DISCRETE RANDOM FEEDBACK MODELS
IN INDUSTRIAL QUALITY CONTROL

DISSERTATION
Presented in Partial Fulfillment of the Requirements
for the Degree Doctor of Philosophy in the
Graduate School of the Ohio State
University

By

ALBERT BENTLEY BISHOP, B.E.E., M.S.

The Ohio State University
1957

Approved by:

[Signature]
Adviser
Department of Industrial Engineering
PREFACE

The material in this dissertation is offered in an effort to bridge the gap between Industrial Quality Control as practiced in industry today and some of the seemingly applicable theory now available and being developed in the field. I am particularly concerned with probability theory, already the backbone of quality control, and control theory, now primarily applied in the design and analysis of deterministic electro-mechanical control systems. The inherent non-rigidities of industrial systems require use to be made of developments in both of these theoretical fields. In the first three chapters I attempt to describe the functions of Industrial Quality Control and the restrictions placed on the performance of these functions in such a way that the need for theoretical developments is emphasized. In the last five chapters I develop procedures for the design and analysis of a class of discrete feedback-control systems for random processes, a type well suited for control of many industrial processes. This procedure combines the concepts of probability and control theories. It is hoped that the procedure will be of direct use to designers and researchers in all control fields and may find application in Industrial Quality Control. It is also hoped that the formulation of the quality-control function presented and the methods used in our later chapters, may stimulate further activity in this important field.

It is impossible for me to express the full extent of my gratitude for the expert and timely aid so generously given by my advisor, Dr. Loring G. Mitten of the Department of Industrial Engineering, The Ohio State
University. The use of the difference-equation as my basic model and much of the development of the transform solution procedure of Chapter VIII are a direct result of Dr. Mitten's keen insight. His unselfish devotion to his duties as advisor I am sure have rarely, if ever, been equaled. For his advice and friendship, I shall always be grateful.

I also wish to express my appreciation to Dr. P. N. Lehoczky and Dr. F. C. Weimer who served with Dr. Mitten on the reading committee for this work, and to Professor F. M. Mallett who also read the manuscript. Dr. Weimer suggested use of the stability criterion discussed in Chapter VII.

My final debt of gratitude is to my wife, Louise, and my children, John and Sue, for their patience and understanding throughout my work. Without them I could never have survived the disciplines of writing a dissertation, and with them I found sufficient excuses for distraction to make the job pleasant though almost impossible.

Albert B. Bishop
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>1</td>
</tr>
<tr>
<td>INTRODUCTION.</td>
<td>1</td>
</tr>
<tr>
<td>Quality Control Theory and Practice.</td>
<td>1</td>
</tr>
<tr>
<td>Purpose in Writing</td>
<td>2</td>
</tr>
<tr>
<td>The Functions of Industrial Quality Control.</td>
<td>4</td>
</tr>
<tr>
<td>II</td>
<td>11</td>
</tr>
<tr>
<td>QUALITY CONTROL PRACTICE.</td>
<td>11</td>
</tr>
<tr>
<td>Factors in the Evolution of Quality Control.</td>
<td>11</td>
</tr>
<tr>
<td>Statistical Aspects.</td>
<td>16</td>
</tr>
<tr>
<td>Non-statistical Aspects.</td>
<td>29</td>
</tr>
<tr>
<td>III</td>
<td>34</td>
</tr>
<tr>
<td>DECISIONS IN INDUSTRIAL QUALITY CONTROL</td>
<td>34</td>
</tr>
<tr>
<td>Decision Categories.</td>
<td>34</td>
</tr>
<tr>
<td>Measurement Decisions.</td>
<td>35</td>
</tr>
<tr>
<td>Analysis Decisions</td>
<td>37</td>
</tr>
<tr>
<td>Control-Action Decisions</td>
<td>42</td>
</tr>
<tr>
<td>Criterion Selection.</td>
<td>42</td>
</tr>
<tr>
<td>IV</td>
<td>50</td>
</tr>
<tr>
<td>CONTROL-THEORY AND THE INDUSTRIAL PROCESS</td>
<td>50</td>
</tr>
<tr>
<td>Decision Rules in Quality Control.</td>
<td>50</td>
</tr>
<tr>
<td>Control Theory</td>
<td>51</td>
</tr>
<tr>
<td>Critical Characteristics of Industrial Processes</td>
<td>58</td>
</tr>
<tr>
<td>Chapter</td>
<td>DIFFERENCE-EQUATION MODELS - DIRECT SOLUTION</td>
</tr>
<tr>
<td>---------</td>
<td>---------------------------------------------</td>
</tr>
<tr>
<td>V</td>
<td>General Formulation</td>
</tr>
<tr>
<td></td>
<td>Homogeneous Solution</td>
</tr>
<tr>
<td></td>
<td>Particular Solution</td>
</tr>
<tr>
<td></td>
<td>Evaluation of Homogeneous-Solution Coefficients</td>
</tr>
<tr>
<td></td>
<td>Expected Value and Variance of Process Mean</td>
</tr>
<tr>
<td>VI</td>
<td>SPECIFIC DIFFERENCE-EQUATION MODELS</td>
</tr>
<tr>
<td></td>
<td>First-Order Equations, Simple Proportional Control</td>
</tr>
<tr>
<td></td>
<td>Proportional Control With Incomplete Adjustments</td>
</tr>
<tr>
<td></td>
<td>Second-Order Equations - General</td>
</tr>
<tr>
<td></td>
<td>Second-Order Equation - Real, Unequal Roots</td>
</tr>
<tr>
<td></td>
<td>Second-Order Equation - Equal Roots</td>
</tr>
<tr>
<td></td>
<td>Second-Order Equation - Complex Roots</td>
</tr>
<tr>
<td></td>
<td>Stability of General Second-Order System</td>
</tr>
<tr>
<td></td>
<td>Evaluation of Techniques</td>
</tr>
<tr>
<td>VII</td>
<td>STABILITY OF HIGH-ORDER DIFFERENCE-EQUATION MODELS</td>
</tr>
<tr>
<td></td>
<td>Solution Problems - General</td>
</tr>
<tr>
<td></td>
<td>Stability Determination in the L-Domain</td>
</tr>
<tr>
<td>Chapter</td>
<td>Page</td>
</tr>
<tr>
<td>------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>VIII TRANSFORM METHODS FOR DIFFERENCE-EQUATION MODELS</td>
<td>121</td>
</tr>
<tr>
<td>The z-Transform</td>
<td>121</td>
</tr>
<tr>
<td>Simple Proportional Control</td>
<td>122</td>
</tr>
<tr>
<td>Second-Order Systems</td>
<td>131</td>
</tr>
<tr>
<td>General Linear Difference-Equation Models</td>
<td>139</td>
</tr>
<tr>
<td>Comparison of Procedures</td>
<td>140</td>
</tr>
<tr>
<td>CONCLUSION</td>
<td>142</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>143</td>
</tr>
<tr>
<td>AUTOBIOGRAPHY</td>
<td>147</td>
</tr>
</tbody>
</table>
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table Number</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Characteristics of Simple Proportional Control Systems.</td>
<td>86</td>
</tr>
<tr>
<td>2</td>
<td>Characteristics of Type-2 Second-Order Control Systems.</td>
<td>101</td>
</tr>
<tr>
<td>3</td>
<td>Conditions for Stable Operation of Second-Order Control Systems.</td>
<td>111</td>
</tr>
</tbody>
</table>
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure Number</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Operating Functions of Quality Control</td>
<td>7</td>
</tr>
<tr>
<td>II</td>
<td>Schematic Diagram of Discrete Feedback System for Control of a Stochastic Process</td>
<td>62</td>
</tr>
</tbody>
</table>
CHAPTER I

INTRODUCTION

Quality Control - Theory and Practice

The field of Industrial Quality Control first gained autonomy in 1924 with Shewhart's introduction of the concepts of probability theory in the control of industrial processes. Adoption of statistical principles to operational control problems came slowly, however. Applied quality control has been principally involved with labor-saving and explicative techniques, procedures and hardware to facilitate the accomplishment of a fairly restricted set of control tasks. Meanwhile, there have been a wide variety of theoretical developments which, along with probability theory, are potentially of use to quality control. These include control theory, the theory of measurement, communication theory and decision theory, all of which treat broad categories of problems and criterion functions.

The lack of application of potentially useful available theory is by no means surprising, nor is it peculiar to quality control. Like the theories applicable to the areas of production scheduling and forecasting, inventory control, and market analysis, the theories of potential use in quality control are extremely complicated. Successful functioning of the quality-control system, however, involves the understanding and support of office and shop personnel, people with limited formal training but rich in industrial experience. The tendency of such people to rely on what they know and are accustomed to rather than on "complicated" new
methods is a natural reaction. The quality-control engineer, himself only recently possessed of any significant formal statistical training, has had a career basically devoted to employee education to gain acceptance for already-proven methods. The result has been an emphasis on treating a few restricted problems more cheaply and efficiently, thus leaving little time for theoretical developments or the broadening of the types of problems attacked. Theoretical developments have thus evolved for the most part from theoretical and applied mathematicians such as Wald, Weiner, and Von Neumann; and, although communications between groups has been relatively good, only the simplest theoretical advances have been given application. As a result, many useful aspects of probability theory and essentially the entire realm of control and other theories are foreign to Industrial Quality Control.

Purpose in Writing

It is the purpose of the work presented herein to attempt to bridge a portion of the gap between Industrial Quality Control and some of the vast store of seemingly applicable theory now available and under study for use in the field. Our principal concern is with probability and feedback-control theories, although it is hoped that the formulation given the quality-control function in the early chapters of this work may display the field in such a way as to suggest potential applications of other theoretical developments.

We will first discuss the evolutionary factors leading up to the emergence of quality control as a recognized entity on the industrial scene, and trace the development of the field and its environment up
to the present. By considering these historical and environmental factors, we hope to gain a fuller understanding of the existant functions and problems of Industrial Quality Control and perhaps offer some guidance for theoretical applications in the field.

Two basic categories of operational quality-control problems will be identified: (1) sorting problems, i.e., the separation or filtering of commodities into categories, and (2) process-control problems, i.e., the maintaining of the operating level of a process at some desired level. Our primary concern relative to control-theory applications will be the latter. This of course omits the very important development and implementation problems involved in measuring, recording, information flow, equipment and instrument design, and education. This is not to infer any relative importance of one type problem over another. All are important and all must be solved if any integrated theory of quality control is ever to be evolved. The emphasis placed on these two types of problem is a more restricted interpretation of the scope of quality control than that held by many current enthusiasts who feel quality control is the application of statistical methods to any industrial problem (Ref. 1). We will furthermore give only cursory attention to the basic concepts of criterion-function formulation, which lies in the areas of decision theory and value theory. The criteria used in

---

1The letters "Ref." and a number in parentheses refer to the number of the reference as listed in the Bibliography.
the developments which follow have been derived through experience
with industrial problems or assumed for illustrative purposes.

Later chapters will discuss the elementary concepts and procedures
of control theory and develop and illustrate applications thereof to
industrial process control. The main developments are with respect to
discrete processes.

To illustrate the importance of probability and control theories
in quality control, let us at this point examine the industrial process
and the functions of quality control as related to it.

The Functions of Industrial Quality Control

The purpose of industrial quality control is the regulation of
appropriate characteristics of the materials flowing into, through,
and out of the processes within the industrial enterprise. The problems
encountered in the performance of this task can best be appreciated
after a brief look at the characteristics of industrial processes them­
selves. These processes, generally being composed of a complexity of
men, machines, and materials, each subject to random variations, shortages,
and breakdowns, are in general significantly non-rigid.

This non-rigidity is of two types. One is the time-independent
stochastic type, predictable in a probability sense, which results from
the interaction of a multiplicity of factors, none of which predominate.
Even though every controllable process parameter is held constant,
variations in process output will occur. Probability theory is parti­
cularly applicable in dealing with this type of non-rigidity. The other
manifestation of system non-rigidity is the time-dependent shift or
drift in one or more parameters of the process output distribution. This may result from numerous causes such as tool wear, machine breakdown, and material shortages. The overall performance of the process is affected in various ways by all of these factors, so that rarely can such a process be simply "turned on" and expected to perform in any given predetermined manner. Even if, by chance, the initial conditions be correct, the probability of a shift away from desired conditions can be extremely high. Most industrial processes must, therefore, be constantly or periodically observed to detect and eliminate such shifts, and it is this necessity for feedback control which makes control theory applicable to the field of Industrial Quality Control.

Actually all systems exhibit both phases of non-rigidity to a degree. Even electro-mechanical servomechanisms, usually regarded as deterministic systems, are subject to some stochastic variation in addition to shifts of output mean. The distinguishing feature is, however, the degree of each phase of non-rigidity. In studying simple machines such as the lever, an assumption of rigidity will often suffice. On the other hand the designer of the classical servomechanism need consider only shifts in output mean. To evaluate adequately the performance of a control system for an industrial process, however, both types of non-rigidity will significantly affect performance. In these situations, parameters of the output distribution in addition to the mean must be considered. Our discussions in later chapters will deal with mean and variance.
The evolution of sorting and process control as basic operating functions of industrial quality control also stems from the stochastic and time-varying non-rigidities of industrial processes. It is possible to have each of these functions take place at any stage of the process, although economic considerations usually dictate the extent and location of control activity. Problems associated with locating control activity constitute a set of control-system-programming problems the solutions for which are a necessary part of an integrated theory of quality control.

Figure I illustrates the relationships between the industrial process and the sorting and process-control functions of quality control. Operations A and B can be any successive individual operations in a process or can be generalized to being, respectively, the set of all operations underway or completed on a given group of materials and the set of all operations still to be done. The term "operations" is meant to include not only direct production activities but also purchasing, transporting, storing, shipping, and all sorting operations other than the particular one isolated between sets A and B.

Figure I shows the flow of materials from the set of operations A to the set B. Materials may occur either as discrete units or in amorphous form. Flow may be either continuous or intermittent. It is not meant to be inferred that all materials entering set B pass through the same set of previous operations. Each material must pass through some previous set, however, which permits a possible application of control activity relative to it. Each set of operations is characterized by a set of measurable parameters such as power consumption, stop settings, reaction temperatures, acceptance limits, or flow rate.
FIG. 1. - OPERATING FUNCTIONS OF QUALITY CONTROL.
PROCESS PARAMETERS

FILTER (Isolated)

Stage 1

Stage 2

Stage 3

Scrap

Rework

PARAMETER ADJUSTMENTS

PROCESS PARAMETERS

OPERATION B

LEGEND

PRODUCT AND PROCESS INFORMATION

DECISION INFORMATION

CAUSE AND EFFECT RELATIONS

MATERIALS FLOW

FUNCTIONS OF QUALITY CONTROL.
The values of these parameters of course affect the outputs of each particular set of operations. Adequate means of adjustment of these parameters are of course necessary and are assumed for our purposes.

The following measurements can be taken: (1) the set A parameters, (2) the characteristics of the materials leaving set A, and (3) the set B parameters. The measurements shown in the figure as part of the isolated filtering operation are for sorting procedures calling for multiple filtering stages, so actually are a special case of (2). Naturally only those measurements applicable to a particular situation will be used at any one time, but all categories are included in the figure for illustrative purposes. No distinction is made at this time among kinds of measurements such as variables, attributes, or even subjective descriptions. It is furthermore assumed that adequate instrumentation exists.

Three kinds of operating decisions can be made on the basis of the data collected. Feedback and feedforward are process-control decisions; filtering involves the sorting of material. The decision process must yield sufficient information to effect each type of control desired. A technology defining the relationships among material characteristics, process parameters, and parameter adjustments is of course required.

Feedback control is effected by adjustment of the parameters of set A in such a way as to eliminate any observed deviations between actual and desired output. This could mean raising a generator rpm., adjusting or changing a tool, or installing a sampling plan at the beginning of or somewhere within set A.
The function of filtering is the separating of groups of material into subgroups according to the value of some material characteristic of each subgroup. For illustrative purposes, the dichotomous separation common to attributes testing is depicted, but the number of subgroups is not meant to be restricted in any way. Likewise, the number and application of filtering stages is intended to be in no way limited; although Figure I shows just three stages as might be used when screening initially-rejected lots. An additional kind of filtering decision arises when the decision to use subsequent filtering stages depends on the mix of subgroups resulting from one or more prior stages (as in the case of double or sequential sampling). This type of decision is indicated by the decision-information line from the filter back to the measuring devices.

Feedforward control involves the same kinds of activity as feedback control, except the adjustments are made in the parameters of set B instead of set A. These adjustments are made in such a way that set B can "best" handle the materials about to enter it.

Three steps in the effecting of control can be identified in the preceding discussion. They are measurement, decision (which we shall refer to as "analysis" in later chapters), and control action. The portions of Figure I involving each of these have been designated by 1, 2, and 3, respectively. In our discussions of the applications of probability and control theories we will be primarily interested in a portion of step 2; namely the selection and evaluation of decision rules to dictate routine control action on the basis of certain properties
of the measurements taken. These control steps will be considered more fully in Chapter III, where particular attention is afforded the concept of decision rules. In Chapter IV, the concept of the decision rule will be introduced, and developments of certain types of such rules taken up in the remainder of the dissertation.
CHAPTER II

QUALITY CONTROL PRACTICE

Factors in the Evolution of Quality Control

Before examining current Industrial Quality Control in any detail, it is perhaps worthwhile to delineate some of the factors which influenced the emergence of quality control as a recognized function in industry. The application of probability theory and statistical techniques to control problems is of course the factor which finally provided autonomy for the field of quality control; but actually a number of technological and social developments served to set the stage for this to occur.

Chronologically, the first of these developments was the rapid technological advance in production machines and processes marking the late eighteenth and the nineteenth centuries. Prior to this era, most manufactured products had been produced by independent craftsmen by fairly simple processes. Each item of product was essentially individually made; and the craftsman's financial success depended largely on the customer acceptance of his output, often a fairly subjective measure. Raw materials were also handled on an individual basis or even self-produced. With such simplicity and flexibility of operations and the individual attention given each item of product, problems of product sorting and process control were almost non-existent. Unacceptable raw materials rarely found their way into manufacturing processes and those processes incapable of a consistently acceptable output were scrapped or revised.
Technological advances, particularly in the textile and associated industries, and the resulting cost and complexity of manufacturing necessitated the coordination and combination of many previously independent operations. The outcome was the emergence of the factory in the early nineteenth century as the major producer of manufactured goods. With the factory and the accompanying higher production rates and transfers of skills, came problems of coordination, interchangeability, product acceptance, process control, and training of labor. There also resulted, for the first time, long enough production runs to permit utilization of feedback-control principles. The factory system thus created situations in which the functions of quality control, as defined in Chapter I, could be gainfully performed.

The dynamic nature of the Industrial Revolution, which reached its peak in the early 1800's, precluded activity of this sort despite the availability of many of the statistical postulates now used as a base for statistical quality control. The rapid progress in machine and process development permitted tremendous savings to be realized by the simple expedient of keeping mechanically up to date. Managerial efforts were, therefore, devoted to exploiting sources of the latest technological developments and any industrial "research" was usually a pragmatic effort along the same lines. The overabundance of labor created by the use of machines provided factories with an extremely low-cost resource, which meant factory operating and overhead costs were at most moderate. There was, therefore, little time or reason to attempt improvements in plant efficiency. With respect to the quality
of output, the lack of objective standards for a rapidly growing and changing array of products made many types of final inspection impossible. There was also opportunity for marginal producers to pass off a wide range of quality on an uninformed and, for a while, unsuspecting public.

Toward the close of the nineteenth century, however, machine and process technology began to stabilize. Progress of course continued, but the rate of development and the levels of cost and complexity attained by newer machines made continual machine turnover unfeasible in many cases. Labor had become a higher-cost and more regulated resource through its own organization and through legislation involving child labor, minimum wages, etc. In addition, chemical, physical and biological technologies had advanced to the point where simple means of testing and standardizing certain types of products were available. Some of these developments found their way into pure-food-and-drug laws and weights-and-measures legislation in the early 1900's. Emphasis on the efficient production of a quality product was, therefore, beginning to exist.

There was during the same general period a shift in outlook toward science and mathematics. Little knowledge of or interest in science on the part of the masses had existed during the early part of the Industrial Revolution. Even had extreme need for process control and product inspection existed in those days, it is doubtful that use would have been made of the mathematically well-known principles of sampling from discrete distributions, the law of large numbers, the binomial distribution, and the Theorem of Bayes. Science and, in particular, mathematics had not yet been thought of as holding anything of use to
the industrial world. General interest in science, however, received
a startling awakening from Darwin in 1854. Industrial interest was
soon heightened by the contributions of chemistry and metallurgy to the
steel industry in the form of the open hearth and electric furnace.
Numerous early inventors, although largely pragmatists, helped advance
the banner of science. Probability theory itself was brought to the
public eye by the regression relationships developed by Galton and,
at the turn of the century, by the agricultural experiments of R. A.
Fisher.

With a profit-oriented emphasis on efficient plant operations and
a general awareness of the potential advantages in the use of scientific
principles, the scientific management concepts of Taylor et al. were a
natural outgrowth of the early twentieth century. Regardless of how
"scientific" the approaches may have been, Taylor and his associates
were successful in finally focusing formal attention on the internal
workings of the factory and introducing the concept of engineering to
industry.

Taylor's approach was, however, entirely deterministic and it was
not until 1924, some 20 years later, that Shewhart introduced statistical
concepts to the industrial scene. Shewhart's work was actually the
product of another development which has since had a tremendous impact
on all phases of industry, not the least of which is Quality Control.
That factor is the inception of company-sponsored research, actually
pioneered by Du Pont in 1902 and later by the Bell Telephone Laboratories
where Shewhart, Dodge and Romig developed the first statistical tools
for industry.
Statistics itself, which had fallen into ill-repute among mathematicians a hundred years earlier, regained mathematical stature as the result of the axiomatic, measure-theoretic treatment given it in 1933 by Kolmogorov (Ref. 4). The interests of the applied mathematician in probability theory were thereby revived. The result has been the stimulation of activity in various combinatorial problems and hypothesis tests of interest to industry. As is well-known, however, the acceptance and use of statistical methods by industry advanced fairly slowly after its noble beginnings. This was due to a great extent to a lack of understanding of probability notions by industrial management. It was actually World War II and succeeding conflicts which provided impetus toward the adoption of not only the concepts of quality control but of the application of the scientific method to many phases of industrial and military operations. Since this time, the trend toward a systems-oriented and technically advised management has been more rapid.

We find a wide range in the extent to which quality control is utilized in industry today. The degree of use in any given plant is primarily a function of the stage of evolution of the company and the demands placed on the company because of its competitive and market positions. From the foregoing it should be clear; however, that for Industrial Quality Control to exist at all as a recognized entity the following factors are necessary:

1) the factory system with its relatively complex production technology and high production rates,

2) customer, legal, and economic pressures for a quality

---

\(^2\) See for example, Feller (Ref. 2), Mood (Ref. 5), Wald (Refs. 6,7), Duncan (Ref. 8), Burr (Ref. 9), and Cowden (Ref. 10).
product plus an adequate means of determining quality,

3) financial incentive for adequate control of the manufacturing process,

4) a progressive, well-informed management, open-minded with regard to possible gains from utilization of scientifically developed management tools, and

5) a basic scientific understanding (or research to try to gain an understanding) of the nature of the systems and processes involved.

These five factors have, as has been pointed out, just recently reached levels permitting significant gains in Industrial Quality Control. Meanwhile, control-theory and probability-theory development has been accelerating in other fields such as the design and development of electro-mechanical control devices. This pair of factors explains to some extent the divergence between current quality-control practice and theories of probability and control. As has also been mentioned, recent developments in all five of the areas listed seem to predict accelerating advances in Industrial Quality Control in the near future with an increasing number of practical applications of theoretical advances in probability and control. Furthermore, as this advance continues, it is not unlikely that the control of quality and of other aspects of the industrial enterprise be combined into a central industrial control activity.

Statistical Aspects

Since statistics is the very life blood of Industrial Quality-Control, it is fitting to begin our examination of the field with a
look at the history and development of statistics in quality-control. It is interesting to note that as early as 1911 a Frenchman by the name of Provost successfully carried out one of the first documented applications of statistics to the control of an industrial process (Ref. 11). As manager of the Havre factory of the French Tobacco Monopoly he was faced with controlling the amount of tobacco in cigarettes, keeping in mind both the costs of tobacco and customer reaction to loosely-packed cigarettes. Using the assumption of a normal distribution, Provost obtained estimates of each worker's daily average and variability by sampling methods. A bonus was paid for the proper mean and a low dispersion. The reported results were (Ref. 12):

1. a great reduction in the amount of inspection,
2. practical elimination of underweight packages, and
3. an annual saving of 1,000,000 Kg of tobacco (from overweight).

Although the statistical concepts he used were of the simplest kind, Provost appears to be one of the first to recognize the inherent stochastic nature of industrial processes. He also recognized the general behavior of the "constant cause system" as yielding asymptotically normal outputs.

After Provost, essentially no industrial applications of statistics are recorded until Walter Shewhart produced the first control chart for process control on May 16, 1924, at Bell Telephone Laboratories. The first explanations of the principles involved appeared in the Bell System Technical Journal (Refs. 13, 14) and other journals (Ref. 15) over the next several years. The first broad scale, self-explanatory, and self-supporting work in the field was Shewhart's book Economic Control of Quality of Manufactured Product (Ref. 16), published in 1931.
Here for the first time the stochastic nature of industrial processes was explained and "control" defined in probabilistic terms:

---a phenomenon will be said to be controlled when, through the use of past experience, we can predict, at least within limits, how the phenomenon may be expected to vary in the future. Here it is understood that prediction within limits means that we can state, at least approximately, the probability that the observed phenomenon will fall within the given limits.

The notions of "chance cause systems" and "assignable causes" were introduced and the following postulates set forth:

Postulate 1. - All chance systems of causes are not alike in the sense that they enable us to predict the future in terms of the past.

Postulate 2. - Constant systems of chance causes do exist in nature.

Postulate 3. - Assignable causes of variation may be found and eliminated.

These postulates form the basis of current Statistical Quality Control, the name by which the quality-control field is most commonly called.

In essence they state that the stochastic nature of systems is due to the interaction of a multitude of factors, none of which predominate, and the effects of which are constant in a probability sense. In general it is assumed that the isolation and elimination of such factors is either impossible or uneconomical. Additional factors may also affect the process, however, forcing it out of acceptable limits. Such factors are, given a reasonable technology, eliminatable. The ultimate of control activity, "maximum control" is, therefore, defined as:

the condition reached when the chance fluctuations in a phenomenon are produced by a constant system of a large number of chance causes in which no cause produces a predominating effect.
In other words, "maximum" control exists when all assignable causes have been eliminated. Fluctuations still exist, but these are statistically predictable.

Shewhart's book, being the pioneering effort in the field, devotes a relatively large portion of space to definitions. "Quality" is defined as each of the following:

1) a set of characteristics.
2) an attribute.
3) a number of the same kind of things.
4) a distribution.
5) a rate.
6) a relationship.

Means of describing and representing these aspects of quality are discussed in terms of the statistics needed for determination of quality. The importance of representative data is stressed and the principles of data presentation outlined. The effectiveness of graphical presentation is particularly emphasized. The important notions of random and stratified sampling are introduced and the sampling distributions of mean, median, standard deviation, range, and correlation coefficients discussed. A comparison of the efficiency of each as an estimator of population parameters is also made. The use of both the distribution and the moments of statistics in the determination of action limits is described. Shewhart explicitly mentions the economic aspects of setting limits so that an adequate compromise between the risks due to false alarms and failure to detect lack of control is effected. Without formal treatment, he offers \( \pm 3\sigma \), where \( \sigma \) is the statistic being con-
sidered, as an "economic" limit. The relationships between control limits and tolerances are also pointed out.

The statistics recommended for use are (1) sample fraction defective for attributes control and (2) sample mean and standard deviation for variables control. The use of the skewness factor in variables control is also recommended when sufficient knowledge of the parent population warrants. The use of the range in place of standard deviation when the universe is normal is discussed, but a preference for standard deviation expressed because of the greater dependence of the distribution of the range on the functional form of the distribution of the population. Five criteria for detection of lack of control are listed:

Criterion I - Given a set of $N$ data, divide the group into $m$ rational subgroups. For each statistic, $\theta$, to be used, use $\overline{\theta}$ and $\sigma^2_{\theta}$ which are, if possible, unbiased estimates of the appropriate population parameters. Construct control charts with limits $\overline{\theta} \pm 3\sigma^2_{\theta}$ for each statistic and treat any point falling outside these limits as evidence of lack of control.

Criterion II - Divide the $N$ given data into $m$ subgroups of $n$ each. Compute:

$$\sigma^2_x = \frac{\sum_{i=1}^{m} x_i^2}{m} - \overline{x}^2$$  \hspace{1cm} (1)$$

$$\overline{\sigma}^2 = \frac{\sum_{i=1}^{m} \sigma_i^2}{m}$$  \hspace{1cm} (2)$$
If:
\[
\left| \frac{d}{\sigma_d} \right| > 3
\]

consider the process producing the samples as being out of control.

Large positive values of \( d \) indicate Poisson sampling, i.e., the changing of conditions within subgroups but the same pattern recurring during each subgroup. Large negative values of \( d \) indicate Lexian sampling, i.e., changes occurring between two or more subgroups but constancy within. The expected value of \( d \) is zero if sampling takes place in Bernoulli fashion, i.e., with the same constant cause system in effect throughout.

Criterion III - Pick out some controlled variable, \( Y \), which is suspected of being a predominant cause in the constant cause system. Take \( n \) simultaneously observed pairs of data, \( X_i Y_i \) \((i=1, \ldots, n)\) and determine the correlation coefficient, \( r \). If
\[
|r| > \frac{3}{\sqrt{n-1}}
\]

consider this as evidence that \( Y \) is an assignable cause.

Criterion IV - Obtain \( n \) observations of \( X \) and compute the statistic \( \tilde{X}_1 \). Select a variable \( Y \), suspected of being an assignable cause, and, controlling \( Y \) so that it cannot possibly influence \( X \), take \( n \)
more readings of \( X \). Compute the same statistic using the new set of readings. Call it \( \Theta_2 \). If

\[
|\Theta_1 - \Theta_2| > 3 \sigma_{\Theta_1 - \Theta_2}
\]  

(7)

consider this evidence that \( Y \) is an assignable cause.

Criterion V - (Specifically for cases where data cannot be divided into rational subgroups or variables of interest actually controlled.) Calculate the mean, \( \bar{X} \), standard deviation, \( \sigma \), and skewness factor, \( k \), from the \( n \) observations available. For each of the \( m \) cells into which the data have been grouped, compute the theoretical cell frequencies \( y_{\Theta_i} \) \((i=1, \ldots, m)\) using,

\[
y_{\Theta_i} = \int_{x_1}^{x_i+1} \frac{1}{\sqrt{2\pi} \sigma} \left[ 1 - \frac{k}{2} \left( \frac{x - \bar{X}}{\frac{1}{3} \sigma^2} \right) \right] e^{-\frac{x^2}{2\sigma^2}} \, dx
\]  

(8)

where

\[
x = X - \bar{X}.
\]  

(9)

Calculate

\[
\chi^2 = \sum_{i=1}^{m} \frac{(y_i - y_{\Theta_i})^2}{y_{\Theta_i}}
\]  

(10)

where the \( y_i \) are the observed cell frequencies. If the computed \( \chi^2 \) is greater than the value tabulated in the \( \chi^2 \) table for \( P = .001 \) at \( m \) degrees of freedom, take the fact as an indication of lack of control.

Thus Shewhart has in formal fashion covered fairly completely the possible treatments of data as an overall group. In addition he gives informal
treatment to the time pattern of observations in a section entitled "Role of Judgment in Choice of Criteria". He is thorough in treating several drastically different types of distributions.

Shewhart's work involved exclusively the operational problem of process control. In fact, until well into World War II, process control and quality control were interchangeable terms. Nevertheless, Shewhart formulated the basic structure and justification of quality control in its broadest definition.

The application of statistics to the problems of "product control" or, in terms of the trade, "acceptance sampling" was spearheaded by two other Bell Laboratories men, H. F. Dodge and H. G. Romig. The first articles on this subject appeared in 1928 (Ref. 18) and 1929 (Ref. 19), but the major contribution was made in January of 1941 with the publication of the Dodge and Romig "Single and Double Sampling Inspection Tables" (Ref. 20). Here the procedures for single and double sampling plans, by attributes, used for fifteen years in the Bell System, were set forth with methods for deriving plans of each type which would satisfy one of three types of conditions. The extremely useful Thorndike Chart was included therein. The first type of condition was the passage of the Operating Characteristic Curve of the plan through (or as close as possible to) a specified point. "O.-C. Curves", the inversion of the statistically well-known power curve, had already been discussed relative to acceptance sampling in an article by Dodge in 1935 (Ref. 21). The other types of condition require detailing of rejected lots and involve (1) limiting the poorest quality lot which will pass through inspection on the average and (2) the minimization of the average amount
of inspection for a specified quality of submitted lots. The notion of double sampling, which results in lower average inspection for certain types of submitted lots than comparable (same O-C Curve) single-sampling plans, was a complicated but interesting innovation. Variables sampling and extensions to larger possible numbers of samples under the title of Multiple Sampling (Ref. 22) and the ultimate, Sequential Sampling (Refs. 23, 7) followed a few years later under the stimulus of World War II. Both extensions have been made only with respect to the O-C Curve type of requirement (except by trial and error methods). Sequential sampling is of course based on the powerful probability ratio test and can be extended to the testing of almost any hypothesis against a single alternative for given specified risks (O-C Curve).

At about this time the few users of Statistical Quality Control were beginning to add a few flourishes to the initial concepts. One was the use of the range in place of the standard deviation for small samples in variables control-chart work. Another was the formulation of the control chart for defects per unit of product by the Ballistic Research Laboratories at Aberdeen Proving Ground, Maryland.

World War II is responsible for much activity in the quality control field. Shewhart's original work had been essentially unnoticed in this country outside of the Bell System, although the English responded enthusiastically to a series of lectures by Shewhart at University College, Cambridge, in 1931. In 1937 only a handful of American companies employed statistics, although Bell Laboratories, the American Society for Testing Materials and the American Society for Mechanical Engineers
had been jointly trying to popularize Statistical Quality Control for several years. It was really not until 1939 when the government became an important purchaser of a large variety of high-quality items that proper incentive was created for large-scale industrial advances. The Ordinance Department had for some time been using a variation of the control chart system called "Analysis of Acceptance Tests and Inspections". The success of these techniques had given the Ordinance quality-control men a voice in the War mobilization preparations. Official formal status for quality control was decided against, however, and the adoption of acceptance sampling techniques agreed upon instead. The first plans were drawn up in 1942. (Ref. 24). Suppliers of military equipment thus were put under pressure to produce consistently high-quality goods. The Navy (Ref. 25), Quartermaster Corps and Signal Corps soon followed suit.

In 1940 the War Department began a two-part educational program for American Industry. The first effort was the preparation of American War Standards, Z1.1-1941 and Z1.2-1942, "Guide for Quality Control and Control Chart Method of Analyzing Data", (Ref. 26), and Z1.3-1942, "Control Chart Method of Controlling Quality During Production" (Ref. 27). These were prepared by the American Standards Association and resulted in the formation of the Association's Committee on Quality Control. A notable contribution of these documents was the inclusion of probability limits for control charts in addition to the previously exclusive 3σ-limits. The other governmental effort was the establishment of a training course for industrial executives, managers, and government procurement personnel to teach them the practical advantages of
Statistical Quality Control. Originally a ten-day course held at Stanford University, the program was shortened to eight days and subsequently offered at 35 different locations around the country. The initial backing came from the Engineering, Science, and Management War Training Program (ESMWT) of the United States Office of Education. Later activity was supported by the Office of Production Research and Development of the War Production Board.

General Leslie E. Simon's book *An Engineer's Manual of Statistical Methods* (Ref. 28) was published in 1941. He was at the time an Army Major and assistant director of the Ballistic Research Laboratories. In this book the relationship between acceptance sampling and "quality control" (process control) is pointed out, thus tying together these two facets of present-day industrial quality control for the first time. Both attributes and variables are considered. Also provided is a packet of charts of the incomplete beta-function for use in determining limits for attributes control and testing. Simon also discusses statistical tests of differences and prediction techniques as useful adjuncts to the quality control repertoire. His "Grand Lot Scheme", if properly carried out, takes advantage of added precision afforded by large samples.

Government-sponsored research in statistics, primarily at Columbia University, provided added stimulation to the Statistical Quality Control field. The Statistical Research Group was organized in July, 1942, at Columbia and brought together many of the nation's leading mathematicians and statisticians. Work emanating from this group includes Wald's individual and group sequential sampling plans, the Wald and Wolfowitz sampling plans for continuous production (Ref. 28) and the notions of curtailed inspection and normal, loosened, and tightened inspection on
the basis of a supplier's past performance (Ref. 29). Many of these concepts were adopted into military specifications by the Navy and eventually in the Department of Defense Military Standard 105A (Ref. 22), which is still in use.

The contributions of Harold Dodge in the formulation of sampling plans for continuous production bear mentioning. A series of plans, based on AOQL, predated Wald and Wolfowitz by two years (Refs. 30, 31). Since that time, over a dozen such plans treating both attributes and variables have been devised by some ten different authors.

With the War also came the necessity for life-testing, a subject of long-standing importance to the Bell System. Poisson and other theories of failure (Ref. 2) became of interest in quality control. Complex equipments using large numbers of the same component part, such as a particular vacuum tube in electronic gear, also required control of the distribution of certain component performance characteristics for proper equipment operation. Such controls are specified in Military Standard 1B (Ref. 33) wherein both tolerance limits for individuals and for sample mean are specified and a limit placed on standard deviation. Dorian Shainin's Lot Plot plan is designed for essentially the same purpose (Ref. 34).

The development of Statistical Quality Control was of course aided materially and placed on a fairly permanent basis with the organization

---

3 For an excellent summary of sampling plans for continuous production see Storer (Ref. 32).
of the American Society for Quality Control on February 16, 1946. The Society took over the publication of *Industrial Quality Control* which had previously been published jointly by the University of Buffalo and the Society for Quality Control Engineers since July 1944. The Quality Control and Applied Statistics Abstract Service which began a monthly publication in January, 1956, (Ref. 35), should be a further stimulus for the field.

Today we find at the disposal of the quality control field the entire store of mathematical and statistical tools thus far developed. We also find pure and applied research in the development of such tools. The vast theory and know-how of point and interval estimation, hypothesis testing, sample statistics, and frequency distributions are potentially applicable to analysis of industrial data and, in particular, quality-control data. Many applications have been made and many developments initiated within the applied quality-control field. Numerous non-parametric statistics have been found useful, in particular the analysis of runs for use in process control. The efficient design of experiments utilizing such techniques as analysis of variance, latin squares, and stratified sampling is a well-known science. Sampling schemes involving single, double, multiple, and unit and group sequential procedures for batch inspection and continuous plans for continuous-production inspection have been formulated. Many types and varieties of process-control devices and means of analyzing data are available. Sampling and control plans can be formulated which meet risk requirements, AOQL specifications, or minimize inspection. Theoretical developments for automatic control procedures for stochastic processes have also
been accomplished (Refs. 36, 37), although much remains to be accomplished. In fact, the entire field of control theory is almost directly applicable to quality control problems but requires some modification and evaluation before adoption can be effected. Nevertheless, Industrial Quality Control is not lacking of theoretical structure and guidance, particularly in statistics.

Non-statistical Aspects

As has been mentioned, across-the-board application of statistical theories and techniques to control quality has not been possible. The painfully slow adoption of such techniques is evidence of this fact. The advances which have been made, however, are due in large measure to the all-important quality-control function of selling itself. Shewhart devoted a relatively large portion of his book to explaining the purpose of and potential economic savings from quality control. Over half of the articles in the early issues of Industrial Quality Control and almost half of the current issues involve success stories or praises of some variety for statistical quality control. The very emergence of the ASQC indicates a need for an organized front for the field.

From the very beginning, statistical concepts had to be sold. They had to be sold to owners and managers to secure investments in statistical control. They had to be sold and explained to workmen to gain the very necessary cooperation needed to run an effective program. They had to be sold to experienced supervisors and tool setters who felt their experience was superior to any kind of mathematics. Superiority of sampling methods over 100% inspection in particular had to be
demonstrated as did the control chart's ability to detect subtle changes in process level before real trouble actually occurred (Refs 16, 38). Thus, salesmanship, including quality reports, publications, and all-around good public relations and technical competence has been and is of utmost importance to Industrial Quality Control.

Allied to the selling problem is the problem of effecting sufficient understanding on the workmen's part to allow proper performance of his job. This is an educational problem which is, of course, partly psychological in nature. In essence it means that measurement, recording, analysis, decision, and control-action tasks must be made extremely simple and their purpose clear in a practical sense. This explains the common use of two or three-sigma control chart limits in place of probability limits, R-charts in place of $\sigma$-charts, sample sizes of four to simplify finding $\sqrt{N}$ or five to permit finding $\bar{X}$ by doubling the total. It explains the predominance of single-sampling plans instead of the often more efficient multiple plans, and the use of "quick-and-dirty" significance tests. Much effort has been devoted to the construction of tables, charts, graphs, and nomograms to aid otherwise difficult computations. Notable among these are Molina's Tables, the Thorndike Chart, normal probability paper, binomial probability paper, and the numerous tables of various statistical distributions. Even with such devices, the quality-control engineer must specify every detail of the inspection-action sequence including how samples are to be drawn, forms filled out, and action taken. A survey of Industrial Quality Control articles shows a significant portion of space devoted to computational aids, data forms, and information channeling. Detailing of procedures
is also a necessary part of any acceptance or process specification\textsuperscript{4}.

Teaching aids also play a fairly significant role in quality control. Various random-signal generators, urns with colored balls, and Western Electric's quincunx all find wide use.

The technical problems of actual measurement have not usually been thought of as a primary function of quality control. Instrumentation engineers are usually responsible for development and provision of adequate measuring devices. With increasing complexity of industrial processes and demands for higher precision outputs, however, the reliability and validity of measurements and the speed of recording are becoming of increasing concern to quality control. Some integration of these activities may be a necessary future development.

The final facet of Industrial Quality Control considered here is perhaps the most important; yet for this facet to be realistically pursued the previously listed functions must all be performed with complete adequacy. This is the facet of economic design and economic evaluation of quality-control procedures. It is a facet which only recently has been meaningful except in a gross sense and even now is hindered by problems of collection of meaningful cost data and of computational complication. In a gross way, savings from utilization of any control or sampling procedure can be estimated and weighed against the cost of effecting such control if cost records can be obtained. Greater detail is needed, however, to compare alternative procedures.

\textsuperscript{4} See, for example, Military Standard 105A (Ref. 22) or Discovery Sampling (Ref. 39), a report by James R. Crawford on a rather unique sampling system developed at Lockheed Aircraft Corporation.
In a restricted sense, AOQL plans limit the maximum possible costs due to defectives getting past inspection, and minimum inspection plans provide certain protection while minimizing inspection cost. Complete economic analysis, however, involves consideration of all cost-generating factors affecting a given process or product. Theoretical consideration of this fact was given by Wald in his formulation of risk functions, which include costs of errors plus cost of sampling (Ref. 6). Satterthwaite has derived expressions for minimum-total-cost single (Ref. 40) and continuous (Ref. 41) attributes' sampling plans for the case of a known frequency ratio between each of two quality lots being submitted. The cost equations that are minimized are fairly complete, but minimization involves numerous assumptions. This work illustrates, however, how complicated a process economic design of quality-control procedures is, even if cost data are available. Breakwell has devised a scheme for computation of single and sequential acceptance plans for both attributes and normally-distributed variables which minimizes the maximum value of a Wald-type risk function of the form (for attributes):

\[
 r = \begin{cases} 
 C(p-p_c) P_r \{\text{accept}\} + N \\
 C'(p_o-p) P_r \{\text{reject}\} + N 
\end{cases} 
\]  

(11)

where (Refs. 42, 43):

- \( r \) = the risk function
- \( p_c \) = critical fraction defective
- \( p \) = actual fraction defective of a submitted lot
- \( C(p-p_c) \) = loss due to acceptance of lot with \( p > p_c \)
- \( C'(p_o-p) \) = loss due to rejection of lot with \( p < p_o \)
- \( P_r \{\text{accept}\} \) = probability of accepting a lot with fraction defective \( p \)
- \( P_r \{\text{reject}\} \) = probability of rejecting a lot with fraction defective \( p \)
- \( N \) = sample number = measure of sampling cost.
Duncan (Ref. 44) and Cowden⁵ consider the economic balance between sampling costs, costs of wrong decisions and general operating and engineering costs in determination of sample size, sampling interval, and control limits for minimum-cost control of a stochastic process. Both solutions require simplifying assumptions but indicate general trends. Cowden's cost equation, however, is extremely restrictive. It is worth while to note that for a significant number of sets of conditions both Breakwell and Cowden find the best solution is no inspection at all.

Very little additional work of a general nature treating economic balances of this sort has been published, indicating a need for a tremendous amount of research in this field. Economic design and analysis of quality-control procedures is, however, essential for the field to keep pace with current trends in systems' analysis and Operations Research. It is, therefore, an important facet of Industrial Quality Control.

⁵ See Chapter 29 of Cowden's Statistical Methods in Quality Control, Reference 10.
CHAPTER III

DECISIONS IN INDUSTRIAL QUALITY CONTROL

Decision Categories

In Chapter I we defined the purpose of Industrial Quality Control as the regulation of appropriate characteristics of the materials flowing into, through, and out of the processes within the industrial enterprise. We stated that in dealing with the two general categories of sorting and process-control problems the performance of feedback, filtering, and feedforward functions are performed. Each such function requires measurement, decisions for action and control action. Chapter II provided a summary of some of the procedures and techniques in industrial use for the accomplishment of quality-control tasks and indicated the variety of such techniques that exist.

In the present chapter the technical decisions which are required to set up and carry out the functions of quality control are delineated and some criteria against which to evaluate these decisions are discussed. Several of the techniques from Chapter II are referenced as examples of possible outcomes of certain of these decisions. Although the decisions involved are highly interrelated, we will consider them in the categories of:

1. Measurement,
2. Analysis,
3. Control action, and
The scope of each category will be explained as the category is discussed.

In general, the decisions to be made in Industrial Quality Control are those needed to obtain and transform observations of materials and processes into control activity.

**Measurement Decisions**

Measurement decisions are those involved in obtaining material and process data. Referring to Figure I, these decisions involve the establishment and operation of the function-blocks and flow-paths labeled (1). Such decisions may result in a permanent set of data-gathering procedures or in the specification of an initial set and one or more subsequent sets, the use of which depends on the outcomes of prior measurements, analyses, and controls.

The types of decisions involved are:

1. **Selection of items to be measured:** Lot or batch acceptance sampling plans in general specify a random sampling from the entire lot. By "random" is usually meant assuring that each item in the lot has an essentially equal chance of being included in the sample. When sub-lots within a large lot can be identified it is sometimes advantageous to resort to stratified sampling, i.e., the drawing of random samples of certain relative sizes from each identifiable sub-lot. In continuous acceptance sampling and process control where the temporal order of items is known, samples can be drawn according to a specified time or item-of-production pattern.

2. **Number of items to be measured:** In single acceptance sampling this is a fixed number specified once and for all.
when the plan is adopted. In double, multiple, and sequential plans an initial sample size is designated, often unity in sequential sampling, and sizes of possible subsequent samples listed for various results from analysis of the first sample. Sometimes the entire remainder of a lot constitutes the second sample, e.g., when screening or detailing is called for. In continuous sampling the sampling intervals for various combinations of past results must be decided upon even though the usual procedure is to take one item at a time. Rarely is the sample size or interval changed from a preset standard in process control; however, a few such procedures are in use (Ref. 9). Sample-size variation is, nevertheless, a decision inherent in process control. In each case, the number of items required and the spacing between samples is dependent on the precision required of the analysis, the characteristics of the lot or process, and the types, frequency of occurrence, and patterns of assignable causes expected.

(3) Dimensions and parameters to be measured—The measurements to be made depend on the properties of the material that are being controlled and the ability to actually make certain types of measurements. Thus, the material dimensions and process parameters to be measured are those which affect the properties of interest and can be obtained with available means. Selection of critical properties depends on numerous factors of engineering know-how, ultimate product use, customer relations, and legal restrictions.
(4) Measurement characteristics:-The principal characteristic of the measurements that must be decided upon is the precision. This decision usually is a direct outgrowth of the method of analysis to be used but is also affected by the costs of obtaining various levels of precision. In practice a dichotomy is made between variables and attributes tests and measurements; although, in a sense, an attributes measurement is a coded low-precision variables measurement. Nevertheless, the choice between variables and attributes measurements should be given explicit attention.

(5) Measuring instruments:-The instruments used depend directly upon the type, precision, and speed of measurement required. In many cases, instrument cost and availability is, conversely, an important if not conclusive factor in determining the measurements to be made. An interesting survey of measuring devices for control of various types of continuous processes has been compiled by Dudenbostal and Priestly (Ref. 45).

Analysis Decisions

The word "analysis" is used here in an attempt to avoid confusion in following through the measurement, decision, control sequence. The "decision" portion of this sequence involves transforming a given set of observations into some set of control actions. In the sense we are using "decision" in this chapter, however, the word has much broader meaning and in fact includes as only one of its facets the very important decision of specifying "decision rules" for the above transformations.
We, therefore, prefer "analysis" to describe the transformations of observations into actions. It is further felt that in quality control, where, as was explained in Chapter II, the emphasis is on routinized procedures, the transformation stage consists of analysis and routine comparison against standards and is not really a decision-stage of any significant level. In Figure I, the analysis function takes place essentially within the "Decision Maker" block; although it does jointly involve the path by which measurements arrive from the "Measuring Devices" block. These areas are labeled as \( \text{2} \) in the figure. The "analysis" decisions to be made are:

1. **Hypothesis to be tested:** Because of the stochastic nature of industrial processes, the analysis of observations of such processes usually involves a statistical test of some sort rather than direct prediction of the stage of the process. To do this, a hypothesis is tested against one or more alternatives on the basis of the probability of occurrence of the observations at hand should the hypothesis be true or false. In acceptance sampling, the hypothesis usually tested is that the lot has a given fraction defective or, if the distribution form of a given variable is known, that the distribution has a certain mean or variance. In process control, the hypothesis usually tested is that the process mean coincides with some desired value. Other hypotheses involving the form of distributions and differences between lot characteristics are also widely used.
(2) Statistic used: To test the hypothesis selected, a test statistic must be computed on the basis of the sample drawn. The array of statistics available to test the given hypothesis and the extent of knowledge of the properties of the populations involved will influence the choice of the statistic to be used. The desired precision and the costs of obtaining data and computing the statistic also have an important effect. In attributes tests of all kinds the sample fraction defective is the statistic used. Variables tests may use any or all of the following: sample mean, variance, range, and skewness. As has been mentioned, the emphasis on speed and simplicity in Industrial Quality Control has placed a growing interest in the simply-computed non-parametric and other "rough and ready" forms of statistics. A number of these are reviewed by Wallis in Reference 46.

(3) Limits: The limits defining the critical region of the hypothesis test are usually determined by applicable criteria, often an arbitrarily-selected significance level or other point on the power curve. Applicable criteria will be discussed later in this chapter. Double, multiple, and sequential sampling plans require definition of more than one critical region, one to determine rejection and one to determine the need for additional observations. Usually in process control utilizing control-chart techniques, a single critical region is defined by the control limits of the chart. (Other tests, such as runs, will have their own critical regions.)
In practice, however, we might think of the existence of two different regions, because the type of action taken or the kind of trouble looked for will often depend on whether data points fall above the upper limit or below the lower limit. In automatic control systems in which the characteristics of control action are functions of the actual value of the statistic, essentially an infinite number of critical regions exist. We may then broadly define the selection of limits for our tests as the selection of all standards of comparison and the selection of all constants and coefficients for control system mechanisms.

(4) Decision rules: The three preceding decision steps when completed determine a decision rule, i.e., a set of procedures for a production worker, clerk, or control device to follow on the basis of the value of the statistic computed. A given decision rule may be formulated from the test of a single hypothesis or appropriate combinations of several hypothesis tests. It may merely say that when a shift in a process parameter is indicated that a search for the cause thereof is to be made. It may, however, completely specify the direction, magnitude, and velocity of the particular adjustment to be made given a certain value of a statistic or combination of values of statistics. Decision rules in acceptance sampling may involve complete acceptance or rejection, changes in sample size or rate, or changes of hypothesis
test or limits, (e.g., tightened and loosened inspection in Reference 22). Thus, we find a wide range in the directed activity and in the specificity of decision rules.

(5) Data recording—Once each measurement is taken it must be either acted on directly or stored for computational or combinatorial purposes. Computations can often be facilitated by judicious arrangements of data sheets or by graphical or pictorial presentations, e.g., the control chart. The form in which data are recorded and presented, therefore, depends directly on the form in which measurements are made and on the computations necessary to evaluate the statistics to be tested. The factors to be decided upon are the items of data to be used and the functional arrangement of these data to best aid the computational procedures to follow. In the case of automatic control, measurements are either acted upon instantly or combined with other measurements for control purposes. In the former case no storage at all takes place; in the latter case, measurements are stored only until necessary computations are completed. In neither case are measurements usually recorded.

(6) Computational procedure—The computational procedure is the actual conversion of raw data into the required statistics. Decisions to be made involve the selection of computational aids and shortcuts and calculating equipment. A number of these were mentioned in Chapter II. As has been previously stated, this set of decisions assumes a relatively important
position in quality control because of the requirements of simplicity and definiteness for the successful operation of a quality-control system.

Control-Action Decisions

The decision-rules selected to govern the analysis (or, it will be remembered, the actual "decision") function of the control process will dictate the type and sometimes the amount of control action to be taken. The actual devices used to accomplish this action, however, depend for the most part on administrative procedures and the technology of the products and processes involved. In Figure I, the "Parameter Adjustment" blocks and the branch points in the "Filter" block are the points at which control action is usually effected. The portions of the diagram thus involved are labeled 3. Although the actual accomplishment of control action is an essential part of the quality-control function, it is of only passing interest here.

Criterion Selection

Some criterion or set of criteria are needed to evaluate any decision or action sequence. The problem of selection of meaningful criteria to be used to design, run, and evaluate a company's quality-control system is, therefore, an all-important one in quality control. Solutions to this problem depend, in a broad sense, on the overall objectives of the company. The quality-control system as a whole is continually judged by management as to the effects of the system on the attainment of company objectives. In general such judgment is based on financial
return on investment in the control system and the effects of quality control on worker morale, customer good will, and other factors. The actual weighting given each such factor depends on the relative importance of the various components of the company's objectives to company management.

The operational or technical criteria used by the quality-control department for actual design and operation of acceptance-sampling and process-control procedures must of course reflect the company objectives. Optimum formulation of control procedures is often a difficult, if not impossible task, however, because of the difficulties involved in quantitative expression of many of the applicable criteria. Determination of values for intangible factors such as morale and cooperation is at best subjective, and even direct cost factors such as the cost of a defective item passing through a sampling inspection plan can usually be evaluated only approximately. This fact has undoubtedly deterred the development of useable criterion functions in the field. As a result, it is very common to find in use only those criteria involving the risk of making a wrong decision, criteria which have been borrowed directly from the field of statistical hypothesis testing. These latter types of criteria, though useful, involve rather limited suboptimizations in control-system problems. In the light of current trends toward automation, system analysis, and the potential integration of company control systems, a more general type of criterion function is needed despite formulation difficulties. Wald's statistical decision functions, (Ref. 6), constitute a theoretical effort in this direction and prove the existence of optimum solutions to a certain class of criterion
problems. The work of Saterthwaite, Breakwell, Duncan, and Cowden in the development of cost criteria are steps in the same direction. The ultimate is of course maximization of a functional consisting of a formal statement of company objectives.

The word "risk" was used above as the probability of a type-one or type-two error in a hypothesis test. Wald, on the other hand, defines a "risk function" in terms of the sum of the expected cost of inspection plus the expected costs of type-one and type-two errors. We offer a working definition of risk that is even broader than Wald's but offers extreme flexibility in discussing control systems. We define risk as the amount of any unwanted, detrimental factor by which a control system is evaluated. The amounts of such factors may be measured in terms of any one of the following:

(1) Probability risks - the probability that one of two types of wrong decision will be made in testing a hypothesis.

(2) Quantity or dimensional risks - risks expressed as the expected value of some product or process variable such as number of items inspected, number of defective items in production, or pounds of excess material in a finished product. The use of minimum average inspection for a given type of lot is a common application of this category of risk.

(3) Cost - the expected cost or sum of expected costs of various factors inherent in the control system. A wide variety of subtypes exists since numerous different cost bases can be applied.
Some restriction concerning risk, regardless of how the risk is expressed, is a basic part of any criterion. Three such restrictions are:

(1) Specified risk - the requirement that a certain risk assume a given value.

(2) Limited risk - the requirement that a given risk fall within a given range of values, usually zero and a specified maximum limit.

(3) Minimum risk - the requirement that a given risk be minimized.

For synthesis purposes, (3) would seem always preferable to (2) and (2) to (1). The restriction used in any given case, however, depends greatly on what is included in the particular risk and the extent of available process knowledge. For instance, a specified probability risk for a given quality lot that has been found empirically to reduce the number of defective items in production without causing significant increases in inspection and scrap costs may be the safest possible type criterion to use in establishing an acceptance inspection plan in the absence of specific cost data. When accurate cost data become available, it is time enough to attempt minimization. Specified and limited risks are also most useful in comparing two or more control systems or in checking a system against some standard.

Having now defined risk and noted ways of restricting risks, we are in a position to categorize criterion functions for the design and
operation of components of a quality-control system. The exact form of criterion to be used in any given situation actually depends on the amount of system information available and the decision situation involved (i.e., synthesis, rating, or comparison of control systems).

We use as our basic distinction among categories of criteria the amount of product or process information that the quality-control engineer possesses or is willing to use to design or evaluate his system. The following six categories are felt to cover the possible levels of system knowledge:

1. **Complete ignorance** - The occurrence of any logically possible population must be allowed for in system evaluation and design.

2. **Bounded set of populations** - Only those populations from a given bounded population set are considered in system evaluation and design. Selection of this set may often be somewhat arbitrary but should be based on the best possible knowledge of the products and processes involved.

3. **Given population** - A specified population which is either highly likely to occur or is critical with respect to system performance is considered alone for design and evaluation purposes.

4. **Population distribution** - The frequency distribution of the occurrence of a given set of populations is known and considered in design and evaluation of the control system.
(5) Population time-series autocorrelation - Sufficient knowledge of the temporal order of the populations involved is available to permit computation of meaningful autocorrelation functions which can be used for system design and evaluation considerations.

(6) Complete knowledge of population time-series - The exact population to be encountered by the system at every instant of time is known and system design and evaluation is based accordingly. The control problem is here essentially reduced to a deterministic scheduling problem. Also, as will be seen in the next chapter, the classical control theory techniques for control system studies involve the assumption of complete time-series knowledge except in a few cases where autocorrelation functions are employed.

Each of these six system-knowledge categories can now be combined with each of the three risk-restriction categories to form a complete yet fairly compact set of control-system criteria. Actual control system synthesis or analysis can be based on any one or combinations of as many of these criteria as may be desired. In such studies the risk itself may be expressed in any of the terms previously listed. A word of explanation of several of the criteria formed from the above combinations is in order.

First, it is necessary to point out that in general all pairs of criteria involving specified and limited risk restrictions are subject to some interchange for practical reasons. First, when using a limited
risk criterion for design purposes, the limit itself is often used in the system equations. This computational convenience usually results in a maximum risk equal to the limit. In effect, therefore, the system is designed according to a specified risk. Conversely, the discrete nature of many of the variables in industrial systems prevents exact fulfillment of many specifications. In such cases, risk limits must be substituted for exact risk specifications.

In the face of complete ignorance of the populations with which one is dealing or when these populations can be identified only as belonging to a certain set, the usual procedure is to restrict the maximum risk incurred from any of the possible populations. The procedure of minimizing this maximum risk is well known as the "minimax" procedure and is the one used in Wald's developments and in the sampling plans devised by Breakwell. The limited risk restriction, which we propose to call "limimax" in keeping with other title contractions, is illustrated in the designation of sampling plans by AOQL.

A complete review of the entire list of criteria reveals some categories that may seem to have little practical significance. Nevertheless, a complete categorization of this type is felt to be useful for two reasons. One is to bring order to an otherwise completely heterogeneous field. It is felt, for instance, that any criterion now in use or that may be proposed can be meaningfully placed in one of the categories available here with a resulting gain system understanding. The other reason for complete categorization is to stimulate theoretical and practical advances in the development of a broader
class of criterion functions by suggesting heretofore untried utilizations of system information. The result should be an ability to handle a wider range of more complex control problems, and would constitute a useful step toward actual coincidence of company objectives and control-system design and analysis criteria.
CHAPTER IV

CONTROL THEORY AND THE INDUSTRIAL PROCESS

Decision Rules in Quality Control

In earlier Chapters of this work the importance of simply-followed, routinized procedures in quality control was emphasized. A concise set of simple rules is needed to guide factory workers and quality-control clerks in carrying out manual phases of control. Where control procedures have been automated, these "rules" must be built into the control-system machinery. The simpler the rules, the less complex the machinery. With either manual or machine performance of control tasks, however, the more situations that can be handled without intervention of supervision, engineers, or maintenance personnel, the more smoothly the operation will run. There is, of course, an economic limit to the types and diversity of situations the control system should be designed to handle in routine fashion. This limit, moreover, can be quantitatively established only by careful evaluation of control-system performance under appropriate service conditions. It is also important to design the control system for as efficient operation as possible for the set of operations which can be economically handled as routine. This requires careful selection and evaluation of the "rules" given to clerks or built into control equipment.

In Chapter III we called rules of this kind "decision rules" and noted their place in the measurement-analysis-action control sequence.
We also listed criteria which might be useful in evaluating such rules. We now offer some procedures for evaluating various decision rules that one might wish to apply and for finding optimum values for the decision-rule constants in certain situations. These procedures are essentially the ones followed in classical control-system design and evaluation which have been revised to account for the stochastic nature of industrial processes. We will illustrate the procedures by investigating a few specific forms of decision rules.

Unfortunately, the procedures illustrated are limited almost exclusively to the study of linear systems and, therefore, linear decision rules. Procedures do exist for investigations of a few restricted classes of non-linear systems such as those involving dead bands, saturation, and hysteresis, but limitations on time and space preclude their inclusion herein. Non-linear decision rules are at present, of course, in wide use in industrial quality control in connection with standard control-chart procedures, and so are of vital interest to the field. It is, therefore, hoped and recommended that the research begun here can eventually be extended into this practical but complex field.

Control Theory

To describe adequately control theory with all its facets and ramifications would be the work of a multi-volume technical series. We will,

\footnote{Two such methods are "describing function analysis" and "phase plane analysis" discussed in Chapters 10 and 11, respectively, of Automatic Feedback Control System Synthesis by John G. Truxal, Reference 47.}
therefore, simply describe Control Theory as that set of theories, procedures, and techniques useful in the synthesis and analysis of control systems. The field of control-system design and analysis emerged from the complex of electronic, mechanical, and hydraulic controls used by the Military in World War II and has enjoyed an accelerating development ever since. Today the field encompasses the mathematical areas of the theory of equations, differential equations, operational calculus, Laplace transform theory, and functions of a complex variable. The techniques and procedures for application of these mathematics are manifold and in many cases several approaches in studying a single problem are required to gain adequate understanding of the system involved. Despite seeming diversity in the field, a basic underlying structure exists in problem treatment. The variety of mathematical methods and applicative techniques is called upon only when system complexity renders direct approaches impossible or unfeasible. It so happens, however, that the vast majority of practical systems are of such a nature as to make such "indirect" techniques economically applicable.

The direct approach, which is common to almost all phases of deterministic engineering analysis, is to represent the system under study or proposed for use by a time differential equation. System stability (transient behavior) is determined from the homogeneous solution of the equation, and the steady-state performance for any given input function (forcing function) from the particular solution obtained from substitution of the forcing function in the basic equation. The usual solution method for linear homogeneous equations is to assume a solution of the form

\[ \theta_o (t) = e^{rt}, \]  

(12)
where $\Theta_0$ is the system output at time $t$ and $r$ is a real or complex number to be evaluated. This form reduces the homogeneous differential equation to an algebraic equation, the solution of which is a linear combination of $t^b e^{r_i t} (i=1,2,\ldots,n)$ where $n$ is the order of the system equation, $r_i$ is the $i$th root of the resulting algebraic equation, and $b$ is an integer between zero and one less than the multiplicity of the root. The combinatorial constants are then evaluated from initial or other boundary conditions. Solutions of the particular equation depend on the form of the forcing function so will be given no general treatment here. Solution methods can be found in any differential equations text.

The differential equations representing many practical systems prove to be extremely difficult to solve by direct means. In such cases, one of two general approaches may be followed either singly or in combination. The first of these is essentially analytic and involves the use of the Laplace transform. The second, an empirical approach, is to construct an experimental model and observe its performance under given test conditions. Very often iconic representation is not feasible, however, and analogues are sought for system study. In such cases, which are actually fairly common, the Laplace transform is first employed to provide system representation of in terms of the complex frequency.

$$s = \gamma + j\omega, \quad (13)$$

and analogues determined from established relationships in the $s$-plane. In equation (13), "$j$" is equal to $\sqrt{-1}$.

The Laplace transform procedure for either analytic or analogue solution follows the same general lines. After transformation to the $s$-plane, which for linear systems results in algebraic equations, the
equation is rearranged to isolate either the error or the output in terms of system parameters and forcing functions. The denominator of the error or output expression is factored and a partial fraction expansion effected which results in a linear series of functions of "s". Ideally the inverse transformations of these terms are well-known so that transformation back to the time domain is readily accomplished. The resulting expression represents the system error or output as a function of time; that is, the solution to the original system differential equation.

Once the system equation has been solved, the results can be compared with given performance requirements. Criteria in wide use include the following.

(1) For transient solution;
   (a) convergence as "t" increases (this of course requires negative exponents for all exponential terms and factors for positive "t")
   (b) time constants of all exponential terms and factors (usually the longest of these is of most interest)
   (c) amplitudes of oscillatory terms
   (d) damping characteristics (under-, critical-, or over-damped)
   (e) the natural frequency (frequency of oscillation with zero damping)

(2) For unit step-function input:
   (a) the steady-state error
(b) time until first occurrence of zero error
(c) time to peak overshoot (if any)
(d) amplitude of peak overshoot (if any)

(3) For unit velocity-step-function input, the steady-state displacement error (the "velocity-error constant")

(4) For unit acceleration-step-function input, the steady-state displacement error (the "acceleration-error constant")

(5) For unit-amplitude sine wave:
   (a) the maximum error or output amplitude and the frequency at which it occurs.
   (b) the error or output amplitude at which the corresponding phase shift is \( -180^\circ \)
   (c) the limit of open and closed-loop response as the sine-wave frequency approaches zero
   (d) the maximum phase shift incurred

(6) For random inputs (usually characterized by an autocorrelation function), some function of the error such as root-mean-square

Other criteria and means of comparing and evaluating systems can of course be applied in particular situations.

As may be seen from example problems in most servomechanism texts, completion of the transform method can become extremely tedious\(^7\).

\(^7\) See for example Brown and Campbell, Principles of Servomechanisms, Chapter 3, Sections 8 and 9, pp 74-83 (Ref. 48). The system under study is an elementary remote-control positional servomechanism, the system equation for which is among the least complex found for practical systems.
especially for complicated multi-loop systems. In such cases the usual procedure is to sacrifice some precision of analysis for computational convenience, feasibility, or even possibility. Therefore, rather than actually attempt complete solution of the system equation, techniques are employed which will yield information concerning certain critical performance parameters. These may be compared with the appropriate performance criterion. Use of various combinations of such techniques will often provide adequate information for design or analysis purposes.

Chief among the above-mentioned critical performance parameters is that of stability or the convergence of the transient response. A system in which transients grow or fail to die out is immediately eliminated from further practical consideration since it is impossible for such a system to compensate for any error induced in the system. Two procedures by which system stability can be determined directly from the transformed system equation are in common use. These are known as the Routh-Hurwitz Stability Criteria and the Nyquist Stability Criterion. The Routh-Hurwitz criteria are based on the relationships between the form and coefficients of the system output equation and the location of the singularities (poles in the case of lumped-constant real systems) of the equation. A system having poles in the right half s-plane or on the imaginary axis other than the origin is unstable. The Nyquist Criterion utilizes the similarity between the algebraic form of the homogeneous system equation

---

8 Both of these criteria are discussed in Bronwell, Advanced Mathematics in Physics and Engineering, Chapter 16 (Ref. 49). An extremely complete yet concise account of the Nyquist Criterion and the associated mapping theorems may be found in James, Nichols, and Phillips, Theory of Servomechanisms, pp. 62-75 (Ref. 50).
and the particular solution for a sinusoidal forcing function.

Stability is thus determined from the steady-state solution for inputs of the form $e^{j\omega t}$, where $\omega$ is the angular frequency of the sinusoid. This criterion is also based on the locations of the poles of the system equation and involves a mapping of the imaginary axis of the $s$-plane into the plane of either the open or closed-loop transfer function. An additional procedure, the Root-Locus Method\(^9\), permits determination of closed-loop performance from open-loop performance by utilizing the relationships between the poles and zeroes of the open-loop equation and the poles of the closed-loop equation. In many design problems the basic form of the open-loop system is almost completely dictated by the physical characteristics of the process to be controlled and the components available for control-system use. For this reason, the form of the open-loop transfer function is often essentially fixed and the Root-Locus method can be directly applied. The method involves the tracing of the movement of the open-loop zeroes and poles with changes in the frequency-insensitive gain. Acceptable combinations are thus found. The effects of compensating networks can be evaluated by the same procedures. This method has the additional advantage of simultaneously yielding several other critical performance criteria such as natural frequency, damping ratio, and principal time constant.

Next to stability, one of the performance parameters of most interest in the control of most industrial processes is the steady-state error from a step input. It may be easily shown for stable systems, the open-

\(^9\) See Truxal, op. cit., Chapter 4.
loop transfer function of which has a pole of unit order or greater at the origin of the s-plane, that the steady-state error is zero. The majority of decision rules of interest in industrial control will be in this category. In many industrial systems the time constant and oscillatory behavior are also of interest. When this is true, it no longer suffices to know only that a system is stable, the actual pole-zero configuration must be known or the original equation completely solved.

Cyclic variations caused by worker fatigue and similar factors are common in industrial processes. Therefore, system response to sinusoidal excitation is important in such cases.

As has been previously stated, detailed accounts of servomechanism techniques may be found in the texts in the field. References have already been cited.

Critical Characteristics of Industrial Processes

Having reviewed control theory and the associated approaches and techniques of deterministic control-system design, we now take up the problem of routine or automatic control of the industrial process. In Chapter I we mentioned the two forms of non-rigidity inherent in such processes. We are now seeking to employ control theory to cope with the time-dependent form of non-rigidity, but in so doing we must evaluate the effects of continued routine or automatic control action on the time invariant, stochastic form of non-rigidity. Control action, as has been said, is based on measurements of one or more product or process characteristics and as such is subject to both sampling and measurement error.

Thus each time control action is effected, some function of these errors
is fed back into the system. Thus the process output becomes a random variable with a distribution dependent on the distributions of the contributing errors, the decision rule employed, and the physical nature of the system. We will consider only those cases in which the distributions of the inherent errors are independent of process inputs and outputs as well as of time. With this type of error, the system output distribution at any given time will have a mean equal to the output found by usual control-theory techniques plus the mean of the error distribution. The variance of the mean at the time of any observation is a function of the accumulated effects of the error variances in all previous adjustments. It is, therefore, important that the variance after an infinitely large number of adjustments be finite. Otherwise, the variance of the output of the closed-loop system would approach infinity, which would reduce to zero the probability of producing a product in any specified tolerance band. In other words, even though the expected value of process output might coincide exactly with the desired value, acceptable items can be produced only when the closed-loop variance is finite. Furthermore, to evaluate a given form of decision rule against some criterion, such as maximizing the time summation of the probability of an item of output falling in a tolerance band, the exact form of the process output distribution must be known.

An additional factor having a major effect on the method of attack is the discrete nature of the control action in the majority of quality-control systems. Whether the product itself is continuous or discrete, the usual procedure is to take periodic rather than continuous readings and effect a sequence of discrete corrective actions. This discreteness permits the use of difference equations instead of the usual differential equations to represent the quality-control system. Determination of
whether or not the procedures to be set forth here apply to the growing number of continuously controlled processes must be made by tests of convergence on strategically selected difference equations. If a difference equation can be found which converges as the sampling rate is increased without bound to the differential equation of the continuous system of interest, this difference equation may be used to approximate the differential equation. The procedures here developed can then be applied directly to this difference equation. It is felt that little difficulty should be experienced in finding such difference-equation approximations for most practical continuous industrial systems. This point, however, has not been investigated. Tests for convergence may be found in several texts in applied mathematics which treat difference equations.\(^{10}\)

It should be pointed out that expected performance of continuously-controlled systems can obviously be studied by direct application of the methods discussed in the last section of this Chapter. In so doing, one treats these systems as being deterministic, the assumption habitually made in classical control-system design. Whenever the variance or other parameters of the process-output distribution play a significant role in the evaluation of a system against a given criterion, however, procedures of the type here set forth must be employed.

The following discussions will involve only discrete systems. Extensions to continuous ones may be carried out by the reader and is

---

\(^{10}\) See, for example, Hildebrand, Methods of Applied Mathematics, pp 319-328 (Ref. 51). Chapter 3 of this book gives an excellent presentation of the basic concepts and applications of difference equations.
suggested as a topic of future research. We will also assume sampling and measurement error to be independent of the state of the system, system inputs, system outputs, and time. With these fairly realistic restrictions, the class of systems with which we are to concern ourselves can be schematically represented by the diagram in Figure II. Although the legend indicates the meaning of the symbols used; some additional explanation of the figure is in order.

For simplicity it will be assumed that the sampling switch, B, is closed periodically for an infinitesimally small length of time. The assumption of an infinitesimally short closing period enables us to ignore averaging effects which would occur if the output changed while the sample was being drawn. As a result, we need often describe the system performance only at the beginning of each of a sequence of discrete time periods. We will define the $k^{th}$ period as beginning at the time of the $k^{th}$ closing of $B$, where $k = 0,1,2,\ldots$. The remaining symbols take on special meaning in the light of this treatment. We define $m_k$ as the actual value of process mean at the instant of the $k^{th}$ closing of $B$. Likewise, $\xi_k$ is the value of the sampling and measurement error in the $k^{th}$ sample. Since $x_k$ is the actual value of the $k^{th}$ sample we have the useful relationship represented by differential device labeled 3 in Figure II,

$$x_k = m_k + \xi_k.$$  

In industrial processes, one normally considers the input, $I$, as representing a design bogie for the process mean which is extremely likely to remain fixed for the life of a given process or at least for periods that are long relative to the sampling period. For this reason we will consider $I$
PHYSICAL PROPERTIES OF SYSTEM

LEGEND:

I = Input or desired level for process mean
m = Output or actual level of process mean
ζ = A random variable representing all errors inherent in making a measurement
B = Sampling switch
x = Observed value of process output at instant B is closed
D = Observed deviation between I and m or simply x-I
G_D = The decision rule
S = Control-action signal
G_S = The effects on S of system physical properties
A = Adjustment actually made prior to next observation of m
r = Time-dependent shift in process mean
Ω = Differentials to detect differences

FIG. II. - SCHEMATIC DIAGRAM OF DISCRETE FEEDBACK SYSTEM FOR CONTROL OF A STOCHASTIC PROCESS.
as fixed at a value of zero, which we can do without violation of the
system representation by considering all measurements directly as devia-
tions between observed and desired process output. By so doing, we make

\[ x_k = D_k \]  \hspace{1cm} (15) 

and can eliminate differential number 1 from further consideration.

The decision rule, represented in transfer-function notation by
\( G_D \), relates the observed deviations to the control action to be taken. It
is a function of any or all of the \( x \)'s computed prior to making the \( k \)th
decision. Thus the \( k \)th decision could conceivably be based on \( x_0 \) through
\( x_k \); although it is customary to base the rules on a fixed number of
previous observations. We will consider, as has been said, only linear
decision rules, i.e., rules involving only linear combinations of the
\( x \)'s. We will furthermore assume that the rule remains fixed for the
duration of the process and that the only control actions called for are
displacements in process mean. Consideration of multistage rules or
those involving changes in sampling procedures or rate of change of
process mean is not provided in this work. Like so many other facets
of this vast field, however, these types of decision rules constitute
fascinating and seemingly useful areas of research.

In Figure II, \( S_k \) is the control action decided upon as a consequence
of making observation \( x_k \). Constraints imposed by the physical properties
of the production process and control mechanism impose a continuous non-
zero adjustment period on any control action to be taken. We define \( A_k \)
as the actual amount of the \( k \)th adjustment that has been completed at the
instant the \((k+1)\)st sample is drawn. Obviously \( A_k \) is not related to \( S_k \)
through direct application of the transfer function $G_g$. This fact should cause no confusion, however, as long as specific definitions are kept in mind. To have a workable system independent of sampling rate undoubtedly certain restrictions must be placed on $G_g$. A slightly less severe set of restrictions on $G_g$ will yield a workable system for given ranges of sampling period but not for others. We will not consider these restrictions for the present, but will assume open-loop process performance to suit our needs in specific cases.

At differential number 2, $A_k + m_k$ is added to $r_{k+1}$, where $r_{k+1}$ is defined as the net effect of all time-dependent ("assignable") shifts in process mean during the $k$th sampling period. Thus, at the time of the $k$th closing of $B$,

$$m_k = (m_{k-1} + A_{k-1}) + r_k.$$  \hspace{1cm} (16)

Having now formally defined the control system to be considered, we are now in a position to develop our analysis techniques.
CHAPTER V

DIFFERENCE EQUATION MODELS - DIRECT SOLUTION

General Formulation

Let us consider the general $n^{th}$-order linear decision rule

$$G_d(x) = a_0 x_k + a_1 x_{k-1} + \cdots + a_{n-1} x_{k-n+1}$$

$$= \sum_{i=0}^{n-1} a_i x_{k-i}, \quad (17)$$

where the $a_i$ are constants to be evaluated. We will assume for the present that $G$, the process transfer function, has open-loop stability with a dominant time constant less than a fourth of the sampling period. Thus, for all practical purposes, the control action can be treated as complete by the time the next sample is drawn, and

$$A_k = S_k. \quad (18)$$

The process mean at the time of the $(k+1)^{st}$ observation can, therefore, be expressed as

$$m_{k+1} = m_k + \sum_{i=0}^{n-1} a_i x_{k-i} + r_{k+1}$$

$$= m_k + \sum_{i=0}^{n-1} a_i m_{k-i} + \sum_{i=0}^{n-1} a_i e_{k-i} + r_{k+1}$$

$$= (1+a_0) m_k + \sum_{i=1}^{n-1} a_i m_{k-i} + \sum_{i=0}^{n-1} a_i e_{k-i} + r_{k+1}. \quad (19)$$

The expression for the process mean at the time of the $(k+n)^{th}$ observation is, by the same procedure,
A rearrangement of (20) gives

$$m_{k+n} = (1+a_o) m_{k+n-1} + \sum_{i=1}^{n-1} a_i m_{k+n-1-i} + \sum_{i=0}^{n-1} a_i e_{k+n-l-i} + r_{k+n} \quad (21)$$

which is an $n$th-order linear difference equation in terms of the process mean.

The complete solution of equation (21) for $m_k$ is

$$m_k = m_k^{(H)} + m_k^{(P)}, \quad (22)$$

where $m_k^{(H)}$ is the general solution of the homogeneous equation

$$m_{k+n} - (1+a_o) m_{k+n-1} - \sum_{i=1}^{n-1} a_i m_{k+n-1-i} = 0 \quad (23)$$

and $m_k^{(P)}$ is any particular solution of equation (21).

**Homogeneous Solution**

Let us first consider the homogeneous equation and assume a solution of the form

$$m_k = y^k. \quad (24)$$

By use of the shifting operator,

$$E^n m_k = m_{k+n}, \quad (25)$$

(23) may be written

$$E^n m_k - (1+a_o)E^{n-1} m_k - \sum_{i=1}^{n-1} a_i E^{n-1-i} m_k = 0. \quad (26)$$
If we now define the "linear difference operator"

\[ L = E^n - (1+a_0)E^{n-1} - \sum_{i=1}^{n-1} a_i E^{n-i} \]  \hspace{1cm} (27)

(26) can be expressed simply as

\[ L m_k = 0 \]  \hspace{1cm} (28)

If we substitute our assumed solution form as expressed by (24) into (25), we find

\[ E^k y = y^{k+n} = y^n y^k \]  \hspace{1cm} (29)

so that

\[ L y^k = \left[ y^n - (1+a_0) y^{n-1} - \sum_{i=1}^{n-1} a_i y^{n-1-i} \right] y^k \]  \hspace{1cm} (30)

Therefore, (24) will satisfy the homogeneous equation if \( y \) is a root of

\[ y^n - (1+a_0) y^{n-1} - \sum_{i=1}^{n-1} a_i y^{n-1-i} = 0. \]  \hspace{1cm} (31)

As long as

\[ a_{n-1} \neq 0, \]  \hspace{1cm} (32)

zero roots are excluded and (31) will have \( n \) non-zero roots. Let these be represented by \( y_i \) (i=1,2,...,n). Each root may be classified both by its order, i.e., the number of occurrences of the root, and by whether it is real or complex. Complex roots must of course occur in conjugate pairs for real coefficients in (31). Equation (31) can now be written in factored form, i.e.,

\[ (y-y_1)(y-y_2)\cdots(y-y_i)\cdots(y-y_n) = 0, \]  \hspace{1cm} (33)
and (26) as
\[ L y^k = (y-y_1)(y-y_2)\cdots(y-y_i)\cdots(y-y_n) y^k. \] (34)

The part of the general solution of the homogeneous equation corresponding to single-order roots is thus
\[ m_k = \sum_{i=1}^{(1)} c_i y_1^k \] (35)

where the one in parentheses indicates summation only over the single roots of (31) and the \( c_i \) are combinatorial constants to be evaluated from initial conditions.

Suppose now that (31) has a double root, say \( y_1 \). Equation (34) would then become
\[ L y^k = (y-y_1)^2(y-y_3)\cdots(y-y_i)\cdots(y-y_n) y^k. \] (36)

In this case we have not only
\[ [L y^k]_{y=y_1} = 0 \] (37)

but also
\[ \left[ \frac{\partial}{\partial y} (L y^k) \right]_{y=y_1} = L \left[ \frac{\partial}{\partial y} (y^k) \right]_{y=y_1} = L \left[ k y^{k-1} \right]_{y=y_1} = 0. \] (38)

Since \( y_1 \) is a constant, the results of (38) can be expressed as \( k y_1^k \). Thus the part of the general solution of the homogeneous equation corresponding to double roots is
\[ m_k = \sum_{i=1}^{(2)} \left( c_i + c_{i1} k \right) y_1^k \] (39)

where the two in parentheses indicates summation only over double roots. The \( c_i \)'s are again constants to be evaluated from initial conditions.

By continued application of the procedures just used, it is easily shown that the part of the general homogeneous solution corresponding to \( p \)-fold roots is
Summing over all possible orders of roots gives the general solution
of the homogeneous equation as

$$m_k = \sum_{p=1}^{n} \left( \sum_{i=1}^{(p)} c_i + c_{12} k^2 + \ldots + c_{i(p-1)} k^{p-1} \right) y_i^k$$

$$= \sum_{i=1}^{(p)} \sum_{q=0}^{p-1} c_{iq} k^q y_i^k \tag{40}$$

where again the $p$ in parentheses indicates the order of the root. There
naturally cannot be roots of every order present.

This expression for $m_k$ is perfectly general in that it applies to
real and complex roots alike. For any given real root, the physical
significance of $y_i^k$ is readily apparent. In the case of complex roots,
this significance can better be understood from the following treatment.

Let $y_1$ and $y_2$ constitute a conjugate pair of complex roots, then

$$y_1 = u + jv$$

and

$$y_2 = u - jv. \tag{42}$$

The part of the general solution of the homogeneous equation corresponding
to this pair can be written as

$$m_k = c_1 (u + jv)^k + c_2 (u - jv)^k \tag{44}$$

or, in polar form,

$$m_k = c_1 (|y| \ e^{j\phi})^k + c_2 (|y| \ e^{-j\phi})^k$$

$$= (|y|)^k (c_1 \ e^{j\phi} + c_2 \ e^{-j\phi})^k$$

$$= (|y|)^k c_1 \cos \phi \ k + c_1 \ j \sin \phi \ k + c_2 \ cos \phi \ k - c_2 \ j \sin \phi \ k$$

$$= (|y|)^k (c_3 \cos \phi \ k + c_4 \sin \phi \ k). \tag{45}$$
Here we have let

\[ y = \sqrt{u^2 + (iv)^2} = |y_1| = |y_2|, \quad (46) \]

\[ \phi = \tan^{-1} \frac{v}{u}, \quad (47) \]

\[ c_3 = c_1 + c_2, \quad (48) \]

and

\[ c_4 = j (c_1 - c_2). \quad (49) \]

Either \( c_1 \) and \( c_2 \) or \( c_3 \) and \( c_4 \) are to be evaluated from initial conditions. Thus, the linear combination of each pair of conjugate roots results in a linear combination of sinusoidal terms (with frequency varying linearly with \( k \)) multiplied by the absolute value of the root raised to the \( k^{\text{th}} \) power. For multiple complex pairs, \((45)\) would simply be multiplied by the factor \( \prod_{q=0}^{p-1} c_{1q} k^q \), as would a \( p \)-fold real root, giving for the part of the general solution of the homogeneous equation corresponding to \( p \)-fold complex pairs

\[ m_k = \left[ \sum_{i} \sum_{q=0}^{p-1} c_{1q} k^q \left[ y_j \right]^k (c_{3q} \cos \phi_1 k + c_{4q} \sin \phi_1 k) \right]. \quad (50) \]

The \( p^c \) in parentheses indicates summation is carried out only over \( p \)-fold conjugate complex pairs of roots. This expression may be simplified slightly by the combination of constants to yield

\[ m_k = \left[ \sum_{i} \sum_{q=0}^{p-1} k^q \left[ y_j \right]^k (c_{1q} \cos \phi_1 k + c_{1q}^i \sin \phi_1 k) \right]. \quad (51) \]

Equation \((51)\) can now be combined with the results indicated by \((41)\) to give the general solution of the homogeneous equation
\[
    m_k(H) = \sum_{p=1}^{n} \left[ \sum_{q=0}^{p-1} c_{piq} y_i^k + \sum_{q=0}^{p-1} k^q y_i^k \left( c_{piq} \cos \phi_{pi}^k + c_{piq}^\prime \sin \phi_{pi}^k \right) \right]
\]

where \((p^o)\), as before, indicates summation over complex \(p\)-fold pairs and \((p^R)\) indicates summation over \(p\)-fold real roots. Since, as has been said, the total number of real plus complex roots must equal \(n\), not every category of root will exist even though the expression derived accounts for all possible orders of real and complex roots. Categories not represented for a particular decision rule will naturally not appear.

**Particular Solution**

The next step is to find a particular solution to the system equation (21). The right hand side, RHS, of this equation consists of a linear combination of \(n\) terms in \(\epsilon\) and the term \(r_{k+n}\). We have already stated that the \(\epsilon\)'s are mutually independent random functions of \(k\). The series of shifts \(r_k\) constitute an as-yet-unspecified function of \(k\).

In many industrial systems, shifts in process mean, measured in displacement per time period, assume certain time patterns of special interest. Long term trends due to tool wear or the clogging of air filters may produce a shift pattern of the form \(\alpha k^p\), where \(\alpha\) is a constant and \(p\) is zero or any positive integer, or a pattern which is a linear combination of several terms of the same general form. Too low a flow of coolant in a cutting operation could cause a temperature rise characterized by \(\alpha k\) or \(e^{\alpha k}\). Cyclic variations due to hourly changes in worker efficiency, unevenly worn bearings or bent shafts could result in shifts described by a linear combination of terms of the form \(\sin \alpha k\) and \(\cos \alpha k\).
To evaluate the effects of such shifts alone without regard to the random effects of the $\varepsilon$'s, it is merely necessary to represent $m_k^{(P)}$ as a linear combination of the terms describing the shift patterns. Should the form of any forcing-function term be represented by a root of the homogeneous equation, it is necessary to multiply all terms with that particular form by $k^p$, where $p$ is the order of the root of the homogeneous equation. To illustrate, let the shift pattern being investigated be of the form $\sin \alpha_k$ or $\cos \alpha_k$ (amplitude is neglected). We would then be required to check all pairs of complex roots of the homogeneous equation to determine whether any of the $\varphi_i$ were equal to $\alpha$. If any such roots were found, the $\sin \alpha_k$ and $\cos \alpha_k$ terms in the particular solution would have to be multiplied by $k^p$, where $p$ is the order of the complex pair of homogeneous-equation roots with $\varphi_i$ equal to $\alpha$.

The procedures outlined in the preceding paragraph, though somewhat restricted by requirements placed on the form of the forcing function, are nevertheless handy for determination of the expected value of process mean at any observation instant. For such an evaluation, the $\varepsilon$'s are replaced by their expected value, which we have assumed to be independent of $k$, which simply adds a term of the form $\alpha k^0$ to the linear combination of forcing-function terms. The constant $\alpha$ here represents the mean value of sampling and measurement errors, which often in practice equal zero.

A more general approach to the determination of the particular solution of linear difference equations is available, however, and is indeed required to provide means of evaluating the stochastic properties of industrial systems. We start with the general homogeneous solution which, for our purposes here, we write in the form
The $w_i$ include the corresponding $y_i$ and any sines, cosines, or powers of $k$ by which it is multiplied in the general expression in (52). There must be of course exactly $n$ such terms. This may be verified from equation (40) which indicates that a $p$-fold root appears $p$ times in the linear combination, being multiplied, respectively, by the zeroth through the $(n-1)^{st}$ powers of $k$. Equation (45) shows that each pair of conjugate complex roots results in both a sine term and a cosine term. Therefore, regardless of the nature of the roots, exactly $n$ terms can appear in (53). The $c_i$ in (53) represent the same combinatorial constants appearing in (52). We now assume a particular solution of the form

$$m_k^{(P)} = \sum_{i=1}^{n} c_k^{(i)} w_i,$$  \hspace{1cm} (54)

where the $c_k^{(i)}$ are functions of $k$ to be determined. We can now write

$$m_{k+1} = \sum_{i=1}^{n} c_k^{(i)} w_i(k+1)$$

$$= \sum_{i=1}^{n} c_k^{(i)} w_i(k+1) + \sum_{i=1}^{n} \left[ c_k^{(i)} - c_k^{(i)} \right] w_i(k+1)$$

$$= \sum_{i=1}^{n} c_k^{(i)} w_i(k+1) + \sum_{i=1}^{n} w_i(k+1) \cdot \Delta c_k^{(i)}.$$  \hspace{1cm} (55)

We of course require $n$ independent conditions to determine the $n$ functions $c_k^{(i)}$. The first of these will be to require the second summation of (55) to vanish. Thus,

$$\sum_{i=1}^{n} w_i(k+1) \cdot \Delta c_k^{(i)} = 0,$$  \hspace{1cm} (56)

and

$$m_{k+1} = \sum_{i=1}^{n} c_k^{(i)} w_i(k+1).$$  \hspace{1cm} (57)
By the procedure indicated in equation (55) we get from (57)

\[ m_{k+2} = \sum_{i=1}^{n} c_k^{(i)} w_i^{(k+2)} + \sum_{i=1}^{n} w_i^{(k+2)} \Delta c_k^{(i)}. \]  

(58)

As our second condition, we will require the second summation of (58) to vanish, which yields

\[ \sum_{i=1}^{n} w_i^{(k+2)} \Delta c_k^{(i)} = 0 \]  

(59)

and

\[ m_{k+2} = \sum_{i=1}^{n} c_k^{(i)} w_i^{(k+2)}. \]  

(60)

This same procedure is repeated for a total of (n-1) times which gives

the (n-1) conditions

\[ \sum_{i=1}^{n} w_i^{(k+p)} \Delta c_k^{(i)} = 0 \quad (p=1,2,\ldots,n-1), \]  

(61)

and the n equations

\[ m_{k+p} = \sum_{i=1}^{n} c_k^{(i)} w_i^{(k+p)} \quad (p=1,2,\ldots,n-1) \]  

(62)

and

\[ m_{k+n} = \sum_{i=1}^{n} c_k^{(i)} w_i^{(k+n)} + \sum_{i=1}^{n} w_i^{(k+n)} \Delta c_k^{(i)}. \]  

(63)

When the expressions for the various \( m_k \)'s given by (54), (62), and (63) are substituted in the original system equation the following result is obtained:

\[ \sum_{i=1}^{n} w_i^{(k+n)} \Delta c_k^{(i)} = \sum_{i=0}^{n-1} a_{i+k-n-1} + r_{k+n}. \]  

(64)

This expression results by virtue of the fact that

\[ L w_i^{(k)} = 0 \quad (i=1,2,\ldots,n). \]  

(65)

Equation (64) and the (n-1) conditions in (61) give us n linear equations in terms of the n unknown \( \Delta c_k^{(i)} \). The matrix of these equations can be inverted by standard methods to solve for the \( \Delta c_k^{(i)} \) in terms of known quantities. Such solutions will be of the general form
\[ \Delta c_k^{(i)} = f_k \cdot D_k^{(i)} \quad (i=1,2,\ldots,n) \]  \hfill (66)

where

\[ f_k = \sum_{i=0}^{n-1} a_i \equiv k+n-l-i + r_{k+n} \]  \hfill (67)

and the \( D_k^{(i)} \) are coefficients determined during the matrix inversion.

The \( D_k^{(i)} \) are functions of the homogeneous solutions and of \( k \). The \( c_k^{(i)} \) can now be determined from the corresponding \( \Delta c_k^{(i)} \) by use of the well-known relationship that, if

\[ \Delta y_k = F_k, \]  \hfill (68)

then

\[ y_k = \sum_{p=1}^{k} F_{p-1} = \sum_{i=0}^{k-1} F_{p}. \]  \hfill (69)

Thus

\[ c_k^{(i)} = \sum_{p=1}^{n} f_{p-1} \cdot D_p^{(i)} = \sum_{p=0}^{n-1} f_p \cdot D_p^{(i)}. \]  \hfill (70)

From (70)

\[ m_k^{(p)} = \sum_{i=1}^{n} \left[ \sum_{p=0}^{n-1} f_p D_p^{(i)} \right] w_{ik}, \]  \hfill (71)

and the resulting complete solution for \( m_k \) is the sum of (52) and (71).

**Evaluation of Homogeneous-Solution Coefficients**

The next step in our difference-equation solution is the evaluation of the combinatorial coefficients in the homogeneous solution given by (52). As has been stated, there are exactly \( n \) coefficients to be evaluated so that \( n \) boundary conditions must be used in the evaluation. For our purposes here, the most readily obtainable set of such conditions is the first \( n \) values of process mean, \( m_k, k=0,1,\ldots,n-1 \). These may be obtained by repeated application of the decision rule during the stages of the
process where an accumulation of a full n previous samples does not exist.
The manner in which this is handled is arbitrary in the case of manual systems but may be dictated by the characteristics or economics of the control equipment in automated ones. Numerous "initial-decision" rules could be employed or built into control machinery. One rule might be to do nothing until n observations had been made. Another might be to act on all available samples and assign arbitrary values to fill in the remaining variables in the decision rule. In any case, for n greater than two or three, n repetitions of computing successive process means can become extremely tedious. In such situations some help may be obtained from a recursion formula which directly relates the component terms of $m_k$ to appropriate terms in previously computed expressions of process mean. These are best developed by actual derivation of enough terms of the sequence of $m_k$'s to observe the inherent relationships. Proof is usually by induction. Success in recognizing relationships depends greatly on the experience and ingenuity of the system-designer in properly grouping the symbols involved.

Let us consider the initial-decision rule whereby samples corresponding to negative values of $k$ will be considered as having a value of zero. We, therefore, act on the available samples using as many of the decision-rule constants as is necessary. Such a procedure is felt to have both logical and intuitive appeal. The process mean at the time of the $k^{th}$ observation can, under these circumstances, be expressed as
\[
m_k = r_o \left[ (1+a_o) b_o^{k-1} + \sum_{i=1}^{k-1} a_i b_o^{k-1-i} \right] + \sum_{i=1}^{k} r_i b_o^{k-i} + \sum_{i=1}^{k} \epsilon_{k-i} \left[ \sum_{p=0}^{i-1} a_p + \sum_{q=1}^{i-1} \epsilon_o^{k-q} \cdot \sum_{p=0}^{i-1-q} a_p \right],
\]

where \( b_i^p \) and \( g_i^p \) represent, respectively, the coefficients of \( r_i \) and \( \epsilon_i \) in the expression for \( m_i \). The coefficient of each \( r \) and each \( \epsilon \) in \( m_k \) is thus expressed in terms of combinations of multiples of certain coefficients in previously computed expressions for process mean. From (72) we find, specifically,

\[
m_0 = r_o,
\]

\[
m_1 = r_o (1+a_o) + r_1 b_o + \epsilon_o a_o
\]

\[
= r_o (1+a_o) + r_1 + \epsilon_o a_o,
\]

and

\[
m_2 = r_o \left[ (1+a_o) b_o^1 + a_1 b_o^0 \right] + r_1 b_o^1 + r_2 b_o^0 + \epsilon_o \left[ a_o + a_1 + b_o^1 \right] + \epsilon_1 a_o
\]

\[
= r_o \left[ (1+a_o)^2 + a_1 \right] + r_1 (1+a_o) + r_2
\]

\[
+ \epsilon_o \left[ a_o + a_1 + a_o^2 \right] + \epsilon_1 a_o,
\]

which agree with the expressions derived by direct application of the

---

The "system-designer" whose "experience and ingenuity" resulted in an arrangement of symbols permitting formulation of this recursion relationship is the author's advisor, Dr. Loring G. Mitten, Associate Professor, Department of Industrial Engineering, The Ohio State University.
decision rule. The recursion formula is thus shown to hold for \( k = 0, 1, \text{and} 2 \).

To prove it holds for any \( k \), we must show that if it holds for \( k \) it will hold for \((k+1)\) also. We will not attempt proof here, but the procedure would be to substitute (72) into the system equation as given in (19) and show that the resulting expression for \( m_{k+1} \) was identical to (72) with \((k+1)\) substituted for \( k \).

An ideal situation to arise in determining recursion relationships is to gain sufficient insight to write a general expression for \( m_k \) directly in terms of the \( r_i \) and \( \varepsilon_i \) with coefficients which are explicit functions of \( k \) and the decision rule constants. Any desired value of \( m_k \) as well as limiting conditions as \( k \) increases without bound are readily determined without the use of formal difference-equation techniques. Though such situations are felt to be rare, we were able to use this procedure for the first-order equation representing simple proportional control with zero time delay (Ref. 36).

Once \( m_0 \) through \( m_{n-1} \) are obtained in terms of shifts, \( r_i \), and sampling and measuring errors, \( \varepsilon_i \), the \( n \) coefficients in (52) are computed from the \( n \) simultaneous equations

\[
m_k^{(H)} = \hat{m}_k \quad (k=0,1,\ldots,n-1),
\]

where \( m_k^{(H)} \) is the specific form of (52) corresponding to the given system equation and \( \hat{m}_k \) the actual known value of process mean. The solution follows standard methods but is by no means a simple task for \( n \) greater than two or three. Since the exact form of the resulting expressions depends so heavily on the specific system involved, no attempt is made here at a general representation of results.
The Expected Value and Variance of Process Mean

To evaluate the interaction between feedback control and the stochastic properties of the industrial process, some measure of the probability distribution of the process mean at the beginning of each adjustment period is required. Theoretically this may be found by calculating the joint density function of the combination of $\epsilon$'s in any $m_k$ and shifting the computed mean of this joint density by the accumulated shifts due to the $r_i$ in the same $m_k$ expression. In general, determination of joint densities is a difficult task so that more easily obtained measures are sought. The mean and variance usually suffice. In the special case of the normal distribution (which ironically is one distribution for which joint densities can be computed by straight-forward methods), the mean and variance completely describe the distribution function. With reference to the discussion of Shewhart's work in Chapter II, we would expect the $\epsilon$'s to be approximately normally distributed because they result from a multiplicity of chance causes none of which are dominant. We would expect the resulting joint densities to approximate normality even more closely because of the operation of the law of large numbers as $k$ increases.

Evaluation is by no means limited to normally distributed error terms, but the normal distribution should be kept in mind as a possible useful and realistic approximating distribution. We are furthermore not limited to use of the mean and variance as measures of the distribution. Any distribution moment or combination of moments desired can be used. It is only necessary to determine how each combines in the formation of joint densities. The combinatorial aspects of mean and variance are well-known, however, and together they describe approximately normal distributions with almost 100% accuracy.
The expected value or mean of the process mean, $\overline{m}_k$, is simply equal to the sum of the expected values of the various terms in the general expression defining $m_k$. Since we have not explicitly determined $m_k$, we will not attempt such for $\overline{m}_k$. In general, however,

$$\overline{m}_k = \sum_{i=1}^{k} b_i^k r_i + \sum_{i=1}^{k-1} g_i^k E(\varepsilon_i), \quad (77)$$

where $b_i^k$ and $g_i^k$ have the same meaning as in equation (72) and $E(\varepsilon_i)$ stands for the expected value of $\varepsilon_i$.

The variance of $m_k$ is determined from the well-known relationship

$$\sigma^2 = \sum_{i=1}^{\infty} a_i^2 x_i^2 \geq a_1^2 \sigma_x^2, \quad (78)$$

where the $a_i$'s are constants and the $x_i$'s independent random variables. We have already limited our discussions to systems in which the distribution of sampling and measurement errors are independent of time and the state of the system. The $\varepsilon_k$ are, therefore, mutually independent random variables with the same probability distribution and, consequently, equal variance. To find the variance of $m_k$ it is first necessary to put the expression for $m_k$ into the standard form expressed by

$$m_k = \sum_{i=1}^{k} b_i^k r_i + \sum_{i=1}^{k-1} g_i^k \varepsilon_i, \quad (79)$$

which is a linear combination of the $r_i$ and $\varepsilon_i$. To do this, the $r_i$ and $\varepsilon_i$ in the initial conditions, $m_k^{(H)}$, and $m_k^{(P)}$ must be combined which may
involve some manipulation in certain cases. The form presented in (79) is also the basis for the expression in (77). In the case of the expected value, placing the expression in standard form is not as important as when dealing with the variance (where the independence of constituent terms is essential). From (78) and (79), the variance of \( m_k \) is found to be

\[
\sigma^2_{m_k} = \sum_{i=1}^{k-1} (\varepsilon_i^k)^2 \sigma^2_{\varepsilon_i} = \sigma^2 \sum_{i=1}^{k-1} (\varepsilon_i^k)^2,
\]

(80)

where \( \sigma^2 \) is the common variance of the \( \varepsilon_i^1 \).

The important question of system stability and the general problem of performance evaluation will be taken up in the following Chapter in connection with some discussions of specific types of systems.
First-Order Equations, Simple Proportional Control

The general procedures outlined in the previous chapter can best be understood by applying them to a few simple systems. System evaluation is also more simply discussed with regard to a specific situation.

In this section the simple proportional controller or first-order difference equation will be considered. In the next section the general case of the second-order difference equation will be discussed.

By simple proportional control we mean control effected by an adjustment which is a fixed proportion of the single observed deviation between desired and actual process mean at the time of that particular observation. If control can be initiated and the adjustment completed before the next observation is made, the decision equation can be written simply

\[ m_{k+1} = m_k + a_0 x_k + r_{k+1} \]

\[ = m_k + a_0 (m_k + \epsilon_k) + r_{k+1} \]  
\[ = (1 + a_0) m_k + a_0 \epsilon_k + r_{k+1}. \]  

The resulting difference equation becomes

\[ m_{k+1} - (1 + a_0) m_k = a_0 \epsilon_k + r_{k+1}. \]  

Proceeding as indicated in the previous chapter, we first find the solution to the homogeneous equation
We assume a solution of the form
\[ m_k = y^k \]
and substitute in (83) giving
\[ y^{k+1} - (1+a_o) y^k = [y - (1+a_o)] y^k = 0 . \]  \hspace{1cm} (85)
Equation (85) is satisfied by a linear function of the root of
\[ y - (1+a_o) = 0, \]  \hspace{1cm} (86)
which obviously is
\[ y = (1+a_o). \]  \hspace{1cm} (87)
Thus
\[ m_k^{(H)} = c_o (1+a_o)^k \]  \hspace{1cm} (88)
where \( c_o \) is a constant to be determined from initial conditions.

First, according to the order of the solution procedure previously outlined, we will find a particular solution to difference equation (82). Let us assume a solution of the form
\[ m_k^{(P)} = C_k (1+a_o)^k . \]  \hspace{1cm} (89)
Omission of the parenthetical superscript on \( C_k \) should cause no confusion in this simple case. From (89) we get
\[ m_{k+1} = C_{k+1} (1+a_o)^{k+1} = C_k (1+a_o)^{k+1} + [C_{k+1} - C_k] (1+a_o)^{k+1} = C_k (1+a_o)^{k+1} + (1+a_o)^{k+1} \cdot \Delta C_k . \]  \hspace{1cm} (90)
Substituting (89) and (90) in (82) yields
\[ C_k (1+a_o)^{k+1} + (1+a_o)^{k+1} \cdot \Delta C_k = (1+a_o) C_k (1+a_o)^k \]
\[ = a_o c_k + r_{k+1} . \]  \hspace{1cm} (91)
which reduces to
\[(l+a_0)^{k+1} \cdot \Delta c_k = a_0^{e_k} + r_{k+1} \] (92)
or
\[\Delta c_k = \frac{a_0^{e_k}}{(1+a_0)^{k+1}} + \frac{r_{k+1}}{(1+a_0)^{k+1}} .\] (93)

As will be remembered, however, if
\[\Delta c_k = f_k,\] (94)
then
\[c_k = \sum_{i=0}^{k} f_{i-1}.\] (95)

Utilizing the relationship between (94) and (98), we get, from equation (93),
\[c_k = \sum_{i=1}^{k} \frac{a_0^{e_i-1}}{(1+a_0)^i} + \sum_{i=1}^{k} \frac{r_i}{(1+a_0)^i}.\] (96)

Multiplying (96) by \((1+a_0)^k\) gives us for \(m_k^P\)
\[m_k^P = \sum_{i=1}^{k} a_0^{e_i-1} (1+a_0)^{k-i} + \sum_{i=1}^{k} r_i (1+a_0)^{k-i}.\] (97)

Evaluation of the constant \(a_o\) in the expression for \(m_k^H\) is accomplished readily from the condition
\[m_o = c_o (1+a_0)^0 = c_o = r_o.\] (98)
The complete solution is then
\[m_k = m_k^H + m_k^P = r_o (1+a_0)^k + \sum_{i=1}^{k} a_0^{e_i-1} (1+a_0)^{k-i} + \sum_{i=1}^{k} r_i (1+a_0)^{k-i} = \sum_{i=0}^{k} r_i (1+a_0)^{k-i} + a_o \sum_{i=0}^{k-1} i (1+a_0)^{k-1-i}.\] (99)
From (99) the expected value and variance of the process mean are easily found to be

\[ \overline{m_k} = \sum_{i=0}^{k} r_i (1+a_o)^{k-i} + a_o \sum_{i=0}^{k-1} E(\varepsilon) (1+a_o)^{k-1-i} \]  

(100)

and

\[ \sigma_{m_k}^2 = \sum_{i=0}^{k-1} (1+a_o)^{2(k-1-i)} \overline{m_k} \]  

(101)

respectively. The variance expressed by (101) is obviously the variance about the expected value, \( \overline{m_k} \).

In evaluating any control system, the first aspect to be considered is stability. Stability is the ability of a system to return to a desired level after some disruption has taken place. In linear systems, stability has nothing to do with the nature of the disruption (unless of course the disruption affects the linearity of the system) and, therefore, can be determined completely from the homogeneous solution of the system equation. Stability, of course, does not guarantee the ability of the system to follow any type of forcing function or to meet other types of design criteria such as limits on dominant time constant, peak overshoot, and the like. A system must be stable, as was indicated in Chapter III, however, to respond properly to any forcing function including the randomly distributed \( \varepsilon \)'s. Therefore, even in the absence of shifts, \( r_i \), and for an \( E(\varepsilon) \) of zero (which would make \( \overline{m_k} \) equal to zero), the unstable system would be unsatisfactory because of the effect of any non-zero \( \varepsilon \) which might occur. The situation mentioned would result, in the limit, in an \( \overline{m_k} \) of zero and an infinite \( \sigma_{m_k}^2 \).
From (88), the homogeneous solution for \( m^k \), we see that as long as 
\[(1 + a_o) \text{ is less than one in absolute value } m^k(H) \text{ will approach zero as } k \text{ increases.} \]
Values of \( a_o \) which result in a stable proportional controller are, therefore,

\[-2 < a_o < 0. \tag{102}\]

For the special case of a zero \( a_o \), i.e., no control action, the process mean will be subject to any forcing function which may occur and will not converge to zero even though \( \sigma_{m_k}^2 \) is zero. A fairly complete discussion of the relationships among \( a_o \), step-function response, and the limiting variance of \( m_k \) is presented in Reference 36 and is summarized in Table 1.

**TABLE 1**

Characteristics of Simple Proportional Control Systems

| \( a_o \) | \( \lim_{k \to \infty} m_k^2 \) | \( \sigma_{m_k}^2 / \sigma^2 \) | *Expected Fraction of \( r \) remaining after 1 adj 2 adj 3 adj \( \infty \) adj |
|---|---|---|---|---|
| > 0 | \( \infty \) | -- | -- | -- | \( \infty \) |
| 0 | 0 | 1.00 | 1.00 | 1.00 | 1.00 |
| -0.5 | 0.33 | 0.50 | 0.25 | 0.125 | 0 |
| -1.0 | 1.00 | 0 | 0 | 0 | 0 |
| -1.5 | 3.00 | -0.50 | 0.25 | -0.125 | 0 |
| \( \leq -2.0 \) | \( \infty \) | -- | -- | -- | \( \infty \) |

* Assuming \( E(\xi) = 0. \)
Procedures are also discussed for finding optimum values of $a_o$ given certain tolerance limits on individual items of system output and for determining, for a ramp-type forcing function, the $a_o$ which will minimize the mean-square deviation of items of product from bogie. A summary of these procedures can also be found in Reference 52.

**Proportional Control With Incomplete Adjustments**

Before considering higher-order difference equations, let us investigate the effects of simple proportional control on a system in which the transient from one adjustment does not die out completely before the next observation is made. Let $b$ equal the portion of the intended adjustment that is completed at the time of the next observation. We put no restrictions on $b$ as yet; although, if the process exhibits open-loop stability, $b$ will be positive but less than two regardless of the sampling rate.

Neglecting, as we have all along, the time derivatives of the process mean, we may write the system equation as

$$m_{k+1} = m_k + b \cdot a_o \cdot x_k + r_{k+1}$$

which is observed to be of the same form as (81) except for the presence of the factor $b$. The complete solution is easily found to be

$$m_k = \sum_{i=0}^{k} r_i \left(1+a_o b\right)^{k-i} + a_o b \sum_{i=0}^{k-1} \varepsilon_i \left(1+a_o b\right)^{k-1-i}$$

in which

$$m_k^{(H)} = r_o \left(1+a_o b\right)^k.$$  (105)

From (105), we see that the system will be stable for

$$|1 + a_o \cdot b| < 1$$  (106)

or

$$-\frac{2}{b} < a_o < 0 \quad (b < 0)$$  (107)
and

\[ 0 < a_0 < -\frac{2}{b} \quad (b > 0). \tag{108} \]

The control system is ineffective if \( b = 0 \) as it would be for \( a_0 = 0 \). Expressions (107) and (108) indicate the relationship between \( a_0 \) and \( b \) and allow us to draw some general conclusions regarding control systems depending on discrete measurement and adjustment procedures. First, \( b \) can be negative only if the open-loop transient of the physical system is unstable and contains dominant sinusoidal terms. In addition, the frequency of these terms must be an integral multiple of the sampling frequency and the phase angle such that each control action causes a net change in the process mean at the time of the next observation that is opposite from the direction of the control action. In such cases we must actually initiate control action in a direction away from the desired level. Should the open-loop transient be unable to assume negative values or if the sampling rate precludes measurement when the transient is negative, \( b \) will always be positive and (107) will apply. For \( b \) less than one, which would always occur for critically damped or over-damped systems (and may occur for underdamped or even non-stable systems), the allowable values of \( a_0 \) are extended below \(-2\). For \( b > 1 \), \( a_0 \) is more closely restricted and must be between zero and \(-1\) for \( b = 2 \). Even though \( m_x^H \) can be made to converge, there is no assurance of satisfactory control if sampling periods are at all lengthy unless the open-loop physical process is a stable one. Violent fluctuations between observations could occur which would assume, at the instants observations are made, the proper values to make (105) converge. Very little good product would be produced and severe equipment damage could result. In actual practice,
however, most industrial processes, especially those involving humans, are inherently stable. It is presumed and could probably be experimentally determined that many such systems are approximately critically damped. This would restrict $b$ to a positive range which would probably not exceed 1.4. In summary, to control systems which are inherently unstable, either continuous control or periodic control with very short sampling periods is required\textsuperscript{12}.

Other variations of proportional control, such as time delays in effecting control action, in general lead to higher-order difference equations which can be treated as special cases of the material to follow.

**Second-Order Equations - General**

The second-order linear decision rule forms an excellent basis to illustrate the difference-equation approach to discrete-random-feedback-control-system evaluation. The roots of the homogeneous equation are readily obtainable from the quadratic formula, but sufficient complication is encountered in determining constants, computing variance, etc., that a bit of realism is added that is not present in the first-order case. On the practical side, many commonly used decision rules such as proportional control with a one-period time delay, derivative control, and derivative plus proportional control, result in second-order equations.

The general second-order decision rule can be expressed by

$$m_{k+1} = m_k + a_0 x_k + a_1 x_{k-1} + r_{k+1}$$

$$= (1+a_0) m_k + a_1 m_{k-1} + a_0 e_k + a_1 e_{k-1} r_{k+1}$$

(109)

\textsuperscript{12} For a discussion of the precise relationship between sampling period and system characteristics see Truxal, op. cit., Chapter 9 (Ref. 47).
Rearranging gives us the desired difference equation

\[ m_{k+1} - (1+a_0) m_k - a_1 m_{k-1} = a_0 \epsilon_{k} + a_1 \epsilon_{k-1} + r_{k+1} \]  \hspace{1cm} (110)

or

\[ m_{k+2} - (1+a_0) m_{k+1} - a_1 m_k = a_0 \epsilon_{k+1} + a_1 \epsilon_k + r_{k+2} \]  \hspace{1cm} (111)

Considering first the homogeneous equation

\[ m_{k+2} - (1+a_0) m_{k+1} - a_1 m_k = 0 \]  \hspace{1cm} (112)

we assume a solution of the usual form

\[ m_k = y^k \]  \hspace{1cm} (113)

which gives

\[ y^{k+2} - (1+a_0) y^{k+1} - a_1 y^k = \left[ y^2 - (1+a_0) y - a_1 \right] y^k = 0. \]  \hspace{1cm} (114)

The homogeneous solution is, therefore, a linear combination of the roots of

\[ y^2 - (1+a_0) y - a_1 = 0, \]  \hspace{1cm} (115)

which are found by use of the quadratic formula to be

\[ y = \frac{(1+a_0) \pm \sqrt{(1+a_0)^2 + 4a_1}}{2}. \]  \hspace{1cm} (116)

Depending on the values of \( a_0 \) and \( a_1 \), these roots may be

(1) real and unequal,

(2) real and equal, or

(3) a conjugate complex pair.

It is not necessary to differentiate among these types at this point, although this will have to be done to describe and evaluate the system specifically. For the present we can simply express the homogeneous solution as

\[ m_k^{(H)} = c_1 w_{1k} + c_2 w_{2k} \]  \hspace{1cm} (117)
where the w's are appropriate functions of k and the roots of (116).

To find a particular solution of (111) we proceed as before by assuming a solution of the form

\[ m_k^{(P)} = c_k^{(1)} w_{1k} + c_k^{(2)} w_{2k}. \] (118)

Therefore

\[ m_{k+1} = c_k^{(1)} w_{1(k+1)} + c_k^{(2)} w_{2(k+1)} + \left[ w_{1(k+1)} \cdot \Delta c_k^{(1)} + w_{2(k+1)} \Delta c_k^{(2)} \right]. \] (119)

We now require the bracketed expression to vanish so that

\[ m_{k+1} = c_k^{(1)} w_{1(k+1)} + c_k^{(2)} w_{2(k+1)} \] (120)

and

\[ m_{k+2} = c_k^{(1)} w_{1(k+2)} + c_k^{(2)} w_{2(k+2)} + \left[ w_{1(k+2)} \cdot \Delta c_k^{(1)} + w_{2(k+2)} \Delta c_k^{(2)} \right]. \] (121)

Substitution of (121), (120), and (118) yields,

\[ c_k^{(1)} \left[ w_{1(k+2)} - (1+a_o) w_{1(k+1)} - a_1 w_{1k} \right] \]

\[ + c_k^{(2)} \left[ w_{2(k+2)} - (1+a_o) w_{2(k+1)} - a_1 w_{1k} \right] \]

\[ + w_{1(k+2)} \Delta c_k^{(1)} + w_{2(k+2)} \Delta c_k^{(2)} = a_o \epsilon_{k+1} + a_1 \epsilon_k + r_{k+2}. \] (122)

The two expressions in brackets satisfy the homogeneous equation and so equal zero. Equation (122) becomes

\[ w_{1(k+2)} \cdot \Delta c_k^{(1)} + w_{2(k+2)} \cdot \Delta c_k^{(2)} = a_o \epsilon_{k+1} + a_1 \epsilon_k + r_{k+2}. \] (123)

which along with the condition

\[ w_{1(k+1)} \cdot \Delta c_k^{(1)} + w_{2(k+1)} \cdot \Delta c_k^{(2)} = 0 \] (124)
permits solution for the $\Delta C_k$ in terms of known quantities. The resulting expressions are

$$\Delta C_k^{(1)} = \frac{-w_2(k+1)\left[a_0 e_i + a_1 e_{i-1} + r_{i+1}\right]}{w_1(k+1) \cdot w_2(k+2) \cdot w_1(k+2) \cdot w_2(k+1)} \quad (125)$$

and

$$\Delta C_k^{(2)} = \frac{w_1(k+1)\left[a_0 e_i + a_1 e_{i-1} + r_{i+1}\right]}{w_1(k+1) \cdot w_2(k+2) \cdot w_1(k+2) \cdot w_2(k+1)} \quad (126)$$

which gives

$$C_k^{(1)} = -\sum_{i=1}^{k} \frac{w_2[i\left(a_0 e_i + a_1 e_{i-1} + r_{i+1}\right)]}{w_1 \cdot w_2(i+1) \cdot w_1(i+1) \cdot w_2(i)} \quad (127)$$

and

$$C_k^{(2)} = \sum_{i=1}^{k} \frac{w_1[i\left(a_0 e_i + a_1 e_{i-1} + r_{i+1}\right)]}{w_1 \cdot w_2(i+1) \cdot w_1(i+1) \cdot w_2(i)} \quad (128)$$

We will now evaluate (127) and (128) for each of the three types of roots for (115).

Second-Order Equation - Real, Unequal Roots (Type 1)

When the roots of (115) are real and unequal, they can be expressed simply as $y_1$ and $y_2$, where the $y$'s are functions only of $a_0$ and $a_1$. Then

$$m_k(H) = c_1 y_1^k + c_2 y_2^k \quad (129)$$

and

$$w_{ik} = y_i \quad (130)$$

The expressions for the $C_k^{(i)}$ then become

$$C_k^{(1)} = -\sum_{i=1}^{k} \frac{y_i^j \left[a_0 e_i + a_1 e_{i-1} + r_{i+1}\right]}{y_1 \cdot y_1 \cdot y_2 \cdot y_1 \cdot y_1 \cdot y_2} \quad (131)$$
and, similarly,
\[ c_k^{(2)} = \frac{1}{y_2 - y_1} \sum_{i=1}^{k} \frac{a_0 e_i + a_1 e_{i-1} + r_{i+1}}{y_i^2 (y_2 - y_1)} . \] (132)

Substituting (130), (131), and (132) into (118) and rearranging we get
\[ m_k^{(p)} = \frac{1}{y_2 - y_1} \sum_{i=1}^{k} \left[ (y_2^{k-1} - y_1^{k-1})(a_0 e_i + a_1 e_{i-1} + r_{i+1}) \right] . \] (133)

It is now necessary to evaluate \( c_1 \) and \( c_2 \) in (117) from derived
expressions for \( m_0 \) and \( m_1 \). These expressions for \( m_0 \) and \( m_1 \) are given in
(73) and (74), respectively. Using (73) and (74) we obtain
\[ m_0 = c_1 y_1^0 + c_2 y_2^0 = r_0 \] (134)
and
\[ m_1 = c_1 y_1 + c_2 y_2 = r_0(1+a_0) + r_1 + e_o a_o , \] (135)
which solved simultaneously for \( c_1 \) and \( c_2 \) yield
\[ c_1 = - \frac{r_0(1+a_0) + r_1 + e_o a_o - r_0 y_2}{y_2 - y_1} \] (136)
and
\[ c_2 = \frac{r_0 (1+a_0) + r_1 + e_o a_o - r_0 y_1}{y_2 - y_1} . \] (137)

These expressions may be simplified by noting from (116) that
\[ (1+a_o) = y_1 + y_2 . \] (138)
The coefficients can now be written
\[ c_1 = - \frac{r_0 y_1 + r_1 + e_o a_o}{y_2 - y_1} \] (139)
and
\[ c_2 = \frac{r_0 y_2 + r_1 + e_o a_o}{y_2 - y_1} , \]
which gives the following expression for \( m_k^{(H)} \) when substituted in (129):
\[ m_k^{(H)} = \frac{1}{y_2-y_1} \left[ r_o (y_2^{k+1} - y_1^{k+1}) + (e_0 a_o + r_1) (y_2^k - y_1^k) \right]. \quad (140) \]

The complete solution may be written as the sum of (140) and (133). Thus, with a change of summation index,

\[ m_k = \frac{1}{y_2-y_1} \sum_{i=0}^{k} (y_2^{k+1-i} - y_1^{k+1-i}) (a_o e_{i-1} + a_1 e_{i-2} + r_i) \quad (141) \]

where \( e \)'s with negative subscripts are taken as zero. The upper limit of the summation was left at \( k \) because the term corresponding to \( i = k+1 \) is zero for all \( k \). The expected value of \( m_k \) is easily determined as

\[ \bar{m}_k = \frac{1}{y_2-y_1} \sum_{i=0}^{k} (y_2^{k+1-i} - y_1^{k+1-i}) \left[ (a_o + a_1) E(e) + r_i \right]. \quad (142) \]

To find the variance of \( m_k \) it is necessary first to put (141) into the standard form expressed by (79). Minor manipulation of (141) yields

\[ m_k = \frac{1}{y_2-y_1} \sum_{i=0}^{k} r_i (y_2^{k+1-i} - y_1^{k+1-i}) + \frac{1}{y_2-y_1} \sum_{i=0}^{k-1} e_i \left[ a_o (y_2^{k-i} - y_1^{k-i}) + a_1 (y_2^{k-i} - y_1^{k-i}) \right] (143) \]

from which the variance of the process mean is readily found to be

\[ \sigma_m^2 = \frac{\sigma^2}{(y_2-y_1)^2} \sum_{i=1}^{k} \left[ a_o (y_2^{i-1} - y_1^{i-1}) + a_1 (y_2^{i-1} - y_1^{i-1}) \right]^2. \quad (144) \]

To obtain a closed expression, we first square the term in brackets which gives, after grouping of terms,

\[ \sigma_m^2 = \frac{\sigma^2}{(y_2-y_1)^2} \sum_{i=1}^{k} \left[ a_o^2 + 2a_o a_1 y_2^{i-1} + a_1^2 y_2^{-2} \right] \]
The three series in the brackets of (145) may now be summed independently by use of the well-known expression for the sum of a geometric series. When this is done, we obtain

\[
\sigma_{m_k}^2 = \frac{\sigma^2}{(y_2 - y_1)^2} \left[ a_o^2 y_2^2 + 2a_o a_1 y_2 + a_1^2 \right] \left[ \frac{1 - y_2^k}{1 - y_2} \right]
\]

\[
-2 \left[ a_o^2 (y_2 y_1) + a_o a_1 (y_1 y_2) + a_1^2 \right] \left[ \frac{1 - (y_2 y_1)^k}{1 - y_1 y_2} \right]
\]

\[
+ \left[ a_o^2 y_1^2 + 2a_o a_1 y_1 + a_1^2 \right] \left[ \frac{1 - y_1^k}{1 - y_1} \right],
\]

which is useful both for evaluating the variance for any given \( m_k \) and for determining the limiting value of variance as \( k \) is increased without bound.

It is obvious both from the stability criterion, which requires \( m_k(H) \) in equation (129) to vanish as \( k \) approaches infinity, and directly from (146), that for a useful system

\[
|y_i| < 1 \quad (i=1,2).
\]

If (147) is not satisfied the variance of the process mean becomes infinitely large as \( k \) increases causing the probability of good product to approach zero. If (147) is satisfied, however,

\[
\lim_{k \to \infty} \sigma_{m_k}^2 = \frac{\sigma^2}{(y_2 - y_1)^2} \left[ \frac{a_o^2 y_2^2 + 2a_o a_1 y_2 + a_1^2}{1 - y_2} \right]
\]

\[
-2 \frac{a_o^2 (y_2 y_1) + a_o a_1 (y_1 y_2) + a_1^2}{1 - y_1 y_2}
\]

\[
+ \frac{a_o^2 y_1^2 + 2a_o a_1 y_1 + a_1^2}{1 - y_1^2},
\]

(148)
which can be further simplified if such is deemed desirable in any specific case. Substitution for the roots, \( y_1 \), of their values in terms of \( a_0 \) and \( a_1 \) makes (148) a function of decision-rule constants only, providing means of determining optimum or acceptable values of these constants. In establishing these values, (147) must not be violated and the following relationship must be observed:

\[
a_1 > -\frac{(1+a_0)^2}{4},
\]

(149)

for the roots of the determinantal equation to be real and unequal.

**Second-Order Equation- Equal Roots (Type 2)**

When

\[
a_1 = -\frac{(1+a_0)^2}{4}
\]

(150)

the radical in (116) vanishes, giving

\[
y_1 = y_2 = y = \frac{1+a_0}{2}.
\]

(151)

The homogeneous solution in this case becomes

\[
m_k(H) = c_1 y^k + c_2 k y^k
\]

(152)

and

\[
w_{1k} = y^k
\]

(153)

and

\[
w_{2k} = k y^k.
\]

(154)

The resulting expressions for the \( c_k^{(1)} \) are, therefore,

\[
c_k^{(1)} = -\sum_{i=1}^{k} \frac{i y^i (a_0 e_i + a_1 e_{i-1} + r_{i+1})}{y^i (i+1) y^{i+1} - y^{i+1} x_i y^i
\]

\[
= -\sum_{i=1}^{k} i y^{i-1} (a_0 e_i + a_1 e_{i-1} + r_{i+1})
\]

(155)
and
\[ d_k^{(c)} = \sum_{i=1}^{k} y^{-i-1} (s_0 \epsilon_i + a_1 \epsilon_{i-1} + r_{i+1}) \]  
(156)

The expression for the particular solution is thus
\[ m_k^{(P)} = \sum_{i=1}^{k} \left[ (k y^{k-1} + y^{k-1}) (s_0 \epsilon_i + a_1 \epsilon_{i-1} + r_{i+1}) \right] \]  
(157)

The coefficients in the homogeneous solution are determined in straight­forward fashion from
\[ m_0 = c_1 y^0 + c_2 \cdot 0 \cdot y^0 = r_0 \]  
(158)

and
\[ m_1 = c_1 y + c_2 \cdot 1 \cdot y = r_0 (l+a_0) + r_1 + a_0 \]  
(159)

and are found to be
\[ c_1 = r_0 \]

and
\[ c_2 = \frac{r_0 (l+a_0) + r_1 + a_0 - r_0 y}{y} = \frac{r_0 y + r_1 + a_0 y}{y} \]  
(160)

using in (160) the relationship given by (151). The homogeneous solution
is, therefore,
\[ m_k^{(H)} = r_0 y^k + r_0 k y^k + k(r_1 + a_0) y^{k-1} \]  
(161)

Adding (161) to (157) yields the following complete solution
\[ m_k = \sum_{i=0}^{k} y^{k+1} (k-1)(a_0 \epsilon_{i-1} + a_1 \epsilon_{i-2} + r_i) \]  
(162)

where \( \epsilon \) terms with negative subscripts are taken as zero. The expression
for the expected value of \( m_k \) is readily found to be
\[ \bar{m}_k = \sum_{i=0}^{k} y^{k-i} (k-i+1) \left[ a_0 E(\epsilon_{i-1}) + a_1 E(\epsilon_{i-2}) + r_i \right] \]  \hspace{1cm} (163)

where the expected values of \( \epsilon \) terms with negative subscripts are zero and the others some fixed value \( E(\epsilon) \). In order to determine the variance of \( m_k \) we first put (162) in the standard form

\[ m_k = \sum_{i=0}^{k} r_i y^{k-i} (k-i+1) + \sum_{i=0}^{k-1} \epsilon_i y^{k-i-1} \left[ a_0 (k-i) + a_1 (k-1-i) y^{-1} \right], \hspace{1cm} (164) \]

The variance is thus readily determined to be

\[ \sigma^2 m_k = \sigma^2 \sum_{i=0}^{k-1} y^{2i} \left[ a_0 (i+1) + a_1 y^{-1} i \right]^2. \hspace{1cm} (165) \]

To get (165) in closed form, we first square the term in brackets, which yields, after a grouping of terms

\[ \sigma^2 m_k = \sigma^2 \sum_{i=0}^{k-1} y^{2i} \left[ i^2 (a_0^2 + 2a_0 a_1 y^{-1} + a_1 y^{-2}) + 2i (a_0 a_1 y^{-1}) + (a_1^2) \right]. \hspace{1cm} (166) \]

The series in the brackets of (166) can be summed independently using standard methods of the calculus of finite differences. Neglecting coefficients, these sums are of the following forms:

\[ \sum_{i=0}^{k-1} y^{2i} = \frac{y^{2k-1}}{y^2-1}, \hspace{1cm} (167) \]

\[ \sum_{i=0}^{k-1} i y^{2i} = \frac{1}{(y^2-1)^2} \left[ (y^2-1)^k y^{2k} - y^{2(k+1)} + y^2 \right], \hspace{1cm} (168) \]

and

\[ \sum_{i=0}^{k-1} i^2 y^{2i} = \frac{1}{(y^2-1)^3} \left[ (y^2-1)^2 y^{2k} - 2(y^2-1)^k y^{2(k+1)} + y^{2(k+2)} - y^4 \right. \]

\[ \left. + y^{2(k+1)} - y^2 \right]. \hspace{1cm} (169) \]
Because of the complexity of these expressions, we will not substitute them directly into (166). Should the variance of a given $m_k$ be desired, it is recommended that (167), (168), and (169) be solved separately for the given values of $k$ and $\alpha_0$ and the results inserted in (166). In general, however, we will be more interested in the limiting conditions as $k$ is increased indefinitely and in so doing must insist on a stable system. From (152), the expression for $m_k^{(H)}$, the restriction on $y$ for stable system operation is

$$|y| < 1.$$  \hspace{1cm} (170)

We should perhaps be reminded at this point that

$$\lim_{k \to \infty} k^n y^k = 0$$  \hspace{1cm} (171)

where $n$ is any fixed positive number and if (170) is satisfied. This relationship is basic to all of our variance studies as is now apparent.

From (170) and (171), we see that, for a stable system, 

$$\lim_{k \to \infty} \sum_{i=0}^{k-1} y^{2i} = \frac{1}{1-y^2},$$  \hspace{1cm} (172)

and

$$\lim_{k \to \infty} \sum_{i=0}^{k-1} i y^{2i} = \frac{y^2}{(1-y^2)^2},$$  \hspace{1cm} (173)

and

$$\lim_{k \to \infty} \sum_{i=0}^{k-1} i^2 y^{2i} = \frac{y^2 + y^4}{(1-y^2)^3}.$$  \hspace{1cm} (174)

These limits may be substituted into (166) and the limit of $\sigma_{m_k}^2$ found as $k$ increases without bound. Because of the fixed relationship between $\alpha_0$ and $\alpha_1$ for the class of solutions we are now considering, we can furthermore express the variance limit uniquely in terms of $\alpha_0$. We could have, naturally, done so for any of the expressions derived thus far, but found the proce-
dure used more convenient. The following relationships are summarized at this time to facilitate restatement of any expression which one may desire to put in terms of \( a_0 \) alone:

\[
a_1 = -\frac{(1+a_0)^2}{4},
\]

\[
y = \frac{1+a_0}{2}
\]

and

\[
1 - y^2 = \frac{(1-a_0)(3+a_0)}{4}.
\]

Substitution of (172) through (177) in (166) results in the simple expression

\[
\lim_{k \to \infty} \sigma^2 m_k = \frac{a_0^4 + 8a_0^3 + 18a_0^2 + 5}{(1-a_0)(3+a_0)^3}.
\]

The expected performance of the type-2 second-order system is summarized in Table 2 for representative values of \( a_0 \). It will be noticed that for values of \( a_0 \) near the stability limits there is a momentary increase in the deviation between the expected mean and bogie following any shift \( r \). This is due to the fact that only the \( a_0 \) term of the decision rule reflects this shift in the first adjustment. Convergence starts after a few adjustments, however. The variance limit is observed to decrease rapidly from infinity as \( a_0 \) decreases from 1.00, reach a minimum of 0.185 at \( a_0 \) equal to zero, and increase again without bound as \( a_0 \) approaches -3.00. The limit is less than one for \( a_0 \)'s from a little less than 0.70 to -1.00.
### Table 2
Characteristics of Type-2 Second-Order Control Systems

<table>
<thead>
<tr>
<th>$a_0$</th>
<th>$a_1$</th>
<th>$\lim_{k \to \infty} \frac{\sigma_m^2}{\sigma^2}$</th>
<th>Expected Fraction of $\sigma$ Remaining after</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>1 adj.</td>
</tr>
<tr>
<td>$\geq 1.0 \leq -1.00$</td>
<td>$\infty$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.5625</td>
<td>0.493</td>
<td>1.5</td>
</tr>
<tr>
<td>0</td>
<td>-0.25</td>
<td>0.185</td>
<td>1.0</td>
</tr>
<tr>
<td>-0.5</td>
<td>-0.0625</td>
<td>0.365</td>
<td>0.5</td>
</tr>
<tr>
<td>-1.0</td>
<td>0</td>
<td>1.000</td>
<td>0</td>
</tr>
<tr>
<td>-1.5</td>
<td>-0.0625</td>
<td>2.793</td>
<td>-0.5</td>
</tr>
<tr>
<td>-2.0</td>
<td>-0.25</td>
<td>9.667</td>
<td>-1.0</td>
</tr>
<tr>
<td>-2.5</td>
<td>-0.5625</td>
<td>72.143</td>
<td>-1.5</td>
</tr>
<tr>
<td>$\leq -3.0 \leq -1.00$</td>
<td>$\infty$</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

*Assuming $E(\varepsilon) = 0$.

In comparison with the simple proportional controller summarized in Table 1, page 86, the type-2 second-order system has a slightly higher variance limit than the comparable first-order system for $a_0$ greater than -1.00, and slightly lower limit than the first-order system when $a_0$ is less than -1.00. The second-order system, however, exhibits faster corrective action over the entire range of comparison. Therefore, where overadjustments are required in anticipation of future shifts or to overcome non-linearities such as backslash, the type-2 second-order system is in general superior. In other cases, a decision between the two would have to be based on knowledge of the sizes, frequency, and perhaps pattern.
of expected shifts, and the associated economic criteria. To meet requirements of extremely low variance limit at some sacrifice in adjustment speed, the type-2 second-order system with positive $a_0$ might be the best type of controller to use. One would have to be reasonably sure that no large shifts would occur since, as shown in Table 2, the first few adjustments following a shift actually result in increases in the deviation between actual and desired output mean.

**Second-Order Equation - Complex Roots (Type 3)**

When

$$a_1 < -\frac{(1+a_0)^2}{4}$$

the term under the radical in (116) is negative. The roots of (115) are, therefore, of the form

$$y = u + jv$$

where

$$u = \frac{(1+a_0)}{2}$$

and

$$v = \frac{\sqrt{(1+a_0)^2 + 4a_1}}{2}.$$  

The homogeneous solution in this case has the form

$$w_k^{(H)} = |y|^k (c_1 \cos \phi k + c_2 \sin \phi k),$$

where $|y|$ is the absolute magnitude $\sqrt{u^2 + v^2}$. For simplicity of notation, we will use simply $y$ as the magnitude of the roots in the following development. Thus,

$$w_{1k} = y^k \cos \phi k$$

and

$$w_{2k} = y^k \sin \phi k.$$
The resulting expressions for the $C^{(i)}_k$ are, therefore,
\begin{equation}
\phi^{(1)}_k = \sum_{i=1}^k \frac{y^i \sin \phi i \left[ a_0 e_i + a_1 e_{i-1} + r_{i+1} \right]}{y^i \cos \phi i \cdot y^{i+1} \sin \phi (i+1) - y^{i+1} \cos \phi (i+1) \cdot y^i \sin \phi i}
= \sum_{i=1}^k \frac{y^{-i-1} \sin \phi i \left[ a_0 e_{i-1} + a_1 e_{i-1} + r_{i+1} \right]}{\sin \phi i}
\end{equation}

and
\begin{equation}
\phi^{(2)}_k = \sum_{i=1}^k \frac{y^{-i-1} \cos \phi i \left[ a_0 e_{i-1} + a_1 e_{i-1} + r_{i+1} \right]}{\sin \phi}
\end{equation}

The expression for the particular solution may now be written
\begin{equation}
m_k^{(p)} = \frac{1}{\sin \phi} \sum_{i=1}^k y^{k-i-1} \left[ \sin \phi k \cdot \cos \phi i - \cos \phi k \cdot \sin \phi i \right] \left[ a_0 e_i + a_1 e_{i-1} + r_{i+1} \right]
= \frac{1}{\sin \phi} \sum_{i=1}^k y^{k-i-1} \sin \phi (k-i) \left[ a_0 e_i + a_1 e_{i-1} + r_{i+1} \right].
\end{equation}

The coefficients in the homogeneous solution are found, as before, from
\begin{equation}
m_0^{-y^0} (c_1 \cos \phi + c_2 \sin \phi) = r_0
\end{equation}

and
\begin{equation}
m_1 = y (c_1 \cos \phi + c_2 \sin \phi) = r_0 (1+a_0) + r_1 + e \cdot a_0
\end{equation}

The resulting values are
\begin{equation}
c_1 = r_0
\end{equation}

and
\begin{equation}
c_2 = \frac{r_0 y \cos \phi + r_1 + e \cdot a_0}{y \sin \phi}
\end{equation}

having used in (191) the definitional relationship
\begin{equation}
(1+a_0) = 2 y \cos \phi.
\end{equation}
The homogeneous solution is, thus,

\[ m_k^H = y^k r_0 \cos \theta k + y^k r_0 \frac{\cos \theta}{\sin \theta} \sin \theta k + \frac{e^{k-1}}{e^{\sin \theta}} (\sin \theta k)(r_1 + e_0 a_0) \]

\[ = y^k r_0 \left[ \sin \theta \cos \theta k + \cos \theta \sin \theta k \right] \left[ \frac{e^{k-1}}{e^{\sin \theta}} (\sin \theta k)(r_1 + e_0 a_0) \right] \]

\[ = y^k r_0 \sin \theta (k+1) + \frac{e^{k-1}}{e^{\sin \theta}} (\sin \theta k)(r_1 + e_0 a_0) \cdots \] (193)

Adding (193) to (188) gives for the complete solution

\[ m_k = \frac{1}{\sin \theta} \sum_{i=0}^{k} y^{k-i} \sin \theta (k-i+1) \left[ a_0 e^{i-1} + a_1 e^{i-2} + r_1 \right] \]

(194)

in which \( \epsilon \) terms with negative subscripts are considered as zero. The expected value of \( m_k \) is

\[ \overline{m_k} = \sum_{i=0}^{k} y^{k-i} \sin \theta (k-i+1) \left[ a_0 E(\epsilon_{i-1}) + a_1 E(\epsilon_{i-2}) + r_1 \right] \]

(195)

where again the expected values of \( \epsilon \) terms with negative subscripts are equal to zero and the rest to a constant \( E(\epsilon) \). The standard form of the expression for \( m_k \) is

\[ m_k = \frac{1}{\sin \theta} \sum_{i=0}^{k} r_i y^{k-i} \sin \theta (k+1-i) + \frac{1}{\sin \theta} \sum_{i=0}^{k-1} \epsilon_1 y^{k-1-i} \sin \theta (k+1-i) + \sum_{i=0} a_0 \sin \theta (k-i) + a_1 y^{-1} \sin \theta (k-i) \]

(196)

from which

\[ \sigma_m^2_k = \frac{\sigma^2}{\sin^2 \theta} \sum_{i=0}^{k-1} y^{2i} \left[ a_0 \sin \theta (i+1) + a_1 y^{-1} \sin \theta i \right]^2 \]

(197)
Following the usual procedure we square the term in the brackets, which gives, after use of trigonometric identities,

\[ \sigma^2_{m_k} = \frac{\sigma^2}{\sin^2 \phi} \sum_{i=0}^{k-1} \sin^2 \left( \frac{a_0^2}{2} + a_0 a_1 y^{-1} \cos \phi + \frac{a_1^2 y^2}{2} \right) \]

\[ - \left[ \frac{a_0^2 \cos 2 \phi}{2} + a_0 a_1 y^{-1} \cos \phi + \frac{a_1^2 y^2}{2} \right] \cos 2 \phi \]

\[ + \left[ \frac{a_0^2 \sin 2 \phi}{2} + a_0 a_1 y^{-1} \sin \phi \right] \sin 2 \phi. \]  

(198)

The term in brackets in (198) consists of a constant term and terms in cos 2 \( \phi \) and sin 2 \( \phi \). The sinusoidal terms are periodic in \( i \) and, therefore, continue to oscillate between boundaries determined by their amplitude coefficients as \( i \), and therefore \( k \), is increased. The average value of this oscillation will be zero, however, so we could speak of an average limit to the variance of the process mean. We must realize in so doing that for any particular large value of \( k \) that the actual variance may actually lie anywhere in a symmetrical band around the average limit.

We first notice from (183) that, for a type-3 second-order system to be stable, the magnitude of the two complex roots must satisfy the condition

\[ |y| < 1. \]  

(199)

Therefore, for a stable type-3 system, the average limit of variance of the process mean is

\[ \text{Ave.} \lim_{k \to \infty} \sigma^2_{m_k} = \frac{\sigma^2}{\sin^2 \phi} \left[ \frac{a_0^2}{2} + a_0 a_1 y^{-1} \cos \phi + \frac{a_1^2 y^2}{2} \right]. \]  

(200)

Minor simplification can be obtained by substituting for \( y \), \( \cos \phi \), and \( \sin 2 \phi \) the equivalent expressions in terms of \( a_0 \) and \( a_1 \). This is not attempted here.
Stability of General Second-Order System

We will now discuss the relationships that must hold between $a_0$ and $a_1$ for stable operation of the second-order system. Our basic root form as expressed by the quadratic formula has been found to be

$$y = \frac{1 + a_0}{2} \pm \sqrt{\frac{(1 + a_0)^2 + 4a_1}{2}}$$

and the criterion of stability found for all three types of second-order system was that the absolute magnitude of both $y_1$ and $y_2$ be less than unity.

For the type-1 system the magnitudes of the $y_i$ are obtained by simple addition of the two terms in (201). Therefore, if we let

$$d = 1 + a_0$$

and

$$g = \sqrt{(1+a_0)^2 + 4a_1},$$

our stability criterion requires the following inequalities to be satisfied:

$$-2 < d \pm g < 2.$$  \hspace{1cm} (204)

To determine the desired relationships, we must give separate consideration to each of four categories of type-1 system.

First we take $a_1$ and $d$ both positive. The symbol $g$, as we have defined it, is always positive, so that the restricting inequality is

$$d + g < 2.$$  \hspace{1cm} (205)

With $a_1$ positive, $g$ is clearly larger than $d$ so that we may immediately limit

$$d < 1$$  \hspace{1cm} (206)
for a stable system in this category. If (206) is satisfied, we may transpose (205) without violating the inequality. Thus

\[ g < 2 - d. \]  \hfill (207)

Squaring, we get

\[ g^2 < 4 - 4d + d^2, \]  \hfill (208)

which, in terms of the decision-rule constants, is

\[ (1+a_0)^2 + 4a_1 < 4 - 4 (1+a_0) + (1+a_0)^2. \]  \hfill (209)

Cancelling common terms and factors and rearranging gives simply

\[ a_1 < -a_0. \]  \hfill (210)

Of course (206) and the requirement that \( d \) be positive limits \( a_0 \) to the range

\[ -1 < a_0 < 0 \]  \hfill (211)

which is compatible with \( a_1 \) being positive. The complete restriction on \( a_1 \) for this category of root is thus

\[ 0 < a_1 < -a_0. \]  \hfill (212)

Let us now consider the category of system in which \( a_1 \) is positive but \( d \) is negative. In this situation, the restricting inequality is

\[ -2 < d - g. \]  \hfill (213)

Again, for \( a_1 \) positive, \( g \) is greater in absolute value than \( d \), which immediately limits \( d \) to

\[ -1 < d < 0, \]  \hfill (214)

and, therefore, \( a_0 \) to

\[ -2 < a_0 < -1. \]  \hfill (215)

Changing the signs of the terms on both sides of (213) also changes the direction of the inequality. This yields the equivalent restriction

\[ g - d < 2, \]  \hfill (216)
which, because of (214), can be transposed to
\[ g < 2 + d. \]  
(217)

Squaring gives
\[ g^2 < 4 + 4d + d^2, \]  
(218)

which is, in terms of \( a_o \) and \( a_1 \),
\[ (1+a_o)^2 + 4a_1 < 4 + 4(1+a_o) + (1+a_o)^2. \]  
(219)

This expression reduces to the restriction on \( a_1 \) of
\[ a_1 < 2 + a_o \]  
(220)

which, from (215), is entirely compatible with the requirement that \( a_1 \) be positive. The complete restriction on \( a_1 \) is, therefore,
\[ 0 < a_1 < 2 + a_o. \]  
(221)

When \( a_1 \) falls in the range
\[ -(1+a_o)^2 < 4a_1 < 0, \]  
(222)
we have essentially the same pair of situations as those just covered.

We must, however, impose an additional restriction in order to satisfy (222). These conditions are easily found to be
\[ a_o > -1 + 2 \sqrt{-a_1} \]  
(223)

and
\[ a_o < -1 - 2 \sqrt{-a_1}. \]  
(224)

For \( d \) positive, (223) must apply; for \( d \) negative, (224) applies. Otherwise, the restrictions imposed are the same as those for \( a_1 \) positive. Thus when \( d \) is positive
\[ -1 + 2 \sqrt{-a_1} < a_o < -a_1, \]  
(225)

the right hand condition being obtained by changing the signs and the direction of the inequality in (210). The fairly obvious restriction that, for \( d \) positive,
\[ d < 2 \]  
(226)
and, for $d$ negative,

\[-2 < d. \tag{227}\]

Therefore,

\[a_0 < 1, \tag{228}\]

which limits $a_1$ to the range

\[-1 < a_1 < 0. \tag{229}\]

The type-2 second-order system has already been considered in detail. Briefly, the conditions for the stability of this type of system can be expressed by replacing the inequalities in (224) and (225) by equalities. The stable range was found to be

\[-3 < a_o < 1 \tag{230}\]

and, subject to the functional relationship between $a_o$ and $a_1$,

\[-1 < a_1 < 0. \tag{231}\]

In the case of the type-3 system, we will use $d$ as before but will redefine $g$ as

\[g = \sqrt{-4a_1 - (1+a_o)^2}. \tag{232}\]

Under the conditions for the type-3 system, namely that

\[4a_1 < (1+a_o)^2, \tag{233}\]

g will always be positive real and

\[\begin{bmatrix} y \end{bmatrix} = \frac{1}{2} \sqrt{d^2 + g^2} = \sqrt{-a_1}. \tag{234}\]

Therefore, to satisfy the stability criterion,

\[\begin{bmatrix} y \end{bmatrix} = \sqrt{-a_1} < 1 \tag{235}\]

or

\[-1 < a_1 < 0. \tag{236}\]

From (233) we get our final defining relationship

\[-1 - 2\sqrt{-a_1} < a_o < -1 + 2\sqrt{-a_1}. \tag{236}\]
Two special cases require mentioning at this point. The first is the situation in which

\[ a_1 = 0, \]  

which of course reduces the system to the first-order simple proportional controller. This type of system has already been covered. The other special case is that in which

\[ a_0 = 0. \]  

This is the form of system equation resulting when the effecting of corrective action decided upon in a proportional control situation is delayed by one observation period. Such a system obviously cannot be stable for \( a_1 \) positive. For \( a_1 \) negative, the case of zero \( a_0 \) is covered by the preceding discussion.

A summary of the stability relationships developed in this section is presented in Table 3. Also listed are brief descriptions of the system behavior during convergence for various combinations of decision-rule constants. The behavior of any system, stable or unstable, is of course determined by the exact difference equation, but the categories here delineated are felt to constitute useful general design information. Should an occasion arise wherein similar categorization might be desired for both stable and unstable systems, i.e., descriptions of kinds of instability as well as kinds of convergence, such categorization is readily obtained by a simple extension of the methods of this section.

Mention should perhaps be made at this point of the relationship between our general second-order decision rule and the common derivative plus proportional control. The latter would be characterized by the system equation
### TABLE 3

Conditions for Stable Operation of Second-Order Control Systems

<table>
<thead>
<tr>
<th>$a_1$</th>
<th>$a_o$</th>
<th>System Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 &lt; a_1 &lt; 1$</td>
<td>$-2 + a_1 &lt; a_o &lt; -a_1$</td>
<td>Type 1; roots of opposite sign; diminishing saw tooth caused by alternate change of sign of negative root during convergence</td>
</tr>
<tr>
<td>$a_1 = 0$</td>
<td>$-2 &lt; a_o &lt; 0$</td>
<td>First-order equation. See Table 2.</td>
</tr>
<tr>
<td>$-1 &lt; a_1 &lt; 0$</td>
<td>$-1 + 2 \sqrt{-a_1} &lt; a_o &lt; -a_1$</td>
<td>Type 1; both roots positive; smooth exponential convergence</td>
</tr>
<tr>
<td></td>
<td>$a_o = -1 + 2 \sqrt{-a_1}$</td>
<td>Type 2; positive double root; smooth convergence</td>
</tr>
<tr>
<td></td>
<td>$-1 - 2 \sqrt{-a_1} &lt; a_o &lt; -1 + 2 \sqrt{-a_1}$</td>
<td>Type 3; exponentially decaying sinusoid</td>
</tr>
<tr>
<td></td>
<td>$a_o = -1 - 2 \sqrt{-a_1}$</td>
<td>Type 2; negative double root; diminishing saw tooth during convergence</td>
</tr>
<tr>
<td></td>
<td>$-2 + a_1 &lt; a_o &lt; -1 - 2 \sqrt{-a_1}$</td>
<td>Type 1; both roots negative; diminishing saw tooth during convergence</td>
</tr>
</tbody>
</table>

\[ m_{k+1} = m_k + A_o \, x_k + A_1 \, \nabla x_k + r_{k+1} \]  
(239)

where $A_o$ is the proportionality constant, $A_1$ the derivative constant and $\nabla$ a symbol to represent the first backward difference of $x_k$. Expanding the difference expression yields
\[ m_{k+1} = m_k + A_0 x_k + A_1 x_k - A_1 x_{k-1} + r_{k+1} \]
\[ = m_k + (A_0 + A_1) x_k - A_1 x_{k-1} + r_{k+1} \]
which is in the form of our usual second-order equation. The following correspondence is obvious:
\[ a_0 = A_0 + A_1 \]
and
\[ a_1 = -A_1. \]

Evaluation of Techniques

It should be emphasized in concluding this chapter that the treatment given the simple systems herein is ideally that used for study of any linear discrete random control system. Once the difference equation representing the decision rule has been formulated and the homogeneous solution found, one can proceed exactly as we have illustrated for the simpler systems here. All categories of roots and, therefore, all possible forms of homogeneous solution can be treated by appropriate combinations of the techniques here presented. Obviously computational difficulties will accrue in any practical problem, especially if optimizing conditions are sought. Nevertheless, the approach outlined should serve a useful purpose in system design and evaluation.
CHAPTER VII

STABILITY OF HIGH-ORDER DIFFERENCE-EQUATION MODELS

Solution Problems - General

It should be apparent from the developments in the last two chapters that direct solution of control system difference-equation models involves some potentially severe complications. In particular, the solution of \( n \) simultaneous equations for the computation of the coefficients of both the particular and the homogeneous solutions can be extremely time consuming if \( n \) is large. Standard matrix-inversion procedures for either desk calculator or automatic computing machines may help to ease this situation\(^{15}\). It may also be possible, for certain classes of forcing functions, to determine \( m_k(p) \) by methods which permit more ready computation of the particular-solution coefficients. A few such classes were mentioned in Chapter V. The method we pursued in our solution procedure, however, is the only common direct method which will handle independent random variations of the type common to industrial systems, thereby permitting evaluation of \( \sigma^2_{mk} \). Another cause of some difficulty in direct-solution procedures with high-order equations is the computation of the actual expressions for the first \( n \) of the \( m_k \). The recursive formula given in Chapter V should be of assistance in this regard. For variance determination when multiple

\(^{15}\) Such procedures may be obtained from almost any of the desk-calculator and computing-machine manufacturers or found in most texts on numerical analysis. An example of such a text is Nielsen, Methods in Numerical Analysis, Reference 53.
roots of high order are present, it becomes necessary to sum series with
terms of the form $i^p y^2$, where $p$ assumes successive integral values up to
one less than the order of the root. Unless tables of such sums are
available a tedious (though straightforward) summation procedure is
required.

A problem associated with the solving of difference equations which
we have not mentioned as yet is the important step of determining the roots
of the determinantal equation for the homogeneous solution. The problems
listed above are all capable of solution by analytical methods if one
wishes to attack them with such. The finding of roots of a polynomial
by analytical procedures, however, is possible only through quartic
equations and involves fairly lengthy procedures even for cubic forms.
Classical treatment of higher-order polynomials usually involves numerical
methods consisting of successive approximations (as in Horner's method)
or iterative techniques (such as the Newton-Raphson method). Our entire
problem could in one sense be circumvented by using a numerical-solution
method for our original difference equation. These methods are restricted
to solving specific numerical problems, however, which would limit their
application to analysis of given systems. Design could be accomplished
only by successive solutions using trial sets of values for process and
decision-rule parameters. The accuracy of some of these methods is also
subject to some question.

General knowledge concerning the roots of polynomials can be gained
without actually finding the roots by such devices as Descartes' rule of
signs and the relationships between the roots and the coefficients of the
polynomial. Although neither of the devices mentioned is of general
usefulness here, they do suggest the notion of general system evaluation without direct solution. Such an approach has already been mentioned as an integral part of control theory, most commonly used therein for stability determinations. The remainder of this chapter is, therefore, devoted to the development of a stability criterion similar to the classical Nyquist criterion described earlier; this criterion can be applied without knowledge of the form of \( m_k^H \). Initial concentration on the determination of stability seems entirely logical in that stability is the first requisite for any system of long-range usefulness. In exceptional circumstances wherein a system is to be in use for a fairly short fixed period of time, an unstable system with relatively long dominant time constant may be acceptable. In such cases, a more detailed evaluation than stability determination is needed. By and large, however, stability is the factor which separates useable from useless systems and ideally should be determinable by as simple means as possible.

Before taking up our proposed method of stability determination, it should be pointed out that the root-coefficient relationship mentioned with regard to polynomials offers an extremely gross indication of system instability. In terms of the general \( n^{th} \)-order determinantal equation

\[
y^n - (1 + a_0) y^{n-1} - \sum_{i=1}^{n-1} a_i y^{n-i-1} = 0
\]

with roots \( y_1, \ldots, y_n \), the following relationships hold:

\[
(1 + a_0) = -\sum_{i=1}^{n} y_i
\]

\[
a_1 = \sum_{i=1}^{n} \sum_{p=1}^{i-1} y_i y_p
\]
\[ a_2 = - \sum_{i=1}^{n} \sum_{p=1}^{i-1} \sum_{q=1}^{p-1} y_i y_p y_q \] 

\[ \vdots \]

\[ a_{n-1} = (-1)^n \prod_{i=1}^{n} y_i \]

As has been stated, stable system operation requires that the absolute magnitude of all roots, \( y_i \), be less than one. Therefore, all cross-products of roots must be less than one in absolute magnitude. This results in the necessary but not sufficient condition for system stability that

\[ |(1+a_0)| < n \]

\[ |a_i| < \binom{n}{i+1} \quad (i=1,2,\ldots,n-1) \]  \( (245) \)

where \( \binom{n}{i+1} \) is the standard symbol designating the number of possible combinations of \( n \) items taken \( (i+1) \) at a time.

**Stability Determination in the L-Domain**

In Chapter V we defined \( L \), the linear difference operator, as

\[ L = y^n - (1+a_0) y^{n-1} - \sum_{i=1}^{n-1} a_i y^{n-i-1}. \]  \( (246) \)

We propose a stability criterion based directly on \( L \) which does not necessitate prior determination of the roots of

\[ L y^k = 0. \]  \( (247) \)

We have already found that, regardless of the form of any root of \( (247) \), the system will be stable if and only if the absolute magnitude of all

---

\(^{14}\) The method developed in this section is an adaption of the well-known Nyquist stability criterion and was suggested by Dr. F. C. Weimer, Department of Electrical Engineering, The Ohio State University. For a detailed proof of the Nyquist criterion see Reference 50.
roots is less than one. This is obvious from equation (52), the expression for the homogeneous solution of the general $n$th-order difference equation. We, therefore, require all roots of (247) to be within the unit circle on the $y$-plane with the real part of $y$ as abscissa and the imaginary part as ordinate. We wish to be able to determine whether all roots are within the unit circle in the $y$-plane by examining the behavior of $L$ in the $L$-plane.

The transformation which maps points from the $y$-plane onto the $L$-plane is of course single-valued. That is, for every value of $y$ we may place in (246), $L$ will assume a corresponding value determined by the exact form of the difference operator. Therefore, if we consider a sequence of successive points along some path in the $y$-plane, we can compute, from (246), a corresponding path in the $L$-plane. The $y$-plane path of interest to us from a stability standpoint is of course the unit circle. Our method of attack will be to determine the $L$-plane contour which corresponds to the $y$-plane unit circle and to determine whether the system is stable from the characteristics of this $L$-locus. The locus may be determined by straight-forward substitution of sufficient values of

$$y = 1 \mathcal{L}$$

in (246) to describe the salient features of the $L$ contour.

To develop the criterion for judging system stability from observation of the $L$-plane locus, consider a first-order difference equation with its root at $y_1$. Equation (246) becomes

$$L (y) = y - y_1$$

or, in polar form,

$$L (y) = \rho_1 e^{j\phi_1},$$

(249)
where
\[ \rho_1 = \sqrt{(\text{Re. } y - \text{Re. } y_1)^2 + (\text{Imy} - \text{Imy}_1)^2} \] (251)

and
\[ \phi_1 = \tan^{-1} \frac{\text{Imy} - \text{Imy}_1}{\text{Re. } y - \text{Re. } y_1} \] (252)

We now let \( y \) describe the unit circle in, say, a counter-clockwise direction. If \( y_1 \) lies within the unit circle, the angle \( \phi_1 \) will be swept through an angle of \( 2\pi \) radians. This means that the L-plane locus will circle the L-plane origin once. The L-plane locus must, of course, return to its starting point when \( y \) returns to its starting point. The L-locus is, therefore, closed. If \( y_1 \) lies anywhere on the unit circle, the L-locus will pass through the L-plane origin at the value of \( \phi_1 \) equal to the phase angle of \( y_1 \). If \( y_1 \) lies outside the unit circle, \( \phi_1 \) will not sweep through \( 2\pi \) radians but will increase (or decrease) an amount depending on the exact position of \( y_1 \) and return to its original value (after perhaps changing slightly in the opposite direction). In this case the contour in the L-plane will not encircle the origin.

Consider now the second-order linear difference operator with the roots of (247) at \( y_1 \) and \( y_2 \). Thus,
\[ L(y) = (y - y_1)(y - y_2), \] (253)

which, in polar form, is
\[ L(y) = \rho_1 \rho_2 e^{j(\phi_1 + \phi_2)}. \] (254)

The quantities \( \rho_1, \rho_2, \phi_1, \) and \( \phi_2 \) are adequately defined by (251) and (252), respectively, using the appropriate subscripts. If both \( y_1 \) and \( y_2 \) lie within the unit circle, the L-locus will encircle the origin twice.
since both $\phi_1$ and $\phi_2$ will be swept through $2\pi$ radians. If one root lies within and one without the unit circle, $L$ will encircle its origin just once. If both $y_1$ and $y_2$ lie outside the unit circle, $L$ will not encircle its origin at all. If one root is within and one root on the unit circle, $L$ will circle the origin once and pass through the origin once as $y$ describes the unit circle. Other possibilities also exist. The above discussion holds equally well for double roots, that is, for $y_1$ and $y_2$ equal, although the combinations of root placement is limited by their equality.

We can easily extend our considerations to any order linear difference operator and, therefore, any number of roots. In every case, the $L$-locus corresponding to the unit circle in the $y$-plane will encircle the $L$-plane origin once for every root of the determinantal equation, (247), which lies within the unit circle in the $y$-plane. It will furthermore pass through the origin once for every root of (247) which lies on the $y$-plane unit circle.

Our stability criterion is thus simply stated:

For an $n^{th}$-order system to be stable, the locus of the linear difference operator corresponding to the unit circle in the plane of the roots of the determinantal equation must encircle the $L$-plane origin $n$ times. It may not pass through the origin.

Therefore, we need only to plot the values of $L$ for sufficient values of $y$ on the unit circle to determine the number of times the $L$-locus encircles the $L$-plane origin. From this plot we can, by simply counting the encirclements, determine whether or not a system is stable without having to find the roots of the determinantal equation. An additional piece of information which may be gained should the $L$-locus pass through the origin is the exact form of all roots lying on the unit circle.
With such knowledge, compensation for the causes of these roots might be made more directly and economically than might otherwise be possible.

It should be obvious at this point that our stability criterion could be greatly simplified by the expedient of defining the form of the homogeneous solution as

\[ m_k = Y^{-k} \tag{255} \]

where, obviously

\[ Y = \frac{1}{Y}. \tag{256} \]

The condition for stability now becomes that all roots of

\[ \triangle = 1 - (1+a_0) Y - \sum_{i=1}^{n-1} a_i Y^{i+1} = 0 \tag{257} \]

lie outside the unit circle in the Y-plane. The resulting stability criterion is simply:

For any linear system to be stable, the locus of the linear difference operator (\(\triangle\)) corresponding to the unit circle in the plane of the roots of the determinantal equation (\(\triangle=0\), i.e., the Y-plane) must not encircle or pass through the \(\triangle\)-plane origin.

Generally, the important features of the \(\triangle\)-plane contour can be determined from fewer actual substitutions than can the corresponding L-plane contour, the differences becoming appreciable as \(n\) increases. A quick check on the \(\triangle\)-plane locus may be made by successively computing \(\triangle [1 \triangle 0]\) and \(\triangle [1 \triangle I]\), both of which are simple combinations of the coefficients of (257) and correspond to points on the real axis of the \(\triangle\)-plane. A necessary but not sufficient condition for stability is that these two points lie on the same side of the \(\triangle\)-plane origin.
The z-Transform

The importance of transform methods and, in particular, the Laplace transform in classical control-system design and evaluation was explained in Chapter IV. A transform approach is also applicable to the study of discrete random feedback systems and the rudiments of such an approach will now be considered.

We are, of necessity, limited to a discrete transform to describe the state of the system at discrete time intervals. The z-transform,

\[ T_t(z) = \sum_{k=0}^{\infty} t_k z^{-k}, \] (258)

where \( t_k \) is any function of the discrete integral valued variable \( k \), is proposed for use here. A useful property of the z-transform is

\[ T_{k+1}(z) = \sum_{k=0}^{\infty} t_{k+1} z^{-k} \]

\[ = z t(z) = z \sum_{k=0}^{\infty} t_k z^{-k} \]

\[ = z^{i-1} T(z) - z^{i-1} \sum_{k=0}^{i-1} t_k z^{-k}, \] (259)

The inspiration and much of the groundwork for this chapter are directly attributable to Dr. L. G. Mitten.
which permits expression of all like terms in the linear difference equation by the same transformation.

The basic procedure to be followed in studying system performance in terms of the mean of its output as a function of \( k \) is begun by transforming the basic system difference equation. The transformed equation is then solved for \( T_m(z) \) and \( m_k \) is found by inverse transformation — which amounts to determination of the coefficient of \( z^{-k} \) in the power series expansion of the expression for \( T_m(z) \). A simple restriction applied to the expression for \( m_k \) removes the necessity for computing initial conditions.

We will illustrate the transform solution procedure by solving difference equations representing the simple proportional controller and the second-order decision rule. Extensions to \( n \)th-order systems will be indicated.

**Simple Proportional Control**

The difference equation for the simple proportional controller given in Chapter VI is

\[
m_{k+1} - (1+a_0) m_k = a_0 e_k + r_{k+1}, \tag{260}
\]

assuming complete decay of transients set up by all adjustments made prior to each observation. For the transform solution method, however, we find it more convenient to write difference equations such that \( k \) is the highest subscript to appear. We, therefore, write (260) as

\[
m_k - (1+a_0) m_{k-1} = a_0 e_{k-1} + r_k, \tag{261}
\]

or, in terms of the shifting operator, \( E \),

\[
m_k - (1+a_0) E^{-1} m_k = a_0 E^{-1} e_k + r_k. \tag{262}
\]
The transform of (262) is

\[
T_m(z) - (1+a_0) \left[ z^{-1} T_m(z) - z^{-1} (0) \right] = a_0 \left[ z^{-1} T_\epsilon(z) - z^{-1}(0) \right] T_r(z),
\]

(263)

the zeros entering in due to the summation restrictions expressed in (259). The vanishing of initial conditions is a result of expressing the difference equation in the form given in (261). This precludes the necessity of evaluating initial conditions at this point in the solution procedure. It will soon become evident that this unpleasant task is, as a result, entirely unnecessary. If we multiply (263) by \( z \) and group terms we get

\[
T_m(z) \left[ z - (1+a_0) \right] = a_0 T_\epsilon(z) + z T_r(z)
\]

(264)

which, when solved for \( T_m(z) \), yields

\[
T_m(z) = \frac{a_0 T_\epsilon(z)}{z-(1+a_0)} + \frac{z T_r(z)}{z-(1+a_0)}.
\]

(265)

The two terms on the right-hand side of (265) represent the performance of the proportional controller under each of two specific types of forcing function, namely the \( \epsilon \)-sequence and the \( r \)-sequence. Were it possible to express these sequences as functions of \( k \) for which the infinite sum of the resulting transform term were known, \( T_\epsilon(z) \) and \( T_r(z) \) could be expressed in closed form. As such they would be independent of \( k \) and would represent the steady-state behavior of the system under the influence of each of the respective forcing functions. This is the method usually followed in classical studies of continuous systems, except that there one transforms from the time domain into the complex frequency or s-domain. There are, of course, relatively few forms of forcing function which will transform
in closed form, which has been a definite limitation to analytical studies of continuous systems. In our work, however, we neither need nor necessarily want to express transformations in closed form. In fact, our variance evaluation is based on the premise that the elements of the ε-sequence be independent. We are, therefore, not limited to any particular analytical form of forcing function in the accomplishment of our tasks as previously set forth. These tasks, it will be recalled, are the determination of (1) the expected value and variance of the process mean at the time of any given observation, k, (2) the limiting value of process variance, and (3) stability criteria. Because of the fact that we will rarely have an expression in our transformed equation in which k does not appear, we will put aside the classical control-theory name of "steady-state transform" for terms of this kind. We will instead refer to the term containing $T_ε(z)$ as the "random-error transform" and will call the term containing $T_r(z)$ the "shift transform".

The denominator of each of these transforms is simply the sum of the coefficients of $T_m(z)$ in the transformed equation as expressed in (264). In classical control theory, this combination of coefficients is known as the "characteristic function" of the system, a title that we shall use here. From (263) and (264) it may be seen that the characteristic function of the simple proportional controller is of exactly the same form as the linear difference operator found in Chapter VI for the same type of system. If we let $C(z)$ represent the characteristic function of a system, the equation

$$C(z) = 0$$  (266)
is called the "characteristic equation" of the system. This equation is analogous to and of the same form as the determinantal equation used to determine the homogeneous solution for direct solution of linear difference equations. Continuing this analogy, one would say that the criterion of stability to be used in transform methods of solution is that the roots of the system characteristic equation be less than one in absolute magnitude. We shall see presently that this is in fact the case.

The next step in our solution procedure is to express (265) as a power series in $z^{-1}$. To do this, we utilize the fact that

$$\frac{1}{z-c} = \sum_{k=1}^{\infty} c^{k-1} z^{-k}, \quad (267)$$

where $c$ is any constant. This relationship can be verified by simple long division. Equation (265) can now be written

$$T_m(z) = a_0 T_\mathcal{E}(z) \sum_{k=1}^{\infty} (1+a_0)^{k-1} z^{-k} + z T_r(z) \sum_{k=0}^{\infty} (1+a_0)^{k-1} z^{-k}. \quad (268)$$

We may proceed from this point by either of two methods. The direct procedure, included here only for illustrative purposes, is to substitute for $T_\mathcal{E}(z)$ and $T_r(z)$ their respective defining power series given by (258). When this is done, (268) becomes
For $k$ equal to zero, the expression in the brackets of the first term of (269) vanishes. We may, therefore, change the lower limit of the summation over $k$ to zero without affecting final results. If this limit is changed in this way we can write (269) in transform notation as

$$T_m(z) = a_0 \left[ \sum_{k=0}^{\infty} \epsilon_k z^{-k} \right] \left[ \sum_{k=1}^{\infty} (1+a_0)^{k-1} z^{-k} \right]$$

$$+ z \left[ \sum_{k=0}^{\infty} r_k z^{-k} \right] \left[ \sum_{k=1}^{\infty} (1+a_0)^{k-1} z^{-k} \right]$$

$$= a_0 \sum_{k=1}^{\infty} \left[ z^{-k} \sum_{i=0}^{k-1} \epsilon_i (1+a_0)^{k-i-1} \right]$$

$$+ \sum_{k=0}^{\infty} \left[ z^{-k} \sum_{i=0}^{k} r_i (1+a_0)^{k-i} \right].$$  (269)

Taking the inverse transform of (270), we get

$$m_k = a_0 \sum_{i=0}^{k-1} \epsilon_i (1+a_0)^{k-i-1} + \sum_{i=0}^{k} r_i (1+a_0)^{k-i},$$  (271)

which agrees explicitly with (99), the expression found by direct methods in Chapter VI. Since (271) is already in standard form, the mean and variance of $m_k$ for any given $k$, and the limit of the variance as $k$ approaches infinity can be found directly by methods previously discussed. From (267) it is obvious that, to have a stable system, the constant in
the denominator must be less than one in absolute value. From (266),
however, this constant is found to be the root of the system characteristic
equation. Therefore, our supposed stability criterion, that the roots
of the characteristic equation must be less than one in absolute magnitude
for stable system operation, is in fact true.

The procedure followed in deriving equation (271) is completely
general and may be applied under any circumstances for any type of linear
system. In considering more complex systems, however, a slight deviation
from this procedure will simplify both calculation and notation. From
(267), it is apparent that

\[
\frac{t_i z^{-i}}{z-c} = t_i z^{-i} \sum_{k=1}^{\infty} c^{k-1} z^{-k} = t_i \sum_{k=1}^{\infty} c^{k-1-i} z^{-k}, \tag{272}
\]

where \( t_i \) is a constant depending on \( i \). From the definition of the z-
transform,

\[
\frac{T_t(z)}{z-c} = \sum_{i=0}^{\infty} \frac{t_i z^{-i}}{z-c}, \tag{273}
\]

which, with the results of (272), yields

\[
\frac{T_t(z)}{z-c} = \sum_{i=0}^{\infty} t_i \sum_{k=1}^{\infty} c^{k-1-i} z^{-k}. \tag{274}
\]

Noting that

\[
c^{k-1-i} = c^{k-1} \cdot c^{-i}, \tag{275}
\]

we may write (274)

\[
\frac{T_t(z)}{z-c} = \left[ \sum_{k=1}^{\infty} c^{k-1} z^{-k} \right] \left[ \sum_{i=0}^{\infty} t_i c^{-i} \right] = \frac{T_t(z)}{z-c}. \tag{276}
\]
On the basis of (276) we may now restate our basic equation for $T_m(z)$ as follows:

$$T_m(z) = a_o \frac{T_e (1+a_o) + z T_r (1+a_o)}{(z-(1+a_o)) + (1+a_o)z}$$

$$= \left[ a_o T_e (1+a_o) + z T_r (1+a_o) \right] \sum_{k=1}^{\infty} (1+a_o)^{k-1} z^{-k}$$

$$= \sum_{k=0}^{\infty} \left[ a_o (1+a_o)^{k-1} T_e (1+a_o) + (1+a_o)^k T_r (1+a_o) \right].$$

The inverse transform gives

$$m_k = a_o (1+a_o)^{k-1} T_e (1+a_o) + (1+a_o)^k T_r (1+a_o)$$

$$= a_o (1+a_o)^{k-1} \sum_{i=0}^{\infty} \varepsilon_i (1+a_o)^{-i} + (1+a_o)^k \sum_{i=0}^{\infty} r_i (1+a_o)^{-i}$$

$$= a_o \sum_{i=0}^{\infty} \varepsilon_i (1+a_o)^{k-1-i} + \sum_{i=0}^{\infty} r_i (1+a_o)^{k-i}. \quad (278)$$

In order for (278) to represent the actual expression for the process mean, it is necessary to restrict the range of all summations so that negative powers of $(1+a_o)$ are excluded. This results in the obvious restriction that any given $m_k$ can be affected only by shifts, $r_i$, up to the time when a given observation is made and can be affected only by sampling and measuring errors which enter into prior adjustments. This is, in effect, our transform method of handling initial conditions; although, should other types of initial-decision rules than the one we adopted here be used, it would be necessary to start with equation (260) instead of (261) and carry a special initial-condition transform. It is only because we have adopted a kind of "natural initial-decision rule" that the convenience of handling
initial conditions by a blanket restriction on the final result is possible. Actually, even if an initial-condition transform were needed and used, the same restriction would have to be placed on the inverse of the random-error and shift transforms to obtain meaningful results should the latter of our two procedures be used. In other words, the addition of more terms in the transformed equation will not change the results due to those already present. This restriction, however, should cause no difficulty. Applying this restriction to (278) gives

\[ m_k = a_0 \sum_{i=0}^{k-1} \varepsilon_i (1+a_o)^{k-1-i} + \sum_{i=0}^{k} r_i (1+a_o)^{k-i}, \quad (279) \]

which is identical to (271) and, therefore, to (99) in Chapter VI.

In passing, we would like to offer, as a standard term for use in discrete control-system parlance, the expression "natural initial-decision rule", which we define as an initial-decision rule which causes the initial-condition transform in the transformed equation to vanish.

Although it may seem odd to discuss partial-fraction expansions in connection with first-power characteristic equations, it is convenient at this point to show the equivalence of our second method above (which led to (278)) and the effects of applying the partial fraction expansion technique. Let us consider the expression for \( T_m(z) \) given in (265), which we will write in the form

\[ T_m(z) = \left[ a_o \right] \frac{T_e(z)}{z-(1+a_o)} + \left[ z \right] \frac{T_r(z)}{z-(1+a_o)}. \quad (280) \]

This form assures that, regardless of the coefficients of \( T_e(z) \) and \( T_r(z) \), the fractions that we are dealing with are proper (a prerequisite for
application of the partial-fraction expansion technique. Assurance of course comes from the fact that the highest power of \( z \) in either \( T_e(z) \) or \( T_r(z) \) is zero.

The partial fraction expansion calls for the expression of each fraction as a linear combination of fractions having for their denominators the factors of the denominator of the original fraction. Since the characteristic equation now under consideration is of the first power, it has only one factor. Therefore, the first step in the partial fraction expansion of (280) is to write

\[
T_m(z) = a_0 \frac{K_e}{z - (1 + a_0)} + \left[ z \right] \frac{K_r}{z - (1 + a_0)},
\]

(281)

where the K's are constants determined from the general expression

\[
K_t = \lim_{z \to c} \left( z - c \right) \left[ \frac{T_t(z)}{z - c} \right] = \left[ T_t(z) \right]_{z=c} = T_t(c),
\]

(282)

where the term in brackets represents the entire original fraction.

Substitution of the appropriate form of (282) into (281) yields

\[
T_m(z) = a_0 \frac{T_e(1 + a_0)}{z - (1 + a_0)} + \left[ z \right] \frac{T_r(1 + a_0)}{z - (1 + a_0)}
\]

(283)

which is identical to the first stage of equation (277). Thus, the equivalence we set out to prove has been demonstrated and we are at liberty to use the partial fraction expansion on terms involving \( T_e(z) \) and \( T_r(z) \).

We will find this most useful in dealing with higher-order systems.
Second-Order Systems

We will now apply the transform solution method to the second-order system equation. As was stated in Chapter VI, the second-order system offers sufficient complexity to permit a realistic evaluation of the advantages and disadvantages of any solution procedure. It also permits investigation of all possible types of roots that characteristic equations may have. We also wish to provide an adequate basis of comparison with the direct solution procedure discussed in detail in Chapter VI.

We first state the basic second-order equation in the form

\[ m_k - (1+a_o) m_{k-1} - a_1 m_{k-2} = a_0 \epsilon_{k-1} + a_1 \epsilon_{k-2} + r_k. \]  

(284)

In terms of the shifting operator, (284) becomes

\[ m_k - (1+a_o) E^{-1} m_k - a_1 E^{-2} m_k = a_0 E^{-1} \epsilon_k + a_1 E^{-2} \epsilon_k + r_k \]

(285)

Keeping in mind the restrictions on the initial-condition summations in equation (259), we may write the transformed equation directly in the form

\[ T_m(z) \left[ 1 - z^{-1} (1+a_o) - z^{-2} a_1 \right] = T_\epsilon(z) \left[ a_0 z^{-1} + a_1 z^{-2} \right] T_r(z). \]

(286)

Multiplication by \(z^2\) gives

\[ T_m(z) \left[ z^2 - z (1+a_o) - a_1 \right] = T_\epsilon(z) \left[ a_0 z + a_1 \right] + z^2 T_r(z), \]

(287)

which, when solved for \(T_m(z)\), yields

\[ T_m(z) = \left[ a_0 z + a_1 \right] \frac{T_\epsilon(z)}{z^2 - (1+a_o)z - a_1} + \left( z^2 \right) \frac{T_r(z)}{z^2 - (1+a_o)z - a_1}. \]

(288)

It may be noted that the characteristic function is identical in form with the linear difference operator defined for second-order systems in Chapter VI (see equation 114)).
We must now express the random-error and shift transforms as power series in $z^{-1}$. One could attempt this series expansion by actually computing the reciprocal of the characteristic function by long division. This results in a power series of the desired form beginning with $z^{-2}$. Unless a general expression for the $k$th term of this series can be discovered, however, the best that can be obtained from this method is a term-by-term computation of $m_k$ for a finite number of terms. Determination of limiting conditions would be impossible. The difficulties in obtaining a general expression for series expansions of this sort are of the same variety as those attendant upon finding a general expression for $m_k$ directly. Therefore, if such an expression could be found, there would be no need for the procedures being discussed herein. In the light of the low probability of such direct determinations, however, we will perform a partial-fraction expansion of (288), which will permit ultimate computation of the expressions desired.

The partial-fraction expansion of course requires knowledge of the factors of the characteristic function, which in turn depend on the roots of the characteristic equation. We are, therefore, faced with the same problem of finding roots of polynomials that we encountered in our direct-solution methods. For second-order equations, however, the roots are readily obtained from the quadratic formula. Thus, for the characteristic function in (288), the roots of the corresponding characteristic equation are

$$z_1 = \frac{(1+a_o)}{2} + \sqrt{\frac{(1+a_o)^2 + 4a_1}{2}}$$

and

$$z_2 = \frac{(1+a_o)}{2} - \sqrt{\frac{(1+a_o)^2 + 4a_1}{2}}$$

(289)
The characteristic function may now be expressed in factored form as
\[ C(z) = (z-z_1) (z-z_2), \]  
where \( z_1 \) and \( z_2 \) are defined by (289) and (290), respectively.

Let us first consider the case of real, unequal roots, which occurs when
\[ 4 a_1 > - (1+a_0)^2. \]  
For roots of this type, (288) may be written in the form
\[ T_m(z) = \left[ a_0 z + a_1 \right] \left[ \frac{K_1 e}{z-z_1} + \frac{K_2 e}{z-z_2} \right] + \left[ z^2 \right] \left[ \frac{K_1}{z-z_1} + \frac{K_2}{z-z_2} \right]. \]  
We now evaluate the \( K \)'s from (282), keeping in mind that the bracketed term represents the entire original fraction. Substitution for the \( K \)'s in (293) yields
\[ T_m(z) = \left[ a_0 z + a_1 \right] \frac{T_{e_1}(z_1)}{(z-z_1)(z-z_2)} + \frac{T_{e_1}(z_2)}{(z-z_2)(z-z_1)} \]
\[ + \left[ z^2 \right] \frac{T_{r_1}(z_1)}{(z_1-z_2)(z-z_1)} + \frac{T_{r_1}(z_2)}{(z_2-z_1)(z-z_2)} \].  
Each of the four terms in (294) may be readily expressed as a power series in \( z^{-1} \) by use of (267). This yields, after some rearrangement of terms and adjustment of summation limits,
\[ T_m(z) = \frac{1}{z_2-z_1} \sum_{k=0}^{\infty} z^{-k} \left\{ a_0 \left[ T_{e_1}(z_2) z_2^k - T_{e_1}(z_1) z_1^k \right] \right. \]
\[ + a_1 \left[ T_{e_1}(z_2) z_2^{k-1} - T_{e_1}(z_1) z_1^{k-1} \right] \]
\[ + \left[ T_{r_1}(z_2) z_2^{k+1} - T_{r_1}(z_1) z_1^{k+1} \right] \} \]  
\[ + \frac{1}{z_2-z_1} \left\{ -a_1 \left[ T_{e_1}(z_2) z_2^{-1} - T_{e_1}(z_1) z_1^{-1} \right] \right. \]
\[ + \left[ T_{r_1}(z_2) z - T_{r_1}(z_1) z \right] \}. \]
The terms in the second bracket are the result of changes made in the ranges of summation of the terms in the first bracket to get them in common powers of \( z^{-1} \). The second bracket is included in (295) only for illustrative purposes. The \( \epsilon \)-terms vanish in the final solution by our requirement that only non-negative powers of the roots of the characteristic equation are to be considered in the expression for \( m_k \). The \( \tau \)-terms vanish in the final solution because they are the coefficients of \( z^1 \), and therefore, correspond to \( r^{-1} \) which we define as zero. In fact all \( m_1 \), \( r_1 \), and \( \epsilon_1 \) with negative subscripts are to be taken as zero. Therefore, we need no longer consider the second bracket in our solution procedure. Inverse transformation thus yields

\[
m_k = \frac{1}{z_2 - z_1} \left\{ a_0 \left[ T_\epsilon (z_2) z^{-1}_2 - T_\epsilon (z_1) z^{-1}_1 \right] + a_1 \left[ T_\epsilon (z_2) z^{-1}_2 - T_\epsilon (z_1) z^{-1}_1 \right] \right. \\
\left. + \left[ T_\tau (z_2) z^{-1}_2 - T_\tau (z_1) z^{-1}_1 \right] \right\} \\
= \frac{1}{z_2 - z_1} \left\{ a_0 \sum_{i=0}^{\infty} \left[ \epsilon_i (z_2^{k-i} - z_1^{k-i}) \right] + a_1 \sum_{i=0}^{\infty} \left[ \epsilon_i (z_2^{k-i} - z_1^{k-i}) \right] \\
+ \sum_{i=0}^{\infty} \left[ r_i (z_2^{k+1-i} - z_1^{k+1-i}) \right] \right\}. \tag{296}
\]

With our restriction on exponents, (296) may be written

\[
m_k = \frac{1}{z_2 - z_1} \left\{ a_0 \sum_{i=0}^{k} \left[ \epsilon_i (z_2^{k-i} - z_1^{k-i}) \right] + a_1 \sum_{i=0}^{k-1} \left[ \epsilon_i (z_2^{k-i} - z_1^{k-i}) \right] \\
+ \sum_{i=0}^{k+1} \left[ r_i (z_2^{k+1-i} - z_1^{k+1-i}) \right] \right\}. \tag{297}
\]
Actually, the upper limit of all three summations could be lowered by one since, for \( i \) equal to this limit, the summand vanishes. If this were done, \((297)\) would be identical to \((143)\), the expression found by direct solution in Chapter VI.

To determine the variance of \( m_k \), we proceed in the same manner outlined in Chapter VI, i.e., convert to standard form, determine the variance as the sum of the variances of all applicable \( \varepsilon \)-terms, and put the resulting expression in closed form. The criterion for stability is again that both roots of the characteristic equation be less than one in absolute magnitude. This is apparent from the fact that each root becomes the constant in the denominator of one or more terms of the partial-fraction expansion and, as such, is raised to the \((k-1)\)th power in the resulting power series (see \((267)\)).

When

\[
z_1 = z_2 = z_d, \tag{298}
\]

where \( z_d \) stands for "double root", the partial fraction expansion of \((288)\) assumes the form

\[
T_m(z) = \left[ a_0 z^2 + b_1 \right] \left[ \frac{K_1 \varepsilon}{(z-z_d)^2} + \frac{K_2 \varepsilon}{(z-z_d)} \right] + \left[ z^2 \right] \left[ \frac{K_1 r}{(z-z_d)^2} + \frac{K_2 r}{(z-z_d)} \right].
\tag{299}
\]

The \( K \)'s are evaluated as follows:

\[
K_{1t} = \lim_{z \to z_d} \left( z-z_d \right)^2 \frac{T_t(z)}{(z-z_d)^2} = T_t(z_d) \tag{300}
\]

\[
K_{2t} = \lim_{z \to z_d} \frac{d}{dz} \left( z-z_d \right)^2 \left( \frac{T_t(z)}{(z-z_d)^2} \right) = \frac{d}{dz} \left[ T(z) \right]_{z=z_d} = T'(z_d). \tag{301}
\]
Substituting (300) and (301) in (299), we get

\[ T_m(z) = \left[ a_0 z + a_1 \right] \left[ \frac{T_\epsilon(z_d)}{(z-z_d)^2} + \frac{T_\epsilon'(z_d)}{(z-z_d)} \right] + \left[ z^2 \right] \left[ \frac{T_r(z_d)}{(z-z_d)^2} + \frac{T_r'(z_d)}{(z-z_d)} \right]. \]  

(302)

It may easily be shown by long division that

\[ \frac{1}{(z-c)^2} = \sum_{k=2}^{\infty} (k-1) c^{k-2} z^{-k}. \]  

(303)

Using the results of (303) and (267), we can express (302) as a power series in \( z^{-1} \). After a grouping of terms and adjustment of summation limits, the resulting expression is

\[ T_m(z) = \sum_{k=0}^{\infty} z^{-k} \left\{ a_0 \left[ T_\epsilon(z_d) \cdot k \cdot z_d^{k-1} + T_\epsilon'(z_d) \cdot z_d^k \right] 
+ a_1 \left[ T_\epsilon(z_d) \cdot (k-1) \cdot z_d^{k-2} + T_\epsilon'(z_d) \cdot z_d^{k-1} \right] 
+ \left[ T_r(z_d) \cdot (k+1) \cdot z_d^{k} + T_r'(z_d) \cdot z_d^{k+1} \right] \right\}. \]  

(304)

The extra terms resulting from the adjustments of summation ranges have all been found to vanish under our restrictions of non-negativity of subscripts and powers of roots of the characteristic equation so are not included. Inverse transformation thus yields

\[ m_k = a_0 \left[ T_\epsilon(z_d) \cdot k \cdot z_d^{k-1} + T_\epsilon'(z_d) \cdot z_d^k \right] 
+ a_1 \left[ T_\epsilon(z_d) \cdot (k-1) \cdot z_d^{k-2} + T_\epsilon'(z_d) \cdot z_d^{k-1} \right] 
+ \left[ T_r(z_d) \cdot (k+1) \cdot z_d^{k} + T_r'(z_d) \cdot z_d^{k+1} \right]. \]  

(305)
It may easily be shown that

$$T_t'(z_d) = \sum_{i=0}^{\infty} -i + t_i z_d^{-(i+1)}.$$  \hspace{1cm} (306)

Substitution of (258) and (306) into (305) gives

$$m_k = a_0 \left[ k \sum_{i=0}^{\infty} \epsilon_i z_d^{k-i} - \sum_{i=0}^{\infty} \epsilon_i z_d^{k-i} \right] + a_1 \left[ (k-1) \sum_{i=0}^{\infty} \epsilon_i z_d^{k-2-i} - \sum_{i=0}^{\infty} \epsilon_i z_d^{k-2-i} \right] + \left[ (k+1) \sum_{i=0}^{\infty} r_i z_d^{k-1} - \sum_{i=0}^{\infty} ir_i z_d^{k-1} \right].$$ \hspace{1cm} (307)

Applying our non-negativity restriction on the exponents of \( z_d \) and regrouping terms yields

$$m_k = a_0 \left[ \sum_{i=0}^{k-1} \epsilon_i (k-1) z_d^{k-i-1} \right] + a_1 \left[ \sum_{i=0}^{k-2} \epsilon_i (k-1) z_d^{k-2-i} \right] + \left[ \sum_{i=0}^{k} r_i (k+1) z_d^{k-1} \right],$$ \hspace{1cm} (308)

which is of the same form as (164), derived by direct methods in Chapter VI. Procedures from this point on are standard. The stability criterion may also be observed as standard by noting the use of (258) and (306).

In the case of complex, or type-3, roots, the same partial-fraction expansion equations hold as for type-1, (real unequal) roots. We will, therefore, follow the same solution procedure letting

$$z_1 = |z| e^{j\phi}$$ \hspace{1cm} (309)

and

$$z_2 = |z| e^{-j\phi},$$ \hspace{1cm} (310)
where \(|z|\) is the absolute magnitude of the roots and

\[ \phi = \tan^{-1} \frac{\text{Im}(z)}{\text{Re}(z)}. \]  \hspace{1cm} (311)

The relationships between \(|z|\) and \(\phi\) are the same as those derived in Chapter VI. There is actually no need to substitute for \(z_1\) and \(z_2\) until the final expression for \(m_k\) is obtained. We, therefore, substitute (309) and (310) in (297) and obtain

\[
m_k = \frac{1}{(|z|)(e^{-j\phi} - e^{j\phi})} \left\{ a_0 \sum_{i=0}^{k} \left[ \epsilon_i \left(|z|\right)^{k-1} e^{-j(k-1)\phi} \right] + a_1 \sum_{i=0}^{k-1} \left[ \epsilon_i \left(|z|\right)^{k-1} e^{-j(k-1+i)\phi} \right] + \sum_{i=0}^{k+1} \left[ r_i \left(|z|\right)^{k+1} e^{-j(k+1+i)\phi} \right] \right\}. \hspace{1cm} (312)
\]

It may be shown from the relationships given in equation (45) that

\[ e^{-jk\phi} - e^{j\phi} = -2j \sin k \phi. \]  \hspace{1cm} (313)

Substituting (313) in (312), we get, after cancelling the common factor 

\[-2j |z|, \]

\[
m_k = \frac{1}{\sin \phi} \left\{ a_0 \sum_{i=0}^{k} \left[ \epsilon_i \left(|z|\right)^{k-1} \sin (k-i) \phi \right] + a_1 \sum_{i=0}^{k-1} \left[ \epsilon_i \left(|z|\right)^{k-2} \sin (k-1-i) \phi \right] + \sum_{i=0}^{k+1} \left[ r_i \left(|z|\right)^{k} \sin (k+1-i) \phi \right] \right\}. \hspace{1cm} (314)
\]

Again, it is possible to lower the upper summation limit by one without affecting final results. If this is done, (314) is in the same form as (196), the expression derived for \(m_k\) with complex roots in Chapter VI.
Procedures from this point on are standard. Furthermore, the stability criterion that the absolute magnitude of the roots of the characteristic equation must be less than one must hold in this case with the roots defined by (309) and (310).

It might be noted that in Chapter VI the development for the type-3 second-order equation could have been combined with the development for the type-1 equation as was done here.

**General Linear Difference - Equation Models**

In the light of the preceding discussion, extensions to the general \( n \)-th order system is straightforward. The only changes in the equations will be the addition of terms to the random-variation transform, each term having a new decision-rule constant and a power of \( z \) one less than the preceding term. All possible types of roots have already been considered. For triple and higher repeating roots, the same methods given here for double roots apply except that repeated differentiation is required to evaluate the \( K \)'s. The principal difficulty to be encountered will of course be the determination of the roots of the characteristic equation.

In general, however, the procedure to be followed for the \( n \)-th order decision rule is as follows. We first write the difference equation in the form

\[
m_k - (1+a_0) m_{k-1} - \sum_{i=1}^{n-1} a_i m_{k-1-i} = \sum_{i=0}^{n} a_i \xi_{k-1-i} + r_k
\]  

(315)

which, when transformed yields
\[ T_m(z) \left[ z^n - (1+a_0) z^{n-1} - \sum_{i=1}^{n-1} a_i z^{k-1-i} \right] = T_C(z) \sum_{i=0}^{n} a_i z^{k-1-i} + z^n T_F(z). \]  

(316)

If the roots of the characteristic equation can be found, the procedures previously outlined (or some procedure) must be used to obtain the coefficient of the general term, \( z^{-k} \), in the power series \( \sum_{k=0}^{\infty} t_k z^{-k} \). This coefficient is then the desired expression for \( m_k \).

When complete evaluation of a system is impossible because of not being able to find the roots of the characteristic equation or computations prove unfeasible, the methods suggested in Chapter VII may be used for gross determinations of stability and the like. The adoption of the Nyquist stability criterion developed in Chapter VII is of course directly applicable to transform-solution methods because of the coincidence of the direct solution determinantal equation and the characteristic equation of the transform.

Comparison of Procedures

Basically the transform procedure involves the same general steps as direct methods of solving our system equations. In the direct method we had to find a homogeneous solution and a particular solution and then evaluate combinatorial constants from initial conditions. With the transform procedure, we have to find the roots of the characteristic equation, which is identical to determining the homogeneous solution involved in the direct approach. We are saved the task, most of the time at least, of computing initial conditions and evaluating combinatorial constants, which is perhaps the main advantage of the transform method. In turn,
however, one has to compute the constants involved in the partial-fraction expansion. Some manipulation follows to prepare for the inverse transformation.

It is doubtful that a solution to any particular problem that could be reached by one method could not be reached by the other. However, with time it is conceivable that a store of recognizable transforms might be built up which would facilitate transform solution.

We hold a preference for the transform method mostly because it completely precludes having to solve sets of simultaneous equations. It is felt that, as a whole, the calculations involved in the transform procedure are more straight-forward and, of great importance to the field of Industrial Quality Control, more easily routinized than those accompanying direct solution. Of course the event that would be the greatest boon to the entire field of analytical control-system design would be a method for ready determination of the roots of polynomials.
CONCLUSION

It is hoped that the formulation given herein to Industrial Quality Control and the associated decisions and problems encountered in routinized and automatic operations may, in some small way at least, provoke attempts at answers and solutions. Some such attempts will be successful. It is also hoped that the tie-in between probability theory and control theory which we have indicated will help to bridge the gap between these fields and stimulate other research dedicated to the same purpose. In conclusion, we hope that the methods presented herein for the design and analysis of discrete control systems subject to random fluctuations in product, process, or measurement parameters will be of direct use to designers and researchers in control fields of all kinds.


I, Albert Bentley Bishop 3rd, was born in Philadelphia, Pennsylvania, on April 7, 1929. I received my secondary-school education at the William Penn Charter School in Philadelphia and my undergraduate training at Cornell University, from which I received the Bachelor of Electrical Engineering Degree in June, 1951. After serving two years in the United States Air Force, during which time I was accepted by the Graduate School of The Ohio State University, I received the Master of Science Degree in Industrial Engineering in December, 1953. I left a job as a Member of the Technical Staff of the Bell Telephone Laboratories in October, 1954, to become an Instructor in the Department of Industrial Engineering at Ohio State University. I concurrently began work as a Research Associate with the Ohio State University Operations Research Group. I have held both of these positions for three years while completing the requirements for the degree Doctor of Philosophy.