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Transversally Elliptic Operators

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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* * * * *

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ABSTRACT

We study the index theory of transversally elliptic pseudodifferential operators in the framework of noncommutative geometry.

For such an operator we construct a spectral triple in the sense of A. Connes and H. Moscovici (“The local index formula in noncommutative geometry” Geom. Funct. Anal. 5(2):174–243. 1995). We prove that this spectral triple satisfies the conditions which ensure the Connes-Moscovici local index formula applies.

We show that the spectral triple has discrete dimensional spectrum. A notable feature of the spectral triple is that its corresponding zeta functions have multiple poles, while in the classical elliptic cases only simple poles appear for the zeta functions. We show that the multiplicities of the poles of the zeta functions have an upper bound, which is the sum of dimensions of the base manifold and the acting compact Lie group. Moreover for our spectral triple the Connes-Moscovici local index formula involves only local transverse symbol of the operator.
Dedicated to my family.
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CHAPTER 1

INTRODUCTION

Atiyah introduced in [2] the index of a transversally elliptic pseudo-differential operator relative to a compact Lie group $G$ acting on a compact manifold $M$, which generalizes the index of an elliptic operator. Although the kernel and cokernel of a transversally elliptic operator are no longer finite dimensional as in the elliptic case, the index

$$\text{index}_a^G(\sigma(P)) = \text{ker}(P) - \ker(P^*) \in C^{-\infty}(G)^G$$

makes sense as a central distribution on $G$, depending on the $K_G$-theory class of the transversal part of its principal symbol. An explicit index formula was given in [2] for torus acting with finite isotropy.

This thesis deals with the index theory for transversally elliptic pseudodifferential operators in the framework of noncommutative geometry. As in the elliptic case, the operator algebraic method is efficient and gives deep insight to the index theory. We compute, in local terms, the index formula for a transversally elliptic operator in periodic cyclic cohomology.

In the context of $K$-homology for operator algebras, an elliptic system of pseudo-differential operators gives a $K$-homology class in $KK(C^\infty(M), \mathbb{C})$ (see, for example
Similarly, a transversally elliptic system of pseudo-differential operators naturally induces a $K$-homology class in $KK(\mathcal{A}, \mathbb{C})$ (see, for example, [21], [23], and [24]), where $\mathcal{A}$ is the smooth crossed product algebra $C^\infty(M) \rtimes G$. The elements of $\mathcal{A}$ are smooth functions (denoted by $a, b$) on $M \times G$, with the product
\[
(a \ast b)(x, g) = \int_G a(x, h)b(h^{-1}x, h^{-1}g)d\mu(h).
\]
$\mathcal{A}$ can be viewed as the smooth convolution algebra of the groupoid $M \rtimes G$ induced by the group action, which is the replacement for (the algebra of functions on) the quotient space $M/G$.

To a transversally elliptic pseudo-differential operator $P$, we will associate (in section 5.1) a natural spectral triple $(\mathcal{A}, \mathcal{H}, D)$ representing the same $K$-cycle as defined in [21]. Here $\mathcal{H}$ is the graded Hilbert space constructed from the direct sum of $L^2$-sections of the complex vector bundles on which $P$ acts. $\mathcal{A}$ acts upon $\mathcal{H}$ by extending the $G$-action $\rho$ on the fibers of the complex bundles:
\[
(\rho(a) \cdot s)(x) = \int_G a(x, g)(\rho(g)s)(g^{-1}x)dg.
\]
Let $\epsilon$ be the grading operator of $\mathcal{H}$. From $P$ we make a graded, self-adjoint operator $D$, which can be assumed to have order 1 without loss of generality. Further we assume $D^2$ has scalar symbol. Thus for any $a \in \mathcal{A}$, $\rho(a) \in B(\mathcal{H})$, $\rho(a)\epsilon = \epsilon\rho(a)$, and $D\epsilon = -\epsilon D$. A pseudodifferential operator $K$ on $\mathcal{H}$ is called transversally smoothing (notationally $K \in \mathcal{K}_\mathcal{A}$) if for any $a \in \mathcal{A}$, $\rho(a)K$ is a smoothing operator (in particular trace class).

In this thesis we shall first prove that $(\mathcal{A}, \mathcal{H}, D)$ is an even spectral triple such that (i) for all $a \in \mathcal{A}$, $[D, \rho(a)] \in B(\mathcal{H})$, (ii) for any $a \in \mathcal{A}$, $\rho(a)(1 + |D|)^{-1}$ is compact, and (iii) there is a $K \in \mathcal{K}_\mathcal{A}$ such that $|D| + K$ is invertible, $(|D| + K)^{-1}$ is in the class $\mathcal{C}_\mathcal{A}$.
$\mathcal{L}^{(\dim M, \infty)}$ for some $p = \dim M$, where $\mathcal{L}^{(p, \infty)}$ is the ideal of $B(\mathcal{H})$ consisting of those compact operators $T$ whose $n$-th characteristic value satisfies

$$\mu_n(\|T\|) = O(n^{-1/p})$$

(cf. [12] for details).

Before we present our main result we need to introduce some notations. For our spectral triple $(\mathcal{A}, \mathcal{H}, D)$, we will introduce (in chapter 3) the crossed product algebra of pseudodifferential operator and the group $G$:

$$\Psi^\infty(E) \times G = \bigcup_k \Psi^k(E) \times G,$$

in which the following operations are closed: for any operators $A \in \mathcal{A}$,

$$dA = [D, A], \quad \nabla(A) = [D^2, A], \quad A^{(k)} = \nabla^k(A)$$

are in $\Psi^\infty(E) \times G$. In particular, the operators

$$P(a^0, a^1, \ldots, a^n) = a^0(da^1)^{(k_1)} \ldots (da^n)^{(k_n)}$$

are in $\Psi^\infty(E) \times G$, where $a^0, \ldots, a^n \in \mathcal{A}$, acting on $\mathcal{H}$ by $\rho$. We denote by $\mathcal{A}_D$ the subspace of $\Psi^\infty(E) \times G$ generated by $P(a^0, a^1, \ldots, a^n)$. Operators in $\Psi^0(E) \times G$ are our main interest since they are bounded linear operators on $B(\mathcal{H})$. For a spectral triple with $|D|$ having scalar symbol, $\mathcal{A}_D \subset \Psi^0(E) \times G$ and, in particular, $\mathcal{A}_D \subset B(\mathcal{H})$. We will show that there are similar filtration and symbol calculus as for ordinary pseudodifferential operators.

For $P \in \Psi^k(E) \times G$, the zeta function

$$\zeta_{P,D}(z) = \text{Trace}(P|D|^{-2z})$$

3
is at least defined and analytic for \( \text{Re}(z) > k + \dim M \).

We call the *dimension spectrum* of our spectral triple is the minimal closed subset of \( \mathbb{C} \) on the complement of which \( \zeta_{\mathcal{P},D} \) can be extended to a holomorphic function for any \( \mathcal{P} \in \mathcal{A}_D \).

The index formula of \((\mathcal{A}, \mathcal{H}, D)\) follows from the Connes-Chern character formula in cyclic cohomology theory. In the case \( G \) is trivial, we have a spectral triple defined by an elliptic pseudo-differential operators, all the zeta function \( \zeta_{\mathcal{P},D} \) have simple poles at integral points, \( \tau_0 \) is, up to a constant, the Wodzicki residue, which is computable in terms of symbols.

The main goal of the thesis is to express the cyclic cohomology character formula in terms of local data such as the symbol of the operator and characteristic classes of the bundle, by means of the operator-theoretical version of the local index formula given in Connes-Moscovici [10]. The formula for the Connes-Chern character given in [10] assumes that the spectral triple has discrete dimension spectrum, with finitely many nonzero \( \tau_k \). In this thesis we show that in the transversally elliptic case, non-simple poles for the function \( \zeta_{\mathcal{P},D} \) appear, so those \( \tau_k \) are the generalized versions of the Wodzicki residue.

We now state the main result.

**Theorem.** (1) For the even spectral triple \((\mathcal{A}, \mathcal{H}, D)\) defined by a transversally elliptic pseudodifferential operator as above, the dimension spectrum is a discrete subset of \( \mathbb{Q} \), and the orders of the poles for \( \zeta_{\mathcal{P},D} \) are no higher than \( \dim M + \dim G \).

(2) For any \( \mathcal{P} \in \Psi^0(E) \times G \) and \( q = -1, 0, 1, \ldots, \dim M + \dim G - 1 \), let

\[
\tau_q(\mathcal{P}) = \tau_q^D(\mathcal{P}) = \text{Res}_{z=0} z^q \zeta_{\mathcal{P},D}
\] (1.1)
be the residues of \( z^q \zeta_{P,D} \) at \( z = 0 \). The Connes character \( ch(\mathcal{A}, \mathcal{H}, D) = [\phi_{2m}] \) in periodic cyclic cohomology can be given by the following even cocycle in the periodic cyclic cohomology:

\[
\phi_{2m}(a_0, \ldots, a_{2m}) = \sum_{k \in \mathbb{Z}^{2m}, q \geq 0} c_{2m,k,q} \tau_q(a^0(da_1)^{(k_1)} \cdots (da_{2m})^{(k_{2m})}|D|^{-(2|k|+2m)}),
\]

for \( m > 0 \) and

\[
\phi_0(a^0) = \tau^{-1}(\gamma a^0).
\]

In the above formula \( k = (k_1, \ldots, k_{2m}) \) are multi-indices and \( c_{2m,k,q} \) are universal constants given by

\[
c_{2m,k,q} = \frac{(-1)^{|k|}}{k!\tilde{k}!} \sigma_q(|k|+m).
\]

where \( k! = k_1! \cdots k_{2m}! \), \( \tilde{k}! = (k_1 + 1)(k_1 + k_2 + 2) \cdots (k_1 + \ldots + k_{2m} + 2m) \) and for any \( N \in \mathbb{N}, \sigma_q(N) \) is the \( q \)-th elementary polynomial of the set \( \{1, 2, \ldots, N-1\} \).

(3) Moreover, the above cocyle can be computed solely in terms of the transversal part of the symbol of \( D \).

Note in the theorem (2) follows from (1) and Connes-Moscovici [10]. (3) shows that in our case the Connes-Moscovici formulas can be expressed in purely local data.

In conclusion we get general local index formula for transversally elliptic operators: the index of the spectral triple depends only on local data such as the symbol of \( D \), characteristic classes of the bundles on which \( D \) acts, and characters of \( G \). Unlike the previously studied spectral triples, (for example in [10]) which were associated to elliptic or hypoelliptic pseudodifferential operators and commutative algebras, ours is based on a noncommutative algebra \( \mathcal{A} \) with multiple poles for the zeta functions.
\( \zeta_{P,D} \), but the multiplicities of the poles of all these zeta functions are no higher than \( \dim M + \dim G \).

We now give the outline of the thesis. In chapter 2 we introduce the preliminaries for the convenience of the reader. These include pseudodifferential operators and transversally elliptic operators. The algebra of crossed products comes naturally into the picture when we deal with operators generated by commutators among pseudodifferential operators, scalar functions, and group action. The groupoid definition will be needed in our later arguments about crossed product algebras. We also present the basic facts about wave front sets since wave front set calculation is the way to sort out the insignificant operators. In chapter 3 we introduce a new algebra of crossed product with pseudodifferential operators alluded above. We also introduce weakly polyhomogeneous symbols needed in the spectral asymptotics. In chapter 4 we discuss the \( K \)-theoretic aspect of the index of the the transversally elliptic operators, the goal is to show that the character in \( K \)-theory is related to the transversal index as defined by Atiyah [2], in the process we establish some basic facts about transversally elliptic operators. In chapter 5 we introduce the spectral triple, and by studying the asymptotics of operators in the proper class, we prove our main results.
CHAPTER 2

PRELIMINARIES

For the clarity of discussion, we set our conventions for the notation. Unless otherwise specified, we assume $G$ is a compact Lie group acting smoothly on a compact manifold $M$. Let $\mathfrak{g}$ denote the Lie algebra of $G$. In most but not all situations we assume $M$ is a Riemannian manifold with an invariant Riemannian metric and $G$ acts by orientation preserving isometries. In the Riemannian case we assume $M$ is orientable, with a $G$-invariant volume form. For an element $g \in G$, let $M^g = M(g)$ be the fixed point set of $g$, which in the presence of invariant Riemannian metric is a disjoint union of totally geodesic submanifolds of $M$. We will consider complex hermitian $G$-bundles over $M$, which are assumed here to be complex bundles with smooth hermitian metric preserved by the $G$-action.

2.1 Symbols

A conic manifold is a $C^\infty$ paracompact manifold $V$ together with a proper and free $C^\infty$ action of $(\mathbb{R}_+, \cdot)$ (as a multiplicative group) on $V$. It follows that the orbit space $V' = V/\mathbb{R}_+$ has a $C^\infty$ structure, and $V \to V'$ is trivializable. A function $f$ on a conic manifold $V$ is called homogeneous of degree $k$ if $\tau^*f = \tau^k f$ for all $\tau \in \mathbb{R}_+$, where the pull back by the action of $\tau$ is by definition $\tau^* f(x) = f(\tau x)$ for any $x \in V$.

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For a vector field \( L \) on \( V \), the pull back \( \tau^*L \) is defined by
\[
\tau^*L = \tau^*L(\tau^{-1})^* \tag{2.1}
\]
A vector field \( L \) on \( V \) is called homogeneous of degree \( l \) if \( \tau^*L = \tau^lL \) for all \( \tau \in \mathbb{R}_+ \).

It follows that \( Lf \) is homogeneous of degree \( k+l \) if \( f \) is homogeneous of degree \( k \) and \( L \) is homogeneous of degree \( l \). Homogeneous vector field of degree zero induces flows in \( V \) that commute with the \( \mathbb{R}_+ \) action.

**Definition 2.1.1.** Let \( V \) be a conic manifold, \( \mu, \rho \in \mathbb{R}, \ 0 \leq \rho \leq 1 \). A symbol of order \( \mu \) and of type \( \rho \) is a smooth function \( a \in C^\infty(V) \) such that
\[
\tau^*(L_k \ldots L_1 a) = O(\tau^{\mu-k\rho}) \quad \text{for } \tau \to \infty \tag{2.2}
\]
locally uniformly in \( V \) and for all vector fields homogeneous of degree \(-1\). The space of these symbols is denoted by \( S^\mu_\rho(V) \). The case \( \rho = 1 \) is the main interest, so \( S^\mu_1(V) \) is usually written as \( S^\mu(V) \).

We now list some basic properties of the symbols. \( S^\mu_\rho(V) \subset S^{\mu'}_\rho'(V) \) if \( \mu \leq \mu' \). The multiplication sends
\[
S^\mu_\rho(V) \times S^{\mu'}_\rho'(V) \to S^{\mu+\mu'}_\rho'(V); \tag{2.3}
\]
and for a smooth map between conic manifolds commuting with \( \mathbb{R}_+ \) actions, the pull back map preserves symbol space. We denote, as usual,
\[
S^\infty_\rho = \bigcup_k S^k_\rho, \quad S^{-\infty}_\rho = \bigcap_k S^k_\rho.
\]
Then for a sequence of symbols \( a_j \in S^\mu_j_\rho(V) \) of decreasing order \( \mu_j \downarrow -\infty \), there exists an \( a \in S^{\mu_0}_\rho(V) \) such that
\[
a - \sum_{j<k} a_j \in S^\mu_k_\rho(V) \quad \forall k = 0, 1, 2, \ldots \tag{2.4}
\]
and we call \( a \sim \sum a_j \) the asymptotic sum of the series \( \sum a_j \).

For a vector bundle \( E \) over a manifold \( X \) with rank \( N \), \( E \setminus \{0\} \), with zero section excluded, is a conic manifold. Functions on \( E \) restricted on \( E \setminus \{0\} \) gives a symbol in the general sense. A symbol with support near the zero section is in \( S^{-\infty}(E) \) so the zero section is irrelevant in symbol calculus. We denote \( S^\mu_p(E) \) as the space of symbols. For \( X = \mathbb{R}^n \), \( S^\mu_p(\mathbb{R}^n \times \mathbb{R}^N) \) is the space of symbols in the classical sense: \( \sigma(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^m) \) satisfying for all multiple indices \( \alpha \) and \( \beta \),

\[
\frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial \xi^\beta} \sigma(x, \xi) = O \left( (1 + |\xi|)^{k-\rho|\beta|+(1-\rho)|\alpha|} \right).
\] (2.5)

locally uniformly.

The general definition of the symbol helps to define the symbol of families when we take a parametric space \( Y \) to be symbols on \( V \times Y \). A weaker version of families of symbols requires the homogeneous vector fields \( L_1, \ldots, L_k \) are pull backs from those on \( V \).

### 2.2 Pseudodifferential operators

Pseudodifferential operators are typically introduced locally, defined by a certain type of Fourier integral operators with a local symbol, the extended globally. By showing the invariance of principal symbol, the notion of pseudodifferential operator is essentially local. It extends to a manifold \( M \). With this approach the symbol is not specified except for the principal part which is known to be a symbol on the cotangent bundle \( T^*M \).

Pseudodifferential operators can be defined invariantly (Hörmander [19]) as follows.
Definition 2.2.1. Let $P$ be a linear operator from $C_0^\infty(M)$ to $C^\infty(M)$, $P$ is a pseudodifferential operator if there is a decreasing sequence $\mu_0 > \mu_1 > \ldots \rightarrow -\infty$ of reals such that, for all $f \in C_0^\infty(M)$ and $g \in C^\infty(M, \mathbb{R})$ with $dg \neq 0$ on supp($f$), there is an asymptotic expansion

$$e^{-\tau g} P(f e^{\tau g}) \sim \sum_{j=0}^{\infty} P_j(f, g) \tau^\mu_j, \quad \tau \rightarrow \infty$$

(2.6)

where $P_j(f, g)$ are smooth functions on $M$ and the right hand side of equation (2.6) is an asymptotic expansion as symbols (in the general sense) in $S^\mu_0(M \times \mathbb{R}_+)$. We denote by $\Psi^k(M)$ the set of pseudodifferential operators with order no greater than $k$.

It follows that, $\mu_0$, the order of $P$, and the principal symbol are well defined (the later being a function on $T^*M \setminus \{0\}$ by choosing a particular function $g_\xi(x) = x \cdot \xi$).

The symbol is more complicated, it has to be defined, for example, in one way, as a function on the bulky infinite jet bundle with complicated composition rules, or another, by restricting to some admissible charts which may not be pleasant. We adopt Widom’s approach [36],[37], by choosing a connection and a special function related to the connection, the symbol is to be defined as a function on the cotangent bundle.

For a manifold $M$ (not necessarily compact), with a linear connection $\nabla$, let $\partial^k$ be the symmetrized $k$-th covariant derivative associated to $\nabla$ ([37],pp.23–27), we choose a function $l \in C^\infty(T^*M \times M)$ such that $l(v, y)$ is linear on the fibers of first factor and the symmetrized derivative $\partial^k l$ satisfies

$$\partial^k l(v, y)|_{y=\pi(v)} = \begin{cases} 0, & k \neq 1, \\ v, & k = 1. \end{cases}$$

(2.7)
Such functions $l$ are called linear with respect to $\nabla$.

The function $l$ is used to choose local directions that are "locally linear", and $l(\cdot, x)$ can be viewed as a family (with $x$ as the parameter) of vector fields on on $M$. And by using $l$ the Taylor's Theorem on the manifold $M$ may not be stated as an equation of functions on $M$, see [36], [37].

**Definition 2.2.2.** For $P \in \Psi^k(M)$ and chosen connection $\nabla$ and function $l$, there exists a function $f \in C^\infty(M \times M)$ which equals 1 on a neighborhood of the diagonal and with support so close to diagonal that for all $v \neq 0 \in T^*_{\pi(v)} M$

$$d_x l(v, x) \neq 0 \text{ if } (\pi(v), x) \in \text{supp}(f).$$  \hspace{1cm} (2.8)

The symbol of $P$ is a function on $T^*M$

$$\sigma_P(v) = P(\phi e^{il(v,x)})|_{x=\pi(v)}.$$  \hspace{1cm} (2.9)

The symbol is unique up to $S^*_\rho(T^* M \setminus \{0\})$. The value at $v = 0$, defined or not, is irrelevant.

With a fixed choice of $\nabla$ and $l$, Widom showed the composition rule of pseudodifferential operators and that the symbol map $P \mapsto \sigma_P$ is a linear isomorphism of

$$\Psi^k(M)/\Psi^{-\infty}(M) \to S^k(T^* M)/S^{-\infty}(T^* M).$$  \hspace{1cm} (2.10)

There are plenty of choices of both the connection and the function $l$ affecting the symbol. For Riemannian manifold one example is

$$l(v_x, y) = g(exp^{-1}_x(y), v_x)$$  \hspace{1cm} (2.11)
where the right hand side is defined when \( y \) is closed to \( x \), and the definition can be extended by a cutoff function on \( y \).

We do not assume the fixed Riemannian connection, but with the \( G \) action it is convenient to work with a connection that is \( G \)-invariant by averaging on the connection form. And similarly we may choose \( l \) such that

\[
 l(g_*v, gy) = l(v, y) \tag{2.12}
\]

by averaging the function \( l \) on \( G \).

It follows immediately that for \( g \in G \) (or any diffeomorphism on \( M \)), we have \( \sigma_P(v) = \sigma_{g P g^{-1}}(g_*v) \). In particular, a \( G \)-invariant pseudodifferential operator has an invariant symbol.

**Lemma 2.2.3.** For a \( G \)-invariant connection \( \partial \), and any function \( l \) satisfying \( (2.7) \),

\[
 l(g_*v - v, y) \text{ is independent of the connection and the choice of the function } l.
\]

**Proof.** Since \( G \) is compact, for any \( g \in G \) choose \( g = \exp X \) from \( X \in g \).

\[
 l(g_*v - v, y) = \int_0^1 \frac{d}{dt} l((\exp tX)_*v, y) dt
 = \int_0^1 \frac{d}{dt} l(v, (\exp tX)y) dt
 = \int_0^1 \partial l(v, (\exp tX)y) \cdot X_{M,(\exp tX)y} dt
 = \int_0^1 \nabla_{(\exp -tX)y} \cdot X_{M,(\exp tX)y} dt.
\]

And the last expression is independent of the choice of the function \( l \). \( \square \)

By a family of pseudodifferential operators \( \{P(y), y \in Y\} \) over a compact parametric space \( Y \) we require the asymptotic expansion \( (2.6) \) being uniform in \( Y \).
We have so far only mentioned symbols and pseudodifferential operators with scalar values, the definition and techniques moves to bundles naturally as follows. For a bundle $E$ over $V$, $S^\nu_P(V; E) = S^\nu_P(V) \otimes \Gamma(E)$ will be called symbols.

**Definition 2.2.4.** Let $E, F$ be hermitian bundles over a manifold $X$, a linear operator

$$P : \Gamma^\infty_0(E) \rightarrow \Gamma^\infty(F)$$

(2.13)

is called a pseudodifferential operator if (instead) in the definition (2.2.1) we assume $f$ to be any compactly supported smooth sections, and asymptotic expansion (2.6) holds as symbols in $S^\nu_P(M \times \mathbb{R}_+) \otimes \Gamma(Hom(E, F))$. We write $\Psi^k(X; E, F)$ or simply $\Psi^k(E, F)$ all such operators of order up to $k$.

For a bundle $E$ on $M$ we choose a connection $\nabla^E$, and denote the covariant derivative by $\partial^k$ associated to $\nabla^E$, similar to the linear function $l$ we find a $C^\infty$ function

$$c : E \times M \rightarrow E$$

such that

1. for any $x \in M$, $c(\cdot, x)$ is a bundle mapping;
2. for any $e \in E$, $c(e, \cdot)$ is a section of $E$ satisfying

$$\partial^k_x c(e, x)|_{x=\pi(e)} = \begin{cases} 0, & k \neq 1, \\ e, & k = 1. \end{cases}$$

(2.14)

The map $c$ defines "constant" coefficients.

**Definition 2.2.5.** For a pseudodifferential operator $P \in \Psi^k(E, F)$, with above notations the symbol $\sigma_P$ is defined as a symbol in $S^k(T^*M; Hom(E, F))$, for $v \in T^*M$
and $e \in E$ with $\pi(v) = \pi(e) = x_0$,

$$\sigma_P(v, e) = P \left( \psi(\pi(v), x)c(e, x)e^{it(v,x)} \right) \bigg|_{x=x_0}. \quad (2.15)$$

We will use the abbreviation $S^k(E, F)$ for $S^k(T^*M; \text{Hom}(E, F))$. $\sigma_P$ is well-defined on $S^k(E, F)$ and the symbol map gives

$$\Psi^k(E, F) / \Psi^{-\infty}(E, F) \to S^k(E, F) / S^{-\infty}(E, F). \quad (2.16)$$

There are similar composition rules in the bundles case.

### 2.3 Transversally ellipticity

In the case when $M$ is compact, $P \in \Psi^k(E, F)$ is a bounded linear operator

$$P : \Gamma_0^\infty(E) \to \Gamma^\infty(F), \quad (2.17)$$

extending to a bounded linear operator between Sobolev spaces of the sections

$$P : H^s(M, E) \to H^{s-k}(M, F). \quad (2.18)$$

**Definition 2.3.1.** For $X \in \mathfrak{g}$, we denote by $X_M$ the *fundamental vector field* generated by the action of the one parameter group corresponding to $X$:

$$X_M f(x) = \frac{d}{dt} \bigg|_{t=0} f(\exp(-tX)x) \quad (2.19)$$

for any smooth function $f$ on $M$ and $x \in M$.

The $G$-action decides a map $X \mapsto X_M$ from $\mathfrak{g}$ to $\Gamma(TM)$.

**Definition 2.3.2.** Let

$$\mu^{M,G}_* : T^*M \to \mathfrak{g}^* \quad (2.20)$$
be the adjoint of this map for the group action, i.e., for any \((x, \xi) \in T^* M\)

\[
\langle \mu_{x}^{M,G}(x, \xi), X \rangle = \langle \xi, X_M(x) \rangle.
\]  

(2.21)

The kernel for the moment map is usually denoted by

\[
T_G^* M = (\mu_{x}^{M,G})^{-1}(0) = \{ (x, \xi) \in T^* M | \langle \xi, X_M \rangle = 0 \forall X \in \mathfrak{g} \}.
\]  

(2.22)

**Definition 2.3.3.** Let \( P \in \Psi^k(E, F) \) be a \( G \)-invariant pseudodifferential operator. \( P \) is called *transversally elliptic* relative to the action of \( G \) if its principal symbol

\[
\sigma_{P,|k|}(x, \xi) : \pi^* E \to \pi^* F
\]

is invertible for all nonzero \( \xi \in T_G^* M \setminus \{0\} \). It is called strongly transversally elliptic if there is a conic neighborhood \( U \) of \( T_G^* M \setminus \{0\} \) in \( T^* M \setminus \{0\} \) such that there is a \( C > 0 \), for all \( (x, \xi) \in U \).

\[
|| (\sigma_{P,|k|}(x, \xi))^{-1} || \geq C(1 + ||\xi||)^{-k}.
\]

(2.23)

**2.4 Wave front set of a distribution**

In this section we recall the standard definition and main properties of the wave front set which is an efficient tool here. The results are commented but their proofs are not reproduced (cf. for example [16]).

**Definition 2.4.1.** For a hermitian bundle \( E \) on \( M \), let \( u \in \Gamma^{-\infty}(M, E) \) be a distributional section of \( E \). The wave front set \( WF(u) \) is a subset of \( T^* M \setminus \{0\} \), such that for any \( (x, \xi) \in T^* M \setminus \{0\}, \xi_x \notin WF(u) \) if and only if for any (phase function) \( \psi \in C_0^\infty(M \times \mathbb{R}^p, \mathbb{R}) \), \( d\psi(\cdot, y)_x = \xi_x \), there is an \( s \in \Gamma_0^\infty(E) \), \( s(x) \neq 0 \), and a neighborhood \( U_y \) of \( y \) in \( \mathbb{R}^p \), such that for all \( n \in \mathbb{N} \).

\[
|\langle u, \exp(-it\psi(x, y'))s \rangle| = O(t^{-n})
\]

(2.24)
for all \( y' \in U_y \) and \( t \to \infty \).

This is a less intuitive but invariant definition. The more intuitive definition starts a distribution \( u \) on \( \mathbb{R}^n \), take its Fourier transform \( \hat{u} \), the directions along which \( \hat{u} \) is rapid decaying are called the nice ones, the rest are in the wave front set, then it is extended to vector-valued distributions in the obvious way. This definition can be verified to be invariant on the cotangent bundle.

Among the first things, wave front set is a refinement of the singular support. This can be shown basically by the key the fact that a distribution with rapid decaying Fourier transform must be smooth.

**Proposition 2.4.2.** \( WF(u) \) is a closed conic subset. The projection of \( WF(u) \) onto \( M \) is the singular support of \( u \).

For distributions, push-forward (the adjoint operator of pull-back) are natural as pull-back is for functions. When \( M \) is compact (which is our case) or when \( f \) is proper it is defined.

**Definition 2.4.3.** Let \( f : M \to N \) be a smooth map. Let \( S \) be a conic subset of \( T^*M \setminus \{0\} \). then the push-forward \( f_*S \) of \( S \) is defined to be

\[
  f_*S = \{(y, \eta) \in T^*N \setminus \{0\} : \exists x \in M, y = f(x), (x, f^*(\eta)) \in (S \cup 0_{T^*M})\}. \tag{2.25}
\]

**Theorem 2.4.4.** Let \( f : M \to N \) be a smooth map, \( u \in C^{-\infty}(M) \) such that \( f|_{\text{supp}(u)} \) is proper. Then the push-forward \( u \mapsto f_*(u) \) is a well-defined continuous linear map

\[
  f_* : C^{-\infty}(M) \to C^{-\infty}(N). \tag{2.26}
\]

Moreover,

\[
  WF(f_*u) \subset f_*(WF(u)). \tag{2.27}
\]
We list some important examples that we will need.

**Example 2.4.5.** (1) Take $M = \{p\}$, the 0-dimensional manifold of a single point, $f$ is simply decided by its image $y = f(p)$ in $N$. Let $1_M$ be the constant function on $M$ with value 1. Then $f_*(1_M)$ is simply $\delta_y$, the delta function of $N$ at the point $y$. Simple computations right from definitions show that $WF(\delta_y) = T^*_yN \setminus \{0\}$, $WF(1_M) = \emptyset$, and $f_*(\emptyset) = T^*_yN \setminus \{0\}$.

(2) A little more generally, let $f$ be an embedding, let $\alpha$ be a smooth function on $M$. compactly supported if $M$ is noncompact. $f_*(\alpha) = \alpha \mu_M$ is usually called the density of $M$ in $N$. direct computation shows $WF(f_*\alpha) = \{(x, \xi) \in T^*N, x \in supp(\alpha), \xi \in N^*_M(N) \setminus \{0\}\}$, where $N^*_M(N) = ker(f^*)$ is the conormal bundle of $M$ in $T^*N$. Compare $WF(\alpha) = \emptyset$ and $f_*(\emptyset) = ker(f^*)$.

(3) Let $\pi : M \times N \to N$ be a the projection to $N$. $M$ needs to be compact to have push-forward of $\pi$ defined. For $v \in C^\infty_0(N)$ and $u \in C^{-\infty}(M \times N)$,

$$
(\pi_*u, v) = (u, \pi^*v) = (u, 1_M \otimes v) = \int_M (u(x, \cdot), v(\cdot))dx. \tag{2.28}
$$

the last integral is only formal. It is true for smooth $u$ where a volume form (of our -choice) is implicit involved. Computation shows $WF(\pi_*u)$ is the projection of $WF(u)$ onto $T^*N$.

In fact, the push-forward operation contains the integration for a distribution as a special case. Let $N = \{pt\}$ be a single point manifold so the functions and distributions on it are just complex numbers. Let $c$ be the unique map $M \to N$, $c_*(u)$ is defined when $M$ is compact and it is just the number

$$
c_*(u) = (u, 1_M) = \int_M u(x)dx. \tag{2.29}
$$

where the last integral is formal.
(4) Now let $f : M \to N$ be a submersion, $M$ and $N$ both compact. Let $u$ be a smooth function on $M$, so $WF(u) = \emptyset$. Then the push-forward $f_* u$ is still an integration along the vertical fibers with our chosen volume form.

$$WF(f_* u) = f_*(WF(u)) = \emptyset,$$

so as we know $f_* u$ is smooth. When $u$ is a distribution $f_*(WF(u))$ is again the projection along vertical directions.

It is straightforward to extend the above theorem to maps between bundles, since the wave front set is a local notion. We restate the theorem in the form we will use most of the time.

**Theorem 2.4.6.** Let $f : (E, M) \to (F, N)$ be a smooth bundle map between two hermitian bundles, $u \in \Gamma^{-\infty}(E)$ such that $f|_{supp(u)}$ is proper. Then the push-forward $u \mapsto f_*(u)$ is a well-defined continuous linear map

$$f_* : \Gamma^{-\infty}(M) \to \Gamma^{-\infty}(N).$$

Moreover,

$$WF(f_* u) \subseteq f_*(WF(u)).$$

Next we consider linear operators of a particular type. Let $(E_1, M_1), (E_2, M_2)$ be two hermitian bundles, and let

$$A : \Gamma^{\infty}(E_1) \to \Gamma^{-\infty}(E_2)$$

be a linear and continuous operator (such as a pseudodifferential operators and more generally a Fourier integral operator). It is obvious they can be viewed as a distribution on the exterior tensor product bundle $E_2 \boxtimes E_1$, that is, with a distributional
Definition 2.4.7. The wave front relation of $\mathcal{A}$ is defined as

$$WF'(\mathcal{A}) = \{(y, x, 0_y, 0_x) \in T^*\mathcal{A} \setminus \{0\} : \exists (y, x, 0_y, 0_x) \in WF(K)\}. \quad (2.35)$$

And we define $WF'_{M_i}(\mathcal{A})$, called $WF'(\mathcal{A})$'s projection on $T^*M_i$, $i = 1, 2$, as follows:

$$WF'_{M_i}(\mathcal{A}) = \{(x, \xi_x) \in T^*M_i \setminus \{0\} : \exists y \in M_2, (y, x, 0_y, \xi_x) \in WF'(\mathcal{A})\}. \quad (2.36)$$

and

$$WF'_{M_2}(\mathcal{A}) = \{(y, \eta_y) \in T^*M_2 \setminus \{0\} : \exists x \in M_1, (y, x, \eta_y, 0_x) \in WF'(\mathcal{A})\}. \quad (2.37)$$

In the event $M_1 = M_2 = M$, which is our main case of concern here, there is a confusion between the above two notations. We then use an extra label, that is, $WF'_{M,1}$ and $WF'_{M,2}$ respectively.

Let $(E_i, M_i), i = 1, 2, 3$ be hermitian bundles over compact manifolds. $A$ as described above and

$$B : \Gamma^\infty(E_2) \to \Gamma^{-\infty}(E_3). \quad (2.38)$$

both linear and continuous operators.

Theorem 2.4.8. If $WF'_{M_2}(\mathcal{A}) \cap WF'_{M_2}(B) = \emptyset$, then

$$B \circ A : \Gamma^\infty(E_1) \to \Gamma^{-\infty}(E_3) \quad (2.39)$$

is a well defined, linear and continuous operator. Moreover,

$$WF'(B \circ A) \subset WF'(B) \circ WF'(\mathcal{A})$$

$$\cup (WF'(B)_{M_2} \times 0_{T^*M_1}) \cup (0_{T^*M_3} \times WF'(\mathcal{A})_{M_1}). \quad (2.40)$$
(By definition WF' is a binary relation between $T^*M_2$ and $T^*M_1$, the composition between wave front relations $\circ$ are that of binary relations.)

**Remark 2.4.9.** Some formulas here about wave front sets (or wave front relations) could be stated shorter if we include the zero section into the wave front set. For example if we use the notation $WF_0 = WF \cup 0$ (zero section), then in theorem 2.4.8 we have

$$WF_0'(B \circ A) \subset WF_0(B) \circ WF_0(A). \quad (2.41)$$

$WF = \emptyset$ is equivalent to $WF_0 = 0$. We shall follow the traditional notation.

We will need the following obvious corollary. Assume now $(E_i, M_i) = (E, M)$ are all the same.

**Corollary 2.4.10.** If $WF'_{M,2}(A) \cap WF'_M(B) = \emptyset$ and $WF'(B) \circ WF'(A) = \emptyset$, then $B \circ A$ is a smoothing operator, in particular, it is trace class.

**Theorem 2.4.11.** In theorem 2.4.6. $f_*$ has a distribution kernel, and

$$WF'(f_*) \subset \{(\xi_x, \eta_y) \in T^*(M \times N) \setminus \{0\} : f^*(\eta_y) = -\xi_x \text{ or } f^*(\eta_y) = 0\}. \quad (2.42)$$

*Proof.* Apply theorem 2.4.6 to the embedding $M \to M \times N, x \mapsto (x, f(x))$. □

### 2.5 Wave front sets of pseudodifferential operators

In this section we recall some important results that we need found in standard textbooks such as [30], [16]). The proofs are not reproduced.

**Definition 2.5.1.** Let $P : \Gamma_{0}^{\infty}(E) \to \Gamma^{-\infty}(F)$ be a pseudodifferential operator. The essential support $\text{ess.supp}(P)$ of $P$ is the compliment in $T^*M \setminus \{0\}$ of the largest open conic subset of the cotangent bundle on which the symbol has order $-\infty$).
In particular, \( P \) is a smoothing operator if and only if \( \text{ess.supp}(P) = \emptyset \).

**Definition 2.5.2.** When \( E = F \), the characteristic set \( \text{char}(P) \) of \( P \) is the subset of \( T^*M \) where the principal symbol is not invertible as bundle morphism.

\( P \) is elliptic if and only if \( \text{char}(P) = 0_{T^*M} \), \( P \) is transversally elliptic relative to a \( G \) action if \( \text{char}(P) \cap T^*_G M = \mu^{-1}(0) \subset 0_{T^*M} \).

**Theorem 2.5.3.** *(Strong Pseudo-local Property)*

\[
WF(Pu) \subset WF(u) \cap \text{ess.supp}(P). \tag{2.43}
\]

It is easy to see that \( WF'(P) \) is contained in the diagonal. And when \( \text{ess.supp}(P) \) is empty \( WF(Pu) = \emptyset \) so \( P \) is smoothing.

**Theorem 2.5.4.** *(Regularity)*

\[
WF(u) \subset WF(Pu) \cup \text{char}(P). \tag{2.44}
\]

This is a generalization of the regularity theorem for the elliptic operators, and that is why we need it. In particular when \( P \) is elliptic, where \( \text{char}(P) = 0_{T^*M} \) and \( \text{ess.supp}(P) = T^*M \setminus \{0\} \), combining the previous two theorems we have \( WF(Pu) = WF(u) \).

### 2.6 Crossed product algebra and action groupoid

**Definition 2.6.1.** \( \mathcal{A} = C_c^\infty(M) \ltimes G \) is defined to be the \(*\)-algebra of smooth crossed product. That is, \( \phi, \psi \in \mathcal{A} = C_c^\infty(M) \ltimes G \) means \( \phi, \psi \in C_c^\infty(M \times G) \) with product

\[
\phi \ast \psi(x, g) = \int_G \phi(x, h) \psi(h^{-1} x, h^{-1} g) d\mu(h) \tag{2.45}
\]

\[
\phi^*(x, g) = \overline{\phi(g^{-1} x, g^{-1})}. \tag{2.46}
\]
To show that the above product forms a $\ast$-algebra the only nontrivial part is the 
associativity of the product, which can be computed directly. Instead we will only 
show that this a priori binary operation is a special case of the convolution product 
of the action groupoid, a fact that helps our similar arguments later on.

A concise definition of a groupoid $\mathcal{G}$ is that a groupoid is a small category (in which 
all morphisms form a set) in which every morphism is invertible. A more detailed 
definition in [27] is

$$\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_0; \tau, \sigma, \iota, \cdot, (\cdot)^{-1}) \tag{2.47}$$

with the five maps satisfying the axioms for a small category. We will not elaborate 
on the general groupoids, our example below explains our notation in detail.

**Example 2.6.2.** The action groupoid is a groupoid $\mathcal{G}_\rho$, where the set of unities $\mathcal{G}_0$ (or 
objects) is $M$, and the set of arrows $\mathcal{G}_1$ (or morphisms) is $M \times G$, and with the five 
maps as follows:

(1) The target map $\tau : M \times G \to M$: $\tau(x, g) = x$.

(2) The source map $\sigma : M \times G \to M$: $\sigma(x, g) = \rho(g^{-1})x = g^{-1}x$ (we will sometimes 
omit the action $\rho$ if no confusion will be caused).

(3) The unity map $\iota : M \to M \times G$: $\iota(x) = (x, e)$.

(4) The partially defined multiplication defined on the subset of $\mathcal{G}_1 \times \mathcal{G}_1$ where the 
source of the first component matches the target of the second component, is given 
simply by the multiplication of the group $G$, That is $(y, h) \cdot (x, g)$ is defined only when $h^{-1}y = x$ hence $y = hx$ and

$$ (hx, h) \cdot (x, g) = (hx, hg). \tag{2.48} $$

(5) The inverse map $(\cdot)^{-1} : M \times G \to M \times G$: $(x, g)^{-1} = (g^{-1}x, g^{-1})$. 

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Remark: for our purpose, our definition of the action groupoid is not exactly the same as the one commonly used in the literature, where typically an arrow \((x, g)\) has source \(x\) and target \(gx\). It is easy to check that the map \((x, g) \mapsto (g^{-1}x, g)\) gives a groupoid isomorphism between them.

**Definition 2.6.3.** A groupoid \(\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_0; \tau, \sigma, \iota, \cdot, (\cdot)^{-1})\) is a topological groupoid if \(\mathcal{G}_1, \mathcal{G}_0\) are topological spaces. all five maps are continuous, \(\tau, \sigma\) are open maps, and \(\iota\) is a homeomorphism onto its image. It is called locally compact if \(\mathcal{G}_1\) is locally compact (so is \(\mathcal{G}_0\) as a consequence). \(\mathcal{G}\) is a smooth groupoid when \(\mathcal{G}_1, \mathcal{G}_0\) are both smooth manifolds. all five maps are smooth, \(\tau, \sigma\) are both submersions, and \(\iota\) is an embedding.

In our case \(\mathcal{G}\) is obviously smooth because \(\mathcal{G}\) are smooth manifolds, \(\tau, \sigma\) are submersions, \(\iota\) is an embedding, multiplication map and the inverse map are smooth.

For a locally compact topological groupoid, it is shown that a Haar measure gives a convolution algebra. both definition introduced for our case.

**Definition 2.6.4.** A smooth Haar system on a smooth groupoid \(\mathcal{G}\) is a family of positive Radon measures \(\lambda^* = \{\lambda^x : x \in \mathcal{G}_0\}\) on \(\mathcal{G}_1\) satisfying the following conditions:

1. For any \(x \in \mathcal{G}_0\), the support of \(\lambda^x\) is in \(\mathcal{G}^x = \{\alpha \in \mathcal{G}_1 : \tau(\alpha) = x\}\) and \(\lambda^x\) is a smooth measure on \(\mathcal{G}^x\).

2. (Left invariance) For any \(x \in \mathcal{G}_0\) and any continuous function \(f : \mathcal{G}^x \to \mathbb{C}\) and any \(\alpha \in \mathcal{G}^x\) we have

\[
\int_{\mathcal{G}^x} f(\beta) d\lambda^x(\beta) = \int_{\mathcal{G}^x} f(\alpha \beta) d\lambda^{\sigma(\alpha)}(\beta) \tag{2.49}
\]

(in other words, \(\alpha_*(\lambda^{\sigma(\alpha)}) = \lambda^{\tau(\alpha)}\)).
(3) (Smoothness) For any $\phi \in C_\infty^c(G)$, the map

$$x \mapsto \int_{G^x} \phi(\beta) d\lambda^x(\beta)$$

is a smooth function on $G_0$.

**Proposition 2.6.5.** A smooth Haar system on a smooth groupoid $G$ defines a convolution product on $C_\infty^c(G)$ by

$$(\phi \ast \psi)(\alpha) = \int_{G^\tau(\alpha)} \phi(\beta) \psi(\beta^{-1}\alpha) d\lambda^\tau(\alpha).$$

And it is a $*$-algebra with

$$\phi^*(\alpha) = \overline{\phi(\alpha^{-1})}. \hspace{2cm} (2.52)$$

In the case of action groupoid $G_\rho$, let $\lambda$ be the Haar measure on $G$. $G^x = \{(x, g) : g \in G\}$ is homeomorphic to $G$ by the projection $\pi$ on the second component. So we simply define $\lambda^x = \pi^*\lambda$. This means $x$ is just a label for the copy of $G$. Left invariance of $\{\lambda^x : x \in M\}$ simply follows that of $G$, and the rest in the Haar system definition are also obvious.

Now in proposition 2.6.5 let $\alpha = (x, g)$. $\beta = (x, h)$ since we need $\beta \in G^\tau(\alpha)$, then $\beta^{-1}\alpha = (h^{-1}x, h^{-1}g)$. It is clear that definition 2.6.1 is just a special case definition 2.6.5. So we have justified definition 2.6.1.

**Lemma 2.6.6.** Let $E$ be a $G$-bundle over $M$ with the action of an element $g \in G$ denoted by $\rho(g) : E_x \to E_{gx}$. Then $\Gamma(M, E)$ is a $\mathcal{A}$-module, with the actions $\phi \in \mathcal{A}$ denoted by $\rho$, for any $s \in \Gamma(M, E)$,

$$(\rho(\phi)s)(x) = \int_G \phi(x, g)\rho(g)(s(g^{-1}x))d\mu(g). \hspace{2cm} (2.53)$$
In the special case when $M$ is a single point space, $\rho$ is nothing but the induction of the representation from the identity subgroup to $G$.

This lemma can be proven by direct computation. But we amount it to some standard results of groupoid action in the following discussion.

**Definition 2.6.7.** Let $G$ be a smooth groupoid, a left $G$ action a smooth manifold $X$ is a pair $(\rho, t)$, where $t : X \to G_0$ is a smooth map and

$$\rho : G_1 \times_{G_0} X \to X$$

(2.54)

where

$$G_1 \times_{G_0} X = \{(\alpha, x) \in G_1 \times X : \sigma(\alpha) = t(x)\}, \quad (2.55)$$

with the following properties

1. for any $z \in G_0$, $\iota(z) = t(x)$, we have $\rho(\iota(z), x) = x$.
2. for all $\alpha$, $\beta$ with $\sigma(\alpha) = \tau(\beta)$ and $\sigma(\beta) = t(x)$. we have

$$\rho(\alpha \beta, x) = \rho(\alpha, \rho(\beta, x)). \quad (2.56)$$

As in the case of an ordinary group action, we abbreviate $\rho(\alpha, x)$ as $\rho(\alpha)x$ or even $\alpha x$ when $\rho$ is clear under context.

Let $E$ be a hermitian bundle over $M$ and $\rho$ a unitary $G$ action on $E$, which means for each $x \in M$, $\rho(g) : E|_x \to E|_{gx}$ is unitary. $\rho$ introduces a $G_{\rho}$ action on $U(E)$ (the bundle of fiber-wise unitary transformations on $E$). We will denote this groupoid action still by $\rho$. The groupoid action is the pair $(\rho, t)$ where $t : U(E) \to M$ is the projection along the fiber and for $u_x \in U(E)|_x = U(E_x)$,

$$\rho((gx, g), u_x) = \rho(g)u_x \quad (2.57)$$

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is an element in $U(E)|_{gx}$.

The proof of the following is standard, see [27].

**Proposition 2.6.8.** Suppose $G$ acts on $X$ from the left, then $C_c^\infty(X)$ is a left $C_c^\infty(G)$-module by the following action: for $\phi \in C_c^\infty(G)$ and $f \in C_c^\infty(X)$,

$$(\phi f)(x) = \int_{G^x} \phi(\alpha)f(\alpha^{-1}x)d\lambda^x(\alpha)$$

(2.58)

From the natural action of $C_c^\infty(U(E))$ on $\Gamma(M, E)$, and combined with the action of $A$ on $C_c^\infty(U(E))$, the action in lemma 2.6.6 is shown to be the composition.

**Lemma 2.6.9.** Let $E$ be a $G$-vector bundle, for any $\phi \in A$, the linear continuous operator

$$\rho(\phi) : \Gamma^\infty(E) \to \Gamma^\infty(E)$$

(2.59)

extends to a bounded linear operator for any $s \in \mathbb{R}$.

$$\rho(\phi) : H^s(E) \to H^s(E).$$

(2.60)

**Proof.** The statement is obvious on $L^2(E) = H^0(E)$ by the definition.

Recall that for any $s \in \mathbb{R}$ let

$$\Lambda_s = (1 + \nabla_E^* \nabla_E)^{s/2}.$$  

(2.61)

where $\nabla_E$ is a covariant derivation on $E$, and $\nabla_E^*$ is its dual with respect to the norm on $L^2(E)$. $\Lambda_s$ is a pseudodifferential operator of order $s$, and by standard Sobolev space theorem $\Lambda_s$ gives an isomorphism $L^2(E) \to H^{-s}(E)$, and also an isomorphism $H^s(E) \to L^2(E)$.
So we need only to show that for any \( \phi \in \mathcal{A} \Lambda_s \rho(\phi) \Lambda_{-s} \) is bounded on \( L^2(E) \).

For the moment regard \( \rho(\phi) \) as \( \rho(\phi) 1_{U(E)} \) in \( C_c^\infty(U(E)) \), acting on \( L^2(E) \) as a linear operator,

\[
\Lambda_s \rho(\phi) \Lambda_{-s} = \int_G \phi((x,g)) \Lambda_s (\rho(g) \Lambda_{-s} \rho(g^{-1})) \rho(g) d\mu(g). \tag{2.62}
\]

For any pseudodifferential operator \( P \) with principal symbol \( \sigma(P) \), \( \rho(g) P \rho(g^{-1}) \) is a pseudodifferential operator with principal symbol \( \rho(g) \cdot \sigma(P) \), in particular, with the same order as that of \( P \). So \( \Lambda_s (\rho(g) \Lambda_{-s} \rho(g^{-1})) \) is a pseudodifferential operator of order 0, uniformly bounded on the compact support of \( \phi \). \( \phi \) is in particular absolutely integrable, so \( \Lambda_s \rho(\phi) \Lambda_{-s} \) is bounded. \( \square \)
CHAPTER 3

SYMBOLIC CALCULUS FOR A CROSSED PRODUCT ALGEBRA

3.1 The crossed product $\Psi^k(E) \rtimes G$

For a manifold with a group action, we will often use operators which are not in the algebra of pseudodifferential operators. For example, in our purpose we frequently use the commutator of a pseudodifferential operator and the action of $\rho(\phi)$, which is in general not pseudo-local. We will define a large enough algebra $\Psi^k(E) \rtimes G$ which is, in short, the one generated by both the group action and pseudodifferential operators. $\Psi^k(E) \rtimes G$ may also be viewed as groupoid algebra as introduced in the previous chapter. Here the main advantage of the groupoid viewpoint is that many properties of the new algebra is easily reproduced.

Similar to definition 2.6.1, we introduce:

Definition 3.1.1. Let $E$ be an hermitian vector bundle over $M$. $\Psi^k(E) \rtimes G$ is the algebra of families of pseudodifferential operators $P(g) \in \Psi^k(E)$, with product

$$ (P \star Q)(g) = \int_G P(h) \cdot (h^{-1}Q(h^{-1}g)h)d\mu(h), $$

(3.1)

$$ P^*(g) = g^{-1}P^*(g^{-1})g. $$

(3.2)
Since the composition, the conjugation with $g, h \in G$, and the integration on the parameter space are closed on $\Psi^k(E)$, the above operation gives elements within $\Psi^k(E) \times G$. And to show that it gives an algebra we make use the groupoid argument almost identically.

It follows directly that $A \subset \Psi^0(E) \times G$. $\Psi^\ast(E) \times G$ has no identity, and $\Psi^\ast(E) \times G$ is a $\Psi^\ast(E)$-bimodule with additive degrees. In fact, $\Psi^k(E) \times G$ is a class of Fourier integral operators, but we do not exploit this fact.

An important fact is that we may identify an element of $\Psi^k(E) \times G$ with its action on $\mathcal{H} = L^2(E)$ or in general, any Sobolev space $H^s(E)$.

**Proposition 3.1.2.** With notations as in lemma 2.6.6. for $P = P(g) \in \Psi^0(E) \times G$, for any $s \in \mathbb{R}$, elements of $\Psi^k(E) \times G$ acting on $H^s(E)$ to $H^{s+k}$ by

$$ (Ps)(x) = \int_G P(g)\rho(g)(s(g^{-1}x))d\mu(g), \quad (3.3) $$

are bounded. This implies $H^\infty(E)$ is a $\mathbb{Z}$-graded $\Psi^\infty(E) \times G$-module; in particular, for any $s \in \mathbb{R}$, $H^s(E)$ is a $\Psi^0(E) \times G$-module.

**Proof.** By Lemma 2.6.9 and the uniformness of the family $P = P(g)$, and integrate on $G$, we reach all the above conclusion. □

From now on we may identify an element of $\Psi^k(E) \times G$ with its action on $\mathcal{H} = L^2(E)$. The composition of two such operators are given by the $\ast$-product. The algebra of pseudodifferential operators is not a subalgebra of $\Psi^\infty(E) \times G$ since the composition rules are different. However the following compositions between pseudodifferential operators and elements in $\Psi^\infty(E) \times G$ make $\Psi^\infty(E) \times G$ a bimodule of
pseudodifferential algebra $\Psi^k(E)$: for $P = P(g) \in \Psi^k(E) \times G$ and $Q \in \Psi^k(E)$

\[(P \ast Q)(g) = P(g) \cdot Q \quad \text{(3.4)}\]
\[(Q \ast P)(g) = (gQ g^{-1}) \cdot P(g).\]

It is straightforward to check that the above gives a $\Psi^\infty(E)$ action on $\Psi^\infty(E) \times G$ and

\[(\Psi^k(E) \times G) \cdot \Psi^l(E) \subset \Psi^{k+l}(E) \times G \quad \text{(3.5)}\]

$P \in \Psi^k(E) \times G$ has a distributional kernel, so we may compute its wave front relation. The next lemma is key to the definition of a transversally elliptic pseudodifferential operator.

**Lemma 3.1.3.** For any $P \in \Psi^k(E) \times G$,

\[WF'(P) \subset \{(g,\xi_x,\xi_x) : (x,\xi) \in \text{ess. supp}(P(g)), \xi_x \in (T^*_G M)_x\} \quad \text{(3.6)}\]

**Proof.** Recall that under the notation, $\rho : M \times G \to M$ is the action of $G$ on $M$. $\rho^* E$ is the pull-back bundle of $E$ on $M \times G$. As an operator, $P$ is the composition of

\[A : H^s(E) \to H^s(\rho^* E) \quad \text{(3.7)}\]

\[A_s(x,g) = \rho(g)(u(g^{-1}x)). \quad \text{(3.8)}\]

the family $P(g)$, and

\[B : H^{s+k}(\rho^* E) \to H^{s+k}(E), \quad \text{(3.9)}\]

\[Bu(x) = \int_G u(g,x) d\mu(g). \quad \text{(3.10)}\]
We now show that $A = (f_A)_*$ and $B = (f_B)_*$ are both push-forward operators, an embedding and a submersion respectively. $f_A : M \times G \to M \times M \times G$

$$f_A(x,g) = (gx, x, g)$$ (3.11)

which is the lower part of the a bundle map

$$\tilde{f}_A : \rho^*E \to E \otimes \rho^*E$$ (3.12)

$$\tilde{f}_A : e_{(x,g)} \mapsto \rho(g)e_{gx};$$ (3.13)

$f_B : \rho^*E \to E$ is the projection to the first component, which is the lower part of the bundle map

$$\tilde{f}_B : E \otimes \rho^*E \to E$$ (3.14)

$$\tilde{f}_B : (e_y, e'_{x,g}) \mapsto e_y.$$ (3.15)

It is easy to check that $A = (f_A)_*$ and $B = (f_B)_*$.

By theorem 2.4.11 we have

$$WF'(A) \subset \{ (g_\xi, \xi, \gamma_g) : \mu_*(\xi) = \gamma_g \}$$ (3.16)

where $\mu_* = \mu_*^{M,G}$ is the moment map defined earlier (with kernel $\{ \xi_x \in (T^*_G M)_x \}$). In particular both projections $WF'_M(A) = \emptyset$ and $WF'_{M \times G}(A) = \emptyset$. And for the wave relation of $P = P(g)$ we apply theorem 2.5.3, which has empty projections. At last by theorem 2.4.11 we have

$$WF'(B) \subset N_{M \times M}^*(M \times M \times G) = \{(\eta_y, \xi_x, 0)\}.$$ (3.17)
Although it has nonempty right projections, the intersection with the (empty) left projection of $P$ is empty which makes the composition friendly as far as wave front sets are concerned.

Apply theorem 2.4.8 twice we have the stated conclusion. □

**Corollary 3.1.4.** For any $\phi \in \mathcal{A},$

$$WF'(\rho(\phi)) \subset \{(g,\xi_x,\xi_x) : (x,g) \in supp(\phi), \xi_x \in (T^*_GM)_x\}.$$  

### 3.2 Weakly Parametric pseudodifferential operators

For a self-adjoint, elliptic, positive ordered pseudodifferential operator $A$, the analysis of the resolvant $(A - \lambda)^{-1}$ for $\lambda$ in a certain sector $\Gamma$ of $\mathbb{C}\backslash \mathbb{R}_+$ plays a very important part in spectral asymptotics. In this section we introduce some basic concept and results by Grubb and Seeley [18] in order to apply them to families of pseudodifferential operators.

Let $\Gamma$ be a sector of $\mathbb{C}\backslash \mathbb{R}_+$ of the form

$$\{\lambda = re^{i\theta}|r > 0. \theta \in I \subset (0, 2\pi)\}. \quad (3.18)$$

where $I$ is a subinterval, let $\Gamma^o$ be its image under conjugacy which has the same form. We are mainly interested in the case when $\Gamma$ is near the negative real axis but some discussion apply to any sector on $\mathbb{C}\backslash \{0\}.$

For simplicity we start with symbols in $M = \mathbb{R}^n$ and with scalar values. What matters for us is the behavior of $r = |\mu| \to \infty$ for $\mu \in \Gamma^o$ or equivalently, the behavior of $z = 1/\mu \to 0$ for $z \in \Gamma,$ which we use more often.
Definition 3.2.1. The weakly parametric symbol space $S^{m,0}(\mathbb{R}^n \times \mathbb{R}^n, \Gamma)$ consists of functions $p(x, \xi, \mu)$ that are holomorphic in $\mu = 1/z \in \Gamma^0$ for

$$|\xi, \mu| = (|\xi|^2 + |\mu|^2)^{1/2} \geq \epsilon$$

(3.19)

for some $\epsilon$ and satisfy, for all $j \in \mathbb{N}$, $1/z \in \Gamma$,

$$\partial_j^j p(\cdot, \cdot, 1/z) \in S^{m+j}(\mathbb{R}^n \times \mathbb{R}^n)$$

(3.20)

locally uniformly in $|z| \leq 1$. Moreover, for any $d \in \mathbb{C}$, define

$$S^{m,d} = \mu^d S^{m,0}.$$  

(3.21)

That is, $p(\cdot, \cdot, 1/z) \in S^{m,d}$ if $z^d p(\cdot, \cdot, 1/z)$ satisfies (3.20).

$S^{m,d}$ are Fréchet spaces with semi-norms implied in the definition.

We now list some properties:

1. As symbols constant on $\Gamma$, $S^{m} \subset S^{m,0}$.
2. If $m < 0$, then $f(x, \xi) \in S^{m}$ implies $p(x, \xi, \mu) = f(x, \xi/\mu) \in S^{0,0}$.
3. $S^{m,d} \subset S^{m',d'}$ for $m \leq m'$, $d' - d \in \mathbb{N}$.
4. The following maps are continuous 

$$\partial_\xi^\alpha : S^{m,d} \to S^{m-|\alpha|,d}$$

$$\partial_z^j : S^{m,d} \to S^{m,d}$$

$$z^k : S^{m,d} \to S^{m,d-k}$$

$$\partial_z^l : S^{m,0} \to S^{m,0}$$

$$\partial_z : S^{m,0} \to S^{m+1,0} + S^{m,d+1} \quad d \neq 0$$

(3.22)

5. $S^{m,d} : S^{m',d'} \subset S^{m+m',d+d'}$.

As usual, we denote

$$S^{\infty,d} = \bigcup_{m \in \mathbb{R}} S^{m,d}, \quad S^{-\infty,d} = \bigcap_{m \in \mathbb{R}} S^{m,d}.$$  

(3.23)
**Definition 3.2.2.** Let $p_j, j \in \mathbb{N}$ be a sequence of symbols in $S^{m_j, d}$, where $m_j \downarrow 0$. Then an asymptotic expansion in $S^{m_0, d}$ (or $S^{\infty, d}$) for $p \in S^{m_0, d}$

$$p \sim \sum_{j \in \mathbb{N}} p_j$$

(3.24)

means that for any $N \in \mathbb{N}$,

$$p - \sum_{j=0}^{N} p_j \in S^{m_{N+1}, d}.$$  

(3.25)

As usual, asymptotic sum always exists for such a sequence $\{p_j, j \in \mathbb{N}, m_j \downarrow 0\}$.

The following is a generalization of classical polyhomogeneous symbols, which has asymptotic expansion of homogeneous in $\xi$ of integer-degree. whose corresponding pseudodifferential operators are called classical pseudodifferential operators (e.g. [32]).

**Definition 3.2.3.** $p \in S^{\infty, d}(\mathbb{R}^n \times \mathbb{R}^n, \Gamma)$ is called weakly polyhomogeneous if there exists a sequence of symbols $p_j \in S^{m_j, -d, d}$, homogeneous in $(\xi, \mu)$ for $|\xi| \geq 1$ of degree $m_j \downarrow -\infty$, such that $p \sim \sum p_j \in S^{\infty, d}$.

**Theorem 3.2.4.** For $p \in S^{m, d}(\mathbb{R}^n \times \mathbb{R}^n, \Gamma)$, the limits

$$p_{(d,k)}(x, \xi) = \lim_{z \to 0} \partial_z^k (z^d p(x, \xi, 1/z)) \in S^{m-k}(\mathbb{R}^n \times \mathbb{R}^n)$$

(3.26)

exists and for any $N \in \mathbb{N}$,

$$p(x, \xi, \mu) \sim \sum_{k=0}^{N} \mu^{-k} p_{(d,k)} \in S^{m+N, d-N-1}(\mathbb{R}^n \times \mathbb{R}^n, \Gamma).$$

(3.27)

This asymptotic expansion grows worse in $|\xi|$, as shown in the example

$$\frac{1}{|\xi|^2 + \mu} = \mu^{-1} \left( 1 - \frac{|\xi|^2}{\mu} + \frac{|\xi|^4}{\mu^2} - \ldots \right)$$

(3.28)
This new asymptotic expansion plays a basic supporting role in Grubb-Seeley $S^{m,d}$ symbol calculus. For instance, the following results about the asymptotic expansion of smoothing kernels and the invariance of symbol are direct consequences of theorem 3.2.4.

**Proposition 3.2.5.** For a parameterized pseudodifferential operator $P(\lambda)$ on $\mathbb{R}^n$ with symbol $p(x, \xi, \lambda) \in S^{-\infty,d}$, its kernel has an expansion

$$K(x, y, \lambda) \sim \sum_{k \in \mathbb{N}} \lambda^{d-k} K_k(x, y).$$

with $K_k$ smooth on $\mathbb{R}^n \times \mathbb{R}^n$, and

$$K - \sum_{k=0}^{N} K_k(x, y) \mu^{d-k} \in C^\infty(\mathbb{R}^{2n} \times \Gamma).$$  (3.30)

holomorphic in $\mu \in \Gamma^o$ for $|\mu| \geq 1$.

In particular for smooth functions $\phi$ and $\psi$ with disjoint support, any $P(\lambda)$ with symbol in $S^{m,d}$, $\phi P(\lambda) \psi$ is such an example.

**Proposition 3.2.6.** Let $\chi : U \rightarrow V$ be a diffeomorphism of open subsets of $\mathbb{R}^n$. Let $P(\lambda)$ be a weakly parametrized pseudodifferential operator on $V$ with symbol $p(x, \xi, \mu)$ in $S^{m,d}(V \times \mathbb{R}^n \times \Gamma)$. Then $Q = \chi^* \cdot P \cdot \chi_*$ is a weakly parametrized pseudodifferential operator on $U$ with symbol $q(x', \xi', \mu)$ in $S^{m,d}(U \times \mathbb{R}^n \times \Gamma)$ with

$$q(x', \xi', \mu) - p(\chi(x'), t(d\chi)^{-1}(\xi'), \mu) \in S^{m-\epsilon,d}(U \times \mathbb{R}^n \times \Gamma)$$  (3.31)

for some sufficiently small $\epsilon > 0$.

Now we can extend the definition of weakly parametrized symbols and pseudodifferential operators to functions on a manifold and to sections of vector bundles.
**Definition 3.2.7.** A parametrized pseudodifferential operator $P(\mu)$ on $M$, $\mu \in \Gamma^\circ$ is in $\Psi^{m,d}(M) \times \Gamma$ if for any $\phi, \psi \in C^\infty_c(M)$, with support in a common coordinate system $U$, $\phi P(\mu) \psi$ has a symbol in $S^{m,d}(U \times \mathbb{R}^n \times \Gamma)$.

**Definition 3.2.8.** $\Psi^{-\infty,d}(M) \times \Gamma$, consists of operators parametrized in $\Gamma$ with kernel $K(x, y, \mu)$, smooth in $(x, y)$, holomorphic in $\mu$ for $\mu \in \Gamma^\circ$, $|\mu| \geq 1$, and has expansion

$$K(x, y, \mu) \sim \sum K_k(x, y, \mu)\mu^{d-k}$$

with

$$\partial^k_x \partial^k_\xi (K - \sum_{k=0}^J K_k\mu^{d-k}) = O(\mu^{d-j-1})$$

for $|\mu| \to \infty$ locally uniformly.

For $P(\mu)$, we can define the symbol first in local chart, piecing together with a proper partition of unity, the result may be considered as a symbol on so called $S^{\infty,d}(T^*M \times \Gamma)$, unique modulo $S^{-\infty,d}(T^*M \times \Gamma)$. When we choose local charts that are with exponential maps of the chosen connection, we may also use Widom's symbol for $P(\mu)$ which is globally defined on $S^{\infty,d}(T^*M \times \Gamma)$. Then the symbol decides $P(\mu)$ modulo an operator in $\Psi^{-\infty,d}(M) \times \Gamma$.

As we see, as usual, the symbols $S^{m,d}$ are basically local so we can extend them to symbols and operators between bundles, denoted by $S^{m,d}(T^*M \times \Gamma; \text{Hom}(E, F))$ and $\Psi^{m,d}(M; E, F) \times \Gamma$, by a partition of unity, since vector bundles are locally trivial, it is is reduced to the trivial bundle case. For trivial bundles the symbols are finite ranked matrix-valued function we require all entries satisfying $S^{m,d}$ condition.

**Theorem 3.2.9.** Let $p(x, \xi) \in S^r(T^*M; E)$ be a weakly homogeneous symbol of order $m \in \mathbb{N}_+$ (i.e., $m$-homogeneous for $|\xi| \geq 1$), and assume $(p(x, \xi) + \mu^m)$ invertible on $36$
a closed sector \( \Gamma \), then it is weakly polyhomogeneous on \( \mu \), with weakly parametrized symbols

\[
(p(x, \xi) + \mu^m)^{-1} \in S^{-m,0}(\Gamma) \cap S^{0,-m}(\Gamma).
\] (3.34)

For example when \( p \) is strictly positive definite, there is a section \( \Gamma \) near negative real axis such that \( \mu = (-\lambda)^{1/m} \) has a unique root for \( \lambda \in \Gamma \), satisfying the invertibility condition. In this case this theorem gives an estimate for the symbol of the resolvant \((p-\lambda)^{-1}\) on \( \Gamma \), which is weakly homogeneous in \((\xi, \mu)\). This special case is our primary interest. it is a generalization of Shubin’s argument [32], where we are almost restricted only to differential operators.

### 3.3 Resolution of Singularities and the Asymptotic behavior of Oscillatory integrals

In pseudodifferential operators, we deal with oscillatory integrals with a very simple phase function, locally of the form

\[
\varphi(x, y; \xi) = \langle x - y, \xi \rangle.
\] (3.35)

Asymptotic analysis of the oscillatory integral with this phase function can be explicitly computable by using the polar coordinates on \( \xi \) plane.

Now in our case where \( G \) acts on \( M \) with a given a connection and a choice of the Widom’s function \( l \), we shall deal with oscillatory integral with a phase function

\[
\varphi(x, y, g; \xi) = l(x - g_\ast y, \xi).
\] (3.36)
Asymptotic behaviors of the above oscillatory integrals is dependent on the group action. Since it is in general not a Morse function, explicit results may not be feasible in general. However, qualitative results exists for analytic phase function, using resolution of singularities originated from [1].

By recent results of [20] and [25], we may assume that the action of $G$ is analytic on a compatible analytic atlas of $\mathcal{M}$. This atlas exists as shown in [20], and the uniqueness is proven in [25].

We will use results that follows the arguments by Bernstein, Melgrange, etc., in oscillatory integrals. We now quote a technical theorem from Melgrange.

**Theorem 3.3.1.** Let $\phi$ be a real valued nonzero analytic function on $\mathbb{R}^n$. For $u \in C^\infty_c(\mathbb{R}^n)$ (real or complex valued) a test function, let

$$I_{\phi,u}(\tau) = \int_{\mathbb{R}^n} e^{i\tau\phi(x)} u(x) d\text{vol}(x).$$

(3.37)

Then for $\tau \to \infty$

$$I(\tau) = \sum_{a,p,q} c_{a,p,q}(u) \tau^{a-p}(\ln \tau)^q,$$

(3.38)

where $\alpha \leq 0$ runs through a finite set of rational numbers. $p, q \in \mathbb{N}$ and $0 \leq q < n$. Moreover $c_{a,p,q}$ are distributions with support inside

$$S_{\phi} = \{ x \in \mathbb{R}^n : d\phi(x) = 0 \}.$$  

(3.39)

and with finite orders not exceeding $n$.

### 3.4 Trace asymptotics for $\Psi^{-n,-1}(E, \Gamma) \rtimes G$

For a family of pseudodifferential operators with parameters $(g, \mu)$, uniformly in $G$, and and weakly parametric in $\mu$ in a sector $\Gamma$, we find the asymptotics the kernel and trace of an operator in $\Psi^{-n}(E, \Gamma) \rtimes G$. 38
First, we may reduce the asymptotic into operators locally defined in a chart, by taking partition of unity, and consider the operators $\phi P(g, \mu)\psi$, and this only affect the asymptotic expansion with a difference of negative power expansions in $\mu$. Next, we observe that in the case of weakly polyhomogeneous symbols in $\mu$, by Widom's argument of symbol transition formula between different choices of connections, the kernel function keep the same asymptotic expansion in $\mu$ among different choices of connections, despite the symbol depends on the choice of connection. Under local coordinate system and the trivial connection on it, a pseudodifferential operator can be written as the Fourier integral form

$$OP(p)u(x) = \int \int e^{i(x-y)\cdot\xi} p(y, \xi) u(y) dyd\xi,$$  \hspace{1cm} (3.40)

and the distributional kernel is the integral on $\mathbb{R}^n(\xi)$, which has possible singularities only on the diagonal.

The next reduction is to the the scalar case.

**Proposition 3.4.1.** Let $U \subset \mathbb{R}^n$ be a relative compact open chart of $M$.

$$p(x, \xi, g, \mu) \sim \sum_{j=0}^{\infty} p_j(x, \xi, g, \mu) \in S^{-n+d\cdot d}(U \times \mathbb{R}^n, \Gamma, G)$$ \hspace{1cm} (3.41)

be a weakly polyhomogeneous, in $(\xi, \mu)$, uniformly in $G$, with $p_j$ homogeneous with degrees $m_j \downarrow -\infty$. Then $\int_G OP(p)p(g)dg$ has a continuous kernel $K_p(x, y, \mu)$ with a diagonal expansion

$$K_p(x, x, \mu) \sim \sum_{j=0}^{\infty} c_j(x)\mu^{m_j+n} + \sum_{\alpha, p, q} c'_{\alpha, p, q}(p)(x)\mu^{d+\alpha-q}(\ln \mu)^q$$ \hspace{1cm} (3.42)

for $|\mu| \rightarrow \infty$, locally uniformly in $\Gamma$. All terms are decided by the symbol expect $c'_{\alpha, p, 0}$.

The contribution to each $c_j, c'_{\alpha, p, q}$ are from finitely many $p_j$'s. The following proof is a adapted from the proof of theorem 2.1 in Grubb-Seeley [18].
Proof. Without loss of generality we may assume \( d = 0 \), the general case follows by considering \( \mu^d p \). We can assume \( U \) is a convex neighborhood of origin. We may assume the \( G \)-orbit sends \( \frac{1}{2}U \) to \( U \), and the partition of unity is subordinate to a covering by such \( \frac{1}{2}U \). The part of the kernel outside \( U \) is smooth, so in the following proof we may replace the integral over \( G \) by the integral on the closed subset of \( G \) that send \( \frac{1}{2}U \) to \( U \).

Let \( r_J \) be remainder terms:

\[
 r_J = \sum_{j=0}^{J} p_j. \tag{3.43}
\]

\[
 \int G OP(r_J)p(g) \, dg \text{ has kernel}
\]

\[
 K_{r_J}(x, x, \mu) = \int_{\mathbb{R}^n} \int_{G} e^{i(x-gy)\xi} r_J(x, \xi, \mu) \, dg \, d\xi \tag{3.44}
\]

so by theorem 3.2.4, for any \( N \in \mathbb{N}_+ \)

\[
 r_J(x, \xi, \mu) + \sum_{k=0}^{N} s_k(x, \xi) \mu^{-k} + O(|\xi|^{m_j-N} \mu^{-N}), \tag{3.45}
\]

where \( s_k \in S^{m_j+k} \). For any \( N \), taking large enough \( J \) will yield

\[
 K_{r_J}(x, x, \mu) = \sum_{k=0}^{N} c_{N, J, k}(x) \mu^{-k} + O(\mu^{-N}) \tag{3.46}
\]

so this will contribute like smoothing operators. But the contribution goes to possibly any non-positive integer regardless of how large \( N \) and \( J \) are.
Now we discuss the contribution of each homogeneous symbols $p_j \in S^{m_j, 0}$ which is homogeneous in $(\xi, \mu)$ of degree $m_j$. As usual, we split the integral into three parts:

$$K_{p_j}(x, x, \mu) = \int_{R^n} \int_G e^{i(x-y) \cdot \xi} p_j(x, \xi, \mu) dg d\xi$$

$$= \int_{|\xi| \geq |\mu|} \int_G e^{i(x-y) \cdot \xi} p_j(x, \xi, \mu) dg d\xi$$

$$+ \int_{|\xi| \leq 1} \int_G e^{i(x-y) \cdot \xi} p_j(x, \xi, \mu) dg d\xi$$

$$+ \int_{1 \leq |\xi| \leq |\mu|} \int_G e^{i(x-y) \cdot \xi} p_j(x, \xi, \mu) dg d\xi. \tag{3.47}$$

For the first integral, since $|\mu| \geq 1$ is the only interested situation, $p_j$ is homogeneous, so

$$\int_{|\xi| \geq |\mu|} \int_G e^{i(x-y) \cdot \xi} p_j(x, \xi, \mu) dg d\xi$$

$$= \mu^{m_j + n} \int_{|\xi| \geq 1} \int_G e^{i|\mu|(x-y) \cdot \xi} (|\mu|^{-n-m_j} p_j)(x, \xi, \mu/|\mu|) dg d\xi, \tag{3.48}$$

which gives the logarithm term in the estimate.

For the second and third integral, we use theorem 3.2.4 again.

$$p_j(x, \xi, \mu) = \sum_{k=0}^{M} \mu^{-k} q_k(x, \xi) + R_M(x, \xi, \mu), \tag{3.49}$$

where

$$q_k(x, \xi) = \frac{1}{k!} \partial_x^k p(x, \xi, 1/z) \in S^{m_j+k} \tag{3.50}$$

and homogeneous in $\xi$ for $|\xi| \geq 1$ of degree $m_j + k$, and $R_M = O(|\xi|^{m_j + M} \mu^{-M})$.

The second integral only contribute to non-positive powers of $\mu$.

The third integral break down into integrals of for those of $q_k$, and $R_M$ which are all homogeneous,

$$\mu^{-k} \int_{1 \leq |\xi| \leq |\mu|} \int_G e^{i(x-y) \cdot \xi} q_k(x, \xi) dg d\xi$$

$$= \mu^{-k} \int_{1}^{|\mu|} \int_{S^{m_j+k+n-1}} \int_G e^{i\tau(x-y) \cdot \xi} q_k(x, \xi') dg d\xi' d\tau \tag{3.51}$$
using polar coordinates in $\xi$-plane, we get another part that contribute to these logarithm expansions. For $R_M$, which can be extended to homogeneous in $\xi$ for all $\xi \neq 0$, with a difference of the second integral type, choose large enough $M > -n - m_j$, so that $r_M$ gives terms just as $q_k$, except also for non-positive powers of $\mu$.

The conclusion follows from the combination of three integrals.
CHAPTER 4

THE INDEX OF A TRANSVERSALLY ELLIPTIC OPERATOR

Let $P \in \Psi^k(E, F)$, $P : \Gamma(E) \to \Gamma(F)$, be a transversally elliptic pseudodifferential operator relative to $G$ action on $E$ and $F$. Let

$$\pi_P : L^2(E, M) \to \ker(P)$$

be the projection to the kernel of $P$. In this section we define the index as a distribution on $G$.

4.1 Definition of the index

Lemma 4.1.1. For any $u \in L^2(E)$, $WF(\pi_P(u)) \subset char(P)$.

Proof. By the regularity theorem 2.5.4,

$$WF(\pi_P(u)) \subset WF(P(\pi_P(u)) \cup char(P).$$

But $P(\pi_P(u)) = (P\pi_P)(u) = 0$ and so has empty wave front set.

Note transversal ellipticity is not assumed in this lemma.
Theorem 4.1.2. Let $P$ be a transversally elliptic operator. For any $\phi \in \mathcal{A}$, $\rho(\phi)\pi_P$ is a smoothing operator. In particular, for any $f \in C_c^\infty(G)$, $\rho(f)\pi_P$ is trace class.

Proof. $\rho(f) = \rho(\pi_2^*(f))$ is a special case of $\rho(\phi)$. By Lemma 4.1.1, Lemma 3.1.4, Theorem 2.4.8, for any $u \in L^2(E),\nabla F(\rho(\phi)\pi_Pu) \subset WF(\rho(\phi)) \circ WF(\pi_Pu)$.

Since $P$ is transversally elliptic, $(T_cG \cdot M) \cap char(P) = \emptyset$, the above composition is empty. So $\rho(\phi)\pi_P$ is a smoothing operator, in particular, trace class. □

Now we are ready to introduce the definition of the index of a transversally elliptic operator. The following is equivalent to Atiyah's original definition [2]. Atiyah pointed out in [2] the wave front set approach by Hörmander (cf., eg. [28]) will show that the index below also makes sense as a distribution on the Lie group even for non-compact Lie groups.

Definition 4.1.3. The index of a transversally elliptic operator $P$ is defined to be the distribution on $G$ such that for any $f \in C_c^\infty(G)$,

$$index^G_a(P)(f) = Trace(\rho(f)\pi_P) - Trace(\rho(f)\pi_P^*) \quad (4.3)$$

We observe that it is possible to define the index without the $G$-invariant condition, cf. [23], [24]. Since $P$ is $G$-invariant under our assumption, the index is a central distribution on $G$.

Definition 4.1.3 naturally extends to a distribution on the groupoid $\mathcal{G}$ induced by the action of $G$ on $M$. 

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Definition 4.1.4. We define the local index density of a transversally elliptic operator $P$ as the distribution on $G_1 = M \times G$ such that for any $\phi \in \mathcal{A} = C^\infty_c(M \times G)$,

$$index^G(P)(\phi) = Trace(\rho(\phi)\pi_P) - Trace(\rho(\phi)\pi_{P^*}).$$ (4.4)

When $G$ is the trivial group, then $index^G$ is a smooth function on $M$ – the index density as in the heat kernel proof of the index theorem for classical operators – and its integral on $M$ is the index of the operator $P$. The index density itself is not a topological invariant, although the index – its integral on $M$ – is.

4.2 Transversal parametrix and transversally smoothing operators

Proposition 4.2.1. Let $P$ as above be a strongly transversally elliptic pseudodifferential operator of order $m$. Then there exists $Q$, a transversally elliptic pseudodifferential operator of order $-m$, with principal symbol $\sigma(Q) = \sigma(P)^{-1}$ on $T^*_G M$, such that $K = 1_F - PQ$ and $K' = 1_E - QP$ satisfy the following properties: for any $A \in \Psi^\infty(E) \times G$, $KA$, $AK'$, and $K'A$ are all smoothing operators.

For brevity we will call such a $Q$ a transversal parametrix for $P$.

Proof. The construction of the transversal parametrix repeats essentially the construction of the parametrix of an elliptic pseudodifferential operator, cf. [30].

In the scope of this proof, We shall use the equal sign \textquoteleft\textquoteleft\equiv\textquoteright\textquoteright\ for the asymptotic equivalence on a conic neighborhood of $T^*_G M$.

Let $\sigma(P)$ be the symbol of $P$, with order $m$ and let $Q_0$ be a pseudodifferential operator of order $-m$ and with inverse symbol on a conic neighborhood of $T^*_G M$. Then $R_0 = Q_0P - 1$ is a pseudodifferential operator, in general not necessarily of
negative order, but it has negative ordered symbol on a conic neighborhood of $T^*_qM$. By applying a cutoff function for a conic neighborhood of $T^*_qM$ to the symbol of $R_0$, we get a symbol for a pseudodifferential operator $R$ of negative order. Consequently,

$$\sigma(R) \equiv \sigma(R_0).$$  \hfill (4.5)

Let $C$ be a pseudodifferential operator with the following asymptotic expansion of its symbol:

$$\sigma(C) \sim \sum_{i=0}^{\infty} (-1)^i \sigma(R^i)$$  \hfill (4.6)

where $\sigma(R^i)$ is the symbol of the $i$-th power $R^i$ of $R$. Thus

$$\sigma(C(1 + R)) \equiv \sigma(1).$$  \hfill (4.7)

We claim that $Q = CQ_0$ is a pseudodifferential with properties we need. First we check the properties about $1 - QP$. We have

$$\sigma(QP) = \sigma(CQ_0P) = \sigma(C(1 + R_0)) \equiv \sigma(C(1 + R)) \equiv \sigma(1).$$  \hfill (4.8)

From above discussion, for any $a \in \Psi^\infty(M) \rtimes G$ $WF'(QP - 1_E) \circ WF'_{(\rho(a))} = \emptyset$, and $WF'_{(\rho(a))} \circ WF'(QP) = \emptyset$ so the composition either way is smoothing.

To prove to properties about $1_F - PQ$, we may construct $Q'$ similarly so $1 - PQ'$ composed with $\rho(a)$ for any $a \in \Psi^\infty(M) \rtimes G$ either is smoothing. Note that

$$\sigma(Q) - \sigma(Q') \equiv \sigma(QPQ') - \sigma(QPQ') = 0.$$  \hfill (4.9)

Since

$$\sigma(1 - PQ) = \sigma(1 - PQ') + \sigma(P(Q' - Q)) \equiv 0.$$  \hfill (4.10)

In conclusion, the properties for $1 - PQ$ can be proved similarly by applying the wave front arguments. \hfill \square
Definition 4.2.2. Let $\mathcal{K}_G \subset \Psi^\infty(E)$ be the set of pseudodifferential operators that annihilate $\Psi^\infty(E) \times G$ modulo $\mathcal{K}$, the ideal of smoothing operators, with composition product. We shall call elements of $\mathcal{K}_G$ transversally smoothing operators.

As we have seen, $\mathcal{K}_G$ can be described in terms of symbol condition, pseudodifferential operators with asymptotically zero symbol on $T_G^*M$ are in $\mathcal{K}_G$. $\mathcal{K}$ is a large ideal, it contains $\mathcal{K}$ and some pseudodifferential operators of arbitrarily high order.

Example 4.2.3. 1. Take a scaler function $\chi$ on $T^*M \setminus \{0\}$ which vanishes on the intersection of a conic neighborhood of $T^*_qM$ and

$$\{\xi \in T^*M : |\xi| > 1\},$$

and equals 1 constantly outside a conic neighborhood $U$ of $T^*_qM$. and let $P_\chi$ be a pseudodifferential operator with $\chi$ as the symbol. $\sigma(P_\chi)$ vanishes on a conic neighborhood of $T^*_G M$. hence $P_\chi \in \mathcal{K}_G$.

2. For any $Y \in \mathfrak{g}$, $Y_E : L^2(E) \to L^2(E)$, defined by the infinitesimal action of $\exp(tY)$ on $E$, as $t \to 0$, is a first order differential operator on $E$. Let $\chi$ and $P_\chi$ be as in the above example. Then $P_\chi Y_E, Y_E P_\chi, P_\chi Y_E P_\chi$ are in $\mathcal{K}_G$. Since the symbol of $Y_E$ are not zero outside $U$. $Y_E$ is a first order pseudodifferential operator. By composition we get operators in $\mathcal{K}_G$ with arbitrarily high order.

3. As introduce in Atiyah [2]), for an orthonormal basis $Y_i, i = 1, \ldots, \dim \mathfrak{g}$, of $\mathfrak{g}$, let $W'$ be the differential operator

$$W' = 1 - \sum_{i=1}^{\dim \mathfrak{g}} Y_E^2. \quad (4.11)$$

Then $W = W'P_\chi$ is in $\mathcal{K}_G$ as in the above example. Additionally, for a first order strongly transversally elliptic pseudodifferential operator $P$, by choosing $U$ small
enough to be contained in the conic neighborhood where the principal symbol \( \sigma(P) \) is isomorphic, then \( P^*P + W \) is strongly elliptic. We shall use this as an operator to disturb nonnegative transversally elliptic pseudodifferential operators into elliptic ones.

### 4.3 \( K \)-homology of \( \mathcal{A} \)

We recall some definitions. A \( p \)-summable pre-Fredholm module over \( \mathcal{A} \) is a pair \((\mathcal{H}, F)\) where

1. \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \) is a \( \mathbb{Z}_2 \)-graded Hilbert space with grading \( \epsilon = 1_{\mathcal{H}_+} \oplus (-1_{\mathcal{H}_-}) \), which is a \( \mathbb{Z}_2 \)-graded left \( \mathcal{A} \)-module.

2. \( F \in B(\mathcal{H}) \), \( F \epsilon = \epsilon F \), \( [F, \phi] = 0 \) for any \( \phi \in \mathcal{A} \).

3. for any \( \phi \in \mathcal{A} \), \( \phi(F^2 - 1) \in \mathcal{L}^p(\mathcal{H}) \), the \( p \)-Schatten ideal of compact operators.

A \( p \)-summable pre-Fredholm \( \mathcal{A} \)-module is called a \( p \)-summable Fredholm \( \mathcal{A} \)-module if in addition \( F^2 = 1 \).

Let

\[
P: \Gamma(M, E) \rightarrow \Gamma^{-\infty}(M, F),
\]

be a \( G \)-invariant transversally elliptic pseudodifferential operator of order 0. \( P \) has a transversal parametrix \( Q \), which can be assumed to be \( G \)-invariant as well (by averaging on \( G \)). Let \( \mathcal{H} = L^2(E) \oplus L^2(F) \) with grading \( \epsilon = 1_{L^2(E)} \oplus (-1_{L^2(F)}) \), it is a graded \( \mathcal{A} \)-module through the action \( \rho_E \oplus \rho_F \). Let

\[
F = \begin{bmatrix} 0 & Q \\ P & 0 \end{bmatrix}
\]

(4.13)
Theorem 4.3.1. Given a $G$-invariant transversally elliptic pseudodifferential operator $P$ of order 0, $(\mathcal{H}, F)$ introduced as above is a $p$-summable pre-Fredholm module for any $p > \dim M$.

Proof. Condition (1) is obvious; (2) follows from pseudo-local property of pseudodifferential operators and $G$-invariance of $P$ and $Q$; and (3) follows from the properties of the transversal parametrix. □

We now recall the standard process to transform a pre-Fredholm $\mathcal{A}$-module into a Fredholm $\mathcal{A}$-module, preserving $p$-summability (for details see [11], Appendix II of part I).

Given a pre-Fredholm $\mathcal{A}$ module $(\mathcal{H}, F)$, let $\hat{\mathcal{H}} = \mathcal{H} \hat{\otimes} \mathcal{C}$ be the graded tensor product of $\mathcal{H}$ with a $1 + 1$ dimensional graded Hilbert space $\mathcal{C} = \mathcal{C}_+ \oplus \mathcal{C}_-$, with $\mathcal{C}_\pm = \mathbb{C}$. Then $\hat{\mathcal{H}}_+ = \mathcal{H}_+ \oplus \mathcal{H}_-$ and $\hat{\mathcal{H}}_- = \mathcal{H}_- \oplus \mathcal{H}_+$. The $\mathcal{A}$-module structure on $\hat{\mathcal{H}}$ is given by

$$\tilde{\rho}(\phi)(\xi \hat{\otimes} \eta) = (\rho(\phi)\xi) \hat{\otimes} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \eta, \quad (4.14)$$

for any $\phi \in \mathcal{A}, \xi \in \mathcal{H}, \eta \in \mathcal{C}$ ($\mathcal{A}$ acts only non-trivially on $\hat{\mathcal{H}}_+$ as in $\mathcal{H}$). Since $F$ is of odd order, $\epsilon F = -F \epsilon$. $F$ is always of the form

$$F = \begin{bmatrix} 0 & Q \\ P & 0 \end{bmatrix}. \quad (4.15)$$

so we define

$$\tilde{F} = \begin{bmatrix} 0 & \tilde{Q} \\ \tilde{P} & 0 \end{bmatrix} \quad (4.16)$$

where

$$\tilde{P} = \begin{bmatrix} P & 1 - PQ \\ 1 - QP & (QP - 2)Q \end{bmatrix}, \quad \tilde{Q} = \begin{bmatrix} (2 - QP)Q & 1 - QP \\ 1 - PQ & -P \end{bmatrix}. \quad (4.17)$$
Proposition 4.3.2. For a pre-Fredholm $\mathcal{A}$-module $(\mathcal{H}, F)$, $(\mathcal{H}, \tilde{F})$ is a Fredholm $\mathcal{A}$-module, and there exists a pre-Fredholm $\mathcal{A}$-module $(\mathcal{H}_0, 0)$ with zero $\mathcal{A}$-action such that

1. $\mathcal{H} = \mathcal{H} \oplus \mathcal{H}_0$,
2. for any $\phi \in \mathcal{A}$, $\phi(\tilde{F} - F \oplus 0)$ is compact.

Moreover, if $(\mathcal{H}, F)$ is $p$-summable so is $(\mathcal{H}, \tilde{F})$.

In [11] (Part I, section 1), Connes showed that there is a trace $\tau : \mathcal{A} \to \mathbb{C}$,

$$\tau(\phi) = \frac{1}{2} \text{Trace}(\epsilon F[F, \phi]) \tag{4.18}$$

which gives the index map $K_0(\mathcal{A}) \to \mathbb{Z}$,

$$\text{index} F^+_e = (\tau \otimes \text{Trace})(e) \tag{4.19}$$

for any projection $e$ ($e = e^* = e^2$) in the finite matrix algebra $M_q(\mathcal{A})$ for arbitrary $q$.

Definition 4.3.3. (Connes, [11]) The trace $\tau$ (which we denote by $\text{char}(\mathcal{H}, \tilde{F})$) is called the Connes character of the 1-summable Fredholm $\mathcal{A}$-module $(\mathcal{H}, \tilde{F})$.

Next we show that the character $\tau$ of $(\mathcal{H}, \tilde{F})$ is just like the local index density. This fact allows us to get transversal index formula by computing the Connes character.

Proposition 4.3.4.

$$\text{index}^G(P) = \pi_* \text{char}(\mathcal{H}, \tilde{F}) \tag{4.20}$$

where $(\pi)_* = (\pi_G)_*$ is the push-forward by the projection $\pi = \pi_G : M \times G \to G$. 

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Proof. In fact, for a $G$-invariant transversally elliptic pseudodifferential operator $P$, let $G$ be the Green operator on $L^2(F)$, i.e.,

$$GP = 1 - \pi_P, \quad PG = 1 - \pi_{P^*}. \quad (4.21)$$

So the local index density can be written as

$$\text{index}^G(f) = \text{Trace}(\rho(\pi^* f)(1 - GP)) - \text{Trace}(\pi^* f(1 - PG)). \quad (4.22)$$

For a transversal parametrix $Q$ of $P$, let

$$\tau'(\rho(\pi^* f)) = \text{Trace}(\rho(\phi)(1 - QP)) - \text{Trace}(\rho(\pi^* f)(1 - PQ)) \quad (4.23)$$

which is well defined by proposition 4.2.1. First, we show that $\text{index}^G(P) = \tau \cdot \tau'$. From

$$\pi_P = (1 - QP) \pi_P, \quad \pi_{P^*} = \pi_{P^*}(1 - PQ), \quad (4.24)$$

we have

$$\text{index}^G(P)(f) = \text{Trace}(\rho(\pi^* f)\pi_P) - \text{Trace}(\rho(\pi^* f)\pi_{P^*})$$

$$= \text{Trace}(\rho(\pi^* f)(1 - QP)\pi_P) - \text{Trace}(\rho(\pi^* f)\pi_{P^*}(1 - PQ))$$

$$= \tau'(\pi^* f) + \text{Trace}(\rho(\pi^* f)(1 - QP)GP) - \text{Trace}(\rho(\pi^* f)PG(1 - PQ))$$

$$= \tau'(\pi^* f). \quad (4.25)$$

The last equality in the above equation holds since $PG = 1 - \pi_{P^*}$ is the projection on the image of $P$ so

$$\rho_F(\pi^* f)PG = GP \rho_E(\pi^* f). \quad (4.26)$$

Repeating the above process we conclude that for any integer $n$,

$$\text{index}^G(P)(\pi^* f) = \text{Trace}(\rho(\pi^* f)(1 - QP)^n) - \text{Trace}(\rho(\pi^* f)(1 - PQ)^n). \quad (4.27)$$

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This is useful for the verification of a $p$-summable Fredholm module for $p > 1$, we took 1-summable case we merely for simplicity of arguments.

Direct computation shows that:

\[
\tau(\pi^* f) = \frac{1}{2} \text{Trace} \left( \begin{bmatrix}
\rho_E(\pi^* f) + 0_F & 0 \\
0 & -\rho_F(\pi^* f) + 0_E
\end{bmatrix}
\right)
- \begin{bmatrix}
0 & \tilde{Q} \\
\tilde{P} & 0
\end{bmatrix}
\begin{bmatrix}
\rho_E(\pi^* f) + 0_F & 0 \\
0 & -\rho_F(\pi^* f) + 0_E
\end{bmatrix}
\begin{bmatrix}
0 & \tilde{Q} \\
\tilde{P} & 0
\end{bmatrix}
\]

\begin{equation}
= \text{Trace}(\rho_E(\pi^* f)(1 - QP)^2) - \text{Trace}(\rho_F(\pi^* f)(1 - PQ)^2)
= \tau'(\pi^* f).
\end{equation}

The constant factor $1/2$ is not essential, it depends on our particular choice of the way a pre-Fredholm is transformed into a Fredholm module.

\[\square\]

### 4.4 Connes-Chern character in the periodic cyclic cohomology of $\mathcal{A}$

As we have seen in the previous discussion, the computation of the index for transversally elliptic pseudodifferential operators amounts to the computation of the Connes character of a finitely summable Fredholm module. The Connes character evolved into its new version that takes values in cyclic cohomology.

We recall the basic fact we will use here. For a pre-$C^*$-algebra $\mathcal{A}$, and a $p$-summable Fredholm $\mathcal{A}$-module $(H, F)$, let

\[Tr'(T) = \frac{1}{2} \text{Trace}(F[F,T]) \quad (4.29)\]

as the Connes character $(\tau(\pi^* f) = Tr'(\epsilon \pi^* f))$. The Connes-Chern character in the periodic cyclic cohomology

\[ch^*(H, F) \in HP^*(\mathcal{A}) \quad (4.30)\]
is defined to be
\[ ch^*(H, F)(a^0, \ldots, a^n) = (-1)^{n(n-1)/2} \Gamma \left( \frac{n}{2} + 1 \right) Tr' \left( e a^0[F, a^1] \cdots [F, a^n] \right) \] (4.31)
for \( n \) even and
\[ ch^*(H, F)(a^0, \ldots, a^n) = \sqrt{2i}(-1)^{n(n-1)/2} \Gamma \left( \frac{n}{2} + 1 \right) Tr' \left( a^0[F, a^1] \cdots [F, a^n] \right) \] (4.32)
for \( n \) odd (Here \( \Gamma \) is the Gamma function).

As before, let \( e \in M_q(A) \) be a projection (which is equivalent to a finitely generated projective module on \( A \) in the \( K \)-theory \( K_* (A) \) for operator algebras). There is a well understood Chern character (cf. [12],[5]) from \( K_* (A) \) to the periodic cyclic homology of \( A \):
\[ ch_* (e) \in HP_* (A). \] (4.33)

The Connes character can be viewed the dual of this Chern character. The Connes-Chern character gives the index formula (for example, in the even case) in the following fashion. For an element \([e] \in K(A)\), then the index of \( F^+_e \) the twisted operator \( F^+ \) by the the projection \( e \) is
\[ \text{Index} F^+_e = \langle ch^*(H, F), ch_*(e) \rangle. \] (4.34)

In conclusion, to find the index of \( P \) and its twisted versions, we may compute the Connes-Chern character of a Fredholm module \( F \) associated with it, in the periodic cyclic cohomology.
CHAPTER 5

LOCAL INDEX FORMULA IN NONCOMMUTATIVE GEOMETRY

5.1 The Connes-Moscovici Local index formula

In this section we briefly recall the definitions and results from [10] (also see [12],[17]).

Let \( \mathcal{A} \) be a \(*\)-algebra, which is a dense \(*\)-subalgebra of a pre-C* algebra \( \mathcal{A} \).

**Definition 5.1.1.** An spectral triple is a triple \( (\mathcal{A}, \mathcal{H}, D) \) where

1. \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \) is a \( \mathbb{Z}_2 \)-graded Hilbert space and left \( \mathcal{A} \)-module, with grading \( \epsilon \);
2. \( D \) is an unbounded self-adjoint operator on \( \mathcal{H} \) such that \( D\epsilon = -\epsilon D \);
3. for all \( a \in \mathcal{A} \), \([D,a] \in B(\mathcal{H})\);
4. for all \( a \in \mathcal{A} \), \( a(1 + D^*D)^{-1} \) is compact.

An odd spectral triple over \( \mathcal{A} \) is similarly defined except without grading and grading related conditions in the above.

Starting with a spectral triple, by the observation in [3] by Baaj and Julg, the following assignment

\[
D \mapsto D(1 + D^*D)^{-1/2}
\]  

(5.1)
determines a pre-Fredholm module. In fact it is shown in [3] that all $K$-homology classes of a pre-$C^*$ algebra can be obtained this way. We may switch to the computation of the Connes-Chern character of a spectral triple for the following reasons: the index of $D$ is preserved in the Baaj-Julg assignment; the Connes-Chern character is an invariant of the $K$-homology class; and when $D$ is a pseudodifferential operator, so is $D(1 + D^*D)^{-1/2}$ whose symbol can be computed in terms of of the symbol of $D$.

Let $F$ and $|D|$ be the elements of the polar decomposition of $D$:

$$D = F|D|,$$  \hfill (5.2)

where $F = \text{sign} D$ is unitary and $|D| = (D^2)^{1/2}$ is positive. $(\mathcal{H}, F)$ is a bounded pre-Fredholm module, representing the same class in $K$-homology determined by the spectral triple. Following the literature ([10], [17]), we shall use the notation $|D|^{-1}$ as zero by definition on the kernel of $D$, and inverse of $|D|$ on $\ker D^\perp$:

$$|D||D|^{-1} = |D|^{-1}|D| = 1 - \pi_{\ker D}.$$  \hfill (5.3)

In particular, the notation $|D|^{-1}$ does not imply the assumption that $|D|$ is invertible.

We also showed the relation between constants.

**Definition 5.1.2.** For some $p \geq 1$, a $p^+$-summable spectral triple is a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ such that for any $\phi \in \mathcal{A}$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}$,

$$\phi(\lambda - |D|)^{-1} \in \mathcal{L}^{p^+}(\mathcal{H}) = \mathcal{L}^{(p,\infty)}(\mathcal{H}),$$  \hfill (5.4)

where $\mathcal{L}^{(p,\infty)}$ is the ideal of $B(\mathcal{H})$ consisting of those compact operators $T$ whose $n$-th characteristic value $\mu_n(|T|) = \min\{ \|T|_{E^\perp}\|; \dim E = n \}$ satisfies

$$\mu_n(|T|) = O(n^{-1/p}).$$
Let $\delta$ be the derivation operator $ad(|D|)$:

$$\delta(T) = ad(|D|)(T) = [|D|, T]$$

(5.5)

defined on the bounded operators $B(\mathcal{H})$ and takes values as unbounded operators on $\mathcal{H}$. Let $Dom(\delta) \subset B(\mathcal{H})$ be the domain of $\delta$; that is, $A \in B(\mathcal{H})$ is in $Dom(\delta)$ if and only if $[|D|, A]$ extends to a bounded operator on $\mathcal{H}$. And we denote

$$Dom^\infty(\delta) = \bigcap_{k \geq 1} Dom(\delta^k).$$

(5.6)

**Definition 5.1.3.** A $p^+$-summable spectral triple is regular if

$$A \cup [D, A] \subset Dom^\infty(\delta)$$

(5.7)

For a $n^+$-summable spectral triple $(A, H, D)$ which satisfies

$$A \cup [D, A] \subset Dom(\delta^2),$$

(5.8)

Connes character formula says the $n$-cocycle in Hochschild cohomology of $A$ is, for $a_i \in A$.

$$\phi_\omega(a^0, \ldots, a^n) = \lambda_n Tr_\omega(e a^0[D, a^1] \cdots [D, a^n]|D|^{-n}),$$

(5.9)

where $\lambda_n$ is a universal constant, and $Tr_\omega$ is the Dixmier trace (see [12] IV.2. for details). In the odd case the same is true with $\epsilon = id$.

Connes-Moscovici [10] went further to solve the general problem for local character formula for spectral triples with discrete dimension spectrums. We now to introduce more notations. Let $A_D$ be a subspace of $B(\mathcal{H})$ generated by the following operations: for any operators $A \in A$,

$$dA = [D, A], \ \nabla(A) = [D^2, A], \ A^{(k)} = \nabla^k(A)$$

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are in $\mathcal{A}_D$. moreover the operators

$$P(a^0, a^1, \ldots, a^n) = a^0(d^{a^1})^{(k_1)} \cdots (d^{a^n})^{(k_n)}$$

are in $\mathcal{A}_D$, where $a^0, \ldots, a^n \in \mathcal{A}$, acting on $\mathcal{H}$ by $\rho$.

To apply the Connes-Moscovici theorems, we first make the following assumptions in our specific spectral triple. We will prove that these assumptions hold for the spectral triple we construct in the next section.

**Assumption** We assume that $(\mathcal{A}, \mathcal{H}, D)$ is (1) $p^+$-summable for some $p > 1$, (2) regular, (3) with discrete dimension spectrum, (4) for $P \in \mathcal{A}_D$, the zeta function

$$\zeta_{P,D}(z) = \text{Trace}(P|D|^{-2z})$$

is at least defined and analytic for $Re(z) > k + p$. In this case

$$\tau_q(P) = \tau^0_q(P) = \text{Res}_{z=0} z^q \zeta_{P,D}.$$  \hspace{1cm} (5.10)

(5) Only finite many of $\tau_{aq}$, $q = 0, \ldots$ are nonzero in general.

The *dimension spectrum* of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is the minimal closed subset of $\mathbb{C}$ on the complement of which $\zeta_{P,D}$ can be extended to a holomorphic function for any $\mathcal{A}_D$.

We assume that $(\mathcal{A}, \mathcal{H}, D)$ is regular, $p^+$-summable, and the poles of $\zeta_{P,D}$ have multiplicities not exceeding a fixed number.

**Theorem 5.1.4.** (Connes-Moscovici) a) The following formula defines an even cocycle in $(b, B)$ bicomplex of $\mathcal{A}$:

$$\varphi_0(a_0) = \tau_{-1}(\gamma a_0)$$

$$\varphi_{2m}(a_0, \ldots, a_{2m}) = \sum_{k \in \mathbb{N}^{2m}, q \geq 0} c_{2m,k,q} \tau_q(\gamma a_0 (d_{a_1})^{(k_1)} \cdots (d_{a_{2m}})^{k_{2m}} |D|^{-2|k|-2m})$$  \hspace{1cm} (5.11)
for \( m > 0 \), where \( c_{2m,k,q} \) are universal constants given by

\[
c_{2m,k,q} = \frac{(-1)^{|k|}}{k! k!} \sigma_q(|k| + m)
\]  

(5.12)

where \( k! = k! \ldots k_{2m}! \), \( \bar{k}! = (k_1 + 1)(k_1 + k_2 + 2) \ldots (k_1 + \ldots + k_{2m} + 2m) \) and \( \sigma_q(N) \) is the \( q \)-th elementary polynomial of the set \( \{1, 2, \ldots, N - 1\} \).

(b) The cohomology class of the cocycle \( (\phi_{2m})_{m \geq 0} \) in \( HC^{eu}(A) \) coincides with the Connes-Chern character \( ch_*(A, H, D) \).

In the odd case, the Connes-Chern character is computed similarly. Suppose we have an odd spectral triple still called \( (A, H, D) \).

**Theorem 5.1.5.** (Connes-Moscovici) a) The following formula defines an odd cocycle in \( (b, B) \) bicomplex of \( A 
\)

\[
\phi_{2m+1}(a_0, \ldots, a_{2m+1}) = \sqrt{2i} \sum_{k \in \mathbb{N}^{2m+1}, q \geq 0} c_{2m+1,k,q} 
\]

\[
\tau_q(a_0(da_1)^{(k_1)} \ldots (da_{2m+1})^{k_{2m+1}})|D|^{-2|k|-(2m+1)}
\]

(5.13)

where \( c_{2m+1,k,q} \) are universal constants given by

\[
c_{2m+1,k,q} = \frac{(-1)^{|k|}}{k! k! q!} \Gamma^{(q)}(|k| + m + \frac{1}{2})
\]  

(5.14)

where \( \Gamma^{(q)} \) is the \( q \)-th derivative of the Gamma function.

(b) The cohomology class of the cocycle \( (\phi_{2m+1})_{m \geq 0} \) in \( HC^{od}(A) \) coincides with the Connes-Chern character \( ch_*(A, H, D) \).

**Remark 5.1.6.** To illustrate the above statement about our goal, we recall a well-known solved case as a sample, where \( G \) is the trivial group. This is the commutative case since \( A = C^\infty(M) \). \( D \) is an elliptic pseudodifferential operator with scalar principal symbol on a hermitian bundle \( E \) over a smooth manifold \( M \). The spectral
triple is regular, \( \dim M^+ \)-summable, and with discrete dimension spectrum. Since the zeta functions involved have at most simple poles, only \( \tau_0 \) is possibly nonzero. \( \tau_0 \) is up to constant factors the Wodzicki residue, which is computable in terms of symbols of the operators involved.

5.2 The Spectral triple, transversally elliptic case

In this section we construct a spectral triple associated to a transversally elliptic pseudodifferential operator \( P \). We discuss only the even case to simplify our argument, the odd case is similar. First we may assume it is of order 1 as a pseudodifferential operator, because when necessary we may multiply \( D \) with an appropriate power of \( 1 + \Delta \), this operation is an isomorphism of Sobolev spaces hence it does not alter the index of \( D \).

Now let \( \mathcal{H} = L^2 E \oplus L^2 F \), \( \epsilon = 1_E \oplus (-1)_F \), and \( \mathcal{A} \) acts on \( \mathcal{H} \) by \( \rho_E \oplus \rho_F \) as before. Let

\[
D = \begin{bmatrix} 0 & P^* \\ P & 0 \end{bmatrix}.
\]

(5.15)

which is symmetric by construction. We now prove that \( D \) is essentially self-adjoint.

**Lemma 5.2.1.** Let

\[
S = \{ s \in \mathcal{H} \mid \forall \phi \in \mathcal{A}, \rho(\phi)s \in \Gamma^\infty(M, E \oplus F) \}.
\]

(5.16)

then \( S \) is dense in \( \mathcal{H} \) and \( D|_S \) is symmetric.

**Proof.** \( S \) is dense since it contains all smooth sections.

For any \( p \in \mathbb{R} \), \( Ds \) is defined in \( H^{p-1}(M, E \oplus F) \) for \( s \in H^p(M, E \oplus F) \), first we need to show that with the extra conditions it is defined on \( S \) with values in \( \mathcal{H} \). We
observe that since $D$ is $G$-invariant, we have a convergent sequence in $H^{p-1}(M, E)$:

$$Ds = \lim_{f_n \to \delta} \rho(\phi_n)(Ds) = \lim_{f_n \to \delta} D(\rho(\phi_n)s), \quad (5.17)$$

where $\{f_n : n \in \mathbb{N}\}$ is a sequence of functions on $G$ converging to the delta distribution $\delta$ at the identity on $G$. $\rho(\phi_n)$ converges to the identity operator in operator norm as bounded operators on any $H^{p}(E \oplus F)$. For $n$ sufficiently large,

$$\|Ds\|_{L^2} \leq \|D\|_{H^1} \cdot \|\rho(\phi_n)s\|_{H^1} + 1 < \infty. \quad (5.18)$$

Next we need to show that for $s, t \in S$,

$$\langle Ds, t \rangle = \langle s, Dt \rangle. \quad (5.19)$$

As above take approximate delta functions on the group $G$. $f_n \to \delta$

$$\langle Ds, t \rangle = \lim_{f_n \to \delta} \langle \rho(f_n)Ds, t \rangle$$

$$= \lim_{f_n \to \delta} \langle Ds, \rho(f_n^*)t \rangle$$

$$= \lim_{f_n \to \delta} \langle s, D(\rho(f_n^*)t) \rangle$$

$$= \langle s, Dt \rangle \quad (5.20)$$

We may regard $S$ as the set of "transversally smooth" sections. Since $S$ is dense in $\mathcal{H}$ we have well-defined $(D|_S)^*$ and $\overline{D|_S}$, with $D|_S \subset D$. For a symmetric pseudodifferential operator

$$D : \Gamma^\infty_c(M, E \oplus F) \to \Gamma^\infty(M, E \oplus F). \quad (5.21)$$

we have an extension

$$D' : \Gamma^{-\infty}(M, E \oplus F) \to \Gamma^{-\infty}_c(M, E \oplus F), \quad (5.22)$$
defined by the same integral formula. In the following, $D$, $D^*$ and $\hat{D}$ are all restrictions of $D'$. If $D$ has a parametrix, same applies to it so here for convenience we may still use $D$ for $D'$.

**Lemma 5.2.2.** $D$ as an unbounded operator on $H$ is essentially self-adjoint.

*Proof.* To show $S$ is the essentially self-adjoint We need only to show that

$$Ker(D^* \pm iI) \subset Dom(\hat{D}). \quad (5.23)$$

Since

$$Dom(D^*) = \{ s \in H \mid Ds \in H \} \quad (5.24)$$

so it is sufficient to show $Ker(D^* + i) \subset S$. Let $(D^* + i)s = 0$, so $(D + i)s = 0$, $D$ is of positive order. $D + i$ has the same principal symbol, then there is an $G$-invariant $Q$ that is a transversal parametrix for $D + i$: $1 - Q(D + i)$ and $1 - (D + i)Q$ are smoothing when they are composed with $\rho(\phi)$.

$$\rho(\phi)s = \rho(\phi)(1 - Q(D + i))s + 0 \quad (5.25)$$

is smooth, so $s \in S$. The case for $D - i$ is similar. \qed

This shows that from now on we may assume $D$ is self-adjoint by switching to $\hat{D}$.

**Lemma 5.2.3.** For any nonzero real number $\lambda$, $\rho(\phi)(D - \lambda i)^{-1}$ and $(D - \lambda i)^{-1}\rho(\phi)$ are compact operators for all $\phi \in A$.

*Proof.* Let $Q_\lambda$ be a transversal parametrix for $D - \lambda i$. As discussed above, it might not necessarily have negative order, but it can be chosen so by a cutoff on the symbol
in a conic neighborhood of $T^*_q M$. So $1 - Q\lambda(D - \lambda i) = K$ and $1 - (D - \lambda i)Q\lambda = K'$. Apply the inverse to the right hand side of the first parametrix formula, we have

$$(D - \lambda i)^{-1} - Q\lambda = K(D - \lambda i)^{-1}. \tag{5.26}$$

$Q\lambda$ is compact since it is a pseudodifferential operator of negative order. It suffices to show that $\rho(\phi)K(D - \lambda i)^{-1}$ is smoothing. $(D - \lambda i)^{-1}$ is a bounded operator from $H^s$ to $H^{s+1}$ so its composition with a smoothing operator is still smoothing. □

**Proposition 5.2.4.** $(\mathcal{A}, \mathcal{H}, D)$ is a dim $M^+$-summable spectral triple.

**Proof.** Since $D$ commutes with $\rho(g)$. For any $\phi \in \mathcal{A}$.

$$[D, \phi] = \int_G [D, \phi(x, g)]\rho(g)d\mu(g) \tag{5.27}$$

so it is compact, as $[D, \phi(x, g)]$ is a pseudodifferential operator of negative order. Lemma 5.2.3 shows $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple. For any $\phi \in \mathcal{A}$, and $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

$$\phi(\lambda - |D|)^{-1} = \rho Q\lambda + \rho K\lambda(\lambda - |D|)^{-1} \in \mathcal{L}^{(\dim M), \infty} \tag{5.28}$$

since the transversal parametrix can be chosen to be a pseudodifferential operator of order $-1$. $\rho K\lambda$ is trace class and $(\lambda - |D|)^{-1}$ is bounded. □

**Lemma 5.2.5.** When $|D|$ has a scalar principal symbol, the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is regular.

**Proof.** We need to show

$$\mathcal{A} \cup [D, \mathcal{A}] \subset \text{Dom}^\infty(\delta). \tag{5.29}$$

and in fact we will show that

$$\mathcal{A} \cup [D, \mathcal{A}] \subset \Psi^0(E) \times G \subset \text{Dom}^\infty(\delta), \tag{5.30}$$

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and $\delta$ preserves $\Psi^0(E) \rtimes G$.

Elements in $\mathcal{A}$ have scalar symbols, so elements $[D, \mathcal{A}] \in \Psi^1(E) \rtimes G$ have vanishing principal symbols, which implies $[D, \mathcal{A}] \in \Psi^0(E) \rtimes G$.

In general, since $|D|$ has scalar symbols of degree 1, the commutator with any classical pseudodifferential operator has order, at most zero. And since $|D|$ is $G$-invariant, the composition with $\rho(g)$ has no effect. □

5.3 Spectral asymptotics for $(\mathcal{A}, \mathcal{H}, D)$

First we summarize the facts about the algebra $\Psi^k(E) \rtimes G$ and the pseudodifferential algebra $\Psi^k(E)$ that we developed in chapter 3 and chapter 4. For brevity, in the following enumeration only, we use the notation $W^k = \Psi^k(E) \rtimes G$ and $\Psi^k = \Psi^k(E)$.

Recall that $\mathcal{K}_G$ is the algebra of transversally smoothing operators and let $\mathcal{K}$ be the smoothing operators on $L^2(E)$.

1. $W^k \subset W^l$ for $k \leq l$;
2. $W^k \cdot W^l \subset W^{k+l}$;
3. $\Psi^k \cdot W^l \subset W^{k+l}$; $W^k \cdot \Psi^l \subset W^{k+l}$;
4. $W^0 \subset B(\mathcal{H})$;
5. When $r > 0$, $\mathcal{A} \in W^{-r}$ is compact;
6. When $r > \dim M$, $W^{-r} \subset \mathcal{L}^{(p, \infty)}$ for $p = \dim M / r$;
7. $W^k \cdot \mathcal{K}_G \subset \mathcal{K}$; $\mathcal{K}_G \cdot W^k \subset \mathcal{K}$;
8. $W^{-\infty} \subset \mathcal{K}$. 

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With an extra parameter on pseudodifferential operators, that is, for families of pseudodifferential operators, the above properties still hold.

By lemma 5.2.5 we have a regular spectral triple when $|D|$ has scalar principal symbols. As we showed in section 5.1, we need to study the poles of

$$\zeta_A(z) = Tr(A|D|^{-2z})$$

where $A \in \mathcal{A}_D \subset \Psi^0(E) \rtimes G$. For the purpose of estimation, we will relax the condition on $A$, only assuming $A \in \Psi^0(E)$.

Let $P$ be a 2nd-order, weakly polyhomogeneous, self-adjoint, positive, strongly elliptic pseudodifferential operator. For $\lambda$ on a contour along the negative real axis $C$, the resolvant $(P - \lambda)^{-1}$ exists and bounded. Let $\ln(-\lambda)$ denote the unique branch defined on $\mathbb{C} \setminus \mathbb{R}_+$ and with imaginary part $\arg(\ln(-\lambda)) = 0$ on the negative half of the real axis.

**Lemma 5.3.1.** Suppose $N_0$ is a fixed positive integer. $A \in \Psi^k(M) \rtimes G$, $r_j, s_l$ are strictly decreasing sequences in $\mathbb{Q}$. The following three statements are equivalent:

(A) For $N > (n + k)/2$, $A(P - \lambda)^{-N}$ is trace class, and for $\lambda \to \infty$ on any ray of $\mathbb{C} \setminus \mathbb{R}_+$,

$$Tr(A(P - \lambda)^{-N}) \sim \sum_{j \geq 0} c_{N,j}(\lambda)^{-N + r_j} + \sum_{l \geq 0} \left( \sum_{p=0}^{N_0} c'_{N,l,p}(\ln \lambda)^p \right) (\lambda)^{-N + s_l}; \quad (5.32)$$

(B) When $t \to 0^+$

$$Tr(Ae^{-tP}) \sim \sum_{j \geq 0} a_j t^{-r_j} + \sum_{l \geq 0} \left( \sum_{p=0}^{N_0} b_{l,p}(\ln t)^p \right) t^{-s_l}; \quad (5.33)$$

(C) The zeta function $\Gamma(z)Tr(AP^{-z})$ is convergent for $\Re(z) > (n + k)/2$, extending to a meromorphic function on $\mathbb{C}$, with the poles described as follows: up to a
holomorphic function,

\[ \Gamma(z)Tr(AP^{-z}) \sim \sum_{j \geq 0} \frac{a_j}{z - r_j} + \sum_{l \geq 0} \left( \sum_{p=0}^{N_0} \frac{b_{l,p}}{(z - s_l)^{p+1}} \right). \]  \hspace{1cm} (5.34)

Moreover, the coefficient sets \( \{c_{N,j}, c_{N,l,q}\} \), \( \{a_j, b_{l,q}\} \), \( \{a'_j, b'_{l,q}\} \) determine each other by linear combination with constant coefficients.

In our application, we shall have \( r_j = (n + k - j)/2 \), \( s_l = (n + k)/2 - l/q \) for some fixed integer \( q > 0 \) and \( N_0 = \dim G + n - 1 \).

**Proof.** In the proof we will use the following transformation formulas

\[ P^{-s} = \frac{1}{(s-1) \cdots (s-k)} \frac{i}{2\pi} \int_{C_1} \lambda^{k-s} \partial^k \lambda (P - \lambda)^{-1} d\lambda \]

\[ = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-tP} dt. \]  \hspace{1cm} (5.35)

and

\[ e^{-tP} = t^{-k} \frac{i}{2\pi} \int_{C_1} e^{-t\lambda} \partial^k \lambda (P - \lambda)^{-1} d\lambda \]

\[ = \frac{1}{2\pi i} \int_{Re(s) = \varepsilon} t^{-s} \Gamma(s) P^{-s} ds. \]  \hspace{1cm} (5.36)

where \( C_r, r \in [0,1] \) is the counterclockwise contour

\[ C_r = \{ z \in \mathbb{C} : \arg(z) = \pm(\pi/2 - \delta), |z| \geq r \} \]

\[ \cup \{ z \in \mathbb{C} : \arg(z) \in [-\pi/2 + \delta, \pi/2 - \delta], |z| = r \} = rC_1 \]  \hspace{1cm} (5.37)

for some fixed small \( \delta > 0 \). Note that contribution from a different contour only produce a smoothing operator.

(A) \( \iff \) (B): We have

\[ A(P - \lambda)^{-N} \in \Psi^{k-2N}(E) \ltimes G \]

so it is trace class when \( N > (k + n)/2 \). By

\[ \partial^k \lambda (P - \lambda)^{-1} = k!(P - \lambda)^{-k-1}. \]  \hspace{1cm} (5.38)
we get

\[ Ae^{-t^p} = (N - 1)! t^{-N+1} \frac{i}{2\pi} \int_{C_1} e^{-tA(P - \lambda)^{-N}} d\lambda. \]  

\[ (5.39) \]

Direct calculation shows

\[ t^{-N+1} \frac{i}{2\pi} \int_{C_1} e^{-t\lambda} (-\lambda)^{-r_j - N} d\lambda = \frac{t^{-r_j}}{\Gamma(-r_j)} \]  

and similarly

\[ t^{-N+1} \frac{i}{2\pi} \int_{C_1} e^{-t\lambda} (-\lambda)^{s_l - N} (\ln (-\lambda))^p d\lambda = t^{-s_l} \frac{i}{2\pi} \int_{C_1} e^{-\lambda} (-\lambda)^{s_l} (\ln (-\lambda) - \ln t)^p d\lambda \]

\[ \sim \sum_{i=0}^{p} (-1)^i \binom{p}{i} t^{-s_l} (\ln t)^i \frac{i}{2\pi} \int_{C_1} e^{-\lambda} (-\lambda)^{s_l} (\ln (-\lambda))^p d\lambda \]

\[ \sim \sum_{i=0}^{p} (-1)^i \binom{p}{i} t^{-s_l} (\ln t)^i \frac{1}{\Gamma(p-i)} (-s_l). \]  

where \((\frac{1}{\Gamma})^{p-i}\) is the \((p - i)\)-th derivative of the entire function \(1/\Gamma\). By the first equality of (5.36), and the fact that in this integral transform asymptotic expansions in \(-\lambda\) correspond to asymptotic expansions in \(t\), we reach the conclusion with the relation between coefficient coefficients:

\[ a_j = \frac{(N - 1)! c_{N,j}}{\Gamma(-r_j)} \]  

\[ (5.42) \]

and

\[ b_{l,q} = \sum_{i=0}^{p} (-1)^i \binom{q}{i} t^{-s_l} (N - 1)! c_{N,l,i} (\frac{1}{\Gamma})^{(q-i)}(-s_l). \]  

\[ (5.43) \]

The above gives \(b_{l,q}\) recursively.

(B) \(\Leftrightarrow\) (C): (cf. [32]). This part is known as the Mellin integral transform. The argument is very similar to the above \(\square\)
Proposition 5.3.2. Let $A \in \Psi^k(E) \times G$, $P$ be a second order weakly polyhomogeneous, self-adjoint and positive elliptic pseudodifferential operator on $E$, and let be $\Gamma$ a sector near negative real axis. Then for $\text{Re}(z) > (k + n)/2$, $AP^{-1}$ is trace class, $\text{Trace}(AP^{-1})$ is analytic in $z$ on the half plane $\text{Re}(z) > (k + n)/2$, extending to a meromorphic function on $\mathbb{C}$ and up to an entire function in $z \in \mathbb{C}$.

$$\Gamma(z)\text{Tr}(AP^{-z}) \sim \sum_{j \geq 0} \frac{c_j}{z + \frac{j-k-n}{2}} + \sum_{l \geq 0} \left( \sum_{p=0}^{\dim G+n-1} \frac{c_{l,p}^j}{(z - \frac{k+n}{2} + \frac{1}{q})^{p+1}} \right).$$

(5.44)

Where all the coefficients except the first residue at integer points are determined by the symbol.

Proof. By theorem 3.2.9, $A(P + \mu^2)$ is weakly polyhomogeneous, so we apply proposition 3.4.1, for $-\lambda = \mu^2$, and apply the integral transformation we get (C) for (A) in the above list. \qed

If $A'$ is smoothing, then by Corollary 3.2.5 and Theorem 5.3.2

$$\text{Tr}(A'(P - \lambda)^{-N}) \sim \sum_{j \geq 0} c_j (-\lambda)^{-N-j/2}$$

(5.45)

which does not contribute to the pure log and their power terms. That is the generalized noncommutative residues are locally computable.

In our case $P = D^2 + W$, where $W \in \mathcal{K}_G$ is the example given before. In fact, being a differential operator $D^2$ is strongly polyhomogeneous, but only transversally elliptic. $P = D^2 + W$ is strongly elliptic but not a differential operator. it is weakly polyhomogeneous.

Proposition 5.3.3. The residues at $s = 0$ of $z^q \text{Tr}(AD^{-2s})$, $q = 0, 1, \ldots, \dim M + \dim G - 1$ coincide with those of $\text{Tr}(AP^{-s})$ where $P = D^2 + W$. Consequently, they are computable only in term of of the transversal part of the symbol of $D$. 67
Proof. Recall
\[ |D|^{-1} = |D|^\dagger_\dagger = 1 - \pi_{\ker D}, \]
and by computing wave front set of \( \pi_{\ker D} \) using the regularity theorem, as in lemma 4.1.1, \( \pi_{\ker D} \) is in \( K_G \). Thus the effect of \( \pi_{\ker D} \) can be ignored. Then for \( \lambda \in \Gamma \) close to the negative half of real axis, \( |D|^2 - \lambda \) is invertible. Since on \( \Gamma \)
\[ (P - \lambda) - (D^2 - \lambda) = W \in K_G \]
so for any \( A \in \Psi^k(E) \times G \),
\[ A(P - \lambda)^{-1} - A(D^2 - \lambda)^{-1} = -A(P - \lambda)W(D^2 - \lambda) \in \Psi^{-\infty,-1}(E) \times G. \]
This show that \( Tr(A(D^2 - \lambda)^{-1}) \) exists and satisfying (B) for \( Re(s) > (-k - n)/2 \) and has the same pole structure at \( z = 0 \). At last we apply the integral transformation in this section.

Combining all results above in this section, we have completed our argument leading to the theorem below. Our main theorem stated in the introduction also follows from this theorem.

Theorem 5.3.4. Let \((A, \mathcal{H}, D)\) be the even spectral triple defined by a transversally elliptic pseudodifferential operator \( D \) which is a regular spectral triple assuming in addition that \( D^2 \) has scalar symbol on \( T_G^*M \). Then \((A, \mathcal{H}, D)\) has a discrete dimension spectrum contained in \( \mathbb{Q} \). Moreover, the residues \( \tau_q \) are determined by the symbol of \( D \).
APPENDIX A

PERIODIC CYCLIC (CO)HOMOLOGY OF $\mathcal{A} = C_\infty^c(M) \rtimes G$

In this appendix we review the known results about the periodic cyclic homology and cohomology of the cross-product algebra $\mathcal{A} = C_\infty^c(M) \rtimes G$, only to help understanding our results.

We describe how we may view the cyclic cohomology as a cosheaf homology of equivariant de Rham currents, dual to the usual equivariant de Rham cohomology of differential forms.

First we start with what has been done on the cyclic homology of the crossed product algebras, especially as presented by [6], [15] and [4].

Notation A.0.5. In this section the Lie group $G$ will be given the topology of invariant open sets. $G$ action on $G$ itself always means the adjoint action. $G(s)$ be the centralizer of a group element $s \in G$, $\mathfrak{g}(s)$ be its Lie algebra. $G$ acts on $C_\infty^c$ by the push-out $g_* = L_{g^{-1}}^*$ of the left translation by $g^{-1}$, so does it on differential forms by $L_{g^{-1}}^*$ and vector fields by $L_g^*$.

Definition A.0.6. Let $C_G^\infty$ be the (c-soft) sheaf of invariant (central, that is, with the adjoint action) smooth functions on $G$, and let $C_G^{-\infty}$ be the sheaf of germs of invariant distributions on $G$. 
So as usual the notation for the sections is

$$\Gamma(U, C^\infty_G) = C^\infty(U)^G, \quad (A.1)$$

and the stalks of the sheaf are denoted by

$$(C^\infty_G)_x = \lim_{\rightarrow} C^\infty(U)^G, \quad (A.2)$$

where the limit is taken over the partial ordered set of all the open sets $U$ such that $x \in U$ with the partial order being the inclusion.

**Definition A.0.7.** A (local) equivariant differential form on $M$ is the germ of a smooth map $\omega : g \to \Omega^*(M)$ at $0 \in g$ from the Lie algebra $g$ to $\Omega^*(M)$, invariant under the adjoint action of $G$, denoted by

$$\Omega^*_G(M) = C^\infty_0(g, \Omega^*(M))^G \quad (A.3)$$

where the limit is taken over the partial ordered set of all the open sets $U$ of $g$ such that $0 \in U \subset g$ with the partial order being the inclusion.

**Theorem A.0.8.** Let $d_g$ be the linear differential operator on $\Omega^*_G(M)$, defined as follows

$$(d_g \alpha)(X) = d(\alpha(X)) - \iota(X_M)\alpha(X). \quad (A.4)$$

Then $(\Omega^*_G(M), d_g)$ is a $\mathbb{Z}_2$-graded cochain complex whose cohomology is the $\mathbb{Z}_2$-graded $G$-equivariant de Rham cohomology on $M$, and so

$$H^i_G(M) \cong \bigoplus_{k \in \mathbb{Z}} H^{i+2k}(M \times_G BG, \mathbb{C}). \quad (A.5)$$
Definition A.0.9. For $g \in G$, define

$$R(g) = \{ s \in G \mid \exists h \in \text{Ad}(G)(g), M^s \subset M^h, G(s) \subset G(h) \}.$$  \hfill (A.6)

An element $h$ of $G$ is said to be near $g$ if $h \in R(g)$, and a subset $S$ of $G$ is said to be near $g$ if $S \subset R(g)$.

Lemma A.0.10. Let $g \in G$, for $X \in g(g)$ sufficiently small, $g \exp X \in R(g)$, so $R(g)$ is open and invariant.

Definition A.0.11. The differential $\mathbb{Z}_2$-graded sheaf $\mathcal{A}_G^*$ is the sheaf over $G$ (with the topology of invariant open sets) for which

(i) the stalks at $g \in G$ are elements of $\Omega^*_G(M^g)$,

(ii) the projection is, obviously, for $\omega \in \Omega^*_G(M^g)$, $\omega \mapsto g$.

(iii) the topology is generated by the collection of sets $S_{\omega_g,N}$ where $g \in G$, $\omega_g \in \Omega^*_G(M^g)$ and $N$ is an invariant open near neighborhood of $g \in G$ and on which $\omega_g$ is defined.

(iv) the $\mathbb{Z}_2$-grading is the parity stalk-wise and the differential is $d_{g(g)}$.

A global section of $\mathcal{A}_G^*$ is called a global equivariant differential form. And $\mathcal{A}_G^*$ is a $C^*_G$-module, so it is also $c$-soft.

As an immediate consequence of the definition, a section $\omega \in \mathcal{A}_G^*(U)$ is defined for every element $g \in U$ an element $\omega_g \in \Omega^*(M, G)_g = \Omega^*_G(M^g)$, such that:

(i) (invariance) for all $g, s \in G$,

$$\omega_{sgs^{-1}} = s_*(\omega_g);$$  \hfill (A.7)
(ii) (compatibility) in a sufficiently small near neighborhood in $g(g)$ and $X \in g(g \exp Y) \subseteq g(g)$

$$\omega_{g \exp Y}(X) = \omega_g(X + Y)|_{M \exp Y}.$$  \hspace{1cm} (A.8)

When $G$ acts on $M$ trivially (or simply when $M$ is a point), a global 0-form $\omega \in A^*_G$ constant on $M$ direction determines a central smooth function $\theta$ on $G$ by

$$\theta(s \exp X) = \omega_s(X) \hspace{1cm} (A.9)$$

for $s \in G$ and $X$ in a sufficiently small neighborhood of $g(s)$, and vice versa. (It is also true for a global section of $C_G^\infty$.) This is a simple example of the method of descent.

Also by the method of descent we treat the periodic cyclic homology of the crossed product algebra $C^\infty(M) \rtimes G$ as a $C_G^\infty$ module over $G$ with invariant open topology.

**Theorem A.0.12.** (Block-Getzler [6]) There is a quasi-isomorphism between periodic cyclic bi-cochain complex of $\mathcal{A}$ as a sheaf over $G$ with invariant open topology and the de Rham cohomology of the equivariant differential forms on $M$.

In [6], Block-Getzler gave a quasi-isomorphism from $G$-equivariant periodic cyclic homology cochains to $G$-equivariant differential de Rham cochains and it is well known that

$$HP_\ast(\mathcal{A}) \cong HP^G_\ast(C^\infty(M)).$$  \hspace{1cm} (A.10)

But in the dual picture for $HP^G_\ast$ groups, the isomorphism is only true for discrete groups in general.
**Definition A.0.13.** Now we take the sheaf $A_G^*$ with smooth maps replaced by distributions (and temporarily forget the differential $d_g$), call this $c$-soft sheaf $A'_G$. As the standard argument in sheaf theory [7], the compact supported sections of $A'_G$ forms a cosheaf, we will call it $A^G_*$, then $A^*_G = \text{Hom}(A^*_G, \mathbb{C})$ ($A^*_G$ with dual differential $d^*_g$ on $A^*_G$). And we call the homology of the sheaf the $G$-equivariant de Rham homology and the sections of it the equivariant currents.

So by definition, local sections of the cosheaf $A^G_*(M)$ over an invariant open set $U$ are

$$\Gamma(U, A^G_*(M)) = \Gamma_c(U, A'_G(M)).$$  \hspace{1cm} (A.11)

consists of sections of currents with contained in $U$. A local equivariant current near $g \in G$ is a compact supported distribution on $g(g)$ with value in $\Omega_*(M^g)$, a current on $M^g$. And the costalk at a point $g$ is

$$\Omega^G_*(M)_g = C^{-\infty}(g[g], \Omega_*(M^g))^G$$

$$= \lim \ C^{-\infty}(U, \Omega_*(M))^G,$$ \hspace{1cm} (A.12)

where as before the limit is taken over the partial ordered set of all the open sets $U$ of $g$ such that $0 \in U \subset g$. Again a global equivariant current satisfies invariance and compatibility conditions.

From the natural pairing, we have the universal coefficient theorem ([7]), the sequence

$$0 \to \text{Ext}^1(H^{*+1}(A^*_G(M), \mathbb{C}) \to H^*_G(A^*_G) \to \text{Hom}(H^*_G(A^*_G)(M), \mathbb{C}) \to 0$$  \hspace{1cm} (A.13)

is exact.
BIBLIOGRAPHY


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