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FREQUENCY SHAPING AND OTHER DYNAMIC COMPENSATION METHODS FOR SLIDING MODE CONTROL

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the
Graduate School of The Ohio State University

By

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* * * * *

The Ohio State University

2002

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ABSTRACT

In this dissertation, we are mainly interested in designing sliding mode controllers for systems with modelling and parametric uncertainties considering unmodelled actuator and sensor dynamics. Furthermore, the drawbacks of sliding mode control such as high gain, high frequency control inputs are attenuated by extending the controlled system through adding compensator dynamics. This design approach is quite reasonable since most of the analytical models of the plants to be controlled have uncertainties, unmodelled dynamics and limited control capacity. Our approach to designing a sliding mode controller is to filter the discontinuous control inputs using frequency-shaping and dynamic compensation methodologies while satisfying robustness against both matched and unmatched bounded disturbances based on Lyapunov stability. Transient performance improvement and finite time convergence to the switching surface have been assured by this methodology. Dynamic compensator schemes offer more degrees of freedom for controller design and they also improve the transient performance of the closed loop system that may be degraded by the high feedback gains due to conservative choice of the disturbance's upperbounds.
Dedicated to my parents Zühal and Metin Acarman and my sister Ayfer...
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T. Acarman and Ü. Ö zgüner “Frequency Shaping Compensation for Backstepping 
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**FIELDS OF STUDY**

Major Field: Electrical Engineering

Studies in:

- Control Systems
- Power Systems
- Mathematics

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CHAPTER 1

INTRODUCTION

1.1 Problem Statement

Control technology, containing a mixture of theory and applications through the process of analytical modelling for systems analysis is one of the challenging academic and engineering research areas. Control technology named also "hidden technology" has experienced growing popularity and many issues of major interest in different application areas. Theoretical contributions to control literature are projected to modern, complex interrelated systems such as transportation, automotive, communication, chemical systems and robotics. The main goal is to enhance stability and performance results of the dynamical systems considering intelligent trade-offs by using the most feasible actuator and sensor technology. In accordance with performance demanding growth of complex real life applications, and the requirements in achieving optimum performance, control systems technology is progressing in both theoretical and application areas.

In this dissertation we are mainly interested in designing sliding mode controllers for systems with modeling and parametric uncertainties considering unmodelled actuator and sensor dynamics. Furthermore the drawbacks of sliding mode control
such as high gain, high frequency control inputs are attenuated by extending the controlled system through adding compensator dynamics. This design approach is quite reasonable since most of the analytical models of the plants to be controlled have uncertainties, unmodelled dynamics and limited control capacity.

At the first stage of designing a controller, plant dynamics and physical properties of the plant model including the characteristics of the uncertainties, the sensor and actuator dynamics are to be considered. From engineering perspective, complete analytical modelling process representing all the plant dynamic behavior may not be desirable from the amount of work and award point of view. Therefore, some assumptions to simplify the modelling process without degrading the main dynamical behavior of the plant are acceptable. The results of these simplifications have to be considered and compensated at the controller design stage.

Sliding mode control is well known for its insensitivity to bounded uncertainties at the system input channel and reduced order motion on the prespecified manifold once reaching the sliding manifold. The drawbacks of the sliding mode control theory such as high frequency, high gain control terms are to be attenuated without degrading the system performance and robustness properties. Control inputs may be improved by using high order sliding mode controllers and still preserving robustness to the matched disturbances. High order sliding controller design may not be applicable due to the requirement of the high order time derivative of the sliding manifold be equal to zero and also instability problems caused by high order sliding manifold requirement. Another consideration related to conventional sliding mode controllers, the dynamics on the sliding manifold are affected by the disturbances that are not
matched to the input channels. From a robust control point of view, the unmatched disturbances are also to be considered at the controller design stage.

1.2 Literature Survey

This dissertation may be separated into two main parts: the first part is dedicated to show the relations of frequency shaping and dynamic compensator design for sliding mode control and the higher order sliding mode controllers in order to attenuate high-frequency, high-gain control inputs for achieving sliding mode. And the second part presents the use of the dynamic compensation techniques for sliding mode control to stabilize the uncertain systems with modeling uncertainties, unknown disturbances, unmodelled actuator and sensor dynamics. Therefore, we will first review chattering elimination methods, high order sliding mode controllers and compensator dynamics methodologies for sliding mode control then we will review robust controller design methodologies. In this dissertation, we denote sliding manifold \( s(x(t)) \) as function of the states of the controlled system \( x(t) \) and \( \sigma(x_{ext}(t)) \) as dynamic sliding manifold as function of the system extended by adding compensator dynamics.

In Sliding Mode Control Theory Literature, chattering is described as undesired finite frequency, finite amplitude system oscillations that are caused by system imperfections, Utkin, [69], Young and Özgüner, [78]. The chattering phenomenon is generally perceived as motion which oscillates about the sliding manifold. These oscillations are caused by the high-frequency switching of a sliding mode controller exciting unmodelled or parasitic dynamics in the closed loop or directly by the switching nonidealities such as delays.
Continuous approximation of discontinuous control in a boundary layer of a sliding manifold is one of the methods of chattering elimination, [64, 65]. Asymptotic observers are introduced to eliminate the chattering effects caused by unmodelled dynamics, [69]. Another solution to the chattering elimination is the use of higher order sliding modes. They are characterized by a discontinuous control acting on the higher order time derivatives of the sliding manifold, i.e., the second-order sliding mode technique is to use the sliding manifold \( s(x(t)) \) in the original state-space of the plant, with \( s(x(t)) = 0 \) and its first time derivative \( \dot{s}(x(t)) \) identically equal to zero and, enforcing sliding mode by a discontinuous control signal generated according to \( s(x(t)) \) but acting directly on the \( \ddot{s}(x(t)) \), [35, 36].

The prespecified sliding manifold design has been introduced by numerous authors between various approaches, considering pole-placement techniques, [5], to frequency-shaping compensation, [76]. A design method proposed earlier to eliminate chattering and to excite minimally the unmodelled dynamics is to introduce compensator dynamics in sliding mode through a new class of switching which has the interpretation of linear operators. The order of the homogeneous linear time-invariant system of the sliding-mode dynamics is increased by the augmented dimension, [77].

Higher order sliding modes are characterized by a discontinuous control acting on the higher order time derivatives of the sliding variable, i.e., the second-order sliding mode technique is to use the sliding manifold \( s(x(t)) \) in the original state-space of the plant, with \( s(x(t)) = 0 \) and its first time derivative \( \dot{s}(x(t)) \) identically equal to zero and, enforcing sliding mode by a discontinuous control signal generated according to \( s(x(t)) \) but acting directly on the \( \ddot{s}(x(t)) \).
Consider a single-input nonlinear system whose dynamics is defined by the differential system

$$\dot{x}(t) = f(x(t), t, u(t))$$  \hspace{1cm} (1.1)

where $x \in \mathcal{R}^n$ is the state vector, $u \in \mathcal{R}$ is the bounded input, $t$ is the independent variable time, and $f : \mathcal{R}^{n+2} \rightarrow \mathcal{R}^n$ is a sufficiently smooth function. Assume that the control task is fulfilled by constraining the state trajectory on a proper sliding manifold in the state space defined by the vanishing of a corresponding sliding variable $s(t)$, i.e.,

$$s(t) = s(x(t), t) = 0$$  \hspace{1cm} (1.2)

where $s : \mathcal{R}^{n+1} \rightarrow \mathcal{R}^n$ is a known single valued function such that its total time derivatives $s^{(k)}$, $k=0, 1, \cdots, r-1$ along the system trajectories exist and are single valued functions of the system state $x$. The latter assumption means that discontinuity does not appear in the first $r-1$ total time derivatives of the sliding variable $s$. Given the constraint function Eq 1.2, its $r$-th order sliding set is defined by the $r$ equalities

$$s = \dot{s} = \ddot{s} = \cdots = s^{r-1} = 0$$  \hspace{1cm} (1.3)

which constitute an $r$-dimensional condition on the system dynamics. The sliding order characterizes the degree of dynamic smoothness in the vicinity of the sliding mode $s \equiv 0$, [9].

The robust stabilization of uncertain systems with both matched and unmatched modeling uncertainties and unknown disturbances has been one of the most important research area in control theory. Sliding mode control theory has been extensively applied to stabilize the uncertain systems satisfying matched uncertainties, [10, 69]. In the presence of unmatched uncertainty, recursive backstepping based on Lyapunov
design ([55]) has been applied for certain classes of systems. Recursive backstepping combined with robust sliding mode performance has been studied widely in recent years [39, 46]. The Lyapunov function used to stabilize the uncertain systems may lead to high gain and undesirable high frequency, high amplitude chattering. A new Lyapunov function has been proposed to design softer control laws to eliminate the chattering effects [33]. Basically they decomposed the disturbances and designed the Linear Quadratic (LQ) smooth controller making principle minors dominant for positive definiteness.

Controller design for a plant such that the system output tracks a specified reference input in the presence of uncertainties is well known in the control literature. In [17, 19], necessary and sufficient conditions of the robust controller for a linear, time-invariant, multivariable system are given, asymptotic tracking is guaranteed independent of input disturbances and arbitrary perturbations in the plant parameters. The regulation problem considering parametric uncertainty and disturbances and reference signals affecting the system output is examined by [32]. Linear time-invariant decentralized controllers by using a linear quadratic optimal control approach are presented to solve the robust servomechanism problem, [42]. Robust output regulation for nonlinear system subject to parametric uncertainties and high-frequency unmodelled dynamics have been developed and an internal-model based control scheme is used to match the unknown exogeneous disturbances in an adaptive sense by [61]. In [21], a compensation technique for the disturbances generated from a unknown linear exosystem and observer, controller design and stability analysis are given. A robust controller based on a nonlinear observer to estimate the partially or totally unknown and bounded disturbances is designed for nonlinear systems in [58]. A class
of variable structure feedback systems capable of rejecting a persistent disturbance is developed in [74, 75]. Having similar structure to linear multivariable servomechanisms, the dynamics on the switching manifold, the internal stability and output regulation properties of the variable structure servomechanism design are examined using the theory of variable structure systems and sliding mode and high gain feedback systems.

The problem of pole assignment with incomplete state feedback or only using output gain has been extensively studied in control literature. Considering the linear time-invariant multivariable system,

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}
\]  

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \) and \( y \in \mathbb{R}^l \), \( l, m \) are the independent numbers of outputs and inputs with \( m \leq l < n \), such that the pair \( \{A, B\} \) is controllable and the pair \( \{A, C\} \) is observable, \( l \) poles of the closed loop system can be assigned arbitrarily close to their desired values, [16, 20]. The remaining \( n - l \) closed loop poles can not be determined either guaranteed to be in the stable region. This is result was extended to the \( \max(l, m) \) number of closed loop poles can be assigned close to \( \max(l, m) \) desired values in [18] and to the arbitrarily close to \( \min(n, m + l - 1) \) in [20] by using a constant output feedback gain \( K \in \mathbb{R}^{m \times l} \). These results imply that almost all linear, time-invariant multivariable systems can be stabilized by using only output feedback such that the rank condition \( n \leq m + l - 1 \) holds. In [45], the same results were presented and the minimum order of the dynamic compensator required for almost arbitrary pole assignment was shown to be not greater than \( n - m - l + 1 \). In [11],
the order of the dynamic compensator was determined according to the observability and controllability indices to assign the closed loop poles of the extended system by compensator dynamics. The minimum order of the compensator was formulated in terms of the sum of two performance indexes, one considering quadratically the system states and control inputs and the other the error between the desired and assignable pole locations, [63]. Based on these results, in [31, 56], exact pole assignment by using output feedback and some pathological cases causing failure of the methodology were considered. In [13], the necessary and sufficient conditions for output feedback design were presented by minimizing some cost functional.

Output feedback sliding mode controller design has generated considerable interest and output feedback sliding mode existence and reaching conditions have been studied by considering the system properties such as controllability and observability of the reduced order sliding mode dynamics. Since the linear system in regular form is suitable for sliding mode output feedback controller design, existence and reaching conditions of the sliding mode output feedback controller have been investigated by considering the system in regular form, (see for instance, [8, 25, 26, 28, 29, 30, 48, 49, 79]).

1.3 Outline of This Dissertation

In Chapter 2, the use of compensator dynamics in sliding mode through a class of switching surfaces which has the interpretation of linear operators is presented. This approach originally proposed by [76], as dynamic sliding surface design is shown to be related to design of high order sliding mode controllers. The high order sliding mode controllers are observed to be special cases of the dynamic sliding mode design.
Transient performance improvement and finite time convergence to the switching surface have been assured by this methodology and it does not require the higher time derivative of the sliding mode manifold.

In Chapter 3, dynamic sliding surface design combined with recursive backstepping algorithm is introduced. We investigate the elimination of high frequency, high amplitude chattering caused by large state feedback gains derived by Lyapunov based recursive backstepping controller design. Transient performance improvement and finite time convergence to the switching surface have been assured by the proposed frequency shaping compensation for backstepping sliding mode control. The proposed controllers and conventional sliding mode controller combined with recursive backstepping design have been applied to linear systems in regular form with both matched and unmatched time-varying disturbances.

In Chapter 4, a frequency-shaped optimal sliding mode approach based on Lyapunov based recursive backstepping design is presented to solve the robust servomechanism problem. The parametric uncertainties, unmodelled dynamics and uncertain exogenous disturbance belonging to a specified class are attenuated by using compensator dynamics introduced in sliding mode control methodology. Transient performance improvement and finite time convergence to the switching surface have been assured by this proposed backstepping dynamic sliding mode methodology.

In the previous chapters, sliding mode controller design has been limited to full-state feedback of the plant or their estimated values. In Chapter 5, we consider the sliding mode output feedback control, where the state variables are not accessible for direct measurement. We consider the use of compensator dynamics to improve
the transient performance of the closed loop system at the reaching phase and to attenuate high-frequency, high-gain control to achieve sliding mode.

Applications of the developed and considered theoretical approaches are applied to the practical problems. Slosh reduction, robust controller design for drive by wire hydraulic power steering system and output sliding mode control for pneumatic brake systems are given in Chapter 6. The related system models are given in the Appendix section.
CHAPTER 2

RELATION OF DYNAMIC SLIDING SURFACE DESIGN
AND HIGH ORDER SLIDING MODE CONTROLLERS

2.1 Introduction

The use of compensator dynamics in sliding mode through a class of switching surfaces which has the interpretation of linear operators has originally been proposed by [76]. This design method proposed earlier to eliminate chattering and to excite minimally the unmodeled dynamics is shown to be related to design of high order sliding mode controllers.

The sliding manifold proposed in [76]: \( \sigma = \mathcal{L}(x) = 0 \) is interpreted as a dynamic system, \( x \in \mathcal{R}^n \) is the order of the plant. Using a state-space realization of the linear operator \( \mathcal{L}(\bullet) : \mathcal{R}^n \rightarrow \mathcal{R}^m \),

\[
\begin{align*}
\dot{z} &= Fz + Gx, \quad z \in \mathcal{R}^r \\
\sigma &= Hz + Cx, \quad s \in \mathcal{R}^m
\end{align*}
\]  

(2.1)

and the above manifold reduces to

\[
\sigma(x_{\text{ext}}; H, C) = [H \ C]x_{\text{ext}}, \quad x_{\text{ext}}^T = [x \ z]
\]
which is also parametrized by a constant matrix, but defined as a linear manifold of the augmented variables, \([x \ z]\). The order of the homogeneous linear time-invariant system of the sliding-mode dynamics is also increased by the augmented dimension \(r\) to \(n+r-m\), [77].

We propose the same class of switching which has a linear operator representable as a linear time-invariant dynamic system itself, acting on the states. Choosing the number of poles and zeros appropriately, we extend the original system to a higher order time-invariant linear system. The discontinuous signal enforcing sliding mode acts through the dynamic compensator and the output of the dynamic compensator which is a part of the designed switching surface is the continuous input to the original system. The so-called "high order sliding mode controllers" are observed to be special cases of dynamic sliding mode approach that Young and Özbörüner proposed, [76]. Details of the higher order sliding mode algorithms are already given in Introduction chapter. This chapter is organized as follows: we first present dynamic switching surface design for linear time-invariant systems. We apply the same dynamic sliding surface design methodology to nonlinear systems, and we consider the relation of dynamic surface design and second order sliding mode controllers. Finally, simulation results obtained by conventional sliding mode, a second order sliding mode and dynamic sliding mode design and some conclusions of this work is given.

2.2 Linear Systems

Following the work [76], we consider the switching surface as not merely a hypersurface in the original state-space, but a linear operator representable as a linear, time-invariant dynamic system itself, acting on the states. Let the plant be given in
regular form, [10],

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{z}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & 0 \\
A_{21} & A_{22} & B_2 \\
K_2A_{21} & K_1 + K_2A_{22} & -F
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
z
\end{bmatrix} +
\begin{bmatrix}
0 \\
B_2 \\
K_2B_2
\end{bmatrix} u
\tag{2.2}
\]

where \(x_1 \in \mathbb{R}^n, x_2 \in \mathbb{R}^m, u \in \mathbb{R}^m\), the matrices are real, of compatible dimensions and \(B_2\) is of full rank. The switching surface is

\[\sigma = Cx_1 + \mathcal{L}(x_2)\]

where \(C\) is a \(\mathbb{R}^{m \times n}\) constant matrix to define the desired dynamics of the sliding mode and \(\mathcal{L}(\bullet): \mathbb{R}^m \rightarrow \mathbb{R}^m\) is a linear operator which has a realization as a transfer function

\[(sI + F)z = [K_1 + K_2s]x_2\]

where \(K_1, K_2 \in \mathbb{R}^{m \times m}, F \in \mathbb{R}^{m \times m}\). And the linear operator is defined \(\mathcal{L}(\bullet)\) as a dynamic system

\[
\dot{z} = -Fz + K_2A_{21}x_1 + (K_1 + K_2A_{22})x_2 + K_2B_2u
\]

\[y = Hz + x_2\]

The composite system is

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{z}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & 0 \\
A_{21} & A_{22} & B_2 \\
K_2A_{21} & K_1 + K_2A_{22} & -F
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
z
\end{bmatrix} +
\begin{bmatrix}
0 \\
B_2 \\
K_2B_2
\end{bmatrix} u
\tag{2.6}
\]

We have assumed that \(\mathcal{L}(\bullet)\) has an equal number poles and zeros, here the number of poles and zeros is equal to one to be able to introduce the first derivative of the state \(x_2\) and the discontinuous control signal to the augmented state \(z\). And \(z\) is the continuous input to the plant given in Eq. 2.2 (using the same terminology as
in [76] we denote $\sigma(x_{\text{ext}}(t))$ as dynamic switching surfaces, and $s(x(t))$ conventional non-dynamic switching surfaces. Transfer block of the augmented system is given in Fig. 2.1

![System in regular form](image)

Figure 2.1: Transfer block representation of the augmented system

### 2.2.1 Sliding Manifold Design for Linear Systems

The sliding manifold dynamics are described by:

$$
\sigma = Cx_1 + \begin{bmatrix} I & H \end{bmatrix} \begin{bmatrix} x_2 \\ z \end{bmatrix}
$$

(2.7)

and time-derivative of the sliding manifold,

$$
\dot{\sigma} = [A_{21} + CA_{11} + HK_2A_{21}] x_1 + [B_2 - HF] z + [H(K_1 + K_2A_{22}) + A_{22} + CA_{12}] x_2 + HK_2B_2u
$$

(2.8)
The equivalent control on the switching surface may be found from using the conventional sliding mode control design, \( \sigma = 0 \), and \( \dot{\sigma} = 0 \), the equivalent control ([69]) is given,

\[
{u_{eq} = -(HK_2B_2)^{-1}[(A_{21} + CA_{11} + HK_2A_{21})x_1 + (H(K_1 + K_2A_{22}) + A_{22} + CA_{12})x_2 + (B_2 - HF)z]}
\]  

(2.9)

If the sliding mode exists: \( \sigma = 0 \), the reduced order system dynamics is

\[
{z = -H^{-1}C x_1 - H^{-1}x_2}
\]  

(2.10)

\[
\dot{x}_1 = A_{11}x_1 + A_{12}x_2
\]  

(2.11)

\[
\dot{x}_2 = (A_{21} - B_2H^{-1}C)x_1 + (A_{22} - B_2H^{-1})x_2
\]

where

\[
\left\{ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} ; \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \right\} \begin{bmatrix} -H^{-1}C & -H^{-1} \end{bmatrix}
\]

(2.12)

is a controllable pair and closed-loop poles of sliding mode equation can be placed by the selection \( H^{-1}C \) and \( H^{-1} \).

### 2.3 Nonlinear Systems

Let the nonlinear plant be given by

\[
\dot{x}_i = x_{i+1}, \quad i = 1, 2, ..., n - 1 \quad \dot{x}_n = f(x) + g(x)u
\]  

(2.13)

where \( x_i \in \mathbb{R}^m, \ i = 1, 2, ..., n, \ u \in \mathbb{R}^m \), and we assume \( g(x) \neq 0 \) for all \( x \) and \( f(0) = 0 \).

The dynamic switching surface

\[
\sigma = \mathcal{L}(x_n) + s_o(x)
\]  

(2.14)
where \( C_t \) is a \( \mathcal{R}^{m \times m} \) constant matrix, \( s_0(x) \) is a stabilizing continuously differentiable state feedback control of the nonlinear plant given in Eq. 2.13, with \( s_0(0) = 0 \), ([44, 69]) and \( \mathcal{L}(\bullet) : \mathcal{R}^m \rightarrow \mathcal{R}^m \) is a linear operator which has a realization as a transfer function

\[
(sI + F)z = [K_1 + K_2 s]x_n
\]  
(2.15)

where \( K_1, K_2 \in \mathcal{R}^{m \times m}, F \in \mathcal{R}^{m \times m} \). The linear operator \( \mathcal{L}(\bullet) \) is defined as a dynamic system

\[
\dot{z} = -Fz + K_1 x_n + K_2 f(x) + K_2 g(x)u
\]
\[y = z\]
(2.16)

The composite nonlinear system where the extended state \( z(t) \) is the continuous input to the original nonlinear system, and \( u(t) \) is the discontinuous control input, given by,

\[
\begin{align*}
\dot{x}_i &= x_{i+1}, \quad i = 1, 2, ..., n - 1 \\
\dot{x}_n &= f(x) + g(x)z \\
\dot{z} &= -Fz + K_1 x_n + K_2 f(x) + K_2 g(x)u
\end{align*}
\]
(2.17)

### 2.3.1 Sliding Manifold Design for Nonlinear Systems

The sliding manifold dynamics are described by

\[
\sigma = z + s_o(x)
\]
(2.18)

\[
\dot{\sigma} = -Fz + K_1 x_n + K_2 f(x) + K_2 g(x)u + \sum_{i=1}^{n} \frac{\partial s_o(x)}{\partial x_i} \dot{x}_i
\]
(2.19)

The equivalent control on the switching surface

\[
u_{eq} = -(K_2 g)^{-1} \left( -Fz + K_1 x_n + K_2 f(x) + \sum_{i=1}^{n} \frac{\partial s_o(x)}{\partial x_i} \dot{x}_i \right)
\]
(2.20)
If the sliding mode exists: \( \sigma = 0 \), the reduced order system dynamics

\[
\begin{align*}
\dot{x}_i &= x_{i+1}, \quad i = 1, 2, ..., n - 1 \\
\dot{x}_n &= f(x) + g(x)(-s_0(x))
\end{align*}
\tag{2.21}
\]

and the states of the composite system \([x \ z]\) converge to the origin asymptotically.

For certain \( f(x) \) and \( g(x) \) functions, \( u = u_{eq} \) results \( \dot{\sigma} = 0 \), which ensures that the condition \( \sigma = 0 \) can be maintained for all future time.

**Remark:** The second order sliding mode controllers enforce sliding mode according to the sliding manifold \( s(x, t) \) by means of a continuous bounded input \( u(t) \). The continuous control signal \( u(t) \) may be considered as a continuous output of a first-order dynamical system which is driven by a proper discontinuous signal, [34, 53]. Dynamic sliding surface design uses the sliding manifold in the extended state-space of the composite system and it enforces sliding mode according to sliding manifold \( \sigma(x_{ext}(t)) \) acting directly on its first time derivative \( \dot{\sigma}(x_{ext}(t)) \). The discontinuous signal enforcing sliding mode acts through dynamic compensator and the output of the dynamic compensator which is a part of the designed switching surface is the continuous input to the original system.

**Remark:** The constant matrices \( K_1 \) and \( K_2 \) do not affect the dynamics of the sliding mode. They may be considered as a Lead, Lag controller design parameters to improve transient performance response of the dynamic compensator.
2.4 Dynamic Sliding Surface Design of Order 3 and Higher

Without losing generality, we consider linear time-invariant plant but high order dynamic sliding surface design can easily be applied also to nonlinear case. Let the plant be given as in Eq. 2.2 and \( L(\cdot) \) is the linear operator which has a realization as a transfer function

\[
\begin{align*}
\dot{z}_i &= \begin{bmatrix} I \\ s \end{bmatrix} z_{i+1}, \quad i = 1, 2, ..., l - 1 \\
(sI + F)z_l &= [K_1 + K_2 s] x_2 
\end{align*}
\]  

(2.22)

where \( K_1, K_2 \in \mathbb{R}^{m \times m} \), \( F \in \mathbb{R}^{m \times m} \). And we define linear operator \( L(\cdot) \) as dynamic system

\[
\begin{align*}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
&\vdots \\
\dot{z}_l &= -Fz_l + K_2 A_{21} x_1 + (K_1 + K_2 A_{22}) x_2 + K_2 B_2 u \\
y &= Hz + x_2 
\end{align*}
\]  

(2.23)

where \( z^T = [z_1 \ z_2 \ ... \ z_l] \) and \( z_1 \) is the input to the original system given in Eq. 2.2. Sliding manifold dynamics are described by,

\[
\begin{align*}
\sigma &= Hz + Cx_1 + x_2 \\
&= H_1 z_1 + H_2 z_2 + \cdots + z_l + Cx_1 + x_2 
\end{align*}
\]  

(2.24)

\[
\dot{\sigma} = [K_2 A_{21} + CA_{11} + A_{21}] x_1 + [K_1 + K_2 A_{22} + CA_{12} + A_{22}] x_2 \\
&\quad + H_1 z_2 + H_2 z_3 + \cdots + H_{l-1} z_l + K_2 B_2 u 
\]  

(2.25)
where $H_i \in \mathbb{R}^{m \times m}$ for $i = 1, 2, \ldots, l$. The equivalent control by using $\sigma = \dot{\sigma} = 0$,

$$u_{eq} = -(K_2 B_2)^{-1}[(K_2 A_{21} + CA_{11} + A_{21}) x_1 + (K_1 + K_2 A_{22} + CA_{12} + A_{22}) x_2
+ B_2 z_1 + H_1 z_2 + H_2 z_3 + \cdots + H_{l-1} z_l]$$

(2.26)

If the sliding mode exists: $\sigma = 0$, the reduced order system dynamics is

$$z_i = -H_1 z_1 + H_2 z_2 - \cdots - H_{l-1} z_{l-1} - C x_1 - x_2$$

(2.27)

$$\begin{bmatrix}
A_{11} & A_{12} & 0 & 0 & 0 & \cdots & 0 \\
A_{21} & A_{22} & B_2 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & I & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & I & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
-C & -I & -H_1 & -H_2 & -H_3 & \cdots & -H_{l-1}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
z_1 \\
z_2 \\
\vdots \\
z_{l-1}
\end{bmatrix}
= \begin{bmatrix}
\ddot{x}_1 \\
\ddot{x}_2 \\
\ddots \\
\ddots \\
\ddots \\
\ddots
\end{bmatrix}$$

where $\ddot{x} = [x_1 \ x_2 \ z_1 \ \cdots \ z_l]^T$. The reduced order dynamics of the sliding mode may be assigned by a proper choice of the parameters of the switching plane $H_i$ and $C$.

The high order sliding mode uses the sliding mode $s(x(t))$ in the original state-space of the plant with $s(x(t))$ and its $r - 1$ total time derivatives of the sliding manifold $s(x(t))$, $\dot{s}(x(t))$, $\ddot{s}(x(t))$, $\cdots$, $s^{r-1}(x(t))$ equal to zero, which is not trivial for discontinuous dynamic systems. Discontinuous signal acts on $s^r(x(t))$ and it does not appear in the first $r - 1$ total time derivatives of the sliding manifold $s(x(t))$. The $r$-th order sliding mode is determined by the equalities

$$s(x(t)) = \dot{s}(x(t)) = \ddot{s}(x(t)) = \cdots = s^{r-1}(x(t)) = 0$$

(2.28)

In the extended state-space of the composite system using dynamic sliding surface design the discontinuous control signal generated according to $\sigma(\ddot{x}(t))$ acts directly on the first time derivative of $\dot{\sigma}(\ddot{x}(t))$. The discontinuous signal is filtered through $l$ integrator and it is applied to the original system as a continuous input.
2.5 Simulation Results

2.5.1 Linear Systems

Let the linear time-invariant plant be given

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 
\end{bmatrix} = \begin{bmatrix}
1 & 2 \\
3 & 1
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} u
\] (2.29)

The eigenvalues of the system are \( \lambda = \{-1.4495, 3.4495\} \). Three sliding mode controller algorithms are considered, for each controller design the discontinuous input is

\[ u = -(M||x|| + \delta)\text{sign}(s) \] (2.30)

where \( M \) and \( \delta \) are positive constants. For conventional (first order) and the twisting algorithm (second order) sliding manifold is chosen as

\[ s = 15x_1 + 10x_2 \] (2.31)

and the time responses of the states, the plant control input and sliding manifold of the conventional sliding mode (Figure 2.2, Figure 2.3) and second order sliding ([34], [53]) mode controller (Figure 2.4, Figure 2.5) have been simulated.

For dynamic switching surface design, sliding manifold of the extended system and the linear operator are chosen,

\[ \sigma = 15x_1 + 10x_2 + z \]

\[ (s + 1000)z = (s + 1)x_2 \] (2.32)

and the time responses of the states, the plant control input and sliding manifold of dynamic switching surface design (Figure 2.6, Figure 2.7) have been simulated.
Figure 2.2: The time response of the states – (first order)

Figure 2.3: The time response of the control input and the manifold – (first order)
Figure 2.4: The time response of the states – (second order)

Figure 2.5: The time response of the control input and the manifold – (second order)

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Figure 2.6: The time response of the states - (Dynamic switching surface design)

Figure 2.7: The time response of the control input and the manifold - (Dynamic switching surface design)
2.5.2 Nonlinear Systems

The nonlinear system is chosen as

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_1^2 + x_2 + u
\end{align*}
\tag{2.33}
\]

and for each controller, the discontinuous input is

\[
\dot{u} = -(M||x|| + \delta)\text{sign}(s) \tag{2.34}
\]

where \( M \) and \( \delta \) are positive constants.

Sliding manifold is chosen for conventional (first order) and the twisting algorithm (second order) sliding mode

\[
s = 15x_1 + 8x_2 + x_3 \tag{2.35}
\]

The eigenvalues of the reduced order system in sliding mode will be \( \lambda = \{-3, -5\} \). The time responses of the states, the plant control input and the sliding manifold of the conventional sliding mode (Figure 2.8, Figure 2.9) and second order sliding ([34, 53]) mode controller (Figure 2.10, Figure 2.11) have been simulated.

Using dynamic switching surface design, the composite system

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_1^2 + x_2 + z \\
\dot{z} &= -Fz + k_1x_1 + k_2x_1^2 + k_2x_2 + k_2u
\end{align*}
\tag{2.36}
\]
where \( k_1 = 1, k_2 = 1, \) and \( F = 1000. \) Sliding manifold is chosen

\[
\sigma = x_1^2 + 30x_1 + 48x_2 + 12x_3 + z
\]  

(2.37)

The eigenvalues of the reduced order system in sliding mode are \( \lambda = \{-3, -4, -5\}. \)

The time responses of the states, the plant control input and the sliding manifold (Figure 2.12, Figure 2.13) have been simulated.

Simulation results of dynamic surface design methodology applied to linear and nonlinear systems is able to eliminate chattering effects by a continuous output of the compensator system acting on the states of the system. The second order sliding mode algorithm considers the chattering problem by taking account the condition on \( |u|, \) therefore the time responses of the system states converge to the origin by oscillating according to the control input values, [34, 53]. Conventional sliding mode control algorithm satisfies finite time convergence of the states to the sliding manifold by using high frequency control input and high frequency chattering is unavoidable.

2.6 Summary

This chapter contributes to the chattering elimination problem by using the dynamic switching surface design. The high order sliding mode controllers are observed to be special cases of the dynamic sliding mode design. Transient performance improvement and finite time convergence to the switching surface have been assured by this methodology and it does not require the higher time derivative of the sliding mode manifold. Furthermore comparing to second order sliding mode controller parameter tuning complexity, dynamic sliding manifold design using first order sliding
Figure 2.8: The time response of the states – (first order)

Figure 2.9: The time response of the control input and the manifold – (first order)
Figure 2.10: The time response of the states – (second order)

Figure 2.11: The time response of the control input and the manifold – (second order)
Figure 2.12: The time response of the states – (Dynamic switching surface design)

Figure 2.13: The time response of the control input and the manifold – (Dynamic switching surface design)
mode controller design techniques is much convenient to deal with chattering elimination problem. The design method has been applied to linear systems and nonlinear systems and simulation results have been presented.
CHAPTER 3

FREQUENCY SHAPING COMPENSATION FOR BACKSTEPPING SLIDING MODE CONTROL

3.1 Introduction

In this chapter recursive backstepping combined with dynamic sliding surface design is introduced. Lyapunov based recursive backstepping design is applied to robustly stabilize the linear system in the presence of both matched and unmatched uncertainties, and the recursive smooth state feedback control laws are generated by forcing the sliding manifold through compensator dynamics. The compensators are designed to attenuate the frequency contents of the sliding mode dynamics such that high frequency, high amplitude chattering effects are eliminated [3, 4, 76]. Two design methods are presented, the first method is based on pole placement techniques satisfying desired transient performance specifications during sliding mode by using the free parameters of the dynamic compensator. The second method is based on frequency-shaped LQ design techniques, which have been used to minimize high gain and undesirable high frequency, high amplitude chattering effects caused by the Lyapunov function used to stabilize the uncertain systems. This chapter is organized as follows: first, dynamic switching surface is designed for the error dynamics derived
based on recursive backstepping control. Frequency-shaped optimal sliding mode combined by recursive backstepping design to attenuate high gain and minimally excite the unmodelled dynamics is introduced. Simulation results, comparisons and some conclusions of this chapter are presented.

3.2 Dynamic Sliding Mode Control Combined with Recursive Backstepping Design

Let the plant in the regular form with both matched and unmatched disturbances be given,

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
0 \\
B_2
\end{bmatrix} u +
\begin{bmatrix}
D_1(x_1, t) \\
D_2(x_1, x_2, t)
\end{bmatrix}
\]  

(3.1)

where \( x_1 \in \mathcal{R}^n, x_2 \in \mathcal{R}^m, u \in \mathcal{R}^n \), the matrices are real, of compatible dimensions, \( B_2 \) is of full rank and the functions \( D_1(x_1, t), D_2(x_1, x_2, t) \) represent the plant nonlinearities and uncertainties. The control goal is to asymptotically globally stabilize the state \( x_1 \) using sliding mode control design methodology. The dynamic switching surface is chosen as in Eq 2.3, and the realization of the linear operator is given in Eq 2.4, Eq 2.5.

The composite system with matched and unmatched disturbances is given,

\[
\begin{align*}
\dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + D_1(x_1, t) \\
\dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + B_2z + D_2(x_1, x_2, t) \\
\dot{z} &= -Fz + K_2A_{21}x_1 + (K_1 + K_2A_{22})x_2 + K_2B_2u + K_2D_2(x_1, x_2, t)
\end{align*}
\]  

(3.2) (3.3) (3.4)

We have assumed that \( L(\bullet) : \mathcal{R}^m \rightarrow \mathcal{R}^m \) has an equal number poles and zeros, here the number of poles and zeros is equal to one to be able to introduce the first derivative of the state \( x_2 \) and the discontinuous control signal to the augmented state.
$z$. And $z$ is the continuous input to the plant given in Eq. 3.1.

The disturbances are assumed to satisfy,

$$\|D_1(x_1, t)\|_2 \leq d_1\|x_1\|_2$$
$$\|D_2(x_1, x_2, t)\|_2 \leq \|D_1(x_1, t)\|_2 + \|D_1(x_2, t)\|_2$$
$$\leq d_1\|x_1\|_2 + d_2\|x_2\|_2$$

(3.5)

where $\|x\|_2 = (x^T x)^{1/2}$ for all $x \in \mathbb{R}^n$.

Step 1:

Define the error variable $y_1 \equiv x_1$ and $y_2 \equiv x_2 - \alpha_1(x_1)$ where $x_2 = \alpha_1(x_1)$ with $\alpha_1(0) = 0$ is a smooth stabilizing state feedback control of the system given in Eq. 3.2 included to compensate for the uncertainty $D_1(x_1, t)$. A Lyapunov function, $V_1 = \frac{1}{2} y_1^T y_1$ and its first-order time derivative along the trajectory of Eq. 3.2

$$\dot{V}_1(y_1) = \frac{\partial V_1}{\partial y_1} \dot{y}_1 = y_1^T (A_{11} y_1 + A_{12} x_2 + D_1(x_1, t))$$
$$\leq \lambda_{\text{min}}(A_{11} - (\mathbb{K} - d_1))\|y_1\|_2^2 + d_1\|y_1\|_2^2$$
$$\leq -k_1\|y_1\|_2^2$$

(3.6)

for all $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^m$ and $k_1 > 0$ is satisfied by using the state feedback,

$$x_2 = -A_{12}^+(\mathbb{K} - d_1)y_1$$

(3.7)

where $\lambda_{\text{min}}(A_{11} - (\mathbb{K} - d_1)) < -d_1 < 0$ and $A^+$ is the pseudo-inverse of the matrix $A$. Then the time derivative of the error variable $y_1$ and the error variable $y_2$ are
given by

\[
\dot{y}_1 = A_{11}y_1 + A_{12}(y_2 - \alpha_1(x_1)) + D_1(y_1, t) \tag{3.8}
\]

\[
y_2 = x_2 + A_{12}^T(K - d_1)x_1 = x_2 + \alpha_1(x_1) \tag{3.9}
\]

Step 2:

Define \( y_3 \equiv z - \alpha_2(x_1, x_2) \) where \( z = \alpha_2(x_1, x_2) \) with \( \alpha_2(0, 0) = 0 \) is a smooth stabilizing state feedback of the system given by Eq. 3.2 and Eq. 3.3 included to compensate for the uncertainties \( D_1(x_1, t) \) and \( D_2(x_1, x_2, t) \). The Lyapunov function \( V_2(y_1, y_2) = V(y_1) + \frac{1}{2}y_2^Ty_2 \) and its time derivative along the trajectories of Eq. 3.2, Eq. 3.3 is given by

\[
\dot{V}_2 = y_1^T[(A_{11} - (K - d_1))y_1 + A_{12}y_2 + D_1(y_1, t)] + y_2^T[A_{21}x_1 + A_{22}x_2 + B_2z + D_2(x_1, x_2, t) + A_{12}^T(K - d_1)(A_{11}x_1 + A_{12}x_2 + D_1(x_1, t))] \tag{3.10}
\]

using the state feedback,

\[
z = -B_2^+[A_{12}^Ty_1 + A_{21}x_1 + A_{22}x_2 + A_{12}^T(K - d_1)(A_{11}x_1 + A_{12}x_2) - k_2y_2] \tag{3.11}
\]

the time derivative of \( V_2(y_1, y_2) \) is obtained,

\[
\dot{V}_2 \leq -k_1\|y_1\|_2^2 - k_2\|y_2\|_2^2 + \|y_2\|_2(\|D_2(x_1, x_2, t)\|_2 + \|A_{12}^T(K - d_1)\|_2\|D_1(x_1, t)\|_2) \tag{3.12}
\]
and using disturbance inequalities given by Eq. 3.5

\[ \dot{V}_2 \leq -k_1\|y_1\|^2 -k_2\|y_2\|^2 + \|y_2\|^2 (d_1\|x_1\| + d_2\|x_2\|) + d_1\|A_{12}^T(K - d_1)\|y_1\| \]

The change of coordinates \((y_1, y_2, y_3) \leftrightarrow (x_1, x_2, z)\) is well defined, ([33]), and using a change of coordinates \((x_2) \leftrightarrow (y_1, y_2)\) and the triangle inequality,

\[ \|x_2\| \leq \|y_2\| + \|A_{12}^T(K - d_1)\|y_1\| \]

the time derivative of \(V_2(y_1, y_2)\) is given

\[ \dot{V}_2 \leq -k_1\|y_1\|^2 - k_2\|y_2\|^2 + d_2\|y_2\|^2 + \|y_1\|(d_1 + d_2)\|A_{12}^T(K - d_1)\|y_2\| \]

\[ \quad \text{(3.14)} \]

define

\[ \|A_{12}^T(K - d_1)\| \equiv [\lambda_{\max}(A_{12}^T(K - d_1))^T(A_{12}^T(K - d_1))]^{\frac{1}{2}} \equiv \delta \]

\[ \text{(3.15)} \]

by inserting the above definition into Eq. 3.14,

\[ \dot{V}_2 \leq -k_1\|y_1\|^2 - (k_2 - d_2)\|y_2\|^2 + \|y_1\|(d_1 + (d_1 + d_2)\delta)\|y_2\| \]

\[ = -\begin{bmatrix} \|y_1\| \\ \|y_2\| \end{bmatrix}^T \begin{bmatrix} k_1 & -\mathcal{L} \\ -\mathcal{L} & k_2 - d_2 \end{bmatrix} \begin{bmatrix} \|y_1\| \\ \|y_2\| \end{bmatrix} \]

\[ \text{(3.16)} \]

where \(\mathcal{L} = d_1 + (d_1 + d_2)\delta\), the time derivative of \(V_2(y_1, y_2)\) is negative when the state feedback gain is chosen

\[ k_2 > d_2 + \frac{d_1 + (d_1 + d_2)\delta}{4k_1} \]

\[ \text{(3.17)} \]

yields,

\[ \dot{V}_2 \leq -c[\|y_1\|^2 + \|y_2\|^2] \]

\[ \text{(3.18)} \]
for some $c > 0$. Then the time derivative of the error variable $y_2$ and the error variable $y_3$ are given by

$$
\dot{y}_2 = A_{21}x_1 + A_{22}x_2 + B_2z + D_2(x_1, x_2, t) + A_{12}^+(K - d_1) [A_{11}x_1 + A_{12}x_2 + D_1(x_1, t)]
$$
(3.19)

$$
y_3 = z + B_2^+[A_{12}^Ty_1 + A_{21}x_1 + A_{22}x_2 + k_2y_2 + A_{12}^+(K - d_1)(A_{11}x_1 + A_{12}x_2)]
\equiv z + \alpha_2(x_1, x_2)
$$
(3.20)

inserting the state feedback ($z = y_3 - \alpha_2(x_1, x_2)$) Eq. 3.20, the error variable equation becomes,

$$
\dot{y}_2 = -A_{12}^Ty_1 - k_2y_2 + B_2y_3 + D_2(x_1, x_2, t) + A_{12}^+(K - d_1)D_1(x_1, t)
$$
(3.21)

and the third error variable equation is given by taking the first derivative of Eq. 3.20,

$$
\dot{y}_3 = -(F - \frac{\partial \alpha_2}{\partial x_2}B_2)y_3 + (K_1 + K_2A_{21} + \frac{\partial \alpha_2}{\partial x_1}A_{11} + \frac{\partial \alpha_2}{\partial x_2}A_{21})(y_1 - \alpha_1(x_1))
$$

$$
+ (K_2A_{22} + \frac{\partial \alpha_2}{\partial x_1}A_{12} + \frac{\partial \alpha_2}{\partial x_2}A_{22})(y_2 - \alpha_2(x_1, x_2)) + \frac{\partial \alpha_2}{\partial x_1}D_1(x_1, t)
$$

$$
+ \frac{\partial \alpha_2}{\partial x_2}D_2(x_1, x_2, t) + K_2B_2u
$$
(3.22)

3.2.1 Sliding Backstepping Control

Define sliding surface in terms of the error dynamics using

$$
\sigma = Hy_3 + y_2 + Cy_1
$$
(3.23)

where $H$ and $C$ are defined in Eq. 2.3 and Eq. 2.5. The Lyapunov function of the composite error system (Eq. 3.8, Eq. 3.21, Eq. 3.22) is given

$$
V_3 = \frac{1}{2}y_1^Ty_1 + \frac{1}{2}y_2^Ty_2 + \frac{1}{2}\sigma^T\sigma = V_2 + \frac{1}{2}\sigma^T\sigma
$$
(3.24)
the time derivative of the Lyapunov function

\[
\dot{V}_3 = y_1^T [(A_{11} - (\mathcal{K} - d_1))y_1 + A_{12}y_2 + D_1(y_1, t)] + y_3^T\left[ -A_{12}^T y_1 - k_2 y_2 + B_2 y_3 \right. \\
+ D_2(x_1, x_2, t) + A_{12}^T (\mathcal{K} - d_1) D_1(x_1, t)] + \sigma^T \left\{ -H(F - \frac{\partial \alpha_2}{\partial x_2} B_2) y_3 \right. \\
+ \left. H(K_1 + K_2 A_{21} + \frac{\partial \alpha_2}{\partial x_1} A_{11} + \frac{\partial \alpha_2}{\partial x_2} A_{21})(y_1 - \alpha_1(x_1)) \right. \\
+ \left. H((K_1 + K_2 A_{22} + \frac{\partial \alpha_2}{\partial x_1} A_{12} + \frac{\partial \alpha_2}{\partial x_2} A_{22})(y_2 - \alpha_2(x_1, x_2)) + K_2 A_{22} + \frac{\partial \alpha_2}{\partial x_1} A_{12} \right. \\
\left. + (\frac{\partial \alpha_2}{\partial x_2} A_{22})(y_2 - \alpha_2(x_1, x_2)) + \frac{\partial \alpha_2}{\partial x_1} D_1(x_1, t) + \frac{\partial \alpha_2}{\partial x_2} D_2(x_1, x_2, t) + K_2 B_2 u) \right) \\
- A_{12}^T y_1 - k_2 y_2 + B_2 y_3 + D_2(x_1, x_2, t) + A_{12}^T (\mathcal{K} - d_1) D_1(x_1, t) \\
+ C((A_{11} - (\mathcal{K} - d_1))y_1 + A_{12}y_2 + D_1(y_1, t)) \right\} 
\] (3.25)

A discontinuous control input can then be formulated as:

\[
u = - (HK_2 B_2)^{-1} [M\|y\|_2 + \Delta]\text{sign}(\sigma)\] (3.26)

where \(y = [y_1 \ y_2 \ y_3]^T\) and \(\Delta > 0, \ M > 0\) are fairly high gains such that a sliding mode on the sliding surface \(\sigma = 0\) is guaranteed. If the sliding mode exists: \(\sigma = 0, \ y_3 = -H^{-1} y_2 - H^{-1} C y_1\) and the time derivative of the Lyapunov function given in Eq. 3.25 can be derived (through straightforward algebraic manipulations)

\[
\dot{V}_3 \leq -(k_1 - d_1 - \gamma_1)\|y_1\|_2^2 - (k_2 - d_2 - \gamma_2)\|y_2\|_2^2 + \|y_1\|_2 \gamma_3 \|y_2\|_2 
\] (3.27)

where \(\gamma_i, i = 1, 2, 3\) denotes the Euclidean norm of the derived terms in Eq. 3.25 when \(\sigma = 0\).

If the sliding mode exists: \(\sigma = 0\), the state feedback gains \(k_1\) and \(k_2\) can be chosen such that Eq. 3.27 yields

\[
\dot{V}_3 < -c [\|y_1\|_2^2 + \|y_1\|_2^2] \leq -W(y_1, y_2) \] (3.28)
where $W(y_1, y_2)$ is a continuous positive semidefinite function and $c > 0$. Then the error system (Eq. 3.8, Eq. 3.21) is exponentially stable. From Barbalat Lemma, $W(y_1, y_2) \to 0$ as $t \to \infty$ This implies $y_i = 0$, $i = 1, 2, 3$ as $t \to \infty$ and $\sigma = 0$ as $t \to \infty$ Therefore, the stability of the composite system along the dynamic sliding surface $\sigma = 0$ is guaranteed.

Now consider the error dynamics, the switching surface is defined as $\sigma = Hy_3 + y_2 + Cy_1$ where $y_3$ is given in Eq. 3.20. If sliding mode exists on $\sigma = 0$, then the equation of sliding mode is

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} A_{11} - (K - d_1)I \\ A_{12} - B_2H^{-1}L \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$ (3.29)

and the poles of the error dynamics can be placed by the selection of $\{K, H, L\}$ if $(A_{11}, A_{12})$ is a controllable pair. The above system may be written as,

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I \\ -A_{12} + B_2H^{-1}L \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$+ \begin{bmatrix} A_{12} \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ (K - d_1) \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$ (3.30)

and its poles can be placed, if the pair

$$\left( \begin{bmatrix} A_{11} & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} A_{12} & I \\ 0 & 0 \end{bmatrix} \right)$$

is controllable. The controllability of the above pair is evidently the same as of $(A_{11}, A_{12})$ as claimed.

**Theorem 1** By using the discontinuous input (Eq.3.26) sliding manifold is reached in finite time and sliding mode is enforced on the manifold (Eq.3.23). Then, the origin of
the composite error system (Eq.3.8, Eq.3.21, Eq.3.22) is asymptotically stable for the
proposed Lyapunov function (Eq.3.24) and time derivative of the proposed Lyapunov
function (Eq.3.27) is bounded by negative definite function for the proper backstepping
gains $k_1, k_2$.

3.2.2 Frequency-Shaped Optimal Sliding Mode

The idea of frequency shaping, which was proposed by [37], is to introduce fre­
quency dependent weighting matrices in a Linear Quadratic optimal regulator design
formulation. The performance index,

$$J = \int_{-\infty}^{\infty} [x^*(jw)Q(w)x(jw) + u^*(jw)R(w)u(jw)] \, dt \quad (3.31)$$

where $Q(w) \geq 0$ and $R(w) > 0$ for all frequencies $w$ and $x^*$, $u^*$ are the complex con­
jugate transposes of $x$ and $u$ respectively. The frequency-shaping matrices are chosen
based on the argument that if $R(w)$ is chosen to be large over a certain frequency
band, and small outside this band, the control action whose frequencies lie in this
band would be penalized more. Effectively, a reduction of the loop gain of the closed
loop system at high frequencies is achieved over this frequency band. If high fre­
quency in the control action is undesirable, by selecting high-pass characteristics for
the elements of $R(w)$, high frequency control action is minimized. Equally, choosing
low pass characteristics for $Q(w)$ to penalize the low frequency motion of the system
produces a similar effect on the optimal feedback control, [6].

Following the work [76], which converted the frequency-dependent performance
index (Eq. 3.31) into a standard constant weighting matrix through an augmentation
to the original state-space with additional compensator states and dynamics which are defined by the frequency shaping matrices, [37],

\[
\begin{align*}
\dot{y}_1 &= [A_{11} - (\mathcal{K} - d_1)] y_1 + A_{12} y_2 + D_1(x_1, t) \\
\dot{y}_2 &= [A_{21} + A_{12}^T (\mathcal{K} - d_1) A_{11}] y_1 \\
&\quad + [A_{22} + A_{12}^T (\mathcal{K} - d_1) A_{12}] [y_2 - \alpha_1(y_1)] \\
&\quad + B_2 u + A_{12}^T (\mathcal{K} - d_1) [D_1(x_1, t)] + D_2(x_1, x_2, t)
\end{align*}
\]

and the quadratic cost is given for the error dynamics derived to stabilize the system given in Eq. 3.1,

\[
J = \int_{-\infty}^{\infty} \left[ y_1^*(jw) Q_{11} y_1(jw) + y_2^*(jw) Q_{22} y_2(jw) \right] dw
\]

(3.33)
in which, without of loss generality, the cross state and control term has been removed.

The frequency-shaped optimal switching surface is given by,

\[
\sigma(y_1, y_2, z) = y_2 + R_e^{-1} (B_e^T P_e + N_e^T) \begin{bmatrix} z \\ y_1 \end{bmatrix} = 0
\]

(3.34)

where \( z \) is the state of the dynamic compensator realizing the transfer function \( \hat{Q}_{22}(s) \),

\[
\begin{align*}
\dot{z} &= Fz + Gy_2 \\
\eta &= Hz + Dy_2
\end{align*}
\]

(3.35)

\[
\hat{Q}_{22}(s) = D + H (sI - F)^{-1} G
\]

(3.36)

where \( P_e \) is the solution of the Riccati equation

\[
A_e^T P_e + P_e A_e - (P_e B_e + N_e) R_e^{-1} (P_e B_e + N_e)^T + Q_e = 0
\]

(3.37)
with

\[ A_e = \text{diag}(F, A_{11} - (\mathcal{K} - d_1)), \]
\[ B_e = \begin{bmatrix} G & A_{12} \end{bmatrix}, \quad N_e = \begin{bmatrix} H^T D & 0 \end{bmatrix}, \]
\[ Q_e = \text{diag}(H^T H, Q_{11}), \quad R_e = D^T D \]

The frequency-shaped optimal sliding surface (Eq. 3.34) is a linear operator on the states and depending on the weighting matrices, certain frequency band is penalized. Define sliding surface in terms of the error dynamics using

\[ \sigma = y_2 + c_1 y_1 + c_2 z \quad (3.38) \]

where \( c_1 \) and \( c_2 \) are determined by the transfer function \( \hat{Q}_{22}(s) \), by the error dynamics, Eq. 3.32, and the solution of the Riccati Equation given in Eq. 3.37. The Lyapunov function of the system given in Eq. 3.32,

\[ V_2 = \frac{1}{2} y_1^T y_1 + \frac{1}{2} z^T z + \frac{1}{2} \sigma^T \sigma \]

the time derivative of the Lyapunov function

\[ \dot{V}_2 \leq y_1^T [(A_{11} - (\mathcal{K} - d_1))y_1 + A_{12}y_2 + D_1(y_1, t)] + z^T [Fz + Gy_2] + \sigma^T (c_1((A_{11} - (\mathcal{K} - d_1))y_1 + A_{12}y_2 + D_1(x_1, t)) + [A_{21} + A_{12}^T(\mathcal{K} - d_1)A_{11}]y_1 + [A_{22} + A_{12}^T(\mathcal{K} - d_1)A_{12}] [y_2 - \alpha_1(y_1)] + B_2u + A_{12}^T(\mathcal{K} - d_1) [D_1(x_1, t)] + D_2(x_1, x_2, t) + c_2(Fz + Gy_2) \quad (3.39) \]

a discontinuous control input to

\[ u = -B_2^{-1} \left[ M ||y||_2 + \Delta \right] \text{sign}(\sigma) \quad (3.40) \]
where \( y = [y_1 \ y_2]^T \) and \( \tilde{A} > 0, \tilde{M} > 0 \) are fairly high constants such that a sliding mode on the sliding surface \( \sigma = 0 \) is guaranteed. If the sliding mode exists: \( \sigma = 0, \)
\( y_2 = -c_1y_1 - c_2z \) and the time derivative of the Lyapunov function given in Eq. 3.39 can be derived (through straight-forward algebraic manipulations)

\[
\begin{align*}
\dot{V}_2 & \leq -(k_1 - d_1 - \phi_1)\|y_1\|_2^2 + \phi_2\|z\|_2^2 + \|y_1\|_2\|z\|_2 \\
& = - \begin{bmatrix} \|y_1\|_2 \\ \|z\|_2 \end{bmatrix}^T \begin{bmatrix} k_1 - d_1 - \phi_1 & -\phi_3/2 \\ -\phi_3/2 & \phi_2 \end{bmatrix} \begin{bmatrix} \|y_1\|_2 \\ \|z\|_2 \end{bmatrix}
\end{align*}
\]

Choosing \( k_1 > d_1 + \phi_1 + \frac{\phi_3}{4\phi_2} \) yields

\[
\dot{V}_2 < -c \left( \|y_1\|_2^2 + \|z\|_2^2 \right) \leq -W(y_1, z)
\]  \quad (3.41)

where \( \phi_i, i = 1, 2, 3 \) denotes the Euclidean norm of the derived terms in Eq. 3.39 when \( \sigma = 0, c > 0 \) and \( W(y_1, z) \) is a continuous positive semidefinite function.

Following Theorem 1, the state feedback gains \( k_1 \) is chosen such that Eq. 3.41 is satisfied then the error system (Eq. 3.8) is exponentially stable. From Barbalat Lemma, \( W(y_1, z) \to 0 as t \to \infty \) This implies \( y_i = 0, i = 1, 2 as t \to \infty \) and \( \sigma = 0 as t \to \infty \). Therefore, the stability of the composite system along the dynamic sliding surface \( \sigma = 0 \) is guaranteed.

### 3.3 Simulation Results

For simulation purposes, the following plant is chosen.

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\
x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u + \begin{bmatrix} x_1 \sin(10t) \\ x_1 \cos(10t) + x_2 \sin(20t) \end{bmatrix}
\]  \quad (3.42)
The eigenvalues of the system are \( \lambda = \{-1, 3\} \). The discontinuous input to the error dynamics is

\[
 u = -(M||x|| + \Delta)\text{sign}(s) \tag{3.43}
\]

where \( M \) and \( \Delta \) are positive constants.

**Dynamic switching surface design**

For dynamic switching surface design, the linear operator is chosen,

\[
 (s + 1000)z = (s + 1)x_2 \tag{3.44}
\]

Following the recursive (Eq. 3.1 thru Eq. 3.22), the error variables \( y_1 \equiv x_1, \quad y_2 \equiv x_2 + 2x_1 \) and \( y_3 \equiv z + 6x_1 + 5x_2 + 8(x_2 + 2x_1) \), and the error dynamics of the plant given Eq. 3.42 are derived,

\[
 \dot{y}_1 = -3y_1 + 2y_2 + y_1 \sin(10t) \tag{3.45}
\]

\[
 \dot{y}_2 = -6y_1 + 5y_2 + u + y_1 \cos(10t) + y_2 \sin(10t)
\]

\[
 \dot{y}_3 = -4016y_1 + 12890y_2 - 987y_3 + u + 14y_1 \cos(10t) + 14y_2 \sin(10t) - 6y_1 \sin(20t)
\]

The Lyapunov function of the error system (Eq. 3.45) \( V = \frac{1}{2}(y_1^2 + y_2^2 + \sigma^2) \) and the sliding manifold \( \sigma = 1.995y_1 + 5y_2 + y_3 \) are chosen and using the discontinuous input

\[
 u = -(625|x| + 3750|y_2| + 250|y_1| + 10)\text{sign}(\sigma)
\]

finite time convergence to the sliding manifold is guaranteed. During the sliding mode, dynamics of the closed loop system are at \( \{-3 \pm j0.1\} \). The time responses of
the states, the continuous plant control input (Fig 3.1, Fig 3.2) have been simulated.

![Figure 3.1: The time response of the states – (Dynamic switching surface design)](image)

**Frequency-shaped sliding mode design**

Defining the error variables $y_1 \equiv x_1$ and $y_2 \equiv x_2 + 2x_1$ and following the recursive backstepping steps, the error dynamics are obtained,

\[
\dot{y}_1 = -3y_1 + 2y_2 + y_1 \sin(10t) \tag{3.46}
\]

\[
\dot{y}_2 = -6y_1 + 5y_2 + u + y_1 \cos(10t) + y_2 \sin(20t)
\]

A high-pass characteristic for $\dot{Q}_{22}(w)$ with corner frequencies of 1 and 10 $\text{rads}^{-1}$, and a 40dB per decade slope is selected, whereas a unity weighting for $Q_{11}$, i.e.

\[
Q_{11} = 1, \quad \dot{Q}_{22}(w) = \frac{(jw + 2.25)^2}{(jw + 5)^2} \tag{3.47}
\]

The gain value is chosen such that a 40 dB weighting is applied to the high frequencies
in $x_2$, and a 0 dB weighting for low frequencies to ensure that the optimal control action avoids the high frequency disturbances and eliminates high frequency, high gain chattering due to the high gain controller design. The system matrices for the Riccati equation Eq. 3.37 are:

$$A_e = \begin{bmatrix} 0 & 1 & 0 \\ -25 & -10 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad Q_e = \begin{bmatrix} 30.25 & 109.65 & 0 \\ 109.65 & 397.51 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B_e = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad Re = 1 \times 10^4, \quad N_e = \begin{bmatrix} -5.5 \\ -19.93 \\ 0 \end{bmatrix}$$

The resulting shaped optimal switching surface is

$$\sigma = y_2 - 4.7221 \times z_1 - 0.0011 \times z_2 - 0.3224 \times y_1 = 0 \quad (3.48)$$
The states $z_1$ and $z_2$ are defined by a state-space realization of $\dot{Q}_{22}(s)$,

$$
\begin{bmatrix}
  \dot{z}_1 \\
  \dot{z}_2
\end{bmatrix} = \begin{bmatrix}
  0 & 1 \\
  -25 & -10
\end{bmatrix} \begin{bmatrix}
  z_1 \\
  z_2
\end{bmatrix} + \begin{bmatrix}
  0 \\
  1
\end{bmatrix} y_2
$$

The Lyapunov function of the error system (Eq. 6.9) $V = \frac{1}{2}(y_1^2 + \sigma^2)$ using the discontinuous input

$$
u = -(10|y_1| + 5|y_2| + 100|z_1| + 34|z_2| + 10)\text{sign}(\sigma)$$

finite time convergence to the sliding manifold is guaranteed. During the sliding mode, dynamics of the closed loop system are at $\{-3.1448 \pm j0.7558, -7.3564\}$. The time responses of the states (Fig 3.3) have been simulated.

### 3.4 Summary

This chapter contributes to the elimination of high frequency, high amplitude chattering caused by large state feedback gains derived by Lyapunov based recursive
backstepping controller design. Transient performance improvement and finite time convergence to the switching surface have been assured by this proposed backstepping dynamic sliding mode methodology. The proposed controllers and conventional sliding mode controller combined with recursive backstepping design have been applied to linear systems in regular form with both matched and unmatched time-varying disturbances and simulation results have been presented.
CHAPTER 4

SLIDING MODE SERVOMECHANISM DESIGN

4.1 Introduction

The linear multivariable servomechanism design objectives are to regulate a set of error variables to zero for all exogenous disturbances belonging to a specified disturbance class and to improve transient performance of the closed-loop system. This chapter considers servomechanism design using sliding mode for linear time-invariant systems with parametric and modelling uncertainties including unmodelled dynamics. Following the work in [50, 51, 73], the frequency-shaped optimal sliding mode approach based on Lyapunov based recursive backstepping design is presented to solve the robust servomechanism problem. Two fundamental servomechanism design methods are considered, [52]. The first method is based on the feedback of estimates of the disturbance states that is going to be referred as Method A and the second method, Method B, is based on the extended system driven by the output error vector. This chapter is organized as follows: servomechanism and sliding mode observer design has been described, frequency-shaped optimal sliding mode based on Lyapunov stability has been designed to eliminate parametric uncertainties and to attenuate unmodelled
dynamics for two servomechanism design methods and finally, simulation examples and some conclusions are given.

4.2 Formulation of a Servomechanism and Sliding Mode Observer

The unperturbed plant dynamics are given by a linear-time invariant system,

\[
\begin{align*}
\dot{x} &= Ax + Ew + Bu \\
\dot{w} &= Zw \\
y &= Cx + Fw \\
\bar{y} &= Gw \\
e &= y - \bar{y}
\end{align*}
\]  

(4.1)

where \( x \in \mathcal{R}^n \) is the plant state vector, \( u \in \mathcal{R}^m \) is the input vector, \( y \in \mathcal{R}^r \) an output vector, \( \bar{y} \in \mathcal{R}^r \) reference output vector, \( w \in \mathcal{R}^q \) represents a combined state for the exogenous disturbance and output reference and \( e \in \mathcal{R}^r \) represents the error vector.

The pair \((A, B)\) is controllable and the pair \((C, A)\) is observable, the matrices \( B \) and \( C \) are of full rank and the error response \( e \) is readable from system output \( y \). The composite pair,

\[
\left\{ \begin{bmatrix} C & F - G \end{bmatrix}, \begin{bmatrix} A & E \\ \cdots & \cdots & \cdots \\ U & Z \end{bmatrix} \right\}
\]

is observable.
Young et al. ([50, 51, 73]) presented the steady-state state $\bar{x}$ and control input $\bar{u}$ as a linear function of the disturbance $w$,

$$\bar{x} = Xw$$  \hspace{1cm} (4.2)

$$\bar{u} = Uw$$  \hspace{1cm} (4.3)

The matrices $X$ and $U$ satisfy the algebraic equations,

$$AW - XZ + BU = E$$  \hspace{1cm} (4.4)

$$CX = F - G$$  \hspace{1cm} (4.5)

and deviations of the state and the control vector from their respective 'steady-state' values are defined, i.e.,

$$\Delta x \equiv x - \bar{x} = x - Xw$$  \hspace{1cm} (4.6)

$$\Delta u \equiv u - \bar{u} = u - Uw$$  \hspace{1cm} (4.7)

The feedback control law $\Delta u = K\Delta x$ is applied to the system relative to the steady-state system,

$$\Delta \dot{x} = A\Delta x + B\Delta u$$  \hspace{1cm} (4.8)

and the internal stability and output regulation are guaranteed if $\Delta x \rightarrow 0$ and $\Delta u \rightarrow 0$. The control problem can be defined as to regulate the error vector by a feedback controller such that internal stability and output regulation are assured.

4.2.1 Sliding Mode Observer Design

The order of the observer is reduced for the system Eq. 4.1 through a transformation to obtain directly $r$ outputs of the system as a part of its $n$ dimensional state
Using a transformation

\[ T = \begin{bmatrix} C \\ R \end{bmatrix} \]

where \( R \in \mathbb{R}^{n-r \times n} \) is arbitrary provided that \( T \) is invertible. Defining \( x_1 = Rx \), the equation of the composite system (Eq. 4.1) is written in the space \((x_1, y, w)\)

\[
\begin{align*}
\dot{y} &= A_{110} y + A_{120} x_1 + E_1 w + B_{10} u \\
\dot{x}_1 &= A_{210} y + A_{220} x_1 + E_2 w + B_{20} u \\
\dot{w} &= Z w \tag{4.9}
\end{align*}
\]

where

\[
T A T^{-1} = \begin{bmatrix} A_{110} & A_{120} \\ A_{210} & A_{220} \end{bmatrix}, \quad TE = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \quad \text{and} \quad TB = \begin{bmatrix} B_{10} \\ B_{20} \end{bmatrix} \tag{4.10}
\]

A sliding mode observer to reconstruct the state vector and exogenous disturbance \( w \) is given as,

\[
\begin{align*}
\dot{\tilde{y}} &= A_{110} \tilde{y} + A_{12} \eta + B_1 u + L_1 sgn(\tilde{y}) \\
\dot{\eta} &= A_{21} \tilde{y} + A_{22} \eta + B_2 u + L_2 sgn(\tilde{y})
\end{align*}
\]

where

\[
\begin{align*}
A_{12} &= \begin{bmatrix} A_{120} & E_1 \end{bmatrix}, & A_{22} &= \begin{bmatrix} A_{220} & E_2 \\ 0 & Z \end{bmatrix}, \\
B_2 &= \begin{bmatrix} B_{20} \\ 0 \end{bmatrix}, & A_{21} &= \begin{bmatrix} A_{210} \\ 0 \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\eta &= \begin{bmatrix} x_1 \\ w \end{bmatrix}^T, & \tilde{y} &= y - \tilde{y}, & \tilde{\eta} &= \eta - \tilde{\eta}
\end{align*}
\]

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The error dynamics is given by,

\[
\begin{align*}
\dot{y} &= A_{110}\ddot{y} + A_{12}\ddot{n} - L_1 \text{sgn}(\ddot{y}) \\
\dot{n} &= A_{21}\ddot{y} + A_{22}\ddot{n} - L_2 \text{sgn}(\ddot{y})
\end{align*}
\]  

(4.11)

(4.12)

An observer sliding manifold is introduced,

\[ S = \{\dot{n} = y - \dot{y} = 0\} \]  

(4.13)

Stability of the observer dynamics is guaranteed by a candidate Lyapunov function,

\[ V = \frac{1}{2}\ddot{y}^2 \geq 0 \]

Choosing \( L_1 \geq ||A_{110}||_2||\ddot{y}||_2 + ||A_{12}||_2||\ddot{n}||_2, \)

\[
\frac{dV}{dt} = \ddot{y}^2 = \ddot{y} (A_{110}\ddot{y} + A_{12}\ddot{n} - L_1 \text{sgn}(\ddot{y})) \leq 0
\]

(4.14)

along the error trajectories, sliding mode is established and \( \ddot{y} \rightarrow 0 \). In sliding mode the equivalent value of the first observer input is calculated as follows,

\[ [L_1 \text{sgn}(\ddot{y})]_{eq} = A_{12}\ddot{n} \]

substituting \( \ddot{y} = 0 \) and \( \ddot{y} = 0 \) into Eq. 4.12, the error dynamics in sliding mode is given by,

\[
\begin{align*}
\dot{n} &= A_{21}\ddot{y} + A_{22}\ddot{n} - L_2 L_1^{-1} [L_1 \text{sgn}(\ddot{y})] \\
&= A_{22}\ddot{n} - L_2 L_1^{-1} A_{12}\ddot{n} \\
&= (A_{22} - L_2 L_1^{-1} A_{12}) \ddot{n}
\end{align*}
\]

(4.15)

The eigenvalues of the error dynamics \((A_{22} - L_2 L_1^{-1} A_{12})\) can be assigned arbitrarily by appropriate selection of the \( L_2 \) matrix. The error dynamics converge to zero in the sliding mode sense and the plant state \( x \) and exogenous disturbance \( w \) are observed.
4.3 Frequency-Shaped Optimal Sliding Mode Servomechanism

Let the plant parameters $A$, $B$, $C$ of Eq. 4.1 be perturbed, i.e., $A \rightarrow A + \delta A$, $B \rightarrow B + \delta B$, $C \rightarrow C + \delta C$ for a bounded parameter variation $\delta$. And the perturbed system relative to the steady-state system (Eq. 4.8) can be represented,

$$\Delta \dot{x} = A \Delta x + B \Delta u + D(\Delta x, t)$$  \hspace{1cm} (4.16)$$

The system (Eq. 4.16) can be given in the regular form since $\text{rank}(B) = m$, and matrix $B$ can be partitioned as $[54, 69, 70]$,

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

where $B_1 \in \mathcal{R}^{n-m \times m}$, $B_2 \in \mathcal{R}^{m \times m}$ with $\det(B_2) \neq 0$. The nonsingular coordinate transformation

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Tx \quad T = \begin{bmatrix} I_{n-m} & -B_1 B_2^{-1} \\ 0 & B_2^{-1} \end{bmatrix}$$

reduce the system equations Eq. 4.16 into regular form:

$$\begin{bmatrix} \Delta \dot{x}_1 \\ \Delta \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \Delta u + \begin{bmatrix} D_1(\Delta x_1, \Delta x_2, t) \\ D_2(\Delta x_1, \Delta x_2, t) \end{bmatrix}$$  \hspace{1cm} (4.17)$$

where $\Delta x_1 \in \mathcal{R}^{n-m}$, $\Delta x_2 \in \mathcal{R}^m$, $\Delta u \in \mathcal{R}^m$, the matrices are real, of compatible dimensions, $B_2$ is of full rank and the functions $D_1(\Delta x_1, \Delta x_2, t)$, $D_2(\Delta x_1, \Delta x_2, t)$ represent the system nonlinearities and uncertainties. The control goal is to make the states asymptotically globally stable $\Delta x_1$, $\Delta x_2$ using the sliding mode control design methodology.

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The disturbances are assumed to satisfy,

\[ \| D_1(\Delta x_1, \Delta x_2, t) \|_2 \leq d_{11} \| \Delta x_1 \|_2 + d_{12} \| \Delta x_2 \|_2 \]

\[ \| D_2(\Delta x_1, \Delta x_2, t) \|_2 \leq d_{21} \| \Delta x_1 \|_2 + d_{22} \| \Delta x_2 \|_2 \quad (4.18) \]

where \( \| x \|_2 = (x^T x)^{1/2} \) for all \( x \in \mathcal{R}^n \).

4.3.1 Method A: The Feedback of Estimates of The Disturbance States

Define the error variables \( y_1 \equiv \Delta x_1 \) and \( y_2 \equiv \Delta x_2 - \alpha_1(\Delta x_1) \) where \( \Delta x_2 = \alpha_1(\Delta x_1) \) with \( \alpha_1(0) = 0 \), is a smooth stabilizing state feedback control for the system given in Eq. 4.16. It is included to compensate for the uncertainty \( D_1(\Delta x_1, \Delta x_2, t) \). The proposed Lyapunov function, \( V_1 = \frac{1}{2} y_1^T y_1 \) and its first-order time derivative along the trajectory of Eq. 4.17 are

\[
\dot{V}_1(y_1) = \frac{\partial V_1}{\partial y_1} y_1 = y_1^T (A_{11} y_1 + A_{12} \Delta x_2 + D_1(\Delta x_1, \Delta x_2, t)) \\
\leq \lambda_{\min}(A_{11} - (\mathcal{K} - d_{11})) \| y_1 \|_2^2 + d_{11} \| y_1 \|_2 + d_{12} d \| y_1 \|_2^2 \\
\leq -k_1 \| y_1 \|_2^2 \quad (4.19)
\]

for all \( \Delta x_1 \in \mathcal{R}^n \), \( \Delta x_2 \in \mathcal{R}^m \) and \( k_1 > 0 \) is satisfied by using the state feedback,

\[ \Delta x_2 = -A_{12}^+(\mathcal{K} - d_{11}) y_1 \quad (4.20) \]

where \( \lambda_{\min}(A_{11} - (\mathcal{K} - d_{11})) < -d_{11} - d_{12} d < 0 \), \( A^+ \) is the pseudo-inverse of the matrix \( A \) and \( d \equiv \| A_{12}^+(\mathcal{K} - d_{11}) \|_2 \). Then the time derivative of the error variable \( y_1 \)
and the error variable $y_2$ are given by

$$
\dot{y}_1 = A_{11}y_1 + A_{12}(y_2 - \alpha_1(x_1)) + D_1(y_1, t) \tag{4.21}
$$

$$
= (A_{11} - (\mathcal{K} - d_{11}))y_1 + A_{12}y_2 + D_1(y_1, t)
$$

Using a change of coordinates $(\Delta x_2) \leftrightarrow (y_1, y_2)$ and the triangle inequality,

$$
||x_2||_2 \leq ||y_2||_2 + ||A^+_{12}(\mathcal{K} - d_{11})||_2||y_1||_2
$$

The error variable $y_2 \equiv \Delta x_2 - A^+_{12}(\mathcal{K} - d_{11})y_1$ and its time derivative

$$
\dot{y}_2 = [A_{21} + A^+_{12}(\mathcal{K} - d_{11})A_{11}] y_1 + [A_{22} + A^+_{12}(\mathcal{K} - d_{11})A_{12}] [y_2 - \alpha_1(y_1)]
$$

$$
+ B_2 u + A^+_{12}(\mathcal{K} - d_{11})[D_1(\Delta x_1, \Delta x_2, t)] + D_2(\Delta x_1, \Delta x_2, t)
$$

the error dynamics derived to stabilize the system given in Eq. 4.17,

$$
\dot{y}_1 = [A_{11} - (\mathcal{K} - d_{11})]y_1 + A_{12}y_2 + D_1(\Delta x_1, \Delta x_2, t) \tag{4.22}
$$

$$
\dot{y}_2 = [A_{21} + A^+_{12}(\mathcal{K} - d_{11})A_{11}] y_1 + [A_{22} + A^+_{12}(\mathcal{K} - d_{11})A_{12}] [y_2 - \alpha_1(y_1)]
$$

$$
+ B_2 u + A^+_{12}(\mathcal{K} - d_{11})[D_1(\Delta x_1, \Delta x_2, t)] + D_2(x_1, x_2, t) \tag{4.23}
$$

The frequency-shaped optimal switching surface is considered as in Section 3, and defining sliding surface in terms of the error dynamics using

$$
\sigma = y_2 + c_1 y_1 + c_2 z \tag{4.24}
$$

where $c_1$ and $c_2$ are determined by the transfer function $\hat{Q}_{22}(s)$, by the error dynamics, Eq. 4.22, and the solution of the Riccati Equation given in Eq. 3.37 The Lyapunov function of the system given in Eq. 4.22

$$
V_2 = \frac{1}{2}y_1^T y_1 + \frac{1}{2}z^T z + \frac{1}{2}\sigma^T \sigma
$$

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the time derivative of the Lyapunov function

\[ \dot{V}_2 \leq y_1^T (A_{11} - (K - d_{11})) y_1 + A_{12} y_2 + D_1(y_1, t) + z^T [Fz + Gy_2] \]

\[ + \sigma^T(c_1((A_{11} - (K - d_{11})) y_1 + A_{12} y_2 + D_1(x_1, t)) \]

\[ + [A_{21} + A_{12}^T(K - d_{11})A_{11}] y_1 + [A_{22} + A_{12}^T(K - d_{11})A_{12}] [y_2 - \alpha_1(y_1)] \]

\[ + B_2 u + A_{12}^T(K - d_{11}) [D_1(x_1, t)] + D_2(x_1, t, t) + c_2(Fz + Gy_2) \] (4.25)

a discontinuous control input to

\[ u = -B_2^{-1} \left[ M \| y \|_2 + \Delta \right] \text{sign}(\sigma) \] (4.26)

where \( y = [y_1 \ y_2]^T \) and \( \Delta > 0, \tilde{M} > 0 \) are fairly high constants such that a sliding mode on the sliding surface \( \sigma = 0 \) is guaranteed. If the sliding mode exists: \( \sigma = 0, y_2 = -c_1 y_1 - c_2 z \) and the time derivative of the Lyapunov function given in Eq. 4.35 can be derived (through straight-forward algebraic manipulations)

\[ \dot{V}_2 \leq -(k_1 - d_{11} - \phi_1 - \phi_2) \| y_1 \|_2^2 + \phi_3 \| z \|_2^2 \]

\[ = - \left[ \begin{array}{cc} \| y_1 \|_2^2 & k_1 - d_{11} - \phi_1 - \phi_2 \\ \| z \|_2 & -\phi_3 \end{array} \right] \left[ \begin{array}{c} y_1 \\ z \end{array} \right] \]

Choosing \( k_1 > d_{11} + \phi_1 + \phi_2 + \frac{\phi_4}{4\phi_3} \) yields

\[ \dot{V}_2 < -c \left[ \| y_1 \|_2^2 + \| z \|_2^2 \right] \leq -W(y_1, z) \] (4.27)

where \( \phi_i, i = 1, 2, 3, 4 \) denotes the Euclidean norm of the derived terms in Eq. 4.35 when \( \sigma = 0, c > 0 \) and \( W(y_1, z) \) is a continuous positive semidefinite function.

The state feedback gain \( k_1 \) is chosen such that Eq. 4.37 is satisfied then the error system (Eq. 4.31) is exponentially stable. From Barbalat Lemma, \( W(y_1, z) \to 0 \) as \( t \to \infty \) This implies \( y_i = 0, i = 1, 2 \) as \( t \to \infty \) and \( \sigma = 0 \) as \( t \to \infty \)
Therefore, the stability of the composite system along the dynamic sliding surface \( \sigma = 0 \) is guaranteed.

### 4.3.2 Method B: The Extended System Driven by The Output Error Vector

Error driven dynamical system is defined as \( q \)-dimensional system in [52],

\[
\Delta \dot{x}_c = Z \Delta x_c + JC \Delta x
\]

where \( J \) is chosen such that the pair \( (J, Z) \) is controllable. The error driven composite system may be given in the regular form,

\[
\begin{bmatrix}
\Delta \dot{x}_c \\
\Delta \dot{x}
\end{bmatrix} =
\begin{bmatrix}
Z & -JC \\
-BU & A
\end{bmatrix}
\begin{bmatrix}
\Delta x_c \\
\Delta x
\end{bmatrix} +
\begin{bmatrix}
0 \\
B
\end{bmatrix} \Delta u +
\begin{bmatrix}
D_1(\Delta x_c, \Delta x, t) \\
D_2(\Delta x_c, \Delta x, t)
\end{bmatrix}
\]

(4.28)

where \( \Delta x \in \mathcal{R}^n, \Delta x_c \in \mathcal{R}^q, \Delta u \in \mathcal{R}^m \), the matrices are real, of compatible dimensions, \( B \) is of full rank and the functions \( D_1(\Delta x_c, \Delta x, t), D_2(\Delta x_c, \Delta x, t) \) represent the system nonlinearities and uncertainties. The control goal is to make asymptotically globally stable the state \( \Delta x, \Delta x_c \) using sliding mode control design methodology.

Define the error variable \( y_1 = \Delta x_c \) and \( y_2 = \Delta x - \alpha_1(\Delta x_c) \) where \( \Delta x = \alpha_1(\Delta x_c) \) with \( \alpha_1(0) = 0 \) is a smooth stabilizing state feedback control of the system given in Eq. 4.28 included to compensate for the uncertainty \( D_1(\Delta x_c, \Delta x, t) \). A Lyapunov function, \( V_1 = \frac{1}{2} y_1^T y_1 \) and its first-order time derivative along the trajectory of Eq. 4.17

\[
\dot{V}_1(y_1) = \frac{\partial V_1}{\partial y_1} y_1 = y_1^T (Zy_1 + JC\Delta x + D_1(\Delta x_c, \Delta x, t))
\]

\[
\leq \lambda_{\min}(Z - (K - d_{12}))||y_1||_2^2 + d_{12}||y_1||_2^2 + d_{12}\delta||y_1||_2^2
\]

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\[ \leq -k_1 \|y_1\|^2 \quad (4.29) \]

for all \( \Delta x \in \mathbb{R}^n, \Delta x \in \mathbb{R}^n \) and \( k_1 > 0 \) is satisfied by using the state feedback,

\[ \Delta x = -(JC)^T_d (K - d_{12}) y_1 \quad (4.30) \]

where \( \lambda_{\text{min}}(Z - (K - d_{12})) < -d_{11} - d_{12} \delta < 0 \) and \( A^+ \) is the pseudo-inverse of the matrix \( A \). Then the time derivative of the error variable \( y_1 \) and the error variable \( y_2 \) are given by

\[
\dot{y}_1 = Z\Delta x + JC\Delta x + D_1(\Delta x_c, \Delta x, t) \\
= Zy_1 + JC(y_2 - \alpha_1(y_1)) + D_1(\Delta x_c, \Delta x, t) \\
= (Z - (K - d_{12})) y_1 + JCy_2 + D_1(\Delta x_c, \Delta x, t)
\]

The error variable \( y_2 \equiv \Delta x - (JC)^+(K - d_{12}) y_1 \) and its time derivative

\[
\dot{y}_2 = A\Delta x + Bu + D_2(\Delta x_c, \Delta x, t) \\
+ (JC)^+(K - d_{12}) [JC\Delta x + Z\Delta x_c + D_1(\Delta x_c, \Delta x, t)] \\
= (A + (JC)^+(K - d_{12})JC) \Delta x + (JC)^+(K - d_{12})Z\Delta x_c + D_2(\Delta x_c, \Delta x, t) \\
+ (JC)^+(K - d_{12})D_1(\Delta x_c, \Delta x, t) + Bu
\]

Define the sliding surface in terms of the error dynamics using

\[ \sigma = y_2 + c_1 y_1 + c_2 z \quad (4.33) \]

where \( c_1 \) and \( c_2 \) are determined by the transfer function \( \bar{Q}_{22}(s) \), by the error dynamics, Eq. 4.22, and the solution of the Riccati Equation given in Eq. 3.37 The Lyapunov function of the system given in Eq. 4.31

\[ V_2 = \frac{1}{2} y_1^T y_1 + \frac{1}{2} z^T z + \frac{1}{2} \sigma^T \sigma \quad (4.34) \]
the time derivative of the Lyapunov function

\[
\dot{V}_2 \leq y_1^T [(Z - (K - d_{12}))y_1 + JCy_2 + D_1(\Delta x_c, \Delta x, t)]
+ z^T [Fz + Gy_2] + \sigma^T(c_1((Z - (K - d_{12}))y_1 + JCy_2 + D_1(\Delta x_c, \Delta x, t))
+ (A + (JC)^+(K - d_{12})JC) \Delta x + (JC)^+(K - d_{12})Z \Delta x_c + D_2(\Delta x_c, \Delta x, t)
+ (JC)^+(K - d_{12})D_1(\Delta x_c, \Delta x, t) + B\Delta u + c_2(Fz + Gy_2)) \tag{4.35}
\]

a discontinuous control input to

\[
u = -B_2^{-1} \left[ \tilde{M} \|y\|_2 + \tilde{\Delta} \right] \text{sign}(\sigma) \tag{4.36}
\]

where \(y = [y_1 \ y_2]^T\) and \(\tilde{\Delta} > 0, \tilde{M} > 0\) are fairly high constants such that a sliding mode on the sliding surface \(\sigma = 0\) is guaranteed. If the sliding mode exists: \(\sigma = 0, y_2 = -c_1y_1 - c_2z\) and the time derivative of the Lyapunov function given in Eq. 4.35 can be derived (through straight-forward algebraic manipulations)

\[
\dot{V}_2 \leq -(k_1 - d_{12} - \phi_1) - \phi_2)\|y_1\|_2^2 - \phi_3\|z\|_2^2 + \|y_1\|_2(\phi_4 + \phi_5)\|z\|_2
= - \begin{bmatrix} \|y_1\|_2 \\ \|z\|_2 \end{bmatrix}^T \begin{bmatrix} k_1 - d_{12} - \phi_1 - \phi_2 & -(\phi_4 + \phi_5)/2 \\ -(\phi_4 + \phi_5)/2 & \phi_3 \end{bmatrix} \begin{bmatrix} \|y_1\|_2 \\ \|z\|_2 \end{bmatrix}
\]

Choosing \(k_1 > d_{12} + \phi_1 + \phi_2 + \frac{(\phi_4 + \phi_5)^2}{4\phi_3}\) yields

\[
\dot{V}_2 < -c \left(\|y_1\|_2^2 + \|z\|_2^2 \right) \leq -W(y_1, z) \tag{4.37}
\]

where \(\phi_i, i = 1, 2, 3, 4, 5\) denotes the Euclidean norm of the derived terms in Eq. 4.35 when \(\sigma = 0, c > 0\) and \(W(y_1, z)\) is a continuous positive semidefinite function.

The state feedback gain \(k_1\) is chosen such that Eq. 4.37 is satisfied then the error system (Eq. 4.31) is exponentially stable. From Barbalat Lemma, \(W(y_1, z)\) \(\to 0\) as \(t \to \infty\) This implies \(y_1 = 0, i = 1, 2\) as \(t \to \infty\) and \(\sigma = 0\) as \(t \to \infty\)
Therefore, the stability of the composite system along the dynamic sliding surface \( \sigma = 0 \) is guaranteed.

4.4 Simulation Results

For example purposes, the following servomechanism problem is chosen,

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-2 & -3
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} u + \begin{bmatrix}
0 \\
1
\end{bmatrix} w + \begin{bmatrix}
0.5x_1 + 0.3x_2 \\
-0.3x_1 + 0.7x_2
\end{bmatrix} \tag{4.38}
\]

\[
y = [1 1] \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}^T, \quad \dot{\eta}_1 = 0, \quad w = \eta_1, \quad \bar{y} = [1 0] \eta_2
\]

\[
\dot{\eta}_2 = \begin{bmatrix}
0 & 0 \\
-1 & 0
\end{bmatrix} \eta_2
\]

The steady-state gains for the unperturbed system Eq. 4.38 from the Eq. 4.2, Eq. 4.4, Eq. 4.6 are determined as \( X = [1 1]^T, \quad U = 5 \).

4.4.1 Method A

Defining the error variables \( y_1 = \Delta x_1 \) and \( y_2 = \Delta x_2 + 5\Delta x_1 \) and following the recursive backstepping steps, the error dynamics are obtained,

\[
\begin{align*}
\dot{y}_1 &= -5y_1 + 2y_2 \\
\dot{y}_2 &= 13y_1 - 3y_2 + \Delta u
\end{align*} \tag{4.39}
\]

A high-pass characteristic for \( \hat{Q}_{22}(w) \) with corner frequencies of 1 and 10 rad s\(^{-1}\), and a 40dB per decade slope is selected, whereas a unity weighting for \( Q_{11} \), i.e.

\[
Q_{11} = 1, \quad \hat{Q}_{22}(w) = \frac{(jw + 2.25)^2}{(jw + 5)^2} \tag{4.40}
\]

The gain value is chosen such that a 40 dB weighting is applied to the high frequencies.
Figure 4.1: The time response of the states

in $x_2$, and a 0 dB weighting for low frequencies to ensure that the optimal control action avoids the high frequency disturbances and eliminates high frequency, high gain chattering due to the high gain controller design. The system matrices for the Riccati equation Eq. 3.37 are:

$$
A_e = \begin{bmatrix}
  0 & 1 & 0 \\
  -25 & -10 & 0 \\
  0 & 0 & -5
\end{bmatrix}, \quad Q_e = \begin{bmatrix}
  30.25 & 109.65 & 0 \\
  109.65 & 397.51 & 0 \\
  0 & 0 & 1
\end{bmatrix}
$$

$$
B_e = \begin{bmatrix}
  0 \\
  1 \\
  1
\end{bmatrix}, \quad R_e = 1 \times 10^4, N_e = \begin{bmatrix}
  -5.5 \\
  -19.93 \\
  0
\end{bmatrix}
$$

The resulting shaped optimal switching surface is

$$
\sigma = y_2 - 5.3932 \times z_1 - 0.0464 \times z_2 + 0.1048 \times y_1 = 0
$$
The states \( z_1 \) and \( z_2 \) are defined by a state-space realization of \( \tilde{Q}_{22}(s) \),

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-25 & -10
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2
\end{bmatrix} +
\begin{bmatrix}
0 \\
1
\end{bmatrix} y_2
\]

The Lyapunov function of the error system (Eq. 4.39) \( V = \frac{1}{2}(y_1^2 + \sigma^2) \) using the discontinuous input

\[
u = -(10|y_1| + 5|y_2| + 10|z_1| + 25|z_2| + 10) \text{sign}(\sigma)
\]

finite time convergence to the sliding manifold is guaranteed.

### 4.4.2 Method B

Choosing \( J = [1 \ 1]^T \) controllability condition of \((J, Z)\) can be satisfied for the given system Eq. 4.38.

Defining the error variables

\[
y_1 \equiv \eta \ , \ y_2 \equiv x + \begin{bmatrix}
5 & 1 \\
-1 & 6
\end{bmatrix} y_1
\]

the error dynamics are obtained,

\[
\dot{y}_1 = \begin{bmatrix}
-5 & 0 \\
0 & -6
\end{bmatrix} y_1 + \begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix} y_2
\]

\[
\dot{y}_2 = \begin{bmatrix}
4 & 7 \\
-8 & -13
\end{bmatrix} y_1 + \begin{bmatrix}
0 & 1 \\
-2 & -3
\end{bmatrix} y_2 + \begin{bmatrix}
0 \\
1
\end{bmatrix} \Delta u
\]

Choosing a high-pass characteristic for \( \tilde{Q}_{22}(w) \) with corner frequencies of 1 and 10 \( \text{rads}^{-1} \), and a 40dB per decade slope is selected, whereas a unity weighting for \( Q_{11} \),

\[
Q_{11} = 1, \quad \tilde{Q}_{22}(w) = \frac{(jw + 2.25)^2}{(jw + 5)^2}
\]
The gain value is chosen such that a 40 dB weighting is applied to the high frequencies in $x_2$, and a 0 dB weighting for low frequencies to ensure that the optimal control action avoids the high frequency disturbances and eliminates high frequency, high gain chattering due to the high gain controller design. The system matrices for the Riccati equation Eq. 3.37 are:

$$A_e = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -25 & -10 & 0 & 0 & 0 \\ 0 & 0 & -5 & 0 & 1 \\ 0 & 0 & 0 & -6 & 1 \\ 0 & 0 & 4 & 7 & 0 \end{bmatrix}, \quad Q_e = \begin{bmatrix} 30.25 & 109.65 & 0 & 0 & 0 \\ 109.65 & 397.51 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B_e = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad Re = 1 \times 10^4, \quad N_e = \begin{bmatrix} -5.5 \\ -19.93 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The resulting shaped optimal switching surface is

$$\sigma = y_2 - 0.4819 \times z_1 + 0.3528 \times z_2 + 0.9502 \times y_1 + 1.4237 \times y_2 + 1.6353 \times y_3 = 0$$

The Lyapunov function of the error system (Eq. 4.39) $V = \frac{1}{2}(y_1^2 + \sigma^2)$ using the discontinuous input

$$u = -(10\|y_1\|_2 + 5\|y_2\|_2 + 10|z_1| + 25|z_2| + 10)sign(\sigma) \quad (4.45)$$

finite time convergence to the sliding manifold is guaranteed.

4.5 Summary

This chapter considers the robust servomechanism design based on Lyapunov stability. Transient performance improvement and finite time convergence to the switching surface have been assured by this proposed backstepping dynamic sliding mode
methodology. The parametric uncertainties, unmodeled dynamics and uncertain exogenous disturbance belonging to a specified class are attenuated by using compensator dynamics introduced in sliding mode through a class of switching surfaces which has the interpretation of linear operators.
CHAPTER 5

OUTPUT FEEDBACK CONTROL

5.1 Introduction

In the previous chapters, sliding mode controller design has been limited to full-state feedback of the plant or their estimated values. In many practical applications, the state variables are not accessible for direct measurement or the number of sensors is limited. At controller design stage asymptotic observers ([47]) and sliding mode observers ([38, 69, 70]) have been proposed to reconstruct the state vector from their estimated state feedback. Even the state vectors are available from the observer outputs, their error dynamics between estimated and real states convergence to the zero asymptotically or exponentially.

5.2 Output Feedback Sliding Mode Design

Output feedback sliding mode controller design has generated considerable interest and output feedback sliding mode existence and reaching conditions have been studied by considering the system properties such as controllability and observability of the reduced order sliding mode dynamics. Consider the linear time-invariant multivariable system (Eq. 1.5) and the sliding manifold \( s = Gy \), using the equivalent
control method ([69]), the sliding mode dynamics can be given,

\[ \dot{x} = (A - B(GCB)^{-1}GCA) \]

The sliding manifold dynamics can be assigned without any constraint as in the full state feedback case when \( \text{rank}[C] = l = n \). On the other hand, if \( \text{rank}[C] = l = m \) such that the independent number of outputs is equal to the number of inputs, the sliding surface and dynamics are determined by the system matrices \( \{A, B, C\} \) and the sliding manifold design constant matrix \( G \) does not have any effect on its dynamics. This analysis concludes that the design freedom in designing sliding mode dynamics using output feedback is constrained by the \( (l - m) \) order, the rest of the reduced order dynamics can not be preassigned by any design freedom, [41, 70].

Since the linear system in regular form is suitable for sliding mode output feedback controller design, existence and reaching conditions of the sliding mode output feedback controller have been investigated by considering the system given by Eq. 1.5 in regular form, (see for instance, [8, 25, 26, 28, 29, 30, 48, 49, 79]).

In [79], the geometric approaches have been introduced to determine an output gain \( G \) such that a desired set of system zero locations govern the sliding mode dynamics. For existence conditions it has been introduced an appropriate transformation matrix such that \( GCA = MC \), (Eq. 5.1), and the reaching conditions have been satisfied in terms of the available outputs. Following this work, in [48, 49] different matched disturbances as function of the states instead of the outputs were considered and the discontinuous controller term was modified taking into consideration the upperbounds of the decay rate of the unmeasurable states. Sliding mode output feedback controller design for mismatched uncertainties have been addressed in [62] by considering the transformations in [79] and determining the unmatched disturbance's
upperbounds by less than the norm of the minimum eigenvalue of the reduced order system.

In [22, 23, 24], referring to the Eq. 5.1, same number of independent inputs and outputs were considered and since the rank condition does not satisfy the closed loop pole assignability, it has been shown that through adding the dynamic compensator, a closed loop system can be stabilized if the plant is minimum-phase. Reaching conditions were addressed as an extension of the work [71, 41].

The observability and the controllability of the reduced order system in the sliding mode and reaching conditions have been considered in [28, 30]. Using the system in regular form, it has been shown that the controllability of the reduced order system can be satisfied due to the controllability of the system (see, [69]) but the observability is not guaranteed and it should be checked. They proposed also to use the dynamic compensator in case Kimura-Davison condition is not satisfied for the reduced order system under the assumption that the nominal linear system \( \{A, B, C\} \) has no invariant zeros and the sub-block of the system in regular form \( A_{12} \) satisfies some rank conditions, [29].

In [8, 25, 26], the reduced order dynamics were partitioned into the uncompensated and compensated parts when Kimura-Davison rank condition does not hold. The unobservable (uncompensated) part is constituted by the system zeros and it has been required that the invariant zeros of the system under consideration is stable. Multivariable nonlinear systems with no zero dynamics and with some parametric uncertainty are transformed into a normal form and using high-gain observer, stabilization about an equilibrium point at the origin was considered in [57]. Sliding
manifolds have been determined by using the solution linear matrix inequalities existence conditions for a system with unmatched disturbances under the condition that the upperbounds of the disturbances are to be less than the minimum eigenvalue of the reduced order system [15].

5.3 Output Feedback Variable Structure Control Design Using Dynamic Compensation for Linear Systems

In this chapter, the dynamic output sliding mode controller design is presented. Let the plant,

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
0 \\
0
\end{bmatrix} u +
\begin{bmatrix}
0 \\
D(x_1, x_2, t)
\end{bmatrix}
\]

\[y = C_1 x_1 + C_2 x_2 \tag{5.2}\]

where \(x_1 \in \mathbb{R}^n\), \(x_2 \in \mathbb{R}^m\), \(u \in \mathbb{R}^m\), \(y \in \mathbb{R}^l\) the matrices are real, of compatible dimensions, \(B_2\) and \(C_2\) are full rank matrices and the functions \(D(x_1, x_2, t)\) represent the system nonlinearities and uncertainties.

If the system is observable, controllable and satisfies Kimura-Davison rank condition \(n \leq m + l - 1\) then the system is pole assignable by output feedback control. Otherwise, additional dynamics, inputs and outputs may be used [25, 29, 30, 63] if the uncompensated part of the system is stable or as a more restrictive condition if the system is minimum-phase.

**Assumption 1** The pair \(\{A, B\}\) is controllable and the pair \(\{A, C\}\) is observable. The input and output matrices \(B, C\) are full rank. Any invariant zeros present in the system are stable.
We consider the sliding surface for the output feedback,

\[ \sigma = SC_1x_1 + L(x_2) \]  

(5.3)

where \( L(\cdot) : \mathcal{R}^m \to \mathcal{R}^m \) is a linear operator which has a realization as a transfer function

\[ (sI + F)z = v \]  

(5.4)

as a dynamic system

\[ \dot{z} = Fz + v \]

\[ v = Hz + SC_2x_2 \]  

(5.5)

The composite system is given,

\[ \dot{z} = (F + H)z + SC_2x_2 \]  

(5.6)

\[ \dot{x}_1 = A_{11}x_1 + A_{12}x_2 \]  

(5.7)

\[ \dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2u + D(x_1, x_2, t) \]  

(5.8)

The disturbances are assumed to satisfy,

\[ \|D(x_1, x_2, t)\|_2 \leq \|D(x_1, t)\|_2 + \|D(x_2, t)\|_2 \]

\[ \leq d_1\|x_1\|_2 + d_2\|x_2\|_2 \]  

(5.9)

where \( \|x\|_2=(x^T x)^{1/2} \) for all \( x \in \mathcal{R}^n \).

Define sliding surface in terms of the outputs

\[ \sigma = SC_1x_1 + SC_2x_2 + Hz \]  

(5.10)

If the sliding mode exists: \( \sigma = 0 \),

\[ x_2 = -(SC_2)^{-1} [SC_1x_1 + Hz] \]  

(5.11)
The equation of sliding mode dynamics,

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{z}
\end{bmatrix} = \begin{bmatrix}
A_{11} - A_{12}(SC_2)^{-1}SC_1 & A_{12}(SC_2)^{-1}F \\
SC_1 & H
\end{bmatrix} \begin{bmatrix}
x_1 \\
z
\end{bmatrix}
\]  
(5.12)

and the poles of the error dynamics can be placed by the selection of \{S, H, F\} if \((A_{11}, A_{12})\) is a controllable pair. The above system may be written as,

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{z}
\end{bmatrix} = \begin{bmatrix}
A_{11} & I \\
0 & 0
\end{bmatrix} \times \begin{bmatrix}
I & 0 \\
0 & -A_{12}(SC_2)^{-1}H
\end{bmatrix} \begin{bmatrix}
x_1 \\
z
\end{bmatrix}
\]
\[
+ \begin{bmatrix}
A_{12} & 0 \\
0 & I
\end{bmatrix} \begin{bmatrix}
(SC_2)^{-1}SC_1 & 0 \\
SC_1 & F
\end{bmatrix} \begin{bmatrix}
x_1 \\
z
\end{bmatrix}
\]

and its poles can be placed, if the pair

\[
\begin{bmatrix}
A_{11} & I \\
0 & 0
\end{bmatrix}, \begin{bmatrix}
A_{12} & 0 \\
0 & I
\end{bmatrix}
\]

is controllable. The controllability of the above pair is evidently the same as of \((A_{11}, A_{12})\) as claimed, [69]. But the observability of \((A_{11}, C_1)\) can not be determined by the observability condition of the original pair \((A, C)\) and the observability of the reduced order system has to be checked.

**Assumption 2** [25, 30, 29] The pair \((A_{11}, C_1)\) is observable.

Stability of the sliding motion may be shown,

\[
V = \frac{1}{2}x_1^T x + \frac{1}{2}z^T z
\]

(5.13)

time-derivative of the proposed Lyapunov function Eq. 5.13,

\[
\dot{V} = x_1^T (A_{11} - A_{12}(SC_2)^{-1}SC_1) x_1 + z^T F z
\]
\[
- x_1^T A_{12}(SC_2)^{-1}Hz - z^T SC_1 x_1
\]
\[
\leq -k_1\|x_1\|_2^2 - k_2\|z\|_2^2 + \phi\|x_1\|_2\|z\|_2
\]
\[
= - \begin{bmatrix}
\|x_1\|_2 \\
\|z\|_2
\end{bmatrix}^T \begin{bmatrix}
k_1 & -\phi/2 \\
-\phi/2 & k_2
\end{bmatrix} \begin{bmatrix}
\|x_1\|_2 \\
\|z\|_2
\end{bmatrix}
\]  
(5.14)
Choosing,

\[ k_2 > \frac{\phi^2}{4k_1} \]  \hspace{1cm} (5.15)

yields

\[ \dot{V} < -c \left[ ||x_1||_2^2 + ||z||_2^2 \right] \]  \hspace{1cm} (5.16)

for some \( c \) positive constant.

**Remark:** Since it has been assumed that if the system has any invariant zeros, they are stable, Eq. 5.14 can be always derived bounded by some positive definite function.

### 5.3.1 Controller Design

The output sliding mode control input has to be designed in terms of the available output signals. The control input may be realized by using the knowledge of sliding manifold vector, [25, 30]. The sliding manifold is considered as in Eq. 5.10 and the state \( x_2 \) can be given in terms of the other variables,

\[ x_2 = -(SC_2)^{-1}SC_1x_1 - (SC_2)^{-1}Hz + (SC_2)^{-1}\sigma \]  \hspace{1cm} (5.17)

the time-derivative of the sliding manifold is derived,

\[
\dot{\sigma} = [SC_1A_{11} + SC_2A_{21} - (SC_1A_{12} + SC_2A_{22})(SC_2)^{-1}SC_1]x_1 \\
+ [H(F - H) - (SC_1A_{12} + SC_2A_{22})(SC_2)^{-1}H]z \\
+ [(SC_1A_{12} + SC_2A_{22})(SC_2)^{-1} + H]\sigma \\
+ SC_2D(x_1, x_2, t) + SC_2B_2u
\]  \hspace{1cm} (5.18)
the augmented system,

\[ \begin{align*}
\dot{x}_1 &= (A_{11} - A_{12}(SC_2)^{-1}SC_1)x_1 - A_{12}(SC_2)^{-1}Hz + A_{12}(SC_2)^{-1}\sigma \\
\dot{z} &= Fz - SC_1x_1 + \sigma \\
\dot{\sigma} &= [SC_1A_{11} + SC_2A_{21} - (SC_1A_{12} + SC_2A_{22})(SC_2)^{-1}SC_1]x_1 \\
&\quad + [H(F - H) - (SC_1A_{12} + SC_2A_{22})(SC_2)^{-1}H]z \\
&\quad + [(SC_1A_{12} + SC_2A_{22})(SC_2)^{-1} + H]\sigma \\
&\quad + SC_2D(x_1, x_2, t) + SC_2B_2u
\end{align*} \]  

(5.19)  
(5.20)  
(5.21)  

The proposed Lyapunov function may be proposed,

\[ V = \frac{1}{2}x_1^T x_1 + \frac{1}{2}z^T z + \frac{1}{2}\sigma^T \sigma \]  

(5.22)  

its time-derivative can be given,

\[ \dot{V} = -\left[ \begin{array}{c} x_1 \\ z \end{array} \right]^T \left[ \begin{array}{cc} A_{11} - A_{12}(SC_2)^{-1}SC_1 & A_{12}(SC_2)^{-1}H \\ SC_1 & F \end{array} \right] \left[ \begin{array}{c} x_1 \\ z \end{array} \right] \\
+ \sigma^T (H(F - H) - (SC_1A_{12} + SC_2A_{22})(SC_2)^{-1}H + I) z \\
+ \sigma^T (SC_1A_{11} + SC_2A_{21} - (SC_1A_{12} + SC_2A_{22})(SC_2)^{-1}SC_1) x_1 \\
+ x_1^T A_{12}(SC_2)^{-1}\sigma \\
+ \sigma^T ((SC_1A_{12} + SC_2A_{22})(SC_2)^{-1} + H) \sigma \\
+ \sigma^T SC_2D(x_1, x_2, t) + \sigma^T SC_2B_2u \]  

(5.23)  
(5.24)  
(5.25)  
(5.26)  
(5.27)  
(5.28)

Define

\[ \begin{align*}
\Gamma_{33} &\equiv (SC_1A_{12} + SC_2A_{22})(SC_2)^{-1} + H \\
\Gamma_{23} &\equiv (H(F - H) - (SC_1A_{12} + SC_2A_{22})(SC_2)^{-1}H + I) \\
\Gamma_{13} &\equiv A_{12}(SC_2)^{-1} \\
\Gamma_{31} &\equiv (SC_1A_{11} + SC_2A_{21} - (SC_1A_{12} - SC_2A_{22})(SC_2)^{-1}SC_1)
\end{align*} \]  

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The output feedback control input satisfying the asymptotical global stability of the augmented system may be considered in terms of the available outputs,

\[ u = -(SC_2B_2)^{-1} \left[ \Gamma_{33}\sigma + \Gamma_{23}z + \frac{\phi_{31}\phi_{13}\sigma_{\text{max}}}{\lambda_{\text{min}}\sigma_{\text{min}}} + M\text{sign}(\sigma) \right] \quad (5.29) \]

where \( \phi_{i,j} = ||\Gamma_{i,j}||_2 \) for \( i, j = 1, 2, 3 \) and \( \sigma_{\text{max}}, \sigma_{\text{min}} \) denote the maximum and minimum singular values of a nonsingular matrix \( T \in \mathbb{R}^{(n+p-m)\times(n+p-m)} \) that diagonalizes the reduced order system in sliding mode given by Eq. 5.12, Eq. 5.23, and \( M > ||SC_2D(x_1, x_2, t)|| + \Psi \) and \( \Psi > 0 \) are chosen to satisfy asymptotical stability of the augmented system given by Eq. 5.19, Eq. 5.20 and Eq. 5.21.

5.3.2 Example

The example is chosen from [70],

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
1 \\
0
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix} \quad (5.30)
\]

\[
\begin{bmatrix}
y_1 \\
y_2 \\
y_3
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} \quad (5.31)
\]

In [70], it is shown that even the original system (Eq. 5.33, Eq. 5.34) is controllable, observable and Kimura-Davison rank condition for pole assignability is satisfied. But the reduced order dynamics do not satisfy Kimura-Davison condition and it is not pole assignable.

Using this system (Eq. 5.33), we will show that even adding compensator dynamics to the reduced order system, arbitrarily pole assignability is not possible. The minimum dynamic compensator order may be found to be \( p \geq 1 \), [63],

\[ p \geq \frac{n - ml}{m + l - 1} \quad (5.32) \]
The original system (Eq. 5.33, Eq. 5.34) is going to be extended by second order compensator dynamics as given in Eq. 5.5, the compensated system is derived,

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2 \\
\dot{z}_3 \\
\dot{z}_4
\end{bmatrix} = 
\begin{bmatrix}
-10 & 0 & 0 & 0 & 1 & 0 \\
0 & -10 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} 
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} + 
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} 
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix} \tag{5.33}
\]

\[
\begin{bmatrix}
y_{z1} \\
y_{z2} \\
y_1 \\
y_2 \\
y_3
\end{bmatrix} = 
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} 
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} \tag{5.34}
\]

Choose the dynamic sliding surface as,

\[
\sigma = \begin{bmatrix} S_z : S_1 : I_m \end{bmatrix} \tag{5.35}
\]

when the sliding mode exists; \( \sigma = 0, x_2 = -S_z z - S_1 x_1 \), the reduced order sliding mode dynamics can be given,

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = 
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} 
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + 
\begin{bmatrix}
1 & 0 \\
2 & 0
\end{bmatrix} 
\begin{bmatrix}
s_1 & 0 \\
0 & s_2
\end{bmatrix} 
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + 
\begin{bmatrix}
1 & 0 \\
2 & 0
\end{bmatrix} 
\begin{bmatrix}
s_{z1} & 0 \\
0 & s_{z2}
\end{bmatrix} 
\begin{bmatrix}
z_1 \\
z_2
\end{bmatrix} \tag{5.36}
\]

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{bmatrix} = 
\begin{bmatrix}
-10 & 0 \\
0 & -10
\end{bmatrix} 
\begin{bmatrix}
z_1 \\
z_2
\end{bmatrix} + 
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} 
\begin{bmatrix}
s_{z1} & 0 \\
0 & s_{z2}
\end{bmatrix} 
\begin{bmatrix}
z_1 \\
z_2
\end{bmatrix} + 
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} 
\begin{bmatrix}
s_1 & 0 \\
0 & s_2
\end{bmatrix} 
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} \tag{5.37}
\]
The reduced order dynamics can be written,
\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2 \\
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix}
= \begin{bmatrix}
-10 & 0 & 0 & 0 \\
0 & -10 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2 \\
x_1 \\
x_2
\end{bmatrix}
+ \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
2 & 0 & 2 & 0
\end{bmatrix}
\begin{bmatrix}
s_{z1} & 0 & 0 & 0 \\
0 & s_{z2} & 0 & 0 \\
0 & 0 & s_1 & 0 \\
0 & 0 & 0 & s_2
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2 \\
x_1 \\
x_2
\end{bmatrix}
\] (5.38)

The augmented reduced order sliding mode dynamics with output feedback gains \(\bar{A}\) is derived,
\[
\bar{A} = \begin{bmatrix}
-10+s_{z1} & 0 & s_1 & 0 \\
0 & -10+s_{z2} & 0 & s_2 \\
s_{z1} & 0 & s_1 & 0 \\
2s_{z1} & 0 & 1+2s_1 & 0
\end{bmatrix}
\] (5.39)

The closed-loop poles are the solutions of the algebraic equation,
\[
det(\lambda I - \bar{A}) = 0
\] (5.40)

This equation has the form,
\[
\lambda^4 + \beta_1 \lambda^3 + \beta_2 \lambda^2 + \beta_3 \lambda + \beta_4 = 0
\] (5.41)

where
\[
\begin{align*}
\beta_1 &= 20 - s_{z1} - s_{z2} - s_1 \\
\beta_2 &= 99 - 12s_1 - 10s_{z1} - 10s_{z2} + s_{z1}s_{z2} \\
\beta_3 &= -20 - 40s_1 + s_{z1} + s_{z2} - 8s_1s_{z2} + s_1s_{z2}^2 \\
\beta_4 &= -100 - 200s_1 + 10s_{z1} + 10s_{z2} - s_{z1}s_{z2} + 20s_1s_{z2}
\end{align*}
\] (5.42)

In this equation set (Eq. 5.42) only \(s_1\), \(s_{z1}\) and \(s_{z2}\) are free parameters, only three poles may be assigned arbitrarily.
5.4 Concluding Remarks

If the reduced order sliding mode dynamics do not satisfy Kimura-Davison rank condition, by adding compensator dynamics, the poles of the sliding mode motion can not be assigned arbitrarily. The degrees of freedom for pole assignability can not be increased to full state order through expanding the sliding mode dynamics. In terms of the class of plants to be controlled, the dynamic compensation design does not improve stability of the closed system compared to the static output feedback sliding mode control. Dynamic compensator design may improve the transient performance of the closed loop system and attenuate high-frequency, high-gain control inputs for achieving sliding mode.
6.1 Slosh Reduction

6.1.1 Problem Motivation

Control and handling of heavy commercial vehicles carrying liquid cargo are influenced by the moving liquid within the partially filled tank. During steering and braking maneuvering tasks, truck may exhibit unstable behavior at lateral acceleration levels of 0.3 to 0.4g [m/sec^2]. The fluid slosh forces and dynamic load transfer in lateral and longitudinal directions and parameter variations and uncertainties caused by moving liquid cargo affect the overall dynamics of the vehicle.

6.1.2 Controller Design

The proposed frequency-shaped backstepping sliding mode algorithm is designed to stabilize and to attenuate the sloshing effects of the moving cargo by properly choosing the crossover frequencies of the dynamic compensators in accordance with the fundamental frequencies of the slosh dynamics. It is assumed that the front and rear brakes are controlled independently which is very common vehicle dynamics control technique for heavy trucks (see for instance [14, 27]). At controller design stage,
front traction torque is considered as input to attenuate lateral sloshing effects and rear traction input is considered to attenuate longitudinal slosh effects while reference longitudinal velocity tracking is satisfied through a Proportional-Integral speed controller. The slosh dynamics integrated into the vehicle dynamics and related parameter's variations are given in Appendix A.

The slosh dynamics with lateral acceleration of the vehicle as a input is given,

\[ \dot{\xi}_1 = \xi_2 + f(\xi_1, t) \]  
\[ \dot{\xi}_2 = -w_1^2 \xi_1 - (\lambda_1 a B_1/A_1) [-U r + \dot{Z}_p + \frac{1}{m + m_t} ((F_{y1} + F_{y2})\cos(\delta) \]  
\[ + (F_{y3} + F_{y4}) - (F_{x1} + F_{x2})\sin(\delta) - A_{p} V^2 \text{sgn}(V))] \]  

where \( \xi_1, \xi_2 \) represent the slosh height and velocity, respectively and \( f(\xi_1, t) = \xi_1 (\sin(w_1 t) + \sin(w_2 t) + \sin(w_3 t) + \sin(w_4 t)) \) represent the higher frequencies of slosh height as an additive disturbance. \( U \) is the longitudinal, \( \dot{Z} \) vertical velocity, \( p \) pitch, \( r \) yaw rate of the vehicle, respectively. \( F_y \) represent lateral tire force, \( F_x \) represents traction force and \( \delta \) is the steering input, \( A_{p} \) is the aerodynamic resistance of the vehicle.

A tanker filled \( 50 \% \) by diesel oil of density \( \rho = 0.0103 \text{ kg/m}^3 \), the vehicle and the equivalent model parameters given in Table 6.1 chosen for controller design. For the lateral slosh dynamics, the first three sloshing modes in lateral directions are calculated in Table 6.2, (see [12, 59]).

The control goal is defined as to attenuate the lateral and longitudinal slosh dynamics integrated into the vehicle by independently actuating front and rear brakes to improve control and handling of the vehicle. The lateral and longitudinal slosh
Table 6.1: Vehicle and spring-mass system parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1.7424 (m)</td>
<td>Distance btw. the tractor CG and the tractor front tires</td>
</tr>
<tr>
<td>b</td>
<td>4.2676 (m)</td>
<td>Distance btw. the tractor CG and the tractor rear tires</td>
</tr>
<tr>
<td>e</td>
<td>1.2 (m)</td>
<td>Distance btw. the pinpoint and the tractor tires</td>
</tr>
<tr>
<td>R</td>
<td>1.1 (m)</td>
<td>Tank radius</td>
</tr>
<tr>
<td>$m_p$</td>
<td>7277 (kg)</td>
<td>The moving mass inside the tanker</td>
</tr>
<tr>
<td>$m_0$</td>
<td>3566 (kg)</td>
<td>The stationary liquid mass</td>
</tr>
<tr>
<td>$k_p$</td>
<td>42636 (N/m)</td>
<td>The spring stiffness of the liquid mass</td>
</tr>
<tr>
<td>$c_p$</td>
<td>5000 (N/(m.sec^2))</td>
<td>The viscous friction of the liquid inside the tank</td>
</tr>
<tr>
<td>$z_p$</td>
<td>0.9105 (m)</td>
<td>The vertical height of the moving liquid mass</td>
</tr>
<tr>
<td>$z_0$</td>
<td>0.01 (m)</td>
<td>The vertical height of the stationary liquid mass</td>
</tr>
<tr>
<td>$I_{10}^l$</td>
<td>5915 (kgm^2)</td>
<td>Moment of inertia in the longitudinal axis</td>
</tr>
<tr>
<td>$I_{10}^l$</td>
<td>13515 (kgm^2)</td>
<td>Moment of inertia in the lateral axis</td>
</tr>
<tr>
<td>$I_{10}^z$</td>
<td>13067 (kgm^2)</td>
<td>Moment of inertia in the vertical axis</td>
</tr>
</tbody>
</table>

Table 6.2: Lateral pendulum model parameters for the first three sloshing modes.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>3.4826 (rad/sec)</td>
<td>The natural frequency for the 1th slosh mode</td>
</tr>
<tr>
<td>$w_2$</td>
<td>6.4742 (rad/sec)</td>
<td>The natural frequency for the 2th slosh mode</td>
</tr>
<tr>
<td>$w_3$</td>
<td>8.4255 (rad/sec)</td>
<td>The natural frequency for the 3th slosh mode</td>
</tr>
<tr>
<td>$\lambda_1a$</td>
<td>1.36, $A_1=0.48$</td>
<td>Parameters determined by the liquid fill level for the 1th slosh mode</td>
</tr>
<tr>
<td>$B_1$</td>
<td>0.4</td>
<td></td>
</tr>
<tr>
<td>$\lambda_2a$</td>
<td>4.70, $A_2=0.50$</td>
<td>Parameters determined by the liquid fill level for the 2th slosh mode</td>
</tr>
<tr>
<td>$B_2$</td>
<td>0.04</td>
<td></td>
</tr>
<tr>
<td>$\lambda_3a$</td>
<td>7.96, $A_3=0.50$</td>
<td>Parameters determined by the liquid fill level for the 3th slosh mode</td>
</tr>
<tr>
<td>$B_3$</td>
<td>0.02</td>
<td></td>
</tr>
</tbody>
</table>

Dynamics are described in the previous chapters as unmodelled dynamics and using frequency-shaped backstepping controller design, robustness and asymptotically globally stability against both matched and unmatched uncertainties are satisfied while minimally exciting the unmodelled dynamics. The lateral and longitudinal dynamics are excited with lateral and longitudinal acceleration or deceleration of the vehicle.

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Frequency-shaping backstepping design is considered to attenuate the slosh dynamics integrated into vehicle dynamics. Considering lateral slosh dynamics, the error dynamics are obtained by defining the error variables $y_1 \equiv \xi_1$ and $y_2 \equiv \xi_2 + 100z_1$. Following the recursive backstepping steps,

$$
\dot{y}_1 = -100y_1 + y_2 + y_1 (\sin(w_1 t) + \sin(w_2 t) + \sin(w_3 t) + \sin(w_4 t)) \quad (6.3)
$$

$$
\dot{y}_2 = -10012.12y_1 + 100y_2 + 1.13(-U\tau + \dot{Z}_p) + \frac{1.13}{16923} [(F_{x1} + F_{x2})\sin(\delta) + (F_{y1} + F_{y2})\cos(\delta) + F_{y3} + F_{y4} - 0.5V^2\text{sgn}(V)] \quad (6.4)
$$

A high-pass characteristic for $\tilde{Q}_{22}(w)$ with corner frequencies of 0.5 and 1.5 rad$s^{-1}$, and a 40dB per decade slope is selected, whereas a unity weighting for $Q_{11}$, i.e.

$$
Q_{11} = 1, \quad \tilde{Q}_{22}(w) = \frac{100(jw + 0.5)^2}{(jw + 1.5)^2} \quad (6.5)
$$

The gain value is chosen such that a 40 dB weighting is applied to the high frequencies in $x_2$, and a 0 dB weighting for low frequencies to ensure that the optimal control action avoids the high frequency disturbances and eliminates high frequency, high gain chattering due to the high gain controller design. The system matrices for the Riccati equation Eq.3.37 are:

$$
A_e = \begin{bmatrix}
0 & 1 & 0 \\
-2.25 & -3 & 0 \\
0 & 0 & -100
\end{bmatrix}, \quad Q_e = \begin{bmatrix}
40000 & 40000 & 0 \\
40000 & 40000 & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

$$
B_e = \begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix}, \quad Re = 1 \times 10^4, \quad N_e = \begin{bmatrix}
-20000 \\
-20000 \\
0
\end{bmatrix}
$$

The resulting shaped optimal switching surface is

$$
\sigma = y_2 + 1.9118 \times z_1 + 1.9215 \times z_2 + 196.0785 \times y_1 = 0 \quad (6.6)
$$
The states $z_1$ and $z_2$ are defined by a state-space realization of $\mathcal{Q}_{22}(s)$,
\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-2.25 & -3
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2
\end{bmatrix} +
\begin{bmatrix}
0 \\
1
\end{bmatrix} y_2
\]

The Lyapunov function of the error system (Eq.6.9) $V = \frac{1}{2}(y_1^2 + \sigma^2)$ using the discontinuous input
\[
\sigma = -(1200|y_1| + 1200|y_2| + 700|z_1| + 700|z_2| + 100)\text{sgn}(\sigma)
\]
finite time convergence to the sliding manifold is guaranteed. During the sliding mode, dynamics of the closed loop system are at $\{-1.1671, -2.4701, -297.2843\}$. The time responses of the slosh height, lateral velocity, roll rate and frequency responses are plotted in Fig 6.1, Fig 6.2, Fig 6.4, Fig 6.9 and Fig 6.10, respectively.

For longitudinal slosh dynamics, the sloshing angular frequency $\omega = 0.6379$ rad/sec. The desired velocity trajectory is followed using a Proportional-Integral (PI) controller, the sloshing effects are attenuated through the frequency-shaped nonlinear controller.

\begin{align}
\dot{x}_{p1} &= x_{p2} \\
\dot{x}_{p2} &= -\frac{c_p}{m_p} x_{p2} - \frac{k_p}{m_p} x_{p1} - V r + \ddot{z} q + \frac{1}{m + m_l} (F_{x1} + F_{x2}) \\
&\quad + \frac{1}{m + m_l} (F_{x3} + F_{x4} + k_p x_{p1} - A_p U^2 \text{sgn}(U) + g \sin(\Theta_T))
\end{align}

Defining the error variables $y_1 \equiv x_{p1}$ and $y_2 \equiv x_{p2} + 100 x_{p1}$ and following the recursive backstepping steps, the error dynamics are obtained,
\[
\dot{y}_1 = -100 y_1 + y_2
\]
\[
\dot{y}_2 = 62.85y_1 - 0.68y_2 + \frac{1}{16923}(F_{x3} + F_{x4}) + Vr - \dot{Z}_r
\]

\[
-\frac{1}{16923}[-(F_{x1} + F_{x2}) + (F_{y1} + F_{y2})\sin(\delta) + k_p x_p + 0.5U^2 \text{sgn}(U)]
\]

A high-pass characteristic for \(\tilde{Q}_{22}(w)\) with corner frequencies of 0.1 and 0.35 \(\text{rads}^{-1}\), and a 40dB per decade slope is selected, whereas a unity weighting for \(Q_{11}\), i.e.

\[
Q_{11} = 1, \quad \tilde{Q}_{22}(w) = \frac{100(jw + 0.1)^2}{(jw + 0.35)^2}
\]

The system matrices for the Riccati equation Eq.3.37 are:

\[
A_e = \begin{bmatrix}
0 & 1 & 0 \\
-0.1225 & -0.7 & 0 \\
0 & 0 & -100
\end{bmatrix}, \quad Q_e = \begin{bmatrix}
126.56 & 562.5 & 0 \\
562.5 & 126.56 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
B_e = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad Re = 1 \times 10^4, Ne = \begin{bmatrix} -1125 \\ -5000 \\ 0 \end{bmatrix}
\]

The resulting shaped optimal switching surface is

\[
\sigma = y_2 + 0.1113 \times z_1 + 0.4953 \times z_2 + 199.0047 \times y_1 = 0
\]

The states \(z_1\) and \(z_2\) are defined by a state-space realization of \(\tilde{Q}_{22}(s)\),

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-0.1225 & -0.7
\end{bmatrix} \begin{bmatrix}
z_1 \\
z_2
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} y_2
\]

The Lyapunov function of the error system (Eq.6.9) \(V = \frac{1}{2}(y_1^2 + \sigma^2)\) using the discontinuous input

\[
u = -(1750|y_1| + 1750|y_2| + 500|z_1| + 500|z_2| + 200) \text{sgn}(\sigma)
\]

finite time convergence to the sliding manifold is guaranteed. During the sliding mode, dynamics of the closed loop system are at \{-0.2666, -0.5984, -299.3351\}. The time
responses of the longitudinal velocity, suspension forces, front traction torque inputs are simulated and plotted together with the conventional cases, when there is not controller, Fig 6.3, Fig 6.6, Fig 6.7, Fig 6.8.

The simulation scenario is steering and braking maneuvering task of the tanker vehicle filled by %50 diesel oil. The longitudinal velocity profile is chosen as braking \((0.1 \, g/\text{sec}^2\) constant deceleration) command is applied at \(t=12\, \text{sec}\) of the simulation scenario followed by steering input of \(\delta=15\, \text{deg}\) to represent typical maneuvering tasks of heavy commercial vehicle such as taking highway exits, collision avoidance.

![Graph showing lateral slosh height with sliding mode controller and without controller.](image)

Figure 6.1: Lateral slosh height of the liquid with sliding mode controller (solid) and without controller (dashed)
Figure 6.2: Lateral velocity of the vehicle with sliding mode controller (solid) and without controller (dashed)

Figure 6.3: Longitudinal velocity of the vehicle with sliding mode controller (solid) and without controller (dashed): Longitudinal speed reference with constant deceleration (solid)
Figure 6.4: Roll rate of the vehicle with sliding mode controller (solid) and without controller (dashed)

Figure 6.5: Corner heights of the vehicle with sliding mode controller (solid) and without controller (dashed)
Figure 6.6: Suspension forces of the vehicle with sliding mode controller (solid) and without controller (dashed)

Figure 6.7: Front traction torque of the vehicle with sliding mode controller (solid) and without controller (dashed)
Figure 6.8: Rear traction of the vehicle with sliding mode controller (solid) and without controller (dashed)

Figure 6.9: Frequency responses of slosh height of liquid and lateral velocity of the vehicle with sliding mode controller (solid) and without controller (dashed)
Figure 6.10: Frequency responses of roll angle and longitudinal velocity of the vehicle with sliding mode controller (solid) and without controller (dashed)

6.2 Drive by Wire Hydraulic Power Steering System

6.2.1 Problem Motivation

The National Highway Transportation Safety Administration (NHTSA) Variable Dynamics Test Vehicle (VDTV) was designed to be a test-bed for the study of driver behavior, driver-vehicle interactions, crash avoidance maneuvers, advanced safety systems, and Intelligent Transportation system technologies. It is equipped with a range of computer controlled subsystems, the performance of each which can be adjusted such that the dynamics of the VDTV matches a range (from economy to luxury class) of production vehicles. The original specifications for the VDTV related to steering and lateral vehicle dynamics can be grouped into two categories: specifications on vehicle dynamics and specifications on steering subsystem performance.
To satisfy the performance, functionality, and safety issues a modified hydraulic rack and off-the-shelf motor system with sufficient torque capacity will be integrated into the existing VDTV steering assembly, and a robust rack position control loop integrated into the existing VDTV control software. As a controlled mechanical subsystem, hydraulic power steering rack position control loop design is challenging. The hydraulic power steering system model is uncertain and it involves nonlinearities such as Coulomb friction and power steering pressure boost curves. Additionally, the steering system has the steering wheel torque input alone as the control input for the control of the corresponding eight states. The control objective is to satisfy the rack displacement robustly track a reference signal derived proportionally to the steering wheel angle during emergency maneuvering tasks.

6.2.2 Controller Design

The vehicle model and the drive by wire hydraulic power steering system model used for simulation purposes is given Appendix B, and the related parameters are given in Table 6.3. The vehicle model is a nonlinear two-wheel (bicycle) model, and the steering system model, the controller and the observer are tested using this model. In this application, a sliding mode controller cascaded by the frequency shaped optimal controller for drive by wire hydraulic power steering system utilizing sliding mode and exponentially convergent observers is presented. Using the robustness implications of the sliding mode control theory and the structural properties of the hydraulic power steering system, a nonlinear controller cascaded by an optimal linear controller is designed to stabilize the steering system dynamics and track the steering wheel reference. Lyapunov based controller design satisfies strong robustness with respect to
bounded modeling and parameter uncertainties. The controller and the observer are based on an eight-order nonlinear state-space model of the hydraulic power steering system which is developed, validated numerically with test data. Simulation results are included to demonstrate the effectiveness of the observer and the performance of the tracking controller.

The proposed controller is formed by two parts: a sliding mode controller as an outer loop controller which satisfies robust stabilization and reference tracking of the steering rack displacement state versus the bounded modeling and parametric uncertainties and a frequency shaped optimal controller is proposed as an inner loop controller to satisfy reference tracking of the steering wheel angle. The reference for the steering rack displacement is obtained to be proportional to the reference steering wheel angle by,

$$\Delta R_{ref} = R_p \delta_{wref}$$  \hspace{1cm} (6.13)

where $R_p$ is the pinion radius. With this reference generation, all the nonlinearities caused by the gear and the other disturbances are omitted and an ideal reference rack displacement proportional to the steer wheel reference is derived for the steering rack dynamics. Sliding controller is designed by choosing the sliding manifold,

$$s = (\Delta R_1 - \Delta R_{1ref}) + (\dot{\Delta} R_1 - \dot{\Delta} R_{1ref})$$  \hspace{1cm} (6.14)

where $\Delta R_{1ref}, \dot{\Delta} R_{1ref}$ are assumed to be bounded. The candidate Lyapunov function $V = \frac{1}{2} s^2$ and its time derivative are given,

$$\dot{V} = \left[ (\Delta R_1 - \Delta R_{1ref}) + (\dot{\Delta} R_1 - \dot{\Delta} R_{1ref}) \right]$$

$$\times \left[ (\Delta R_2 - \Delta R_{1ref}) + (\dot{\Delta} R_2 - \dot{\Delta} R_{1ref}) \right]$$

$$= s \left( (\Delta R_2 - \Delta R_{1ref} + a \delta_{sw1} - b \Delta R_1 \right)$$
\[ -c\Delta R_2 - d\phi + e(\delta_{wl} + \delta_{wr}) + e(\delta_{wl} + \delta_{wr}) + \eta_{ps} F_{ps}(\delta_{value}) + \eta_{ps} F_{ps}(\delta_{value}) - C_{FR} \frac{2}{\pi} \tan (\mathcal{K}\Delta R_2) + \Phi(\cdot) \] (6.15)

where \( \hat{\Delta R}_{1\text{ref}} \), \( \hat{\Delta R}_{1\text{ref}} \) are assumed to be bounded, \( \Phi(\cdot) \) represents lumped parametric uncertainties, \( \mathcal{K} \) is a high constant to approximate the \( \text{sgn}(\cdot) \) function [40], \( \bar{x} \) represents the observer error of the system states and the constant are defined,

\[ a \equiv \eta_{F} \frac{K_{sc} K_{ib}}{M R (K_{sc} + K_{ib})}, \quad b \equiv \frac{1}{M R} \left( \eta_{F} \frac{K_{sc} K_{ib}}{R_p (K_{sc} + K_{ib})} + 2\eta_{sc} \frac{K_{sc}}{N_L^2} \right), \quad c \equiv \frac{B_R}{M R}, \quad d \equiv \frac{2\eta_{sc} K_{sc} e}{M R N_L} \text{ and } e \equiv \eta_{sc} \frac{K_{sc}}{M R N_L}. \]

If \( \delta_{sw1} \) is chosen as a virtual input, the stabilizing control law is obtained by,

\[ \delta_{sw1\text{ref}} = -\frac{1}{a} \left[ -c\Delta R_2 - b\Delta R_1 - d\phi + e(\delta_{wl} + \delta_{wr}) + \eta_{ps} F_{ps}(\delta_{value}) - \mathcal{M}_1 s^3 \right. \]

\[ \left. -C_{FR} \frac{2}{\pi} \tan (\mathcal{K}\Delta R_2) - \mathcal{M}_2 \text{sgn}(s) + Ca \left( \Delta R_2 - \hat{\Delta R}_{1\text{ref}} \right) \right] \] (6.16)

where \( \mathcal{M}_1 s^3 \) is a nonlinear damping term to prevent finite escape time and \( \mathcal{M}_2 \text{sgn}(s) \) is the discontinuous input stabilizing the steering rack dynamics to follow reference signals against observer error terms and parametric uncertainties. A frequency shaped linear quadratic regulator is used as an inner loop controller to track the desired steering wheel angle generated by the sliding mode controller while eliminating high frequency, high gain steering wheel torque. Since the steering wheel torque is generated by a dc motor, it has to be bounded by its torque capacity and time constant.

The frequency-dependent performance index (Eq. 3.33) can be converted into a standard constant weighting matrix through an augmentation to the steering wheel.
dynamics Eq. B.1 with additional compensator states and dynamics which are defined by the frequency shaping matrices, [37, 76]. The linear time-invariant system associated with the frequency shaping cost be

\[ \dot{\delta_{sw}} = A\delta_{sw} + BT_{sw} + E\Delta_R \]  

(6.17)

where \( \delta_{sw} \) represent steering wheel position and angular velocity, \( T_{sw} \) steering wheel torque and \( \Delta_R \) rack displacement and velocity. An equivalent quadratic cost with \( Q \) constant weighting and \( R(w) \) frequency dependent matrices are defined with respect to an augmented state system, which is composed of Eq. 3.35 and the system,

\[ \dot{z}_2 = A_2z_2 + B_2T_{sw} \]

\[ y_2 = C_2z_2 + D_2T_{sw} = \tilde{T}_{sw} \]  

(6.18)
which defines a state-space realization of the transfer function

\[ \tilde{R}(s) = D_2 + C_2 (s I - A_2)^{-1} B_2 \]  \hspace{1cm} (6.19)

where \( \tilde{T}_{sw} = \tilde{R}(s) T_{sw} \), and \( \tilde{R}(w) \equiv \tilde{R}(s) \bigg|_{s=jw} \) is the frequency shaping for \( T_{sw} \).

A high-pass characteristic for \( R(w) \) with corner frequencies of 5 and 100 \( \text{rads}^{-1} \), and a 40dB per decade slope is selected, whereas a unity weighting for \( Q \), i.e.

\[ Q = 1, \quad R(w) = \frac{225(jw + 5)^2}{(jw + 100)^2} \]  \hspace{1cm} (6.20)

The gain value is chosen such that a 40 dB weighting is applied to the high frequencies and a 0 dB weighting for low frequencies to ensure that the optimal control action avoids the high frequency disturbances causing high-frequency oscillations.

The equivalent cost and augmented state system are:

\[ J = \int_0^\infty \left[ x_e^T Q_e x_e + 2x_e^T N_e u + u^T R_e u \right] dt \]  \hspace{1cm} (6.21)

where

\[ \dot{x}_e = A_e x_e + B_e u \]
\[ x_e \equiv [z_1 \ z_2 \ \delta_{sw1} \ \delta_{sw2}]^T, \]
\[ A_e = \text{diag} (A_2, A), \quad B_e = \begin{bmatrix} B_2 \\ B \end{bmatrix}, \quad R_e = D_2^T D_2 \]
\[ Q_e = \text{diag} (C_2^T C_2, Q), \quad N_e = \begin{bmatrix} C_2^T D_2 \\ 0 \end{bmatrix} \]

The LQ optimal control solution for a frequency dependent cost function is obtained by solving the Riccati equation for a quadratic cost with cross state and control weighting term. The optimal control solution is found to be,

\[ u = - (-9972.1 z_1 - 189.9 z_2 - 0.0236 \delta_{sw1} - 2 \times 10^{-4} \delta_{sw2}) \]  \hspace{1cm} (6.22)
6.2.3 Observer Design

Observer Design for The Left and Right Road-Wheel

The Left and Right Road-Wheel systems are minimum phase system and state measurements are not available. The road-wheel observer structure is given as a copy of the road-wheel state equations given in Eq. B.3, or Eq. B.4,

\[ \dot{\delta}_{wrl} = \dot{\delta}_{wrl} \]
\[ \dot{\delta}_{wrl} = \frac{K_{sl}}{I_{sl} N_l} \Delta R_l - \frac{K_{sl}}{I_{sl}} \delta_{wrl} + \frac{K_{sl}}{I_{sl}} \frac{X_T C}{I_{sl}} \left( \delta_{wrl} - \tan^{-1} \left( \frac{V_y + \gamma \phi}{V_x} \right) \right) \]
\[ - \frac{B_{sl}}{I_{sl}} \dot{\delta}_{wrl} - \frac{CF_w}{I_{sl}} \text{sgn} \left( \delta_{wrl} \right) \]

The error variables are defined by,

\[ \tilde{\delta}_{wrl} = \delta_{wrl} - \dot{\delta}_{wrl} \]
\[ \tilde{\delta}_{wrl} = \delta_{wrl} - \dot{\delta}_{wrl} \]

The error dynamics are given,

\[ \dot{\tilde{\delta}}_{wrl} = \tilde{\delta}_{wrl} \]
\[ \dot{\tilde{\delta}}_{wrl} = \frac{K_{sl}}{I_{sl}} \tilde{\delta}_{wrl} - \frac{X_T C}{I_{sl}} \tilde{\delta}_{wrl} - \frac{B_{sl}}{I_{sl}} \tilde{\delta}_{wrl} - CF_w \left[ \text{sgn} \left( \delta_{wrl} \right) - \text{sgn} \left( \tilde{\delta}_{wrl} \right) \right] \]

The Lyapunov function for the error dynamics,

\[ V_{obs}(\tilde{\delta}_{wrl}) = \left( \frac{K_{sl}}{I_{sl}} + \frac{X_T C}{I_{sl}} \right) \tilde{\delta}_{wrl}^2 + \tilde{\delta}_{wrl}^2 \]

The time derivative of the Lyapunov function,

\[ \dot{V}_{obs}(\tilde{\delta}_{wrl}) = \left( \frac{K_{sl}}{I_{sl}} + \frac{X_T C}{I_{sl}} \right) \tilde{\delta}_{wrl} \tilde{\delta}_{wrl} - \left( \frac{K_{sl}}{I_{sl}} + \frac{X_T C}{I_{sl}} \right) \tilde{\delta}_{wrl} \tilde{\delta}_{wrl} \]
\[ - \frac{B_{sl}}{I_{sl}} \tilde{\delta}_{wrl}^2 - CF_w \left[ \text{sgn} \left( \delta_{wrl} \right) - \text{sgn} \left( \tilde{\delta}_{wrl} \right) \right] \tilde{\delta}_{wrl} \]
\[ = - \frac{B_{sl}}{I_{sl}} \tilde{\delta}_{wrl}^2 - CF_w \left( \delta_{wrl} - \tilde{\delta}_{wrl} \right) [\text{sgn} \left( \delta_{wrl} \right) - \text{sgn} \left( \tilde{\delta}_{wrl} \right)] \]
\[ \leq - \frac{B_{sl}}{I_{sl}} \tilde{\delta}_{wrl}^2 \]
Since $|\delta_{wr2}| \geq \delta_{wr2}sgn(\delta_{wr2})$, $|\hat{\delta}_{wr2}| \geq \delta_{wr2}sgn(\delta_{wr2})$. Using Barbalat Lemma, $\dot{V}(\delta_{wr1}, \delta_{wr2})_{obs} \leq W(\delta_{wr2}) \leq 0$ where $W(\delta_{wr2})$ is positive definite function, it can be guaranteed an exponentially convergent observer error by $\hat{\delta}_{wr2} \to 0$, $\delta_{wr1} \to 0$ when $t \to \infty$. The road-wheel observer convergence rate can not be affected by any state measurement and it is determined by its parameter values. Simulation results for right road-wheel angle and its estimate are plotted in Fig. 6.14

**Observer Design for The Steering Rack Dynamics**

The following observer structure is constructed for the steering rack dynamics,

$$
\dot{\hat{\Delta}}_{R1} = \hat{\Delta}_{R2} + k_1 w_{\hat{\Delta}_{R1}} \\
\dot{\hat{\Delta}}_{R2} = \eta_F \frac{K_{sc}K_{ib}}{M_R R_p(K_{sc} + K_{ib})} \delta_{sw1} - \frac{1}{M_R} \left( \eta_F \frac{K_{sc}K_{ib}}{R_p^2(K_{sc} + K_{ib})} + 2\eta_s \frac{K_{sl}}{N_L^2} \right) \hat{\Delta}_{R1} \\
- \frac{B_R}{M_R} \hat{\Delta}_{R2} + \eta_s \frac{K_{sl}}{M_R N_L} (\hat{\delta}_{wl} + \hat{\delta}_{wr}) + \eta_s \frac{K_{sl}}{M_R N_L} (\delta_{wl} + \delta_{wr}) - \frac{2\eta_s}{M_R N_L} \epsilon \phi \\
+ \eta_{ps} F_{ps}(\hat{\delta}_{value}) + \eta_{ps} F_{ps}(\delta_{value}) - C_{FR} sgn (\hat{\Delta}_{R2}) + k_2 w_{\Delta_{R2}}
$$

(6.27)

The error variables are defined by $\Delta_{R1} = \hat{\Delta}_{R1} - \Delta_{R1}$, $\Delta_{R2} = \Delta_{R2} - \hat{\Delta}_{R2}$ And the error dynamics are given,

$$
\dot{\hat{\Delta}}_{R1} = \hat{\Delta}_{R2} - k_1 w_{\Delta_{R1}} \\
\dot{\hat{\Delta}}_{R2} = \eta_F \frac{K_{sc}K_{ib}}{M_R R_p(K_{sc} + K_{ib})} \delta_{sw1} - \frac{1}{M_R} \left( \eta_F \frac{K_{sc}K_{ib}}{R_p^2(K_{sc} + K_{ib})} + 2\eta_s \frac{K_{sl}}{N_L^2} \right) \hat{\Delta}_{R1} \\
+ \eta_s \frac{K_{sl}}{M_R N_L} (\delta_{wl} + \delta_{wr}) + \eta_{ps} F_{ps}(\delta_{value}) - C_{FR} \left[ sgn (\Delta_{R2}) - sgn (\hat{\Delta}_{R2}) \right] \\
- k_2 w_{\Delta_{R2}}
$$

(6.29)

Choosing $w_{\Delta_{R1}} = sgn(\Delta_{R1})$ and $k_1 > 0$ such that $k_1 > |\Delta_{R2}|$ then $\dot{\Delta}_{R1}\Delta_{R1} < 0$ can be enforced resulting in sliding mode on the manifold described by $\hat{\Delta}_{R1} = 0$. Then corresponding equivalent control can be calculated by letting, $\hat{\Delta}_{R1} = \hat{\Delta}_{R1} = 0$ as
\[ [\text{sgn}(\Delta_{R1})]_{eq} = \frac{\Delta_{R2}}{k_1} \]  

(6.30)

Let \( w_{\Delta_{R2}} = \text{sgn}\left([\text{sgn}(\Delta_{R1})]_{eq}\right) \). Then choosing \( k_2 \)

\[
k_2 > \eta_F \frac{K_{sc}K_{tb}}{M_R R_p(K_{sc} + K_{tb})}|\dot{\delta}_{sw1}|\eta_{ps}|F_{ps}(\delta_{value})| + \eta_{\beta} \frac{K_{st}}{M_R N_L} \left(|\dot{\delta}_{wl}| + |\delta_{wrl}|\right) + 2C_{FR}
\]

This observer gain choice ensures \( \dot{\Delta}_{R2}\Delta_{R2} < 0 \) while resulting in finite time convergence to \( \Delta_{R2} = 0 \).

Following the same observer design technique, the sliding mode observer for the steering wheel can be constructed satisfying the error converge to zero in finite time, [2, 40]. The performance of the cascaded controller and the observer combination is shown in Fig. 6.12 thru Fig. 6.18. The proposed sliding mode controller (Eq. 6.16) cascaded by LQ regulator (Eq. 6.21) and exponentially convergent observer for left, right road-wheel (Eq. 6.23) and sliding mode observer (Eq. 6.27) are simulated with the eight-order nonlinear state-space model of the drive by wire hydraulic power steering system (Eq. B.1 thru Eq. B.4) in Matlab Simulink environment. Fig. 6.12 thru Fig. 6.18 illustrate the simulation results for an emergency maneuvering operation experimental results, the simulation results are plotted as solid line, experimental test results are plotted as dashed line.

In this application, a sliding mode controller cascaded by a frequency shaped optimal controller and observer design procedure has been proposed for drive by wire hydraulic power steering system. The frequency shaped optimal controller has been used as an inner controller satisfying tracking of the steering wheel angle reference generated by a sliding mode controller and through compensator dynamics the high
Figure 6.12: Rack displacement and its estimate

Figure 6.13: Steering wheel angle and its estimate
Figure 6.14: Right (left) road-wheel angle and its estimate

Figure 6.15: Rack displacement versus experimental result
Figure 6.16: Steering wheel position versus experimental result

Figure 6.17: Hand-wheel torque versus experimental result
frequency components are attenuated while satisfying a desired reference tracking. By a proper design of a sliding mode controller, robustness with respect to bounded modeling and parameter uncertainties has been guaranteed. An exponentially convergent observer for the road-wheels and sliding mode observers for the steering wheel and the rack have been constructed.

Further studies rely more on performing the rack position control loop design and implementation using the VDTV dSpace computer system.
Table 6.3: Steering model parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_{sw}$</td>
<td>0.0344 N.m/(rad/sec^2)</td>
<td>Inertia of steering wheel</td>
</tr>
<tr>
<td>$K_{sc}$</td>
<td>42709 N.m/rad</td>
<td>Steering column stiffness</td>
</tr>
<tr>
<td>$B_{sc}$</td>
<td>0.36042 N.m/(rad/s)</td>
<td>Steering column damping</td>
</tr>
<tr>
<td>$K_{tb}$</td>
<td>83 N.m/rad</td>
<td>Torsion bar stiffness</td>
</tr>
<tr>
<td>$B_{tb}$</td>
<td>0.0 N.m/(rad/s)</td>
<td>Torsion bar damping</td>
</tr>
<tr>
<td>$M_R$</td>
<td>2 kg</td>
<td>Mass of rack</td>
</tr>
<tr>
<td>$R_p$</td>
<td>0.007367 m</td>
<td>Radius of pinion</td>
</tr>
<tr>
<td>$B_R$</td>
<td>88.128 N.m/(rad/s)</td>
<td>Rack damping</td>
</tr>
<tr>
<td>$CF_R$</td>
<td>261 N (at 22 m/s)</td>
<td>Coulomb friction on rack</td>
</tr>
<tr>
<td>$CF_R$</td>
<td>169 N (at 33.5 m/s)</td>
<td>Coulomb friction on rack</td>
</tr>
<tr>
<td>$N_L$</td>
<td>0.11816 m</td>
<td>Steering linkage ration</td>
</tr>
<tr>
<td>$K_{sl}$</td>
<td>14878 N.m/rad</td>
<td>Steering linkage stiffness</td>
</tr>
<tr>
<td>$B_{sl}$</td>
<td>40 N.m/(rad/s)</td>
<td>Steering linkage damping (25-65)</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>0.01 rad/rad</td>
<td>Roll steer coefficient</td>
</tr>
<tr>
<td>$\eta_F$</td>
<td>0.985</td>
<td>Efficiency of forward torque transm.</td>
</tr>
<tr>
<td>$\eta_B$</td>
<td>0.985</td>
<td>Efficiency of backward torque transm.</td>
</tr>
<tr>
<td>$\eta_{ps}$</td>
<td>0.95</td>
<td>Efficiency of power steering transm.</td>
</tr>
<tr>
<td>$I_{sl}$</td>
<td>0.61 N.m/(rad/sec^2)</td>
<td>Inertia of steering linkage and tire</td>
</tr>
<tr>
<td>$CF_w$</td>
<td>261 N (at 22 m/s)</td>
<td>Coulomb friction on wheel linkages</td>
</tr>
<tr>
<td>$CF_w$</td>
<td>169 N (at 33.5 m/s)</td>
<td>Coulomb friction on wheel linkages</td>
</tr>
<tr>
<td>$X_T$</td>
<td>-0.0305 m</td>
<td>Total tire trail length</td>
</tr>
<tr>
<td>$C$</td>
<td>7000 N/rad</td>
<td>Cornering stiffness (front)</td>
</tr>
</tbody>
</table>

6.3 Pneumatic Brake Output Feedback Control

6.3.1 Problem Motivation

Pneumatic actuators are characterized by high-order, time-variant actuator dynamics, nonlinearities due to compressibility of air, external disturbances such as static and Coulomb friction, and wide range of payload variations. The compressibility of air results in highly nonlinear characteristics of the pressure dynamics. As a result, conventional PID feedback controllers are not so effective for control tasks because high-order systems with more than third-order lag tend to become unstable if
the feedback gains are made fairly high, which is necessarily the case, if the nonlinear
dynamics are not compensated for.

Moreover, pneumatic drives have a limited bandwidth restricting the high gains
which can be applied. Combined with their poor damping and low stiffness which
arises from the compressibility of air, as well as significant static and Coulomb fric-
tion from moving parts, the accuracy and repeatability are limited under variations
of payload and supply pressure.

6.3.2 Controller Design

A robust output feedback sliding mode controller for pneumatic brake system is
developed. The model of pneumatic brake system and its properties are given in
Appendix section. The control goal is to follow reference pressure trajectory. The
linearized model of the pneumatic brake system is given,

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{P}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
-KI_{m} & -\frac{\nu_u}{m} & \frac{\Delta P}{m} \\
0 & 0 & x_{20}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_{20}
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
g(x, P)
\end{bmatrix} u +
\begin{bmatrix}
0 \\
-\frac{\nu_u}{m} \tanh(x_2) \\
\Delta x_2 P
\end{bmatrix}
\]

\[
y =
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
P
\end{bmatrix}
\]

(6.31)

where \( x = [x_1, x_2]^T \) and \( x_1, x_2, P \) are the displacement, velocity of the piston and
\( P \) is the pressure inside the pressure chamber. \( x_{20} \) is the linearized velocity and \( \Delta x_2 \)
is the deviation of the linearized state.

The linearized pneumatic brake system is controllable, observable and the reduced
order sliding mode dynamics are minimum phase system. The original system does
not satisfy Kimura-Davison rank condition for output feedback control design. Dy-
namic compensator design is going to be used to arbitrarily assign poles of the closed
loop system. Dynamic compensator's minimum order given in Eq. 5.32 is computed to be \( p \geq 1 \), ([63]). Dynamic compensator's order is chosen to \( p = 1 \).

Sliding manifold is chosen,

\[
\sigma = x_2 + (S_1 C_{11} + C_{21}) x_1 + Hz
\]  

(6.32)

where \( C_{21} = [0 \; 0] \), \( C_{11} = [1 \; 0] \) and \( A_{12} = [0 \; \frac{A_2}{m}]^T \).

The reduced order sliding dynamics,

\[
\begin{bmatrix}
\dot{z} \\
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
F & 0 & 0 \\
0 & 0 & 1 \\
0 & -\frac{K_1}{m} & -\frac{K_2}{m}
\end{bmatrix} \begin{bmatrix}
z \\
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
0 & F_1 & 0 \\
0 & 0 & 0 \\
H \frac{A_p}{m} & F \frac{A_p}{m} & 0
\end{bmatrix} \begin{bmatrix}
z \\
x_1 \\
x_2
\end{bmatrix}
\]  

(6.33)

where \( F, H \) are the compensator dynamics system and output matrices, respectively and \( S_1 \) is the sliding manifold design matrix. The reduced order system is observable, controllable and Kimura-Davison condition is satisfied with \( n \leq l + m^* - 1 \) where \( n = 3, l = 2 \) and \( m^* = rank[\bar{A}_{12}] = 2 \),

\[
\bar{A}_{12} = \begin{bmatrix}
0 & F_1 & 0 \\
0 & 0 & 0 \\
H \frac{A_p}{m} & F \frac{A_p}{m} & 0
\end{bmatrix}
\]  

(6.34)

6.3.3 Simulation Results

Output sliding mode controller and dynamic compensator design parameters are chosen, \( F = -10, H = 1 \) and \( S_1 = 1 \). The goal is to follow the pressure reference. Fig. 6.19 and Fig. 6.20 are the simulation results of the brake pressure and piston displacement inside the brake chamber. The reference for pressure is chosen to be a sinusoidal wave. Fig. 6.21 and Fig. 6.22 are the pressure and displacement of the...
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_t$</td>
<td>effective valve spool clearance area</td>
</tr>
<tr>
<td>$A_p$</td>
<td>area of piston face</td>
</tr>
<tr>
<td>$C_d$</td>
<td>valve orifice discharge coefficient</td>
</tr>
<tr>
<td>$\lambda_1, \lambda_2$</td>
<td>compressible orifice flow constants</td>
</tr>
<tr>
<td>$P$</td>
<td>absolute pressure</td>
</tr>
<tr>
<td>$P_R$</td>
<td>reservoir pressure: $P_R = 951$ kPa</td>
</tr>
<tr>
<td>$P_{atm}$</td>
<td>atmosphere pressure: $P_{atm} = 101$ kPa</td>
</tr>
<tr>
<td>$R$</td>
<td>ideal gas constant</td>
</tr>
<tr>
<td>$R_c$</td>
<td>effective valve spool radial clearance</td>
</tr>
<tr>
<td>$u$</td>
<td>valve command</td>
</tr>
<tr>
<td>$T$</td>
<td>absolute temperature: $T_{atm} = 294$ K</td>
</tr>
<tr>
<td>$T_R$</td>
<td>reservoir temperature</td>
</tr>
<tr>
<td>$T_{atm}$</td>
<td>atmosphere temperature</td>
</tr>
<tr>
<td>$V$</td>
<td>volume</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>ratio of specific heats of gaseous medium: $\gamma = 1.4$</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>closed-loop error</td>
</tr>
<tr>
<td>$\mu_c$</td>
<td>rod Coulomb friction coefficient</td>
</tr>
<tr>
<td>$\mu_u$</td>
<td>rod viscous-drag coefficient</td>
</tr>
</tbody>
</table>

Table 6.4: Pneumatic brake model parameters

piston, in this case reference is chosen as square wave to show the effectiveness of the output sliding mode control design for automotive brake control applications. The pneumatic brake model parameters are given in Table 6.4.
Figure 6.19: The time response of pressure inside the brake chamber and its reference (dashed)

Figure 6.20: The time response of piston position inside the brake chamber
Figure 6.21: The time response of pressure inside the brake chamber and its reference (dashed)

Figure 6.22: The time response of piston position inside the brake chamber
CHAPTER 7

SUMMARY AND FUTURE DIRECTIONS

In this work, we have considered sliding mode controller design for uncertain linear systems in regular form with unmodelled actuator and sensor dynamics. Furthermore the drawbacks of sliding mode control such as high gain, high frequency control inputs are attenuated by designing the sliding surface which has the interpretation of linear operators, as a linear, time-invariant dynamic system itself and defined as a linear manifold in the extended state space. This design approach is quite reasonable since most of the analytical models of the plants to be controlled have uncertainties, unmodelled dynamics and limited control capacity.

We first showed that the high order sliding mode controllers are observed to be special cases of the dynamic sliding mode design and dynamic sliding mode controller design may improve transient performance and they assure finite time convergence to the switching surface. Secondly, we introduced dynamic sliding surface design combined with recursive backstepping algorithm and we investigated the elimination of high frequency, high amplitude chattering caused by large state feedback gains derived by Lyapunov based recursive backstepping controller design. Sliding mode servomechanism problem and output feedback sliding mode control problems were
considered by applying same dynamic and frequency-shaping compensation methodologies. Finally, we applied these controllers to the slosh reduction problems related to heavy trucks carrying liquid cargo, drive by wire hydraulic power steering system and output feedback control of the pneumatic brake system.

Frequency-shaping and dynamic compensation methods for sliding mode control improve the control inputs by attenuating the high frequency, high gain terms. Dynamic sliding manifold design has the advantages of filtering the control input to the system through the desired dynamics and improve the transient performance of the time responses of the states, control inputs and sliding manifolds.

In an unified frame, frequency-shaping and dynamic compensation methods are applied to the high order sliding mode controller design, backstepping controller design, sliding mode servomechanism design and output feedback sliding mode controller design for systems with modelling and parametric uncertainties considering unmodelled actuator and sensor dynamics.

These results may be extended to nonlinear systems by using transformation into a normal system with no zero dynamics or with stable zero dynamics. The linear system in regular form is a special case of the normal form with no zero dynamics and all the results are applicable to this class of nonlinear systems. Another promising area may be in terms of using State-Dependent Riccati Equation (SDRE) method and expand the actual work to nonlinear systems. The SDRE method consists of first using direct parameterization to bring the nonlinear system to a linear structure having state-dependent coefficients. A SDRE is then solved at each point $x$ along the trajectory to obtain a nonlinear feedback of the form $u = -R^{-1}(x)B^T(x)P(x)x$, where $P(x)$ is the solution of the SDRE. Sliding mode control design based on pole placement technique

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was successfully applied to nonlinear systems by using continuously differentiable state feedback control to stabilize the system. SDRE technique combined with the contents of this work may allow state dependent dynamic optimal frequency-shaped sliding mode controller design.
APPENDIX A

THE SLOSHING DYNAMICS

The longitudinal and lateral sloshing dynamics are modeled through an equivalent model as a spring-mass system and a pendulum system, respectively. A predetermined amount of liquid mass is assumed to remain attached to the tanker coordinate systems without deviating and the rest creates a sloshing effect to represent the total forces developed by the mechanical system model match those generated by liquid dynamics by deviating with the longitudinal and lateral excitation of the vehicle. The value of the constant and moving masses, their locations in the tanker and the mechanical model values such as stiffness, damping, the length of the pendulum is estimated based on slosh frequencies related to the fill level, density and the geometrical shape of the tanker are derived by [59, 60],

\[ m_f = \rho_{had} \]  (A.1)

\[ m_p = m_f \left( \frac{a}{3.87h} \right) \tanh \left( \frac{h}{l} \right) \]  (A.2)

\[ k_p = m_f \frac{g}{1.23h} \left( \tanh \left( \frac{h}{l} \right) \right)^2 \]  (A.3)

\[ z_p = \frac{a}{\pi} \tanh \left( \frac{h}{l} \right) \]  (A.4)

\[ z_o = \frac{h}{2} + \frac{m_p}{m_o} \left( \frac{h}{2} - z_p \right) \]  (A.5)
where $m_f$ is the full liquid mass, $m_p$ is the moving mass, $k_p$ is the spring stiffness modelling slosh dynamics, $z_p$ and $z_o$ are the vertical heights of the constant and moving part of liquid cargo, respectively and constant mass is $m_o = m_f - m_p$. The longitudinal dynamic load transfer is derived by analogy to D'Alembert's principle,

$$m_p \ddot{x}_p + c_p \dot{x}_p + k_p x_p = -m_p \ddot{U}$$

(A.6)

where $c_p$ represents the viscous friction of the liquid inside the tank and $\ddot{U}$ is the longitudinal acceleration of the vehicle. The relation between the partially filled liquid level and the natural frequency of sloshing is given (see [72]),

$$f_n = \sqrt{\frac{g \pi h \tanh \left( \frac{\pi h}{l} \right)}{2\pi}}$$

(A.7)

where $l$ is the longitudinal length of the tanker and $h$ is the liquid level inside the tanker. With lateral excitation of the vehicle, the fluid slosh for various mode shapes

Figure A.1: Side view of the vehicle with a spring-mass model representing liquid cargo

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Figure A.2: Rear view of the vehicle with a pendulum model representing liquid cargo

is given by using the superposition approach (see [12])

\[ \ddot{\xi} + w_n^2 \xi_n = - (\lambda_n a) (B_n/A_n) \dot{V} \]  

(A.8)

where \( \xi_n \) is the slosh height, \( w_n \) is the natural frequency of the \( n \)th slosh mode, \( \lambda_n \), \( B_n \) and \( A_n \) are non-dimensional parameters determined by the liquid fill level for the \( n \)th slosh mode and \( \dot{V} \) is the lateral acceleration of the vehicle. Total slosh force generated with respect to lateral excitation,

\[ F_v = -m_l \dot{V} - 2\rho_l (aR)^2 + \sum_{n=1}^{\infty} B_n \xi_n \]  

(A.9)

where \( m_l \) and \( \rho_l \) represent the mass and density of the liquid inside the tank with radius \( R \).

The liquid moments of inertia with respect to the center of gravity of the vehicle sprung mass are approximated,

\[ I_z^l = I_z^0 + m_l (e_i^2 + h_i^2) \]  

(A.10)
Figure A.3: Top view of the vehicle with traction and lateral tire forces
APPENDIX B

DRIVE BY WIRE HYDRAULIC POWER STEERING SYSTEM MODEL

To analyze the specifications and performance requirements for the VDTV steering system replacement, a mathematical model, suitable for simulation, has been developed. The mechanical subsystem model is described by the angular rate and position of the steering wheel, the linear velocity and displacement of the steering rack, the angular rate and position of the left road-wheel, the angular rate and position of the right road-wheel and the analytical equations of the torque transmitted through the left and right steering linkages, the aligning torque related to tire slip and the open angle of the steering. A simple hydraulic speed dependent power steering pressure boost curve model has been used.

The steering wheel dynamics are given by,

\[
\dot{\delta}_{sw1} = \delta_{sw2} \\
\dot{\delta}_{sw2} = -\frac{K_{sc}K_{tb}}{I_{sw}(K_{sc} + K_{tb})}\delta_{sw1} + \frac{K_{sc}K_{tb}}{I_{sw}(K_{sc} + K_{tb})}\frac{1}{R_p}\Delta R_1 \\
-\frac{B_{sc}K_{tb}}{I_{sw}(K_{sc} + K_{tb})}\delta_{sw2} + \frac{B_{sc}K_{tb}}{I_{sw}(K_{sc} + K_{tb})}\frac{1}{R_p}\Delta R_2 + \frac{1}{I_{sw}}T_{sw}
\]

the system parameters are given in the appendix section. The steering rack dynamics,
\[ \dot{\Delta}_{R1} = \Delta_{R2} \]  \hfill (B.2)

\[ \dot{\Delta}_{R2} = \eta F \frac{K_{sc}K_{tb}}{M_{R}R_p(K_{sc} + K_{tb})} \delta_{sw1} - \frac{B_{R}}{M_{R}} \Delta_{R2} - \eta_{\beta} \frac{K_{sl}}{M_{R}N_{L}} (\delta_{wl} + \delta_{wr}) - \frac{2\eta_{\beta} K_{sl} \epsilon_{\phi}}{M_{R}N_{L}} - \frac{1}{M_{R}} \left( \eta F \frac{K_{sc}K_{tb}}{R_p^2(K_{sc} + K_{tb})} + 2\eta_{\beta} \frac{K_{sl}}{N_{L}^2} \right) \Delta_{R1} + \eta_{ps} F_{ps}(\delta_{value}) - C_{FR} \text{sgn}(\Delta_{R2}) \]

Figure B.1: Block diagram of the controller

The power steering force is generated by an eight order polynomial curve fit to provide sufficient match of the experimental data 

\[ F_{ps} = \{ C_1 \delta_v^8 + C_2 \delta_v^7 + C_3 \delta_v^6 + C_4 \delta_v^5 + C_5 \delta_v^4 + C_6 \delta_v^3 + C_7 \delta_v^2 + C_8 \delta_v + C_9 \} A_{pistonarea} \]

where \( \delta_{value} = \frac{K_{sc}}{K_{sc} + K_{tb}} (\delta_{sw1} - \frac{\Delta R1}{R_p}) \).

The left road-wheel dynamics,

\[ \dot{\delta}_{wl1} = \delta_{wl2} \]  \hfill (B.3)

\[ \dot{\delta}_{wl2} = \frac{1}{I_{st}} [K_{sl} \left( \frac{\Delta R1}{N_{L}} - \delta_{wl1} + \epsilon_{\phi} \right) - X_T C_{\alpha t} - B_{sl} \delta_{wl2} - C_{Fw} \text{sgn}(\delta_{wl2})] \]
The right road-wheel dynamics,

\[
\begin{align*}
\dot{\delta}_{wr1} &= \delta_{wr2} \\
\dot{\delta}_{wr2} &= \frac{1}{I_{sl}} \left[ K_{sl} \left( \frac{\Delta R_1}{N_L} - \delta_{wr1} + \varepsilon \phi \right) - X_T C \alpha_l - B_{sl} \delta_{wr2} - CF_w \text{sgn} \left( \delta_{wr2} \right) \right]
\end{align*}
\]  

(B.4)

where \( V_y \) represents the lateral velocity, \( V_x \) the longitudinal velocity, \( \phi \) the yaw rate of the bicycle model and the left and right slip angles are given by,

\[
\begin{align*}
\alpha_l &= \left( \delta_{wl} - \tan^{-1} \left( \frac{V_y + \gamma \phi}{V_x} \right) \right) \\
\alpha_r &= \left( \delta_{wr} - \tan^{-1} \left( \frac{V_y + \gamma \phi}{V_x} \right) \right)
\end{align*}
\]
APPENDIX C

PNEUMATIC BRAKE SYSTEM MODELLING FOR SYSTEMS ANALYSIS

The objective of this section is to present a model that accurately represents the dynamics of air flowing through the components of a pneumatic system configuration, which is common in many heavy vehicle applications, that eventually translates into braking force. This objective is met using the dynamic compressible airflow equations, which describe flow through an orifice. These equations are coordinated to describe the directional motion of dynamic airflow as commanded by the driver at the foot-pedal and as modified downstream by a modulator to facilitate ABS activity. The solenoid actuated relay valve also includes the motion dynamics of a piston in the existence of hysteresis and coulomb friction type built-in non-smooth nonlinearities. The adoption of an isentropic process, as opposed to the more general case of polytropic behavior, is experimentally determined to suffice for accuracy while yielding significant mathematical convenience. Various validation studies are conducted to match the simulation data to those gathered from experiments, [1].

One of the primary challenges in modeling heavy-duty vehicle brake systems is the existence of pneumatics in the loop. Although there are undeniable maintenance, handling, compatibility and robustness (against reasonable leaks) advantages over

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the hydraulic systems from an applications viewpoint, the compressibility of air introduces significant nonlinearities into a pneumatically actuated brake system from an analysis/design standpoint.

A typical pneumatic brake system involves a compressor, primary and secondary tanks, a foot brake pedal, brake chambers, ABS modulators and the plumbing lines in-between. The dynamics of air flowing through this system determines how the brake force is attained at the wheel-ends as referenced by the demand of the driver at the foot brake pedal, (see for instance, Fig. C.1).

Figure C.1: Typical components of a pneumatic brake system

It is very common for most of the components in the brake system to be modified or replaced in time for various reasons such as upgrading, modularization, standardization or simply cost reduction. Therefore, it is essential to be able to accurately predict the effects of changes in the components to the overall brake system performance. An analytical pneumatic model for individual components -combined at the
systems level, which is based on laws of physics rather than numerical data collected through experimentation, can serve as a substitute (or even a replacement when enough confidence is built) for most of the development, improvement and testing phases of components & system design. For instance, an accurate model of a pneumatic braking system is highly advantageous during the preliminary design stages of a new modulator. Due to the fact that the system parameters can be modified in the model, verification of their effects on the overall ABS system performance, or timing compliance laws can be readily determined. The overall brake system performance can be monitored as suggestions for component improvements as discussed.

The following gives a detailed explanation of the pneumatic theory that controls the system operation.

For compressible fluids, the instantaneous mass flow rate through an orifice is expressed by quasi-steady-state isentropic flow process [7, 43]:

\[
m = C_d a P_R \sqrt{\frac{2}{R T_R}} f \left( \frac{P_M}{P_R} \right)
\]  

(C.1)

where \(m\) is the mass flow rate liberated through an orifice of aperture \(a\) in the reservoir (subscript \(R\)) toward a medium lower pressure \(P_M\) and \(T_R\) is the reservoir temperature. The discharge coefficient \(C_d\) is included in Eq. C.1 to account for friction and fluid contractions exhibited by real orifices. For compressible fluids, a general orifice flow expression that handles both the subsonic and the supersonic regimes is accomplished by defining a piecewise flow function \(f(\cdot)\), as,

\[
f(r) = \begin{cases} 
\sqrt{\frac{\gamma}{\gamma-1}} \left( \frac{2}{\gamma} - r^{\frac{\gamma+1}{\gamma}} \right), & r_c \leq r \leq 1 \\
\sqrt{\frac{\gamma}{\gamma+1}} \left( \frac{2}{\gamma+1} \right)^{\frac{2}{\gamma+1}}, & 0 \leq r \leq r_c 
\end{cases}
\]  

(C.2)
with the condition that \( f(r_C = 0) \), where \( \gamma \) is the ratio of the pressure on either side of the orifice and \( r_C \) is the critical ratio where subsonic and supersonic regions are separate and defined as 

\[
 r_C = \left( \frac{2}{1 + \gamma} \right)^{\frac{2-1}{7}}.
\]

The nonlinear state-space model of the presented pneumatic brake system is based on the following assumptions:

- an ideal equation of state
- isentropic valve orifice flow
- isentropic control volume behavior

The brake actuation system is push-pull, and involves translational motion inside of a pressure chamber, where the piston position \( x \) varies between \( 0 \leq x \leq L \) and \( L \) is the maximum available stroke. Variables used in describing the pressure chamber gas dynamics are the chamber pressure \( P_C \), its temperature \( T_C \), specific density \( \rho_C \), the reservoir pressure \( T_R \), specific density \( \rho_R \) and the expansion chamber, (here simply, exterior atmosphere) pressure \( P_{atm} \), its temperature \( T_{atm} \) and specific density \( \rho_{atm} \).

The pressure chamber can be connected by two ports respectively toward reservoir with orifice \( a_R \) and toward exterior \( a_{atm} \) for raising or lowering pressure level in the pressure chamber. The position of a piston of cross section \( A_p \) is regulated by a spring-damper of stiffness \( K_s \). The damping effects stem from general characteristics which include both linear (e.g. viscous) and nonlinear terms (dry friction, stiction or quadratic type effects). Fig. C.2 shows the experimental system at the brake chamber level: The modulator is driven by on-off solenoids which can be described as a single parameter \( u \), such that air flows in or out of the pressure chamber. then, due to
mass rate $\dot{m}$, the inside pressure denoted by $P_C$ is changed and the actuator system generates the force $F_A$.

![Diagram of a typical brake chamber](image)

Figure C.2: A typical brake chamber

A schematic model of pneumatic actuator system is given in Fig. C.3. The mass flow rate from reservoir into pressure chamber and from pressure chamber to the exterior is given by

\[
\dot{m}_{RC} = C_d a_R P_R \sqrt{\frac{2}{RT_R}} \gamma \left( \frac{P_C}{P_R} \right) \\
\dot{m}_{CE} = C_d a_E P_C \sqrt{\frac{2}{RT_C}} \gamma \left( \frac{P_E}{P_C} \right)
\]  

(C.3)  

(C.4)

Inside the pressure chamber, the local mass flow rate may be derived from the usual work and energy conservation concepts given by,

\[
\dot{m}_C = \frac{A_P x}{\gamma RT_C} \frac{dP_C}{dt} + \frac{A_P P_C}{RT_C} C_d \frac{dx}{dt}
\]  

(C.5)
Figure C.3: Schematics of a typical pneumatic actuation system

The equation governing the load motion of the rod is characterized by viscous coefficient $\mu_u$, Coulomb term $\mu_c$ and linear spring $K_s$ by,

\[
\begin{align*}
\frac{dx}{dt} &= \dot{x} \\
\frac{d\dot{x}}{dt} &= \frac{1}{m} [A_p P - \mu_u \dot{x} - \mu_c \text{sign}(\dot{x}) - K_s x]
\end{align*}
\]  
(C.6)

To model the non-ideal behavior of the real orifices, a discharge coefficient $C_d$ is typically defined in terms of the velocity coefficient $C_v$ and the contraction coefficient $C_c$.

\[ C_d \approx C_v C_c \]  
(C.7)

In most cases, $C_v=1.0$ and $C_c=0.982$ may be used, [7, 43]. A basic premise of the model is that the contraction coefficient, denoted here with $C_i$, for the orifice when transmitting an incompressible fluid be known a priori. The results appear in the form of two equations, one for each flow regime. These equations provide $C_c$ as a function of the pressure ratio $r = \frac{P_{downstream}}{P_{upstream}}$ and a force defect coefficient $f_d$ given by the following explicit function of $C_i$,
\[ f_d = \frac{1}{C_i} - \frac{1}{2C_i^2} \]  

(C.8)

\[ C_c \text{ then, can be represented by,} \]

\[ C_c = \begin{cases} 
\frac{1}{2da r^\frac{1}{2}} \left[ 1 - \sqrt{1 - \frac{(2r^\frac{1}{2})^2 (1-r)f}{(\lambda_2 f_1(r))^2}} \right], & r_c < r \\
\frac{1}{2da r^\frac{1}{2}} \left[ 1 + \frac{(r_c-r)r_c^\frac{1}{2}}{(\lambda_2 f_1(r))^2} - \sqrt{1 + \frac{(r_c-r)r_c^\frac{1}{2}}{(\lambda_2 f_1(r))^2}} - \frac{\left(2r_c^\frac{1}{2}\right)^2 (1-r)f}{(\lambda_2 f_1(r))^2} \right], & r \leq r_c 
\end{cases} \]  

(C.9)

and where \( f_1(r), \lambda_2 \) and \( r_c \) are predetermined. Selecting

\[ V(x) = V_D + A_p x \]  

(C.10)

as the volume, where \( V_D \) is the dead volume that exists at the end of the piston, accounting for plumbing, the valve manifold, etc. and \( A_p \) as the piston area, and \( \lambda_1 \) and \( \lambda_2 \) are dimensionless constants

\[ \lambda_1 = \sqrt{\frac{2\gamma}{\gamma - 1}}, \quad \lambda_2 = \sqrt{\gamma \left( \frac{2}{\gamma + 1} \right)^{\frac{1}{\gamma - 1}}} \]  

(C.11)

which are used in the following equations.

The pneumatic actuation system, which is comprised of a solenoid actuated 3-state modulator, has the capability of building \((u = 1)\), holding \((u = 0)\) or exhausting \((u = -1)\) the pressure in the brake chambers via controlling the air flow between the reservoir and the chambers or the brake chambers or the brake chambers and the atmosphere by modulating the associated solenoids. This generic behavior still yields different responses for different generation modulators since there are internal
design variations. The behavior of air flowing through pneumatic transmission lines is also an integral part of the investigation. Given this introduction, the modulator dynamics operates as follows (where $u$ is the control state)

For $u = 1$, If \( \frac{P}{P_R} > \left( \frac{2}{\gamma + 1} \right)^{\frac{2}{\gamma - 1}} \)

\[
\begin{align*}
F &= \frac{\lambda_1}{\lambda_2} \sqrt{\left( \frac{P}{P_R} \right)^{\frac{2}{\gamma}} - \left( \frac{P}{P_R} \right)^{\frac{\gamma + 1}{\gamma}}} \\
\dot{P} &= \frac{\gamma}{V(x)} \left( \frac{\sqrt{R}}{\sqrt{T_R}} T_C C_d \lambda_2 P_R F u - P_A \dot{x} \right)
\end{align*}
\]

If \( \frac{P}{P_R} \leq \left( \frac{2}{\gamma + 1} \right)^{\frac{2}{\gamma - 1}} \)

\[
\begin{align*}
F &= \sqrt{\frac{\gamma}{\gamma + 1} \left( \frac{2}{\gamma + 1} \right)^{\frac{2}{\gamma - 1}}} \\
\dot{P} &= \frac{\gamma}{V(x)} \left( \frac{\sqrt{R}}{\sqrt{T_R}} T_C C_d \lambda_2 P_R F u - P_A \dot{x} \right)
\end{align*}
\]

For $u = 0$,

\[
\dot{P} = \frac{\gamma}{V(x)} (-P_A \dot{x})
\]

For $u = -1$, If \( \frac{P_{atm}}{P} > \left( \frac{2}{\gamma + 1} \right)^{\frac{2}{\gamma - 1}} \)

\[
\begin{align*}
F &= \frac{\lambda_1}{\lambda_2} \sqrt{\left( \frac{P_{atm}}{P} \right)^{\frac{2}{\gamma}} - \left( \frac{P_{atm}}{P} \right)^{\frac{\gamma + 1}{\gamma}}} \\
\dot{P} &= \frac{\gamma}{V(x)} \left( \frac{\sqrt{R}}{\sqrt{T_R}} T_{atm} C_d \lambda_2 P F u + P_{atm} A \dot{x} \right)
\end{align*}
\]

If \( \frac{P_{atm}}{P} \leq \left( \frac{2}{\gamma + 1} \right)^{\frac{2}{\gamma - 1}} \)
\[
\begin{align*}
F &= \sqrt{\frac{\gamma}{\gamma+1}\left(\frac{2}{\gamma+1}\right)^{\frac{2}{\gamma-1}}} \\
\dot{P} &= \frac{\gamma P}{V(x)} \left(\frac{R}{\gamma P_0} T_{atm} C_d A_{p} \lambda_2 P_F u + P_{atm} A_p \dot{x}\right)
\end{align*}
\]

where \( P \) is the chamber pressure, \( P_R \) reservoir pressure, \( P_{atm} \) atmospheric pressure, \( x \) and \( \dot{x} \) are the displacement and velocity of the rod, \( R \) is the ideal gas constant, \( \gamma \) is ratio of specific heats of gaseous medium for air, \( \gamma = 1.4 \). The temperature of the air inside the chamber is given in terms of initial conditions \( P \) by

\[
T_C = T_0 \left[\frac{P}{P_0}\right]^{\frac{\gamma-1}{\gamma}} \quad (C.12)
\]

The simplified block diagram of the pneumatic brake actuator is given in Fig. C.4.

Figure C.4: Simplified block diagram of the pneumatic simulation model

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