 29.77.

In his discussion of grouping and counting, Whiteaker makes the following observation:

No doubt the child first sees two as a pair and three as a group of three and would develop his conception to include a group of four; but before he gets a chance to understand them they are lost in the serial relationships forced upon him by benevolent elders.

This statement appears a plausible description of what happens. From the findings of the studies made on the counting abilities of young children, it is clear that most of them receive some home instruction in counting prior to school age. Moreover, it is quite probable

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44. Angela Priore, "Achievement by Pupils Entering the First Grade," The Arithmetic Teacher, IV (March, 1957), 56.

that they receive little or no home instruction in perceiving the number of objects in groups, since parents and teachers alike look upon counting as the first indication of number learning.

Implications for the Teaching of Arithmetic

Apparentl;y any tendency on the part of children to develop their abilities to perceive the number of objects in a group is stymied by the early and rather persistent efforts to have them count the number of objects in groups. Stressing counting first probably accounts for the analyses of the development of number concepts made by McLaughlin, Judd, Delacroix, and Riess, in which all concluded that a counting ability necessarily precedes any ability to recognize the number of objects in a group.

Since first-grade children have already attained some ability in counting, particularly within the range of numbers with which they will be working in that grade, there appears to be little need for the extra emphasis on counting advocated by some authorities. More attention should be given early in arithmetic programs to the development of abilities in subitizing the number of objects in groups and in using this ability in various grouping procedures. The groups with which the child beginning the first grade works are small groups, many of them small enough that he can subitize the entire group. The ability to count is not a prerequisite for working with these small groups, and it is not the most efficient means for dealing with such groups.
Summary of Studies with Children in the Perception of the Number of Objects in a Group

The results of the studies with children reported in this chapter may be summarized as follows:

1. Most pre-school children, from approximately four to six years of age, can subitize at least 4 objects.

2. It is possible for children to perceive the number of objects in small groups even before they can count the objects in those groups.

3. Pre-school children tend to use subitizing more than children in the primary grades who have been exposed to the emphasis which is placed on counting in those grades.

4. Number pictures which operate for ease of apprehension and thus encourage the use of grouping techniques are composed of simple geometric figures in regular patterns imbedded in simple contexts.

5. The ability to count is not a prerequisite for dealing with small groups; and, in particular, it is not necessary to be able to count the number of objects in small groups before one can subitize them.

6. The majority of pupils enter the first grade with considerable counting ability; hence, there is less need to stress the teaching of counting than many writers suggest.

Implications for the Teaching of Arithmetic

Two major implications for the teaching of arithmetic in the primary grades may be drawn from the results of the studies with children as reported in this chapter.
1. The emphasis usually placed on counting as the first number experience in the arithmetic of the primary grades is unjustified. More attention should be given to the development of abilities in subitizing and using grouping techniques.

2. The use of small congruent objects and simple number pictures aids in the development of abilities to subitize and to use grouping techniques. The greatest simplicity is obtained in number pictures when the units are of geometric design and placed in regular patterns on plain backgrounds. If the child is to make a maximum development of subitizing and grouping techniques, however, he should have experiences with number pictures in which the units are pictures of objects, and with number pictures in which the units are arranged in irregular patterns.
Chapter IV

NUMBER STUDIES WITH ANIMALS

Introduction

Purpose of the Chapter

Messenger (see Chapter II) concluded that a subject comes to think he perceives the number of objects in a group directly only after he has had many experiences in counting the number of objects in groups of that size. Many authors of current arithmetic programs appear to have accepted this analysis, since instruction in grouping usually follows instruction in counting in arithmetic textbooks. It is obvious that the question of the precedence of counting or subitizing cannot be answered by reference to the adult studies; and of the studies with children, only Binet's was performed with children who could not count beyond three. The results of Binet's investigations indicated that the number of objects in a small group may be perceived by a subject before he can count them. A discussion of the results of studies with animals might clarify and extend the information on this point.

As recently as fifty years ago, most of the information about animal behavior was anecdotal and anthropomorphic in character. While such information persists today, there has been a large number
of carefully controlled experiments with animals. In several experiments investigators have attempted to determine whether animals can be taught anything about number.

The main purpose of this chapter is to review studies in which the object has been to determine the extent to which animals can learn to make discriminations on the basis of the number of objects in a group. Accepting the assumption that animals do not count, the findings might shed light on the assumed need for counting as a prerequisite for finding the number of objects in a group.

The Criteria for "Having Learned" in Number Experiments with Animals

In some experiments the investigators consider success in "having learned" as being indicated by the animal when it responds correctly on a predetermined number of successive trials of the problem. The more common measure of "having learned," however, is that of better-than-chance accuracy. In most problems, for success, the animal must select the one correct choice out of several possibilities. At that point in a series of training trials when the animal starts making the correct choice in more cases than it does all possible incorrect choices, it is said to be successful in the learning. One or the other of these two criteria for success applies in each of the studies reported on herein, unless specifically stated otherwise.

Types of Number Studies with Animals

Honigmann\(^2\) divided number studies with animals into four categories: (1) identification of an objective by relative position, (2) the multiple-choice method, (3) the alternation problem method, and (4) the discrimination method.

A number of investigators have studied the behavior of various animals in experimental situations which may be classified under Honigmann's first three categories. Both Kinnaman's\(^3\) study with monkeys and Porter's\(^4\) study with English sparrows were concerned with the ability of those animals to locate an object by its relative position in a series of objects.

Yerkes developed the experimental method commonly known as the multiple-choice method for testing human behavior,\(^5\) and then adapted

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it for testing crows, pigs, monkeys and apes, and chimpanzees. Spence carried on Yerkes' experiments with chimpanzees, and Sadovinkova adapted the multiple-choice method for testing siskins (a species of finch) and canaries.

Investigations employing alternation-problem methods fall into two classes. In problems involving alternation of an action, the animal is required to perform alternately two actions which are equivalent or similar but which are oppositely oriented with respect to the animal. Heron gives a good theoretical treatment of such


problems. Hunter has experimented extensively with rats,\textsuperscript{13-16} and
raccoons,\textsuperscript{17} Karn has experimented with cats\textsuperscript{18} and dogs,\textsuperscript{19} and
Gellerman has experimented with monkeys\textsuperscript{20-21} and humans,\textsuperscript{22} using
situations involving alternation of an action. A second type of
alternation problem is one in which an animal is taught to take

13. Walter S. Hunter, "Kinaesthetic Sensory Processes in the White
Rat," \textit{Psychological Bulletin}, XV (February, 1918), 36-37.

Processes in the White Rat," \textit{Psychobiology}, II (February, 1920),
1-17.

15. Walter S. Hunter, "The Sensory Control of Maze Habit in the White

16. Walter S. Hunter and Joseph W. Nagge, "The White Rat and the
Double Alternation Temporal Maze," \textit{Journal of Genetic Psychology},
XXXIX (September, 1931), 303-19.

17. Walter S. Hunter, "The Behavior of Raccoons in a Double Alter-
nation Temporal Maze," \textit{Journal of Genetic Psychology}, XXXV
(September, 1928), 374-85.

Problem in the Temporal Maze," \textit{Journal of Comparative Psychology},
XXVI (August, 1928), 201-8.

19. Harry W. Karn and Harold R. Malamud, "The Behavior of Dogs in
the Double Alternation Problem in the Temporal Maze," \textit{Journal of
Comparative Psychology}, XXVII (June, 1939), 161-66.

20. Louis W. Gellerman, "The Double Alternation Problem: I. The
Behavior of Monkeys in a Double Alternation Temporal Maze,"
\textit{Journal of Genetic Psychology}, XXXIX (March, 1931), 50-72.

21. Louis W. Gellerman, "The Double Alternation Problem: III. The
Behavior of Monkeys in a Double Alternation Box Apparatus,"
\textit{Journal of Genetic Psychology}, XXXIX (September, 1931), 359-92.

22. Louis W. Gellerman, "The Double Alternation Problem: II. The
Behavior of Children and Human Adults in a Double Alternation
Temporal Maze," \textit{Journal of Genetic Psychology}, XXXIX (June, 1931),
197-226.
every second, every third, or every fourth piece of food in a series of pieces of food. Révész\textsuperscript{23-24} and Honigmann\textsuperscript{25} have used this type of alternation problem in experiments with chickens.

In general, investigators who have used any one of the first three experimental methods listed by Honigmann have taken the position that if an animal was successful in learning the appropriate responses in the ways prescribed, then it was forming concepts which are commonly considered number concepts and was, therefore, learning about numbers. Even though animals have achieved some successes in all of these studies, authorities have generally been reluctant to interpret the successes as being indicative that the animals were making their choices on the basis of any learning about number. In particular, it appears that animals do not use any kind of process which could be called counting in the human sense. If animals can learn to react to the number of objects in a group, they must use some other method or methods. The results of discrimination studies with animals, to be discussed in the next section, should clarify this point.

\begin{itemize}
\item \textsuperscript{24} Géza Révész, "Zur Analyse der tierschen Handlung," Archives N'eerlandaises de Physiologie de l'Homme et des Animaux, VII (1922a), 169-77.
\item \textsuperscript{25} H. Honigmann, "The Alternation Problem in Animal Psychology," Journal of Experimental Biology, XIX (October, 1942), 141-57.
\end{itemize}
Number Studies with Animals by the Discrimination Method

In the discrimination method animals have been tested generally in three different ways: (1) discrimination of successively presented stimuli, (2) discrimination between two quantities of objects presented simultaneously, and (3) discrimination in repeating an action a fixed number of times.

Discrimination of Successively-Presented Stimuli

Explanation of the problem and report of studies. In 1929, Woodrow reported a study which he made on monkeys to determine the "longest series of sounds which can be distinguished from a series containing but one additional sound." The most successful monkey learned to differentiate with a high degree of reliability between series of 1 and 3 sounds, 2 and 3, and 3 and 4, and with low reliability between 4 and 5. In a case where one of the monkeys was able to discriminate between 3 and 1, and 3 and 2, Woodrow felt he did this on the basis of the longer of the two series of taps rather than on the basis of the number of taps. Therefore, on the forty-ninth day of the trials, he doubled the time between the 2 taps and left the time between 3 taps unchanged, thus making the time for both series the same. The monkey had to learn all over again, but by the fifty-eighth day he was back up to 87.5 per cent accuracy.


27. Ibid., p. 123.
Woodrow felt it was possible but not certain that the monkey then discriminating on the basis of "twoness" and "threeness" rather than on length of series.

When Woodrow changed the quality of the sound, or when he changed the procedure so that the monkeys could not see the movements which produced the sounds, their ability to distinguish between two sets of sounds showed a marked decrease. He concluded that the monkeys' decisions were not based on the judgment of number alone. He writes that "the most plausible assumption, on the whole, seems to be that the monkeys discriminated on the basis of the totality of attributes possessed by the stimulus-pattern." 28

In 1931, Kuroda 29 attempted to teach a Macacus monkey to associate audible number and position in a series. Kuroda had a series of holes, numbered from left to right, in a board with food behind one of them. The monkey was able, after 700 trials, to associate 1 sound of a bell with the first hole and 2 sounds of a bell with the second hole. But with 3 sounds of a bell, he not only failed to learn to locate the proper hole but actually forgot his previous training.

Kuroda felt that if the monkey had been able to learn this association, even for the first few holes, it would have been evidence of counting in a certain sense of the term on the part of the monkey. His experiment was well designed, however, to point out that the monkey

28. Ibid., p. 142.

cannot learn to count, and that 2 was the limit of the monkey's ability to perceive the number of successively presented sounds.

Several authorities have implied that the need for counting is greater in determining a number of successively presented stimuli or in determining position in a series than it is in determining a number of simultaneously presented stimuli. The monkey was confronted with both the successively presented stimuli and the position in a series problem here. Lay's studies referred to in Chapter III indicated that first-grade children would have been able to do little better than to associate 3 successively presented sound stimuli with the third position in a series without the aid of counting.

In 1941, Douglas and Whitty\(^30\) were able to train some of the five primates they experimented with to discriminate between 1 and 2 successively presented stimuli, and one of them to discriminate between 3 and 4 simultaneously presented dots. The conclusion, however, was that these discriminations were dependent upon bases other than number.\(^31\)

**Summary: Successively-presented stimuli.**—Primates have achieved some successes in problems of discriminating between two series of successively-presented stimuli, although it appears that the animals may have made their decisions on the basis of a composite of factors in the stimulus fields. If the most favorable conclusion is accepted

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31. Ibid., p. 142.
from these experiments, that the animals reacted on the basis of number, the maximum success was the discrimination between 3 and 4 sounds, and the usual success was the discrimination between 1 and 2 sounds. Further discussion of these findings in relation to the teaching of arithmetic will be presented after relevant information concerning perception of the number of stimuli presented simultaneously is reported.

**Discrimination between Groups of Visual Stimuli Presented Simultaneously**

*Explanation of the problem.*—The experimental determination of the upper limit of the ability of adults to perceive visually the number of objects in a group is usually made absolutely. This means that in such experiments the subject is presented with a single stimulus field, usually of dots or other geometric figures, and he is to report, record, or indicate by some sign, the number of objects he sees in the field.

If the subjects being tested are such that it is difficult or impossible to devise a reliable method by which they can directly report, record, or indicate by some sign the number of objects they see in a single group, then a testing method may be used which determines the subject's ability to perceive the number of objects relatively. In this method a subject is usually presented two stimulus fields, and he indicates his choice by reacting to one of the fields in preference to the other. If the two fields do not differ in size by more than one object, and if other factors which may provide him clues by which the subject may make his choice—such as relative
areas occupied by the two groups—are controlled, then it is assumed the subject is reacting relatively to the number of objects in each of the two groups. This method is often, though by no means exclusively, the method used when testing the ability of animals to perceive the number of objects in a group.

The symbolism commonly used in psychological journals to designate that a subject can make the discrimination between a group of x objects and a group of y objects is \( x : y \). The \( x : y \) symbolism will be used henceforth in this study.

Report of studies.—Révész\textsuperscript{32} taught a fowl he used to choose 3 grains of corn over 2 grains, 4 over 3, 5 over 4, 8 over 6, 10 over 7, and sometimes 7 over 6 grains. He remarked that his subject reacted in the same way whether the grains of corn in the piles were in irregular or regular, horizontal or vertical, order. Research with his other chickens showed that the limit from animal to animal was somewhat variable. He felt that chickens made their discriminations because they preferred a larger amount of food to a smaller amount, and added that the limit attained by his subjects agreed approximately with the limit for a small child on like problems.

In 1926, Werner Fischel\textsuperscript{33} reported experiments with a goldfinch, two hedge-sparrows, and some pigeons. The goldfinch was able to

\textsuperscript{32} Geza Révész, "Tierpsychologische Untersuchungen," Zeitschrift für Psychologie, XXXVIII (1922b), 135.

\textsuperscript{33} Werner Fischel, "Haben Vögel ein 'Zahlengedächtnis'?", Zeitschrift für vergleichende Physiologie, IV (August 9, 1926), 345-69.
learn by the gluing method to discriminate between 3 grains of
hempseed and 1, 3 and 2, 4 and 2, 5 and 3, and 6 and 3, but not
between 5 and 4, 7 and 5, and 6 and 10 grains. The pigeons learned
to distinguish between two cards, one of which had 1 dot on it and
the other 2 dots. They also learned to distinguish 3 dots from
2 dots, when the 3 was in a triangular pattern; however, only one
pigeon learned to make the discrimination 2:3 when the 3 dots were
arranged linearly.

Nine years later, Koehler, Müller, and Wachholtz\(^{34}\) reported
that the pigeons they used learned to discriminate between 5 and 4,
6 and 4, and 6 and 5 grains. Since the pigeons had to learn the
discrimination for every group of numbers, the investigators felt
the birds had not made choices simply on the basis of the larger
of the two heaps of grain. Arndt\(^{35}\) succeeded in teaching pigeons
to distinguish between 000 and 00 by starting with 000 and 00
and gradually changing this pattern to 000 and 00.

In 1934, Tellier\(^{36}\) reported a series of investigations she
had performed at the University of Liége with an exceptionally tame,
and apparently quite intelligent, male Bonnet monkey (Macacus Sinicus),

\(^{34}\) O. Koehler, O. Müller, und R. Wachholtz, "Kann die Taube Anzahlen
erfassen?" Verhandlung der Deutschen Zoologischen Gesellschaft,
E. V. \textit{XXVII} (1935b), 39-54.

\(^{35}\) W. Arndt, "Abschliessende Versuche zur Frage des 'Zähl'-vermögens
der Haustabe," \textit{Zeitschrift für Tierpsychologie}, III (1939), 88-
114,2.

\(^{36}\) Mlle. Marieette Tellier, "L'intelligence des singes inférieurs,
là vision des formes et la généralisation," \textit{Memoires de la
(1934), 1-47.
named Coco. In one of the tests, Tellier used cards with different numbers of small black squares on them. She had several cards for each of the numbers from two to twelve with a different arrangement of the small squares on each card. Coco learned to discriminate a card that had 3 squares on it from cards that had either 2 squares or 4 squares, 5 squares from 4 and 6, etc., up to 10 from 11 squares. Tellier felt that Coco's mistakes on this test resembled those made by a person when he was judging the relative concentration of small black squares without counting them.\textsuperscript{37} As a result of these and a second series of tests,\textsuperscript{38} Tellier concluded that Coco was capable of a kind of primordial generalization.

The successes of Coco were so astounding and so out-of-line with other experimental results that many psychologists have seriously questioned the authenticity of the controls used in the experiments. Lashley,\textsuperscript{39} in particular, has criticized Tellier's experiments and other studies made with Coco; he classifies them in the same category as those with famous computing horses and dogs.

Dr. Loh Seng Tsai\textsuperscript{40} taught a rat to leap from a table through a small swinging door for food. There were three such doors marked

\begin{itemize}
\item \textsuperscript{37} \textit{Ibid.}, pp. 51-53.
\item \textsuperscript{39} K. S. Lashley, "Studies of Siam Intelligence from the University of Liége," \textit{Psychological Bulletin}, XXXVII (April, 1940), 237-48.
\item \textsuperscript{40} "Rationating Rodents," \textit{Life}, XXXVI (January 11, 1954), 65, 66, 68.
\end{itemize}
with a picture of 1 object, a picture of 2 objects, and a picture of 3 objects, respectively. The position of the "two"-door was constantly changed, and the units which made up the number pictures were continually altered. In this way, Dr. Tsai believes he held constant only the number of images involved, since the rat could not discriminate by position of the door, nor by color, patterns, nor even by size of the images on the door. After several months, the rat learned that he could obtain food through the door marked with 2, and only 2, images.

In 1956, Hicks\textsuperscript{41} sought to teach eight rhesus monkeys to discriminate 3 objects from 1, 2, 4, or 5 objects. The stimulus objects were cards portraying geometric figures of various sizes and shapes. There were six testing situations, preceded and interspersed with training periods. The performance level of Hicks' subjects remained significantly above chance in all testing situations, although as he changed more and more of the characteristics of the stimulus-patterns, the monkeys' performance level decreased. Hicks concluded that his subjects responded on the basis of differences in number of figures.\textsuperscript{42}

Katz discussed the unusual successes of some animals on various discrimination problems. In connection with the ability of animals to discriminate on the basis of number, he commented that "there is

\textsuperscript{41} Leslie H. Hicks, "An Analysis of Number-Concept Formation in the Rhesus Monkey," Journal of Comparative and Physiological Psychology, XLIX (1956), 212-18.

\textsuperscript{42} Ibid., p. 217.
every reason to assume at least many animals share with the child the capacity to comprehend quantities of similar things perceptually either absolutely or relatively. On the other hand, Moore and Bierens de Haan are of the opinion that animals can make discriminations on the basis of number only in cases where the differences in the number of objects in the two groups or the differences in the areas occupied by the two groups are quite noticeable, and that animals make the choices they do largely because they prefer a larger to a smaller amount of food.

Summary: visual stimuli presented simultaneously.— Révész and Koehler succeeded in teaching the fowls they experimented with to make the discrimination when two groups of visual stimuli were presented simultaneously. In other experiments reported here, with the exception of Tellier's, the maximum discrimination was 3:2. Tellier's successes with the monkey Coco appear questionable. Hicks had some success in teaching rhesus monkeys to discriminate 3 objects from 1, 2, 4, or 5 objects.

In none of the experiments reviewed has there been any indication that the animals learned to count. On the other hand, in the more


45. J. A. Bierens de Haan, "Notion du nombre et faculté de compte chez les animaux," Journal de Psychologie Normale et Pathologique, XXXIII (1936a), 400.
carefully designed experiments, the investigators succeeded in controlling secondary clues to the extent that the number of objects in the situation was the dominant factor. It appears that some animals can learn to make discriminations on the basis of number. This ability to do so varies with the species and even with the individual members of a species.

**Discrimination in Repeating an Action a Fixed Number of Times**

**Explanation of the problem.**—The problem generally requires the animal to repeat an action, such as pecking or reaching for food, a fixed number of times when some clue has been presented to it or when, by some method of punishment used during the training period, it has learned to stop after taking the desired number. Thus, if a fowl learns to eat 4, and only 4, grains of corn when some arrangement of 4 dots is the clue, then the animal is said to have learned to make the proper discrimination. Other methods of testing animal ability to repeat an action a fixed number of times will appear in the discussion of studies which follows.

**Report of studies.**—Koehler, Müller, and Wachholtz\(^46\) succeeded in teaching a pigeon to repeat an action a fixed number of times. The limit for this bird, in learning to peck for food, was 6 grains. In a later experiment Koehler and Wachholtz\(^47\) attempted to eliminate

\(^{46}\) Koehler, Müller, and Wachholtz, op. cit., pp. 39-54.

all figural and positional clues. The pigeon learned to eat a fixed number of grains up to 6; but since the bird had less than 50 per cent accuracy on 6 grains, 7 was not attempted.

Arndt used a chute to offer peas to pigeons. By varying the time intervals between peas from one to sixty seconds, Arndt was able to control the rhythm factor. Three pigeons were able to learn to eat from 2 to 5 peas, depending on the number being trained for at the time, under these conditions. Arndt controlled the figural factor by offering small boxes, attached to a turntable, which the pigeons had to open to get the peas. When Arndt put one pea in each of two boxes and left the third empty, the pigeon learned to eat 2 peas and stop. When Arndt put the second pea in the third box, the pigeon opened all three boxes and ate 2 peas. This experiment was later extended to taking 3, 4, 5, 6, and 7 peas out of a varying number of boxes. The 22 birds used in the 1163 trials for a problem involving 6 peas placed in varying groups in six boxes attained an overall accuracy of about 62 per cent. The limit, however, was 6; no reliable results were obtained when 7 peas were used. Arndt tried premature frightening. The pigeons were trained to eat 4 peas and were frightened away after having eaten only 3. However, the next time they ate 4 peas again. The experimenter also had some success in teaching his pigeons to associate colors with the number of times to repeat an action.

Marold used budgerigars, a species of small parrots. In training experiments these birds learned to eat 6 grains from a heap of 10 to 15, but were unable to learn to eat 7 grains. Marold's experiments with premature punishment at first agreed with Arnlt's. Later he used a well-fed bird and offered the grains after the punishment on a new board of unusual form and color. The bird then took only the number, or nearly the number, of grains it had eaten during the punishment trial. Marold trained one bird in a triple task: the bird learned to eat 2 grains, 3 grains, or 4 grains respectively on the basis of which of three differently painted cardboards was used as a clue. Marold also had some success in teaching his birds to associate numbers with the words dyo and tries, where the bird had to learn to take 2 grains on the signal dyo and 3 on the signal tries.

Schiemann found that 6 was the limit of repeating an action for the jackdaws he used. When the task was to open a varying number of dishes containing different numbers of baits, the limit was usually 4, although there were some successes with 5 baits. In premature punishment tests, the birds were trained to eat a certain number of baits and then were punished after taking a number of baits.

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49. Erhard Marold, "Versuche an Wellensittichen zur Frage des 'Zähl-vermögens'," Zeitschrift für Tierpsychologie, III (1939), 170-223.

fewer than the training number. Thereafter, with considerable accuracy for several trials, they took only the number of baits they had obtained on the punishment trial. In associating colors with numbers of baits to be taken, one jackdaw succeeded for up to 5 baits. One bird was able to associate the number of dots on a lid with the number of baits to be eaten with better-than-chance accuracy up to and including 4 baits. Some pattern help was necessary at first, but later the bird achieved independence from this clue.

Iashley\(^51\) describes an experiment by Gallis who taught Coco, the monkey used in Tellier's\(^52\) research, to open his clenched fist twice in succession to obtain a mealworm hidden there. Later this was extended to 3, but not to 4 in succession. Gallis altered the conditions so that the monkey sometimes found 2 mealworms when he opened the hand the first time; then he would not open it the second time. In the extension to 3 baits, Gallis would sometimes put 2 mealworms in the first time and 1 in the second. Coco would not open the hand the third time. Gallis felt the monkey understood 1 + 2 = 2 + 1 = 3 and concluded that Coco had concepts of two and three. Iashley dismisses Gallis' experiments as being "remarkable only for the neglect of every experimental control."\(^53\)

Bierens de Haan\(^54\) also suspected the results of some of these studies and repeated Gallis' experiment in 1935 with the same result.

\(^{51}\) Iashley, \textit{op. cit.}, p. 240.

\(^{52}\) Tellier, \textit{loc. cit.}

\(^{53}\) Iashley, \textit{op. cit.}, p. 240.

\(^{54}\) J. A. Bierens de Haan, \textit{op. cit.}, pp. 406-8.
But then Bierens de Haan instituted a control not used by Gallis. He offered the hand three times in a row with food in it each time. The monkey took all 3 pieces of food rather than stopping at 2, as he had been trained to do. The experimenter felt this indicated that the monkey could in some way tell whether there was food in the hand or not. Then he extended the time between the three single tests from 10 seconds to 20 and then to 30 seconds. The per cent of errors increased to 90 at 30 seconds. Bierens de Haan concluded that the monkey had not learned number in any sense but only a certain rhythm of action.

A different and apparently more difficult problem in repetition of an action was reported by Pastore55 in 1955. Pastore worked with canaries. He taught one of his birds to pull a toy truck along a track until it was lined up with an elevated bin so that the bird could reach the food in the bin. Pastore then placed an opaque partition between the bird and the truck so that she could not see when she had pulled the truck alongside the elevated bin, but had to look around the partition to check.

Pastore first placed the truck in such a position that 1 tug on the string by the canary would bring it alongside the bin. After the canary had learned this task, he then taught her to use 2 tugs by placing the toy truck 2 tug-lengths from the bin. Then he

interspersed the two; and after 2600 trials the canary learned to look around the partition, judge whether the truck was 1 tug-length or 2 tug-lengths away from the bin, match the two different distances with the appropriate number of tugs, and make the appropriate response. After 6000 trials this particular bird learned to correlate as many as four types of tugging responses with four different distances. One observer suggested that the canary had found a concept of number. Pastore says this was not actually proved by the experiments. He writes that "the bird might have learned four different types of motor or muscular rhythms. In an analogous situation, a musician runs off an appropriate series of notes without, of course, counting the individual notes."56

Salman57 was of the opinion that cases in which animals have been taught to eat 2, 3, 4, or 5 pieces of grain does indicate some ability to discriminate between x and y pieces of food; and therefore, that animals do possess some capacity for number discrimination.

Koehler58 gave some of his own interpretations of the findings in the experiments performed by Arndt, Marold, Schiemann, Müller, Wachholtz, and himself. Koehler was of the opinion that not even

56. Ibid., p. 79.


the cleverest of the birds showed inborn capacities for learning any sequence of signs resembling named numbers, as human beings do, but that some of them did learn "un-named numbers" presented to them simultaneously and successively.

Thinking un-named numbers, according to Koehler, consists of thinking in terms of qualitatively "equal marks" in much the same way as a human does who nods his head once for one or twice for two. On the other hand, Koehler says that thinking named numbers is a process of thinking qualitatively different marks or syllables in a fixed order as a human does in counting when he thinks "one, two, three, four, five." Koehler observes that "there is no sign of 'number intelligence' at all, no obvious progress of learning ability as in the case of children . . ." He concluded that a given species of bird shows the same ability to grasp un-named numbers whether they are presented simultaneously or successively; with pigeons this is 5 or 6 depending on the experimental conditions, 6 with jackdaws, and 7 with ravens and parrots. Koehler considered the similarity between birds and humans in these abilities remarkable; when stimuli are presented to humans tachistoscopically, not allowing enough time to count, few persons can perceive 8 and many persons can do no better than to perceive 5 objects.

60. Ibid., p. 44.
61. Ibid., p. 44.
Summary of discrimination studies in repeating an action a fixed number of times. -- In problems in which the birds used by Arndt, Koehler, Warold, and Schismann were to repeat an action a certain number of times, there was a limit of 6 in nearly every case. Pastore's canary did not do so well, but the problem Pastore presented to his bird appears to be more difficult in view of the fact that the canary had to make a discrimination of distance and then "translate" that into a number of tugging responses. In general, the monkeys used by Gallis and Bierens de Haan did not do so well on repetition-of-an-action problems as did the birds.

Summary of Animal Studies

The purpose of this chapter was to review animal studies in the discrimination of number. Physically, there is an overlap and continuum between humans and animals. Within this overlapping area there is a type of learning psychologists call "discrimination learning," which is limited largely by the acuity of the sense or senses being used. Animals have achieved noticeable successes in various experiments requiring "discrimination learning."

It appears that at least some animals can learn to make discriminations on the basis of number, that is, to perceive a number of objects both relatively and absolutely. In discrimination studies in which the stimuli were presented successively, the usual success animals have achieved is the discrimination between 1 and 2 and the maximum success is the discrimination between 3 and 4 objects.
Animals have achieved greater successes than that in studies involving stimuli presented simultaneously and in studies involving the repetition of an action a fixed number of times. Some birds, in particular, have learned the discrimination 4:5 when the stimuli were presented simultaneously, and the upper limit for these birds in the problem of repeating an action a fixed number of times, was about 6 objects. These upper limits are remarkably similar to the upper limits of subitizing for adults (see Chapter II).

It was assumed at the first of this chapter that animals do not learn "to count" in order to aid them in making discriminations on the basis of number. The findings of the studies reported in this chapter corroborate that point of view. Animal successes in the more carefully designed experiments, then, indicate that the assumption that counting is prerequisite to the perception of the number of objects in a group is in error.

Suggestions for the Teaching of Arithmetic

Certain of the implications for the teaching of arithmetic derived from the results of number-perception studies with adults and children (Chapters II and III) appear to be supported in some degree by the findings in animal studies. These may be listed as follows:

1. The assumption that counting is prerequisite to discriminating between two groups of objects on the basis of number and to perceiving the number of objects in a group is not supported by the findings in animal studies.
2. There is a limit to the ability to discriminate on the basis of number. This limit depends, probably to a great extent, on the acuity of the sense involved.

3. The ability to discriminate on the basis of number is not automatic. It must be learned and it can be improved through practice within the limits of sensory acuity.

4. It is more difficult, in general, to perceive a number of stimuli when they are presented simultaneously. If early number experiences deal largely with events or objects presented successively, there is greater need to develop a counting technique than there is if early number experiences deal largely with stimuli presented simultaneously.

5. From the results of number-perception studies, there appears to be no justification for a program that would require children to expend time and energy in attempting to increase their ability to perceive the number of objects in a group—subitize—beyond 6 or 7 objects.

6. It appears to be more difficult to learn to make discriminations on the basis of number than it is to do it on such bases as color, area covered, design, pattern of objects, etc. There is a possibility that the perception of the number of objects in a group differs from other perceptions in that it may be made through various senses, the particular aspect of number gained being dependent upon the sense or senses used. This may account, in part at least, for the greater difficulty experienced by both humans and animals in learning to discriminate on the basis of number as compared with
learning to discriminate on other bases. Number is not automatically attained through a process of finding the common element in several equal-sized groups. A concerted effort appears necessary if the subject is to learn to attend to the number of objects in a group rather than to other characteristics of the group.
Chapter V

NUMBER IDEAS AMONG PRIMITIVE PEOPLES

Purpose of the Chapter

The purpose of this chapter is to report on anthropological studies concerning the uses made of number and the extent of the development of number ideas among primitive peoples. Such information may be indicative of the manner in which number ideas originate and of the extent of the development of quantitative ideas when there is a limited need for numbers and when little, if any, formal instruction in arithmetic has taken place. More particularly, anthropological studies should provide some insights into two matters: (1) the extent of the development of number ideas when a written number system is lacking, and in some cases when a verbal number system is lacking; and (2) the relationship of number ideas to physical operations and physical things. Such information should clarify the issues of the early introduction of number symbols in the arithmetic of the primary grades and should be indicative of the extent to which number ideas are dependent on physical operations with concrete objects.
Are Number-Words Indicative of the Extent of the Ability to Develop Number Ideas?

For over one hundred years anthropologists and linguists have been carefully recording the attitudes, beliefs, social activities, language—in fact, almost all aspects—of primitive peoples. It is well that this material is available since the number of tribes which are purely primitive (in the sense that they are completely or even relatively untouched by the ideas, languages, and materials of civilized man) is rapidly decreasing.\^ Number-words of primitives have been of particular interest to linguists, since number-words tend to change more slowly than others, and are thus often used as keys to link related linguistic groups. For the anthropologist, number-words often provide clues to the primitive's economy, and are also used for tracing linguistically related groups.

The Paucity of Number-Words among Some Primitive Tribes

Many tribes have been found to be quite limited in their use of number-words. Blackwood\^ reported in 1935 that several tribes of the Northwest Solomon Islands had number-words up to ten, but that some of the groups did not go beyond five. Schmidt\^ wrote that


the Bakairi Indians on the Upper Xinger had no number-words beyond six. Murdock reported that a few Tasmanian tribes and the Arunda of Australia had number-words to five. The Semang of the Malay Peninsula were discovered to be unable to count beyond three. It was found by John Murdoch at Point Barrow in 1890 that the Eskimos in ordinary conversations specified no numbers above five; six and higher being designated by amadraktuk, meaning many.

According to Taylor, in the original language of the Island Carib of British Honduras, there were words to express only four numbers, although the natives did use their word for hand to mean five and their word for man to mean twenty. Fabrega observed that the Brunka, a tribe of Costa Rican Indians, had number-words to eight; the Tirub tribe of the same region, to seven; and the Guatuso, to five, although this latter group had a word for ten which seemed to be derived from the word meaning "two." In the wealth of information Conant collected on primitive number systems, he listed a

5. Ibid., p. 21.
6. Ibid., p. 86.
number of tribes which had words for only the first two, the first three, the first four, or the first five numbers.  

Earl, in 1853, listed four tribes in New Guinea which had number-words for ten, three Australian tribes with four number-words, and two Australian tribes with only three number-words. Mathews reported only three number-words for the Kamilaroi and Darkinung dialects of New South Wales in 1903. Dorman found, in 1917, that the Masarwas, a tribe of Bushmen in and near Southern Rhodesia, had numeral and ordinal adjectives up to and including five, with a few compound number-words above that. In 1902, Armandale and Robinson reported that the Sakai Tanjong, a tribe in Malay, had no number names but replied to questions requiring small numerical answers by holding up the proper number of fingers in silence. Pruner-Bey, writing in

12. R. H. Mathews, "Language of the Kamilaroi and Other Aboriginal Tribes of New South Wales," Journal of the Anthropological Institute of Great Britain and Ireland, XXXIII (July to December, 1903), 268, 274.
1861, referred to natives of the interior of New Guinea, the Mairassis, who had no number-words but simply showed the number of fingers necessary to indicate the number in a group and repeated the same word, awari, with each showing of the fingers.

The Ability of Primitive Peoples to Extend Their Number Ideas and Number Vocabularies

Many people have interpreted the paucity of number-words among certain primitive tribes to be indicative of a poorly developed number concept, and some have even interpreted it to mean that many primitives are intellectually incapable of much further development of number ideas. The facts do not substantiate either of these conclusions. In later sections of this chapter, illustrations of the extent to which primitive man used quantitative ideas without the aid of number-words will be given. For the present, the ability of primitive peoples to extend their number ideas and number vocabularies will be treated.

Many primitive peoples, particularly those with occasion to trade and barter with their neighbors, developed rather extensive number systems. Bancroft reported an extensive counting vocabulary for the Apache;16 and Murdock found extensive vocabularies among the Crow,17 the Haida,18 and the Iroquois19 Indians. Murdock also

17. George Peter Murdock, op. cit., p. 266.
18. Ibid., p. 222.
19. Ibid., p. 293.
observed extensive number vocabularies among the Todas\textsuperscript{20} of Southern India, the Ainus\textsuperscript{21} of Northern Japan, the Ganda\textsuperscript{22} of Uganda, and the Dohomeans\textsuperscript{23} of West Africa. Undoubtedly, all of these peoples at one time had less extensive systems, but as they developed economic intercourse with neighbors, and as they came into contact with civilized man, they readily evolved numeration systems. Often, the number-words they already had were compounded to form number-words for larger quantities. This method of extending number vocabularies will be elaborated upon later in this chapter under the heading Number Bases.

In some cases primitive peoples have adopted part or all of the numeration systems of their neighbors or of civilized man. Lévy-Bruhl found that the Bergdama, one of the oldest peoples of South Africa, were originally limited in numeration to twenty; but later Bergdama spoke a kind of pidgin-nama, adopted from the neighboring Namas, and easily counted to one thousand.\textsuperscript{24} Fábrega found that the Brunka Indians of Costa Rica, who had number words of their own only to eight, quickly adopted the Spanish numerals beyond eight.\textsuperscript{25} Delafosse recorded that a great many of the primitive tribes of

\begin{itemize}
\item \textsuperscript{20} Ibid., p. 108.
\item \textsuperscript{21} Ibid., p. 65.
\item \textsuperscript{22} Ibid., p. 509.
\item \textsuperscript{23} Ibid., p. 553.
\item \textsuperscript{24} Lucien Lévy-Bruhl, "La numération chez les Bergdama," \textit{Africa}, II April, 1929), 163.
\item \textsuperscript{25} Fábrega, op. cit., p. 454.
\end{itemize}
Africa readily extended their numeration systems into the hundreds and thousands by adopting number words from the Arabs or the Berbers. Farris wrote that a tribe living at the Equator, near the Upper Congo River had a numeral system running as far as a million. In view of the fact that the natives seemed to have no particular use for such an extensive system, he felt that it must have been used at a time when the tribe had a large commerce in beads.27 The list of primitive tribes which have readily extended their number vocabularies when the necessity has arisen, or when the tribes have come in contact with other peoples, may be expanded. Knocker,28 Ray,29 Boaz,30 Gallatin,31 Pruner-Bey,32 Herkovits,33 and Richards34 have all recorded similar examples.


Most authorities agree that the limit of a tribe's number-words is no indication of the limit of the concept of number among the members of that tribe. It is true, of course, that the primitive's need for number is not nearly so extensive as is civilized man's. In some groups, living in very primitive conditions, there may be little use for number-words beyond one, two, three, and many.

Conant, Schmidt, Lévy-Bruhl, and Gow were of the opinion, although they expressed it somewhat differently, that the extent of a people's number vocabulary was little indication of that people's skill with numbers and their understandings of quantity, and practically no indication of their ability to extend their number ideas and vocabulary.

Summary: Number Words

The paucity of number words among primitive peoples is no indication of their ability to extend their number vocabularies and their use of numbers. In every case where a group of people has found it an economic or social necessity to extend its number vocabulary and its use of number ideas, it has done so readily. Nor is the paucity of number words a reliable indication of the extent to which a group of people make use of number ideas. Primitive man has found

35. Conant, op. cit., p. 31.


ways to think about and to communicate quantitative ideas without resorting either to spoken or written number symbols. An excellent example is gesture counting which is to be elaborated upon in the next section.

**Implications for the Teaching of Arithmetic**

Primitive people can make considerable use of quantitative thinking without the need of written symbols. In all probability, young children can be taught considerable arithmetic before they encounter the written symbols of arithmetic. In a present-day civilized society, ideas, including quantitative ideas, are commonly communicated via spoken and written symbols. The child must eventually learn these symbols and their meanings. The fault, though, with a great deal of arithmetic teaching is that the written symbols are introduced before the child has had experiences with the concrete quantities which are the referents for the symbols he is attempting to learn. This procedure appears to be based upon the dubious assumption that learning arithmetic means learning to manipulate the written symbols of arithmetic. The ease with which primitive peoples have extended their number vocabularies when the need arose indicates that children may have less trouble in learning the spoken and written symbols of arithmetic if these symbols are introduced after the children are well acquainted with groups of objects of various sizes and see the need for symbolizing the number of objects in those groups.
Reports of Gesture Counting among Primitive Peoples

Primitive man is not restricted in his concept of number by the limit of his vocabulary of number-words. In fact, he is not even restricted in the operation of counting by such limitations, if counting be viewed in the larger sense of setting up a one-to-one correspondence between the set of objects being counted and some other set, either objects or words, for which a definite order has been established.

The fingers have, of course, been the most widely used such set whether or not number-words are lacking. Among primitive peoples the most common practice in counting with the fingers appears to be to start with the little finger on the left hand. In 1929, Lévy-Bruhl gave excellent examples of such counting among the Bergdama of South Africa. For one tribe the count proceeded as follows:

<table>
<thead>
<tr>
<th>Spoken word</th>
<th>Gesture</th>
<th>Literal meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>kari gaoneb</td>
<td>touch little finger, left hand</td>
<td>&quot;little chief&quot;</td>
</tr>
<tr>
<td>kari gaoneb gab</td>
<td>touch little annular, left hand</td>
<td>&quot;brother of little chief&quot;</td>
</tr>
<tr>
<td>siga mabeb</td>
<td>touch middle finger, left hand</td>
<td>&quot;that which is in middle&quot;</td>
</tr>
<tr>
<td>naibeb</td>
<td>touch index finger, left hand</td>
<td>&quot;that one which points&quot;</td>
</tr>
<tr>
<td>gei khei khaabes</td>
<td>touch thumb, left hand</td>
<td>&quot;the important man (one) who (which) is last (or behind)&quot;</td>
</tr>
</tbody>
</table>

The counting then went to the right hand, starting with the thumb and proceeding to the little finger while the same words were repeated in reverse order, since they were actually names of fingers. Thus, the words *gei kheī khaābes* were spoken when either the left thumb or the right thumb was touched; however, the count when the left thumb was touched was five and the count when the right thumb was touched was six. After the little finger on the left hand was reached, the extremities of the fingers of the two hands (but not the palms since the fingers only were used for the count) were brought together and the words *gām ha* pronounced. The literal meaning of *gām ha* is "they are killed (made away with)." If members of this tribe continued to count to twenty, they gave the toes the same names as the fingers and proceeded in the count, but started with the right foot, little toe. When twenty was reached, they pronounced the words *hoakīha gama*, in which *gama* is a contraction of *gām ha*.

Lévy-Bruhl was of the opinion the words used were not true number-words, but names of the fingers functioning as numbers. The name alone did not suffice to designate a number. Thus, the word *naibāb* could mean four, seven, fourteen, or seventeen, and presumably even higher numbers if the Bergdama counted higher. It was necessary, of course, to watch the gestures to see if the count had ended on the index finger left hand, index finger, right hand, the toe next to the big toe on the right foot, or the toe next to the big toe on the left foot.
In the same article, Lévy-Bruhl also gave examples of the counting of other Bergdama tribes as recorded by M. Vedder, where the counting proceeded in the same way except that the words spoken and the gestures made included two fingers at a time. In other words, the natives counted by two's. In these tribes a word or phrase was repeated with each gesture. If it was necessary to stop at an odd number, the person doing the counting added another word to include the additional object to the pairs he had counted.

Chalmers, in 1903, reported on a group of New Guinea natives where, as the count proceeded from one to ten, the gestures consisted of touching the following parts of the body in order: little finger of the left hand, ring finger, middle finger, index finger, thumb, wrist, elbow, shoulder, neck or left breast, and ear or right breast. Weeks, in 1909, found Bengala tribesmen of the Upper Congo River showing, rather than touching, the fingers as they counted. He reported that the Bengala seldom used their toes in counting. They were capable of performing addition and subtraction mentally, but invariably checked with the use of palm nuts or some other objects for visual proof that their computations were correct. Weeks further

40. Ibid., p. 166.


observed that the Bengala always counted by fives and tens. They would consistently combine a group of three and a group of two to make five, then another group of three and a group of two to make another five, altogether ten, etc.

In 1902, Roscoe recorded a method of gesture counting used by the Baganda, an African tribe whose language was that of the Bantu family;\(^{13}\) and in 1907 he reported the gestures used by the Bahima tribe of the Uganda Protectorate.\(^{14}\) Lyons\(^{45}\) described gesture counting among natives of the Gogodara tribe of Western Papau; Codrington,\(^{16}\) gesture counting among the natives of Bank’s Islands; Cole,\(^{17}\) the gestures he found in a tribe in German East Africa; and Stannus,\(^{18}\) the gesture counting among the Nyasa peoples of Central Africa. All of


\(^{15}\) A. P. Lyons, "Notes on the Gogodara Tribe of Western Papau," *Journal of the Royal Anthropological Institute of Great Britain and Ireland*, IV (July to December, 1926), 346.


\(^{18}\) H. S. Stannus, "Notes on Some Tribes of British Central Africa," *Journal of the Royal Anthropological Institute of Great Britain and Ireland*, XL (July to December, 1910), 289.
these people used the fingers in some way. Lowie⁴⁹ found that the Yuki Indians of California did not use the fingers, but instead the four spaces between them. Tylor⁵⁰ noticed that some tribes of Brazil counted the finger joints. Greenburg⁵¹ came upon eight examples of African languages in which the word for hand was used to express five. Likewise, Conant⁵² recorded several dialects in which the word five was identical with or clearly derived from the word for hand.

Tylor⁵³ wrote that among many Indian tribes of South America the answer to the question of "how many?" was accomplished by simultaneously showing the number of raised fingers with the spoken word, rather than by touching the fingers with the other hand. If the natives said "five," they showed one hand with all fingers raised; if they said "ten," they showed both hands. How the fingers were raised varied with different tribes. To show three, the Otomacs united the thumb, index finger, and middle finger, and closed the other two. The Maipure, to indicate the same number, displayed the index finger, the middle finger, and the ring finger.


⁵². Conant, op. cit., pp. 52-68.

⁵³. Tylor, op. cit., p. 245.
Frazer reported that the Inuits, Eskimos of Hudson's Strait, had number-words only to ten, but that they could and did indicate larger numbers. To indicate thirty-seven, for example, they would show both hands open three times to indicate the number of tens and then say their number-word for seven. Frazer also observed that the Roucouyennes of South America had no number-words beyond one meaning "three." From there on, number was demonstrated simply by showing fingers and toes. Beyond twenty the people said cole psi, meaning "much" or cole, cole, meaning "much, much."  

Among the most peculiar forms of gesture counting are those recorded by Moggridge, Man, and Elwin. Moggridge revealed in 1902 that the best counter he knew in the Anguru tribe (of the Portuguese territory southeast of British Central Africa) was a man who stowed the ends of his fingers one by one into his mouth, counting up to ten as he did so. In 1932, Man found the Andaman Islanders had number words for only one and two. If they wished to express nine, for example, they would tap the nose in order with the fingers, starting


55. Ibid., p. 272.


with either little finger. They used their number words for the first
two taps and from there on simply repeated the word an-ka, meaning
"and this," with each succeeding tap. Peculiarly enough, these
people had verbal ways of expressing the ordinal idea of number up
through the last of six things taken in order of six events happening
sequentially.

Elwin's recent report on the Hill Saoras of the Indian state of
Orissa includes an interesting account of the counting methods of these
people. It is taboo for a Saora to count beyond twelve. Elwin records
how a chief counted to twenty. The chief began on the left foot (he
was squatting) by counting five; then by using the left hand for five
more and two fingers on the right hand he reached twelve, the limit
by taboo. With the thumb and the other two fingers of the right hand
and the toes of the right foot, he made eight more. Since the Saoras
have number names only to twelve, they count and show twenty as twelve
and eight.\footnote{58}

Summary: Gesture Counting

It appears possible that gestures were used to point out the
"how many" of a group before number-words evolved. Man carried his
own arms, legs, eyes, ears, fingers, and toes with him as model groups.
Besides, there were numerous model groups in nature. Primitive man
had but to point to or touch the appropriate model group, the indi-
vidual objects of which could be put into a one-one correspondence

\footnote{58. Verrier Elwin, The Religion of an Indian Tribe (London: Oxford
University Press, 1955), p. 529.}
with the objects of the unknown group. Even when he started the
operation of counting, which is at least one step removed from the
immediately concrete (since it requires the idea of an established
sequence), he found it easier to utilize a sequence of objects, parts
of the body, than sounds.

Implications for the Teaching of Arithmetic

The use of gestures is by no means absent from civilized man's
use of numbers. The use young children make of their fingers in
counting and in showing the "how many" of a group may well be
encouraged rather than discouraged, since the fingers present
readily accessible and easily recognized model groups as referents
for the number-words and number symbols for which children are to
learn the meanings.

Model Groups and Number-Words

Opinions of Authorities

There are examples in many languages, both primitive and civilized,
in which the reference is not to the fingers but to other small model
groups. It was indicated also in the previous sections that the
fingers and other appendages are not always referred to as counters,
but sometimes as model groups themselves. Cow gave the following
examples of number-words which refer directly to small model groups:
<table>
<thead>
<tr>
<th>Word</th>
<th>People</th>
<th>Number</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>ny or ceul</td>
<td>Chinese</td>
<td>2</td>
<td>ears</td>
</tr>
<tr>
<td>paksha</td>
<td>Tibet</td>
<td>2</td>
<td>wing</td>
</tr>
<tr>
<td>t' Koan</td>
<td>Hottentot</td>
<td>2</td>
<td>hand</td>
</tr>
<tr>
<td>geyenkñaté</td>
<td>Abipones</td>
<td>4</td>
<td>ostrich-toes</td>
</tr>
<tr>
<td>neenhalék</td>
<td>Abipones</td>
<td>5</td>
<td>a hide (with 5 spots on it)</td>
</tr>
<tr>
<td>pona</td>
<td>Marquesans</td>
<td>4</td>
<td>knot; a bunch of 4 fruits tied in a knot</td>
</tr>
</tbody>
</table>

In an article appearing in book form in 1939, Wertheimer expressed the opinion that counting by the use of number words is not the only means for a development of number concepts. He wrote thus:

Counting, for example, in the sense of repeating additions of unity, does not constitute the only factor in the genesis of number. The ideal of universal transferability (i.e., abstractions) of thought structures is not necessarily the aim of all thinking in this sphere. There are structures (Gebilde) which, less abstract than our numbers, nevertheless serve analogous ends or can be used in place of numbers. These structures do not abstract from their natural context and natural relationship; they may occasionally be abstract with regard to the form or arrangement of materials, less often, however, with regard to material itself.60

Wertheimer maintains that in experiencing wholes as geometric configurations, form, arrangement, and organization are involved. Thus, even though the direct discrimination of number may terminate at five or six, this limit may be extended considerably in special arrangements of objects. All meaningful counting or use of number


does not consist of the addition of one more, Wertheimer contends. It is possible, he holds, for a person to add one for a long time and still have only one concept of number. That concept is that any given number is one more than its predecessor. In the same way, one may know that "d" follows "c" in the a b c's. This concept of number would probably be of little value to primitive man, whose use of number is very directly tied to the concrete. He is not interested in inventing a beautifully ordered and structured number system. Wertheimer thinks it very probable that number originated through references to natural groups and structures, not through counting. Thus, primitive man did not conceive of number as one and additions of one to attain the number involved, but rather as groups, articulated wholes. Counting was a later development.61

Dantzig is of the same opinion.62 In a book published in 1939, he held that man learned to aid his limited perception of number by use of the artifice of counting. He then described how he felt it was possible to create the idea of number through the use of model collections. Primitive man finds many such model groups in his environment, such as the two wings of a bird, the three leaves of a clover, the four legs of an animal, the five fingers of the hand. To determine the number in a group, then, consists of comparing that group with known model-groups. This comparison may be made, particularly

61. Ibid., p. 272.

for small numbers, simply by noting that the number of objects in the 
undetermined group is the same as the number of objects in one of the 
model-groups. If this is impossible by direct discrimination, Dantzig 
then believes a comparison of the groups may be made by the process of 
one-to-one correspondence.

Summary and Implications for the Teaching of Arithmetic

There is little evidence extant to indicate whether primitive 
man first used subitizing, one-to-one matching, or counting to deter-
mine the sizes of groups. It seems plausible, however, that he subi-
tized first and then created the techniques of matching and counting 
to aid him in his discrimination when the size of the group was beyond 
his ability to perceive the number of objects directly. Perhaps 
subitizing should be the first means taught to young children for 
determining the number of objects in a group. Then the necessity for 
learning such operations as one-to-one matching and counting to aid 
the limited ability of perception could be made more apparent to the 
pupils.

Number Bases

Reports of Studies

The literature abounds with reports of various number bases used 
among primitive peoples. Conant devoted a chapter of his book to
"Miscellaneous Number Bases," a chapter to "The Quinary System," and a chapter to "The Vigesimal System." It is highly improbable, however, that primitive peoples are conscious of any idea of base. Conant pointed out that the compounding of number-words begins at two or three rather commonly among Australian and Tasmanian tribes. The count proceeds by pairs, rather than ones, in many tribes of that area. Moreover, it is common in all number systems to use a repetition of the number-words in some way. It is as though one would say, "two apples, two apples, apple" for five apples or "pair of people, pair of people, pair of people" for six people. Conant gave several examples of such "binary systems." The quaternary system was found among Hawaiian groups, the Iuli of Paraguay and the Mocobi of the Panama region. Some use of base six was found among the Bolans of Western Africa, the Sundas of Java, and others.

Safford found base six operating in the numeration system of the Yaps of Guam:

<table>
<thead>
<tr>
<th>word</th>
<th>number</th>
<th>meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>me-de-lip</td>
<td>7</td>
<td>six-and-one</td>
</tr>
<tr>
<td>me-rug</td>
<td>8</td>
<td>six-and-two</td>
</tr>
<tr>
<td>me-reb</td>
<td>9</td>
<td>six-and-three</td>
</tr>
</tbody>
</table>

63. Conant, op. cit., Chapters V, VI, and VII.
64. Ibid., pp. 106-12.
65. Ibid., pp. 117-20.
The number bases most extensively used are the quinary, decimal, and vigesimal. The reason for the references to the fingers and toes as these bases came into use is obvious. Lévy-Bruhl wrote that there were many cases where the tribes used fingers in counting and still had no idea of the quinary base. Delafosse was of the opinion that it is erroneous to speak of quinary or decimal numeration in the Negro languages of Africa. What really should be said is that these languages have a special word for each of the first five numbers or the first ten numbers, and then the compounding of the words begins to form higher number-words. While a great many of the compounds are of the form \( b \times n \), where \( b \) equals either 5 or 10, and \( n = 1, 2, 3, \ldots (b - 1) \), this is by no means universally true. Other compounds, such as "two threes" for six, "two fours" for eight, and "ten less one" for nine, often appear. Number-words above ten are often compounded in the form \( 10 \times n \) or \( 10 k \times n \); but in some, particularly where there is a special word for twenty, compounds in the higher teens are of the form \( 10 - n \). Lounsbury maintains that in many cases among primitives the compounding of number-words is not the result of a conscious use of a base number, but rather the result of the primitive's inability to express higher numbers linguistically without using complex circumlocutions. The primitive's need for numbers often surpasses his vocabulary of number-words; consequently, he uses the number-words


he has in some combination to express these higher numbers. Thus
the Bororoa of the Amazon region say, "This two and this one which
lacks a partner." In the Carib language of the Rio Branco region
the following expression is common: "This pair and this pair and this
one without a partner." 69

Many compound number-words appear, by the way in which they are
compounded, to indicate that they originated as primitive man saw
larger groups as being composed of easily recognizable subgroups.

Some examples of these follow:

<table>
<thead>
<tr>
<th>word</th>
<th>tribe</th>
<th>number</th>
<th>meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>dildnendnmon</td>
<td>Radack, Guam</td>
<td>7</td>
<td>three-three-one</td>
</tr>
<tr>
<td>eidinu</td>
<td>Radack, Guam</td>
<td>8</td>
<td>double four</td>
</tr>
<tr>
<td>eidinemdvnon</td>
<td>Radack, Guam</td>
<td>9</td>
<td>double-four-and-one</td>
</tr>
<tr>
<td>caramute uameta</td>
<td>Piaraq, Orinoco River</td>
<td>8</td>
<td>five and three</td>
</tr>
<tr>
<td>buluman bata</td>
<td>Kurnai, Australia</td>
<td>4</td>
<td>two and two</td>
</tr>
<tr>
<td>bulum</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>boolla-boolla-kalim</td>
<td>Kabi, Australia</td>
<td>5</td>
<td>two-two-one</td>
</tr>
</tbody>
</table>

69. Floyd G. Loumsbury in An Appraisal of Anthropology Today, ed. by
Sol Tax, Loren C. Eiseley, Irving House, and Carl F. Boegelin

70. Safford, op. cit., p. 96.

71. A. Ernst, "Upper Orinoco Vocabularies," American Anthropologist,
Old Series VIII (October, 1895), 401.

72. Lorimer Fison and A. W. Howitt, Kamilaroi and Kurnai (Melbourne;
George Robertson, 1830), footnote, p. 255.

73. John Mathew, Eaglehawk and Crow, A Study of the Australian
The subtractive principle appears to some extent in very primitive systems. For example, Conant\(^74\) found Green Island natives indicating eight objects by the word andra-lua, meaning "less two"; and Murdoch\(^75\) reported that the Eskimo at Point Barrow said skimiexolaituyuna for fourteen, meaning "I have not fifteen." Other compounds involving parts or multiples of model groups have been used. Conant found the Nicaobar Islanders saying heam-umjome ruktei, meaning "a man and a half" for thirty,\(^76\) and the Affadeh tribe of Africa saying dukungadegokang for ninety, meaning "20 x 4 ≠ 10."\(^77\) Safford reported that the Chamorra of Guam said mguanmaise, meaning "a pair and a half," for three.\(^78\)

Summary: Number Bases

Almost all of the smaller whole numbers, up to and including 12 and 20, have at one time or another in the language of some group of people appeared as a base or semi-base. Undoubtedly the predominance of quinary, decimal, and vigesimal bases can be attributed directly to the number of fingers and toes the human possesses. It is unlikely, though, that primitive man made any conscious use of a number base. It is more probable that he simply compounded the number-words he knew

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74. Conant, op. cit., p. 122.
76. Conant, op. cit., p. 78.
77. Ibid., p. 134.
to express larger numbers. Perhaps the idea of bases for number systems grew out of these linguistic circumlocutions. Probably 10 appears as a base often because primitive man had names for the ten fingers. As he began to use higher numbers, he compounded the ten finger-words he already knew.

Implications for the Teaching of Arithmetic

The young child who knows the number-words only to five may use the technique of compounding the words he knows to express the number of objects in groups larger than five. In fact, if he is encouraged to compound the words for smaller numbers, it would appear that he would be able to appreciate better the use of base 10 in the Hindu-Arabic number system and to understand better the meanings of the words for numbers greater than ten. Thus, if a child has had experiences in expressing "eight" as "five and three," it would appear that he would be better prepared to understand "thirteen" as "three and ten."

The Concrete Basis of Number

The Derivation of Number Words

Malinowski is of the opinion "that language, and all linguistic processes derive their power only from real processes taking place in man's relation to his surroundings." In its primitive uses,

language is a mode of action, not an instrument of reflection. Therefore, in studying a primitive language, Malinowski believes, one should study it against the background of human activities and view it as a mode of human behavior in practical matters.

Hartman referred to the close relationship of numbers to reality among primitive peoples. He wrote that "to us, number is a property of groups of objects which is unrelated to their material aspects, but its integration with reality is much closer among primitives." 81 Conant held that "as it originates among savage races, number is entirely concrete." 82 Thus, the word for hand came, in many cases, to be used to stand for the size of any group which has as many objects in it as the hand had fingers. Greenburg 83 and Conant 84 gave examples of languages in which the words for five were clearly derived from the words for hand. Safford 85 listed 18 groups of people from Pacific Ocean islands who used words for five derived from a stem word, lima, meaning "hand." Although the original meanings of the number-words in English appear to be lost, there are analogous situations in the derivations of other English words. Thus, the color "rose" has come

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80. Ibid., p. 312.
82. Conant, op. cit., p. 72.
84. Conant, op. cit., pp. 52-68.
to mean a color similar to that of the flower by that name.

Hough has also referred to the concrete use of numbers by primitive peoples:

Modern, more exact observations on tribes show that numbers are specific things, and that there is no generic term to represent them divorced from things. Ten are ten fingers, not ten as an abstraction. Ten fingers may, as an advance, be expanded to twenty digits by employing the toes, and that may be the limit, and instead of proceeding, the savage must begin again with visual and tangible things up to the number he is capable of seeing.®®

Communication of Number Ideas without the Use of Number Words

In speaking of the Bergdama, Lévy-Bruhl pointed out that they did not say a number-name in answer to how many, but actually represented with toes and fingers the multiplicity of objects. They demonstrated number rather than said it. Lévy-Bruhl called this démonstratifs numériques.®® He further pointed out that primitive man often determines the number of objects simply by noticing whether or not there has been a modification in a group with which he is familiar:

It is already known that "primitives," without possessing number names other than "one," "two," and "three," can estimate exactly the amount of objects or individuals in a given multiplicity. If the amount is at all familiar to them, they see immediately if it has remained similar to

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itself, or if it has undergone a modification by increase or decrease. 88

In a similar vein, Boaz 89 pointed out that for small groups of objects it is just as satisfactory to the primitive to know the objects of the group by names or characteristics. He may have no desire to count them.

Summary.— Authorities generally hold that primitive man finds or communicates answers to many of his quantitative questions without the aid of number-words or any elaborate system of symbolic arithmetic. In general, primitive man accomplishes this in two ways: (1) he demonstrates rather than says numbers, and (2) he associates quantity closely with the objects in question and uses his knowledge of the structure of a group of objects to answer his quantitative questions concerning the group.

Implications for the teaching of arithmetic.— While primitive man may resort to demonstrative methods to indicate number a larger proportion of the time than does civilized man, such means are not restricted in their use to primitive peoples. Civilized man often finds such methods ample to answer his quantitative questions. In particular, young children use such means. There are probably very

88. Ibid., p. 171. "On savait déjà que les 'primitifs' peuvent, sans posséder d'autres noms de nombres que 'un,' 'deux,' et 'trois,' estimer exactement la somme d'objets ou d'êtres contenue dans une multiplicité donnée. Pour peu que cette somme leur soit familière, ils voient aussitôt si elle est restée semblable à elle-même, ou si elle a subi une modification en plus ou en moins."

few pre-school children who do not demonstrate number with their fingers, and very few who cannot determine whether some toy of theirs is missing by means other than counting. In fact, young children learn a number of ways by which to find answers to quantitative questions and to communicate their answers before they learn much formal arithmetic. This has been pointed out by such writers as Carper, Brownell, and Clark and Eads. Yet, adults often insist that the child use counting and other arithmetic operations even when he knows easier and more meaningful ways to find the answers to questions involving small quantities.

The Use of Multiple Sets of Number-Words

There are many examples of languages and dialects in which there are or were more than one set of number-words, with the set to be used depending on the kind of objects to be counted. Thus, the cardinal number-words in the Chamorra language differ according to the nature of the thing counted. Days, months, and years are counted by the simple number-words of the language. Measurements are expressed by the numerals with the prefix tak or tag. In counting living things, there is a tendency for reduplication. A suffix is appended in counting inanimate objects.

The Bribri tribe of Costa Rica had six distinct modes of counting, depending on the shape or nature of the objects counted. The six systems were for people, round objects, small animals, long objects

90. Safford, op. cit., p. 97.
and large animals, trees and plants, and houses.\(^91\) The Cabecara language had a set of number words for people, round objects, birds and other small animals, preceded by the name of the thing counted. Separate suffixes were used in counting houses, long objects, and trees.\(^92\) The terraba applied two definite series of number words. The prefix \(kro\), meaning "tree," was used in counting long objects; the prefix \(kuo\), meaning "round," was used in counting round objects.\(^93\)

In the Carrier, one of the Dene dialects of western Canada, Conant found eight different sets of number-words.\(^94\)

Boaz found, in 1890, that the language of the Chemakum Indians of the State of Washington had different sets of number-words for six different classes of nouns, although one of these sets operated as a general set because it was used to count many items not included in the other five.\(^95\) Boaz also reported, ten years later, that the Vancouver Indians speaking the Kwakiutl language used number-words which took classifying suffixes to distinguish among such classes as human beings, round objects, and flat objects.\(^96\)

A rather elaborate use of numbers and number-words was described.

\(^{91}\) de Fábrega, \(op.\ cit.,\) pp. 450-51.

\(^{92}\) \(Ibid.,\) p. 451.

\(^{93}\) \(Ibid.,\) p. 452.

\(^{94}\) Conant, \(op.\ cit.,\) p. 86.


\(^{96}\) Franz Boaz, "Sketch of the Kwakiutl Language," American Anthropologist, II (October-December, 1900), 720.
by Harrington for the Tiwa language, a dialect of the Taos Indians of New Mexico, in 1910. Numeral classifiers did not occur, but the Indians used several sets of number-words as follows: (1) enumerating, used in counting; (2) responsive, used in answering; (3) adjectival, used with nouns and having endings denoting animate and inanimate gender, and grammatical number; (4) substantival, used as nouns with endings denoting animate and inanimate gender, and number; (5) ordinal, used to denote relative position or sequence; (6) multiplicative, used to designate number of times, such as once, twice, etc.; and (7) fractional, of which there were only two, one for a half and one for any part smaller than a half.

In 1925, Reichard reported that the Wiyot Indians near Humboldt Bay, in Nevada, used at least 14 different sets of counting words. Their first four number-words closely paralleled adjectives in their use. By changing suffixes, the Wiyots achieved various number-words depending on the words being modified.

In the Tsimashian Indian language of British Columbia, there were seven distinct sets of number-words, one each for (1) no definite object referred to; (2) flat objects and animals; (3) round objects and divisions of time; (4) men; (5) long objects; (6) canoes; and (7) measures. This multiplicity of numeral sets still exists in the


language of several of the tribes of British Columbia. Conant listed
number-words for 16 different numeral sets in the Heiltsuk dialect.
The multiplicity of sets of number-words was employed to some extent
by the Aztecs, the Japanese, the Malaysians, the Hebrew and other
Semitic peoples such as the Phoenicians. 101

Wertheimer writes that there are several South Sea Island languages
in which there are different ways of counting such things as fruit,
money, animals, and men. He points out that when higher numbers are
reached by multiplication or addition operations, they still remain
attached to the material objects. The author says, "Thus several
specific structures combined together yield specific (not arbitrary)
larger structures." 102

In more recent times, Oliver 103 reported on the Sinai group of
Bougainville Island who speak the Motuna dialect. Motuna has 40 or
more noun classes, these being distinguished by the use of different
sets of numerical adjectives. Some of them occur so often in everyday
communication that the Sinai find it sufficient to name only the
numerical adjective without appending the noun. The numerical ad-
jectives for one noun class—that including shell money, areca nuts,
wooden slit-gongs, and eggs—function almost as an abstract numeration
system because of the frequency of talk about large numbers of those items.

100. Ibid., p. 88.
101. Ibid., pp. 88-90.
102. Wertheimer, op. cit., p. 270.
Summary.---Thus it seems that primitive man tends to associate his number thinking closely with the objects, events, or ideas which he is enumerating. To many primitive peoples three canoes do not constitute an instance of the same quality as three nuts. The two groups are essentially different as far as primitive man is concerned, and he makes a clear distinction between them. The primitive, though, is not totally unaware of the number property of the two groups; in fact, he may be quite aware that the number of canoes and the number of nuts are the same. This is clearly indicated by the number of cases in which the various sets of number-words used actually are derived from a basic set and are changed in form by the adding of suffixes or prefixes in their use as numerical modifiers. The primitive's use of various sets of number-words does indicate, however, that in most cases the other properties of the groups he deals with are at least as important to him as the number property is. Number-words are most often used as modifiers to express one of the properties of groups.

Summary and Implications for the Teaching of Arithmetic

There are some important parallels between primitive man's use of numbers and that of young children and even of civilized adults. The most significant of these is the clear indication that number ideas originate in and remain closely related to concrete reality. This is indicated in the studies of primitive man through (1) his methods of answering quantitative questions and communicating the
answers without resorting to extensive number vocabularies or arithmetic systems; and (2) by the close relationship he maintains between his number-words and the objects or ideas to which they refer. Such methods of dealing with quantities are by no means restricted to primitive groups. Civilized man often deals with quantities in similar fashion, and children, in particular, can and do learn a great deal about quantity before they encounter any adult-imposed vocabulary and formal arithmetic. There is little argument for divorcing arithmetic from concrete objects for young children and teaching it as an "abstract" system. Children have little use for numbers divorced from the objects they are used to modify.
Chapter VI

OPERATIONISM

Introduction

Some Uses of the Word "Meaning"

There has been an emphasis on teaching arithmetic meaningfully for more than a quarter of a century. Authorities differ, however, in their use of the word meaning. Some appear to hold that a word, symbol, or sentence is meaningful to a person if he can employ it correctly in a social situation. Wilson's book on the teaching of arithmetic is largely an elaboration of this point of view. Buckingham distinguished between the social meaning, which he called significance, and the mathematical meaning in arithmetic. The social meaning, according to Buckingham, refers to the importance, the necessity of arithmetic in the social order. A term or operation of arithmetic is meaningful to one in this sense if he knows when it is applicable in a social situation. The mathematical meaning, according to Buckingham, refers to the logical structure of arithmetic. According to this theory, a term or sentence in arithmetic is meaningful


to a person when he sees its logical relations to other terms or sentences; and an arithmetical process or algorithm (algorithm) is meaningful to him when he can rationalize it in terms of other rules, assumptions, and symbolic operations of arithmetic.

Van Engen\(^3\) considered three dimensions of the meaning of words and symbols: the syntactic, the pragmatic, and the semantic. Van Engen's three dimensions were discussed in Chapter I. Van Engen's syntactic and pragmatic dimensions of meaning refer respectively to the variations in meanings produced by context and emotional overtones.

The findings of the anthropological studies reported in Chapter V indicate that number ideas originate in and remain closely related to concrete reality. It appears, then, that an arithmetic program for the primary grades which develops the meanings of the terms and statements of arithmetic through physical operations on concrete objects would represent a fruitful approach to the teaching of quantitative ideas. The operational point of view to meaning is accepted for the purposes of this study. In brief, the meaning of a term or statement according to the operational point of view is in terms of a set of operations which one performs in applying that term or statement in a concrete situation.

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The Use of the Word "Operation"

The word *operation* will be used in this study in accordance with its use in the philosophy of operationism; and in the discussion of operational meanings in arithmetic it will be used to refer to physical operations. This is not meant to imply that a meaningful operation is merely physical, however, the purpose in this study is to describe those physical operations by which a child may gain meanings for the terms, symbols, and statements of arithmetic. The word *physical* will be left undefined in this study; however, the following illustrations of physical operations may be given; moving objects with the hands; seeing, such as subitizing the number of objects in a group; and hearing, such as hearing a number of successively presented sounds. On the other hand, computation with numerals in an arithmetic algorithm is not classified as a physical operation in this study. This use of the term *operation* is totally different from its use in the expression "arithmetical operation" in which *operation* is commonly meant to refer to one of the four symbolic "operations" of arithmetic—addition, subtraction, multiplication and division. In this study the symbolic "operations" of arithmetic will be referred to as arithmetic processes or symbolic processes. As an example, the physical operation of taking three objects from a group of seven objects will be called an *operation* and is different from the subtraction process, $7 - 3$; the latter is an example of an arithmetic or symbolic process.

The Purpose of the Chapter

The purpose of this chapter is to make explicit the main ideas of operationism and the use of the word *meaning* in operational theory.
Since operationism is a refinement of the theory of meaning in the philosophical movement known as logical positivism, a brief history of this movement will be given first as background for the discussion of operational theory which will constitute the major part of this chapter.

The Setting for Operationism

Rationalism and Empiricism

What are the sources of knowledge? Two propositions may be stated regarding these sources: (1) there cannot be knowledge without a "knower"; (2) there cannot be knowledge without a "known thing."

History has recorded two extremes as mankind has contemplated the relative importance of the "knower," usually designated as mind, that part of the "knower" that supposedly does the knowing, and the "thing known," usually designated as matter or experience.

On one extreme is the rationalist who is awed by the power of the mind. The rationalist believes that "knowledge" of the structure of the universe comes about through thought. He holds that the absolute principles of truth and the only reality that can possibly be known may be discovered through the sheer power of thought. This idea of the source of knowledge received considerable impetus from the Greeks and has left its mark indelibly on Western culture.

On the other extreme, are those John Dewey called "sensational empiricists" who proclaim "the necessity of direct, first-hand contact with things as the source of all knowledge." If the word

A proposition about the nature of meaning, ideas, concepts, or universals: that they (and thus, some contend, knowledge) 'consist of' or 'are reducible to' references to directly presented data or content of experience; or that signs standing for meanings, ideas, concepts, or universals refer to experienced content only or primarily; or that the meaning of a term consists simply of the sum of its possible consequences in experience; or that if all possible experimental consequences of two propositions are identical, their meanings are identical. 5

It has been popular to associate philosophers with those who give their allegiance to the power of the "mind" and scientists with those who incline toward empiricism. And, in fact, philosophy in the tradition of Plato and science in the tradition of Galileo may be taken to represent the two poles, mind versus matter. The first half of the twentieth century, however, has seen a definite movement to unite the empiricism of modern science and the formalistic, or rational, logic of philosophy.

The Vienna Circle and Its Influence

S. S. Stevens sees the central theme of this movement to be as follows:

...that science seeks to generate confirmable propositions by fitting a formal system of symbols (language, mathematics, logic) to empirical observations, and that the propositions of science have empirical significance only when their truth can be demonstrated by a set of concrete operations. 7


Stevens says it has become the business of modern philosophy to discover and perfect the rules of scientific language. The scientist applies the formal symbolic model to the observable world, testing operationally the ideas created by the symbolic model.

The movement to unite philosophy and science, spelling out the relationship of one to the other, received considerable momentum from certain men in Europe whose preoccupation was with philosophy as analysis. Among those who concerned themselves with this philosophy of science was a group of men which came to be known as Der Wiener Kreis, the Vienna Circle. In 1924 the Vienna Circle originated as a discussion group under the leadership of Moritz Schlick at the University of Vienna.

The thinking of the men in the Circle was influenced by the earlier empiricism of Hume, Mill, and Comte, and by the methodology of empirical science as developed since the middle of the nineteenth century by such men as Helmholtz, Mach, Poincare, Duhem, Boltzman, and Einstein. Two other groups interested in the philosophy of science were centered at Berlin, under the leadership of Hans Reichenbach, and at Cambridge, where the Cambridge School of Analysis included men who had been influenced by G. E. Moore, Russell, Wittgenstein, and Frege. There was considerable liaison of ideas among

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8. Ibid., p. 223
these three groups, a liaison which also included men from smaller centers such as Lvow, Warsaw, and Prague.9

The Circle included men from the fields of philosophy, law, physics, mathematics, medicine, and the social sciences. The views developed by the Vienna Circle were called logical positivism. Later many of those interested in the ideas of the Circle came to prefer the term logical empiricism or the term scientific empiricism as being more indicative of their interest in integrating the techniques of science and logic.

During the 1930's, a number of the men from the Continental groups immigrated to the United States. Here they found a favorable intellectual climate for a philosophy of science, distinct from metaphysical philosophy, in the pragmatist tradition as it had been developed by Charles Peirce, William James, and John Dewey. Certain

9. For short histories of the Vienna Circle and other European groups interested in the philosophy of science see:


For a short history of men of positivistic persuasion since the time of the Greeks see:

American philosophers, such as Benjamin, Hook, C. I. Lewis, and Nagel, and American scientists, such as Bloomfield, Boring, Bridgman, Pratt, Tolman, Skinner, and Stevens, had been thinking and writing along related lines.¹⁰

Nagel maintains that the aim of logical empiricism is "to perfect principles of logic and method and to exercise them upon the various statements occurring in everyday discourse and the sciences."¹¹ Stevens says logical empiricism seeks to clarify the language of science and to investigate the conditions under which empirical propositions are meaningful.¹² The theory of meaning held a central place in the discussions of the Vienna Circle. Nagel says the men in the Circle took for granted the body of knowledge of the special sciences; they were interested in clarifying the meaning and the implications of that body of knowledge.¹³ The logical empiricists of the Circle originally held that the meaning of a statement is constituted by the methods used in verifying it. This was their position in answer to the question: How does one distinguish between meaningful statements and meaningless ones? If a statement could be definitely established as true or false by sensory observation then

¹⁰ Feigl, op. cit., pp. 405-9.
¹² Stevens, op. cit., p. 237.
it was meaningful. If it could not be established as definitely true or false by sensory observation, then it was meaningless.

A more accurate statement of the proper question for logical empiricism later became this: What are the conditions necessary for a statement to have meaning? Nagel submits the following as both a necessary and sufficient condition that a statement have empirical meaning:

If a statement does not consist entirely of observation terms it is empirically meaningful if, and only if, with the help of observation data and rules of logic, other statements are deducible from it so that statements capable of direct confirmation are finally obtained.14

Logical empiricism, for example, rejects metaphysical propositions, not because they are false, but because in this theory of meaning they are meaningless.15

Such a procedure as stated by Nagel is what the logical empiricist calls philosophic analysis of a statement. For the moment the term "abstract" as applied to words and other symbols is taken to mean that they do not refer to some thing, some event, or some action of immediate experience. A statement containing one or more words or other symbols which are "abstract" is not directly confirmable. In such a case, other statements which are confirmable directly in ex-


perience are deduced from that one by the rules of logic. Thus, the crux of the meaning of terms, symbols, and statements is in things and the operations performed on things.

To say however that the meaning of a proposition is in its verification does not imply that it is verifiable directly nor even that it must be verified now in the present. Moritz Schlick pointed out that verifiability means possibility of verification; it must be theoretically verifiable; one must stipulate how the verification shall be done.¹⁶ Blumberg and Feigl emphasize this aspect of meaning. One works back from complex propositions to simple propositions and arrives "ultimately at the immediate facts whose being-the-case constitutes the meaning of the proposition."¹⁷ While he may not actually perform the act or acts producing verification, if a complex proposition is given, one always asks how it can be verified.

Rudolph Carnap discussed verification and confirmation somewhat as follows: Verification is the procedure used in finding out whether a sentence is true or false, whether or not it corresponds to reality.¹⁸


The word *sentence* is used here in the same sense as it was used by Tarski to mean a declarative sentence.\(^{19}\) The procedure need not be performed in fact, but it must be possible of performance under suitable conditions. Many persons who are inclined toward scientific empiricism hold that verification is replaced by the concept of confirmation. An hypothesis is said to be confirmed to a certain degree by a certain amount of evidence. The concept of confirmation is closely connected with the statistical concept of probability. A sentence is confirmable if there are possible suitable experiences which could contribute positively or negatively to its confirmation. The probability rather than the certainty of scientific statements has been recognized only in relatively recent times.

**Summary: The Setting for Operationism**

The movement of scientific empiricism has been the harmonizing of the rational and the empirical. The dichotomy of old between philosophy and science has become a complementary relationship. Stevens says: "This [the harmonizing of the rational and the empirical] has proved disastrous for metaphysics, challenging for logic, and salutary for science."\(^{20}\) Certainly among its major accomplishments has been the ridding of both philosophy and science of pseudo-problems,

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that is, problems which are meaningless because they are impossible of verification; and the reformulation of some of the traditional problems so that they are now meaningful.

The Meaning Theory of Operationism

Definitions of Operationism (Operationalism)

Stevens defines operationism as follows:

Operationism: The doctrine that the meaning of a concept is given by a set of operations.
1. The operational meaning of a term (word or symbol) is given by a semantical rule relating the term to some concrete process, object or event, or to a class of such processes, objects or events.
2. Sentences formed by combining operationally defined terms into propositions are operationally meaningful when the assertions are testable by means of performable operations. Thus, under oper­
a tional rules, terms have semantical significance; propositions have empirical significance.21

The foremost proponent of operationism has been the physicist P. W. Bridgman. In his book, The Logic of Modern Physics, published in 1927, Professor Bridgman makes the statement: "In general, we mean by any concept nothing more than a set of operations; the concept is sy nonomous with the corresponding set of operations."22 In another place, almost a quarter of a century later, Bridgman expresses the same thesis:

The fundamental idea back of an operational analysis is simple enough; namely that we do not know the meaning of a concept unless we can specify the operations which were used by us or our neighbor in applying the concept in any concrete situation.  

In still another place, he writes as follows:

The essence of the attitude is that the meanings of one's terms are to be found by an analysis of the operations which one performs in applying the term to concrete situations or in verifying the truth of statements or in finding the answers to questions.

This third statement is the most useful one since in it the words terms, statements, and questions are employed instead of the word concept. Hullfish has discussed the vagueness of the word concept, and points out that there is no measure of a concept but that "there is, simply, a measurement of products, of reactions to... stimulating conditions."

The terms, statements, and questions of arithmetic may be analyzed operationally in the way suggested by Bridgman. For example, the term "take-away" has operational meaning for a child when he can relate it to the actual physical process of taking away or removing a given number of objects from a group of objects. The symbol has meaning for him when he can relate it to two groups of ten objects.


each and a group of three objects. A sentence such as "23 and 15 are equal to 38" is meaningful to the child if he can show that when he combines two groups of ten objects each and one group of ten objects and a group of three objects with a group of five objects he gets three groups of ten objects each and eight objects for a total of thirty-eight objects.

The Importance of an Operational Analysis in Providing Meanings for the Terms and Statements One Uses

Bridgman maintains that the operational aspect is not the only aspect of meaning, but that it is often the most important aspect. In an operational analysis, he says, one deals with necessary rather than sufficient conditions for meaning. This viewpoint is somewhat in contrast to Nagel's statement; and, in fact, Bridgman maintains that in operationism he is only interested in making explicit the necessary conditions for a term or a statement to have meaning. He maintains that he is not attempting to set up a theory of meaning nor to hold that meaning involves nothing more than operations, but he does say that unless one knows the operations he does not know the meaning. 26

Professor Bridgman is of the opinion that the operational analysis has a two-fold importance: When a person is confronted with a situation in which obscurity of meanings is a factor, such an analysis clarifies the situation and gives precision, articulateness, and

26. Ibid., pp. 4-5.
stability to his concept. When two or more persons are concerned, such an analysis secures unambiguous recording and communication, for (1) if persons agree in their operational description of the meaning of a term, there will be no confusion in their communication of meaning; (2) when the same situation is repeated, everyone will apply the same term; and (3) if the term is given, the situation can be reconstructed from it. 27

The "Concepts" of "Operation" and "Operational Analysis"

Questions may be raised about the "concepts" of operation and operational analysis. For example, is there a fundamental operation of arithmetic, such as counting or the perception of the number of objects in a group? What are the requirements for a good operational analysis? How far does an operational analysis need to be continued before the meaning of the term or statement is made clear?

Professor Bridgman leaves the concept of operation unanalyzed. He says that it is a matter of common experience that a person can perform certain operations and that others can perform the same operations; operations have a certain "repeatability" and a certain objectivity. 28 In another place, Bridgman points out that one of the requirements of a good operational analysis is that the operations


specified shall be repeatable and performable on demand. It is a fact that no operation can be repeated with absolute precision. There are operations, however, which can be repeated with the precision necessary for the problem at hand. Professor Bridgman concludes that it is in terms of such operations that definitions should be framed.\textsuperscript{29}

As an example in arithmetic, two different children combining two groups of objects to obtain the total will not perform the physical operations involved in precisely the same way; but each child can learn to perform these operations in such a way that it is evident to others that he is performing the operation of combining groups.

It is also a fact that the person who is asked or who wants to repeat an operation must have acquired the physical and mental maturity necessary to learn that operation; and, furthermore, he must have learned it. In performing operations, the use of various senses is required and one acquires perceptual abilities only through maturity and practice. This latter fact, in its relation to perceiving the number of objects in a group, was brought out in Chapters II and III concerning adults and children.

An operational analysis needs continue only to that point where those who are interested in the meaning of a term or statement involved are familiar with the operations called into play. Bridgman is of the opinion that an operational analysis neither needs to be nor can

be an ultimate analysis. It needs only to be adequate for the purpose at hand in the immediate context. He points out that a physicist explaining some new research to other physicists needs to analyze only those parts of his experiments which are unfamiliar to his audience, even though the operations which he specifies may be totally unfamiliar to the layman. 30 On the other hand, an operational analysis of addition for the first-grade child will need to be made in terms of physical operations the child can perform with objects.

In Chapter II, it was pointed out that many writers consider counting to be the fundamental "operation" of arithmetic. The perception of the number of objects in a group may be even more fundamental. It may be a questionable procedure, however, to call discrimination and perception, particularly as they refer to the number of objects in a group, fundamental operations of arithmetic; and the findings regarding subitizing indicate that it is erroneous to consider counting the fundamental operation of arithmetic. Bridgman's point of view—that there is no need to carry an operational analysis to an ultimate or fundamental operation—may be more sound here. At least, the analysis of an arithmetic term, symbol, or statement for a child needs to be carried only to that point where he is familiar with the terms and operations being used in the analysis.

Physical and Mental Operations

Must an operational analysis always be in terms of physical operations? Feigl feels that the meaning of operationism should be

30. Ibid., p. 89.
confined in its application to the definition of "concepts" which arise directly from physical operations. As examples of "concepts" which are not definable in terms of physical operations, he gives those of pure mathematics and logic.31 Feigl's position regarding pure mathematics and logic appears to be that of many logical positivists. Ludwig Wittgenstein in his *Tractatus Logico-Philosophicus*, written in 1922, maintained that theorems of pure mathematics or of logic say nothing about the experienceable, observable world, but are actually tautologies. The Continental group of logical positivists read Wittgenstein widely; Bergmann is of the opinion that the tautological nature of pure mathematics and logic is one major point of agreement among most logical positivists.32

Bergmann and Spence are of the same opinion as Feigl, that operationism is properly concerned with "concepts" which arise directly from physical operations; therefore, it is more applicable to such fields as physics and chemistry than it is to such fields as the social sciences. It is even more practical in the physical sciences than in the life sciences, they affirm. The physical sciences are less complex; there are fewer variables to control in these fields.33

On the other side of the issue, Benjamin, a philosopher, says there are physical and mental operations. He calls "extensional" those definitions which are made in terms of physical operations performed upon objects. A symbol used in such definitions he calls a "descriptive" symbol. The meaning of a descriptive symbol is a function of its referent and also of the operational route followed in passing from the referent to the symbol. Thus, the symbol \( \frac{1}{4} \) refers to four objects which may have been seen or felt, or to four sounds which may have been heard. That is to say, the meaning of \( \frac{1}{4} \) is dependent not only upon the "thing" perceived, but also upon the way in which it is perceived.

Benjamin calls definitions "intensional" which are expressed in terms of mental operations performed upon symbols to produce other symbols. Symbols arising in this manner he calls "suppositional." As examples of purely intensional symbols and propositions, he gives those of pure mathematics and logic; they have intensional meaning by virtue of their participation in symbolic systems, deriving that meaning and their truth from their analytic character. These symbols have extensional meanings also, he says, for they are always referrable for their meanings to other symbols which are themselves descriptive.

Benjamin concludes that "suppositional symbols differ from denotatively defined symbols only in their 'distance' from the given."36

John Dewey recognized both physical and mental, or symbolic, operations:

By means of symbols, whether gestures, words, or more elaborate constructions, we act without acting. That is, we perform experiments by means of symbols which have results which are themselves only symbolized, and which do not therefore commit us to actual or existential consequences.37

For example, Dewey maintained that the operations denoted by mathematical symbols are in the main symbolic, but that there is a mathematics closely related to the physical and that that part of mathematics can be shown operationally. For Dewey, mathematical ideas are concrete when they are employed exclusively in doing something in the physical world; but they become abstract when they are freed from any such use.38 Operations indicated by symbols and performed only by symbols suggest further operations which may be performed. Furthermore, this abstractness in mathematics accounts for its wide applicability.

Dewey considered that

... a logical theory of mathematics must account for both that absence of necessity of existential reference which renders mathematical propositions capable of formal certification, and for the generalized possibility of such reference.39

38. Ibid., p. 154.
According to Bridgman, the operational criterion of meaning does not demand that even the operations which give meaning to a physical concept be physical. He recognizes two kinds of operations: (1) the operations of the laboratory made with instruments, in which the discriminatory capabilities of the sense organs play a major role; (2) mental operations which may be verbal or paper-and-pencil or a combination of the two. These operations are usually with symbols, mathematical or otherwise. Bridgman contends that physical and mental operations are used in such a way as mutually to reinforce and supplement each other, and that it seems impossible and meaningless to attempt to draw any sharp line of separation between the two.

One may summarize the foregoing discussion by saying that some writers maintain there are two kinds of operations, physical and mental. Other writers feel that the theory of operationism is limited largely, if not solely, in its application to physical operations. Still other writers maintain that the difference in symbols as regards the operations to which they refer is one of degree rather than of kind. This issue, while an important one philosophically, is not a crucial one in elementary arithmetic, because both historically and psychologically the terms and processes of elementary arithmetic are non-tautological in nature. The symbols of elementary arithmetic are used to make statements about observable phenomena. The concrete

41. Ibid., p. 61
basis of the terms and processes of arithmetic has already been elaborated upon in the chapter on primitive peoples.

Regress in Operationism

In the theory of operationism, the method of providing meaning for terms and propositions somewhat removed from the immediately empirical is called the method of regression. In brief, regression is the procedure of defining and explaining a new symbol or statement in terms of other symbols, statements, and operations which are more familiar to the person attempting to learn its meaning. Thus, the word ratio in arithmetic may be explained as "a comparison of two groups of objects by division." If the child knows what is meant by the symbolic process of division and if he knows what it means physically to compare two groups of objects, then he knows the meaning of the term ratio. If not, however, then the regress must continue, explaining and showing what is meant by "division" and by "comparison" as they relate to other symbols and operations.

Bridgman points out that an operational analysis is valueless without a background of experience. Operational definitions may and often do form a regress, he believes; but it is not an infinite regress. In practice, it will terminate at that point where those people interested are already familiar with the operations which would follow in the regress.

I believe that examination will show that the essence of an explanation consists in reducing a situation to elements with which we are so familiar that we accept them as a matter of course, so that our curiosity rests.\(^3\)

Stevens maintains that when one does not understand the operations and does not know how they can be carried out, then the operations should be described in terms of others which appear simpler through personal experience. Regress may be continued, defining each successive term by simpler operations. The process is not an infinite one, however; even if carried to the ultimate, the regress would terminate in operations of discrimination, which Stevens calls the most basic of operations. In practice, the operational regress need be pursued only until agreement is reached, that is, when the other person understands the terms and operations which are being described.\(^4\)

The process of giving meaning to "suppositional" symbols through the "intensional route" described by Benjamin is equivalent to the regress of Stevens and Bridgman. Suppositional or intensional symbols are always referrable for their meanings to symbols which can be defined extensionally, and differ from the extensionally defined symbols only in their distance from the empirically given.\(^5\)

Spiker and McCandless, in discussing regress or "reduction" in relation to the "abstract", have the following to say:


Let us use the term "abstract" to refer to words whose definitional chains are long in the sense that numerous statements are required for defining them solely in terms of the primitive predicates. We may then describe scientific practice in this regard as that which utilizes explicit definitions only for the more abstract concepts; (even in those cases the reduction process is carried down only so far as is necessary to avoid serious ambiguity.)

These writers hold that Rudolph Carnap had used the term "primitive predicates" to refer to very simple terms where the understanding can be obtained only through acquaintance with concrete referents.

It is significant to notice that Spiker and McCandless do not specify any particular length of a definitional chain defining a "concept" before they would call it an "abstract concept." There may be an advantage in thinking of such a concept as being simply "less concrete" or as being "further removed" from concrete referents.

One may think of this process of giving meaning to "abstract concepts" as the "definitional chains" of Spiker and McCandless, the "regress" of Stevens or Bridgman, or in the "intensional to the extensional" language of Benjamin. In any event, it is important to recognize the procedure of defining and explaining in simpler terms (simpler by virtue of their being less far removed from the empirically given) and simpler operations (simpler in that they are more familiar to persons interested) until agreement and understanding are reached.

In the arithmetic of the primary grades the meanings of the

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terms, symbols, and statements may be made evident through regresional routes. Thus, the chain of meanings for the symbol 27 may be shown as follows:

1. 27 means
2. \(2(10) \neq 7\), which means
3. 2 tens and 7 ones, which means
4. "two tens and seven ones" (this meaning spoken).

A young child, if he knows the meaning of "two tens and seven ones" as \(0000000000\) \(0000\), can then learn the meaning of the symbol 27 through such a regresional chain.

As a second example of such a regresional chain, consider the meanings involved in the statement \(8 \div 2 = 4\). Evidently, there are, at least, two meanings to this expression (which will be discussed more thoroughly in Chapter VIII). One meaning is that if 8 objects are regrouped in groups of 2 objects each, there will be 4 such groups. The other meaning is that if 8 objects are divided into 2 equal groups, there will be 4 objects in each group. If the child has performed these physical operations with groups of objects, then the symbols, \(8 \div 2 = 4\), will be meaningful to him.

In both of these examples it should be noticed that the child may not need to retrace the entire regresional chain. Thus, if he knows what "20 \(\neq 7\)", or what "2 tens and 7 ones" means, the symbol 27 may be explained in terms of these meanings without further regress. Actually, the regresional chains for the terms, symbols, and statements of arithmetic in the primary grades are short. In fact, in most cases the terms used may be shown to refer directly to concrete objects.
or to physical operations on those objects. Thus, "take-away" and "add" are arithmetic terms with direct physical operations as referents. The written symbols refer to the spoken terms, and thus usually require only two steps in the regresional chain to make their meanings evident. One meaning of the symbol $-$ is "take-away" and a meaning of the symbol $+$ is "add." Both of these terms, as pointed out above, refer directly to physical operations. There will be a further elaboration of these points under the heading "Stages of Symbolization" in Chapter VII.

**Equivalence of Operationally Defined Constructs**

If a term is defined by two different sets of operations, should it be said that there are then really two different ideas involved? For example, length may be measured directly or indirectly. Do these different methods of determining length lead to two different ideas of length? Bridgman answers this question in the affirmative; strictly speaking, he says, there are two "concepts." In fact, it was a very similar situation which led Bridgman to consider the importance of operational definitions. He points out that Einstein had recognized that such apparently simple terms as length and time have multiple meanings; relativity forced upon physicists the realization that the precise meaning of "concepts" involves knowing the procedures used in obtaining them in concrete instances.

A child knows the meaning of an arithmetic problem if he knows

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how to obtain the answer in a concrete situation. For example, the symbolic process of subtraction is used to find answers for such varied questions as these: (1) If 3 objects are removed from a group of 7 objects, how many objects are left?; (2) How many more objects are there in a group of 7 objects than there are in a group of 3 objects?; (3) If a person has 3 objects and needs 7, how many more does he need? According to operational theory, these three uses for the symbolic process or subtraction constitute three different meanings; and the child should know subtraction as three different sets of physical operations necessary to solve these problems in concrete situations.

If two sets of operations lead to identical results, they should at first be classified as two different "concepts"; and it may be best to apply different names to them, Bridgman maintains. One of the values of operationism is in emphasizing that it is never safe to assume that two numerically equivalent results will continue to be equivalent. In fact, in experimental procedure, equivalence can be determined only within a certain margin of error. As the procedures are refined or as the range of experience widens, a difference may be detected in the results. 49

The chief value of operationism in this respect, however, is that a careful analysis of the different sets of operations which produce the same results is the best way of providing understanding. It is

best at first to employ problems deliberately framed so that those differences in procedure will be apparent. Thus, in arithmetic a child should encounter take-away problems which he solves by performing physical operations on groups of concrete objects, comparison problems, and problems in how-many-are-needed which he solves by performing physical operations on groups of concrete objects. When he has observed the relationships as well as the differences among these three problems during the operational solutions, and when he sees that the arithmetic process of subtraction may be used to solve the problems, then he may use subtraction to solve all three, utilizing the thinking involved in any one of them. This procedure is just the reverse of a teaching method in which the symbolic process of subtraction is taught before its application to the three different problem situations is discussed.

For practical purposes, the same name is usually given to ideas arising from two or more different sets of operations which consistently yield equivalent results—in arithmetic, the same numerical results. Bridgman says that the fusing of results of two different sets of operations under a single name should follow a clear distinction between the two, and that it is never safe to forget completely what one has done in the two different sets.50 For example, after a person understands and has used two different names for the two different sets of operations in the direct measurement and the indirect measurement of a line segment, the common term "length" may be applied with

reference to the line segment regardless of which method is used to
determine it. In a second example, taken from elementary arithmetic,
there are at least two different meanings of division with whole
numbers, partition and measurement. The two sets of physical oper-
ations performed on groups of objects which give rise to these two
meanings are recognizably different. It is only after the child
understands these two meanings thoroughly that he may then safely
solve either symbolically, thinking by whichever method is easier for
him. And if he is to use the symbolic process of division intelli-
gently in problem situations, he must never forget the different sets
of operations which he first performed with objects.

The foregoing discussion should not be interpreted to mean that the
criterion of numerical equivalence is the only one to apply to ascertain
whether two different sets of operations are equivalent. In fact, the
partition and the measurement meanings of division are never operation-
ally equivalent. The fact that problems involving either of these may
be performed symbolically by thinking in terms of the other does not de-
pend solely upon the fact that both problems, if they involve the same
numerical data, yield the same numerical answers. There is a relation-
ship between the solution to the partition problem and the solution to
the measurement problem which may be shown operationally. It is this
relationship between the operational solutions, rather than the fact
that they yield the same numerical answers, which permits either problem
to be solved symbolically by thinking in terms of the other. This point
will be discussed further in Chapters VII and VIII where the relationships
among various sets of physical operations will be shown.
Summary

The movement to unite the empiricism of science and the rationalism of philosophy has crystallized over the past half-century into logical or scientific empiricism. In the main, this movement has sought (1) to clarify the language of science and (2) to investigate the conditions under which statements are meaningful.

The question of meaning has been central in logical empiricism. What is meant by meaning has been made explicit in the theory of operationism. Briefly, according to operationism, the meaning of a "concept" is not known unless the operations which are used in applying it in a concrete situation can be specified. In the theory of operationism, then, one may be said to know the meaning of a term, symbol, or statement which names or stands for a "concept" if the set of operations one performs in applying it in concrete situations can be demonstrated or specified.

In view of the discussions in this chapter regarding operationism, the following statements may be made about meaning in the operational sense and about the set of operations in an operational analysis.

1. The meaning of a term, symbol, or statement is in relation to a set of operations one performs in applying the term, symbol, or statement in a concrete situation.

2. The operations specified must be repeatable and performable on demand. In lieu of the actual performance of a set of operations, however, a person may demonstrate that he knows the meaning of the term, symbol, or statement by describing the operations necessary.
3. Some authorities recognize both physical and mental operations. The terms physical operation and mental operation are analogous in many respects to the more popular terms of concrete and abstract. Actually, physical operations and mental operations reinforce and supplement each other, and it seems meaningless to attempt to draw any sharp distinction between the two.

4. Regress is the procedure of defining and explaining terms, symbols, and statements in terms of simpler ones (simpler by virtue of their being less far removed from the empirically given) and simpler operations (simpler in that they are more familiar to persons interested) until agreement and understanding are reached.

5. Two different sets of operations lead to two different ideas. If the two sets of operations continually lead to identical results, then for practical purposes the same name is applied to both. This fusing of results of different sets of operations under a single name, however, should follow rather than precede a clear distinction between them; and it is never safe to forget completely the two different sets of operations involved.

Implications for the Teaching of Arithmetic

Operational analysis is not limited in its application to the physical sciences, although its application in the physical sciences appears more direct than in other areas of learning. It was pointed out in Chapter V that, as they originate, the concepts of arithmetic are closely tied to the concrete, and consequently are more nearly
like those of the physical sciences than they are like the tautological structures of higher mathematics and other abstract systems.

The operational approach appears to be a clear-cut, precise, and readily adaptable approach for giving meanings to the terms, symbols, and statements of arithmetic. In Chapters VII and VIII, the implications of the operational approach for the teaching of the meanings of terms, symbols, and statements associated with whole numbers in the arithmetic of the primary grades will be dealt with in detail.
Chapter VII

THE OPERATIONAL APPROACH TO TEACHING

ARITHMETIC IN THE PRIMARY GRADES: I

Introduction

Purposes of Chapters VII and VIII

The purposes of Chapters VII and VIII are these: (1) to make an analysis of the physical operations with objects in groups which provide meanings for the arithmetical words, symbols, and rules associated with whole numbers; and (2) to point out the relationships among various sets of physical operations which permit them to be performed symbolically by the various arithmetic processes. In Chapter VII, grouping, matching, counting, addition, and multiplication will be considered; in Chapter VIII subtraction and division will be discussed.

The Use of the Word "Meaning" in this Study

The word meaning is used in this study in the operational sense. The meaning of a term, symbol, or statement is in relation to a set of operations which one performs in applying that term or statement in a concrete situation. In lieu of the actual performance of the operations, a description of the performance may be substituted.

The basic assumption made at this point is that all of the terms, symbols, and sentences of arithmetic in the elementary school have
meanings either (1) through direct reference to concrete objects and
physical operations with those objects, or (2) through regressional
routines which terminate in concrete objects and physical operations.
Consequently, no term, symbol, or sentence of elementary arithmetic
is "abstract" in the sense that it cannot eventually be made to make
contact with concrete objects and physical operations. It is further
assumed that for the terms, symbols, and sentences in the arithmetic
of the primary grades this relation to concrete referents is direct,
or, at most, the regressional chains are short. One purpose of this
chapter will be to demonstrate that the above statement holds when
young children are being introduced to whole numbers. The kinds of
problems proposed will be largely those which are commonly thought of
as being solved by a use of one or more of the four fundamental
arithmetic processes, commonly called operations—addition, subtraction,
multiplication, and division—performed on whole numbers. The basic
ideas and the symbols are in the main those commonly introduced during
the first three or four grades.

Sets of Operations

The set of physical operations involved in solving a problem
situation in elementary arithmetic consists of the following: (1) an
operation of determining the number of objects in the groups being
considered, or of reproducing those groups; (2) an operation of
moving the objects in some manner, such as combining and taking away
groups or regrouping and matching them, in order to portray the groups
so that the answer to the quantitative question asked about the
situation may be found; (3) an operation, the last one in every set of operations, of determining the number of objects or groups resulting, that is, of obtaining the answer to the quantitative question asked.

As an example of a set of operations involved in answering a question in arithmetic, consider the following problem: If one divides (or regroups) a group of 6 blocks into 3 equal groups, how many blocks will there be in each of the 3 groups (Fig. 1)?

The child has a set of three operations to perform. First, he must reproduce a group of 6 blocks. Second, he must set up three areas and assign the blocks, usually one by one, to them; that is, he must actually do the regrouping. Third, he must determine the number of blocks in any one of the 3 equal groups.

The first and third steps listed above are operations designed to determine "how many" objects or groups there are. Such operations include counting, comparing with model groups, grouping, and subitizing. The operation or operations in the second step, such as those described in the example above, depend on the kind of problem involved. These operations of moving the objects in some manner, such as combining, taking away, regrouping, and matching, may be thought of as
the operations which rearrange or display the group or groups involved so that the number question asked may be determined by reference to the new display. The second step in the set of operations for the problem above is illustrated in Figure 1. The rectangles A, B, and C, represent the three areas set up by the child. He then moves the block numbered 1 in the figure into area A, the block numbered 2 into area B, the block numbered 3 into area C, etc., until all of the blocks have been divided equally among the three areas. Thus, a regrouping of the blocks is attained so that the child, by referring to any one of the three equal areas, can determine the answer to the question: How many blocks are there in each of the 3 groups? The meaning of the problem, then, is in terms of the set of operations necessary to solve it.

Establishing Correspondences between Symbols and Their Referents

The wording of the question.—It should be emphasized that if the child is to establish a correspondence between the terms, symbols, and sentences of arithmetic and the objects and operations to which they refer, the wording of arithmetical questions is important. The problem of the example given in the last section may be thought of as consisting of two parts. The first part of the problem—"If one divides (or regroups) a group of 6 blocks into 3 equal groups. . ."

specifically tells the pupil what operations he is to perform on the objects. The second part of the problem—"how many blocks will there be in each of the 3 equal groups?"—calls for the pupil to use one of the operations for the determination of the size of a group. In
early number work, the second part of the problem, the question, should specifically point out what group or groups are to be considered in the pupil's answer and should give some indication of the kind of answer desired.

The kind of answer desired. — The last point requires further elaboration. Consider this example: If one brings a group of 7 blocks and a group of 8 blocks together, how many will there be in all? A young child who has not yet progressed to a study of two-place numbers might give any one of several correct responses. He may say, "There are 7 blocks and 8 blocks," or "There are 9 blocks and 6 blocks," or "There are 3 groups of 5 blocks each," etc. These answers are not wrong; they are simply not in the form desired. In arithmetic the child must learn, among the many other things, the forms of answers expected. The expected form here is so many tens of blocks and so many ones of blocks, specifically 1 ten and 5 ones, or 15. Since children cannot be expected to know the desired forms of answers unless they have been given opportunities to learn them, a number question should indicate in some way the kind of answer expected. The cited example, when given to beginning school children, should be worded somewhat as follows: If one brings a group of 7 blocks and a group of 8 blocks together, how many groups of 10 blocks each can be made from the total and how many single blocks will there be in addition?
Methods of Determining or Reproducing the Number of Objects in a Given Group

The Operation of Subitizing

Kaufman (see Chapter II) gave an operational definition of subitizing which may be paraphrased as follows:

Definition 1: Subitizing is that which occurs in the visual discrimination of numerosness when the stimulus-number is less than six.

Kaufman's definition is stated in terms of a range of the number of objects which his adult subjects could subitize; such a definition is quite restrictive and not desirable. The range is variable for adults. The average range for subitizing with young children four to six years old appears to be one or two objects less than that for adults and appears to vary with age and mental maturity. The range for animals varies widely with the species and even among the individual members of a given species.

The design and findings of Kaufman's experiment permitted him to conclude that in subitizing a subject reports the number of objects in a group immediately and accurately with a high degree of confidence in his report. However, the use of the word subitizing would be restricted if it were defined in terms of time and confidence. Young children who cannot yet count and animals which do not count can learn to discriminate on the basis of number; moreover, while confidence is an important aspect in subitizing, the degree of confidence
may be difficult to determine in the case of either young children or, particularly, animals.

Thus, in order to permit a wider use of the term subitizing while retaining its essential features in operational form, the following definition is proposed by the present writer:

Definition 2: Subitizing is the operation of recognizing directly the number of objects in a group.

When the number of objects in a group is beyond the limit of subitizing, a person may then use a procedure most authorities call regrouping or sub-grouping; that is, breaking up the large group into smaller sub-groups, each of which may be subitized. The person may then determine the size of the whole group with the aid of known number combinations. Thus, one may regroup an unknown number of marbles into two groups of 3 marbles and 5 marbles, each of which can be subitized. Knowing the number combination 3 4 5 = 8, one then knows there are 8 marbles altogether. Carper (see Chapter III) called all such operations of regrouping or subgrouping by a more general term, grouping procedures. Actually, in order to determine the number of objects in a large group by use of an effective grouping procedure, one of two operations must be performed. One of these operations consists in an identification of groups which are already effective distinct subgroups of the large group. The other operation consists in physically rearranging or marking off the objects so that the subgroups are distinct and identifiable, or in identifying subgroups before they are actually discrete and distinct from the rest of the large group.
In Figure 2, the 9 dots are distinguishable immediately as subgroups of 4 dots and 5 dots each. With the arrangement in Figure 3, however, there are no distinct subgroups within the 9 dots. When the child first encounters the problem of finding the number of objects in a group as portrayed in Figure 3, he should be using physical objects that he may move to make a new arrangement with subgroups he can subitize. Thus he may rearrange the 9 objects as 3 groups of 3 objects each (Figure 4), as a group of 3 objects and a group of 6 objects (Figure 5), or as a group of 4 objects and a group of 5 objects (Figure 6). His arrangements may or may not be regular patterns, depending on his ability in subtitizing and in recognizing patterns. Later the child may create a spatial separation of the subgroups simply by drawing lines in such a way as to make the subgroups appear distinct (Figure 7 and Figure 8).

Still later the child may use a grouping procedure to determine the number of dots in Figure 3, creating his subgroups mentally by
attending to certain parts of a group at a time. Thus, he may mentally subgroup the dots in Figure 3 and see the 9 dots as being composed of subgroups of 4 dots and 5 dots each. Obviously, this latter operation is more difficult than either of the first two operations described.

Full advantage of regrouping into subgroups which may be subitized comes with mathematical maturity, since the person uses number combinations extensively to find the size of the total group. A few examples of how the knowledge of number combinations and arithmetic processes may be utilized with subitzing follow:

1. Subitzing objects and subitzing groups: Thus, . . . . . . may be seen as 3 groups of 3 dots each for a total of 9 dots.

2. Subitzing and adding: Thus, . . . . . may be seen as 4 dots and (plus) 3 dots, or 7 dots.

3. Subitzing and subtracting: Thus, . . . . . may be seen as 5 and 1 less than 5 or as 1 less than the familiar pattern of 10 of the domino arrangement.

4. (a) Subitzing and counting. I: Thus, in . . . . . the group of 3 may be seen immediately and the other dots counted. In this combination of operations one says, "three, four, five . . . ."

This procedure is often referred to as partial counting.

(b) Subitzizing and counting. II: This is the set of operations used in counting by twos, threes, fives, etc.

Although arithmetical processes and the number facts are of
considerable aid in these grouping procedures, there is no reason why instruction in these procedures should be deferred until the arithmetical processes and many number facts are known; for, in fact, the relationship is reciprocal. An understanding of the arithmetical processes is aided considerably by grouping procedures which utilize subitizing, and the child can learn number facts through the use of these procedures. This accords with the findings in such research studies as those done by Carper, Freeman, Russell, and others reported in Chapter III of this study. For example, if the child rearranges a group of 9 objects into two subgroups, such as 5 and 4, which he can subitize, then he can see that 5 objects and 4 objects are 9 objects; or, more briefly, he has had an experience involving the number combination $5 + 4 = 9$. This procedure appears to have two values: (1) it is quicker and easier than the procedure in which the child counts the group of 5 objects, the group of 4 objects, and then the total group of 9 objects; (2) the child learns to see and think in terms of groups instead of individual objects.

The Operation of Matching

Another method of determining the size of a group is matching. Suppose that a group (Group A, Figure 9) has an unknown number of objects in it. If the objects of group A can be matched with the objects of some other group (Group B, Figure 9) in such a way that to each object of group A there corresponds one, and only one, object of group B, and to each object of group B there corresponds one, and only one, object of group A, then group A is known to contain the
same number of objects as does group B. If the number of objects in
group B is known, then the number of objects in group A has been
determined by this operation.

\[
\text{Group A} \quad \text{\includegraphics[width=1cm]{three_objects}}
\]

\[
\text{Group B} \quad \text{\includegraphics[width=1cm]{two_objects}}
\]

Fig. 9

If group B in Figure 9 represents the child's fingers, then, after setting-up the one-to-one correspondence, the child knows there are just as many objects in group A as there are fingers on one of his hands. If he knows that the word *five* is associated with the number of fingers he has on his hand, then he may use the same word *five* for the number of objects in group A. The actual physical operation of matching in this case may be performed by placing one, and only one, finger on each one of the 5 objects in group A. If group B consists of objects other than fingers, the child may establish the correspondence by pairing—placing side by side—an object of group B with an object of group A, a second object of group B with a second object of group A, etc. This operation of pairing is indicated in Figure 9 by the arrows drawn between corresponding objects in the two groups.

The process of matching is not limited in its use to determining the number of objects in a group nor to establishing the equality of two groups. Matching procedures can be and are used to aid in obtaining answers to several different questions, such as these: "Which of two groups is the larger (smaller)?" and "How much larger (smaller)
is one group than a second group?" The use of matching procedures in determining the answers to such questions will be elaborated upon in the section of Chapter VIII concerned with subtraction.

The Operations Associated with Counting

James and James\(^1\) define "to count" as follows: "To name a set of consecutive integers in order of their size, usually beginning with 1." The simple process of naming a set of consecutive integers with no reference to objects counted is sometimes called _rote counting_.

**Rational counting** is the operation of associating the number names—one, two, three, etc.—taken in serial order with the objects of a group. This operation is usually performed by pointing to, or looking at, each object of the group in turn as a number word is said or thought, the number words being taken in serial order starting with the word _one_.

The operation of counting may be used to obtain the answers to two different types of questions. The first question is that of "how many?" In this case, counting is used to determine the number of objects in a group. This operation may be termed _cardinal counting_, since its purpose is to ascertain the cardinal number of the group, that is, the number of objects in the group. The second type of question involves finding the relative position of any object in a series of objects with reference to a starting point. This operation may be called _ordinal counting_. Although only the use of cardinal

---

counting is applicable in this discussion of methods for determining the sizes of groups, in order to contrast them both counting operations will be discussed at this point.

It is not uncommon for young children to be confused in their understanding of the referents for the number words they use when counting. Children are usually taught to say the number words in order as they point in turn to each object in a group they are counting (Figure 10).

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\text{one two three four five six seven}

Fig. 10

Some children when asked to reproduce or show seven objects will point only to the one object with which they have associated the word \textit{seven}. If operational theory is adhered to, the two sets of operations employed for cardinal counting and ordinal counting respectively should be noticeably different so that the operations used will help children to remember why they are using counting.

\textbf{Cardinal counting.} -- The operations the child should perform in cardinal counting as outlined in the paragraph above may be portrayed as in Figure 11, when he is counting the objects in group B, one by one. At any one step in the counting sequence, group A contains all of the objects the child has counted up to that point; and group B at any one step in the counting sequence contains all of the uncounted objects up to that point.
At the beginning of the operations there is an unknown number of objects in group B. In Step 1 the child moves one object from group B to group A as he says, "One" or "I now have one." At the completion of this step, group A contains 1 object and group B contains the objects as yet uncounted. In Step 2 the child moves a second object from group B to group A as he says, "Two" or "I now have two," at the same time indicating with fingers or arms that the word two refers not only to the object being moved but also to the object that was there before. At the completion of this step, there are 2 objects in group A. This process continues until all of the objects originally in group B
have been transferred to group A. With the last such object transferred, the word _five_ is said.

Later, of course, the child will not move the objects as he counts them; and, in fact, he will count objects which he cannot move physically. If early experiences in cardinal counting proceed as outlined above, however, the child will come to associate the number-word used at any step with the total number of objects counted up to that point and not solely with the one object attended to at that step.

**Ordinal counting.**—In counting the series of objects in Figure 12 to determine the position of object "X," the proper ordinal number-words should be associated in turn with the objects, starting with the first object (here assumed to be the one to the extreme left) and terminating with the position occupied by the "X" in the series.

0 0 0 X 0
Number 1 Number 2 Number 3 Number 4
First Second Third Fourth

**Fig. 12**

A clear distinction should be made in the early stages between ordinal and cardinal counting. The sets of operations for the two are different; and if operational theory is adhered to, two different names should be given to the two sets, such as: (1) counting to find "which one" or "what place"; and (2) counting to find "how many."

It would seem advisable at first, then, for the child to make this
distinction by the number-words he uses. In ordinal counting, at first, he should say, "first, second, third, . . ." or "number one, number two, number three, . . ." In cardinal counting he should say, "I now have one, I now have two, . . ." or simply "one, two, . . ." In ordinal counting he should touch or point to the objects in turn as he repeats the ordinal number-words, so that each ordinal number-word refers to one, and only one, object. In cardinal counting, as he counts, he should move the objects into a new group, or indicate this movement by some gesture of the fingers or arms, so that at all times the number-word being used refers to the number of objects in the new group he is creating.

The Operations of Addition

Sutherland's Study of One-Step Problem Patterns in Arithmetic

In 1947 Ethel Sutherland, made a study to determine the patterns of verbal problems that can be solved by means of one step, that is, by the use of one of the four arithmetic processes—addition, subtraction, multiplication, and division. In order to make this determination, she examined four widely-used series of textbooks in arithmetic covering grades three through six. In her study, Sutherland classified 15,000 word problems appearing in the sixteen textbooks into various addition, subtraction, multiplication, and division

2. Ethel Sutherland, "One-Step Problem Patterns and Their Relation to Problem Solving in Arithmetic," Contributions to Education, No. 925 (New York: Teachers College, Columbia University, 1947).
patterns. Miss Sutherland's analysis of problem patterns in arithmetic is one of the most thorough and detailed analyses made to date. Her analysis will be used as a guide throughout the remainder of this chapter as various physical operations with groups are examined.

Sutherland appears to make her categories of the one-step problem patterns of arithmetic on several different bases. At some points she makes a distinction between two different problems because the two are operationally different; at other places she makes the distinction because the two problems are linguistically different or involve different social applications; and at still other points she appears to make a distinction because the two problems are symbolically solved by different methods.

Sutherland lists four different problem patterns for addition:

(1) patterns in which the phraseology helps to emphasize the idea of finding a sum; (2) patterns in which the phraseology does not contain any characteristic words or expressions, such as those in (1) above; (3) patterns in which the phraseology is that of buying and selling; (4) patterns in which the phraseology is similar to that used in certain subtraction problems. Sutherland, therefore, categorizes the problem patterns of addition only on a linguistic basis. The purpose of this section will be to analyze addition problems on the basis of the physical operations with objects necessary to give meaning to addition.

The Operations of Addition (No Use Made of the Idea of Place Value)

An example of addition may be worded as follows: If one brings together (or combines) a group of 2 counters and a group of 3 counters, what single number-word tells how many counters there will be altogether? The child starts with the two separate groups (Step 1, Figure 13).

Group A 0 0 0 0
Group B 0 0 0 0

Step 1

Fig. 13

He has two operations to perform. There is a physical operation of bringing the two groups together (Step 2). This operation is the referent for the spoken words "bring together" or "combine." The second operation is that of determining the number of counters in the new group (A 4 B).

A second problem in addition for young children may be worded as follows: If a group of 2 counters is added to a group of 3 counters, what single number-word tells how many counters there will be altogether? The physical operation of moving the group of 2 counters into conjunction with the group of 3 counters differs only slightly from the operation of moving two groups together into a new location as described above. This question, however, involves the use of the word add and provides a physical referent for it. Later the mathematical symbol 4 will have as referents such words as bring together.
combine, and add; and these words, in turn, will have physical operations as referents.

The word and is often used in addition problems and is, therefore, one of the referents for the sign +. The child may have the problem: What is the sum of 7 houses and 5 houses. And may imply that the 7 houses are in one grouping, the 5 houses are in another; and that it is physically impossible to bring the two groups together. Consequently, the word and may have as a referent "if these two groups were (or could) be brought together." As such, the word and should enter at a later stage of symbolization than such words as bring together, move together, combine, and add to. The child must imagine the combining in situations where and is used, or else he must use objects which can be combined physically, such as counters, bottle caps, beads, etc., to "stand for" those which cannot.

A child may add groups which total more than 10 or even groups which are larger than 10 prior to any knowledge on his part of place value or of grouping in tens if he knows: (1) the meaning of addition as a physical operation, and (2) how to count objects at least up to the total of the two groups. Thus, he may add group A of 8 objects and group B of 6 objects (Figure 14, Step 1) simply by moving the two groups together (Step 2) and counting the new group (A + B).

\[
\begin{align*}
A & : 00000000 & \quad (A + B) & : 000000 \\
B & : 000000 & & \quad 000000 \\
& \text{Step 1} & & \quad \text{Step 2}
\end{align*}
\]

Fig. 14
This he may do and give the single answer "fourteen" if he can count that many objects. It is conceivable that even though he uses the word fourteen he may be totally unaware that it means "four and ten" and that the number system provides for giving such answers in terms of "so many tens and so many ones."

**Successive Stages of Symbolization**

As the child goes from (a) physical operation of bringing together, to (b) spoken words associated with the operation, to (c) written words, to (d) the symbol 4, he goes through stages which may be termed successive stages of symbolization.

An illustration of successive stages of symbolization is given here. The child may have the problem of finding out how many chairs there are altogether in a group of 2 chairs and a group of 3 chairs. In the first stage, he would use real chairs to solve the problem. In the second stage, he would use toy chairs "to stand for," as symbols for, real chairs. In the third stage, he may use counters such as checkers, blocks, or buttons, which do not look like chairs but which he has agreed to let "stand for," or symbolize, chairs.

In the fourth stage of symbolizing this situation, the child may use pictures of chairs (hh hhh) and later other marks on paper, such as tally marks (/// ///) or circles (00 000), to "stand for" chairs. Thus, the bringing together of two groups of chairs may be pictured in two steps (Figure 15).

```
  hh     hhh     hhhhh
Step 1  Step 2
```

*Fig. 15*
In Step 1 the two separate groups are shown. In Step 2 the two groups are shown after they have been brought into a new group of 5 chairs.

The first use of paper and pencil, then, would not make use of written words or arithmetical symbols. The child would symbolize the chairs, the checkers, the blocks, or the counters by making pictures of them. Thus, paper-and-pencil drawings may be used to symbolize three-dimensional objects; therefore, solutions with paper-and-pencil drawings represent a later stage of symbolization than do solutions with three-dimensional objects.

As the child proceeds in the stages of symbolization, the problem discussed above may take the following forms:

1. Three chairs and (or combined with) (or plus) two chairs make (are equal to) (are) five chairs, (usually given in verbal form only).

2. 3 chairs plus 2 chairs are 5 chairs.

3. 3 chairs + 2 chairs = 5 chairs.

4. $3 + 2 = 5$.

What is meant by a regressive chain in operational theory as discussed in Chapter VI is that for its meaning each new symbol or sentence encountered may be referred to preceding stages of symbolization or to real objects and physical operations. If the person seeking the meaning of the new symbol or sentence knows the meaning of these referents, then he knows the meaning of the new symbol or sentence. If necessary, more than one step in the regression is made. This regression continues "backwards" through the stages of
symbolization until the person arrives at a stage in which he does know the meanings of the symbols and sentences involved, either in terms of other words or symbols or in terms of actual or imaginary operations with things. It seems reasonable to assume that if arithmetic is built up carefully and meaningfully through the various stages of symbolization, as each new stage is introduced the child will not have to proceed more than one or two steps down the regressive route before he knows the meanings of the new symbols used.

The Operations Involved in Adding Two-Place Numbers (No Carrying)

A child may add 15 counters and 15 counters if he knows (1) the meaning of addition as a combining operation, and (2) how to count objects at least up to the total of the two groups. This addition may be accomplished prior to any knowledge of place value or grouping in tens, as has been previously pointed out. The essentially new ideas entering at this point, however, are those of grouping in tens and the use of place value.

In beginning the concept of place value, the child will at first express each of the two groups as so many tens and so many ones with the counters (Step 1, Figure 16). This operation simply regroups the original group into two or more subgroups, one of the subgroups containing less than 10 objects and the other subgroups containing 10 objects each. To find the total, which the child will also express as so many tens and so many ones, he will combine the groups of ones and combine the groups of tens, keeping, however, each ten separate
from the others so that he may recognize or count easily the number
of groups of ten (Step 2, Figure 16).

```
00000  000  00000  00000
00000  00  00000  000
00000  000  00000
00000  00000
```

Step 1  Step 2

Fig. 16

The groups of tens and the ones need not be arranged in any
particular pattern. The linear pattern of fives is used here simply
for convenience. The child will give his answer as 2 tens and 8 ones,
or 28. A verbalization of these operations might be as follows:
When one ten and five ones are combined with one ten and three ones,
the total is two tens and eight ones. It will be noted in the dis-
cussion so far that the use of positional value has not been necessary.
Fifteen objects may be correctly symbolized in various ways and still
make use of grouping in tens. The next step is for the child to gain
an operational meaning for substituting a quantity for its equal.

The Operation of Substituting or Exchanging

The child is now at the stage where he should learn to represent
a group of objects by a single object. The operation of substituting
a quantity for its equal is initiated through permitting one object
or mark to "stand for," "represent," a group of objects. A black
counter may, by agreement, be allowed to "stand for" a group of 5
white counters. If the child has 15 white counters, he may arrange
them in groups of 5 each as follows: White counters 0 0 0 0 0 0 0 0 0 0. He may then "substitute" or "exchange" one black counter for each group of 5 white counters as follows: Black counters 0 0 0. The pupil should have experiences in representing various-sized groups of objects with single objects, in exchanging or substituting one object for a group of n objects, and in exchanging or substituting a group of n objects for a single object.

The ultimate objective here is to establish an operational meaning for "substituting" or "exchanging" one object for a group of 10 objects. Physically, this is accomplished by removing a group of 10 objects and replacing the entire group with a single object which "stands for" the group of ten. If the new symbolism is to be meaningful, two requirements must be met: (1) the new symbol, the substitution object which will stand for 10 of the original objects, must be different in some way from the objects for which it is substituted, else people will not be able to distinguish it from the objects which have a value of one; (2) everybody must know and agree on the value or meaning of the new symbol, that is, all must know it refers to a group of ten of the original objects.

Condition (1) above could be met by making the substitution object different in size, color, or some other characteristic from the original objects. Thus, 15 may be portrayed as 0 0 0 0, where the large counter stands for 10 and the small counters for 1 each; or as 0 0 0 0, in which the black counter stands for
10 and the white counters for 1 each. The order of arrangement of the ten-counter and the unit counters is unimportant, since the differences in the sizes or colors are distinguishable regardless of where the counters are. One solution of the problem, $15 + 13$, then may be portrayed as in Figure 17.

$$\begin{array}{c}
\text{Combine the two groups.} \\
+ \\
\text{Fig. 17}
\end{array}$$

It would seem advisable, however, to let the difference in the values of the counters depend on position rather than on physical characteristics, since that is the essence of the meaning of two- or more-figure numerals. A device employing place value, such as a place-value chart or the open abacus portrayed in Figure 18, is helpful at this point. The open abacus is a device providing for a units column, a tens column, a hundreds column, etc. As many counters as desired may be placed in any column. A counter in the units column represents one unit; a counter in the tens column represents one ten; etc. The open abacus may be symbolized by a paper-and-pencil open abacus. This first of many similar illustrations in this chapter is intended to show various steps in the physical solutions.
of arithmetic examples with three-dimensional open abaci. In Step 1, Figure 18, the 15 is portrayed as 1 in the tens column and 5 in the ones column. The 13 is portrayed in the same way. The physical operation is one of sliding the 2 markers in the tens column and the 2 groups of objects in the ones column together. Step 2, Figure 18, portrays the result of the combining operations.

<table>
<thead>
<tr>
<th>tens</th>
<th>ones</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>00000</td>
</tr>
<tr>
<td>0</td>
<td>000</td>
</tr>
</tbody>
</table>

Step 1

<table>
<thead>
<tr>
<th>tens</th>
<th>ones</th>
</tr>
</thead>
<tbody>
<tr>
<td>00</td>
<td>00000</td>
</tr>
<tr>
<td>0000</td>
<td></td>
</tr>
</tbody>
</table>

Step 2

Fig. 18

The Operations Involved in Adding with "Carrying"

An example involving "carrying" is one in which the pupil is to find the total when a group of 17 counters is combined with a group of 15 counters. When the answer to this question is found by means of counters, checkers, or buttons, accompanied by verbal symbols, it takes the following form (the correspondences with both words and one form of expanded algorithm are shown):

Step 1. The two groups divided into subgroups are displayed (Figure 19).
Fig. 19

Step 2. The groups of tens are combined and the groups of ones are combined, the two tens groups being kept as distinct subgroups so that the number of groups of tens may be easily determined (Figure 20).

Fig. 20

Step 3. This step may be called the "grouping in tens" operation. Ten of the counters from the group of 12 are moved out as a separate group. In other words, the 12 counters are regrouped as 10 counters and 2 counters, since the answer is to be given as so many tens and so many ones. (Figure 21).
On a place-value device such as an open abacus, the steps may be displayed as in Figure 22. In Step 1 the two numbers, 17 and 15, are represented. In Step 2 the counters representing 15 are moved upward toward those representing 17.

<table>
<thead>
<tr>
<th>Tens</th>
<th>Ones</th>
<th>Tens</th>
<th>Ones</th>
<th>Tens</th>
<th>Ones</th>
<th>Tens</th>
<th>Ones</th>
<th>Tens</th>
<th>Ones</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0000</td>
<td>0</td>
<td>0000</td>
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<td>0000</td>
</tr>
</tbody>
</table>

Step 1 Step 2 Step 3 Step 4 Step 5

\[ \begin{array}{c}
17 \\
+15 \\
12 \\
2 \\
32 \\
\end{array} \]

Fig. 22

The result is shown in Step 3. The symbol $12$ in the algorithm refers to the 12 counters in the ones column, and the symbol $2$ refers to the 2 counters in the tens column. In Step 4, 10 of the counters in the ones column are exchanged for 1 counter in the tens column by pushing 10 ones down and replacing them with 1 ten. The result is portrayed in Step 5. In the final "symbolic" answer, the $3$ refers to the 3 counters in the tens column and the $2$ refers to the 2 counters in the ones column. "Carrying" in hundreds, thousands, etc., is essentially the same.

**Summary of Addition Operations**

There are three kinds of operations which appear in problems commonly thought of as addition problems, and which constitute the meaning of addition.

1. The first of these is a physical operation of adding to, combining, or bringing together counters of the same denomination or value. Thus, the counters representing ones are combined, those representing tens are combined, etc.

2. An operation is used to determine the number-word or numeral associated with the total, that is, the answer. Such operations include counting, matching, subitizing, and grouping.

3. The operation of substituting or exchanging becomes of considerable importance in addition when the so-called process of carrying is involved. This operation is a fundamental one in mathematics, and early in arithmetic the child should have many opportunities to substitute a quantity for its equal.
The Operations for Multiplication

Definitions for the Multiplication of Whole Numbers

The multiplication of whole numbers is generally considered to be closely related to addition. In fact, multiplication is often thought of as a quicker way of finding the total when a number of equal groups are combined than is addition. For example, Hickerson writes the following:

It has been learned already that adding is a quicker way than counting to determine the total number of objects in a group. Now it can be learned that certain kinds of addition can be accomplished much more quickly by using the process called multiplication. When the groups to be combined are of the same size, or contain the same number of items, multiplication can be used instead of addition. 5

Clark and Eads define the multiplication of whole numbers in much the same way as does Hickerson, but they point out more clearly than he does what is known and what the problem is:

In multiplication we put together groups of the same size; it is a special kind of addition. The size of each of the equal groups is known; the problem is to find the size of the resulting total group (the product). 6

Wheat is even more explicit. In pointing out the differences between addition and multiplication, he maintains that because of noticeable differences, multiplication is not short-cut addition.


He writes thus:

The quantities we add are seldom equal, and in adding we do not give explicit attention to the number of the quantities. The quantities we multiply are always equal, and in multiplying we must always give attention to the number of the quantities. Even when we add equal quantities, the procedure is not the same as when we multiply. Multiplying is not short-cut adding, though for some additions there is an apparent similarity. 7

For example, Wheat points out that in the addition 6 + 6 + 6, the person doing the adding does not have to be concerned with how many 6's there are; but in the multiplication 3 x 6 the specifically tells him how many 6's there are.

Sutherland's Problem Situations for Multiplication

Sutherland gives five major groups of multiplication patterns and a total of eight in all. These patterns, in abridged form, are included here:

1. The pattern requiring the total amount to be found.
   a. Given the number in one group and the number of groups; find the total.
   b. Given the cost of one item and the number of items; find the total cost.

2. The pattern requiring the total distance.
   a. Given rate and time; find the total distance.
   b. Given mileage per gallon and number of gallons; find the total distance.

3. The pattern requiring a fractional part of a whole quantity.
   a. When the fraction is a unit fraction (1/2, 1/3, 1/4, etc.).
   b. When the numerator of the fraction is greater than 1
      (2/3, 3/4, 5/6, etc.).

4. The pattern requiring a per cent of any number.

5. The pattern requiring the whole quantity when a part of it
   is given. 3

It is evident that Sutherland's classification was made largely on
the basis of the social situations involved and the vocabularies
associated with those situations.

Problem Situations for Multiplication

The following problem is an example of the multiplication situa-
tion. If one combines 3 groups, each of which has 2 objects in it,
how many will there be in all? According to the requirements stated
in this situation, the child has two operations to perform. First
he must combine (bring together) the groups A, B, and C (Figure 23).

\[ \text{Group A} \quad \text{Group B} \quad \text{Group C} \]
\[ \quad \quad 00 \quad 00 \quad 00 \]
\[ \text{Group (A + B + C)} \quad 000000 \]

Fig. 23

The second operation is that of determining the number of objects in
the new group \((A + B + C)\). This set of operations is identical with
those for addition. This is true, of course, because the child is

directed to do the same two things as he does in an addition situation. He is directed to combine the groups and to determine the number of objects in the new or combination group.

A multiplication question may be asked in various forms. Some examples with their appropriate operational solutions follow:

1. If 2 pieces of a certain kind of candy cost 1 cent, how many pieces of candy can be bought for 3 cents? Although it is advisable for the child to set up the correspondence indicated in Step 1 of Figure 24, when he first encounters the problem, the essential operations are combining the pairs of pieces of candy into a new group of six (Step 2) and then determining the total.

\[
\begin{array}{c}
\text{Pennies} \\
\text{Candy}
\end{array}
\]

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

Step 1

\[
\begin{array}{c}
\text{Candy}
\end{array}
\]

000000

Step 2

Fig. 24

2. If a man can walk at an average rate of speed of 4 miles an hour, how far can he walk in 2 hours? This situation may be portrayed as in Figure 25.
Again the operation is that of combining the distances traveled in the two-hours' time.

3. What is the area of a rectangle 3 inches long and 2 inches wide? The statement is commonly made that inches multiplied by inches produces square inches. Inches are not multiplied by inches; numbers are multiplied by numbers, and the meaning of the numbers must be portrayed. Since this example is concerned with area, the units to be employed are the commonly used square units. The problem is to determine the number of square inches in the rectangle, and the solution may be portrayed in such a way as to show the use of multiplication to determine the number of square inches. If the rectangle is divided into square inches, it may be seen either as 2 rows of 3 square inches each (Figure 26) or as 3 columns of 2 square inches each (Figure 27). In either case, the problem is to determine the total number of square inches altogether.
In all of the three problems discussed, two elements are present:

1. there is a given number of groups, lengths, rows, or columns;
2. there is given an equal number of objects or units in each of the groups, lengths, rows, or columns. The child is to determine the number of objects or units in the total. The differences among the three problems are in their applications, not in their operational situations.

Thus, the multiplication of whole numbers has one meaning operationally. The meaning is nearly identical with that of addition. Wheat, referred to earlier, appears to be correct in his analysis of a difference between addition and multiplication in terms of how the data are given. In fact, that is the major difference. The child must determine the number of equal groups and the number of objects in each of the equal groups. In an addition situation, the child need not be, and often is not, concerned with the number of groups; but since the groups may differ in size, he must determine the number of objects in each separate group. After the proper determination
has been made, the idea in both addition and multiplication, operationally speaking, is that the groups be combined and the size of the combination group be determined.

The Multiplication Algorithm and "Carrying"

This section concerning multiplication will be concluded with a discussion of physicalization for a multiplication algorithm where "carrying is involved. The example $25 \times 13$ will be used. Three twenty-fives means 3 twenties and 3 fives. If a group of 10 is portrayed by a large circle (to preserve space) and the ones are portrayed by small circles, 3 twenty-fives may be shown as in Figure 28, accompanied by an expanded algorithm.

\[
\begin{array}{c}
0 \quad 0 \quad 00000 \\
0 \quad 0 \quad 00000 \\
0 \quad 0 \quad 00000
\end{array}
\]
\[
\begin{array}{c}
2 \ (10) \times 5 \\
3
\end{array}
\]
\[
\frac{6 \ (10) \times 15}
\]

Fig. 28

If the 15 ones are regrouped as 1 ten and 5 ones, the picture then takes the pattern of Figure 29. The partial product in the algorithm is now shown as 7 tens and 5 ones.

\[
\begin{array}{c}
0 \quad 0 \quad 00000 \\
0 \quad 0 \quad 00000 \\
0 \quad 0 \\
0
\end{array}
\]
\[
\begin{array}{c}
2 \ (10) \times 5 \\
3
\end{array}
\]
\[
\frac{7 \ (10) \times 5}
\]

Fig. 29
The next step is to portray 10 fives and 10 twenties (Figure 30).

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\quad 0000
\]

\[
2 \ (10) \times 5
\]

\[
\frac{10}{20} \ (10) \times 50
\]

Fig. 30

The \(2 \ (10) + 5\) refers to the number of tens and the number of ones in each group, and the \(10\) refers to the number of groups. The \(50\) in the partial product refers to the 50 singles and the \(20\) refers to the 20 groups of tens. Exchanging 1 ten for 10 ones and 1 hundred (represented here by the symbol \(C\)) for 10 tens, the arrangement appears as in Figure 31.

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\quad 2 \ (10) + 5
\]

\[
\frac{10}{2} \ (100) + 5 \ (10)
\]

Fig. 31

The \(5\) refers to 5 tens and the \(2\) to 2 hundreds. Combining this
product with the product of 3 twenty-fives, the picture appears as in Figure 32.

\[
\begin{array}{ccc}
25 & \times & 13 \\
\hline
75 & \text{or} & 2 \times 10 + 5 \\
25 & \times & 13 \\
325 & \text{or} & 2 \times 100 + 5 \\
\end{array}
\]

Fig. 32

The 3 twenty-fives and the 10 twenty-fives are now combined. This step is the one of adding the partial products in the algorithm. After exchanging 10 tens for 1 hundred the picture appears as in Figure 33.

\[
\begin{array}{ccc}
25 & \times & 13 \\
\hline
75 & \text{or} & 2 \times 10 + 5 \\
25 & \times & 13 \\
325 & \text{or} & 2 \times 100 + 5 \\
\end{array}
\]

Fig. 33

The solution of the example 25 \times 13 may be portrayed on an open abacus. The actual movement of the counters can be shown only on a three-dimensional abacus or with motion pictures, but the various steps may be shown as in Figure 34.
In the first step, 3 twenty-fives are shown. Step 2 portrays the abacus after the 3 twenty-fives have been moved together and 1 ten has been substituted for 10 ones. The abacus in Step 2 portrays the first partial product 75 which appears in the algorithm. Step 3 portrays 10 fives in addition to the 2 twenty-fives after 5 tens have been substituted for 10 fives (if the abacus has only a limited number of beads, such as 20, on it, the substitutions will have to be made successively each time 10 beads appear on the units rod). Step 4 portrays 10 twenties after 2 hundreds have been substituted for 20 tens (again, if the abacus has only 20 beads, the substitutions must be made successively), and after one hundred has been substituted for 10 of the 12 counters already on the tens rod. The final answer, then, is 325.
Summary of Multiplication

The difference between addition and multiplication rests in the way in which the data appear. In a multiplication example, the data include the number of equal groups and the number of objects in each of the equal groups. In an addition example, the child needs to know the sizes of the various groups. The groups need not be, and usually are not, the same size, and they may be added successfully without attention being given to the number of groups involved. After the data are known, the operations in both addition and multiplication are those of bringing together the separate groups and determining the total number of objects in the new grouping.

The discussion of the operational approach to teaching arithmetic will be continued in Chapter VIII. A composite summary of Chapters VII and VIII will be included at the end of Chapter VIII.
Chapter VIII

THE OPERATIONAL APPROACH TO TEACHING

ARITHMETIC IN THE PRIMARY GRADES: II

This chapter is a continuation of the discussion of the operational approach to teaching arithmetic and will deal with the teaching of subtraction and division.

The Operations of Subtraction

Analyses of Subtraction Situations

The analysis of subtraction situations is somewhat more complex than is the analysis of addition or of multiplication situations because of the number of different types of problems which can be solved symbolically by subtraction, as well as the number of different ways in which the problem situations may be stated in words. Authorities in the teaching of arithmetic are cognizant of this multiplicity of uses for the arithmetical process of subtraction; and, as a result, generally give attention in their writing to such an analysis. Some examples are given here in order to indicate the variety of opinions which exist among various authorities regarding the analysis of subtraction situations.

Clark and Eads.—Clark and Eads believe that with beginning school children subtraction has one of the following three meanings:
1. The "how-many-are-left" meaning. In these situations, the child sees subtraction as "taking away" and the question involved is "how many are left?"

2. The "comparison" or "difference" meaning. In these situations, there is a difference in the numbers of objects in two groups. The question asked is "how many more, or how many less, are there in one group than in the other?"

3. The "how-many-more-are-needed" meaning. In these situations, a number of objects is desired. The question is "how many more are needed?"

Hickerson.— Hickerson is of the opinion that there are three fundamentally different concepts of subtraction. He lists these as follows:

(1) How much is left when I take one number from another? \( (8 - 6 = ?) \)
(2) How many must I add to one number to make another number? \( (6 + 1 = 8) \)
(3) How much must I take away from one number to make another? \( (8 - 1 = 6) \)

Brueckner and Grossnickle.— Brueckner and Grossnickle find four different kinds of situations which give rise to subtraction:

1. A given amount is decreased and the result of the subtraction shows the remainder.

2. Two amounts are compared and the result of the subtraction shows the difference between the two amounts.


3. A given number of objects is on hand and the result of the subtraction tells how many more objects must be added to those on hand to make a required larger group.

4. The sum of two numbers and one of them are known and the result of the subtraction is the other of the two component parts of the given sum.³

Wheat.—Wheat maintains that there are five subtraction situations which the pupils must study. His list follows in abridged form:

1. The remainder or "take-away" idea. A given amount is decreased, and the purpose is to find how much is left.

2. The "how-much-is-gone" idea. A given amount has been decreased and the amount remaining is known. The purpose is to determine the amount of the decrease.

3. The "how-much-more" idea. A given amount is on hand and a larger amount is desired. The purpose is to determine how much more is needed to make the desired amount.

4. The comparison or difference idea. Two unequal amounts are compared to determine how much more or how much less one amount is than the other.

5. The other number idea. A sum or total amount and one part are given. The purpose is to determine the other part.⁴


Hartung and Van Engen.---Hartung and Van Engen list three "action-process" types of subtraction. The word action-process, as used by these authors refers to the method they advocate in teaching arithmetic, that of associating a physical action in solving an arithmetic problem with the arithmetic process by which the answer to the problem is found symbolically. The Van Engen-Hartung method is an operational approach to the teaching of arithmetic. Their materials and methods will be discussed more thoroughly in Chapters IX and X. Their three types of subtraction are listed here in abbreviated form:

1. "Additive-subtraction." A given amount is desired and a smaller amount is on hand. How much more is needed?

2. "Subtractive-subtraction." A part of an amount is taken away. What is the remainder?

3. "Comparative-subtraction." Two quantities are compared. What is the difference between the two?5

It is evident that the authorities referred to above have made their analyses on different bases. It is possible to make classifications of subtraction situations on the basis of language used, the algorithm, or combinations of these. Hickerson's analysis, for example, appears to be made on the basis of which element is missing in the algorithm. In the next section, the more comprehensive analysis of subtraction situations made by Sutherland will be reviewed.

Sutherland.— In order to examine subtraction situations on an operational basis, the analysis made by Ethel Sutherland⁶ in 1947 will be used as a guide. Her study of 2,998 subtraction problems represents one of the most thorough and detailed analyses of subtraction situations made to date. Miss Sutherland classified "subtraction" problems into four major groups of patterns with a total of ten one-step patterns. Her classifications, with examples, follow in shortened form:

1. The "remainder" or "how-much-is-left" idea.
   a. A given amount is decreased; find how much is left or remains. Example: Jack had 8 pieces of candy. He gave 3 pieces to Joel. How many pieces of candy did he have left?
   b. A given amount of money has been decreased by a purchase; find how much change is received. Example: Jim bought a pencil for 4¢. He gave the clerk a dime. How much change did he receive?
   c. A given amount has been decreased and the amount left is known; find the amount taken away, spent, sold, etc. Example: Jim gave some of his 8 marbles away. He has 5 left. How many did he give away?

2. The "how-much-more" or "building-up" idea. A given amount

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⁶ Ethel Sutherland, "One-Step Problem Patterns and Their Relation to Problem Solving in Arithmetic," Contributions to Education, No. 925 (New York: Teachers College, Columbia University, 1947), pp. 25-34.
is on hand; find how much more is needed to make a desired larger amount. Example: Jack wants to color 3 pictures in his book. He has already colored 5 of them. How many more does he have to color?

3. Problems involving the "comparison" or "difference" idea.

a. Of two unequal amounts given, how much more or how much less is one than the other? Example: Mary has 6 dolls and Jane has 3. How many more dolls has Mary than Jane?

b. Of two unequal amounts given, how much larger, longer, taller, older is one than the other? Example: Jimmy is 8 years old and Mary is 6 years old. How much older than Mary is Jimmy?

c. Of two unequal amounts given, find the difference between them. Example: Jack weighs 60 pounds and Mary weighs 45 pounds. What is the difference between their weights?

d. Of two unequal amounts given, find the difference where some comparison word other than difference is used. Example: World War II ended in 1946. How many years ago was that?

e. Two unequal amounts have been compared. One of the amounts and the difference between them are known. Find the other amount. [Although Sutherland does not say so, if this is to be a subtraction situation, it must be the larger amount which is known] Example: Joe has 3
marbles. That is 2 more marbles than Jim has. How many marbles does Jim have?

4. The pattern involving the "separation-into-parts" idea.
The sum and one part are given; find the other part. Example:
There are 27 children in Miss Jones' room. Fifteen of them are girls. How many are boys?

The question arises: do Sutherland's ten one-step subtraction patterns differ from each other operationally or are some of them operationally the same? Her ten one-step subtraction patterns will be considered in the section which follows. An analysis of each of these patterns will be shown in terms of physical operations, with the aim of detecting those which are operationally different and those which are operationally the same.

The Remainder or How-Much-Is-Left Idea

A given amount is decreased; find how much is left or remains.—A problem of this type may be worded as follows: If from a group of 7 objects, 4 are taken away (removed, lost, etc.) how many objects remain (are left)?

0 0 0 0 0 0 0 0 0 0 0 0
Step 1
Step 2
Step 3
Fig. 35

The child starts with a group of 7 counters (Figure 35). He has

7. Ibid., pp. 13-17
two operations to perform. He removes 4 counters from the group, either one at a time or as a group. He then determines the number of counters left. In this situation 7 or the word seven refers to the number of counters in the original group (Step 1). The words take away, remove, etc., have as referents the physical activities involved in removing the 4 counters that are enclosed in the rectangle (Step 2). The word three or 3 in the answer refers to the number of counters remaining (Step 3). The first physical operation in this problem situation is that of "taking away" or "removing" as portrayed in Step 2. After the child has had experiences in actually "taking away" a group of objects from a larger group, he may simply indicate the "taking away" by some such action as covering the objects to be removed with his hand; crossing them out if they appear on paper; or dividing the "take away" group from the "remainder" group in some way, such as by a line 0 0 0 / 0 0 0 0 or by circling the take-away group 0 0 0 0 0. The specific number question he must answer following the taking away operation is this: How many are left?

A given amount of money has been decreased by a purchase; find how much change is received. -- It is difficult to see why Sutherland distinguished this pattern from her first pattern. In fact, she recognized the close relationship between the two patterns and wrote that the reason for classifying them separately "is that retail buying is a social situation that concerns all of us." It should be pointed out here, however, that this problem situation is seldom solved as a

8. Ibid., p. 22.
subtraction problem by the person making change. It is common to count ahead from the price of the purchase to the denomination tendered in payment. Thus, if an item costs 67 cents and a one-dollar bill is given in payment, the person making the change may give back to the buyer three pennies, one nickel, and one quarter. As he does so, he would count ahead as follows: "sixty-seven, sixty-eight, sixty-nine, seventy, seventy-five, one dollar," starting to give the coins as he says the word "sixty-eight." If subtraction is used, in this example, it is usually used to check the counting.

A given amount has been decreased and the amount left is known; find the amount taken away. — The following is a problem involving the amount taken away: If several counters are removed from a group of 7 counters, leaving 4, how many counters are removed? The set of operations may proceed as in Figure 36. A group of 7 counters is on hand (Step 1). The child picks out 4, either by counting or by subitizing, which are to remain. He may differentiate these counters in some way from the others (Step 2). The other counters are then removed (Step 3). The number of counters removed is determined (Step 4).

When this problem situation is presented and solved in the manner outlined above, it is equivalent operationally to the problem
in which the child takes away part of a group and determines the remainder. This is so, because the child may actually remove from consideration the counters he is to keep and determine the numbers of counters remaining. This solution has the merit that in it the child can see the entire group with which he starts; he can see the group he is to keep; and he can see the group to be taken away.

Some authorities, however, point out that this problem may be, and usually is, presented in a different way. Such a presentation may be worded: If there are now remaining only 4 objects of a group which originally contained 7 objects, how many objects have been removed? In this situation, the child starts with only the group of 4 objects which remains. An operational solution in this case would proceed as follows: The child increases the number of objects up to 7, the size of the original group, as he counts, "four (number of objects he has), five, six, seven." Then he determines the number of objects added, which tells him the number of objects which had been removed.

The latter set of operations is considerably different from the set of operations used in the first solution, and is the same as the set of operations to be described later when Sutherland's category of "the building-up" situation is analyzed. Consequently, a further discussion of this solution and the way in which it may be transformed into a take-away situation will be deferred until that point. It is important to notice that the second operational solution presented here involved an operation of "adding to" or "building up," in spite
of the fact that the problem is couched in the language of take-away situations. This problem situation is an example of one which may be solved operationally in a manner which does not duplicate what happened originally to the objects in the group. Therefore, it is not correct to classify this problem exclusively as a take-away problem as Sutherland does.

Summary of remainder patterns.-- It appears, then, that the essential physical operations involved in the problem situations of Sutherland's first major pattern are those of "taking-away" part of a group from a whole group and determining the number of objects remaining, and, thus, these operations constitute the meaning of the take-away subtraction. The first and third patterns differ only in the known group, the "remainder" group or the "take-away" group. The problem of the third pattern, however, may be solved operationally by a "building-up" operation, and consequently should not be classified exclusively as a take-away, remainder situation. The second pattern is essentially the same as the first one, unless it is solved by the use of the "counting-ahead" operation commonly used in making change.

Problems Involving the Comparison or Difference Idea

This major pattern follows the "how-many-are-needed" or "building-up" idea in Sutherland's analysis. The order of these two patterns is reversed here to show the relationships between them.
Of two unequal amounts given, find how much more or how much less one is than the other. — The following problem involved the comparison idea. If group A which contains 3 counters is compared with group B which contains 7 counters, how many more counters are there in group B than in group A? Each counter of group A is matched with one and only one counter of group B (Figure 37). The number of unmatched counters in group B, here enclosed in a rectangle, is to be determined (Step 1).

![Diagram](#)

Fig. 37

The 3 counters of group B which have been matched with the counters of group A may be "taken away," "removed," from any consideration when the number of unmatched counters still "remaining" in group B is determined. To emphasize the relationship here to the operations involved in finding the remainder, it is advisable at first to remove physically all of the matched counters (Step 2).

The same problem when it involves the "how-many-fewer" question is as follows: How many fewer counters has group A than group B? Each counter of group A is matched with one and only one counter of group B. The number of counters which would be needed to match the remaining unmatched counters in group B is determined (Figure 38).
At first appearance, this problem may seem to be identical with the problem of "how-many-more"; and, in fact, its relation to that problem is close and is readily shown. Closer analysis, however, reveals a difference which makes this situation somewhat more difficult operationally. The problem here, evidently, is to determine a number of counters not, in fact, present; specifically the number of counters which would be necessary to fill the empty rectangle so that group A would have the same number of counters as group B. Operationally, the solution to this problem is preliminary to a consideration of the problem of "how-many-more-are-needed," to be discussed later.

The difficulty may be surmounted by importing counters to add to group A until a one-to-one matching can be set up between all of the counters of group A and all the counters of group B. Since the number of counters imported is precisely the same as the number of unmatched counters in group B, a determination of the number of unmatched counters suffices to answer the question. Thus, the set of operations here is shown to be equivalent to the set of operations for determining how many more objects there are in one group than another, which in turn was shown to be transformable into the set of operations for take-away situations.
Of two unequal amounts given, find how much longer, larger, taller, older, one is than the other.-- A problem of this type is as follows: If rod A is 7 inches long and rod B is 3 inches long, how much longer is rod A than rod B? Physically, the operations here appear to be equivalent to those described for the previous pattern. The child can match the 3 inches on rod B with 3 inches on rod A. This may be done either by matching inch for inch or by simply laying rod B alongside rod A in such a way that an end of rod A coincides with an end of rod B (Figure 39). Rod A is longer than rod B by that amount of rod A unmatched. Similarly rod B is shorter than rod A by the same amount.

![Diagram](attachment:image.png)

**Fig. 39**

This pattern does not appear to be different operationally from Sutherland's first comparison one, since the set of operations involves a matching and a determination of the size of the unmatched portion. The main difference is in the use of words such as longer, longer, taller, and older, words commonly used in comparing continuous quantities. Since the child must think about and describe the situation here in terms of definite measuring units, which he may consider as discrete in the same sense as counters are, it does not appear that this situation is operationally different from the same kind of problem situation involving discrete objects.
Of two unequal amounts given, find the difference between them.—

Sutherland indicates that the only difference between this pattern and the comparison situation is in the use of the technical word difference. She is of the opinion that this word is difficult for young children, and its use should be deferred until the fourth grade. Understandably, the teacher must provide for physical referents for the word difference. However, the operations for determining the difference are not at variance with those of her first comparison pattern.

It is possible, of course, that the two quantities being compared may be the same size, and then the difference would be zero. Such situations should be included in difference problems for young children. In fact, questions of "how many more?" and "how many fewer?" ought to be asked when the two groups being compared have the same number of objects. The child will be led to answer that one group is neither larger, nor smaller, than the other. Equal-sized groups have previously been discussed in the section on matching in Chapter VII.

Of two unequal amounts given, find the difference where some comparison word other than "difference" is used.— Since the basis for distinction for this pattern is that of language used rather than operations involved, the same comments apply to it as to the last pattern discussed.

Two unequal amounts have been compared and the larger amount and the difference are known; find the smaller amount. — Group B, which contains 7 objects, has 4 more objects than group A. How many objects are there in group A? Group A and group B have been matched, and there are found to be 4 more objects in group B than in group A; that is, attention is directed to the difference in the two groups (Figure 40).

The difference of 4 objects is enclosed in a rectangle. Attention is then directed to group A, the unknown group. Since the objects of group A have been matched one-to-one with those objects of group B which are not enclosed in the rectangle, it follows that there are just as many objects in group A as there are in the part of group B outside the rectangle. Hence, it is sufficient to determine the number of counters in group B which are outside the rectangle, that is, the remainder of group B when the group of 4 objects is known.

Sutherland classified this problem situation as a comparison situation. Operationally it appears as a converse of comparison situations. In a comparison situation the child knows the sizes of both groups, he performs the matching operation, and the problem is to determine how many more (or fewer) objects one group has than
another. In this converse situation, the child knows the size of the larger group and how much larger this group is than another. The matching has already been performed, and the child is to determine the size of the smaller group. Therefore, this situation should not be classified operationally as a comparison problem. It appears to be more closely related operationally to the situation in which the size of a whole group and the size of one part are known and the problem is to find the size of the other part. It is interesting to note that the other converse to comparison situations is never a subtraction problem. Thus, if the size of the smaller group and the difference are known, then one must "add" to find the size of the larger group.

Summary of comparison patterns. — Four of the five one-step patterns listed by Sutherland under the major heading of comparison or difference have one physical operation in common. This operation is that of making a one-to-one matching of the objects of the smaller group with the same number of objects in the larger group. There are two problem situations in this group which are not classified separately by Sutherland, but which do differ operationally in the last step. The child should have experiences with each of them. The two are these: (1) How many more objects are there in one group than in another? and (2) How many fewer objects are there in one group than another? The problem of finding the size of the smaller of the two groups when the size of the larger group and the difference between the two groups are known is operationally a converse of comparison situations and should be so classified.
The "How-Much-More" or "Building-Up" Idea

A given amount is on hand; find how much more is needed to make a desired larger amount. A question of this type may be worded as follows: If 7 objects are needed and there are 4 objects on hand, how many more are needed? The difficulties involved in this problem are two in number: (1) the group whose size is to be determined is not present, and (2) there is no comparison group present. One solution to the problem is as follows: Group A contains 4 objects (Figure 41). The child increases the number of objects up to a group of 7 as he counts, "four (number of objects on hand), five, six, seven" (Step 2, indicated here as the group in the rectangle). The number of counters is then determined.

\[\begin{array}{c}
\text{Step 1} \\
\text{Step 2}
\end{array}\]

\[\begin{array}{c}
0000 \\
0000[000]
\end{array}\]

\text{Fig. 41}

It is readily recognized that this solution for the problem involves, in part, the operation of "adding to." There is nothing in the operation of "adding to" to suggest that the situation here is related to other problem situations which the child will perform later by use of the arithmetical process of subtraction. If the relationship to other subtraction situations is to be made clear, this problem will have to be handled in a different manner.

In a second operational solution, the child may start with 7 counters which represent the 7 objects he needs. He differentiates
of these counters as representing the objects he already has. He determines the number of objects remaining. These remaining objects represent the number of objects he needs. Thus, the problem of determining the number of objects needed is shown to be transformable into a problem of determining the remainder of a group when a part of the group has been removed.

In a third solution (Figure 42), the 4 counters on hand are portrayed (Step 1). A comparison group, group B, of 7 counters is imported (Step 2). The problem now is to determine how many more counters must be added to group A to make it equal to group B. This is the problem of determining how many fewer counters there are in group A than in group B (Step 3). The problem has already been dealt with in Sutherland's first comparison situation and has been shown to be transformable into take-away situations.

![Diagram](image)

Step 1  Step 2  Step 3

Fig. 42

This problem situation was introduced by stating two difficulties involved: (1) the group whose size is to be determined is not present, and (2) there is no comparison group present. These difficulties are surmounted in the latter two solutions. Moreover, in either of these two solutions the problem is shown to be transformable into a take-away, remainder situation.
The Pattern Involving the "Separation-Into-Parts" Idea

The sum and one part are given; find the other part.— If group B of 7 objects is divided into two subgroups, one of which has 4 objects, how many objects are there in the other subgroup? Group B of 7 counters is portrayed in Figure 43, Step 1. Group B is divided into two subgroups, A and C, with A containing 4 counters (Step 2).

\[
\begin{array}{cc}
B & 0 0 0 0 0 0 0 0 \\
A & 0 0 0 0 0 0 0 0 \\
C & 0 0 0 0 0 0 0 0 \\
\end{array}
\]

Step 1

Step 2

Fig. 43

The number of counters in group C is then determined. The operation of dividing group B into the two subgroups may be accompanied by removing the known subgroup, A, of 4 counters from group B or by the simple operation of separating the two subgroups. This situation leads to the same operations as those in take-away situations and cannot be said to differ operationally from take-away situations.

The Subtraction Algorithm

Methods of teaching the subtraction algorithm.— In performing a subtraction problem symbolically, such as \(7 - 3 = 4\), a child may think in any one of the following three ways: (1) 7 take away 3 is 4; (2) the difference between 7 and 3 is 4; or (3) 4 is the amount which must be added to 3 to produce 7. Furthermore, when the subtraction example involves "borrowing," such as \(31 - 18 = 13\), the child may use either of two different methods to make the subtraction
possible. These are known as the "decomposition method" and the "equal-additions method."

The equal-additions method.— The equal-additions method is based on the fact that if the same quantity is added to both minuend (top number) and subtrahend (bottom number) in a subtraction example, the difference remains the same. Thus, the difference between 7 and 3 is 4 \((7 - 3 = 4)\). If 5 is added to both the 7 and the 3, the difference between the resulting 12 and 8 is 4 \((12 - 8 = 4)\). Algebraically, this rule may be stated \(a - b = (a + c) - (b + c)\).

In a subtraction algorithm in which borrowing is involved \((34 - 13)\), the equal-additions method proceeds as follows: Since 3 ones cannot be taken from 4 ones (Step 1), 10 ones are added to the minuend, resulting in 30 + 14 (Step 2, here shown in an expanded algorithmic form). At the same time 1 ten is added to the subtrahend (Step 2). Thus, 10 is added to both minuend and subtrahend, and the difference \((30 + 14) - (20 + 8)\) is the same as the difference \((34 - 13)\).

\[
\begin{array}{cccc}
 & 3 & 4 & \text{Step 1} \\
- & 1 & 8 & \text{Step 2} \\
\hline
 & 30 & 14 & -(20 + 8) \\
\end{array}
\]

The difficulty in showing the equal-additions method operationally when the take-away idea is used is apparent when the problem is solved on the abacus. Only the minuend \(\overline{34}\) is portrayed in a form employing place value (Step 1, Figure 44). Ten ones may be added to the minuend (Step 2); but since the subtrahend does not appear as a separate
representation in a take-away situation, there is no way to portray the 1 ten to be added to the subtrahend, although one more ten may be taken away than appears in the subtrahend.

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Step 1

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Step 2

Fig. 44

If the difference idea is being portrayed, then both minuend and subtrahend are represented and the equal-additions method can be physicalized. In Step 1 of Figure 45, both groups are shown. In Step 2, 10 ones have been added to A, and 1 ten has been added to B. The ones and tens of the two groups are compared by a matching process, and A is found to have 1 more ten and 6 more ones than B. Therefore, 16 is the difference between A and B.

A 000 000 (34) A 000 0000000000 (30 + 14)

B 0 000000 (18) B 00 000000 (20 + 8)

Step 1 Step 2

Fig. 45
In the algorithmic solution employing the equal-additions method, once the equal-additions have been made, the child may think in terms of "take-away," "difference," or "what-must-be-added." Thus, in the example, \( \frac{30 + 14}{20 + 8} \), the child, while subtracting the ones, may think, "14 take away 8," "the difference between 14 and 8," or "what must be added to 8 to get 14?"

The equal-additions method, then, may be shown operationally with the comparison problem. However, it appears that the decomposition, take-away method in the algorithmic solution is the best method to employ with young children for the following reasons:

1. The relationships of the comparison and additive operations to the take-away operation are shown to be feasible in the discussion of the operations of subtraction.

2. There is no point in the child learning a "take-away" table, a "difference" table, and an "additive" table once he sees that in algorithmic form all three subtraction problems may be solved by thinking in terms of any one of them.

3. The decomposition method employs the idea of substituting a quantity for its equal, a basic idea in mathematics.

4. The take-away operation is easily portrayed operationally and understood, and, in fact, is usually the first idea of subtraction taught to young children.

The next section will discuss a physical solution for an example of subtraction by the decomposition, take-away method. Only one
example will be shown, since the various stages of subtraction problems
conform closely to those of addition portrayed in the section on the
operations of addition.

The decomposition, take-away method.— The example \(34 - 18 = ?\)
will be used in this discussion of the decomposition, take-away
method. In Step 1, Figure 46, the \(34\) in the algorithm refers to the
3 counters in the tens column and the 4 counters in the ones column.
The example calls for the pupil to remove (take away) 8 counters from
the ones column and 1 counter from the tens column. Since there are
only 4 counters in the ones column, it is impossible to remove 8 ones.

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**Fig. 46**

Step 2 is the step in which 1 counter of the tens column is
exchanged for 10 counters which are added in the ones column. This
is the "decomposition" step, the step in which 10 ones are substituted
for 1 ten. The new form of the minuend (top number) in the algorithm
may now be thought of as \(20 + 14\). The algorithm may be portrayed
as \(\frac{2}{\frac{1}{3}}\). The 2 refers to the number of counters in the tens

\(\frac{-1}{-\frac{8}{\frac{1}{3}}}\).
column, and the 14 refers to the number of counters in the ones column. Step 3 portrays the beginning of the removal of 8 counters from the ones column and 1 counter from the tens column. Step 4 portrays 16, the remainder.

In the algorithmic solution employing the decomposition method, once the decomposition step has been taken, the child may think in one of three ways. Thus, the child, in subtracting, may think "what must be added to 8 to obtain 14 and what must be added to 10 to obtain 20?" He may think "what is the difference between 8 and 14 and what is the difference between 10 and 20?" Or, as a third alternative, he may think "what is 14 take away 8 and 20 take away 10?" However, the physical operations of taking-away 10 and 8 from 30 and 14 are clearly portrayed and understood, and the decomposition method is readily physicalized. Hence, the decomposition, take-away method appears to be the best method to use in portraying the subtraction algorithm physically.

Summary of Subtraction Situations

From the point of view of operationism there are three types of problems which are solved by subtraction, and, hence, three different meanings for subtraction. These three are as follows:

1. The first type includes all problems involving the physical operations of taking away a part of a whole group and determining the number of objects remaining. In each such problem, an operational solution may be effected by means of a separation of the original
group into two subgroups. The size of one group is known, and it may be removed (taken away) from consideration. The size of the other group is unknown; the problem is to determine the number of objects in it.

2. The second type includes all problems involving a comparison or a one-to-one matching between the objects of two groups and a determination of the difference. Sutherland included in this type the situation in which the size of the larger group and the difference are known, and the problem is to find the size of the smaller group. This situation appears to be a converse of comparison situations and more closely related, operationally, to take-away situations.

3. The third type includes all problems utilizing the operations of "building up" or "adding to" and the determination of the number of objects added. There is nothing in the operation of "adding to" to suggest that such problem situations can be solved by subtraction. The solution must be transformed into other sets of operations in order to show the relationship of this situation to other subtraction situations.

When solved using number symbols, \( 7 - 3 = 4 \), all three problems above take the arithmetic form of subtraction. The fact that these problems may be solved symbolically by subtraction does not depend alone upon the fact that 4 is the answer in each case. It also depends upon the fact that it is possible to show operationally that a given solution to any one of the problems is transformable into any other solution (set of operations) for such problems. The take-away,
reminder problem, however, appears to be a basic idea in subtraction in that it is easy to understand and to solve operationally. Moreover, operational solutions for other subtraction situations can be readily transformed into sets of operations for take-away, remainder situations.

Two methods of solving the subtraction algorithm when "borrowing" is necessary were shown in this section—the equal-additions method and the decomposition method. Reasons based on operational solutions were given to show why the decomposition, take-away method appears superior to other methods in teaching the subtraction algorithm to young children.

The Operations for Division

Two Meanings for Division

Most authorities list two basic meanings for division. Brueckner and Grossnickle give these two meanings:

Division has two different meanings; first, how many times one number is as large as another number; and second, the size of the equal parts into which a number is divided.\(^1\)

Clark and Eads point to the interrelationship between multiplication and division for the two meanings of division:

Division is inversely related to multiplication. The inverse of \(4 \times 6\) chairs = 24 chairs are: (1) into how many groups, with 6 chairs in a group, can 24 chairs be divided? (2) If 24 chairs are divided into 4 equal groups, how many chairs will there be in each group?\(^1\)

The Measurement and the Comparison Meanings of Division

The measurement meaning of division.---The first meaning listed by Brueckner and Grossnickle and by Clark and Eads is usually called the measurement or quotient meaning of division. In order to determine how many groups of 2 objects each there are in a group of 8 objects, one "measures," using 2 objects as the measuring unit, the number of groups of 2 objects each in the group of 8 objects. This operation of "measuring" may be portrayed as in Figure 47.

\[00/00/00/00\]

Step 1a

Step 1b

Fig. 47

The first operation is that of regrouping the 8 objects into equal subgroups of 2 objects each. This operation may be accomplished by moving the objects so that there is a spatial separation of the subgroups (Step 1a), or by employing some device such as drawing circles or straight lines to mark off the groups of 2's which makes it easier to determine the number of such groups (Step 1b). The second operation is that of determining the number of groups of 2 objects each which result from the regrouping. The effective way, then, to "measure" the number of 2's in 8 is to mark off the groups of 2's in some way so that they may be easily discriminated and the number of groups readily determined.

The word quotient is derived from the Latin word quotiens meaning "how many times." The situation above may be interpreted
as follows: Two is contained in 8 four times, or groups of 2 objects each occur 4 times in a group of 8 objects. The quotient, the name applied to the answer resulting from a division, tells the number of times the divisor is contained in the dividend, or the number of times the divisor would have to be taken to equal the dividend.

**The comparison meaning of division.**—The term quotient applies as well to another situation which is commonly classified as "measurement division." This problem situation may be worded as follows: How many times as large as a group of 2 objects is a group of 8 objects?

Authorities generally make little or no distinction between this problem and the problem of finding how many 2's there are in 8. For example, Stern treats both as comparison situations. That is, the determination of how many 2's there are in 8 is considered by her to be equivalent to the determination of how many times as large as a group of 2 objects a group of 8 objects is. McLellan and Dewey, over 60 years ago, developed the thesis of measurement advocated by Stern; and, in fact, defined number as the multiplicity of times that a measuring unit is used in making definite the size of an unmeasured and vague quantity. Other writers also include the two situations under one classification, the measurement use of division.


The following is an operational solution for the question: How many times as large as a group of 2 objects is a group of 3 objects (Step 1, Figure 48)?

The first operation is a series of matching operations in which the 2 objects in group B are matched successively with pairs of objects in group A (Step 2). The pairs of objects in group A are marked off in some way (in the example, with rectangles) as they are matched with the objects of group B. Then, it is seen that it has been possible to perform this matching of pairs of objects 4 times (Step 3); hence, there are 4 times as many objects in group A as there are in group B. "Comparison division" involves a series of matching operations between the objects of the smaller group and the objects of subgroups of the larger group. "Measurement division" involves a simple regrouping of the objects of the larger group. This operational distinction should be made and the two words measurement and comparison
used. "Comparison division" is transformable into "measurement division," however, since the first pairing of the objects of the smaller group with an equal number of objects in the larger group establishes the size of the subgroups into which the large group may then be divided.

Sutherland's problem patterns for measurement division.—

Sutherland lists eight patterns under her major classification of the measurement meaning of division. The first six of these patterns are operationally equivalent to the problem of finding how many groups of \( x \) objects each there are in a group of \( y \) objects, and differ only in the social applications being made. The first six patterns fall under two major headings and may be listed in abbreviated form as follows:

1. Simple patterns requiring a given number of equal groups to be formed.
   a. The total number of objects and the number of objects in each group are known; find the number of groups.
   b. The total cost and the cost per unit are known; find the number of units.
   c. The total amount to be done and the rate of doing it are known; find the time required.

2. Patterns involving rate, time, and distance.
   a. The total distance and the rate per unit of time are known; find the time.
   b. The total distance and distance covered per gallon are known; find the number of gallons.
c. The total time and the time per unit of distance are known; find the distance.

The last two measurement patterns in Sutherland's list are comparison problems. These two may be listed as follows:

3. Patterns involving comparisons
   a. How many times as large as one quantity is another?
   b. What fractional part of one quantity is another?p22

Sutherland's second comparison problem necessarily involves a fraction as an answer. No discussion of comparison situations which require fractional answers will be made in this study, since the concern here is with whole numbers. Aside from the special applications she uses, Sutherland's measurement category includes the two main situations discussed previously: (1) measurement division, when a total amount is divided into groups of a given size to ascertain the number of groups thus formed; (2) comparison division, when a larger quantity is compared with a smaller quantity to determine how many times as large as the smaller quantity the larger quantity is.

The Partition or Division Meaning of Division

The partition operation.— The second meaning listed by Brueckner and Grossnickle and by Clark and Eads is commonly called partition or division. A problem which involves this meaning may be worded as follows: If a group of 6 objects is divided into 3 equal subgroups, how many objects are there in each subgroup? The

14. Sutherland, op. cit., pp. 49-68.
actual operations involved in the solution of this problem are as follows: (1) three areas or divisions are set up (the rectangles portrayed in the diagram) to which the objects may be assigned (Figure 49); (2) the objects are moved (assigned) usually, but not necessarily, one at a time to each of the 3 subgroups in order (arrows); (3) the number of objects in any one subgroup is determined.

![Fig. 49](image_url)

The meanings of one-\(n\text{th}\) of a quantity.— The partition use of division is closely related to the idea of a fractional part of a group; and, in fact, some writers introduce this use of division as the problem of finding one-\(n\text{th}\) of a group of objects \((n = 2, 3, \ldots)\). For examples, Upton and Uhlinger\(^{15}\) first use the partition idea in a problem involving one-half of a group; and Durell and Hagaman\(^{16}\) consider one-half of a dozen as an introduction to the partition concept. An operational meaning of \(1/n\) of a group, according to these and some other writers, is this: The group of objects is divided into \(n\) equal parts, and the number of objects in one of the groups is determined.


These operations, when performed in this way for finding $\frac{1}{n}$ of a group, are identical with the set of operations for dividing a group of objects into $n$ equal subgroups in order to find the size of a subgroup.

Another meaning operationally for one-$n^{th}$ of a group is that attained by taking every $n^{th}$ object in a series of objects. For example, in Figure 50, in which the problem is to find one-fourth of 12 objects, the objects may be placed in order and every fourth object taken. This may be done by numbering the objects, 1, 2, 3, 4,

```
0 0 0 0 0 0 0 0 0 0 0 0
1 2 3 4 1 2 3 4 1 2 3 4
```

Fig. 50

and considering or removing physically all those objects under which the number 4 appears. It may be noticed in this case that by taking all objects numbered 2, or 3, or 1, the same objective could be accomplished.

The problem may also be solved operationally by regrouping the 12 objects into subgroups of 4 objects each and then taking one object out of every group of 4 (Figure 51). It is not necessary to take objects from each of the groups of 4 which have the same relative position. Thus, in Figure 51 the third object of the first group, the first object of the second group, and the fourth object of the third group are taken.

```
0 0 0 0 0 0 0 0 0 0 0 0
```

Fig. 51
Summary of partition division.— The operational meaning of partition division is as follows: If a group of y objects is divided into n equal subgroups, how many objects are there in each such subgroup? If y is divisible by n, then the number of objects in any one of the equal subgroups may be called one-n\textsuperscript{th} of y. However, one-n\textsuperscript{th} of a group may be obtained in various ways. If operational theory is adhered to, it would appear better to make a distinction between the two problems: (1) "partition" or "dividing a quantity into n equal parts," and (2) "finding one-n\textsuperscript{th} of a group." The problem of finding one-n\textsuperscript{th} of a group is not exclusively a partition problem.

Sutherland's problem patterns for partition division.— Sutherland lists eight problem patterns under her major category of partition situations. She divides these into three major patterns with certain sub-patterns which may be listed as follows:

1. Patterns requiring the amount, the size, or the cost of each equal part.
   a. The numerator of the fraction is 1; find a fractional part of a quantity.
   b. The total amount is divided into a number of equal parts; determine the size of each part.
   c. The total amount is divided equally among a number of persons; determine the share for each.
   d. The total cost and the number of like items are known; find the cost per item.
2. The pattern in which the total amount and the number of units involved are known to determine the average per unit (patterns involving distance are not included in this category).

3. Patterns involving averages in relation to distance, rate, and time.
   a. The total distance and time are known; find the average rate per unit time.
   b. The total distance and the number of gallons consumed are known; find the average distance per gallon consumed.
   c. The total time and total distance are known; find the time required to cover one unit of distance.17

Sutherland's distinctions in the four patterns about averages are made on the bases of use and language. It remains to be seen, however, whether an operational determination of an average is equivalent to the operations for other partition situations. Consider the example: If 5 boys have a total of 15 marbles, what is the average number of marbles owned by each? The solution may be accomplished by assigning the marbles to the five boys in turn until they have all been assigned and then determining the number each boy has.

The solution of this problem is identical with that of other partition problems. This is what would be expected, since the meaning of the word average as applied to this situation is as follows: If 5 boys have a total of 15 marbles—regardless of how many each of them owns individually—and if the marbles are divided equally among

17. Sutherland, op. cit., pp. 68-72.
the boys, how many marbles will each boy receive? In fact, this is
the operational meaning of average, and children should first encounter
it as a situation in which a number of objects are divided equally
among a group of children. The use of the word average may probably
best be deferred until sometime after the primary grades.

The Relationship Between "Measurement" and "Partition" Division

To answer the question posed in measurement division, one may
proceed from the diagram established to portray the partition idea.
The partition question may be stated again: If 6 is divided into
2 equal parts, how many objects will there be in each part? This is
portrayed in Figure 52 as the first step. There are 3 objects in
each of the 2 equal subgroups. If now, the objects are regrouped
so that the new groups each contain 2 objects, one from each of the

![Figure 52](image)

equal subgroups formed in the partition step, there will be as many
new groups (measurement step) as there are objects in either of the
2 equal groups which resulted from the partition step. That is to
say, the number of 2's in 6 is the same as the number of objects in
each part when a group of 6 objects is divided into 2 equal parts.
Division as Repeated Subtractions

Division is often said to be related to subtraction in much the same way that multiplication is related to addition. Spencer and Brydsgaard maintain that division is a special case of subtraction, when the subtrahends are all equal. Clark and Eads show how the answer to a measurement problem in division may be obtained through successive subtractions. The accompanying example shows how to find out how many 4¢ stamps may be bought for 20¢ by repeated subtractions. After the first stamp is bought, 16¢ remains; after the second stamp is bought, 12¢ remains; etc., until after buying 5 stamps, no money remains. Therefore, the number of 4¢ stamps which can be bought for 20¢ is 5. Clark and Eads point out, however, that a partition problem cannot be solved by repeated subtractions. For example, if 15 apples were to be divided among 5 boys, one could not subtract 5 boys from the 15 apples.

It is readily agreed that the long-division algorithm goes about producing the quotient by subtracting successively various multiples of the divisor. Consider the long-division algorithm for 78 divided by 3. In the first step, if the trial figure in the quotient is the proper one,


the child multiplies 20 (the 2 being in the ten's place) by 3 and subtracts the product 60 (the 0 being understood in the algorithm) from 78. This is simply the process of subtracting 20 of the threes at once, instead of one at a time, from 78. The subtraction leaves a remainder of 18; and it is seen that 6 more threes can be subtracted from the 18 leaving no remainder. Hence, 3 can be subtracted from 78 a total of 26 times; so there are 26 threes in 78.

There are two major points on which the teaching of division as repeated subtractions may be seriously questioned:

1. Pencil-and-paper operations with arithmetical symbols, that is, algorithmic solutions, are not identical with physical operations performed on objects. Although in the long-division algorithm one solves a division problem by subtracting multiples of the divisor from the dividend and from successive remainders, the physical problem involved is usually one of regrouping objects and determining the number of groups or objects in one group which results. This difference between the symbolic and physical solutions of problems is both the bane of arithmetic, since the child cannot always see an exact correspondence between an algorithmic solution with numerals and the actual physical solution, and the power of arithmetic, since it permits the solution of a vast variety of physical problem situations by a small number of arithmetic algorithms.

2. A division problem may be solved by repeated subtractions of equal amounts. It should be emphasized, however, that the question involved is this: How many subtractions are made? This question is
not the same as any of the three major questions answered by subtraction. The three major questions of subtraction are these:

(a) How many remain (are left)? (b) What is the difference? (c) How many must be added? In addition and multiplication the question asked is this: How many are there altogether? Addition and multiplication differ in the manner in which the data are given; they are operationally the same in the step of combining or moving the objects together; and they are both designed to answer the same question—how many are there altogether? Division differs from subtraction in all three of these respects.

Summary of the Meaning of Division

There are three meanings of division which may be clearly portrayed operationally:

1. "Measurement, or quotient, division." If a group of 8 objects is regrouped into groups of 2 objects each, how many groups are there?

2. "Comparison division." If a group of 8 objects is compared with a group of 2 objects, how many times as many objects are there in the larger group as there are in the smaller group?

3. "Partition division." There are two problems here which may be solved operationally in the same way: (a) If a group of 8 objects is divided into 2 equal subgroups, how many objects will there be in each such subgroup? (b) If one-half of a group of 8 objects is being considered, how many objects are being considered? However,
the problem of finding one-nth of a group may be solved in various ways which do not involve partitioning, and therefore should not be classified exclusively as a partition problem.

The Long Division Algorithm

In the foregoing sections it has been pointed out that there are three essentially different operational meanings for division. In this section, physical interpretations for the long division algorithm will be displayed for the measurement and the partition meanings of division in two different examples, and for the partition meaning only in a third example. Regardless of whether the problem is one of partition or one of measurement, the algorithm may be carried on in the language of either.

Example 1.— Thinking in terms of the measurement use of division in the example \( \frac{3}{69} \), a person asks himself this question: How many threes are there in 69? Ordinarily, the thinking would proceed somewhat as follows: There are 2 threes in 6 and 3 threes in 9, hence there are 23 threes in 69. The first step in reality should be stated "there are 20 threes in 60." However, since the 2 only is written in the tens place of the quotient, rather than the symbol 20, the algorithmic method actually states "there are 2 tens of threes in 6 tens." This statement can be verified, since \( 3 \times 2 \text{ tens} = 6 \text{ tens} \). In the completed algorithm accompanying Figure 53, the first step is that of finding that there are 2 tens of threes in 6 tens. In the second step it is determined that there are 3 ones of threes in 9 ones.
The portrayal is as in Figure 54 if one thinks in terms of partition. One-third of 6 tens is 2 tens. One-third of 9 ones is 3 ones. Hence, 1/3 of 69 is 2 tens and 3 ones or 23. In this example, it appears that the use of the partition idea is somewhat easier to portray along with its associated thinking than is the measurement idea. In the next example, the two uses of division will be discussed in relation to an example in which the first figure of the dividend is less than a one-figure divisor.

Example 2.— The example \( \frac{3}{12} \), since 12 means 1 ten and 2 ones, may be written in expanded algorithmic form as \( \frac{3}{10\ +\ 2} \). This form can be interpreted as the question: How many groups of 3 each are there in a group of 10 objects and how many groups of 3 each are there in 2 objects? An algorithmic solution is as follows:

\[
\begin{align*}
\frac{3}{10\ +\ 2} &= 4 \\
\frac{3}{9} &= 3 \\
\frac{1\ +\ 2}{3} &= 3
\end{align*}
\]

The same solution with objects would proceed as in Figure 55. The number of threes in the group of 10 is found to be 3; there is one object remaining. This one object is grouped with the 2 others to produce another 3, or 4 groups in all.
The above solution, however, does not lead to the way in which the algorithm is ordinarily performed. Such a solution may be portrayed with a device employing place value. If the measurement idea is being used, the object is to determine how many tens of threes there are in 1 ten and how many ones of threes there are in 2 ones. Since there is only 1 ten, there is not even as many as 1 ten of threes contained in it. In fact, as shown in the expanded algorithm, there is only slightly more than 1 three of threes in 1 ten, or in other words, about one-third of a ten of threes in 1 ten. This would involve a fraction in the tens place. It is necessary, then, to exchange 1 ten for 10 ones and to think of 12 as representing 12 ones, not 1 ten and 2 ones. When this is done, it is seen that there are 4 ones of threes (in shorter form, 4 threes) in 12 ones (Figure 56).
If the partition idea of division is used, $1/3$ of the 1 ten cannot be found without obtaining a fraction. Hence, again the 1 ten is exchanged for 10 ones, and $1/3$ of 12 ones is found as portrayed in Figure 57. There are 4 objects in $1/3$ of a group of 12 objects.

Again, in this example the use of the partition idea, in giving a physical interpretation to the algorithm, appears to be superior to the type of physical portrayal and the accompanying thinking necessary when the measurement idea is used. Because of the physical relationship between partition division and measurement division, which has been shown previously, it is permissible to think in terms of either one while solving the algorithm, regardless of which one is involved in the physical solution. In fact, the situation in division is like that in subtraction, in which all three subtraction situations may be solved symbolically by the same algorithm, using the decomposition, take-away method. The point is that partition appears to be more easily portrayed and described when giving a physical interpretation to the algorithm that is measurement division, and hence appears to be the better one to use in explaining the way in which the algorithm works. It is recognized that most authors prefer to use measurement, or quotient, in teaching the division algorithm. This may be explained by the emphasis often placed on division as being the inverse of multiplication. Thus, if a child knows that 4 threes are 12, then presumably he already knows the answer to the question: "how many threes are there in 12?" The next example will be portrayed in terms of the partition idea only.
Example 3.— In the example \( 32 \sqrt{128} \) (Figure 58), \( 1/32 \) of 1 hundred results in a fraction of hundreds. Moreover, after the 1 hundred is exchanged for 10 tens to produce a total of 12 tens, \( 1/32 \) of the 12 tens would result in a fraction of tens (Figure 59). Hence, it is necessary in this example to exchange the 12 tens for 120 ones in order to produce a total of 128 ones. Figure 60 portrays 128 ones partitioned to show that \( 1/32 \) of 128 ones is 4 ones.

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<thead>
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Fig. 58

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Fig. 59

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Fig. 60

Summary: The long-division algorithm.— One may portray the physicalizations for the long-division algorithm by using the thinking either of measurement or of partition division, although the partition idea appears to be easier to portray and to describe in explaining the algorithm.
Summary: The Operational Approach in Arithmetic

Chapters VII and VIII reported on an analysis of the physical operations with objects which provide meanings for the arithmetical words, symbols, and rules associated with whole numbers; and pointed out relationships among various sets of physical operations which permit problems solved by those operations to be treated symbolically by the various rules and algorithms of arithmetic.

A set of operations was described as those operations necessary to determine the answers to a quantitative question through the use of concrete objects. The major attention was given to discrete objects arranged in groups. Ordinarily a set of operations involves three steps: (1) arrangement, reproduction, or determination of the given data, although in many arithmetic examples the data is supplied and presented in the required form by the teacher or the teaching materials; (2) physical operations which involve moving the objects in some manner, such as combining groups, taking away groups, regrouping and matching groups, and which are designed to display the objects in such a way that the quantitative question being asked may be determined by attending to the new arrangement or display; (3) the determination of the number of objects or groups resulting, that is, the numerical answer to the quantitative question asked. If the sets of operations necessary to solve two different quantitative questions are different, then two different meanings are involved. If the two sets of operations are the same, then the two meanings are identical.
Three methods of determining the number of objects in a group were discussed in Chapter VII: (1) subitizing is the operation of recognizing directly the number of objects in a group; (2) matching is the operation of establishing a one-to-one correspondence between the objects of two groups; (3) cardinal counting is the operation of associating the number names, taken in serial order, with the objects of a group in order to determine the number of objects in the group.

The physical operation of combining groups leads to the algorithmic process of addition. In Chapter VII, the meaning of "successive stages of symbolization" was illustrated, the substitution operation was discussed, and its use in addition examples was portrayed in physicalizations for the addition algorithm.

There were found to be three quantitative situations which were operationally different but which are ordinarily performed symbolically by subtraction (Chapter VIII): (1) situations involving the physical operations of taking away a part of a group and determining the number of objects remaining; (2) situations involving the comparison of two groups in which the physical operations were those of a one-to-one matching and determining the difference; (3) situations involving the "building-up" operation, in which objects are added to those already on hand to attain a desired number of objects and the number of objects which were added is determined. Physicalizations for the subtraction algorithm were portrayed, showing examples of both the equal-additions method and the decomposition method. Reasons were
given to show why the decomposition, take-away method, from the stand-point of operationism, appears to be the most favorable method in teaching the subtraction algorithm.

Multiplication (Chapter VII) was found to differ operationally from addition in only one major respect—the manner in which the data are given or portrayed. Physicalizations for the multiplication algorithm were demonstrated.

Operationally, there were found to be three meanings for division. In "measurement division" the question is this: "How many x's are there in y?" In "comparison division" the question is this: "How many times as large as a group of x objects is a group of y objects?" In "partition division" the question is as follows: "If a group of y objects is divided equally into x subgroups, how many objects are there in each subgroup?" The problem of finding one-n-th of a group may be solved operationally in various ways and, hence, should not be classified exclusively as a partition problem. Both the measurement and the partition ideas were used in showing physicalizations for the division algorithm. It has been concluded that the partition idea is easier than the measurement idea in portraying and in describing physicalizations of the algorithm.

Implications for the Teaching of Arithmetic

Certain implications for the teaching of arithmetic in the primary grades follow from the analyses made in Chapters VII and VIII.

1. The arithmetic program should provide opportunities for the
children to encounter in physical situations all of the problem situations which were shown here to be operationally different.

2. These problem situations should be solved at first entirely with concrete objects and physical operations performed on those objects. It is in this stage that referents in terms of objects and physical operations are established for the written symbols and algorithms to be introduced later.

3. In the symbolic solutions of quantitative questions, some algorithms are more difficult than others. Division is commonly taught last since a knowledge of multiplication and subtraction is prerequisite for it. Such considerations in the symbolic approach to the teaching of arithmetic have led to the sequence of teaching the "four fundamental processes of arithmetic" in this order: addition, subtraction, multiplication, and division. There appears to be no evidence that such a prescribed order is necessary in an operational approach with concrete objects and physical operations. Children can, and do, solve such problems as take-away, comparison, partition, quotient, combining, and adding to, prior to any knowledge of the symbolic solutions. An operational approach, then, permits a wider range of quantitative situations to be considered in kindergarten and first and second grades.

4. More attention should be given to subitizing and matching procedures than is usually given. Subitizing aids pupils to think in terms of groups of objects rather than in terms of individual objects. Matching is an essential step in the operational solutions
of many quantitative questions. Counting is an important process for the child to learn, but it is often over-emphasized. The need for learning to count can be made more evident by demonstrating the limits of the ability to subitize and to group.

5. The operational approach demands that the child have access to objects which he can manipulate physically. It is difficult to see how this approach could be used successfully in programs in which the materials of instruction are limited to written materials. Moreover, the mere existence of concrete materials in the classroom and casual references by the authors of textbooks regarding the appropriate-ness of concrete materials are not sufficient to insure a meaningful approach. There must be a recognized and persistent effort by the teacher and the authors of arithmetic programs to insure that the use of concrete materials will provide meanings.
Chapter IX

SURVEY OF TEXTBOOK SERIES: I

Introduction

Purpose of Chapters IX and X

The purpose of Chapters IX and X is to report the findings of a survey made of thirteen current arithmetic programs in order to determine the following: (1) the extent to which grouping procedures based upon the findings of psychological studies on the perception of number have been adopted in elementary arithmetic programs; and (2) the extent to which the theory of operationism is used in current programs to provide meanings for arithmetic terms, symbols and statements.

These arithmetic programs were surveyed to determine the status of the following:

1. Early number experiences. Since the kinds of number experiences encountered early in a textbook series are indicative of what the authors consider fundamental to further arithmetic learnings, it seemed advisable to survey the early emphasis on counting and learning the number symbols.

2. Perceptual materials. The purpose here was to evaluate perceptual materials appearing in modern arithmetic programs and to determine to what extent these programs make use of number pictures to develop abilities in subitizing and grouping.
3. Operational meanings. The purpose here was to determine the degree to which modern arithmetic programs make use of an operational approach in providing meanings for terms, symbols, and rules of arithmetic in the primary grades. In particular, the extent of the operational treatment of the following arithmetic processes with whole numbers will be reported: (a) counting, (b) addition, (c) subtraction, (d) multiplication, and (e) division.

In the present chapter, points 1 and 2 above, along with current practices in teaching counting, will be discussed. In Chapter X, the extent of the use of an operational approach in the teaching of the four fundamental processes—addition, subtraction, multiplication, and division—will be considered.

The Arithmetic Series Surveyed

For each of the programs, with but two exceptions, a survey was made of the textbooks or the teacher's editions or both, and of any accompanying workbooks, for grades one through four. The exceptions are the Houghton Mifflin Company series by Catherine Stern and the D. C. Heath Company series by Hollister and Gunderson. Catherine Stern's unique approach is developed in a kindergarten program and in a series of books for the first and second grades. Since Hollister and Gunderson are the authors of only the first two books in the Heath series, it seemed better to limit discussion to those two books.

The criteria used in selecting these arithmetic series are as follows: (1) they are all available in recent editions at the University of Virginia; (2) they are from major publishing companies;
(3) they are written by outstanding leaders in the field and have nation-wide distribution; and (4) they provide a definite set of written materials for at least grades one and two. The thirteen series of textbooks are listed here by publisher and "senior" author or authors. A complete list of all textbooks and workbooks examined appears in Bibliography VII.


Hereafter, in referring to any of the series, the name of the first author listed or the name of the company will be used. The author named first in each case in the above list of series is either the "senior" author or one of the "senior" authors of the series. A series of arithmetic may be expected to reflect in some degree the philosophy of the "senior" author or authors.
Early Number Experiences

Certain implications of the operational approach and of the findings regarding subitizing for the teaching of arithmetic were kept in mind as the thirteen series were surveyed. Four of these implications having to do with the first quantitative ideas to which children are introduced in systematic arithmetic programs follow:

1. Young children entering the first grade can generally subitize up to four objects in a group. There is an opportunity at the first-grade level to extend this ability and to use it in grouping techniques as the child works within the range of numbers ordinarily considered during the first grade.

2. Since most children have some ability in counting objects before they enter the first grade, there is little necessity to emphasize counting at this time. Moreover, if counting is emphasized early in number work, children tend to use counting techniques even in situations in which subitizing and grouping are just as reliable, more mature, and more efficient.

3. Various problem situations which may later be thought of as problems in addition, subtraction, multiplication, and division can be solved by a child early in the work of the first grade without being so designated at that time, if he solves them by means of physical operations on concrete objects.

4. A considerable number of arithmetical concepts may be clearly and firmly established by use of the operational approach prior to any introduction of written symbols. The objects and physical
operations which the child uses in the operational approach provide
distinct and understandable referents for the symbols which will be
introduced later.

The Extent of the Teaching of Counting Early in Arithmetic Programs

In every series surveyed, the child has some contact with count-
ing during the first half of his work in the first grade. Rational
counting (see Chapter VII) as the first means of determining the
number of objects in a group is either emphasized or is tacitly
assumed in eleven of the thirteen series (Table II). In ten of the
series, some work in counting is provided within the first four
pages. The authors suggest that the teacher provide experiences in
counting objects in the room, and they present pictures in the text-
books for counting exercises. In another series, the authors say
the children should be able to count before using the book (Table II).
Some work in one-to-one matching precedes rational counting in the
Silver Burdett series, but this is designed primarily to show count-
ing as a one-to-one matching of number names and objects. In addition
to the information shown in Table II, it was found that in all of
the series except two provision is made for learning to count to at
least 100 by the end of the first grade and that in ten of the thirteen
series the pupils also receive some practice in counting by 2's, 5's,
and 10's in the first grade.

The following series place considerable emphasis on counting as
an operation which is basic to all, or nearly all, other learning in
arithmetic: Row, Peterson; Silver Burdett; the Steck Company; and
Table II
THE TEACHING OF COUNTING AND ONE-TO-ONE MATCHING IN THE FIRST BOOKS OF CERTAIN CURRENT ARITHMETIC SERIES

<table>
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<tr>
<td>Ginn:</td>
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<tr>
<td>Buswell and Brownell</td>
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<tr>
<td>Heath:</td>
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<tr>
<td>Gunderson and Hollister</td>
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<td>Yes</td>
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<tr>
<td>Stern</td>
<td>*</td>
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<td>Van Engen and Hartung</td>
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* Counting is taught during "Level 1" of Book 1, which follows the work with the author's concrete devices described in Experimenting with Numbers.

** The authors of this series suggest the teacher should be certain that children can do some rational counting before they start with the work in the book.

*** For explanation of the methods used in these two series, see text.
Laidlaw Brothers. The value the authors of the Row, Peterson series place on counting is clearly indicated when they speak of combining groups as "counting together" and of separating groups as "counting apart."¹ The authors of the Silver Burdett series are of the opinion that the child cannot learn how to combine groups and determine the total (add) "until he can determine, by using rational counting, the number of items or objects in each of the smaller groups."² The authors of the Steck Company series state that "counting is the fundamental number activity,"³ and they believe that practice in rote counting must precede rational counting. Eight series provide for exercises in rote counting, although rote counting does not necessarily precede rational counting in each of the eight series. For example, exercises in rote counting to 100 are common in several series even after the children have done considerable rational counting. The major purpose of Area I of Book 1 of the Laidlaw Brothers series is the development of counting techniques.⁴


The Macmillan series introduces rational counting first and then teaches it concurrently with the immediate recognition of the number of objects in small groups. The authors of this series make every effort to get the pupils to shorten the process from counting to find how many in a group by using such operations as these: (1) the immediate recognition of the number of objects in the whole group; (2) the immediate recognition of a subgroup and counting ahead to determine the size of the whole group—partial counting; and (3) the immediate recognition of larger groups as being made up of easily perceived subgroups. The emphasis throughout the first book of this series is on developing the ability to recognize the number of objects in groups without counting them. This emphasis is clearly indicated in the tests included in the book; four of the six are tests in the recognition of the number of objects up to ten, when grouped in certain patterns, without counting them.5

The authors of the Scott, Foresman series list counting and the recognition of small groups as being of equal importance in what they term the first of five phases of number readiness. Their five readiness phases for the work of the first year are the following: (1) readiness for the general use of numbers, (2) readiness for the basic number facts, (3) readiness for measuring, (4) readiness for working with the number system, (5) readiness for the use of money.6 A considerable


amount of time in the first book of this series is devoted to the group concept, through such exercises as subitizing groups, forming groups, recognition of subgroups which make up larger groups, combining groups, separating groups, and rearranging groups. The first book of this series begins with lessons devoted to developing ideas of certain indeterminate number words, such as few, many, long, large, too few, too many, etc., without counting; to simple pairing, matching one series of objects with another; and to recognizing the number of objects in small groups without counting. The authors feel that such number ideas lead to readiness for rational counting. Counting, the use of number-words instead of objects as a means of matching, is not introduced until Lesson 16.

The Primer of the Ginn series appears to differ in certain ways from the first books of the other series. It would be safe to say that the authors of this series depend more on the pre-school learnings of the children than do other authors. A minimum amount of time is devoted to counting, apparently on the assumption that the children already know this operation.

The Houghton Mifflin series by Stern is the only arithmetic program of those surveyed which specifically avoids the teaching of counting on the first level of instruction. In the first stage of this program only concrete materials are used. These concrete materials are designed to develop the following concepts of number in the order

listed: (1) the measurement, or ratio, concept; (2) the regular-pattern group concept; and (3) the series concept. The pupils do not learn counting until Level Two of the first stage of the Stern program. They learn to count objects for the purpose of knowing the number names and their order, and later for counting the number of objects in large or irregularly-shaped groups. Stern writes as follows:

It is possible to recognize the total of such groups small regularly-patterned groups at a glance without counting. In other cases, as in determining the number of flowers in a vase, counting the flowers one by one seems to be the only way of getting the total.  

It may be stated, then, that in eleven of the thirteen series of arithmetic examined, one of the first experiences is that of rational counting to determine the number of objects in a group. In one of the eleven series, however, instruction in recognizing the number of objects in a group (subitizing) is introduced early, and efforts are made to get the pupils to shorten the process from counting to subitizing. One series specifically avoids the operation of counting throughout the first level of the work. The authors of the one remaining series take the position that certain quantitative ideas and skills—such as the understanding of some indeterminate number-words, matching, and the immediate recognition of small groups—must precede any meaningful use of counting.

The extent of the teaching of rational counting, and particularly the extent to which it is taught during the early part of the arithmetic

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programs, indicates that most authors believe that counting is a basic skill which must be learned to some extent before further progress can be made with number ideas. The findings reported in Chapter III of this study regarding the ability of children to subitize do not support this belief that counting is prerequisite to all other number activities. In every arithmetic series surveyed, the work during the first half of the first grade is restricted largely to the numbers from 1 to 10 inclusive. Within this range, subitizing and grouping techniques are quite effective. In particular, the time spent by children in counting groups of two objects, three objects, and four objects probably results in little learning, since children can subitize these groups readily; and the need for counting to determine the number of objects in a group can be shown more effectively with larger groups.

The Extent of the Teaching of Numerals and Number-Words in the First Books of Arithmetic Series

In the operational approach, the child's first experiences in a systematic arithmetic program would consist of physical operations performed with concrete objects. He would use spoken number-words and verbal statements to describe his operations and to communicate his number knowledge and his answers to number questions. The numerals (1, 2, 3, ...) and other written symbols (such as two, +, -, etc.) of arithmetic would not be needed at this level; and, in fact, the purpose is to provide for a wide variety of operationally meaningful number experiences which the child will symbolize in various ways.
before he uses the numerals and other signs of arithmetic. A program which stresses "symbol learning" in the first grade may be said to view arithmetic largely as a "symbolic system" to be learned and not as a field which describes the quantitative properties and relationships of observable phenomena.

Table III summarizes the extent to which the various books teach the reading and writing of numerals and number-words. The Scott, Foresman series does not teach the reading of any of the number-words until grade two, and the Macmillan series treats this as optional material in grade one. It is an interesting observation that ten of the thirteen series take up the number symbols (numerals) from 1 to 10 in order. In some cases the pupils learn to read the numerals, to write them, and to associate them with appropriately sized groups at the same time or at nearly the same time. In some series the writing of the numerals comes later in the program; for example, in the Steck series the writing of the first five numerals comes after several pages in which they are read and some recording is done by the use of tally marks. In the World Book series, the writing of the numerals is delayed until after the readiness book. In each of the Ginn, the Merrill, and the Winston series, the first lesson in writing the numerals follows by several pages the first lesson in reading them.

The extent of the reading and writing of the numerals and number-words during the first grade is not shown in Table III. In ten of the thirteen series, the children encounter lessons in reading the
Table III

THE TEACHING OF NUMERALS AND NUMBER WORDS IN THE FIRST BOOKS OF CERTAIN CURRENT ARITHMETIC SERIES

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<td>Carpenter and Swenson</td>
<td>1</td>
</tr>
<tr>
<td>Merrill Books:</td>
<td></td>
</tr>
<tr>
<td>Durrell and Hageman</td>
<td>8</td>
</tr>
<tr>
<td>Row, Peterson:</td>
<td></td>
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<tr>
<td>Wheat</td>
<td>8</td>
</tr>
<tr>
<td>Scott, Foresman:</td>
<td></td>
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<tr>
<td>Van Engen and Hartung</td>
<td>16</td>
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<tr>
<td>Silver Burdett:</td>
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<tr>
<td>Norton</td>
<td>6</td>
</tr>
<tr>
<td>Steck:</td>
<td></td>
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<tr>
<td>Benbrook and Foerster</td>
<td>9</td>
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<tr>
<td>Winston:</td>
<td></td>
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<tr>
<td>Brosnanickle and Merton</td>
<td>2</td>
</tr>
<tr>
<td>World Book:</td>
<td></td>
</tr>
<tr>
<td>Clark</td>
<td>9</td>
</tr>
</tbody>
</table>

* Numbers—markers from 1 to 10 and *, =, and 0 are used in Unit 10 of Experimenting With Numbers to record number facts. Writing the number symbols starts on page 1 of Book One and reading the number-words on page 95 of Book One.

** Reading number-words is included as optional material on the back cover of Book 1.

*** Writing the number symbols is introduced in page 9 of Book 2 and reading the number-words on page 8 of Book 2.
numerals to 100 during the work of the first grade. In the Silver Burdett series, the reading is extended to 150. In the Houghton Mifflin series and the Row, Peterson series, the children read the numerals only to 10. In nine of the series, the children have experiences in writing the numerals to 100. In the Houghton Mifflin series and the Row, Peterson series, the pupils learn to write the numerals to 10 and in the Laidlaw series to 50. The Scott, Foresman series does not teach writing the numerals until the second year. Ten series teach reading the number-words to ten during the first year; two of them teach reading number-words to six; and the Scott, Foresman series introduce the reading of number-words in grade two.

Ordinal number-words are taught in the books for the first grade in eleven of the series (see Table III), the most common terminating point being with the ordinal word fifth. Only in the Winston series is there any evidence that the children ever need to be able to read the ordinal words at this level. The ordinal use of numbers is, of course, evident in exercises employing the series or positional use of the number symbols and number-words. For example, some books include exercises in numbering objects in a given order. In another exercise appearing in several books, the child creates a picture by connecting dots in order with lines.

Summary and Discussion of Early Number Experiences Provided in Current Arithmetic Textbooks

In the work of the first grade of every series examined, the pupils learn to read some of the numerals; they learn to associate
the numerals with appropriately sized groups to some extent; and they learn rational counting. It would appear that most authorities are of the opinion that at least a part of the first experience in a systematic arithmetic program consists in learning to read the numerals and to understand what they mean in terms of groups of discrete objects. In several of the series, the first-grade pupils also learn to associate some of the written number-words with appropriately sized groups, and in twelve of the thirteen series surveyed, the pupils learn to write the numerals. This is a clear indication that most authors believe that learning arithmetic means learning about the symbols of arithmetic and how to perform certain symbolic operations with the numerals. As a consequence, a large portion of the work of the first year in most current arithmetics appears to be the learning of symbols rather than the developing of concepts of quantity through an operational approach.

The emphasis placed on counting in the first books of current arithmetic programs indicates that nearly all authors consider counting a basic arithmetic skill. In brief, counting appears to be rather generally considered the fundamental arithmetic operation. This emphasis on counting as a necessary skill to be learned before further progress can be made in arithmetic is not supported by the results of the investigations regarding subitizing and grouping reported in this study.
Number Pictures in Textbooks

There is evidence that the recommendations regarding the use of number pictures in developing grouping techniques, as made by Howell, Freeman, Carper, and others (see Chapter III), have influenced the authors of modern arithmetic programs to some extent. This influence is particularly evident in the extensive use made of number pictures in current arithmetics. The purpose of this section is to analyse and to evaluate number pictures and the uses made of them.

Kinds of Pictured Materials Appearing in Current Arithmetic Programs

Arithmetic textbooks generally provide four types of picture materials:

1. Pictures which are used to illustrate a story or a set of exercises or to illustrate, develop meanings for, or give practice in the use of comparative or positional terms, such as bigger, long, left, and fifth. Pictures of this type are not used by any of the authors of the series surveyed to develop abilities in subitizing and grouping.

2. Sequences of pictures which portray successive stages of an action taking place and which are intended to provide meanings for the fundamental processes of arithmetic. Pictures of this type will be considered later in this chapter as a part of the discussion of the extent of the use of the operational approach in current arithmetic programs.

3. Number pictures which show numbers in terms of pictures of concrete objects, such as animals, toys, people, or other real things.
4. Number pictures which show numbers in terms of pictures or diagrams of geometric forms, such as circles, squares, rectangles, stars, etc.

Only number pictures of type 3 and 4 will be considered at this point.

Evaluation of Number Pictures and the Uses Made of These Materials in Modern Arithmetic Series

There is a wide variation in the overall quality of the number pictures portrayed in modern arithmetic books; however, this cannot be interpreted to mean that some authors are failing to achieve an objective they all consider important. In fact, the authors of some of the series surveyed are of the opinion that there is little necessity to develop the ability to perceive the number of objects in a group. Writers who feel there is little to be gained from developing abilities in number perception may be expected to give relatively less attention to the perceptual qualities of the number pictures in their books.

Table IV contains an evaluation of the perceptual qualities of number pictures appearing in the first books of the thirteen series surveyed. This evaluation was made according to the following criteria:

1. Are the pictured units, regardless of whether they are of concrete objects or of geometric figures, congruent or nearly so?

2. Are the units imbedded in simple contexts; that is, are they depicted on plain backgrounds, and are they well-spaced so as to appear as discrete objects?
Table IV
EVALUATION OF THE PERCEPTUAL QUALITIES OF THE NUMBER PICTURES IN THE FIRST GRADE BOOKS OF CERTAIN CURRENT ARITHMETIC PROGRAMS

<table>
<thead>
<tr>
<th>Series</th>
<th>Good</th>
<th>Fair</th>
<th>Poor</th>
</tr>
</thead>
<tbody>
<tr>
<td>American Book:</td>
<td></td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>Upton and Unlinger</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Ginn:</td>
<td></td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>Buwell and Brownell</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Heath:</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Gunderson and Hollister</td>
<td></td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>Houghton Mifflin:</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Stern</td>
<td></td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>Laidlaw Brothers:</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>McSwain and Ulrich</td>
<td></td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>Macmillan:</td>
<td></td>
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<tr>
<td>Carpenter and Swenson</td>
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<td>X</td>
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<tr>
<td>Merrill Books:</td>
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<tr>
<td>Durell and Hagaman</td>
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<td>X</td>
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<td>Row, Peterson:</td>
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<td>Wheat</td>
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<td>Scott, Foresman:</td>
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<tr>
<td>Van Engen and Hartung</td>
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<td>Silver Burdett:</td>
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<tr>
<td>Morton</td>
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<td>Steck:</td>
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<tr>
<td>Benbrook and Foerster</td>
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<td>Winston:</td>
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<td>World Book:</td>
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<td>Clark</td>
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<td>X</td>
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</tbody>
</table>
3. Are there provisions for irregular patterns and for various regular patterns?

4. Do the colors used contrast well with the background?

These criteria, which give due consideration to the recommendations of Carper, Dawson, and others (Chapter III), were established as being those which, when adhered to, would provide materials most conducive to developing the ability to subitize and to using this ability in grouping procedures. Table IV lists five series which may be rated as "good," four which may be rated "fair," and four which would have to be rated "poor" according to these criteria.

The authors of every series surveyed make some provisions for teaching children to perceive the number of objects in small groups directly and to use this ability in determining the number of objects in larger groups. These series differ widely, however, in the amount of attention given to this development. Brief résumés of the positions taken by the various authors are given here to clarify the extent of the differences of opinion regarding the importance placed on developing the ability to subitize.

1. The authors of the American Book series recommend some practice in sight recognition, to be given with flash cards. They point out that "this recognition of groups shortens the counting of all the objects in each box."9

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2. The authors of the Ginn series make considerable use of domino patterns. It appears, however, that these authors expect abilities in number perception to develop rather incidentally through practice in counting groups.

3. The authors of the Heath series make definite use of dot patterns to give practice in recognizing small groups. Apparently they expect some, if not all, children to recognize immediately the number of objects in a group up to 5 and to use this ability to some extent to see the subgroups in larger groups. They write that "as groups become larger, children experience more difficulty in recognizing the number in a group without counting."10

4. Catherine Stern, the author of the Houghton Mifflin, series, is of the opinion children will learn to perceive the number of objects in small groups; however, she feels that any consistent ability along these lines is limited largely to regular groups.

5. In the Laidlaw Brothers series, some of the number pictures are rather well designed for developing abilities in number perception. There is little evidence, however, that the authors expect the pupils to use sight recognition of the groups to any extent.

6. The Macmillan series probably makes the most persistent effort to develop the ability to perceive the number of objects in a group of any of the series surveyed. Almost the entire readiness book is designed to further this aim, and one may conclude that the

authors feel that the development of this ability is one of the major purposes of a readiness program. The authors apparently expect the children to extend this ability to 10 in regular patterns; to make use of partial counting to determine the number of objects in the larger groups, 6 to 10, when they are irregularly arranged; and to recognize larger groups by immediate recognition of subgroups which have sums they know. The most serious criticism of this series is the dependency placed on a limited number of regular patterns.

7. The first book of the Merrill series by Durell and Hagaman makes definite provisions for teaching the immediate recognition of groups. The authors feel that children have developed this ability to some extent before they enter school, and there is some evidence that the authors feel this ability can be extended up to an immediate recognition of 10 objects if arranged in regular patterns, such as the domino pattern. The perceptual materials in this series can be judged, at best, fair. Thus, this series appears to be an example in which the authors recognize the importance of developing the ability to subitize, but the implementation of the recognition is not adequate as evaluated by the criteria for judging number pictures used in this study.

8. The authors of the Row, Peterson series feel that pupils should be aided to recognize all groups of twos and threes and, if possible, of fours and fives at a glance. Partial counting, flash cards of regularly arranged dot patterns, and counting by twos and
threes are suggested as excellent practice exercises along these lines. Although the authors of this series place considerable stress on counting at first in the work with groups, it is clear that they feel that as the work in arithmetic progresses, the children will learn to recognize the number of objects in small groups without counting. Wheat writes as follows:

In the beginning children may have to resort to counting to find how many there are in a group, but gradually through such practice in group recognition, all pupils in the first grade should be able to recognize at a glance and without counting groups as large as five when arranged in one or more of the common patterns.

The authors of the Scott, Foresman series believe matching and the immediate recognition of small groups are preliminary to rational counting. After about ten pages, the work on the immediate recognition of small groups is started:

The next step in his the pupil's readiness program is to become thoroughly familiar with the model groups 2, 3, and 4. He should learn to recognize immediately without counting groups of 2, 3, and 4 objects. The child's understanding of these model groups will enable him later to break up and identify larger groups.

The emphasis throughout the readiness book of this series is on groups and the extension of the abilities to use grouping techniques.

The number pictures of the book are well adapted to further this aim.

10. There is some reason to believe the authors of the Silver Burdett series expect children to recognize groups up to at least 4 or 5 objects. For example, at one place they state that "if the teacher finds that some of the children have not yet learned to recognize groups of four, the suggestions for preparatory activities may be repeated." On pages 49 to 55 of Book One, an effort is made to have the pupils recognize the various subgroups within larger groups. The development of the ability to subitize, however, is dealt with in a minor way. Moreover, the number pictures of the Silver Burdett series lack, in general, many of the characteristics of good perceptual materials as judged by the criteria of simplicity.

11. The authors of the Steck Company series are of the opinion that children can already recognize groups of 3 and 4 without counting when they enter school. Although counting is considered the fundamental operation of arithmetic by these authors and most of the exercises early in Book One involve counting, there is a definite effort made later in the book to have children recognize immediately small groups up to 6 objects. The number pictures in this series, as judged by the criteria of simplicity, are good; and it would appear that the authors could promote abilities in number perception much earlier and further than they do.

12. The authors of the Winston series feel that the ability to recognize group patterns is of considerable importance in learning number combinations. The emphasis, therefore, is placed on recognizing a set of number patterns to aid in learning the number facts, rather than on subtitizing.

13. The number pictures in the readiness book and in Book 1 of the World Book series appear in a variety of simple, regular patterns. There appears to be an effort made to help the pupils recognize the various number groups through the use of these patterns.

Summary: Evaluation of and Uses Made of Number Pictures

It may be said that some attention, although slight in some instances, is given to developing the ability to recognize immediately the number of objects in a group in all of the thirteen series surveyed. In ten of the books, however, practice in immediate recognition follows counting instead of preceding it or being concurrent with it. In one series, the measurement rather than the group idea of whole numbers is emphasized first, so that neither counting nor subtitizing is stressed. Only the materials of the Macmillan Company and the Scott, Foresman Company could be said to make an early and persistent effort to develop the ability to subitize.

In general, arithmetic textbooks present a variety of number pictures. As judged by the simplicity criteria of congruency, background complexity, and color, all of the books include some good number pictures to aid in the development of the ability to perceive the number of objects in a group. In four of the series, the number
of such good number pictures is so limited, or the overall presentation so poor, as to rate these series "poor" in an evaluation. Four additional series may be rated "fair," and five of the thirteen series include a sufficient number of high quality number pictures to be rated "good."

It may be said that the materials presented in current arithmetic programs are of such a quality that they could be adapted to furthering the ability to subitize to a greater extent than the authors of the books now make use of them. The failure to do so appears to result from the emphasis placed on counting by most of the authors.

The Extent of the Use of the Operational Approach in Teaching Counting

The purpose of this section is to report findings regarding the extent to which current arithmetic programs provide operational meanings for the various concepts associated with counting.

The Teaching of Counting

In Chapter VII, an operational distinction was made between cardinal counting and ordinal counting. It was pointed out that in first experiences of cardinal counting the child should move concrete objects from a group which he is counting into a new position where a counted group is being formed, saying the number-words as he does so. The operation is designed to help the child keep in mind that the number-word he says as he moves each object is associated with
the total number of objects in the counted group at that time and not with the one object he is moving. On the other hand, in ordinal counting it was suggested that the child point to or touch each object of a series of objects in turn as he says the ordinal number-words first, second, ... or number one, number two, ... This operation is designed so that the child will associate an ordinal number-word with only one object, the one being counted at the time.

The most common way in which authors of the thirteen series teach rational counting to find the number of objects in a group involves two directions: (1) have the child touch, look at, or nod toward each object or picture in turn as he says the number-words in order; (2) make sure the child understands the last number-word said refers to the number in the group counted and not just to the last object to which attention was given. For example, the authors of the Heath readiness book caution the teacher that some children may "not realize that the last number they say tells how many objects are counted." They simply add, however, that such children need special attention. As another example, Upton and Uhlinger advocate the following:

Another step in teaching rational counting is to make clear that in counting 5 objects the word "five" is not a name that belongs exclusively to the last object but a number that tells the total number of objects that have been counted up to that point.15

Their solution to this problem, too, simply repeats their precaution without giving specific operational directions.

Wheat says the child must take cognizance of the number question which has been asked. He writes thus:

The name five at one and the same time distinguishes the fifth object and includes it in a group with the objects counted before it. As we utter the name five, we may point to the fifth object, but we also mean the group we have completed counting. Five has both a cardinal and an ordinal meaning.17

Wheat feels that the meaning attached, whether ordinal or cardinal, is brought out chiefly through the question-answer technique. Thus, if the teacher asks which position a certain object, such as the third, holds in a series, the child counts, "One, two, three," and then, remembering the question asked, replies, "third." The word three distinguishes the third object, Wheat maintains. But it also means the group of three which has been counted, and if the requirement is to determine the "how many" of the group, the child must realize this and reply "three" instead of "third." 18

Wheat's position regarding the importance of the question asked is sound. As is true in all quantitative situations, the question asked determines in large part the quantitative problem involved. Moreover, it is true that adults use the number-words one, two, three, ... interchangeably as ordinals and cardinals. It appears, however, that an operational distinction of the types discussed in Chapter VII should be made at first so that the child may associate

17. Harry Grove Wheat, How to Teach Arithmetic, op. cit., p. 2.
two different sets of operations and two different sets of num-
ber-words with the two different questions. The weakness of the position
stated by Wheat is that in it two number questions are made to refer
to the same procedures. In the operational approach, this could
hardly be said to aid in clarifying the distinction between the two.

Examples of an Operational Approach in Teaching Cardinal Counting
in Current Arithmetic Series

Only a few series make provisions for the child's doing something
operationally to help him keep in mind at all times that it is the
total number in the group with which he is concerned in cardinal
counting. On page 5 of the Workbook for the Row, Peterson Primer,
which Wheat has co-authored, cardinal counting is depicted in the
following fashion: The objects counted are kept in a new group
through an operation of pulling, placing, or dragging the objects one
by one, as the count proceeds, from the group being formed. Thus,
in spite of his emphasis on understanding cardinal or ordinal counting
by means of the questions asked, Wheat makes an operational distinction
between the two types of counting.

On page 35 of Book 1 of the Laidlaw Brothers series, rational
counting to determine the number of objects in a group is shown by
means of an expanding spiral which takes each new object into a group
of those already counted. The operation for five objects would be
shown as follows:
On page 39 of *Ready for Numbers* of the Winston series, the authors provide practice in what they call "controlled counting." They give two instances of different kinds in cardinal counting. In one case the authors suggest placing 9 blocks in a row to count. The child is then told to move the blocks one by one into a new row as he counts. A few lines later, the authors suggest that if the child is using a fact-finder frame (a counting frame) he should actually slide the beads along the wire from one side of the frame to the other, covering those counted with one hand as he moves them with the other hand.

It will be noticed immediately that three of the four examples above involve concrete objects. And, in fact, any operation in which the child moves the items he counts into a new "counted" group to aid him in keeping his attention on the group counted would have to be performed with concrete objects. It is manifestly impossible for him to move pictured items into a new group as he counts them. His only recourse in counting pictured items is to point to or touch each item in turn as he counts it: unless he uses some means, such as his hand, to cover up the group already counted, or some paper-and-pencil device, such as the expanding spiral of the Laidlaw Brothers series. The most productive means, however, seem to be for the child to learn cardinal counting and use it for a length of time with small concrete objects he can move, before he turns to counting objects in pictures.
Summary: The Teaching of Counting

Every series teaches rational counting to determine the number of objects in a group, and every series (eleven of the thirteen in the first year) teaches rational counting to determine the number associated with the position of an item. Apparently, only a relatively few books make an outright attempt to have the child count the number of objects in groups in such a way that some operation (other than nodding, pointing, and saying the number-words) is involved to help the child remember that it is the total group rather than the last item counted in which he is interested.

In Chapter X the discussion of the analysis of textbook series will be continued, with attention being given to the extent of the use of an operational approach in the teaching of the four fundamental processes with whole numbers.
Chapter X

SURVEY OF TEXTBOOK SERIES: II

The purpose of this chapter is to report the extent to which current arithmetic programs make use of an operational approach in teaching addition, subtraction, multiplication, and division of whole numbers.

The Extent of the Use of the Operational Approach in Teaching Addition

In Chapter VII, addition was described as being composed of two physical operations: (1) the actual combining (bringing together) of two or more groups of objects, and (2) the determining in some way of the number of objects in the total group. In order to develop the concept of addition as that of moving together two or more groups of objects, the actual physical combination of the groups would appear to be the first step. All of the series examined provide for this first step in some way. For example, the authors of the Merrill series write as follows:

In beginning addition, children need actually to see two separate groups coming together and to realize that this new larger group is not something different, but is made up of the same units that were in the two smaller groups. 1

Every teacher's manual examined suggests the use of concrete objects and kinaesthetic activities to establish a meaning for addition in terms of combining groups and making a determination of the number of objects in the total group. Even though most authors point out that directions in teacher's manuals are to be considered an integral part of the program, this does not guarantee that every teacher using the materials will follow the directions of the manual; and it is obviously impossible to combine physically the objects pictured in a textbook.

**Attempts to Physicalize the Addition Algorithm**

Several examples were given in Chapters VII and VIII to show how the algorithms for addition, subtraction, multiplication, and division should be physicalized. In no case, however, was it claimed that the physicalization duplicated the algorithm. What was shown were the referents for the symbols of the algorithm at various stages in the physicalization. For example, in physicalizing $3 + 2 = 5$, there is not a group of 3, a group of 2, and a group of 5; there is a group of 3, a group of 2, and a physical operation which combines the two groups into a new group of 5 objects.

Some authors of textbooks appear to feel that meaning is attained through the use of concrete objects when the operations performed on those objects duplicate, or nearly duplicate, the algorithm. For example, Catherine Stern in discussing the use of her Number Track,
a device which emphasizes the serial aspect of the number system, states the following:

This is an exact way to find that 5 cars plus three cars make 8 cars in all—and, still, it is an unsatisfactory way to show addition. This does not demonstrate the equalness of 5 plus 3 and 8, since in the very act of counting or lining up the objects, the child loses the original 5 plus 3 and sees, in the end, only an aggregate of 8 units.2

In essence, Stern attempts to duplicate the statement made by the algorithm, that is, to establish a one-to-one correspondence between the various symbols in the algorithm and the things and processes being symbolized. Thus, according to Stern's analysis the physical meaning of 5 + 3 = 8 is as follows: The 5-block is placed end to end with the 3-block. These two blocks, so placed, are compared by actual juxtaposition with other blocks, and it is found that their combined length is the same as the length of the 8-block. This approach of Stern's is operational; her error is in attempting to make each different symbol of the algorithm refer to a different physical object or process. The algorithm is a combination of symbols for recording the results of what was done; it is not meant to duplicate what was done physically.

The following examples also attempt to draw a close resemblance between the concrete situation and the algorithm, and thus emphasize the relationship of equivalence (no attempt is made here to duplicate the pictures of objects as they actually appear on the pages referred to):

Most of the authors, however, do not attempt to show an equivalence between the number of objects involved and the symbols, taking the position that the algorithm is simply a means of recording the situation. They hold that physicalizations of arithmetic processes are not supposed to attempt to duplicate the algorithmic solutions. A physical duplication of the equivalence relation expressed in an algorithm, such as \( 4 \times 3 = 7 \), does not give the essential meaning of adding groups. Children must be helped to realize that the new group formed when two separate groups are brought together is not something different but is actually made up of the same units.

### The Shortcomings of Pictured Materials in an Operational Approach

There is a noticeable shortcoming of pictured materials when an attempt is being made to provide for operational meanings. Pictured

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objects cannot be manipulated physically. The authors of all of the
texts examined suggest that the teacher use concrete objects before or
concurrently with the text materials. In some cases, such as in the
Steck Company series and the Winston series, the directions for the
use of concrete materials are rather complete and detailed. In some
other series, the directions amount to little more than the suggestion
that such materials be used. In two series, concrete materials are
included as a part of the programs. The entire readiness program of
the Houghton Mifflin Company is carried on with concrete materials
only, and the Macmillan Company books make extensive use of counters
punched out of stencils which accompany the books. As a general rule,
however, concrete materials are not provided as integral parts of
arithmetic programs, and suggestions in teacher's manuals do not
guarantee their use.

The use of sequences of pictures to depict actions taking place.—
The authors of many of the textbooks surveyed attempt to depict the
physical operations of combining, joining, adding to, etc., by the
use of sequences of pictures. In Book One of the Silver Burdett
series, addition is introduced in page 66, and the authors suggest
the use of concrete objects to illustrate the meaning of addition.
On a number of pages throughout the book following page 66, the
authors depict, either in a pair of pictures or in a single picture,
the action taking place as one group joins another or as two groups
are combined. The number question "how many in all?" is asked orally
or in writing; and the number fact is stated in one or two other
ways, leading successively throughout the book toward more and more "abstract" statements of the number facts involved. On page 17 of Book 1 in the Houghton Mifflin series, Catherine Stern pictures four blue birds joining six red birds along with suggestions for the appropriate number questions to be asked by the teacher. Starting on page 22 of the Workbook for Book One, in the Row, Peterson series, the authors depict joining actions along with the appropriate number questions which are supplied orally and in writing.

The above examples are representative of series in which the action depicted is accompanied in nearly every case by appropriate number questions supplied either orally by the teacher or in written form in the book, or both. In other words, the pictures used in these books appear to be used to illustrate number questions which are asked.

There are examples of the use of such pictures in which there appears to be a different purpose. In some series, after the authors have shown sequences of such number action pictures as referents for number questions, the converse situation is then set up. A pictured sequence is shown, and the child is expected to see it as portraying a certain number situation which calls for the answer to a definite number question. Thus, on page 12 of Numbers in Action of the Scott, Foresman series, the following sequence of pictures, among others appears: In the first picture a group of two pigs and a group of three pigs are shown running toward the same trough from different directions. The second picture shows the five pigs eating from the trough. The pupils have already had experiences with combining groups
in answer to definite number questions, and they are expected in this case to see the sequence of pictures as portraying a definite number question, as yet not provided, and which the children are expected to provide.

In view of the fact that the action of combining was completed in the sequence of pictures described, the children may arrive at the question expected. On the next page of this book, however, only single pictures are shown. In each case the authors are attempting to portray a joining or combining action taking place; but the completed action is not shown. The children are expected to imagine the completed action and to see this as a physical action involving the question "how many altogether?" It is evident in such picture situations that questions other than "how many altogether?" could be asked. For example, in one picture on this page, which portrays a group of three chickens and another group of two chickens running toward some grain, several number questions, such as the following, might conceivably be asked: "how many times as many chickens are there in one group as there are in the other?"; "how many more chickens are there in one group than the other?"

There seems to be no justification for taking the position that a given pictured situation implies one and only one number question. It is the question asked which creates the problem; and, in fact, if no number question is asked, the individual has no good reason even to consider that a number question is involved. Pictures of the same type involving number potentially also appear in several series
in connection with subtraction, multiplication, and division. In cases where the authors use them to imply unstated number questions, the same criticisms apply.

**Summary: The Teaching of Addition**

Every series examined provides a meaning for addition in terms of a set of two operations: (1) the operation of joining one group to another or of combining two groups, and (2) the process of determining the number of objects in the total group. The extensive use of textbook materials in the first grade appears to add a difficulty in providing operational meanings for the elementary arithmetic concepts of the fundamental processes with whole numbers. It is impossible to make the physical moves with pictures of objects which are necessary to provide the operational meanings. This inherent shortcoming of the written materials has resulted in a use of sequences of pictures which are intended to depict certain actions meant to be the referents for more symbolic statements. In some cases, this use of the pictured-action sequence appears to be stretched beyond the value it may have. In particular, the pictured sequence does not, in and of itself, portray one and only one number question. The value of such action pictures appears to be limited to cases in which they appear concurrently with specific number statements.

The need for concrete materials appears inescapable if the operational approach is to be utilized. Pictures of objects and of actions taking place are at least one stage of symbolization beyond the actual use of concrete objects and physical operations on them.
The full meaning of pictures can be clear only if the child has had experiences with the objects and activities portrayed in the pictures. In fact, the best set of materials for teaching arithmetic in the first grade would consist of individual kits of concrete objects—small-scale models of animals, furniture, and other familiar objects; counters; measuring units; play money; place-value devices; etc.—accompanied by a teacher's manual describing their use to develop operational meanings.

The Extent of the Use in Current Arithmetic Textbooks of the Operational Approach to Teaching Subtraction

In this section, an analysis will be made of the use of the operational approach in the teaching of subtraction in the thirteen series of textbooks. In Chapter VIII, it was concluded that from the point of view of operationism there are three distinct types of problems which are solved symbolically by subtraction: (1) problems which involve the physical operations of taking away a part of a whole group and determining the number left; (2) problems which involve physical operations of one-to-one matching and determining the difference (how many more, or fewer, objects there are in one group than another); and (3) problems which involve the physical operations of adding to a group to obtain a desired larger group and determining the number added.

It was also pointed out in Chapter VIII that the take-away idea
appears the easiest to illustrate operationally. Moreover, it is also
the basic subtraction idea in the sense that the equivalence of other
subtraction situations to the take-away situation can be shown readily.
It is this equivalence which permits the various problems to be solved
symbolically by the subtraction algorithm. It also permits the child
to learn one set of subtraction facts which he may use later for
symbolic solutions to any of the three subtraction situations.

A further conclusion reached in Chapter VIII was that the oper-
ational approach provides a means by which the child may solve, with
the use of objects and physical operations on those objects, a number
of quantitative problems long before he is able to cope with them by
means of the usual solutions with numerals. In particular, all of
the subtraction situations can be introduced through an operational
approach long before there is need for the symbols, the numerals, and
the algorithm of subtraction.

The Extent of the Use of Problems Involving Ideas of Take-Away,
Comparison, and Number Needed Prior to Teaching the Symbols and the
Algorithm for Subtraction

A child may determine the number of objects left when three of
a group of seven are taken away; he may determine how much larger a
group of seven objects is than a group of three objects; etc., even
before he is familiar with the symbols 7, 3, 4, and 7 - 3 = 4.
If such problems were presented early in an arithmetic program and
their operational solutions pointed out, this would be a clear indication
that the authors of that program were aware of meaning in the oper-
ational sense.
The authors of some series do present problems involving the ideas of take-away, comparison, and number-needed sometime before they present the algorithm for subtraction. Table V is a summary of those series which present these three types of problems early in the program before any reference is made to the word subtraction or to the symbols and the algorithm for subtraction.

Table V shows that the authors of only three series present the idea of "take-away," which is relatively simple to understand operationally, before they are ready to present the symbolic process of subtraction. In fact, only two series could be said to present it early, since the Scott, Foresman series does not use it until the work for the second grade.

All thirteen series use comparison situations early to aid in the development of ideas of the relative size of numbers. The questions generally involved are "which group is the larger?" or "which group is the smaller?" with no attempt being made to determine how much larger or how much smaller one group is than the other. Table V shows that only six of the thirteen series use comparison situations at this stage, in which there is a specific question asked for the amount of the difference in the sizes of the two groups. The authors of the Row, Peterson series give the following operational method to the teacher for use at this pre-symbolic level so that the pupils have a method for determining how many more objects one group has than another: "To compare two groups, we match them one-to-one and count
Table V

CURRENT ARITHMETIC SERIES WHICH PRESENT SITUATIONS INVOLVING TAKE-AWAY, COMPARISON, AND NUMBER-NEEDED PRIOR TO THE TEACHING OF THE SYMBOLS AND THE ALGORITHM FOR SUBTRACTION

<table>
<thead>
<tr>
<th>Series</th>
<th>Take-Away</th>
<th>Comparison</th>
<th>Number-Needed</th>
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<tbody>
<tr>
<td>American Book:</td>
<td></td>
<td></td>
<td>X</td>
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<tr>
<td>Upton and Uhlinger</td>
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<tr>
<td>Ginn:</td>
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<tr>
<td>Buswell and Brownell</td>
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<tr>
<td>Heath:</td>
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<td></td>
<td>X</td>
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<tr>
<td>Gunderson and Hollister</td>
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<tr>
<td>Houghton Mifflin:</td>
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<tr>
<td>Stern</td>
<td></td>
<td></td>
<td>X</td>
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<tr>
<td>Laidlaw Brothers:</td>
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<tr>
<td>McSwain and Ulrich</td>
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<tr>
<td>Macmillan:</td>
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<td></td>
<td>X</td>
</tr>
<tr>
<td>Carpenter and Swenson</td>
<td></td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Merrill Books:</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Barell and Hagaman</td>
<td></td>
<td></td>
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<tr>
<td>Row, Peterson</td>
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<td>X</td>
<td>X</td>
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<td>Wheat</td>
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<tr>
<td>Scott, Foresman:</td>
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<td>X</td>
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<tr>
<td>Van Engen and Hartung</td>
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<tr>
<td>Silver Burdett:</td>
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<tr>
<td>Morton</td>
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<tr>
<td>Steck:</td>
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<tr>
<td>Benbrook and Foerster</td>
<td></td>
<td></td>
<td>X</td>
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<tr>
<td>Winston:</td>
<td></td>
<td></td>
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<tr>
<td>Grossnickle and Merton</td>
<td></td>
<td></td>
<td>X</td>
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<tr>
<td>World Book:</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Clark</td>
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</tbody>
</table>

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the excess in the larger group."In general, however, series other than Row, Peterson, do not treat the questions of "how many more?" or "how many fewer?" in any great detail until subtraction with numerals is introduced.

Table V also shows that four series present questions requiring a determination of the number of objects needed to make up a desired amount prior to the introduction of symbolic subtraction. Such examples generally appear when the child is studying addition and may be symbolized later in the programs, as in the example $2 \rightarrow \underline{\_} = 5$.

It may be said, then, that situations involving take-away, comparison, or the number-needed are not generally introduced in the books of current arithmetic series until the authors are ready to present subtraction as a symbolic process. Moreover, authors who do present such questions prior to the time they take up subtraction with numerals do not present clear-cut operational methods for their solution.

The Treatment of Subtraction in Current Arithmetic Programs

In all thirteen series, the teaching of symbolic subtraction follows the teaching of addition, although the Row, Peterson series presents take-away and comparison situations prior to addition situations. Table VI shows the order in which the various arithmetic series surveyed take up the different concepts of subtraction in their

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Table VI

THE ORDER IN WHICH QUESTIONS ARE SOLVED BY THE PROCESS OF SUBTRACTION IN THE MATERIALS FOR GRADES I AND II OF CERTAIN CURRENT ARITHMETIC SERIES

<table>
<thead>
<tr>
<th>Series</th>
<th>Take-Away</th>
<th>Comparison</th>
<th>Number-Needed</th>
</tr>
</thead>
<tbody>
<tr>
<td>American Book:</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Upton and Unlinger</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Ginn:</td>
<td>1</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>Buswell and Brownell</td>
<td></td>
<td></td>
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<tr>
<td>Heath:</td>
<td>1</td>
<td>2</td>
<td>-</td>
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<tr>
<td>Gunderson and Hollister</td>
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<tr>
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<td>1</td>
<td>2</td>
<td>3</td>
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<tr>
<td>Stern *</td>
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<td>1</td>
<td>2</td>
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<td>2</td>
<td>3</td>
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<tr>
<td>Durell and Hagaman</td>
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<tr>
<td>Row, Peterson:</td>
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<tr>
<td>Wheat **</td>
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<td>1</td>
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* This series emphasizes the comparison idea first in exercises comparing lengths of blocks, but the process of subtraction is introduced in take-away situations.

** This series presents many exercises in comparing groups by one-to-one matching prior to the use of subtraction. The process of subtraction, however, is introduced in take-away situations, and all subtraction facts throughout the material for the first two years are learned in take-away situations.
developments of the process and the symbolism. The treatment by various series of the three different concepts of subtraction will now be discussed.

Take-away situations, involving the question "how many are left?"—

In all thirteen arithmetic series surveyed, the first use of symbolic subtraction is to answer take-away—how-many-are-left—questions. (Table VI). The Houghton Mifflin and the Row, Peterson series emphasize the comparison idea first, but they introduce symbolic subtraction through take-away situations. Moreover, take-away questions are used almost entirely in all series to teach subtraction facts. All of the series attempt to develop the idea of take-away operationally, and all of them make suggestions, although in some cases hardly adequate, for the use of concrete objects or dramatizations. Series of number pictures depicting take-away situations are used to some extent in all books. A rather common exercise in take-away examples is one in which the child crosses out the pictured objects which are "taken-away."

The activities suggested and used in the Heath series are typical of the kinds of experiences used in most of the thirteen series to introduce and to develop the take-away concept. The authors of this series suggest the use of concrete objects with activities which involve the physical operation of "taking-away," and they utilize both sequences of number pictures and crossing-out examples to further the idea. 7 The authors of the Merrill series suggest dramatization

to develop the idea of take-away, and use sequences of pictures in their written materials to portray take-away actions. The authors of the Row, Peterson series introduce take-away as a counting process. All subtraction problems in Book One of the Silver Burdett series arise in take-away situations; some attempt is made to show what take-away means operationally, but the emphasis appears to be on learning the subtraction facts.

The authors of the Scott, Foresman series give considerable practice in their first book to the operation of breaking up larger groups into various arrangements of smaller groups. They do not introduce the remainder idea when the action of separating into two groups is completed, however, until page 15 of their second book, Numbers in Action. The three stages in teaching used by these authors to provide operational meanings are these: (1) the physical action of separating groups, often portrayed in this series by sequences of number pictures; (2) imagined action, sequences of pictures portraying part of the action in which the child is to imagine the completed action; and (3) symbolism, which proceeds through such stages as

5 - 3 is __, 5 - 3 equals __, 5 - 3 = __, and $\frac{5}{3}$.

To summarize, the take-away situation is the first use of symbolic subtraction employed in all thirteen series surveyed. All the


series attempt in varying degrees to show what take-away means operationally. This is accompanied by directions for the use of concrete materials and dramatizations, by the use of sequences of pictures depicting take-away activities, and by paper-and-pencil activities in which the child crosses-out the objects in a group which are taken-away. It appears that most of the authors feel that the take-away idea is the fundamental idea of subtraction and that the child should have a good understanding of it and learn some of the subtraction facts in take-away situations before he encounters other situations commonly solved by subtraction. It may be said, then, that the authors of current arithmetic series treat the take-away situation operationally in much the way recommended in this study; moreover, these authors tend to look upon the take-away-how-many-are-left problem as the easiest and best means to introduce symbolic subtraction. Most children, however, have probably encountered take-away situations even before they enter the first grade. Yet the most common practice in current arithmetic programs is to defer the take-away problem until the authors are ready to present symbolic subtraction, which generally comes in the latter part of the first grade, or in the second grade.

Comparison situations, involving the question "what is the difference?"—In the comparison situation of subtraction, the physical operation of one-to-one matching is one of the important steps in the set of operations (see Chapter VIII). Each of the thirteen series gives some attention to the development of the operation of one-to-one
matching. This is usually done prior to or concurrently with the work in counting and is considered by several of the authors to be especially important for an understanding of rational counting. It is also used in the work of the first grade in many series in examples in which the child is to determine which of two quantities is the larger, or the smaller, and in a few series to determine how much larger, or smaller, one group is than another. The most common exercise in one-to-one matching requires the child to draw lines between the matched objects of the two groups. The authors of the Scott, Foreman series, however, use markers which may be placed on the objects of one group and then moved to the objects of the second group to determine which group has more, or fewer, objects in it.

In Chapter VIII, the comparison situation and its relationship to the take-away situation were shown operationally. If the difference between a group of \( x \) objects and a group of \( y \) objects is to be found \( (x < y) \), the operational procedure is as follows: (1) the \( x \) objects of the first group are matched one-to-one with objects of the second group; (2) the matched objects of the second group may then be removed (taken away) physically; (3) the number of objects remaining in the second group is then determined. The latter two steps show the relationship of the comparison situation to the take-away situation and, therefore, the reason why it is classified as a subtraction example.

When the comparison idea of subtraction is introduced, all authors take the step of showing the one-to-one matching procedure.
They do not, however, go on from that point to show why comparison situations may be solved by subtraction. For example, the authors of the Ginn series give several situations in which the problem is to determine how many more objects there are in one group than another, showing the symbolic solutions as they show the matching. Then they simply state a rule in bold type: "To find how many more, you subtract." 10

The authors of the American Book Company series consider "the important thing to be that subtraction enables one to find how many more or how many less." 11 The authors give no indication as to why subtraction works in this case other than to present it as an application of subtraction and to show that subtraction produces the correct answer.

The authors of the Macmillan series write that "to tell how many more or how many fewer, we subtract." 12 The Laidlaw Brothers series develops comparative subtraction by the use of one-to-one matching, but it does not show how it is transformable into a take-away situation. 13 The authors of the Steck series say that it is


important to show that one computes comparison problems by subtraction, but they do not clarify the relationship of this concept to the take-away idea. To them the important thing is that "when one needs to find the number more or less, he subtracts." Similar instances may be found in the Merrill series and the Silver Burdett series.

The above authors take the position that comparison should be taught as an application of subtraction; that a relation to take-away does not exist or, at most, exists but is too difficult to show to young children; and that the best the teacher can hope for is to show that subtraction produces the correct answer in the comparison situations.

The authors of the Heath series point out that in comparison situations unlike things may be compared:

It would be unwise to get technical about this with second grade children. However, we can tell them that while we surely cannot subtract coats from caps or collars from dogs, we can subtract the number of coats from the number of caps or the number of collars from the number of dogs to find how many more we have of one kind than the other. We can write the example the same way that we write the example for a take-away problem.15

These authors come close to the heart of the matter and then miss it completely by "getting technical" in their explanation. It is much simpler, and operationally correct, to point out that the coats are not subtracted from the caps, but that as many caps as may be


matched one-to-one with the coats may be taken away from the total number of caps to find how many more caps there are than coats.

Only two of the thirteen series examined could be said to give an adequate discussion of the comparison situation in operational terms. The Winston series is one of these:

To introduce the comparing concept of subtraction, how much more is one than the other; to visualize that in comparing by subtraction, you take away from the larger group the same number of objects as are in the smaller group. 16

They then proceed to develop the comparison idea along these lines and to point out its relation to take-away in the manner they suggest.

The authors of the Scott, Foresman series give what is probably the best discussion of the comparison situation in operational terms. On page 63 of Numbers in Action, they show three sets of pictures depicting the operations in situations in which the problem is to determine how many more objects there are in one group than another. Groups A and B are here portrayed in a sequence of three pictures to illustrate the method used by these authors (Figure 63).

![Picture 1]

![Picture 2]

![Picture 3]

The problem is to determine how many more objects there are in group A than in group B. In the first picture, the two groups are shown in irregular arrangements. In the second picture, the two groups are placed in regular linear arrangements so that the one-to-one matching may be more easily established. In the third picture, the authors portray a hand pushing (taking away) all of the objects of group B and those of group A which have been matched with the objects in group B (here portrayed by a line and an arrow indicating the direction of movement). The authors feel that such a portrayal emphasizes the fact that the comparison situation presents a problem which may be solved by subtraction.

To summarize, the authors of the thirteen series surveyed utilize the one-to-one matching procedure to illustrate the comparison idea. Only two series, however, may be said to go on from the matching operation to show that the comparison situation is related to the take-away idea and thus may be solved by the symbolic process of subtraction. Most of the authors do not present an operational solution which shows the relationship of comparison problems to take-away problems. An explanation for this may lie in the fact that in most books the comparison use of subtraction is taught somewhat later than the take-away use and after the child has attained some facility with symbolic subtraction. Most authors appear to take the position that it is only necessary to point out to the pupils that comparison is an application of symbolic subtraction; they do not attempt to show the relationship of the comparison situation to other subtraction situations.
How-many-are-needed situations.— The question involved in additive-subtraction is usually asked in the following form: If a person has \( x \) objects and wishes \( y \) objects (\( x < y \)), how many more does he need? It was pointed out in Chapter VIII that this problem presents a major difficulty to children for a number of reasons: (1) the problem is usually worded in a language associated with addition rather than subtraction; (2) neither the total group nor the subgroup to be determined is present; (3) there is no comparison group present so that the child may use the procedure learned in comparison situations. It was suggested in Chapter VIII that the how-many-are-needed problem should first be portrayed to children in one of three ways: (1) by adding the number of objects needed and then determining the number added; (2) by starting with the total number needed, differentiating those already on hand, and then determining the number remaining; or (3) by using a comparison group.

The relationship of the how-many-are-needed problem to the take-away-remainder problem may be shown through the use of either the second or the third solution suggested here.

In the American Book series, the authors first introduce the problem in a situation involving a comparison group. On page 29 of Book 1, the pupils are to determine how many more saucers are needed to have as many as there are cups portrayed. When the problem is introduced later for more serious consideration on page 66 of Book 2, the child first solves it by adding as many objects as are needed and then determines the number of objects added. The child then
fills in the blanks in the following statements:

- 3 rabbits and _ _ _ _ _ rabbits are 5 rabbits.
- 3 rabbits from 5 rabbits leaves _ _ _ rabbits.
- 3 from 5 is ___.

\[
\begin{array}{c}
5 \\
-3 \\
\end{array}
\]

This approach does not show why the problem is a subtraction problem. The authors are assuming in the second and third statements that the child already recognizes the situation as one requiring subtraction.

The authors of the Merrill series suggest that concrete situations in which there is a comparison group present should be used at first to present the how-many-are-needed problem. They suggest experiences in finding how many more pencils, scissors, etc., are needed for a given number of children. Similar situations in which how-many-are-needed problems are presented with comparison groups present may be found in the Macmillan series (page 35, More About Numbers), the Row, Peterson series, (page 82, Book Three), the Houghton Mifflin series (page 15, Book Two), and the Silver Burdett series (page 93, Book 2).

The authors of the Steck Company series present the following problem situation: If Sue wants a doll which costs five dollars, and she has three dollars, how much more does she need? The authors then give two ways in which it may be portrayed. One of these utilizes

17. Upton and Uhlinger, Book 2, op. cit., p. 70.

a comparison group. The other method may be shown as follows:

\[
\begin{align*}
\text{Doll costs} & \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
\text{Sue has} & \quad 0 \quad 0 \\
\text{Sue needs} & \quad 19
\end{align*}
\]

This solution starts by portraying the cost of the doll, five dollars. Sue has one part, three dollars. The problem is to find the other part of the cost of the doll. The latter method used by these authors is the second of the three methods suggested for solution of this problem in Chapter VIII, and is the solution which is most directly related to the take-away-remainder idea.

The authors of the Scott, Foresman series present this problem by "adding to" the number on hand and then making a determination of the number of objects added. They extend the idea, however, in an attempt to show the relationship of the how-many-are-needed solution to the take-away solution. On page 67 of Numbers in Action, they present the following sequences of number pictures: (1) the first picture portrays a box with 7 square holes into which 7 cubes will fit, and there are 6 cubes present; (2) the second picture shows another cube being brought in; (3) the third picture shows the seventh cube being placed in the empty hole; (4) the fourth picture portrays a hand "taking away" from the 7 cubes the 6 cubes which were shown at first, leaving the one cube.

Most authors, then, present the how-many-are-needed problem in situations which are very similar to comparison situations. Such a

practice is desirable provided the relationships between the comparison situation and the take-away situation have been shown. It was pointed out in the previous section, however, that very few of the authors show operationally the relationships between the comparison situation and the take-away situation.

**Summary: The Teaching of Subtraction**

All of the thirteen series introduce the symbolic process of subtraction in take-away situations. This indicates that most of the authors feel it is the basic subtraction idea, both in the sense that it is the first idea to be learned about subtraction and in the sense that it is the best one to use in learning the subtraction facts; and they attempt to develop the take-away idea operationally. Only two of the thirteen series, however, could be said to give an adequate treatment of the comparison idea in showing its relationship operationally to take-away. For the most part, authors point out that a comparison situation involves a one-to-one matching procedure, then show symbolically that subtraction produces the correct answer, and thus treat comparison as applications of subtraction.

Several authors present the how-many-are-needed problem through situations which are very similar to comparison situations; however, since the operational treatment of comparison situations is seldom complete in modern arithmetic series, this treatment of how-many-are-needed situations could not be called an adequate operational treatment.
The Extent of the Use of the Operational Approach
in Teaching Multiplication

The conclusion was reached in Chapter VII that the multiplication of whole numbers has one meaning operationally. The meaning is nearly identical with that of addition. The set of operations for multiplication, however, differs from the set of operations for addition in the first step. In multiplication the child must first determine, or reproduce, the number of objects of each of a number of equal groups and the number of such equal groups. The second operation is that of combining the groups, and the third operation is that of determining the number of objects in the combination group.

Results of the Survey

The results of the survey of the thirteen arithmetic series indicate that less space is given to developing the meaning of multiplication than is given to developing the meanings for any of the other three arithmetic processes. There appear to be definite reasons why authors give more space to addition, subtraction, and division than they do to multiplication. Ideas of addition are commonly taught in the first grade to very young children. Considerable time is given to developing the concept of addition and to teaching the number combinations. Subtraction and division are both used to answer a variety of quantitative questions which are different in their operational solutions, in their wording, and in their social applications. However, the multiplication of whole numbers is closely related to
addition; it is taught later, after the children have had considerable work with the symbols of arithmetic; and it is not used to answer as wide a variety of questions as may be answered by either subtraction or division. As a result, comparatively little space is expended in textbooks in developing the concept of multiplication, although much space is usually given to teaching the multiplication facts.

A number of authors of modern arithmetic programs maintain that there are three ways to determine the answer to a so-called multiplication problem. For example, the authors of the World Book Company series first use the word multiply in the following manner:

Saying "5, 10, 15" as Billy did is counting to find three 5's.
Saying "5 + 5 + 5 = 15" is adding to find three 5's.
Saying "Three 5's are 15" is multiplying.
Harry did a multiplication example.
He multiplied 5 by 3.

On page 265 of Book 3, the authors of the Silver Burdett series introduce multiplication with these two questions: (1) How many are 7 and 7? and (2) How many are two sevens? The Laidlaw Brothers series takes the same approach on page 197 of Book 3, and the Steck Company series uses the idea on page 87 of Worktext, Book 2. On page 89 of Workbook 3, the authors of the Ginn series use a set of statements similar to that in the World Book Company series. Similar examples may be found in the books of the American Book Company series (page 67,

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Workbook 3), the Merrill series (page 34, Workbook 2), the Macmillan series (page 37, Workbook 3), and the Winston series (page 134, Using Numbers). These examples of the ways in which multiplication is introduced indicate that if the addends are all equal size, then the language of multiplication is a linguistic device for saying in a short way what has to be said in a long way if the addition language is used.

The idea of multiplication as putting equal groups together is emphasized in the Scott, Foresman series (pages 17-18, Book 3), the Heath series (page 142, Book 2), and the Row, Peterson series (page 222, Book 3). The relationship of multiplication to division is brought out in most of the series reviewed.

Summary: The Teaching of Multiplication

All the series relate multiplication closely to addition, and, in fact, explain it as the process of determining the total number of objects when the groups being combined are all equal in size. All the series emphasize the fact that the groups must be equal before multiplication may be used, and that multiplication differs from addition of equal quantities and counting ahead by equal increments essentially in the way the question, or the answer, is worded. In all series, however, the teaching of multiplication comes in the second or third grade after the child has had considerable experience with symbols and symbolic solutions for quantitative questions. As a result, little attempt is made to treat multiplication operationally, either with the use of concrete objects or with pictures. Authors
generally seem to take the position that multiplication is closely related to addition and that since the child is familiar with symbolic addition, it is necessary only to point out the relationship without showing the operational aspect.

It is desirable, of course, to work toward the symbolic level in arithmetic. In fact, the value of the operational approach in arithmetic is that it provides clear and unambiguous referents for the symbols the child will eventually use. The usual treatment of multiplication in current arithmetic series, however, is a good illustration that, even though a new topic may be introduced after the child has had considerable experience with symbols, the operational approach aids in making the new ideas clear and understandable. The meaning of multiplication itself and the commutative nature of multiplication are clearly portrayed with objects, and are made more convincing to the pupil than are verbal or written symbolic statements regarding them.

The Extent of the Use of the Operational Approach in Teaching Division

Introduction

In Chapter VIII three different meanings for division were discussed: (1) measurement or quotition division, in which the question is "how many groups of x objects are there in a group of y objects (x<y)?" (2) comparison division, in which the question is "how many times as large as a group of x objects is a group of y objects
(x < y)\) and (3) partition division, in which the question is "if a group of y objects is divided into x equal parts (x < y), how many objects are there in each equal part." It was shown in Chapter VIII that if a larger group of objects was divided into n equal parts, then one of the equal parts could be called one-n\(^{th}\) of the larger group. Other operational solutions for finding one-n\(^{th}\) of a group were demonstrated; therefore, it would not be correct to say that the problem of finding one-n\(^{th}\) of a group is exclusively a partition problem. It should be treated differently and named differently at first.

In every series surveyed, the teaching of division follows, or is concurrent with, the teaching of multiplication. By the time division is introduced, the pupils have had considerable experience with the symbolization of arithmetic. As a consequence, even though most authors make a distinction between measurement and partition division, there is a tendency to do this through word explanations and arithmetic symbols; and it is common to include comparison as a measurement use of division and the problem of finding one-n\(^{th}\) of a group as a partition problem. The teaching of measurement division will be discussed first.

The Teaching of Measurement Division

Twelve of the thirteen series surveyed introduce the arithmetic process of division through the measurement idea. The exception is the Winston series, which uses questions of both measurement and partition to introduce division (on page 269 of Discovering Numbers).
Each of the series by Ginn, Macmillan, Merrill Books, and Row, Peterson uses situations involving one-n² of a group in the materials for the second grade. The Laidlaw Brothers and the Steck Company series use this idea in materials for the first grade. But in none of these latter six series is the solution of problems involving one-n² of a group called division at these early levels.

The authors of the Steck Company series introduce division in Chapter 9, "Easy Division Facts," of Book 3. They start Chapter 9 with the topic "The Meaning of Division," and present a measurement question of the following type: If lollipops cost three cents each, how many can be bought for 12 cents? The authors present four solutions for the problem. The first solution involves a matching procedure of one lollipop for each three cents, thus:

| lollipops | 0 | 0 | 0 | 0 |
| pennies   | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

The second solution involves the procedure of regrouping the 12 pennies into groups of threes, thus:

| pennies   | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

The third solution involves subtracting 3 from 12 four times; and the fourth solution treats the problem as the inverse of multiplication—since there are four 3's in 12, then four lollipops can be bought for \( \frac{12}{4} \). The authors introduce the symbolism \( \frac{12}{4} \) on the next page.

and two pages later state the rule: "Divide to find the number of equal groups." Measurement is also introduced in other series through the use of situations similar to the one used by the Steck Company. Such situations usually involve some item which costs $x$ cents per unit, and the problem is to determine the number which can be bought for $y$ cents ($x < y$). For example, the authors of the World Book Company series ask on page 211 of Book 3 how many 5-cent balloons can be bought for 35¢. They suggest three methods of solution: (1) successive subtractions of 5's; (2) regrouping in 5's; and (3) the use of multiplication—since $7 \times 5\$ equals 35¢, then there are seven 5's in 35. The operational solution for measurement division discussed in Chapter VIII is the second of the solutions given here by both the Steck series and the World Book series. The errors in considering division as repeated subtractions were pointed out in Chapter VIII. This method of treating division in current arithmetic programs will be discussed later.

Another kind of question used to introduce measurement division asks how many $x$'s there are in $y$. Many series use this type of problem. The problem of how many 2's there are in 8 is considered on page 142, Book 2, of the Row, Peterson series. The authors show an operational solution of regrouping the objects with pictures of blocks. The division algorithm is not used until page 236 of Book 3. The authors of the Silver Burdett series ask how many 3's there are

22. Ibid., p. 98.
in 6 on page 290 of Book 3. A picture of two hats with three flowers on each hat is shown. No attempt is made to indicate the actual regrouping. Consequently, this procedure may more correctly be termed one of showing the relation of multiplication to division, and is not an operational solution for the division problem.

The American Book Company authors show how to find the number of 2's in 14 by circling pairs of objects. They say to the pupil: "When you find how many 2's there are in 14, you are dividing 14 by 2. The answer is 7. This can be written a short way like this: $14 \div 2 = 7$. The sign $\div$ means divided by." The authors of the Merrill series introduce the measurement problem in a similar example. These authors say that examples of this type are "preparation for understanding the meaning of division and multiplication," a remark that is all too common in arithmetic textbooks. Actually, when the authors have the pupils regroup 12 objects into subgroups of 2 objects each, this is the "meaning" of measurement division. The authors may choose to call such activities preparation for symbolic division and multiplication, but they do not constitute "preparation for understanding the meaning of division and multiplication."

The authors of the Scott, Foresman series first introduce the measurement idea of division on page 53 of their second-year textbook,

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**Numbers In Action.** The emphasis in this series is on separating groups into equal subgroups, that is, regrouping a total group into equal subgroups. Moreover, there is a gradual development of symbolism and algorithm which is characteristic of this series as the child continues to solve situations in which he is to find the number of x's in y. For example, the division sign (÷) is not used until page 28 of Book 3, even though the child by that time has been solving measurement problems for more than a year.

**Division as Repeated Subtractions**

The emphasis on division as repeated subtractions appears in a number of books. For example, the authors of the Winston series entitle page 274 of *Discovering Numbers*, "Tonny Discovers Two Meanings of Division." The question Tommy asks himself is "How many groups of 2 Indians can I make from my 8 Indians?" The authors portray Tommy's solution by use of successive subtractions as Tommy takes two Indians away at a time and determines the number of subtractions he makes. In this example, the authors appear to be forcing the analysis of division as repeated subtractions, where simple regrouping and counting the number of groups would be a more direct operational approach.

The authors of the Macmillen Company *How to Use Arithmetic* series describe a situation in which they find the number of 2's in 10 by

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repeated subtractions. In brief, they describe the sequence of events as follows: Jane started with 10 pennies in her hand. The next time she showed her open hand she had only 3 pennies, the next time only 6 pennies, and so on until she had no (0) pennies left. The authors accompany this description with the symbols showing the repeated subtractions. Then they write:

If you have 10 pennies and subtract 2 at a time, how many times can you subtract 2 pennies?
How many 2's are there in 10?
You subtracted to find how many candy canes Jane could buy. You can divide to find the same thing. You can see below that division is quicker.

<table>
<thead>
<tr>
<th>Subtraction</th>
<th>Division</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 - 2 = 8</td>
<td>10 = five 2's</td>
</tr>
<tr>
<td>8 - 2 = 6</td>
<td>4 - 2 = 2</td>
</tr>
<tr>
<td>6 - 2 = 4</td>
<td>2 - 2 = 0</td>
</tr>
</tbody>
</table>

There is a quick way to subtract the same number many times. It is called division.

Although the authors ask the pupils how many times 2 pennies can be subtracted from 10 pennies, they do not emphasize this point. They go on to say simply that "you subtracted to find how many candy canes Jane could buy." But the children must do more than subtract if they are to answer the question in this example. They must determine the number of subtractions made.

The set of operations for take-away subtraction consists of two operations: (1) a part of a group is taken away from the whole group

27. Ibid.
(there is only one physical action of taking away unless the objects are removed one at a time or in other subgroups which are smaller than the take-away group); (2) the number of objects remaining is determined (this operation answers the question "How many are left?"). On the other hand, the set of operations for division shown as repeated subtractions consists of the following. A number of objects is taken away from the group. The same number of objects is taken away from the remaining objects of the original group. This operation is repeated, each time taking away the same number, until there are no objects left in the original group. Contrary to the way in which these take-away steps are commonly described in textbooks, it is not necessary to know the number of objects remaining in the original group after any one action of taking away. Thus, the word, subtraction, is not applicable to this analysis. The number of taking-away actions must be determined, usually by counting them as they take place. Such an operation answers the question, "how many times was the taking-away operation performed?" This question is not the same as the question in take-away subtraction.

The two sets of operations described above are different and are used to answer two totally different questions. If authors of textbooks use the method of repeated subtractions to find answers to questions involving the measurement use of division, they should emphasize the differences between the questions in the two situations and the differences between the two sets of operations used.
The Measurement Idea and the Long-Division Algorithm

In Chapter VIII it was concluded that the thinking of partition division is better adapted for teaching the long-division algorithm than is the thinking of measurement division. Nevertheless, in all the series surveyed the measurement use of division is the one employed in teaching the algorithm.

The difficulty in using the thinking associated with the measurement idea while solving an example by the long-division algorithm is evident when the authors of the Silver Burdett series show the example $6\overline{270}$ on page 222 of Book 4. The authors state that the 2 cannot be divided by the 6, so that it is necessary to divide 27 by 6. Actually if the measurement idea is applied, the thinking should be as follows: There are no hundreds of 6's in 2 hundreds. Hence it is necessary to think that there are four tens of 6's in 27 tens. This thinking is awkward. The partition thinking would proceed as follows: It is impossible to find one-sixth of 2 hundreds (or of 2 of anything) without the use of fractions. Hence, one finds one-sixth of 27 tens. One-sixth of 27 tens is 4 tens. It appears that partition thinking lends itself better to an analysis of such long-division examples than does the thinking associated with the measurement division. This assumes, of course, that an attempt is being made to show the meaning physically of the different steps in the algorithm. Actually, the authors of the Silver Burdett series treat this problem strictly on a symbolic level when they point out that "we cannot divide the 2 by 6."^{28}

Summary and Discussion of Measurement Division

Most modern arithmetic series, then, introduce division through situations involving the measurement (quotient) idea. Division is commonly introduced last of the so-called "four fundamental operations of arithmetic," and generally after the pupils have had considerable experiences with the symbolism and algorithms of arithmetic. As a result, the explanations accompanying measurement situations are often verbal or written and appear in many cases to be used simply to introduce the new symbolism of division.

Even if it is granted that the division algorithm is the most difficult of the four algorithms the child learns to perform with whole numbers in the primary grades, this is still no reason to delay the presentation of measurement and partition situations until he is ready to learn the algorithm. The problem of finding the number of 2's in 6 is no more difficult when the child performs a physical regrouping of 6 objects than is the problem of finding the total of two groups of 7 objects and 8 objects each. Thus, the operational approach provides solutions for problems long before the child may be ready for the algorithm, or even before he is ready for written answers using numerals. In fact, the operational solutions provide the meanings for the written symbols and the algorithms which will be taught later.

The results of the survey of modern arithmetic textbooks indicate that most authors do not present the measurement problem in any form until the third grade and until they feel the pupils are ready to learn such configurations of symbols as \( \frac{1}{4} \sqrt{8} \) and \( 5 \cdot 2 = 4 \). The Scott,
Foresman series was found to be the most noteworthy exception. In this series the measurement problem is presented in the second grade and the symbolic development takes place over a period of a year before such signs as $\ast$ are introduced.

The Teaching of Partition Division

The operational solution for finding the number of objects in each group when $y$ objects are divided into $n$ equal groups is as follows: (1) areas are established to which the $y$ objects are assigned one by one in turn; (2) the number of objects in any one of the equal groups is then determined. In Chapter VIII it was shown that there are various methods for finding one-$n^{\text{th}}$ of a group; and hence this problem is not operationally identical with the partition problem, although the number of objects in one of the $n$ equal groups resulting from the partition division may be called one-$n^{\text{th}}$ of the original group.

Partition is introduced prior to the process of symbolic division in at least seven of the series through situations in which the purpose is to find one-$n^{\text{th}}$ of a group. For example, on page 68 of the second half of Book 2 of the Ginn series, the child encounters examples in which one-half of a group is portrayed. These examples are not related to division at this point, and the child does not find one-half a group. The pictures are of groups already divided in halves, and the child is to determine the number of objects in one of the two equal subgroups.

In twelve of the thirteen series, however, the use of the arithmetic process of division to solve partition questions follows its
use in measurement questions. Approximately ten pages after their introduction of division through measurement examples, the authors of the Steck Company series use a situation involving one-half of 8 objects to introduce the partition idea. The authors show the division of 8 objects into two equal subgroups, without actually showing how the division of 8 objects into two equal subgroups, without actually showing how the division 8 is brought about. They accompany this with two symbolic records, \(1/2 \times 8 = 4\) and \(2 \div 8\), and then state the rule: "Divide to find the number in each group." The authors, in reality, show only the completed picture of a partition operation and present partition as an application of the use of the algorithm which was introduced under measurement division. Such a procedure could not be called operational.

The authors of the Macmillan Company series introduce the partition use of division in the chapter following their introduction of measurement division and direct the pupil: "Finding one-half a group is the same as dividing by 2." This approach, too, is simply one of showing partition as an application of the division algorithm.

The authors of the American Book Company series also introduce the partition idea immediately following the introduction of division through measurement situations and write that "another way to write 1/2 of 6 is 6 \(\div 2\). This means 6 is divided into 2 equal parts. A quick way to find 1/2 of 6 is to think 6 \(\div 2 = 3\)." It is difficult

to understand why a quick way to think of "1/2 of 6" is "6 ÷ 2 = 3," other than that the pupils learned "6 ÷ 2 = 3" before they learned "1/2 of 6." The authors of this series treat division throughout largely in terms of processes with numerals.

The authors of the World Book Company series use situations involving halves, fourths, etc., for some time before situations involving fractional parts are called division problems on page 253 of Book 3. Later, in the work of grade four, these authors review measurement and partition situations with the following examples:

\[
\begin{align*}
\text{A} & \quad \begin{array}{c}
\includegraphics[width=0.2\textwidth]{dots.png}
\end{array} \\
\text{B} & \quad \begin{array}{c}
\includegraphics[width=0.2\textwidth]{zeros.png}
\end{array}
\end{align*}
\]

Here are 14 dots. This picture shows there are \( \frac{1}{2} \)'s in 14.

\[
\frac{14}{2} = 7.
\]

Picture A above shows that 14 ÷ 2 = 7.

Picture B shows that 1/2 of 14 also equals 7. Do you agree?

While the pupils may agree, the authors fail to point out that picture A also shows that 1/7 of 14 is 2 and that picture B shows that 14 ÷ 7 = 2.

In the Winston series, the authors introduce the partition concept on pages 91 and 92 of Using Numbers, prior to the process of division. There they show how to divide a total group into two equal subgroups in two different ways. One of these is the partitioning operation in which one object is given to Bill, one object to Jim, one object...

to Bill, one object to Jim, etc. In the second method, the authors suggest the pupils use disks placed in a row. The child places a pencil between the first and second disks, next between the second and third disks, then between the third and fourth disks, etc., until he has placed the pencil in such a way that it divides the total group equally. The second method suggested by these authors is considerably more difficult when one is trying to find one-third, one-fourth, etc., of a group, since it is more difficult to discriminate when a group is divided in the ratio of 2 to 1, 3 to 1, etc., than when it is divided into two equal parts. However, the Merrill series is one of the few series which show different operational methods for finding one-nth of a quantity. This problem is commonly considered exclusively a partition problem in most of the series surveyed.

Summary of the Treatment of Partition Division

In twelve of the thirteen series, the treatment of partition as an application of division follows by a few pages the introduction of division through measurement situations. In many cases it is treated as an application of the algorithm the child has learned in connection with measurement situations. The partition operation (the operation of dividing groups of objects equally into a given number of subgroups) is probably one of the earliest quantitative situations a child encounters. The idea of dividing several things one by one among so many people or groups is a simple operation to perform and to understand. Yet many arithmetic series delay a treatment of this problem until the child is ready to study division as an arithmetic process;
and, in fact, since in many series partition follows the use of the division algorithm in measurement situations, the treatment of partition is delayed until after the child is familiar with the algorithm. In seven of the thirteen series, in which partition is introduced sometime prior to the process of division, the treatment is often very brief. In the main, partition is presented as an application of the division algorithm which was introduced by means of the measurement aspect of division. It is presented through the problem of one-$\frac{1}{n}$th of a group. The presentation is made largely through a written discussion of symbolic processes rather than through operational solutions. Most modern arithmetic series treat the problem of finding one-$\frac{1}{n}$th of a group as though it were equivalent to partition division.

Summary: Survey of Textbook Series

Chapters IX and X reported the results of a survey made of thirteen modern arithmetic programs for the following purposes:
(1) to determine the extent of the emphasis on counting and learning symbols in early number work; (2) to evaluate the number pictures used in these series and to determine the extent to which such materials are used to develop the number abilities of children; (3) to determine the extent to which these series make use of an operational approach in providing meanings for the terms, symbols, rules, and algorithms of arithmetic in the primary grades.

The findings and conclusions may be summarized as follows:

1. In ten of the thirteen series surveyed, rational counting to
determine the number of objects in a group is one of the first lessons the child encounters. This indicates that most authors of modern arithmetic programs are of the opinion that counting is a basic skill which must be learned to some extent before any further progress can be made with number ideas. The research in subtitizing and grouping does not support this emphasis on counting as a basic arithmetic skill.

2. In the written materials provided for the first grade in all thirteen series, the child learns to read some of the numerals; in twelve series he learns to write the numerals; and in twelve series he learns to read some of the number-words. One may conclude that these authors are of the opinion that knowledge of the written symbols, particularly the numerals, is important, if not indispensable, to advancing quantitative thinking in the primary grades. It has been shown that in the operational approach this reliance upon written symbols is not necessary in early number work.

3. Modern arithmetic series present a variety of number pictures in the materials for the first and second grades. Five of the thirteen series portray high-quality number pictures as judged by the simplicity criteria, while the number pictures in four of the series were judged "poor." Most books use number pictures to illustrate the "group meaning" of the number symbols. While all series provide for some practice in the immediate recognition of the number of objects in a group, only two series could be said to make early and persistent efforts for furthering this ability. A conclusion which appears
in order here is that the materials in many current arithmetics are of such a quality that they could be used for furthering the ability to subitize to a far greater extent than they now are.

4. All thirteen series describe addition as a set of two operations: (1) the actual moving together of two or more groups of objects, and (2) the determination of the number of objects in the new total group. There is evidence, however, that the authors give the major attention to the second operation through the amount of space devoted to illustrating number combinations.

5. In all thirteen series, the first use of the algorithm of subtraction is to find answers to take-away situations, although comparison situations are presented in several books prior to the introduction of subtraction as a formal topic. Moreover, all thirteen series attempt to make take-away subtraction operationally meaningful. After the topic of subtraction has been introduced in take-away situations and some of the subtraction facts learned by the pupils, however, there is a tendency to show comparison situations and number-needed situations as applications of subtraction. Only a few series may be said to show operationally the way in which comparison situations and additive-subtractions are related to take-away situations. One may conclude that, in the main, authors feel that the take-away idea is the basic idea in subtraction and that it is only necessary to show that the arithmetic produces the correct answer for other subtraction situations. Therefore, most authors seem to be in agreement with the findings in this study to the extent that they treat the take-away situations as the basic subtraction idea and attempt to show its
meaning operationally. They are in disagreement, however, with the findings in that they treat comparison situations and number-needed situations largely as applications of the subtraction algorithm, and do not give adequate operational treatments of them.

6. The physical meaning of multiplication receives less attention in modern arithmetic series than the physical meanings of the other three so-called fundamental processes of arithmetic. The fact that multiplication is commonly introduced after the children have had considerable experience with the symbols of arithmetic, plus the close relation of multiplication to addition, probably accounts for this. Authors appear to feel that it is only necessary to point out under what conditions one may multiply in preference to adding.

7. All thirteen series distinguish between the measurement and the partition uses of division. However, they do not in general differentiate comparison division from measurement division, nor do they treat the problem of finding one-$n^{th}$ of a group as distinguishable from partition division. Although partition as one-$n^{th}$ of a group appears quite early in some series, twelve of the thirteen use the measurement concept to introduce the process of division and its algorithm. Partition is then presented as an application of the division algorithm rather than shown operationally. This does not accord with the advantages found in this study for introducing partition first.

8. The most common sequence for introducing the processes of arithmetic in the thirteen series examined is this: (1) counting, (2) adding, (3) subtracting, (4) multiplying, (5) dividing. This
is the traditional sequence, and it developed in this way largely
because of a real or supposed progressive difficulty of the algorithmic
solutions. Although the series of physical operations for finding
answers to such questions as "how much larger is y than x?" and
"how many x's are there in y?" are as easy and understandable as the
series of physical operations for adding, there is a tendency in
modern arithmetic programs to delay such questions until it is felt
that the child is ready to study the algorithms by which such questions
may be answered. Regardless of the emphasis some authors claim to
place on the use of concrete materials, the above is, unfortunately,
clear evidence that they think of the arithmetic of the primary grades
largely in terms of manipulating symbols and not in terms of an oper­
ational approach in which physical operations are performed with
concrete objects.

9. The authors of all thirteen series make suggestions for the
use of concrete materials for teaching arithmetic in the primary
grades. In some cases there are rather detailed directions for the
use of such materials. In other cases, it is simply suggested that
they be used. Only two of the series surveyed provide such concrete
materials as an integral part of their programs. Consequently, one
may say that the materials provided for the arithmetic work of the
early grades consist in the main of written and pictured materials.
This may be both a result and a cause of the early emphasis on learning
the written symbols of arithmetic. In any event, it is impossible
to move and to manipulate written and pictured materials to provide
the operational meanings described in Chapters VII and VIII. To overcome this difficulty, all books make some use, and some books make extensive use, of sequences of number pictures which are devised to show some action taking place. Such sequences of pictures are, of course, valuable, but they should follow as the next step in symbolization after the actual physical manipulation of concrete objects. In this chapter, one misuse of picture sequences is pointed out. A sequence of pictures does not in itself imply a definite quantitative question. A number question must be asked about the groups portrayed in the sequence before arithmetic is involved at all.
Chapter XI

SUMMARY, FINDINGS, AND RECOMMENDATIONS

Summary

In this study, review was made of (1) psychological studies in the discrimination and the perception of the number of objects in a group, (2) anthropological studies which are indicative of the concrete basis of number ideas, and (3) the literature of the theory of operationalism.

The findings of the above sources were used (1) to point out their implications for the teaching of whole numbers in the primary grades, and (2) to analyse the ways in which thirteen current arithmetic series deal with early number experiences.

Psychological studies in the discrimination and the perception of the number of objects in a group. — Authorities in the teaching of arithmetic vary widely in the extent to which they advocate the use of grouping procedures in early number work. In part, this wide variation appears to arise from differences in opinion as to the limits of the ability to perceive the number of objects in a group and as to whether or not this ability is dependent upon the ability to count. Psychological studies of the perception of the number of objects in a group made with adults, children, and animals were reviewed, then, in order to see whether these studies could throw...
any light on three questions. The questions are these: (1) what are the limitations of the ability to perceive the number of objects in a group and to use this ability in more general grouping procedures?; (2) what is the relation between counting and grouping; is one prerequisite to the other?; (3) which process, counting or grouping, is the better foundation upon which to build for the later meanings and skills of arithmetic?

Anthropological studies of number vocabularies and the uses of number among primitive peoples.-- Over the ages man has developed and refined the written symbols and algorithms of arithmetic to such an extent that there is a tendency to forget the concrete basis of number ideas and to view arithmetic as the study of algorithmic solutions, employing numerals, to quantitative questions. In this study anthropological studies were reviewed with the aim of seeing how primitive peoples, who have little or no written number language, solve quantitative problems and communicate number ideas.

The literature dealing with the operational theory of meaning.-- There is little evidence that meaning as it is conceived in the theory of operationism has been worked out in any detail for the teaching of arithmetic. The literature pertaining to the theory of operationism was reviewed in order to point out in theory the essential features of the operational approach to meaning, and to work out the implications of operationism for the teaching of the four fundamental processes--addition, subtraction, multiplication, and division--with whole numbers in the primary grades.
Survey of textbook series.— A survey was made of the printed teaching materials provided by thirteen current arithmetic textbook series in order to determine the extent to which findings from the studies in psychology and anthropology and from the literature on operationism have influenced the writers of these textbooks.

Findings

Subitizing: The Perception of the Number of Objects in a Group

1. The limits of the ability of adults to perceive the number of objects in a group is slightly greater than many authorities hold. Adults can perceive visually up to and including six objects presented simultaneously with a high degree of reliability. They can perceive four or five auditory stimuli presented successively with a high degree of reliability when the time intervals between stimuli are short enough to prevent counting, yet long enough so that the stimuli are perceived as discrete. Adults are not able to do so well in perceiving visual stimuli presented successively.

2. The ability of young children to perceive the number of objects in a group appears to depend on a number of factors, such as maturity, previous experiences with groups of objects, etc. Preschool children, four-to-six-years of age, can perceive visually approximately four objects presented simultaneously.

3. It is more difficult to perceive successively-presented stimuli than it is to perceive simultaneously-presented stimuli. Therefore, it appears that there is a greater need for counting when
determining the number of successively-presented stimuli than when determining the number of simultaneously-presented stimuli.

4. It is possible for a subject to perceive the number of objects in a small group and to discriminate between two groups on the basis of number before he can count the number of objects in a group. Binet found that a four-year-old child he used could discriminate between four objects and five objects even though she could count only to three. Certain animals, particularly some species of birds, which do not count, were nevertheless able to learn to discriminate between four objects and five objects and to repeat an action correctly up to six times.

5. Pre-school children and beginning first-grade children use their abilities to perceive the number of objects in a group in their solutions of quantitative situations. After they have been exposed to the usual emphasis on counting in early number work, however, they tend to use counting procedures even in situations where grouping is easier, just as reliable, and much quicker.

6. Number pictures which operate for ease of apprehension and thus encourage the use of subitizing and grouping procedures are composed of simple geometric figures in regular patterns imbedded in simple contexts. The excessive use of regular patterns, however, would appear to inhibit rather than to abet the development of the ability to subitize, since there is evidence that it is the pattern rather than the number of stimuli in a pattern which is immediately recognized.
7. Most beginning first-grade children can count well beyond
the number of objects which appear in the groups with which they work. The ability to count is not prerequisite for working with these small groups, and counting is not the most efficient means for dealing with such groups. Therefore, there appears to be little need for the excessive emphasis placed on counting procedures early in many arithmetic programs.

8. There is a limit to the ability to perceive the number of objects in a group and to discriminate on the basis of number. This limit appears to depend to a large extent on the acuity of the sense involved. The ability to perceive number and to discriminate on the basis of number must be learned and can be improved through practice within the limits of sensory acuity. In general, it appears to be more difficult to learn to discriminate between two groups on the basis of number than it is to learn to discriminate on other bases such as area covered, color, pattern, etc.

The Concrete Basis of Number Ideas and Operational Meanings

1. There is a clear indication that number ideas originate in and remain closely related to concrete reality. This is indicated by the methods that primitive man uses in answering quantitative questions and in communicating answers without resorting to extensive arithmetic systems or number vocabularies, and by the close relationship he maintains between his quantitative words and the objects or ideas to which they refer.

2. Operational theory provides a clear-cut and readily usable
interpretation for meaning in arithmetic. Meaning in the operational sense may be stated as follows: The meaning of a term, symbol, or statement consists of a set of operations which one performs in applying that term, symbol, or statement in a concrete situation. In lieu of the actual performance of the operations, a description of the performance may be substituted.

3. The terms, symbols, and statements of arithmetic in the primary grades have meanings either through direct reference to concrete objects and physical operations on those objects, or through short regressional routes which terminate in concrete objects and physical operations.

4. There are three operations which may be used to determine the number of objects in a group: (a) subitizing and grouping procedures which utilize subitizing, (b) matching, and (c) cardinal counting. The last operation in any set of operations is designed to determine the answer to a quantitative question and involves the use of one of these three operations.

5. There are three kinds of operations which appear in problems with whole numbers commonly called addition problems. The first of these is a physical operation of adding to, combining, or bringing together objects of the same denomination or value. An operation is used to determine the number or symbol associated with the total, that is, the answer to the quantitative question asked. The operation of substituting a quantity for its equal becomes of considerable importance when the so-called process of carrying is involved.
6. From the point of view of operationism there are three distinct types of problems which are solved symbolically by the arithmetic process of subtraction and, therefore, there are three meanings for subtraction. The three are the following: (a) problems which involve the physical operation of taking away a part of a whole group and determining the number of objects left; (b) problems which involve the physical operation of one-to-one matching (comparison) and determining the difference in the sizes of two groups; (c) problems which involve the physical operations of adding to a group to obtain a desired larger group and determining the number of objects added.

7. The operational meaning of multiplication is to determine the number of objects in a group when it consists of a known number of subgroups all of which contain the same number of objects. In order to determine the total, the groups are combined, or imagined to be combined.

8. There are three meanings operationally for division. (a) In "measurement division" a whole group is regrouped into equal-sized subgroups, and the number of subgroups thus formed is determined. (b) In "comparison division" the number of times the objects of a smaller group can be matched with the objects of a larger group is determined. (c) In "partition division" a whole group is divided equally into a given number of subgroups, and the number of objects in a subgroup thus formed is determined. Although one of the n equal subgroups of a group may be called one-n\(^{th}\) of the total group, one-n\(^{th}\) of a group may be found operationally by other means. Hence, it is
not strictly correct to say that partition and the problem of finding one-nth of a group are equivalent.

9. The algorithmic solution of a quantitative problem does not duplicate the physical solution with objects. From the point of view of operationism, the symbolic solution, however, has meaning for the child if he has progressed through "stages of symbolization" in which the referents for the new symbols used are the objects or symbols with which he is already familiar. In particular, the algorithms commonly used to add, subtract, multiply, and divide whole numbers can be so physicalized that the child can see throughout the referents for the symbols he uses.

The Development of Subitizing and the Use of Grouping Procedures in Current Arithmetic Textbooks

1. In most current arithmetic textbooks the authors consider counting a basic skill which must be learned to some extent before further progress can be made with number ideas. This belief is not in accord with the findings in this study regarding subitizing and grouping.

2. Current arithmetic textbooks for the first grade usually make extensive use of number pictures to develop meanings for the numerals. For the most part, both pictures of concrete objects and pictures of geometric figures are portrayed. The pictures in current arithmetic textbooks are of such a type and quality that they could be utilized for developing the ability to subitize and to group more extensively than the authors now make use of them.
3. Authors differ widely in the emphasis they place on developing the ability to subitize. In only two of the thirteen series surveyed could there be said to be an early and persistent effort made to develop this ability. The relative emphasis placed on subitizing is reflected to some extent in the perceptual materials appearing in textbooks, the better perceptual materials being found in those series which give considerable attention to subitizing and grouping procedures.

4. Most authors are of the opinion that the development of grouping procedures must necessarily follow the development of the ability to count. In ten of the thirteen series, exercises in grouping follow exercises in counting. This procedure tends to lead to an undue emphasis on counting early in number work.

The Development of Meanings Operationally in Current Arithmetic Textbooks

1. In most current arithmetic textbooks no clear operational distinction is made between cardinal and ordinal counting. Only four of the thirteen series were found to provide for some means of making this distinction.

2. Authors of arithmetic textbooks attempt to provide a meaning for addition as being composed of two operations: (a) the actual combining, bringing together, of two or more groups of objects; and (b) the determination in some way of the number of objects in the total group. There appears to be a greater emphasis placed on the learning of number combinations, however, than on understanding the total set of addition operations.
3. In all of the series surveyed an attempt is made to teach take-away situations operationally. Moreover, in most arithmetic series, the algorithmic process of subtraction is introduced in take-away situations, and take-away questions are used almost entirely in all series to teach the subtraction facts.

4. In most cases the authors give only a brief operational explanation of comparison-subtraction. Too often the authors simply show that subtraction produces the correct answer to comparison problems; that is, that comparisons are applications of algorithmic subtraction. In nearly all series some attention is given to matching operations. The use of matching in comparison-subtraction situations is generally treated incompletely, however, and the relation of the solutions of comparison situations to the solution of take-away situations is seldom shown.

5. While additive-subtraction (the how-many-are-needed problem) is usually presented in situations which are very similar to comparison situations, the treatment of additive-subtraction could not be called operational in view of the fact that comparison situations are seldom handled in accord with operational theory.

6. Since the introduction to multiplication usually comes in the second or third grade, after children have had considerable experience with algorithmic solutions, few authors provide adequate physical meanings for multiplication. It is simply pointed out that multiplication is a process for determining the total when the addends are all equal.
7. The process of division is commonly introduced through the use of examples involving measurement division, and the division algorithm is taught almost exclusively using the language of measurement division. Since division is introduced usually in the third grade, the explanation for finding the number of x's in y objects may be said to be operationally adequate in most cases at that level. There is a tendency, however, to introduce solutions for measurement-division situations through the use of a minimum of pictured examples, and there is also a tendency to follow measurement division with partition situations which are shown as applications of algorithmic division. These procedures are not in accord with the advantages pointed out in this study for the use of partition in physicalizing the division algorithm.

8. Most authors make no operational distinction between measurement and comparison division. Moreover, the problem of finding one-n\textsuperscript{th} of a group is usually treated exclusively as a partition problem; authors do not provide for other operational solutions to this problem. In this study it was pointed out that measurement division and comparison division are operationally distinct, and that there are various ways of finding one-n\textsuperscript{th} of a group of objects.

9. Authors generally recognize that the division algorithm proceeds by subtracting multiples of the divisor from the dividend. Because of this recognition there is an unwarranted tendency to explain division as repeated subtractions. In this study it was pointed out that the algorithm does not duplicate the physical situation. There
appears to be no justification for explaining division as a "short way of subtracting."

10. Most authors appear to take the position that instruction of arithmetic in the first grade consists largely of teaching children to read, write, and compute with numerals. This would appear to imply that most authors are of the opinion that children are not using arithmetic until they are using the written symbols of arithmetic. In contrast, it was pointed out in this study that considerable arithmetic can be learned and used meaningfully prior to the introduction of numerals and algorithms.

11. The symbolic arithmetic processes and the problems they are designed to find answers for are taken up in current arithmetic series in the following order: (a) addition, (b) subtraction, (c) multiplication, (d) division. This sequence may well be followed partly because it is the traditional sequence and partly because of a supposed progressive difficulty of these processes. As a consequence, the child does not even consider certain very elementary quantitative questions until he has been in school for two or three years, simply because the algorithmic solutions to these problems are considered more difficult than others and come later in the sequence. An operational approach, however, provides meaningful ways for children to deal with quantitative questions prior to the time when they can deal with those same questions algorithmically.

12. A sequence of pictures depicting various stages of an activity does not necessarily imply any specific arithmetic question.
In general, more than one quantitative question may be asked about such situations. It is the question asked which creates the mathematical problem; and, in fact, if no number question is asked, the individual has no good reason even to consider that a question of quantity is involved.

13. The materials provided for arithmetic experiences in the early grades consist in the main of printed and pictured materials, despite the fact that most authors at least suggest the use of concrete materials. It is impossible to manipulate printed and pictured materials to provide the operational meanings described in this study.

**Recommendations**

In light of the findings in this study recommendations may be made for the following: (a) the teaching of arithmetic in the primary grades, and (b) further studies.

**Recommendations for Teaching Arithmetic in the Primary Grades**

1. Arithmetic programs should provide for more and earlier experiences in subitizing and the use of grouping procedures to determine the number of objects in a group.

2. The usual overemphasis on counting as the first means for determining the size of a group should be reduced, and the importance of the operation of counting made evident by showing its need when the limits of subitizing and grouping are reached.

3. Arithmetic programs should adopt an operational approach
for providing meanings for the terms, symbols, statements, and algorithms of arithmetic.

4. In order to clarify the use of the word meaning in arithmetic, the term should be reserved for use in the operational sense; it should not be used to refer to the other equally important aspects of the teaching of arithmetic, social significance and mathematical reasoning.

5. Arithmetic programs should provide concrete materials rather than printed materials for all or nearly all of the work of the first grade, and the child should have a chance to encounter during his first year in school all of the various problems with whole numbers that he ordinarily meets in the first three years of arithmetic. The written symbolism for computational purposes should be deferred until after he has had considerable experience in solving arithmetic problems by performing physical operations on concrete objects.

Recommendations for Further Study

1. A longitudinal study should be made with pre-school children to determine the development of the ability to subitize.

2. The operational approach should be worked out in detail for the various terms, symbols, statements, and algorithms of arithmetic beyond the work with whole numbers, such as those associated with common and decimal fractions and percentage.

3. A study should be made to explore the possibility of adopting the operational approach to the areas of mathematics beyond the arithmetic of the elementary school.
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