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UMI
THE IMPACT OF DYNAMIC GEOMETRY SOFTWARE ON STUDENTS' ABILITIES TO GENERALIZE IN GEOMETRY

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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* * * * *

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ABSTRACT

The purpose of this study was to investigate whether regular use of dynamic geometry software enhances students' abilities to make generalizations in geometry. Three high school geometry classes participated in the study. The experimental group consisted of two classes taught by the researcher, and used Cabri Geometry II (on individual TI-92 calculators) on a regular basis for exploring concepts in geometry. The third class, taught by another teacher, served as a control group. While both groups used the same textbook and followed the same course of study, the control group did not use dynamic geometry software.

Data sources for the study were: scores on an Entering Geometry Student Test (EGST), a generalization pre- and posttest, task-based interviews, and classroom observations of each group. No significant differences were found between the groups on the EGST or on the generalization pretest. Analysis of covariance (ANCOVA) was used to control for initial differences on the EGST and the generalization pretest. Results of the ANCOVA test did not show any significant differences (p < .05) between the groups on the generalization posttest.

Task-based interviews with a subset of fifteen students from each group were conducted to further investigate differences between the groups on their ability to generalize. Six geometry tasks were posed to the students. The sixth task contained multiple parts. Student responses to the tasks were classified into high, medium, or low
response categories based on criteria developed by the researcher. A chi-square analysis showed that there was a significant relationship ($p < .05$) between group membership and performance in ten of the fifteen categories of the task analysis. The experimental group showed a greater tendency to make and test conjectures during the interviews. Results of the interviews and classroom observations of the experimental group indicate that regular use of dynamic geometry software seems to enhance students' abilities to make generalizations in geometry.
Dedicated to:

My mother and father.

You have both inspired me more than you will ever know.

Dad, I miss you.
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I would like to thank my adviser, Dr. Sigrid Wagner, for her encouragement and support during my years as a doctoral student at The Ohio State University. Her insight and thoughtfulness has always been an inspiration to me. The feedback that she provided during the dissertation process was invaluable. I would like to thank the other members of my dissertation committee, Dr. Douglas Owens and Dr. Stephen Pape, for their help and encouragement during this process. The editorial changes suggested by the dissertation committee resulted in a much-improved final version. I would like to thank Dr. Arthur White for his help in quantitative data analysis. I would also like to thank Dr. Michelle Reed for her help in providing inter-rater reliability in the scoring of the interview tasks.

Many thanks go to all of my colleagues (in the Columbus Public Schools and at The Ohio State University) who have been supportive of my efforts to pursue this degree. I appreciate your encouragement and enthusiasm. Special thanks go to Mr. Andrew Nguyen for all of his help with this study. My heartfelt thanks go to all of the students who participated in this study. Particularly, I wish to thank the thirty students who participated in the interviews. I appreciate your time and effort.

Finally, to my family and friends: Thanks for being there during my journey through this doctoral program. I am sure that many of you wondered if I would ever actually finish this dissertation. The good news: It is done! I share my joy of accomplishment with all of you!
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FIELDS OF STUDY

MAJOR FIELD:  EDUCATION

MATHEMATICS EDUCATION
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CHAPTER 1

STATEMENT OF THE PROBLEM

The ability to generalize in various mathematical situations is a key goal of school mathematics. The importance of generalization, the key place that geometry holds in our curriculum, and advancements in technology set the stage for research in mathematics education that connects all three areas. Abstraction and generalization have long been important processes in school mathematics; the performance of United States students on domestic and international assessments has much room for improvement; and finally, technology provides teachers with new opportunities to enhance student learning in the geometry classroom. This dissertation study connects generalization in the geometry classroom through the use of a particular technological resource, dynamic geometry software.

The chapter begins with a discussion of key changes that have been recommended by the professional organization for mathematics teachers, the National Council of Teachers of Mathematics (NCTM). The chapter continues with data from the Third International Mathematics and Science Study, comparing the performance of U.S. students in geometry to the performance of students from other nations. The importance of generalization and geometry in school mathematics is the focus of the next sections, followed by a discussion of the role of technology in learning mathematics. Subsequent sections familiarize the reader with some of the capabilities of dynamic geometry software programs. The chapter
closes with details of an amazing discovery by a high school student who was working with dynamic geometry software and the particular research questions that guided this study.

**NCTM STANDARDS**

The quality of mathematics education of American students has been a source of discussion in this country for many years. In the recent past, the professional organization for practicing mathematics teachers has become vocal in calling for reform in mathematics curriculum, instruction, and assessment. NCTM has published four documents (*Curriculum and Evaluation Standards for School Mathematics*, 1989; *Professional Standards for Teaching Mathematics*, 1991b; *Assessment Standards for School Mathematics*, 1995; *Principles and Standards for School Mathematics*, 2000) which are meant to guide reform in the teaching and learning of mathematics. NCTM supplemented the *Curriculum and Evaluation Standards for School Mathematics* with an *Addenda Series*, which presents ideas and activities for classroom use. The most recent publication, entitled *Principles and Standards for School Mathematics*, builds upon the strong foundation of the three previous *Standards* publications. Also referred to as *Standards 2000*, the latest publication will continue to provide a foundation for growth, development, and reform in mathematics education in the years to come.

NCTM, in the *Curriculum and Evaluation Standards for School Mathematics*, discusses the need for standards for school mathematics and the need for new goals for society and students as a background to the proposed K-12 recommendations. Throughout history, standards have been important to ensure quality, to indicate goals, and to promote change (NCTM, 1989, p. 2). NCTM argues that we have become an information society, with low-cost technology having had major impacts in the sciences, business, industry, and government. Our schools must adapt and prepare students for the world they will enter.
as young adults. The current organization of schools is more associated with the industrial age than the information age and the number of students preparing for agricultural and industrial careers is greatly diminished. Today’s students must be prepared to adapt in an ever-changing technological society. NCTM lists new social goals for education as follows: 1) mathematically literate workers, 2) lifelong learning, 3) opportunity for all, and 4) an informed electorate (pp. 3-4). To best prepare students to live and be productive in the information society, NCTM suggests that we must have new goals for students. They are as follows: 1) learning to value mathematics, 2) becoming confident in one’s own ability, 3) becoming a mathematical problem solver, 4) learning to communicate mathematically, and 5) learning to reason mathematically (pp. 5-6).

Along with NCTM, the Mathematical Sciences Educational Board (MSEB) and several other organizations with roots in the mathematical sciences have called for change in K-12 classrooms. In Reshaping School Mathematics (MSEB, 1990), MSEB complements NCTM with its discussion of reform in school mathematics. MSEB effectively argues that many changes affect the world of mathematics education. These changes include: changes in the need for mathematics, changes in mathematics and how it is used, changes in the role of technology, changes in American society, changes in understanding how students learn, and changes in international competitiveness (pp. 1-3). As jobs now require analytical rather than mechanical skills, students will need more mathematical power to be able to compete in the job market (p. 1). Finally, a changing American society and changing demands of the workplace make it increasingly important for “an approach to mathematics education that ensures achievement across the demographic spectrum” (MSEB, 1990, p. 3).

Technology has made possible new developments in mathematics and has given us new possibilities as to how students are able to learn mathematics. Various technologies in
mathematics classrooms give students immediate feedback on their conjectures, help students to solve problems, and may assist students in developing reasoning skills. Technology may fundamentally change teaching and learning in mathematics classrooms. Dynamic geometry software, in particular, offers new opportunities for learning geometry. Dynamic geometry software can help students to analyze problem situations and to develop analytical skills.

THIRD INTERNATIONAL MATHEMATICS AND SCIENCE STUDY

Along with recommendations of professional organizations, large-scale data have been collected which compare mathematics achievement in the United States to that of other countries. The Third International Mathematics and Science Study (TIMSS) provides much data for those attempting to make a case for improvements in achievement of American students. Conducted in 1995 by the International Association for the Evaluation of Education (IAEE), TIMSS collected information on a half-million students worldwide, including more than 33,000 students in more than 500 U.S. public and private schools. In mathematics, TIMSS data were collected in the following areas at the fourth-, eighth-, and twelfth-grade levels: fractions and number sense; algebra; geometry; data representation, analysis, and probability; measurement; and proportionality. In the advanced mathematics assessments, data were collected in the areas of numbers and equations, calculus, and geometry. Findings from the eighth and twelfth grades are presented below. Specific attention is drawn to findings in geometry.

Eighth-grade findings. Eighth-grade results showed that U.S. students were outperformed by students from 20 countries, were comparable to students of 13 countries, and outperformed students of only 7 countries. The international standing of U.S. eighth-grade students was stronger in the areas of algebra, fractions, data representation, analysis, probability, and number sense than it was in the areas of geometry, measurement, and
proportionality. Geometry problems included visualization and properties of geometric figures, including symmetry, congruence, and similarity.

In an analysis of eighth-grade textbooks, geometry occupied more space in German and Japanese books than in U.S. texts. Classroom observations indicated that Japanese and German lessons were also more likely to cover geometry. In studying lessons by U.S., German, and Japanese teachers, 30% of Japanese lessons and 23% of German lessons were considered to contain high-quality sequences of mathematical ideas while none of the U.S. lessons were classified in this category.

The TIMSS report (IAEE, 1995) states that a primary goal of the reform movement in mathematics education is to create classrooms where students are engaged in thinking deeply about mathematics. Discovery, understanding, and applications of concepts to new and different situations should be prevalent in mathematics classrooms. TIMSS data suggest that the content of the U.S. eighth-grade mathematics curriculum is not as challenging as that of other countries. United States teachers have tried to implement some of the reform recommendations, according to TIMSS. The changes have been, however, focused on isolated techniques such as cooperative learning rather than the key idea that mathematics teaching and learning should bring about high levels of mathematical thinking in students. Finally, TIMSS data suggest that U.S. students have a relatively lower international standing in mathematics than they do in science.

Twelfth-grade findings. The twelfth-grade items of TIMSS focused on mathematics general knowledge, advanced mathematics, science general knowledge, and physics. The United States was among the lowest performing countries in both the mathematics and science general knowledge assessments. In both areas, the U.S. was below the international average. In mathematics general knowledge, U.S. students were outperformed by 14 out of 20 countries, performed similarly to students in four countries, and outperformed students of only two countries.
Mathematics items at the twelfth-grade level consisted of questions in the following areas: 1) numbers, equations, and functions, 2) calculus, and 3) geometry. The geometry items consisted of basic geometry, coordinate geometry, polygons, and circles, and three-dimensional geometry. Among the content areas, U.S. students' performance was relatively weakest in geometry. No country scored below the United States in geometry.

Other large-scale domestic assessments such as the National Assessment of Educational Progress (U. S. Department of Education, 1997) offer additional data suggesting that U.S. students have much room for improvement in geometry achievement. Clearly, there is both room and reason to conduct research in the area of geometry in American schools.

GENERALIZATION

*Being able to reason is essential to understanding mathematics. By developing ideas, exploring phenomena, justifying results, and using mathematical conjectures in all content areas and—with different expectations of sophistication—at all grade levels, students should see and expect that mathematics makes sense.*

—NCTM, 2000, p. 55.

*We believe that learning should be guided by the search to answer questions—first at an intuitive, empirical level; then by generalizing; and finally by justifying (proving).*


NCTM's *Principles and Standards for School Mathematics* (2000) calls for many changes in mathematics curricula. An emphasis on new topics introduced through problem situations that will encourage students to explore mathematics, to make and test
conjectures, and prove generalizations is evident throughout the document. Conjecturing is a process in which learners make an informed guess as to whether an idea is true. Conjecturing, therefore, is a generalizing process. NCTM (2000) states that “to make conjectures, students need multiple opportunities and rich, engaging contexts for learning” (p. 56).

In the *Curriculum and Evaluation Standards for School Mathematics*, NCTM (1989) describes one of its new goals for students (learning to reason mathematically), by stating that “making conjectures, gathering evidence, and building an argument to support such notions are fundamental to doing mathematics” (p. 6). NCTM (1989) summarizes the discussion on goals by stating that the intent of the goals is for students to become mathematically literate. NCTM defines “mathematically literate” as “an individual’s ability to explore, to conjecture, and to reason logically, as well as to use a variety of mathematical methods to effectively solve problems” (p. 6). Thus, making and testing conjectures is a critical aspect of mathematics learning.

In *Geometry from Multiple Perspectives* (NCTM, 1991a), NCTM points out that “drawing and measuring computer utilities provide learning environments that foster the active process of making and exploring conjectures” (p. vi). This particular publication contains many activities “that require students to experiment, collect data, search for patterns, make conjectures, and give convincing arguments” (p. vi). NCTM further illustrates its commitment to this type of thinking in school mathematics with the following quote: “It is important that provisions be made for students to share their experiences, clarify their thinking, generalize their discoveries, construct convincing arguments, and recognize connections with other topics” (p. vii).

In the *Professional Standards for Teaching Mathematics* (NCTM, 1991b), several shifts in the environment of mathematics classrooms that are important for
achieving the goal of empowering students are discussed. One of these shifts is toward a focus on mathematical reasoning, and away from a focus on memorizing procedures. Another shift is that conjecturing, inventing, and problem solving replace mechanistic answer-finding (p. 3). Many of the vignettes presented in this volume of the Standards series illustrate how reasoning, conjecturing, problem solving, and generalizing can be effectively threaded into mathematics lessons.

MSEB (1990) also describes learning as a quest to answer questions intuitively and empirically, followed by generalization and proof. MSEB states the following: "If mathematics is a science and language of patterns, then to know mathematics is to investigate and express relationships among patterns" (p. 12). Generalization, therefore, is seen as a key goal of school mathematics.

Finally, the College Entrance Examination Board (CEEB) argues that developing the ability to generalize should be an important focus of high school mathematics. In a publication titled Academic Preparation in Mathematics: Teaching for Transition From High School to College (CEEB, 1985), a case is made for examining number patterns with origins in geometry. Students should investigate data, conjecture a general law, and finally test their conjectures for validity.

In this section, the importance of generalization as a process in school mathematics has been discussed. While generalization occurs in every branch of mathematics, geometry provides a rich context for generalization. A brief history of the development of geometry and its role in school mathematics is presented in the following section.
When all is said and done, however, the fact remains that geometry has attracted and held the attention and interest of the human race from prehistoric times until the present, purely for its own sake. Geometry is at the same time a science and an art, mathematics and philosophy. It supplies us the only perfect system of logic and its beautiful completeness is not to be found in any other branch of knowledge.

—J. E. Thompson, 1934, pp. xii-xiv.

Thompson (1934) describes the evolution of geometry from early history to its present status in the school curriculum. With beginnings in the era of the Babylonians, geometry has a rich history of development and use. Ancient Egyptian and early Greek mathematicians contributed to the development of the geometry we know and use today.

Pythagoras (569 – 500 B.C.), a Greek mathematician, was the first to investigate systematically the principles of geometry and to apply the methods of logic to its development. Two more prominent Greek mathematicians, Euclid and Archimedes, furthered knowledge in geometry with their contributions to the field. Euclid, who lived from about 330 B.C. to about 275 B.C., is known for his books on geometry, collectively known as Euclid's Elements. The Elements, written in Euclid's later years, is the basis of school geometry today.

Archimedes (287 – 212 B.C.) is known for perfecting the measurement of the circle, sphere, cylinder, and cone, as well as many other contributions to geometry and mathematics. The development of geometry as a field of study continued beyond Euclid and Archimedes, with many mathematicians making contributions. Appolonius of Perga (260 – 200 B.C.) developed the study of the conic sections, and his work became the basis for modern analytical geometry. Hipparchus of Nicaea is credited with the invention of
trigonometry, an advanced branch of applied geometry. While elementary and secondary students study Euclidean geometry, advanced topics in geometry, including non-Euclidean geometries are generally studied at the post-secondary level.

There is general agreement on the goals of geometry instruction. By studying geometry, students develop logical thinking abilities, spatial intuition about the real world, knowledge needed to study more mathematics, and skills in the reading and interpretation of mathematical arguments (Suydam, 1985, p. 481). NCTM argues that by studying geometry, "students will learn about geometric shapes and structures and how to analyze their characteristics and relationships" (NCTM, 2000, p. 41). NCTM continues by discussing the importance of spatial visualization and how geometry is a natural setting for developing students' reasoning and justification skills. Geometry is important in representing and solving problems in mathematics and in real-world situations.

While tools such as compasses and straigntedges have traditionally been used in the study of geometry at the elementary and secondary level, new tools have emerged in the recent past with the development of various geometry software programs for computers and calculators. These new technological developments have tremendous potential to impact the teaching and learning of geometry in our schools. NCTM acknowledges the important role of dynamic geometry software in the teaching and learning of geometry (NCTM, 2000, p. 25). One particular dynamic geometry software program is described later in this chapter. The researcher has observed in his own classes the power of dynamic geometry software to explore many examples and help students make generalizations in geometry.
DYNAMIC GEOMETRY SOFTWARE

Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning.


Geometry has always been a rich arena in which students can discover patterns and formulate conjectures. The use of dynamic geometry software enables students to examine many cases, thus extending their ability to formulate and explore conjectures.

—NCTM, 2000, p 309.

In Principles and Standards for School Mathematics (NCTM, 2000), NCTM presents its Technology Principle (pp. 24-27). The technology principle states that “electronic technologies—calculators and computers—are essential [italics added] tools for teaching, learning, and doing mathematics” (p. 24). The Technology Principle has three components: 1) technology enhances mathematics learning, 2) technology supports effective mathematics learning, and 3) technology influences what mathematics is taught. NCTM states that the “existence, versatility, and power of technology make it possible and necessary to reexamine what mathematics students should learn as well as how they can best learn it [italics added]” (p. 24). Technology allows students to examine “more examples or representational forms than are feasible by hand, so that they can make and explore conjectures easily” (p. 25). The range of accessible problems is extended by technological tools as students are able to execute routine procedures quickly and accurately, allowing additional time for conceptualizing and modeling. Learning is assisted by feedback, which is supplied immediately by dynamic geometry software programs.
NCTM discusses further the role of dynamic geometry software in mathematics classrooms. Students can use dynamic geometry software to “engage actively with geometric ideas” (NCTM, 2000, p. 41). NCTM points out that dynamic geometry software allows students to have an interactive experience with two-dimensional objects and is useful in modeling. With dynamic geometry software, “students can generate many examples as a way of forming and exploring conjectures” (p. 41).

The teacher is the key to whether or not technology is used effectively in the classroom. Technology alone does not make a difference; it is how the teacher decides to use it that will impact student learning in mathematics. Teachers have opportunities to focus on student thinking as students use technology in the classroom. NCTM clarifies this point as follows: “As students use calculators or computers in the classroom, the teacher has an opportunity to observe the students and to focus on their thinking” (NCTM, 2000, p. 26). Finally, NCTM argues that technology influences what topics are taught and when they appear in the curriculum. Young children can explore problems involving large numbers and can use geometry software to investigate shapes. Technology is useful in helping teachers connect the development of skills and procedures to the more general development of understanding in mathematics. Some skills previously considered to be essential, may now be considered less necessary because of technology. NCTM makes the point that as some skills become less necessary because of the technology, “students can be asked to work at higher levels of generalization or abstraction [italics added]” (p. 26). Clearly, the use of technology can positively impact students’ mathematical learning experiences.

The National Research Council (NRC) reported on the future of mathematics education to the nation in a document called Everybody Counts (National Academy Press, 1989). In its argument for changing the role of technology in mathematics education, NRC makes five key points:
• by utilizing technology, students will better be able to experience mathematics as an exploratory discipline where risking and failing provide clues to success

• technology can make higher mathematics more accessible to students who have weaknesses in algebraic or computational skills

• the learning environment becomes more active and dynamic as students are able to explore a vast number of examples and see the dynamic nature of mathematical processes

• more mathematics can be explored individually by students when they can ask and answer for themselves countless "what if" questions

• innovative instruction based on a new symbiosis of machine calculation and human thinking can shift the balance of learning toward understanding, insight, and mathematical intuition (NRC, 1989, pp. 62-63).

Dynamic geometry software addresses each of the five key points made by NRC. When using dynamic geometry software, students learn geometry in an environment where conjectures can be quickly examined. Students who have weaknesses in computation or in algebra can use the calculation features of dynamic geometry software programs to test their conjectures. Examining large numbers of examples can lead students to making conjectures and generalizations in geometry. Conjecturing and generalizing have already been discussed as being important to student learning. Dynamic geometry software allows students to see many examples in a relatively short amount of time, and thus, can play a key role in conjecturing and generalizing.

In Reshaping School Mathematics: A Philosophy and Framework for Curriculum (MSEB, 1990, p. 18-19), MSEB points out that technology will decrease the value of many paper-and-pencil skills that are traditionally taught and will increase the importance of areas in the mathematics curriculum that are now rarely taught. Technology
will help focus attention on problem formulation as well as problem solving. Lastly, MSEB states that technology may impact the world of teaching and learning in ways that mathematics educators can only dream of. We can only imagine Euclid's amazement if he were to see dynamic geometry software programs and their power to explore the ideas of geometry.

The College Board (CEEB, 1985) points out that computers are changing the discipline of mathematics, and consequently the mathematics curriculum. CEEB states that "computers are making mathematics into a subject that requires much less attention to the mastery of routine procedures and much more to how computer power can be used to explore mathematical ideas" (p. 22). Dynamic geometry software programs (available on computers and on hand-held calculators) offer the capability of learning traditional ideas of geometry in new and different ways, as well as the potential for new discoveries. One such discovery is described later in this chapter.

The Committee for Economic Development (CED), a private, nonprofit, research and education organization made up of corporate executives and university presidents, makes a case for dramatic changes in mathematics and science education due to technological improvements and availability. In *Connecting Students to a Changing World: A Technology Strategy for Improving Mathematics and Science Education*, CED (1995) discusses technology's role in more effective learning as well as presenting a new technology strategy for education. Dynamic geometry programs are highlighted for their potential in helping students to test hypotheses and make conjectures. An example of an active learning environment using a particular dynamic geometry program, *The Geometer's Sketchpad*, is described. Students "can draw, rotate, and measure geometric figures to test hypotheses about what happens when different parameters of a figure change" (p. 4). Technology is described as stimulating student interest and increasing time
spent by students on tasks, which can lead to improved achievement. CED cites evidence that when technology is effectively integrated into mathematics instruction, student learning and achievement can be positively impacted (p. 4).

**CABRI GEOMETRY II**

One particular dynamic geometry software program, *Cabri Geometry II* (Texas Instruments), has been used by the researcher in his geometry classes for several years. *Cabri Geometry II* (hereafter referred to as *Cabri*) offers students a software environment in which geometric objects can be constructed, measured, and manipulated. Basic ideas of geometry as well as advanced ideas are easily explored with *Cabri*. Once a geometric object is created in *Cabri*, students can manipulate the object, perform constructions on the object, and measure angles, lengths, and area. *Cabri* has the potential to impact geometry learning in the manner described by NCTM as well as in each of the five areas addressed by NRC in *Everybody Counts*. With *Cabri*, students have the power to explore many variations of a geometric situation in a short amount of time. With the ability to do calculations easily and quickly, computation takes a secondary role to concepts and ideas. Students have the ability to see mathematics as a dynamic discipline when working with *Cabri*.

An example to show the power of *Cabri* is how a well-known generalization is discovered by students using the software. Early in geometry courses, students discover that sum of the interior angles of any triangle is $180^\circ$. While this result can be attained in an analytic manner or by paper folding, *Cabri* offers an exciting and different way to explore the problem, along with the added advantage of being able to examine many triangles very quickly to develop the idea. Students begin by using the drawing capabilities of *Cabri* to create a triangle. The angle measuring tool can be used to measure each of the three angles. *Cabri* has a calculator feature which allows the user to compute with numerical values that
have been generated by Cabri's measuring tools. The student can add the measures of the three angles together to find out that the sum is $180^\circ$. The real power of software programs such as Cabri lies in what the student is able to do next. Cabri has a feature that allows the user to select an object and move it. This is the reason why software such as Cabri is referred to as dynamic. Figures can be manipulated easily. Anything that depends on the object in any way changes when the object is moved. In this particular example, the student is able to select the vertices of the triangle (one vertex at a time) and drag them to new locations. The shape of the triangle changes, and along with it, the lengths of the sides and the measurements of the angles change. What remains constant, of course, is the sum of the measures of the interior angles. The student is thus able to look at many triangles in a matter of seconds. This is illustrated in Figure 1.1 and Figure 1.2.

\[ m\angle A = 99.4^\circ \]
\[ m\angle B = 47.1^\circ \]
\[ m\angle C = 33.4^\circ \]

Sum of the interior angles = $180.0^\circ$

**Figure 1.1. Triangle sum, initial appearance.**
\[ m\angle B = 48.3^\circ \]
\[ m\angle A = 56.8^\circ \]
\[ m\angle C = 74.9^\circ \]

Sum of the interior angles = 180.0°

**Figure 1.2. Triangle sum, modified appearance.**

The reader should note that *Cabri* has the capability to show differing levels of precision when it reports measurements of lengths, angles, and other computations. In Figures 1.1 and 1.2, the precision for angle measurements and other computations was set to one decimal place. While the sum of the three values shown for the angle measurements in Figure 1.1 is 179.9°, *Cabri* uses more precise values when obtaining the result of the computation of the sum of the measurements. Thus, the correct value of 180.0° is shown as the sum. In Figure 1.2, the sum of the three values shown for the angle measurements is 180.0°.

With generalizations such as these accessible to students in such a short amount of time, it seems clear that experience with software such as *Cabri* has the potential to impact students’ abilities to make generalizations in geometry.
AN AMAZING DISCOVERY

In addition to using the software to help students discover well-known generalizations in geometry, dynamic geometry software has the potential to help students discover brand new generalizations. Dynamic geometry software has, in fact, helped one secondary student, Ryan Morgan, discover a new theorem. The theorem that Ryan Morgan discovered is now named after him (Watanabe, Hanson, & Nowosielski, 1996).

Ryan Morgan's teacher asked his ninth-grade students to verify a theorem known as Marion Walter's theorem, which involves the hexagon formed by connecting trisection points of sides of triangles to the opposite vertex. A sketch of this triangle and the resulting hexagon is shown in Figure 1.3. As Figure 1.3 shows, the ratio of the area of the triangle to the area of the hexagon is ten to one.

\[
\begin{align*}
\text{Area of Triangle ABC} &= 44.60 \text{ cm}^2 \\
\text{Area of Hexagon DEFGHI} &= 4.46 \text{ cm}^2 \\
\frac{\text{Area of Triangle ABC}}{\text{Area of Hexagon DEFGHI}} &= 10.0
\end{align*}
\]

Figure 1.3. Marion Walter's Theorem (based on Watanabe, Hanson, & Nowosielski, 1996).
Marion Walter's theorem, in turn, was prompted by the cover of the February 1992 *Mathematics Teacher* journal (the cover was created by William Johnston). The idea is that if particular trisection points of a side of a triangle are connected to the opposite vertex, the resulting triangle has an area of one-seventh of the area of the original triangle. Using a colorful transformation approach, the reader is convinced of the result by looking at a series of sketches on the journal cover. A *Cabri* sketch illustrating this idea follows, in Figure 1.4. The unlabeled points on the sides of triangle ABC are trisection points of the sides of the triangle.

![Diagram](image)

Area of Triangle ABC = 28.75 cm\(^2\)
Area of Triangle DEF = 4.11 cm\(^2\)

\[
\frac{\text{Area of Triangle ABC}}{\text{Area of Triangle DEF}} = 7.0
\]

*Figure 1.4. *Cabri* sketch based on February 1992 cover of *Mathematics Teacher* journal.*
Marion Walter's Theorem is an extension of this idea. Instead of just three of the segments being drawn in the interior of the triangle, all six possible segments are drawn to form a hexagon. The ratio of the area of the original triangle to the area of the hexagon is ten to one, as discussed earlier.

After using a non-dynamic geometry software program, GeoExplorer (ScottForesman, 1992), Morgan used a dynamic program, The Geometer's Sketchpad (Jackiw, 1991) to extend Walter's Theorem by dividing a side of a triangle into various numbers of congruent segments. Noticing that a pattern emerged when the number of congruent segments was odd, Morgan collected data and used regression to determine a general result. The case of dividing the sides of the triangle into five congruent segments is shown below in Figure 1.5.
Area of Triangle ABC = 50.29 cm² 
Area of Hexagon DEFGHI = 1.80 cm²

\[
\frac{\text{Area of Triangle ABC}}{\text{Area of Hexagon DEFGHI}} = 28.0
\]

Figure 1.5. Morgan's discovery *(based on Watanabe, Hanson, & Nowosielski, 1996).*

Using *The Geometer's Sketchpad*, Morgan showed that when the sides of the triangle are divided into 7, 9, 11, and 13 congruent segments, the corresponding ratios of area of the triangle to the area of the hexagon are 55 : 1, 91 : 1, 136 : 1, and 190 : 1, respectively.
Morgan used regression to determine that the general ratio was
\[ \frac{9n^2 - 1}{8} \text{ to } 1. \]

Full details of this amazing discovery are presented in an article (Watanabe, Hanson, & Nowosielski, 1996) appearing in the May 1996 *Mathematics Teacher* journal.

Examples such as this lead us to believe that there are other unknown results in geometry waiting to be discovered. Dynamic geometry software programs such as *Cabri* have the potential to help us discover many new generalizations in geometry. Dynamic geometry software programs allow students to construct geometric figures, take various measurements, and see the figures and the measurements change dynamically as the original figure is manipulated. Geometry learning has entered a new era with the arrival of dynamic geometry software.

**RESEARCH QUESTION**

This dissertation study investigated a question that is key to the calls for reform in mathematics education. That question is the following:

**RESEARCH QUESTION:** Does experience with *Cabri* enhance students' abilities to make generalizations in geometry?

In this study, I define "to generalize" as "to arrive at a mathematical relationship that is true for all specified variations of an abstraction." An example of a generalization is the previously discussed situation in which the sum of the measures of the angles of any triangle is a constant 180°. To investigate the research question, the generalization abilities of two groups of students were the focus of the study. The experimental group used *Cabri* on a regular basis (see Chapter 3 for a complete list of topics taught using *Cabri*) for the
purpose of exploring and discovering geometric ideas. The control group did not use the software at all, and were taught using a traditional approach. An entering geometry test and a generalization pretest were given to both groups at the beginning of the course for comparison purposes. A generalization posttest was given to compare the groups in the final week of the course. A subset of students from each group participated in task-based geometry interviews. Further research questions which guided the study are as follows:

**RESEARCH SUBQUESTION ONE:** Were there significant differences between the groups in scores on the generalization posttest?

**RESEARCH SUBQUESTION TWO:** Was there a significant relationship between group membership and performance on the interview tasks?

A theoretical framework for the study as well as a review of related research literature is presented in Chapter 2. All pertinent details of the study are described in Chapter 3. Results of the study are presented in Chapter 4. Major findings of the study as well as implications for teaching and research are summarized in Chapter 5.
CHAPTER 2
THEORETICAL FRAMEWORK
AND
REVIEW OF RELATED LITERATURE

The ideas of mathematics education theorist Zoltan P. Dienes provide a theoretical framework for the study. In particular, Dienes' mathematics learning theory and his theory of abstraction and generalization are discussed in relation to dynamic geometry software.

DIENES

I suggest that it is possible to establish fully creative mathematical-learning situations at all the stages of mathematics-learning. When a child has effectively formed a concept from his own experiences, he has really created something that was not there before, and this something is built into his personality in the psychological sense in the same way as essential substances in his food are built into his body. The value of this piece of learning to him will be of a similar kind to the painting of a satisfactory picture or the writing of a good story, or the inventing of an exciting play to be acted with his friends. It will have intrinsic value, as part of the very stuff that life is made of.

This quote from *Building Up Mathematics* (1960), by Dr. Zoltan P. Dienes lays the groundwork for a theoretical framework for the study. As the introductory pages of the book which Dienes entitled “A Survey of the Present Position” are read, one begins to wonder whether the book was indeed written in 1960, or perhaps yesterday. It is both illuminating and somewhat disheartening to read a discussion of the state of affairs in mathematics education from forty years ago that so clearly describes the present situation. Dienes describes the great number of children who dislike mathematics and asserts that the majority of children never succeed in understanding the real meaning of mathematical concepts. Dienes further states that “at best, they [students] become deft technicians in the art of manipulating complicated sets of symbols, at worst they are baffled by the impossible situations into which the present mathematical requirements in schools tend to place them” (p. 1). Dienes suggests that the learning of mathematics can contribute to personal fulfillment as the practice of the arts can. He states that “we need to create mathematical learning situations, partly as if we were practicing an art-form, and partly as if we were devising an original research-situation” (pp. 2-3).

Dienes considers school learning situations as falling into three main categories: mathematical, educational, and psychological. Mathematical learning situations can be thought of as consisting of two aspects: the acquisition of techniques, and the understanding of ideas (p. 3). It can be argued that one aspect, the acquisition of techniques, plays a dominant role in school mathematics even today, despite recommendations to the contrary by leading mathematics professional organizations and mathematical learning theorists. Dienes gives an example that clarifies this situation. He asks the reader to consider if an average person were asked how well he understood the mathematics he learned in school, that he might tell about carrying out the teacher’s instructions but that the reasons often seemed unclear and even mysterious. Children may
be under the impression that they understand mathematics when in fact they do not. It is also easy for teachers, Dienes argues, to be under the impression that students understand ideas when in fact they do not. Children learn to give standard answers to standard questions and true understanding is often never realized. For many students, mathematics ends up as a collection of unrelated procedures.

Teachers are described by Dienes as authorities and dispensers of information. If we use achievement as a measure, then this method of teaching and learning has not fared well. Data from TIMSS and other assessments support the ideas that our present modes of teaching and learning are not totally adequate. Dienes goes so far as to say that "it may be that the whole system of the class-lesson is an unsuitable vehicle for the transmission of mathematical information" (p. 4). Ruling out bad teaching as being a major cause of the situation, Dienes points to the transmission stage of the process as the major issue. While improvements have been suggested and made, Dienes argues that "there are fundamental shortcomings in the present system which cannot be patched up by new methods of communication, however ingeniously devised" (p. 5). Such devices (pictorial aids, models, film, and television) may prove necessary, but not sufficient for an effective teaching and learning situation.

Dienes continues by discussing the psychological situation of mathematics learning. As a society, we have learned much about abstract thinking, but there is much yet to learn. Dienes states that even if all of the psychological knowledge that is known were put to practical use in classrooms immediately, he doubts that it would solve all of the problems in education. As this volume was written in 1960, one may wonder how Dienes would view the situation today. He states that there are three principal areas in which research is conducted: 1) individual differences in ways of forming abstract ideas, and variations in the same individual as he grows up, 2) details of the mechanism of the process of abstraction, and 3) the problem of motivation. As knowledge accumulates in these areas,
Dienes states that it is desirable to make the information available to teachers in a way that can be used in classrooms. He doubts that anything of the kind is happening to the degree to which it is necessary.

**DIENES' MATHEMATICS LEARNING THEORY**

Dienes’ (1960) Mathematics Learning Theory can be summarized into four principles: the dynamic principle, the constructivity principle, the mathematical variability principle, and the perceptual variability or multiple embodiment principle (pp. 30-31).

**Dynamic Principle.** Preliminary, structured practice and/or reflective type of games must be provided as necessary experiences from which mathematical concepts can eventually be built, so long as each type of game is introduced at the appropriate time.

**Constructivity Principle.** In the structuring of the games, construction should always precede analysis.

**Mathematical Variability Principle.** Concepts involving variables should be learned by experiences involving the largest possible number of variables.

**Perceptual Variability Principle or Multiple Embodiment Principle.** To allow as much scope as possible for individual variations in concept-formation, as well as to induce children to gather the mathematical essence of an abstraction, the same conceptual structure should be presented in the form of as many perceptual equivalents as possible.

Dynamic geometry software such as *Cabri* satisfies every principle of Dienes’ Mathematics Learning Theory. As Dienes conceptualized this theory with elementary age students in mind, the researcher will make adaptations of the theory to the high school level. Specifically, the researcher will consider how Dienes’ Mathematics Learning Theory might pertain to a geometry class which utilizes dynamic geometry software. Software such as *Cabri* certainly fits Dienes’ **Dynamic Principle**, as it is dynamic in nature. A sketch of a geometric figure is not fixed; it can be manipulated to look at many cases in a
very short amount of time. Students must also experiment with the software to find out how the various commands work. Regular use of the software helps students develop a knowledge base of how to use the software, so that time is not wasted relearning how to do basic things every time the software is used.

Dienes' next principle, the **Constructivity Principle**, is continually present when students use *Cabri*. Students must construct figures prior to doing analysis on them. Students must learn to differentiate between constructing a figure to meet certain specifications and making a drawing that seems to fit the specifications, when in fact it fails to do so if the figure is manipulated. To clarify this, let us consider the case where the student is asked to investigate properties of rectangles. If a student draws a figure that only *appears* to be a rectangle, it will no longer be a rectangle if one of the vertices is moved to a different location. This is illustrated in Figures 2.1 and 2.2.

![Figure 2.1. A student's "construction" of a rectangle.](image)

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While the polygon in Figure 2.1 *appears* to be a rectangle, consider what happens if any of the vertices are moved to a different location, as in Figure 2.2. In Figure 2.1, the student used the Polygon tool to construct a quadrilateral that *appeared* to be a rectangle, but in fact did not remain a rectangle once the student selected a vertex point and moved it to a new location. The polygon tool allows students to create any irregular convex or concave polygon. If the student wishes to construct a regular polygon, *Cabri* has a separate tool for that task (called Regular Polygon). The polygon tool works in the following way: the student places the first point on the screen and moves the cursor to a new location. As the student places the second point on the screen, a segment is constructed connecting the two points. The student continues placing points until the desired number of sides is reached for the polygon. The last point is placed “on top of” the first point, to indicate to the software that the user does not want any more segments to be constructed. Students learn that they must use construction tools in *Cabri* if they want to construct a rectangle that retains its properties even when a vertex or construction line is moved to a new location. Figure 2.3 illustrates a rectangle that was constructed using the
**Perpendicular Line** command of *Cabri*. A student could construct a true rectangle in the following way: draw any line, then construct lines perpendicular to that line through any two points on the line (or alternatively, not on the line). Finally, construct a fourth line that is perpendicular to either of the two newly constructed lines through any point (on or off the lines), and the outline of the rectangle is formed. Alternatively, *Cabri* has a **Parallel Line** command that could be used to construct the outline of the rectangle in a different manner. Once the outline of the rectangle is constructed, the student can then use the polygon tool to select the four points of intersection (in some consecutive order) and a new object (a rectangle) will exist on the screen. Figure 2.3 shows how the rectangle can be distinguished from the lines by making the rectangle bold using the **Thick** command which is located in the **Hide/Show** menu. The rectangle need not be constructed with a side parallel to the bottom of the screen.

![Figure 2.3. A rectangle constructed using perpendicular lines.](image-url)
Figure 2.4 shows a manipulation of the constructed rectangle, where the size of the rectangle remains unchanged while the orientation of the rectangle is slightly changed. The only change from Figure 2.3 is that line PS was selected and moved to a different location, but point P remained fixed.

![Initial manipulation of constructed rectangle.](image)

Figure 2.4. Initial manipulation of constructed rectangle.

Figure 2.5 shows a second manipulation of the constructed rectangle, where the size of the rectangle changes while the orientation of the rectangle remains unchanged. The only change from Figure 2.4 is that point S was selected and moved to a different place on line PS, thereby increasing the distance from point P to point S. Line SR moved when point S was moved, as it was constructed to go through point S.
Several authors have discussed the importance of the difference between drawing and constructing in a dynamic geometry environment (Finzer & Bennett, 1995; Galindo, 1998; Hazzan & Goldenberg, 1997; Hoyles & Noss, 1994; Scher, 1996; Vincent & McCrae, 1999). Hazzan and Goldenberg (1997) discuss the notion of *successive refinement*, a problem-solving strategy in which the learner solves a problem in a sequence of steps, with each step a refinement of the previous one. Problems that are ideal for this strategy are ones that are non-trivial to the learner. Hazzan and Goldenberg discuss how students refine their construction in stages, so that in each stage, they improve the square by taking into account additional essential information. Being asked to construct a square in a dynamic geometry environment (Gardiner & Hudson, 1998; Hazzan & Goldenberg, 1997; Mariotti & Bartolini Bussi, 1998) is an example of a non-trivial
problem for many students. Vincent and McCrae (1999) note that "Cabri can result in significant progress in understanding of geometric properties and relationships even after relatively few lessons" (p. 507).

Dienes' third and fourth principles, the **Mathematical Variability Principle** and the **Perceptual Variability Principle**, address the notions of variability. Students working with *Cabri* have the opportunity to see many things vary in the same sketch. In the example of the rectangle in Figures 2.3, 2.4, and 2.5, we could move point S or point R around so that the size and shape of the rectangle changes. Size and shape are mathematical variables. What remains invariant is the fact that all of the angles are right angles. The **Mathematical Variability Principle** is embodied by *Cabri* in the sense that a student can change the size and shape of a figure.

The **Perceptual Variability Principle** is embodied by *Cabri* in the sense that a student can change the orientation of the figure. Orientation is a perceptual variable. Line PS was the original object in the sketch. As line PS is moved, the orientation of the rectangle changes. Students can even look at many different figures (space permitting) on the same sketch. For example, another rectangle could be constructed in the same sketch, as shown in Figure 2.6. Dienes (1960) states that "to give the maximum amount of experience, structured so as to encourage the growth of the concept, it seems *a priori* desirable that *all* possible variables should be made to vary while keeping the concept intact" (p. 29).
Figure 2.6. More than one rectangle in the same sketch.

In summary, a dynamic geometry software program such as *Cabri* embodies each of Dienes' principles in unique and important ways. Students have the capability to explore geometry as a dynamic discipline when using software such as *Cabri*. At the same time, they learn important distinctions between a construction and a drawing. Students are able to manipulate mathematical variables, such as size and shape, and perceptual variables, such as orientation. Finally, many figures can be constructed and manipulated on the same screen if desired to further emphasize the variability discussed in Dienes' Mathematics Learning Theory.
DIENES' THEORY OF ABSTRACTION AND GENERALIZATION

Dienes defines the process of abstraction as "the process of drawing from a number of different situations something which is common to them all" (1961, p. 281). For example, Dienes discusses how forming the concept of the natural number two is an abstraction. Students should be presented with many different pairs of objects of the greatest possible diversity. The only thing the pairs of objects would have in common is that there are two of them in each group. In the process of abstraction, a "class is constructed out of some elements which will then be said to belong to the class" (pp. 281-282).

Dienes views generalization as leading from classes to classes instead of leading from elements to classes, as abstraction does. He defines two types of generalization: primitive generalization and mathematical generalization (pp. 282-283). Primitive generalization is defined as the passing from one class to another where the other includes the former as a part. An example is presented in which a child, after having had some experience with adding two numbers in either order and the sum remaining the same, generalizes to the idea that the commutative law of addition must be true for all natural numbers.

Mathematical generalization is defined as follows: a class $B$ is a mathematical generalization of the class $A$ if $B$ includes as a part an isomorphic image of $A$, in relation to all relevant properties. This means that the classes $A$ and $B$ could consist of quite different elements, as long as there was a part of $B$ which was somehow an exact image of $A$, mirroring the properties of $A$ in all relevant respects. Dienes illustrates mathematical generalization by using the class of natural numbers and the classes of positive and negative integers. Although the class of positive integers has the same properties as the class of natural numbers, they form a sub-class of the class of directed numbers to which the natural numbers do not belong. Using the notation from above, the class of directed
numbers is class $B$. The class of natural numbers is class $A$. The image of the class $A$ in the class $B$ is the class of positive integers. An illustration, given in Figure 2.7, may help the reader to better understand Dienes' definition of mathematical generalization.

![Diagram of classes A and B]

**Figure 2.7.** Mathematical generalization.

Dienes summarizes the notion of mathematical generalization by referring to it as discovering a new world and finding out that part of the new world is exactly like our entire old world.

According to Dienes, the ideas of abstraction and generalization are connected in the following way: abstraction can be thought of as *class formation* and generalization as *class extension* (p. 296). A particular instance of this idea, as it relates to geometry, is the following: Consider the abstraction *centroid* as it applies to triangles. A *centroid of a triangle* is the point of concurrency of all three medians of the triangle. In addition to the notion of concurrency, a numerical relationship or generalization exists between the two
smaller segments of each median that lie on either side of the centroid. Specifically, the larger portion of the median is exactly twice as long as the smaller portion of the median. Of course, this same relationship can be expressed in various ways. Figure 2.8 illustrates this idea.

Figure 2.8. Centroid of a triangle.
With *Cabri*, a student can explore the abstraction *centroid of a triangle* very quickly and find out that the medians intersect in a single point in all triangles. Students can also use the measuring and drag capabilities of *Cabri* to see that the numerical relationship between the lengths of the two sections of the median is true in all triangles.

By utilizing the software, we can begin to explore pithier issues in the same arena. We can ask students to investigate whether the notion of centroid is generalizable to other polygons. Does the notion of *centroid of a quadrilateral* have any meaning? If so, what? What about *centroid of a pentagon*? Do any numerical relationships exist? If so, how are they similar to the ones we know about in a triangle? The following *Cabri* sketch (Figure 2.9) illustrates the result for the case of the quadrilateral.

![Diagram of a quadrilateral with medians and centroid labeled.](image)

AE/EF = 3.0  BE/EG = 3.0  CE/EH = 3.0  DE/EI = 3.0

**Figure 2.9.** *Centroid of a quadrilateral* as I have defined it.
The process that was used to create the sketch in Figure 2.9 is as follows. A *macro* (a method used by *Cabri* that enables the user to automate a complicated process) was created that uses three points as initial objects and the centroid of the triangle as the final object. Thus, the user is able to construct the centroid of a triangle once, and then repeat the process on other triangles. For Figure 2.9, a macro called *Centroid of a Triangle* was created and applied to points A, B, and C, to obtain point I, the centroid of triangle ABC. Similarly, the macro was used to find points F, G, and H, the centroids of triangles BCD, CDA, and DAB, respectively. Segments were then drawn to connect the centroid of each triangle to the opposite vertex of the quadrilateral (the one that was not a part of the triangle). For example, the centroid of triangle ABC, point I, is connected to point D to form segment ID. Similarly, segments GB, HC, and FA were drawn. As the reader can see in Figure 2.9, the four segments are concurrent and an interesting numerical relationship exists when we divide the length of the longer section of each segment by the length of the shorter section of the same segment. We can now see that the geometric and numerical generalizations found in a triangle can be extended to quadrilaterals. I will refer to this point as the *centroid of the quadrilateral*. It should be noted that this point is not generally the same as the *center of gravity* for the quadrilateral.

Although many textbooks refer to the *center of gravity* of a quadrilateral as the *centroid*, it is only in a triangle that the *centroid* is the same as the *center of gravity* (as I am using the term *centroid* in the previous discussion). The textbook used by the geometry classes in this study (Larson, Boswell, & Stiff, 1995), defines *centroid of a quadrilateral* (p. 311) to be the *center of gravity* of the quadrilateral. This construction is shown in Figure 2.10.
The centroid of a quadrilateral (as defined by Larson, Boswell, & Stiff) is constructed by locating trisection points of the sides of the quadrilateral. Points Q, R, S, T, U, V, W, and X are trisection points on the sides of quadrilateral ABCD. Segments RS, TU, VW, and XQ are extended, resulting in a new quadrilateral, JKLM, being formed. Quadrilateral JKLM is a parallelogram, and the intersection of its diagonals, MK and LJ, is the center of gravity for quadrilateral ABCD. If this construction is performed...
on a piece of cardboard, quadrilateral ABCD would balance if a pencil was placed beneath point O. In Figure 2.8, the triangle would balance if the pencil was placed beneath point D in a cardboard experiment.

To better illustrate the difference between the *center of gravity* (as defined by Larson, Boswell, & Stiff) and the *centroid* (as I have defined it), Figure 2.11 shows both points on the same quadrilateral. Point O is the *center of gravity* (as shown in Figure 2.10) and Point P is the *centroid* (constructed by the process used in Figure 2.9). It is left to the reader to conjecture for which types of quadrilaterals Point O and Point P do, in fact, coincide.

![Diagram showing center of gravity and centroid](image)

Point O is the *center of gravity* as defined by Larson, Boswell, & Stiff

Point P is the *centroid of the quadrilateral* as I have defined it

**Figure 2.11.** *Center of gravity and centroid of the quadrilateral* on the same sketch.
Figure 2.9 and the subsequent discussion shows how the notion of a centroid can be extended, or generalized, from a triangle to a quadrilateral, thus supporting Dienes’ definition of abstraction as class formation and generalization as class extension. It does, in fact, turn out that this idea can be explored even further, and with very promising results. Cabri can be used at this point to create a macro called Centroid of a Quadrilateral. The next step in the exploration would be to draw a pentagon and use the macro Centroid of a Quadrilateral five times on the each of the five quadrilaterals that can be formed by “leaving out” one of the vertices of the pentagon. Each of the four centroids of the quadrilaterals are then connected with the vertex that was not used to create it. It turns out that these five segments are, indeed, concurrent. Cabri can then be used to measure the longer and shorter sections of each segment and to do the appropriate calculations. As the reader may suppose at this time, a numerical relationship does exist. The longer section is four times as long as the shorter section of each segment. This is illustrated in Figure 2.12.
Figure 2.12. **Centroid of a pentagon.**

This pattern continues as additional macros are created and other polygons are investigated. While the early cases are accessible by paper and pencil techniques, it seems clear that dynamic geometry software gives us power to explore advanced ideas very quickly and easily. This example illustrates how *Cabri* can be used to investigate conjectures rapidly and with the type of power that could have only been a dream before the advent of dynamic
geometry software. Someone with a working knowledge of Cabri could investigate the previous scenario in a matter of minutes. How many ideas of class extension are waiting to be discovered if our students use software such as Cabri on a regular basis?

**GENERALIZING WITH DYNAMIC GEOMETRY SOFTWARE**

Schwartz (1993), one of the developers of The Geometric Supposer (a geometric construction program described later in this chapter), discusses creativity and generalization as they pertain to school mathematics. He claims that in other subject areas, English language arts for example, students are asked to create as well as study the masters such as Shakespeare. In mathematics, however, students are rarely asked to create. Students spend time in mathematics classes learning previously proven results. Schwartz (1993) states that “students must ... be challenged to invent and create” (p. 7) and believes that geometry software programs such as The Geometric Supposer offer that potential. Schwartz (1993) asserts that “the essence of mathematical creativity lies in the making and exploring of mathematical conjectures” (p. 8). Without adequate tools, however, making and exploring conjectures is more difficult, according to Schwartz.

On the topic of generalization, Schwartz (1993) states that “the mental acts of thinking inductively and generalizing are at the heart of what mathematics students ought to learn to do” (p. 9). Schwartz does not claim that using The Geometric Supposer necessarily leads students to be better at making generalizations, but at least it provides “the setting and the occasion” (p. 10).

Schwartz (1993) discusses differences in algebra and geometry with respect to representation of key ideas in the subject. In algebra, we can represent all possible linear relationships by the equation $Ax + By = C$. In geometry, Schwartz (1993) claims it is easy to construct (either by hand, or with software) a regular polygon such as a triangle or even a 17-gon. How do we construct, however, a regular n-gon? Schwartz (1993) states, “one
can construct *any* particular triangle, but one cannot construct a triangle that is *any* triangle" (p. 10). With dynamic geometry software, however, we may be as close as we can get to *variables* for geometric figures. In geometry, we do not have anything that works the same way that letters (and other types of variables) do in algebra. Letters are used to represent numbers that can take on different values. How can we represent a triangle in geometry that can take on different shapes and sizes? With dynamic geometry software, it is possible to create such triangles. Using *Cabri*, a student can draw a triangle and then manipulate it in many ways to visually see "all possible triangles." Figure 2.13 shows triangle ABC in four different configurations.

Figure 2.13. Four versions of triangle ABC.
Yerushalmy and Chazan (1993) discuss the issue of diagrams as an obstacle to student learning in geometry and how *The Geometric Supposer* can be an aid to overcome this obstacle. Three themes serve as the focus for examining these obstacles: 1) diagrams are particular, 2) common usage confuses certain standard diagrams with the classes of objects to which they belong, and 3) a single diagram is often viewed in different ways (Yerushalmy & Chazan, 1993, p. 25). Yerushalmy and Chazan (1993) conclude that *Supposer* students “are able to treat a single diagram as a model for a whole class of diagrams and simultaneously are aware that such models have characteristics not shared by all members of the class” (p. 53).

Yerushalmy (1993) discusses the confusion among the terms *generalization*, *induction*, and *conjecturing*. Yerushalmy (1993) states that in the process of induction, a generalization is a product created by examining a set of examples. Generalization is also a process acted out on either a conjecture or a proven statement, whereby a specific statement becomes a more general statement. Yerushalmy (1993) states that for generalization to occur, three major processes must be involved: the generation of a set of examples, manipulation of the samples, and analysis of ideas to form more general ideas. She concludes by stating that she is “convinced that it is essential to create opportunities to generalize in any domain of mathematics using various software tools” (pp. 82-83).

Dynamic geometry software helps students to generalize because it embodies each of the principles in Dienes’ mathematics learning theory. The next section presents a model that shows how Dienes’ theory is applied to generalization through the use of dynamic geometry software.
A MODEL

In the present study, Dienes’ Mathematics Learning Theory as applied to the process of generalizing was explored through the lens of dynamic geometry software. The following model relates the key aspects of Dienes’ ideas to the study.

<table>
<thead>
<tr>
<th>Dynamic Principle</th>
<th>Constructivity Principle</th>
<th>Mathematical Variability Principle</th>
<th>Perceptual Variability Principle</th>
</tr>
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<tbody>
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![Diagram](image)

**DYNAMIC GEOMETRY SOFTWARE**

*(CABRI GEOMETRY II)*

**GENERALIZATION ABILITY OF STUDENTS**

Figure 2.14. Dienes’ Mathematics Learning Theory explored through the lens of dynamic geometry software.

This study investigated whether the ability of students to make generalizations in geometry is enhanced by regular experience making and testing conjectures with *Cabri*. Students who used *Cabri* on a regular basis were compared to students who studied the same content but did not use the software. *Cabri* not only satisfies Dienes’ four learning principles, but it also helps to create a new and exciting atmosphere in the classroom, one where the instructor can pose more open-ended questions to the students. Students can make conjectures and those conjectures can be quickly verified. *Cabri* adds a new
dimension to the classroom. It should be noted, however, that Cabri does not reduce the need for formal verification. Conjectures are verified with Cabri, and formal proof follows at a later time. Dienes (1960, p. 28) states that "the problem is to devise standard mathematical situations in which ... adventurous kind of thinking can ... take place."

When dynamic geometry software such as Cabri is used, classrooms become a more likely environment for adventurous thinking to take place.

REVIEW OF RELATED LITERATURE

In this section, literature related to the study is presented. There are a great number of articles and research studies that have been published which impact the current study in one way or another. Articles, research studies, and other relevant literature pertaining to the topic of generalization in mathematics are presented, followed by a discussion of articles and research studies pertaining to the role of geometry software in learning.

Generalization. Davydov (1990/1972) discusses generalization as it pertains to various subjects in the school curriculum. He notes the confusion between the process of generalization and the result of this process (p. 10). A search for an invariant in a collection of objects occurs during the generalization process. Davydov discusses how geometry teachers provide numerous examples when trying to get students to generalize results in geometry. Davydov states (p. 13) that generalization is inseparably linked to the operation of abstracting. He discusses an example (p. 13) where students arrive at the generalization that the number three is one more than the number two. At first the student realizes that for any three objects, there is one more than if there were two objects, but subsequently, this feature is separated from the other properties such as size. At this point, the relationship is "regarded as a relationship of abstract numbers, as a particular object of the attention, abstracted from concrete objects" (p. 13).
Mitchelmore (1999) discusses Davydov's theory of generalization and examines three methods of teaching generalizations in light of Davydov's theory. The first method is referred to as the Abstract Before Concrete (ABC) method. Widely used in mathematics teaching, in this method, knowledge is acquired in "context-free" circumstances. This knowledge is then supposed to be available for general applications at later times. Mitchelmore stated that knowledge acquired by the ABC method is often superficial knowledge that cannot be applied in problem situations and is quickly forgotten once examinations are over. The second type of methodology is empirical, where learning proceeds from the concrete to the abstract. An example is given in which students draw squares of dots, representing the square numbers, and use inductive reasoning to find the next number in the sequence. The third method is a problem-solving approach. In this approach, the teacher poses a problem to the students, allows the students to solve the problem, after which the students must convince others that the solution is correct. An example of the problem-solving approach is the following question: Can a scalene triangle tessellate the plane? Mitchelmore's opinion is that students must deal with a great variety of geometry topics while examining this question, and by working with all of these concepts in a problem situation, a deeper level of understanding will result than if the students had been taught each topic separately.

Students' representations of geometric ideas are key to the present study, as representations impact students' abilities to generalize. Vinner and Hershkowitz (1980) report that people use concept images, which are combinations of mental pictures and properties associated with the concept, rather than definitions. Students develop certain images of geometric objects based on what they have seen in textbooks and how their instructor has represented an object on the chalkboard or on paper. Cabri offers the potential to give students a greater variety of representations of geometric objects. A closely related topic is the use of diagrams. Kabanova-Meller (1970) discusses the need
for students to "transform" a diagram when learning geometric theorems. An example of this is when students are learning the theorem about an exterior angle of a triangle, often the diagram might only be drawn with the exterior angle as an obtuse angle. In this case, students may not recognize that the theorem is valid when the exterior angle is acute. The nature of a diagram may influence students' perceptions of when a theorem is applicable. *Cabri* can be used to address this situation. Figure 2.15 shows two different views of an exterior angle of a triangle.

![Figure 2.15. Exterior angles of a triangle.](image)

Several authors discuss the importance of generalization in school mathematics. Cooney and Davis (1976) discuss teaching concepts and generalizations in mathematics and science. Cooney and Davis distinguish between analytical generalizations (verified by logical considerations) and empirical (observed) generalizations. Turnau (1976) stresses the key role of generalization in mathematics: "the process of generalization is one of the most important types of thinking to be developed in mathematical education" (p. 153).
Generalizations that originate from number patterns are the focus of the writing of several authors (Aviv, 1979; Hartman, 1976; Oullette, 1975, 1978; Ranucci, 1974; Smith, 1972; Szetela, 1999). The triangular numbers and other polygonal numbers are the subject of much of this work. Smith (1972) presents a broader generalization on the \( n \)th polygonal number. Ranucci (1974) discusses the need for students to know how to analyze data so that they may work toward generalizations. He describes how high school students used successive differences to arrive at generalizations.

Many research studies that focus on students' abilities to generalize have used patterns and functions as the arena in which to conduct the study. Stacey (1989) studied how students find and use patterns in linear generalizing problems. Stacey investigated several issues in this study: 1) what type of generalizations students make and how they vary with increased schooling, 2) how students explain the patterns they find and the generalizations they make, 3) how consistent students are in the mathematical models used when making generalizations, and 4) how the responses of students who have had some experience generalizing differ from the responses of students who have had little to no experience generalizing. Stacey's study is an example of a study which looks into a deeper issue; it is an analysis of strategies used in generalizing. The subjects of the study were 140 seventh- and eighth-grade students. Stacey reported that a majority of students could recognize the constant difference in the problems and could successfully find the next term in a linear sequence. Students had more difficulty when asked to find terms such as the 100th or 1000th terms. An error that was reported for many students was they moved to a direct proportion when trying to figure out, for example, the 100th term. These students correctly figured the 50th term, but then doubled the result to obtain the 100th term. Stacey also reported that students seem to grasp at relationships without subjecting them to critical thinking.
Orton and Orton (1994) report on four generalization studies. Three of the studies involved 9 to 13-year-old children, while one study involved adults. In all of the studies, the subjects were given the beginning of a pattern (in pictorial form, dots or match sticks) and asked to find subsequent terms such as the 5th term, 10th term, 50th term, or the \( n \)th term. The authors present several obstacles to generalizing, including a fixation with a recursive approach, which can obstruct progress toward finding the universal rule. In the study with adults, the researchers found that success on the 5th, 10th, and 50th terms did not guarantee success on the \( n \)th term. Many of the subjects used differencing even when trying to find the 50th term. The subjects who had the most success on the 50th term had already found the universal rule. In one of the studies with children, a majority were able to explain sequences in terms of differences between successive terms. These students were not as successful finding an algebraic expression for the \( n \)th term, although about half made some attempt to do so. In another study with children, the researchers observed that many students doubled the answer for the 5th term to get their answer for the 10th term, the same type of error noted above by Stacey. In the final study with children, Orton and Orton noted that 70% of the children could find the 20th term of a sequence correctly, but less than half could find the 100th term. The researchers found little evidence that correctly finding the 20th and 100th terms of a sequence helped in the process of generalization.

The generalization ability of secondary students in the context of number patterns was the focus of a study by García-Cruz and Martinón (1998). The subjects for this study were 15- and 16-year-old students. The researchers conducted a teaching experiment with 18 students. The role of the researchers was to facilitate and encourage students' participation in small-group and whole-class discussion. Students were given the first three terms of a number pattern and asked to find the 4th, 5th, 10th, 20th, and \( n \)th terms. The students were asked to explain their solutions and then other students were asked to judge the solutions, with the researchers being careful not to reveal if the solution was
correct or not. The researchers divided the subjects into three levels of generalization ability based on their performance on generalization tasks. These levels were: procedural activity (students merely extended the pattern), procedural understanding/local generalization (students found the \( n \)th term), and conceptual understanding/global generalization (students generalized a strategy).

Many of the studies discussed in this section report on the challenging nature of "finding the \( n \)th term." Students can often extend a pattern but have difficulty when it comes to making a generalization. Taplin and Robertson (1995) discuss the difficulties students have in being able to express generalizations in algebraic terms. The researchers studied the generalization abilities of 12- and 13-year-old students in the setting of number patterns. Students were shown patterns and asked to identify the 5th, 10th, and 100th terms in the sequence. Students were given a variety of materials (matches, blocks, graph paper, and plain paper) and told that they could represent any steps in the pattern in any manner they chose. The students were then asked to describe the pattern and were asked for a generalization for "any term." None of the students in the study could generalize to a rule for "any term."

Some researchers have focused on students' thinking during the generalization process. Linchevski, Olivier, Sasman, and Liebenberg (1998) focused on the moments when students grapple with deciding about the validity of their generalizations. The researchers interviewed 10 seventh-grade students and tried to create cognitive conflict by challenging students' justifications for the methods they used and then documented the students' attempts to resolve the conflicts. The topics for the generalizations were number and picture patterns. The researchers report the same type of error that Stacey (1989) and Orton and Orton (1994) found with their subjects. The error is that subjects correctly found, for example, the 50th term through recursion, but then doubled the result to find the 100th term. Even though the researchers were aware that students make this type of error,
they were surprised at how often it was made and the obstinacy of students to change. Students were shown that this type of strategy produced errors, but this did not seem to prevent them from using the strategy repeatedly.

Generalization problems can be found in many surprising locations, such as a calendar. Iwasaki and Yamaguchi (1997) report on an experiment based on the generalization theory of Dörfler (1991). Eighth-grade students were asked to make generalizations on a calendar by looking at five numbers in a particular formation. Students were also asked to move the formation around and to consider changes in the relationships that they found. Students found such relationships as the horizontal sum is the same as the vertical sum in the formation. The focus of this experiment was to examine generalizing in an algebraic context. A second experiment involved generalizations in a pentagram, thus allowing the researchers to examine generalizing in a geometric context. The activities form a part of what Dörfler calls extensional generalization.

Clinical interviews are often used by researchers to investigate students' abilities to generalize. Gnepp and Maher (1988) report on a study in which 41 seventh-grade students participated in interviews designed to measure their ability to generalize. The students participated in a teaching lesson in the first part of the interview which focused on the acquisition of ability to write mathematical expressions using variables. In the second part of the interview, the students worked on mathematics problems designed to measure their ability to generalize. A sample question asked students to find the total cost of $d$ notebooks and $e$ pencils if one notebook costs $m$ cents and one pencil costs $t$ cents. If students answered this question correctly, they went on to the next question. If not, they were asked the same question with all the variable amounts replaced with numerical values. The student was then asked the original question again. If unsuccessful again, the question with the numerical values was repeated, only with one of the numbers changed to a variable. This process continued, going between the general question and varying levels of
generality in the other question. The student could have been asked up to nine questions for this particular task. After giving an answer, the student was asked to explain how they arrived at their answer. The research study was designed to examine the relationship between Piagetan developmental level and the ability to generalize in mathematics. The subjects were classified according to Piaget’s theory at the beginning of the study. The successful subjects were placed into three groups: spontaneously successful (6 students), successful after at most one intervening variant correctly solved (28 students), and eventually successful (33 students). The reader should note that all of the subjects in the first group were in the second group and all of the subjects in the second group were in the third group. Students’ developmental level and their performance on relative portions of the teaching component were helpful in providing explanations of their success. The researchers concluded that the developmental level of the student should not be used to preclude instruction on generalization in mathematics.

Practicing teachers and pre-service teachers have been the focus of some studies that examine generalization ability. Hitt (1994) studied the generalization abilities of teachers in the context of a computer environment. The study focused on whether the use of computer environments would aid in forming generalizations regarding polygonal numbers. Hitt concluded that spreadsheets do not seem to be an aid in forming generalizations, but that visualization was an aid. Teachers used visualization to write a formula for the nth pentagonal number by relating them to the triangular and squared numbers.

Dossey, Swafford, Langrall, and Kersaint (1997) report on the generalization of patterns and relationships by prospective elementary teachers. The study used the content of arithmetic and geometric sequences and direct and inverse variation. Free-response written items and interviews were used to collect data for the study. Results of the study indicate that the majority of the subjects could make generalizations when problems
involved arithmetic sequences and direct variation. Overall, the group experienced more difficulty with geometric sequences and inverse variation.

**Geometry software.** Generalizations with origins in geometry are plentiful, and are the focus of the writing of several authors. Some topics that provide opportunities for students to generalize in geometric settings are the harmonic triangle (Stones, 1983), area ratios within a triangle (Olson & White, 1989), the silver ratio (Coleman, 1989), area formulas for quadrilaterals (Usnick, Lamphere, & Bright, 1992), Poinset Stars (Hirsch, 1980), Pascal’s Triangle (Truran, 1972), and an interesting problem regarding sums of chord lengths in a unit circle (Oullette & Bennett, 1979). Other authors have also written about generalization in the context of well-known problems in mathematics (Bechem, 1979; Gannon & Martelli, 1993; Higginson, 1981).

The problem proposed by Oullette and Bennett (1979) is quite accessible with dynamic geometry software. The problem begins with two equally spaced points on a unit circle (a circle of radius 1 unit). If a point P is placed anywhere on the circle, a right triangle is formed by point P and the endpoints of the diameter (see Figure 2.16). A relationship exists among the side lengths, which the reader will recognize as the Pythagorean Theorem. Oullette and Bennett wondered if there existed an extension of this result for three equally spaced points on a unit circle. As it turns out, there was an interesting relationship. The researchers did not stop here, and found relationships if there were four, five, and six equally spaced points. The relationship that exists is that the sum of the squares of the lengths from a point P to each of the equally spaced points is a constant which happens to be twice the number of equally spaced points. This situation is illustrated in Figure 2.16. This figure is titled Oullette and Bennett’s discovery as they reported it to be original research for them (even though the result may have already been known to other mathematicians).
(PA)² + (PB)² = 4.00 cm²  \hspace{1cm} (PA)² + (PB)² + (PC)² = 6.00 cm²

(PA)² + (PB)² + (PC)² + (PD)² = 8.00 cm²

Figure 2.16. Oullette and Bennett's discovery  
(*based on Oullette & Bennett, 1979*).

Articles about dynamic geometry software (DGS) in the mathematics education literature are written for various purposes and to various audiences. A few have been mentioned earlier, in the theoretical framework section. Authors have described the capabilities of DGS (Green, 1992; Harris, 2000); classroom activities using DGS (Klein & Hamilton, 1997; Shilgalis, 1998); conjecturing with DGS (Giamati, 1995; Tikoo, 1998; Zbiek, 1996); generalizing (Peterson, 1997); multiple representations with DGS as one representation (Perham, Perham, & Perham, 1997; Vonder Embse & Yoder, 1998);

Clements and Battista (1994) review research on computer environments for geometry learning and describe the unique benefits of constructive computer programs, implications for the design of computer environments, and other educational implications of computer environments. Clements and Battista (1992) highlight the promising role of computers in geometry learning by stating that "computers’ graphic capabilities may also facilitate the construction of geometric representations" (p. 449).

An important issue with regard to generalizations in geometry is the idea of variance versus invariance. Kaput (1992) states that in order for students to recognize invariance, they must be able to see variance. Dynamic geometry software allows students to see variance, as discussed earlier in the chapter. Kaput discusses the differences between static and dynamic computer environments and points out that "dynamic media inherently make variation easier to achieve" (p. 525). Kaput suggests that software programs such as Cabri may soon act as an alternative to the traditional approach to Euclidean geometry (p. 538).
One of the earliest software programs used to explore geometric ideas is *Logo*. In this software program, a turtle moves around the computer screen through a sequence of commands given by the user. Clements and Battista (1989) report that experience with *Logo* encourages students to view objects in geometry in terms of the actions used to create them. Some dissertation studies have used *Logo* as a medium. Scally (1990) studied the impact of experience in a *Logo* learning environment on students' understanding of angle. Students participated in a one-semester *Logo* course to provide experiences that bridge the gap between the middle and high school geometry curriculum. Gains were made between visual and descriptive level thinking on some tasks, generally more by the *Logo* students than the non-*Logo* students. Yusuf (1990) investigated whether middle school students using *Logo* would have a more positive attitude toward learning geometry and a better conceptualization of points, rays, lines, and line segments than students taught the same concepts through lecture and paper and pencil activities. Results indicated that there were significant differences in favor of the *Logo* group. Elmore (1992) created and field tested *Logo* curricular materials to teach certain geometric concepts. Results showed that using *Logo* had little effect on the students' geometric achievement. The students did have a positive attitude toward using *Logo*.

Another type of computer program used in the study of geometry is *The Geometric Supposer* (Schwartz & Yerushalmy, 1985-88). The focus of this type of software is “to facilitate students making and testing conjectures” (Clements & Battista, 1992, p. 452). *The Geometric Supposer: What Is It a Case of?* (Schwartz, Yerushalmy, & Wilson, Eds., 1993) discusses problems of learning, teaching, and implementation associated with the *Supposer*. When using the *Supposer*, students use a menu to select certain geometric figures and then perform constructions on those figures. For example, to investigate the idea of centroid of a triangle with the *Supposer*, a student would select an acute triangle,
perhaps, as a beginning. Using the median command, the student could construct all three medians and observe that they are concurrent. *The Geometric Supposer* remembers constructions as procedures and these procedures can then be repeated on different objects. The *Supposer* can replay the construction on other acute triangles, as well as right triangles and obtuse triangles. In this example, a student could follow the acute triangle exploration with explorations of right and obtuse triangles, and see that the medians of a triangle are concurrent for any type of triangle. Thus, students can make conjectures and arrive at generalizations very quickly using *The Geometric Supposer*. Chazan and Houde (1989) offer advice to teachers in a booklet published by NCTM entitled *How to Use Conjecturing and Microcomputers to Teach Geometry*. Much research has been conducted on student learning using *The Geometric Supposer*.

Yerushalmy (1986) studied student learning with regards to induction and generalization when students use *The Geometric Supposer* software. This was a year-long study of students who learned geometry inductively, using the *Supposer*. Yerushalmy compared *Supposer* students to a comparison group that learned traditional deductive geometry with regards to students’ tendencies to generalize their ideas and their need for convincing arguments in solving geometric problems. The study investigated three aspects of generalization: generalization of empirical information (inducing patterns from numerical data, shifting from drawings to diagrams, and multiple representations), generalization of reasoning, and generalization of reasoning arguments. Data collected were scores on a generalization pre- and post-test, a convincing argument test, analyses of projects in inductive geometry, and observations from the inductive class and from the *Supposer* lab. The study involved four geometry classes with a total of 85 students. Two of the classes were in an urban setting, while the other two were in a suburban setting. The students were deemed to be average or above average by their teachers. The experimental
(Supposer) group did not use a textbook for most of the course. Students concentrated on arriving at their own definitions and conjectures, which became the foundation for the course. The experimental group spent the majority of their time collecting data and generalizing, therefore spending less time on formal proof. After the experimental group learned how to generalize their conjectures, they learned how to prove some of them. The generalization pretest showed no significant difference between the groups. Results of the study indicate: 1) generally, students were not familiar with inductive work, 2) students’ appreciation of data as a source of ideas grew, while their appreciation of data as an argument declined, 3) students shifted the focus of their work from dependence on data to a dependence on qualitative and quantitative manipulation skills, 4) the traditional geometry course reduced the motivation of students to think about richer ideas or to change what were assumed to be key features, while inductive learning helped students consider non-conventional methods of analysis, 5) while different students exhibited various levels of skill and understanding, the highest level of mathematical reasoning, observed toward the end of the study, involved a combination of deduction and induction, and 6) students had difficulty in transferring knowledge between domains of the geometry course, although some topics were more easily transferred than others (pp. vii-viii). The generalization level of the control group decreased significantly while the generalization level of experimental group was either stable or higher when compared with the pretest. The results of the study suggest that the experience with inductive learning “provided students with a rich collection of techniques that increased all types of generalization” (p. 101).

Chazan (1988) analyzed student understanding of similarity before, during, and after instruction using the Supposer. Supposer activities helped identify and deal with student misconceptions. Yerushalmy, Chazan, and Gordon (1987) conducted research on students’ abilities to make conjectures and generalizations while using Supposer software. Supposer students significantly outperformed a comparison group in their ability to develop
generalizations and were equal to and/or somewhat better than the comparison group in their ability to devise informal arguments and traditional formal proofs.

While geometry software can have a major impact on student learning, how teachers use the software determines the level of the impact. Lampert (1988) studied the use of the Supposer as it related to teachers and students. Lampert writes: "the power of The Geometric Supposer lies not in what it can do, but in what it enables teachers to do if they are both able and disposed to use it in the way it was intended" (p.2). Lampert discusses teachers' thinking about geometry content, as well as the role of proof, the role of induction, and the role of technology in geometry learning. Before using the Supposer, the teachers in the study had a very traditional view of geometry teaching and learning: teaching definitions and axioms that led to proving theorems which in turn led to proving more theorems. The teachers knew that this approach did not work for many of their students: only the teachers and the textbooks were doing the thinking. The teachers had not used data collection and conjecturing as a routine part of geometry teaching. The teachers saw the Supposer as providing students with opportunities to do their own thinking, rather than just responding to teachers' questions. Additional Supposer studies (Shepard & Wiske, 1989; Wiske & Houde, 1988) have focused on issues related to teachers, teaching, and implementation of The Geometric Supposer in schools.

Smyser (1994) investigated the effects of using the Supposer on students' spatial visualization ability, van Hiele level of reasoning, and achievement when the software was used with the textbook in geometry classes. The treatment group used Supposer-based activities in conjunction with regular class lessons over a period of approximately five months. The treatment group scored as well as or better, but not significantly better, than the control group on the spatial visualization ability test, the van Hiele test, and the end-of-year achievement test of content knowledge. The progress through the textbook was approximately the same for both groups. A recommendation of the author was that in
future research, subjects should receive more intense exposure to the software during the
treatment period. The present study capitalized on that recommendation, utilizing dynamic
gometry software whenever possible for exploration and discovery purposes.

*Geometry Inventor* (Brock, Cappo, Arvel, & Dromi, 1992), which is comparable
to *The Geometric Supposer*, was the medium for a year-long study (Roberts & Stephens,
1999) of geometry achievement of high school students who used the software varying
amounts of time per week. Three classes were studied: one class used the software twice
per week, a second class used the software once per week, while a third class did not use
the software at all. Mean scores on chapter tests as well as a semester examination and a
final examination were studied to see if the software groups performed significantly better
than the non-software group. Few significant differences were reported. The results
indicate that using the computer software may not be beneficial when teaching certain topics
in geometry. The researchers observed that using the software improved student interest
and participation in geometry.

Several studies have utilized a different dynamic geometry program, *The
Geometer's Sketchpad* (Jackiw, 1991). Galindo, Birgisson, Cenet, Krumpe, and Lutz
(1997) used task-based interviews to study the development of students’ notions of proof
in a dynamic learning environment. The goal of the study was to elicit strategies and
methods used by the students to establish the validity of their propositions. Students were
using *Sketchpad* to create constructions that represented infinitely many examples. This
was seen as progress in their notions of proof. The researchers felt there was a negative
issue related to the use of dynamic geometry software: some students were reluctant to
prove statements for which they had created dynamic constructions. The authors
concluded that appropriate use of dynamic geometry software does help move students
toward meaningful justification of their ideas.
Two versions of The Geometer's Sketchpad were used by Elchuck (1992) to explore students' abilities to make conjectures. Geometry classes that participated in the study were divided into two groups: the dynamic group had access to the full capability of The Geometer's Sketchpad, while the static group had access to all but the drag capabilities of the software. While students in the dynamic group could drag figures to consider additional examples, students in the static group had to do the construction over again on new figures if they wanted to see different examples. A total of 157 ninth-grade students participated in the study. The groups participated in a twenty-class instructional unit using the software. Additionally, students were allowed to spend extra time exploring with the software on a sign-in, sign-out basis. Elchuck felt that there were other factors (besides type of software) that might influence students’ abilities to make conjectures. These factors were: prior mathematical knowledge, locus of control (a measure of perceptions of events being a consequence of a person’s own actions), time spent in independent investigation, spatial reasoning ability, and van Hiele. Elchuck’s conjecture-making ability instrument was written by him, based on criteria described by Yerushalmy, Chazan, and Gordon (1988). The hypothesis that students who use the dynamic version of the software would generate more high-level conjectures was not supported. Neither were the hypotheses regarding spatial visualization, locus of control, or van Hiele level of reasoning. Results of the study did indicate that students with high achievement outperformed students with lower achievement on the conjecture-making test. Additionally, students who spent more time in independent investigation outperformed students with less time in independent investigation on the conjecture-making test.

The nature of students’ inquiry as they use The Geometer's Sketchpad was described by Foletta (1994). The researcher observed how students use the software and characterized their small group interactions. An accelerated ninth-grade student, two
average to above-average tenth-grade students, and a low-achieving tenth-grade student were the subjects of the study. The class in which these students were enrolled used *The Geometer's Sketchpad* on a regular basis for exploration and used a textbook which emphasized an inductive approach (Serra, 1989/1993). Primary data sources were classroom observations, interviews with the subjects, and artifacts from the use of the software (printouts of students' constructions and their corresponding scripts). A script is a feature of *Sketchpad* that records the construction as it was completed by the student and saves the information as a step by step process. The steps can be replayed to see exactly how the student went about the construction. The purpose of the scripts were to document the students' thinking process as they did constructions with the software. The researcher felt, however, that the scripts proved less useful than expected. Secondary data sources were teacher interviews, an interview with the school principal, and a student mathematics belief survey. The study generated four grounded hypotheses: 1) the purposes that characterize students' use of the software are drawing, measuring, or exploring, and students' actions are driven by the goals of the teacher, 2) students tend to use the software as an extension of paper and pencil, rather than as a resource, 3) using the software during investigations enables more involvement and shared responsibility for discourse by the lower achieving student than do activities done without the software, and 4) students tend to conjecture early in investigations, then seek visual and numerical data to confirm their conjecture. Students also are selective about what data they perceive as important as they search for confirming data.

Frerking (1994) investigated the effects of students' use of conjecturing on van Hiele levels and abilities to justify statements or write proofs. The study compared three classes, all taught by the researcher. Varying levels and type of software use was what distinguished the classes from one another. The first class received instruction by
traditional deductive methods. This class used The Geometer's Sketchpad fourteen times throughout the course to explore theorems already discussed or proven in class. The second class developed conjectures with manipulatives, compass and straightedge constructions, and geometry software (The Geometer's Sketchpad and The Geometric Supposer). Students in this group used geometry software a total of twenty times (fourteen Sketchpad sessions and six Supposer sessions). Students were asked to justify or prove some of their conjectures. The third class used The Geometer's Sketchpad twenty-two times to explore ideas and develop conjectures. When in the computer lab, the students used both commercially-developed and researcher-developed activities to make conjectures.

Pretest measures consisted of an Entering Geometry Test (also used by this researcher in the current study) and a van Hiele Geometry Test. Throughout the study, conjectures of the second and third classes were evaluated by the Conjecturing Evaluation Instrument (CEI) and the Justification Evaluation Instrument (JEI). These instruments were modified versions of instruments used by Yerushalmy (1986). Posttest data consisted of the Entering Geometry Test, the van Hiele test, the CDASSG Proof test, and a county geometry achievement test based on material covered in the first semester. Results of the study indicate that the median gain on van Hiele scores was greater for the second and third classes but the gain was not statistically significant at the $p < .05$ level. There were no significant differences among the groups on the net gain on scores on the Entering Geometry Test. There were also no significant differences on achievement in proof writing or on the county achievement test. There was a significant relationship between scores on both the van Hiele pretest and posttest and the county achievement test. There was a significant relationship between scores on the van Hiele posttest and scores on the proof test. There were no significant relationships between scores on the van Hiele pretest and the scores on the CEI or JEI, but there were significant relationships between scores on the van Hiele posttest and scores on the CEI and JEI. Student scores on the CEI did not show
improvement, but some differences were attributed to the design of the activity worksheets. There were no specific improvements in scores on the JEI. Any differences were attributed to the way questions were stated in asking for justifications.

The problem-solving strategies that high-school geometry students use after instruction with *The Geometer's Sketchpad* was investigated by Robinson (1994). Robinson examined students' solution strategies for tendencies to use drawings and dynamic visualization as related to the availability of the software during a locus of points problem-solving session. Data collected were three different measures of spatial visualization: the ETS Card Rotations test, a Cube Comparisons test, and Paper Folding tests. A locus-motion problem inventory was administered. Students were randomly assigned to one of two treatment groups for the locus-motion inventory: students who had the software available and students who did not have the software available. Additionally, the researcher interviewed two students from each class, one from each treatment group. Results of the study indicate that the availability of the computer was not a significant factor for performance on the locus-motion inventory. Spatial ability or mathematics achievement accounted for most of the variance on the locus-motion inventory. Results of the study suggest that strategies learned with the technology are transferable to paper-and-pencil situations, and that active participation in instructional activities is important to successful performance and use of strategies.

Dixon (1995) examined the effects of instruction with *The Geometer's Sketchpad*, students' English proficiency and visualization level on middle school students' construction of the concepts of reflection and rotation. Dixon also investigated the effects of the dynamic geometry environment on students' two- and three-dimensional visualization skills. Two groups were compared, one that exclusively used *The Geometer's Sketchpad*, and one that was taught using traditional deductive methods. Four eighth-grade English language mathematics classes served as the experimental group and
five similar classes served as the control group. The researcher taught all of the treatment classes while the control group classes were taught by other mathematics teachers at the school. The Sketchpad group used lessons designed to allow students to discover properties of reflections and rotations. Results of the study indicate that the students in the experimental group significantly outperformed the control group on content measures of the concepts of reflection and rotation as well as on measures of two-dimensional visualization. No significant differences existed between the groups on measures of three-dimensional visualization. There were no significant differences on any of the dependent variables between limited English proficient students and students who are proficient in English.

The effects of using The Geometer's Sketchpad on high school students' geometric knowledge, constructions, and conjectures were studied by Lester (1996). Forty-seven students were the subjects of the study, with 20 students in the experimental group and 27 in the control group. The experimental group used The Geometer's Sketchpad to explore topics related to circles. The control group used traditional methods and tools to explore the same topics. Both groups were taught by the same teacher. The duration of the study was 20 class days. Data sources were class observations, student work (experimental group students saved their sketches on disks), teacher and researcher meetings, student interviews, and a posttest. The posttest was given to the students following instruction on the unit on circles and was scored on the three dependent variables: geometric knowledge, geometric constructions, and geometric conjectures. Results of the study indicate that differences between the groups were not statistically significant at the $p < .05$ level on the variables geometric knowledge and geometric constructions. A significant difference did exist on the variable geometric conjectures, with the experimental group outperforming the control group.
The effects of problem-solving activities using *The Geometer's Sketchpad* on high school geometry students' readiness for self-directed learning was the subject of a study by Melczarek (1996). Attitude toward the learning of mathematics, the use of computers, and their mediating effects were also explored. Six geometry classes served as the experimental group while a seventh class served as the control group. Students in the experimental group visited the computer laboratory one time per week for a total of six weeks. These students used problem-solving activities designed to be used with *The Geometer's Sketchpad* during the laboratory visits. Data collected were the Self-Directed Learning Readiness Scale, the Computer Attitude Scale, and the Fennema-Sherman Mathematics Attitude Scales. These instruments were used to measure change in students' readiness for self-directed learning, attitude toward mathematics, and attitude toward computers. A measure of students' attitude toward *The Geometer's Sketchpad* was also used. Additionally, class grades of students were used as a data source. Results of the study do not indicate that the problem-solving activities using the Sketchpad had a direct effect on readiness for self-directed learning, on attitude toward mathematics, or attitude toward computers. Attitude toward *The Geometer's Sketchpad* and the interaction between attitude toward computers and attitude toward *The Geometer's Sketchpad* were found to be significant covariates in a model of the change in scores on the Self-Directed Learning Readiness Scale. Results of the study indicate a positive relationship between the use of dynamic geometry software and readiness for self-directed learning through the mediating effects of attitude toward *The Geometer's Sketchpad*.

Students' attitudes toward geometry and toward *The Geometer's Sketchpad* were investigated by Yousif (1997). The researcher was also interested in finding out if any differences existed with respect to gender. Four lower-level geometry classes were the subjects of this study, with 36 students making up the experimental group and 45 making
up the control group. Two different teachers participated in the study, each teaching one class in the experimental group and one class in the control group. Topics studied by the students during the time of the investigation were polygons, areas, the Pythagorean Theorem, and ratios and proportions. Results of the study indicate that attitude scores between the groups were significantly different on the pretest and the posttest, and there was a significant difference between the gain in attitude scores between the two groups. Results did not indicate any significant differences with respect to gender. Interview techniques were used to determine students' attitudes toward the software. The interview results showed positive attitudes toward working with the software and a positive change in students' attitudes toward geometry. The investigator observed that students in the experimental group seemed to enjoy the exploration activities more than students in the control group.

Bell (1998) used The Geometer's Sketchpad to investigate the relationship of an inquiry-based dynamic geometry software environment on students' van Hiele levels, achievement, basic geometry knowledge, conjecturing ability, and dispositions toward mathematics and technology. Students in the experimental group (40 students) used The Geometer's Sketchpad to formulate and test conjectures in a laboratory setting for an average of two classes per week. Laboratory activities were mostly written by the researcher, using a guided inquiry approach (students had to complete a construction, take measurements, manipulate the construction, and formulate conjectures). Students in the control groups (55 students) were taught using traditional deductive methods and then used The Geometer's Sketchpad to reinforce the ideas. Control group students were in the computer lab at most once per week. Data collected were scores on a van Hiele level test and an Entering Geometry test (also used by Frerking, as described earlier, and in the current study), scores on a county semester geometry examination, and Likert-type surveys.
to measure dispositions toward mathematics and technology. Results of the study indicate a significant difference between the groups in the increase on students’ van Hiele level and on the gain in scores on the Entering Geometry test. Both differences favored the experimental group. There were no significant differences between the groups on achievement on the county semester geometry examination. There was a significant relationship between pre- and post- van Hiele levels and scores on the county achievement test and semester grades. Laboratory activities that appeared to be the most effective for the generation of students’ conjectures were those that were step-by-step. Activities that had leading questions and provided space for student responses tended to be more successful than open-ended activities. Qualitative analysis did not suggest improvement in students’ abilities to write conjectures. Significant differences were reported in changes in students’ dispositions toward technology and toward mathematics. The investigator called for more research involving treatment groups using an inductive conjecturing approach to determine if time spent improves conjecturing ability.

Several research studies have used Cabri as the medium through which to look at student learning in geometry. Jones (1998) reports on the mediation of learning in a dynamic geometry environment. Jones discusses choices that the software designers had to make while designing Cabri. One of the things he mentions is the default cursor operation, which is to drag, rather than create, a point. Jones comments that this is not a critique of Cabri, but that certain decisions must be made, and they do, in fact, mediate student learning. Jones reports on results from a longitudinal study, and states that certain outcomes are similar to those that have been reported on student learning using Logo. Jones discusses 1) students’ needs to invent terms, 2) students’ perspectives of certain aspects of the dynamic environment, 3) the relationship of earlier successes in constructing figures to new constructions, 4) the procedural effect of dynamic geometry software on
student learning, 5) students' lack of appreciation of the drag mode's capability to focus on invariance, and 6) challenges that teachers face in providing input that serve the students' communicative needs. In another report on research, Jones (1997) commented on the importance of the role of the teacher in helping students to complete certain constructions using Cabri. Jones observed that a pair of twelve-year-old students successfully completed a construction task, but needed input from the researcher.

Arzarello, Micheletti, Olivero, Robutti, Paola, and Gallino (1998) conducted research with 15-year-old students to analyze the type of dragging that students do when using Cabri to investigate certain topics. The authors distinguish between wandering dragging (dragging randomly to discover something interesting) and lieu muet dragging (a locus is built by dragging a draggable point, in a way which preserves some regularity of certain figures). Students constructed a quadrilateral and each of its angle bisectors. If the original quadrilateral is called ABCD, then a new quadrilateral, HKLM, is formed by the intersections of consecutive angle bisectors. This is illustrated in Figure 2.17.
Figure 2.17. Quadrilateral formed by angle bisectors (based on Arzarello, et al., 1998).

The students noted that when the original quadrilateral was a square, HKLM became a single point. This is shown in Figure 2.18.
Figure 2.18. The case when ABCD is a square (based on Arzarello, et al., 1998).

The students saw this as an interesting fact, and began to wonder if they could drag points A, B, C, and D so that H, K, L, and M remained coincident. This is what the authors termed a lieu muet exploration. The students realized that it was possible to have the points remain coincident even when ABCD appeared to be a very ordinary quadrilateral. This is shown in Figure 2.19.
After taking some measurements, students noted that the sum of opposite sides of ABCD were equal, and recalled this to be a property of quadrilaterals that could be circumscribed about a circle. The result is shown in Figure 2.20.
The role of dynamic geometry software in developing students' notions of construction and proof was studied by Gardiner and Hudson (1998). Twelve- and thirteen-year-old students were given the task of constructing a square using Cabri which was stable under the drag capabilities of dynamic software programs. The authors discuss the ideas of constructions versus drawings in a dynamic environment. These ideas were discussed earlier in this chapter. Gardiner and Hudson concluded that the role of the
teacher is important in assisting students in moving from common conceptions of geometric figures toward more scientific conceptions.

Vincent (1998) reports on research with 11- and 12-year-old students with regard to the effect of dynamic geometry software and changes in students' van Hiele levels. Students completed lessons which were structured Cabri worksheets. Vincent reports that all students in the study increased their van Hiele levels for some or all of the concepts studied. Vincent and McCrae (1999) describe two of the participants in the previous study in more depth and recommend that more research is needed to explore the impact of dynamic geometry software on development of higher levels of geometric understanding. Vincent and McCrae discuss the difficulties students have with constructing particular drag-resistant shapes such as rectangles and equilateral triangles as their constructions seem to be based on visual appearance of the shapes.

*Cabri* software is found as an application on the Texas Instruments TI-92 calculator. The current study and a study by Round (1998) both used TI-92 calculators, but in different circumstances. While the entire experimental group in the current study used individual TI-92 calculators on a regular basis, the study by Round used one TI-92 for demonstration purposes after students had some individual experience with the calculators. Round investigated whether students in a class which used one TI-92 for whole-class demonstration could overcome obstacles to the use of diagrams, become proficient in conjecturing and proof, and whether any differences existed by gender. The study also investigated student attitudes with regards to several factors associated with the learning and doing of mathematics. The study was conducted in two high school geometry classes taught by the researcher. Students used individual TI-92 calculators for one week prior to the calculator being used for demonstration purposes only for the rest of the course. Data collected were attitude surveys, homework assignments, quizzes, tests, and worksheets completed by students, and student interviews. Additionally, student groups
made presentations on theorems related to areas of polygons near the end of the course. Round assessed the initial position of the students regarding the use of diagrams and conjecturing by means of a worksheet, homework assignment, and quiz. Results of the study indicate that some students made progress overcoming the obstacles to the use of diagrams while others continued to exhibit difficulty. The student group presentations demonstrated that a number of students were proficient in constructing and manipulating diagrams even though they had very limited personal experience with the TI-92. Males scored significantly higher on the Pythagorean Theorem worksheet, indicating that the treatment may have had a larger positive impact on males than on females with regard to overcoming the obstacles associated with the use of diagrams in geometry. Females scored significantly higher than males on the worksheet on circles given toward the end of the course. For this topic, the calculator may have had a greater positive impact on females than on males. Round discusses how a number of students still thought that a number of true examples constituted a proof even toward the end of the year. Proof writing abilities were low both at the beginning and the end of the study, a result that the researcher indicated may be associated with students' attitudes toward proof problems. Round stated that hands-on experience with the TI-92 seemed to help students to gain a better understanding of conjecturing. Students fared better on conjecturing problems late in the course when they were asked to make conjectures about specific parts of a diagram.

While many articles and studies have been discussed on the topic of generalization, it is also important to think about situations in which students generalize, when in fact, they should not. One author termed this idea "over-generalization" (Feinstein, 1979). Feinstein points out that for students to be able to discriminate effectively when generalizing, several nonexamples should be included in their experiences. Taback (1996) warns against overgeneralizing, claiming that students need ample opportunity to discover that there exist...
situations that seem as if they will adhere to a general rule, but in fact, after a few instances, they deviate from the rule. Taback gives a numerical example and one from geometry. The geometry example involves the maximum number of regions into which a circle is divided by drawing all possible chords with \( n \) points on the circle. After examining some data from the early cases, students are led to believe that the general rule for the maximum number of regions is \( 2^n - 1 \), where \( n \) is the number of points on the circle. This rule works for two, three, four, and five points on the circle. The corresponding number of regions is two, four, eight, and sixteen. The rule, however, does not work for six points, where the corresponding number of regions is 31 instead of the predicted 32.

Zbiek (1996) discusses a problem in which students may be led to make certain generalizations when, in fact, they should not. This problem is discussed in a dynamic geometry software setting. Dynamic geometry software packages have the capability of controlling the number of decimal places that the user sees when taking measurements. Zbiek presents a problem in which successive midpoints of a pentagon are connected to form a new pentagon. The process is repeated to form a second pentagon, which lies in the interior of the newly created pentagon. A dynamic geometry sketch of the problem is shown in Figure 2.21. Students are asked to take measurements regarding perimeter and area and to make conjectures. Students who observe the data in Figure 2.21 may be led to make certain conjectures that, in fact, are not true. The data in Figure 2.21 suggest that certain perimeter and area ratios are equal. All measurements and computations in Figure 2.21 are rounded to one decimal place, using a feature of the software that allows the user to control the number of decimal places shown in the display. If the same measurements are displayed to two decimal place precision, as shown in Figure 2.22, the reader will see that although the perimeter and area ratios are approximately equal, they are, in fact, not equal to each other.
PERIMETER OF KLMNO = 21.8 cm
PERIMETER OF FGHU = 27.3 cm
PERIMETER OF ABCDE = 33.2 cm

(PERIMETER OF KLMNO) / (PERIMETER OF FGHU) = 0.8
(PERIMETER OF FGHU) / (PERIMETER OF ABCDE) = 0.8

AREA OF KLMNO = 29.9 cm²
AREA OF FGHU = 45.6 cm²
AREA OF ABCDE = 69.3 cm²

(AREA OF KLMNO) / (AREA OF FGHU) = 0.7
(AREA OF FGHU) / (AREA OF ABCDE) = 0.7

Figure 2.21. Pentagon problem [1 decimal place precision]
(based on Zbiek, 1996).

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PERIMETER OF KLMNO = 21.78 cm
PERIMETER OF FGHIJ = 27.35 cm
PERIMETER OF ABCDE = 33.25 cm

(PERIMETER OF KLMNO) / (PERIMETER OF FGHIJ) = 0.80
(PERIMETER OF FGHIJ) / (PERIMETER OF ABCDE) = 0.82

AREA OF KLMNO = 29.86 cm²
AREA OF FGHIJ = 45.58 cm²
AREA OF ABCDE = 69.30 cm²

(AREA OF KLMNO) / (AREA OF FGHIJ) = 0.65
(AREA OF FGHIJ) / (AREA OF ABCDE) = 0.66

Figure 2.22. Pentagon problem [2 decimal place precision]
(based on Zbiek, 1996).
Examples such as this clearly make a case for being careful with certain issues in a
dynamic geometry environment so that students do not make generalizations in cases where
they should not. Laborde (1993) stated that computer use could easily lead to the "triumph
of inductivism." Problems such as these seem to support her position.

SIGNIFICANCE OF THE STUDY

While several dissertation studies have examined the role of geometry software in
conjecturing or generalizing in geometry (Yerushalmy, 1986; Elchuck, 1992; Frerking,
1994; Lester, 1996; Bell, 1998; Round 1998), the present study differs from each of the
listed studies in important ways. While Yerushalmy compared a software class to a non-
software class in the context of induction and generalization, the present study utilized a
different type of geometry software (dynamic vs. static representation) and utilized a
different type of data source (task-based interviews). Elchuck studied conjecturing ability
with varied levels of the software treatment, but did not have a control group in which the
software was not used. Frerking examined conjecturing with three varying levels of
software treatment, but used no non-software group for comparison purposes. Lester
researched conjecturing in a dynamic versus traditional atmosphere, but the present study
differs in length, type of software used, and in the type of data collected (task-based
interviews). Bell studied conjecturing with two different levels of software treatment, but
both groups used dynamic software. Round examined conjecturing in a dynamic
environment, and used TI-92 calculators, like the present study, but conducted research in
a single classroom. Additionally, the present study differs from Round's in the type of
data collected (task-based interviews).

It is critical that high school students develop their abilities to make generalizations.
Although students have been generalizing for many years in geometry without the aid of
geometry software programs, these tools give us an opportunity to look anew at this
important topic. Studies such as the present one can only help to broaden what is known about student learning in geometry. If there exists any possibility that dynamic geometry software programs help students see geometry in a different way and positively impact their abilities to make generalizations, then studies such as the present one are important.

The mathematics learning theory of Dienes provides a theoretical framework for the study. Dynamic geometry software embodies each of Dienes’ principles (the Dynamic Principle, the Constructivity Principle, the Mathematical Variability Principle, and the Perceptual Variability or Multiple Embodiment Principle) in unique and important ways as discussed earlier. Student learning via Dienes’ mathematics learning theory was examined through the lens of dynamic geometry software. In particular, students’ abilities to make generalizations in geometry were studied through the use of a particular dynamic geometry software program, *Cabri Geometry II*. 
CHAPTER 3

METHODOLOGY

This study investigated the differences in students' abilities to make generalizations in geometry when immersed in a dynamic geometry software environment versus a traditional environment. The study was conducted from September 2000 to March 2001, and considered both quantitative and qualitative data. Two groups, one that used dynamic geometry software on a regular basis and one that used traditional methods, were compared to determine if there were differences in their abilities to make generalizations in geometry. The quantitative part of the study used scores on an entering geometry test, a generalization pre- and posttest, and task-based interviews with a subset of students to compare the groups at the beginning of the study and to look for differences in students' abilities to generalize. The qualitative part of the study examined differences between the classrooms of the experimental and control groups and described some comments made by students in the interviews in greater detail. Lesson vignettes of the experimental and control group classrooms are given to familiarize the reader with differences between the two classrooms.

SITE

The study took place at a small high school in a large urban district in central Ohio. The student body consists of approximately six hundred students in grades nine through twelve. The school is one of three alternative high schools in the district. The student
body is completely selected by a lottery. The special focus of the school is two-fold: a
community internship program and a strong college preparatory focus. The internship
program gives students the opportunity to experience learning in a very different way for
three years. Students participate in an internship experience one day a week (for
approximately three-fourths of the school year) during their sophomore, junior, and senior
years. The internship program gives students from this school the opportunity to have a
very different high school experience than their peers in other schools. Students work in a
variety of settings: schools, hospitals, doctors' offices, businesses, law offices,
government offices, and other types of settings.

SUBJECTS

The subjects were enrolled in three Geometry classes during the first semester of
the 2000-2001 school year. As the school operates on a semester block schedule, the entire
Geometry course is completed over a period of one semester. Class periods are 80 minutes
in length and classes meet four days per week for the majority of the semester, due to the
internship program described earlier. Most of the subjects were in the 9th or 10th grade,
with a few subjects in the 11th and 12th grades. Almost all of the subjects were taking
Geometry for the first time.

Two Geometry sections were taught by the researcher and one section was taught
by another mathematics teacher. The sections taught by the researcher will hereafter be
referred to as the experimental group. The experimental group had a total of 67 students.
The section taught by the other mathematics teacher will be hereafter referred to as the
control group. The control group had a total of 26 students. Although both the
experimental group and the control group followed the district-approved course of study,
the experimental group extensively used Cabri for the purpose of exploring geometric
ideas. The control group followed a traditional approach to exploring the ideas of

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geometry. Typical lessons for each group are described later in this chapter. Table 3.1 describes the grade-level distribution of students in the study.

Table 3.1. Grade-Level Distribution of Students in the Study.

<table>
<thead>
<tr>
<th>GRADE LEVEL</th>
<th>EXPERIMENTAL</th>
<th>CONTROL</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>34</td>
<td>4</td>
</tr>
<tr>
<td>10</td>
<td>27</td>
<td>17</td>
</tr>
<tr>
<td>11</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>12</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>TOTALS</td>
<td>67</td>
<td>26</td>
</tr>
</tbody>
</table>

A subset of fifteen students from each group was selected to participate in task-based geometry interviews. All students were given an opportunity to participate in the interviews. Some students volunteered for the interviews early in the course after solicitation by the researcher. The remaining students were asked to participate in the interviews. Students were selected for the interviews to represent a range of ability levels, based on course grades for the geometry course that they just completed. As the course was completed prior to the beginning of the interview process, the course grades could be used as means of selecting the interview candidates. Additional factors (gender and race) were considered when selecting interview candidates. Due to the low number of ninth-grade students in the control group and the high number of ninth-grade students in the experimental group, it was not possible to achieve a balance between the grade levels of the
interview subgroups. No students who volunteered for the interviews were turned down.

The course-grade distribution for the interview subgroups are reported in Table 3.2.

**Table 3.2. Course-Grade Distribution of Interview Subgroups.**

<table>
<thead>
<tr>
<th>GRADE EARNED</th>
<th>EXPERIMENTAL</th>
<th>CONTROL</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>B</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>C</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>F</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Based on grades earned for the course, the groups were similar in their achievement in geometry. Additional information regarding the composition of the interview subgroups is reported in Tables 3.3, 3.4, and 3.5.

**Table 3.3. Gender Distribution of Interview Subgroups.**

<table>
<thead>
<tr>
<th></th>
<th>EXPERIMENTAL</th>
<th>CONTROL</th>
</tr>
</thead>
<tbody>
<tr>
<td>FEMALES</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>MALES</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>
Table 3.4. Racial Distribution of Interview Subgroups.

<table>
<thead>
<tr>
<th></th>
<th>EXPERIMENTAL</th>
<th>CONTROL</th>
</tr>
</thead>
<tbody>
<tr>
<td>BLACK FEMALES</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>BLACK MALES</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>WHITE FEMALES</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>WHITE MALES</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 3.5. Grade-Level Distribution of Interview Subgroups.

<table>
<thead>
<tr>
<th>GRADE LEVEL</th>
<th>EXPERIMENTAL</th>
<th>CONTROL</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>10</td>
<td>4</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>11</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
INSTRUMENTATION

**Entering Geometry Student Test.** In order to compare the experimental group to the control group at the beginning of the course, the Entering Geometry Student Test was administered to both groups. The test is a twenty-item, multiple-choice test containing some geometry content that students may have been introduced to while in middle school. Basic geometry content such as linear pairs of angles, parallel and perpendicular lines, area and perimeter, and circles are among the topics tested. The test was scored on a 100-point scale, with each correct answer worth five points. The Entering Geometry Student Test was originally developed for use by the CDASSG Project (1980) at the University of Chicago (Zalman Usiskin, Director). The complete Entering Geometry Student Test is found in Appendix A.

**Generalization Tests.** Two researcher-developed generalization tests that were parallel in format (Geometry Challenge—Form A and Geometry Challenge—Form B) were given to both groups to compare the generalization abilities of the groups at the beginning and at the end of the study. Complete versions of both of the generalization tests can be found in Appendix B. Half of the students in each group were given Form A at the beginning of the course while the remaining students in each group were given Form B. Students that took Form A (Generalization Pretest) at the beginning of the course took Form B (Generalization Posttest) at the end of the course and students that took Form B (Generalization Pretest) at the beginning of the course took Form A (Generalization Posttest) at the end of the course. Students were selected to take a particular form based on their assigned seat. Alternating rows of students were given each form of the test.

The tests contained five questions that required extending tables and using inductive reasoning to generalize to the \( n \)th term in a sequence. Some of the questions involved problem situations such as the well-known handshake problem, while others had geometric figures from which data were to be collected by the student in order to extend the
information in the table. Generalization to the \( n \)th term was the final item in each of the five questions. Problems such as those on the generalization tests can be found in most high school geometry books. Two books that the researcher is familiar with (Larson, Boswell, & Stiff, 1995; Serra, 1989/1993) contain a few of the problems or similar ones. The researcher created the problems based on his experience with inductive reasoning problems as a high school mathematics teacher (use of various textbooks, articles in professional journals, problems from mathematics contests, attendance at professional conferences). Some sources of problems about number patterns were listed in the Review of Related Literature section of Chapter 2 (Aviv, 1979; Hartman, 1976; Oullette, 1975, 1978; Ranucci, 1974; Smith, 1972; Szetela, 1999). The specific source of each problem on the generalization tests is given in the following discussion.

The specific content and scoring of the generalization tests will now be discussed. The first question on each test asked students to extend a number pattern and find the \( n \)th term in the sequence.

**Form A. Question 1**

You are given the following information regarding a sequence of numbers:

<table>
<thead>
<tr>
<th>TERM IN SEQUENCE (T)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>VALUE OF TERM (V)</td>
<td>8</td>
<td>11</td>
<td>14</td>
<td>17</td>
<td>20</td>
</tr>
</tbody>
</table>

Give the value of the 6th, 7th, and \( n \)th terms in the sequence by filling in the table below.

<table>
<thead>
<tr>
<th>T</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>V</td>
<td>8</td>
<td>11</td>
<td>14</td>
<td>17</td>
<td>20</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
**Form B, Question 1**

You are given the following information regarding a sequence of numbers:

<table>
<thead>
<tr>
<th>TERM IN SEQUENCE (T)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>VALUE OF TERM (V)</td>
<td>0</td>
<td>5</td>
<td>12</td>
<td>21</td>
<td>32</td>
</tr>
</tbody>
</table>

Give the value of the 6th, 7th, and nth terms in the sequence by filling in the table below.

<table>
<thead>
<tr>
<th>T</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>V</td>
<td>0</td>
<td>5</td>
<td>12</td>
<td>21</td>
<td>32</td>
<td>___</td>
<td>___</td>
<td>___</td>
</tr>
</tbody>
</table>

In question 1, Form A was scored as follows: 1 point each for extending the number pattern and 5 points for the generalization. Form B was scored as follows: 1 point each for extending the number pattern and 7 points for the generalization. This decision was made as the generalization required for Form B \((n - 1)(n + 3)\) was considered to be of greater difficulty than the one required for Form A \(3n + 5\). There is also a difference in difficulty level in the generalization for question 4, with Form A containing the more difficult generalization. Question 1 on each form was developed by the researcher. A complete summary of the scoring rubric for each of the tests is presented later in this section as Table 3.6.

The second question on each test involved the same generalization, but the context of the problems was slightly different. Form A considered the well-known handshake problem, while Form B involved the same data in the context of the number of segments required to connect \(n\) points in a plane.
**Form A. Question 2**

If several people are in a room, how many handshakes are possible if each person shakes hands with everyone else once?

Collect data in the following table to help you analyze the situation.

<table>
<thead>
<tr>
<th>PEOPLE</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>HANDSHAKES</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Generalize the situation to predict how many handshakes there will be if there are \( n \) people in the room.

ANSWER: ___________________________

---

**Form B. Question 2**

How many segments are necessary to connect \( n \) points on a plane if each point is connected to all the others?

The following figure will help you to visualize the situation.

![Diagram showing how to connect points](image_url)

Collect data in the following table to help you analyze the situation.

<table>
<thead>
<tr>
<th>POINTS</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>SEGMENTS</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Generalize the situation to predict how many segments there will be if there are \( n \) points.

ANSWER: ___________________________

92
Forms A and B were scored identically, 1 point each for the correct values for the next three cases and 7 points for the generalization. The handshake problem is adapted from Serra (1989, p. 59). The line segment problem is adapted from Serra (1989, p. 57).

The third question on each test asked students to generalize a formula for the sum of the first $n$ odd positive integers (Form A) or the sum of the first $n$ even positive integers (Form B).

**Form A, Question 3**

Find a formula for the sum of the first $n$ odd positive integers.

Collect data in the following table to help you analyze the situation.

<table>
<thead>
<tr>
<th>NUMBER OF ODD INTEGERS</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>SUM</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>HOW TO FIGURE</td>
<td>1</td>
<td>1 + 3</td>
<td>1 + 3 + 5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Generalize the situation to find a formula for the sum of the first $n$ odd positive integers.

**ANSWER:** _______________________________

93
Find a formula for the sum of the first $n$ even positive integers.

Collect data in the following table to help you analyze the situation.

<table>
<thead>
<tr>
<th>NUMBER OF EVEN INTEGERS</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>SUM</td>
<td>2</td>
<td>6</td>
<td>12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>HOW TO FIGURE</td>
<td>2</td>
<td>2 + 4</td>
<td>2 + 4 + 6</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Generalize the situation to find a formula for the sum of the first $n$ even positive integers.

ANSWER: _______________________________

Forms A and B were scored identically, 1 point each for the correct values for the next two cases and 5 points for the generalization. Both of these problems are adapted from Serra (1989, p. 60).

The last two questions on each test asked students to make generalizations based on data collected from geometric situations regarding area, perimeter, and volume.
Consider the following set of figures, each comprised of a right triangle sitting on top of a square.

![Stage 1](image1)  Stage 2  Stage 3

a) Collect data in the following table regarding the area of each figure.

**THE AREA OF A SQUARE IS COMPUTED BY MULTIPLYING ITS LENGTH BY ITS WIDTH.**

**THE AREA FORMULA FOR A TRIANGLE IS**

\[ A = \frac{1}{2} b h \]

<table>
<thead>
<tr>
<th>STAGE</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>AREA</td>
<td>5</td>
<td>12</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) Generalize the situation to predict the area of the figure in the \( n \)th stage.

**ANSWER:**


95
**Form B. Question 4**

Consider the following set of figures.

![Stage 1 Stage 2 Stage 3](image)

**Stage 1**  
![Stage 1 Image]

**Stage 2**  
![Stage 2 Image]

**Stage 3**  
![Stage 3 Image]

a) Collect data in the following table regarding the perimeter of each figure. The length of each small segment is 1. The numbers on the figures in Stages 1 and 2 are to help you see how the numbers in the table below were obtained.

**THE PERIMETER OF A FIGURE IS DEFINED TO BE THE SUM OF THE LENGTHS OF ITS SIDES.**

<table>
<thead>
<tr>
<th>STAGE</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>PERIMETER</td>
<td>4</td>
<td>8</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) Generalize the situation to predict the perimeter of the figure in the \( n \)th stage.

ANSWER: ____________________________

In question 4, Form A was scored as follows: 2 points each for the correct values for the next three cases and 7 points for the generalization. Form B was scored as follows: 2 points each for the correct values for the next three cases and 5 points for the generalization. This decision was made as the generalization required for Form A \([(1/2)(n + 1)(3n + 2)]\) was considered to be of greater difficulty than the one required for Form B \([4n]\). Both of these problems were developed by the researcher.

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Consider the following set of rectangles:

![Rectangles diagram]

Stage 1    Stage 2    Stage 3
3  2  6  4  8

a) Complete the following table regarding the area of each rectangle.

THE AREA OF A RECTANGLE IS COMPUTED BY MULTIPLYING ITS LENGTH BY ITS WIDTH.

<table>
<thead>
<tr>
<th>STAGE</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>AREA</td>
<td>6</td>
<td>24</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) Generalize the situation to predict the area of the rectangle in the nth stage.

ANSWER: ____________________________
Consider the following set of cubes:

Stage 1  Stage 2  Stage 3

Be sure to notice how the length of the sides of the cubes are changing. If the lengths changed from 1 to 2 to 4, think carefully about the length of a side of a cube in stage 4.

a) Complete the following table regarding the volume of each cube.

THE VOLUME OF A CUBE IS COMPUTED BY
THE FORMULA \( V = L \times W \times H \).

<table>
<thead>
<tr>
<th>STAGE</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>VOLUME</td>
<td>1</td>
<td>8</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) Generalize the situation to predict the volume of the cube in the \( n \)th stage.

ANSWER: ________________________________
Forms A and B were scored identically, 2 points each for the correct values for the next three cases and 7 points for the generalization. Both of these problems were developed by the researcher. Table 3.6 summarizes the point distribution for both Form A and Form B of the generalization test.

Table 3.6. Summary of Point Distribution for Generalization Tests.

<table>
<thead>
<tr>
<th>FORM A</th>
<th>QUESTION</th>
<th>POINTS FOR EACH ANSWER</th>
<th>TOTAL POINTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 1 1 5</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1 1 1 7</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1 1 5</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2 2 2 7</td>
<td>13</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>2 2 2 7</td>
<td>13</td>
<td></td>
</tr>
</tbody>
</table>

FORM A TOTAL: 50

<table>
<thead>
<tr>
<th>FORM B</th>
<th>QUESTION</th>
<th>POINTS FOR EACH ANSWER</th>
<th>TOTAL POINTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 1 7</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1 1 1 7</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1 1 5</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2 2 2 5</td>
<td>11</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>2 2 2 7</td>
<td>13</td>
<td></td>
</tr>
</tbody>
</table>

FORM B TOTAL: 50
Interview tasks. The researcher developed some of the interview tasks, while others were adapted from Geometric Explorations on the TI-92 (Keyton, 1996). The interview consisted of six geometry tasks in which students were provided opportunities to make generalizations. Complete versions of the tasks can be found in Appendix C. Condensed versions of the tasks are presented in Chapter 4 along with the analysis of student responses from each task. Task 1 and Task 3 were adapted from Keyton (1996). All of the remaining tasks were developed by the researcher. In Tasks 1 and 3, students were given three geometric figures with measurements of segment lengths on the figures. There was a fourth figure with all of the information except one missing value. Students were asked to study the information on the first three examples and to try to find the missing value in the fourth example. In Task 2 and in Task 4, students were asked to try to relate two theorems. In each case, one theorem could be thought of as a generalization of the other one. Students were asked to interpret the theorems and then relate the two theorems to one another. In Task 5, four examples were given with various measurements indicated on the figures. There was no missing value in the fourth example, as there was in the earlier tasks. Instead, students were asked to study the examples and make any observations that they could make. Task 6 involved the Pythagorean Theorem and was longer than the other tasks. Students were asked to interpret the Pythagorean Theorem in terms of the areas of squares constructed on each side of the triangle. In subsequent parts of Task 6, students explored if the area argument would hold if certain procedures were done to each square. For example, once students confirmed that the areas of the two smaller squares added together yielded the area of the large square, they were asked if the area argument would still hold if each of the squares were cut in half. Various other figures were explored in the task, with ample opportunity for students to suggest figures that they thought would work. The task was designed to see if students would recognize that as long as the figures on each side were similar, the area argument would still work.
INSTRUCTIONAL MATERIALS

The experimental group used Texas Instruments TI-92 calculators on a regular basis (see Table 3.7) throughout the course for exploration and discovery of geometric ideas. The TI-92 has *Cabri Geometry II* as a built-in application. One pedagogically useful feature of the TI-92 is that any TI-92 can be connected to an overhead display unit, making it possible for students to display their work for the entire class. This happened approximately one out of every three times that the experimental group used the TI-92.

Both the experimental and control groups used the same textbook, *Geometry: An Integrated Approach*, 1995, Larson, Boswell, & Stiff. Both the researcher and the control group teacher used various geometry resource materials to supplement the textbook. The researcher did not ask the teacher of the control group to document the use of supplementary resource materials. The researcher does, however, estimate that an extremely high percentage of the supplementary resources used in the two groups were the same. When the researcher used a supplementary resource, the resource was shared with the teacher of the control group. This was done to help control the effect of any mediating influences on the two groups, other than the use of *Cabri*.

As discussed more fully later in this chapter, the control group used traditional methods of geometry instruction, including compass and straightedge constructions. Due to the emphasis of the research study on using *Cabri* for explorations, no compass and straightedge constructions were done with the experimental group. All constructions were done using *Cabri*.

**List of topics.** Table 3.7 provides a list of topics for which the researcher used *Cabri* in doing explorations with the experimental group. The table is organized to show three types of explorations.
### Table 3.7. Topic List for Which the Experimental Group Used *Cabri.*

#### DEFINITIONS
- Midpoint
- Segment bisector
- Angle bisector
- Perpendicular lines
- Linear pair of angles
- Parallel lines
- Corresponding angles
- Alternate interior angles
- Consecutive interior angles
- Circumcenter of a triangle
- Circumscribed circle
- Incenter of a triangle
- Inscribed circle
- Centroid of a triangle
- Orthocenter of a triangle
- Euler line
- Equations of circles

#### THEOREMS
- Vertical angles
- Transitivity of parallel lines
- Triangle sum
- Exterior angle
- Base angles
- Perpendicular bisector
- Angle bisector
- Triangle midsegment
- Triangle inequality
- Hinge
- Polygon interior angles
- Polygon exterior angles
- Trapezoid midsegment
- Quadrilateral midpoints
- Sides of similar figures
- Areas of similar figures
- Proportionality in triangles
- Geometric mean
- Pythagorean Theorem
- Circle theorems

#### PROPERTIES
- Perpendicular lines
- Parallel lines
- Equilateral triangles
- Isosceles triangles
- Scalene triangles
- Acute triangles
- Right triangles
- Obtuse triangles
- Regular polygons
- Parallelograms
- Rhombi
- Rectangles
- Squares
- Trapezoids
- Isosceles right triangles
- Half-equilateral triangles
- Circles
- Tangents/radii of a circle
In this section of the Methodology chapter, lessons are described through the use of vignettes to acquaint the reader with the type of activities the researcher used to explore geometric concepts using *Cabri* with the experimental group.

**Vignette 1—Parallel Lines, Transversals, and Angles.** When two parallel lines are intersected by a transversal, several sets of congruent angles are formed (see Figure 3.1). Students were asked to draw a line, \(l\), and then to use the parallel line command to draw a line, \(m\), parallel to the given line. Then, a transversal, \(r\), was added. Eight angles are formed after the addition of the transversal to the sketch. In geometry, these angles are defined to be corresponding angles, alternate interior angles, alternate exterior angles, and consecutive interior angles. Students were asked to make conjectures comparing the measures of the angles. Students made the following conjectures: corresponding angles are congruent, alternate interior angles are congruent, alternate exterior angles are congruent, consecutive interior angles are supplementary. [NOTE: The students already knew the Vertical Angles Theorem at this time, so there was no need for students to make conjectures regarding vertical angles.] A *Cabri* sketch depicting this exploration is given in Figure 3.2. In order to verify that students' conjectures were true for more situations than the one on their screen, students were asked to drag the original line and observe what happened to the angle measurements. Students learn very early in the course that certain objects in *Cabri* are not draggable. In this case, students could not drag line \(m\) as it was constructed to be parallel to line \(l\). The only way to move line \(m\) is to move line \(l\). As line \(m\) is dependent on line \(l\), it will move anytime line \(l\) is moved. A sketch in which lines \(l, m\) and \(r\) are moved is shown in Figure 3.3. Students were able to observe that their conjectures still held in the modified drawing.
Figure 3.1. Parallel lines, transversals, and angles.

Figure 3.2. Parallel lines and specific angle measures.
Figure 3.3: Modified version of angle measurements.
Vignette 2—Special Case Right Triangles. In isosceles right triangles (45°-45°-90°) and in half-equilateral triangles (30°-60°-90°), well-known relationships exist among the lengths of the sides. In isosceles right triangles, the ratio of the length of the hypotenuse to the length of a leg is \(\sqrt{2}:1\). In half-equilateral triangles, the ratio of the length of the hypotenuse to the length of the shorter leg is 2:1 while the ratio of the length of the longer leg to the length of the shorter leg is \(\sqrt{3}:1\). Students can discover these relationships using Cabri.

Students were asked to construct isosceles right triangles and half-equilateral triangles, and to measure the lengths of the sides. Even though this topic was studied late in the course, many students had some difficulty constructing an isosceles right triangle that is a construction instead of a drawing. Many students also had difficulty constructing the half-equilateral triangle. These difficulties that students had with the constructions are consistent with results reported by studies in Chapter 2. Once the appropriate figures were constructed (with guidance from the researcher), students were asked to make comparisons regarding the lengths of sides and to make any conjectures that they could.

Students noted that relationships did exist between the side lengths, but students did not recognize these numbers to be \(\sqrt{2}\) and \(\sqrt{3}\). Generally, many students are not familiar with decimal approximations to square roots that are irrational numbers. Students tend to be familiar with the decimal approximation for only one irrational number, \(\pi\). Therefore, some class discussion had to take place when students saw decimal approximations for \(\sqrt{2}\) and \(\sqrt{3}\) on their calculator screens.

A Cabri sketch of an isosceles right triangle is given in Figure 3.4 while a half-equilateral triangle is shown in Figure 3.5.
(Length of hypotenuse) / (Length of a leg) = 1.41

Figure 3.4. Isosceles right triangle.

(Length of hypotenuse) / (Length of short leg) = 2.00
(Length of long leg) / (Length of short leg) = 1.73

Figure 3.5. Half-equilateral triangle.
Vignette 3—Circumcenter and Circumscribed Circle of a Triangle. In a triangle, if one constructs the perpendicular bisectors of all three sides, the lines are concurrent. This point is called the circumcenter of the triangle. The distance from the circumcenter to each vertex of the triangle is equal. Therefore, a circle can be constructed which contains all three vertices of the triangles.

To explore this topic with *Cabri*, students were asked to draw a triangle and then to use the perpendicular bisector command to construct the perpendicular bisectors of all three sides of the triangle (see Figure 3.6). When asked to make an observation, the students noted that the lines were concurrent. Students were then asked to do any measuring they would like to do. Once it was discovered that the distance from the point of concurrency to each vertex was the same, the circle tool was used to construct the circumscribed circle (see Figure 3.7).

![Figure 3.6. Concurrency of perpendicular bisectors.](image.png)
Figure 3.7. Circumscribed circle of a triangle.
Vignette 4—Incenter and Inscribed Circle of a Triangle. In a triangle, if one constructs the angle bisectors of all three angles of the triangle, the rays are concurrent. This point is called the incenter of the triangle. The distance from the incenter to each side of the triangle is equal. Therefore, a circle can be constructed which is tangent to all three sides of the triangles.

Students were asked to draw a triangle and then to use the angle bisector command to construct the angle bisectors of all three angles of the triangle (see Figure 3.8). The students noted that the rays were concurrent. Students were then asked to do any measuring they would like to do. Once it was discovered that the distance from this point to each side of the triangle is the same, the circle tool was used to construct the inscribed circle (see Figure 3.9). Students could then modify their sketch to observe that the circle remained tangent to the sides of the triangle even when the original triangle was modified. It should be noted that students needed to be given some guidance on what to measure in this exploration. In the previous vignette, the distances that were equal involved points that were already on the sketch, while in this situation, the equal distances involve points that were not on the original sketch.
Figure 3.8. Angle bisectors of any triangle are concurrent.

Figure 3.9. Incenter and inscribed circle.
Vignette 5—Triangle and Trapezoid Midsegment Theorems. In a triangle, if one constructs the midpoints of two of the sides and connects them with a segment, the segment is called a midsegment. In a trapezoid, the midsegment is defined to be the segment that connects the midpoints of the non-parallel sides. In each situation, the midsegment possesses important properties. The midsegment of a triangle is parallel to the third side of the triangle and it is one-half of the length of the third side of the triangle. The midsegment of a trapezoid is parallel to the bases (the parallel sides) of the trapezoid and its length is the average of the lengths of the bases.

Students were asked to create a triangle, and then to use the midpoint command to construct the midpoints of two of the sides. Students then used the segment tool to construct a segment connecting the two midpoints. Students were asked to make conjectures regarding the midsegment. The conjecture that the midsegment is parallel to the third side of the triangle can be verified using the Check Property command. The Check Property command allows users to select two objects to see if they are parallel to each other. If the two objects are parallel, the software returns a dialog box with the phrase objects are parallel. This exploration is shown in Figure 3.10. For the trapezoid, students were asked to construct a trapezoid, to create the midsegment, and to make any conjectures that they could about the sketch. This exploration is shown in Figure 3.11. [NOTE: These topics were not taught together; they are presented together here for illustrative purposes only. The researcher does not want the reader to assume that the experimental group would have had an advantage on Task 4 of the interview component of the study.]
Figure 3.10. Triangle Midsegment Theorem.

Figure 3.11. Trapezoid Midsegment Theorem.
Vignette 6—Polygon Interior Angles Theorem and Polygon Exterior Angles

Theorem. The formula \(180(n - 2)°\) is used to compute the sum of the interior angles of any convex polygon, where \(n\) is the number of sides of the polygon. This theorem provided opportunities for students to explore the interior angle sum of various polygons and bring their individual TI-92 calculators to the front of the classroom so that the entire class could view their polygon and its interior angle sum. Different students displayed their sketches, so that the class was able to see the interior angle sum for a quadrilateral, a pentagon, a hexagon, a heptagon, an octagon, a nonagon, and even a decagon. Students already knew the Triangle Sum Theorem at this point in the course, which is why a triangle was not constructed as part of this exploration. The researcher collected information regarding the data and a class discussion helped to arrive at a generalization regarding the sum of the interior angles of any convex polygon. Students were asked to move a vertex so that the polygon was concave. They could then see that this result holds only for convex polygons. A Cabri sketch depicting parts of this exploration is shown in Figure 3.12.

The sum of the exterior angles, one at each vertex, for any convex polygon is 360°. Students were asked to volunteer for different polygons to create and measure the exterior angles. Students created polygons, extended each side using the ray tool, and then measured each of the exterior angles. The calculator feature was used to find the sum of the exterior angles. Several students brought their sketches up to show the entire class. The researcher collected data as the students showed their sketches to the entire class and the class arrived at a generalization regarding the sum of the exterior angles of a polygon. A Cabri sketch depicting parts of this exploration is shown in Figure 3.13.
Figure 3.12. Polygon Interior Angles Theorem.
Figure 3.13. Polygon Exterior Angles Theorem.
Vignette 7—Connecting Midpoints of a Quadrilateral. When the consecutive midpoints of any quadrilateral are connected, the resulting figure is a parallelogram. This theorem is valid if the quadrilateral is convex as well as when it is concave. Students are easily able to manipulate figures in *Cabri* to explore which polygon theorems are valid only for convex polygons and which theorems are valid for both convex and concave figures. Students were asked to create a quadrilateral, construct the midpoints, and connect consecutive midpoints. Students were then asked to make any conjectures they could about their sketch. Results of this exploration are given in Figure 3.14.

![Figure 3.14. Connecting midpoints of a quadrilateral.](image)
**Vignette 8—Pythagorean Theorem.** When students are introduced to the Pythagorean Theorem, it is customary to provide them with activities that help them see that the theorem is essentially an argument about the areas of the three squares constructed on the sides of the right triangle. Students were asked to construct a right triangle and then to construct squares on each side of the triangle. As reported by several researchers in the literature review in Chapter 2, the construction of a square is not a trivial problem for many students. The areas of the squares were computed and then students were asked to make a conjecture regarding the areas. The researcher deliberately avoided construction of other similar figures on the sides due to the content of Task 6 of the interview protocol. A *Cabri* sketch depicting this exploration is given in Figure 3.15.

![Figure 3.15. Pythagorean Theorem.](image)

Area (Square IHBA) = 18.3 cm²

Area (Square DACE) = 12.0 cm²

Area (Square CBGF) = 6.3 cm²

**Figure 3.15. Pythagorean Theorem.**
LESSON VIGNETTES (CONTROL GROUP)

In this section of the Methodology chapter, lessons are described through the use of vignettes to acquaint the reader with the experiences of students in the control group during the geometry course. While this is by no means a comprehensive description of the nature of learning in the control group classroom, it should give the reader some idea of how topics were presented by the control group teacher. The reader can make comparisons between these lessons and the same lessons from the experimental group, which were described in the previous section. The researcher observed each of these lessons in the control group classroom.

Vignette 9—Triangle Sum Theorem and Exterior Angle Theorem. The control group teacher chose a hands-on investigation for this lesson. Students were given blank paper, scissors, and tape to do the investigation. The teacher asked students to draw three points anywhere on the paper and to connect the points with segments. The teacher then instructed students to open their notebooks and to draw a line in it. Students were asked to cut off one of the angles and to tape it to the line in their notebooks, as shown in Figure 3.16.

Figure 3.16. Beginning of Triangle Sum Theorem investigation.
Continuing with the investigation, the teacher instructed students to cut off the remaining angles of the triangle and to tape them next to the first one in their notebooks, as shown in Figure 3.17.

Figure 3.17. Conclusion of Triangle Sum Theorem investigation.

The teacher asked the students if they could make a statement about the angles of a triangle. Students responded that the angles of a triangle add up to $180^\circ$.

For the second part of the lesson, the teacher instructed students to draw another line in their notebooks and to place two points on the line. The teacher then instructed the students to place a third point above or below the line and to draw the triangle formed by the three points. The teacher asked the students to label the interior angles of the triangle $A$, $B$, and $C$, and one of the two exterior angles $D$ as shown in Figure 3.18.

Figure 3.18. Exterior Angle Theorem investigation.
The teacher asked students to state the relationship between angles C and D. Students responded that angles C and D form a linear pair. The teacher then used this information to conclude that the measure of the exterior angle is equal to the sum of the two remote interior angles, as shown below in Figure 3.19.

\[
\begin{align*}
    m \angle A + m \angle B + m \angle C &= 180^\circ \\
    m \angle C + m \angle D &= 180^\circ \\
    m \angle A + m \angle B + m \angle C &= m \angle C + m \angle D \\
    m \angle A + m \angle B &= m \angle D
\end{align*}
\]

Figure 3.19. Explanation of Exterior Angle Theorem.
Vignette 10—Special segments in a triangle. The teacher began the lesson by drawing four triangles on the chalkboard, each with a segment drawn inside the triangle, as shown in Figure 3.20.

![Perpendicular Bisector](image1) ![Angle Bisector](image2) ![Median](image3) ![Altitude](image4)

**Figure 3.20. Special segments in a triangle.**

After discussing the meaning of each of the special segments, the teacher drew triangles on the chalkboard to illustrate the results of Theorem 5.5 in the textbook. Theorem 5.5 describes the concurrency properties of the four special segments of a triangle. These triangles are shown in Figure 3.21. After drawing these triangles on the board the teacher instructed the students that certain numerical relationships exist between the two smaller
segments (on either side of the centroid) that make up each median. The teacher told the students that the larger of the two segments is equal to two-thirds of the entire median and that the smaller of the two segments is equal to one-third of the entire median. The teacher also stated that the ratio of the larger segment to the smaller segment is 2:1.

![Circumcenter](image1)

![Incenter](image2)

![Centroid](image3)

![Orthocenter](image4)

**Figure 3.21.** Concurrency properties.
Vignette 11—Triangle Midsegment Theorem. The teacher began the lesson by writing the midpoint and slope formulas on the chalkboard, and asked students to recall the relationships between the slopes of two parallel lines and of two perpendicular lines. Students responded that the slopes of two parallel lines are equal and the slopes of two perpendicular lines are opposite reciprocals of each the other. The teacher presented an example in which the task was to find an equation of a line parallel to a given line and passing through a given point. The teacher then presented an example in which the task was to find an equation of a line perpendicular to a given line and passing through a given point. The teacher then presented the midsegment theorem for triangles. The teacher told the students that if the midpoints of the sides of a triangle are connected, the segment drawn by connecting the midpoints is parallel to the third side of the triangle and is one-half as long as the third side. The teacher drew a triangle on the chalkboard illustrating the situation, as shown in Figure 3.22.

![Figure 3.22. Triangle Midsegment Theorem.](image)

The teacher concluded the lesson by presenting an example in which the task was to find the length of the midsegment and confirm the results of the midsegment theorem in a coordinate geometry setting.
Vignette 12—Polygon Interior Angles Theorem and Polygon Exterior Angles Theorem. The teacher began the lesson by asking the class to consider into how many triangles a quadrilateral is divided if one diagonal is drawn. The class responded that a quadrilateral could be divided into two triangles. The teacher responded that it would not matter which diagonal was drawn; the result was still two triangles. The teacher asked students to collect some data regarding the number of triangles formed in other convex polygons if each possible diagonal was drawn from one vertex. The class responded with the following information, presented in Figure 3.23.

<table>
<thead>
<tr>
<th>Polygon</th>
<th>Sides</th>
<th>Triangles</th>
<th>Interior Angle Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangle</td>
<td>3</td>
<td>1</td>
<td>180°</td>
</tr>
<tr>
<td>Quadrilateral</td>
<td>4</td>
<td>2</td>
<td>360°</td>
</tr>
<tr>
<td>Pentagon</td>
<td>5</td>
<td>3</td>
<td>540°</td>
</tr>
<tr>
<td>Hexagon</td>
<td>6</td>
<td>4</td>
<td>720°</td>
</tr>
<tr>
<td>Heptagon</td>
<td>7</td>
<td>5</td>
<td>900°</td>
</tr>
</tbody>
</table>

Figure 3.23. Data collected for Polygon Interior Angle Theorem.

The teacher asked students if they could generalize a result for the data. Students responded with the formula \([180(n - 2)]°\) for the sum of the interior angles of any convex polygon, where \(n\) represents the number of sides in the polygon. The teacher then discussed a corollary to the theorem, which shows how to find the measure of each interior angle of a regular polygon. The teacher proceeded to present the polygon exterior angle
theorem by drawing a triangle and a rectangle on the chalkboard with their respective exterior angles, as shown in Figure 3.24.

\[ m \angle 1 + m \angle 2 + m \angle 3 = 360^\circ \]

\[ m \angle 1 + m \angle 2 + m \angle 3 + m \angle 4 = 360^\circ \]

Figure 3.24. Polygon Exterior Angle Theorem.
The teacher then presented the corollary to the Polygon Exterior Angle Theorem, which shows how to find the measure of an exterior angle of a regular polygon. The teacher continued the lesson with an example of the Polygon Interior Angle Theorem that involved variables. The teacher concluded the lesson with an example in which students were asked to find how many sides a regular polygon had if they were given the measure of one of its interior angles.

**CONNECTING THE LESSONS TO DIENES' THEORY**

Dienes' Mathematics Learning Theory served as the theoretical basis for the current study. In this section, connections are drawn between the lesson vignettes of the experimental group and Dienes' Theory. Contrasts are made between lessons of the experimental group and the control group.

Four principles constitute Dienes' Mathematics Learning Theory: the Dynamic Principle, the Constructivity Principle, the Mathematical Variability Principle, and the Perceptual Variability or Multiple Embodiment Principle. When the experimental group used *Cabri*, each of these principles was present nearly all of the time. Without exception, the Dynamic Principle was present each time the students created any figure using *Cabri*. Students were routinely asked to drag points and to manipulate figures as they made conjectures using *Cabri*.

The Constructivity Principle was evident in some, but not all of the lessons. While investigating the Triangle Midsegment Theorem, for example, the student could use the Triangle command of *Cabri* to construct a triangle. In cases like this, there is no construction being performed by the student. In the case of the Trapezoid Midsegment Theorem, however, the Constructivity Principle is evident, as students must learn how to construct a trapezoid, rather than draw one. A construction of a trapezoid would be stable under the drag capabilities of the software, whereas a drawing of a trapezoid could be
easily manipulated so that it does not remain a trapezoid once a vertex is moved to another
screen location. Comparing the discussion of the Triangle Midsegment Theorem in the two
groups, it could be stated that the experimental group experienced the topic in a more
dynamic way. The experimental group could manipulate their triangle, watching the
measurements of the midsegment and the third side of the triangle change, but observing
that the ratio of the lengths did not change, no matter what type of triangle they had on their
screen. The control group could only observe one triangle that was drawn on the
chalkboard by the teacher.

The Mathematical Variability Principle was present when students changed the size
and shape of a figure. This principle was present every time students used Cabri. The
same can be said of the Perceptual Variability Principle. This principle was interpreted to
be present when students change the orientation of the figure. Again, this was present
every time students used Cabri. For example, students manipulated the size of isosceles
right triangles when discovering the relationships between the length of the sides. Students
manipulated the shapes of polygons while investigating the Polygon Interior Angle
Theorem and in other situations as well.

Some other comparisons can be made between the lesson vignettes of the two
groups. In the Triangle Sum Theorem vignette of the control group, students discovered
the theorem by cutting out a triangle to see that the angles added up to 180°. In this case,
students were able to see that this result held true for that particular triangle. In contrast,
students in the experimental group could see many triangles by moving vertices on their
Cabri sketch. The Triangle Sum Theorem was not specifically described as a lesson
vignette for the experimental group, but the reader is introduced to Cabri with this idea in
Chapter I. Similarly, the experimental group discovered the Exterior Angle Theorem in a
dynamic way using Cabri, while the control group used a deductive approach. The
deductive approach was also used with the experimental group after using the inductive
approach with *Cabri*. Using *Cabri*, therefore, potentially enhances what students learn, as they are able to see theorems from different perspectives.

Comparing the lessons on the points of concurrency of a triangle, the experimental group was able to see this theorem come to life with *Cabri*. Rather than seeing drawings on the chalkboard drawn by the teacher, the experimental group was able to observe the fact, for example, that the perpendicular bisectors of the sides are concurrent. The experimental group could see the bisectors being drawn and how they intersected at a single point. By manipulating the triangle, they could see that the point of intersection (the circumcenter) sometimes is located inside the triangle, sometimes outside the triangle, and sometimes even on the triangle itself. These observations were not discussed by the control group.

From the lesson vignettes of the experimental and control group, it seems clear that using *Cabri* can only enhance the types of experiences students have when learning concepts in geometry.

**PROCEDURES**

**Recruitment.** Intact geometry classes were used as the experimental and control groups for the study. Students were recruited for participation in the interviews first on a volunteer basis, and then by solicitation according to criteria described earlier in this chapter. All students who volunteered to participate in the interviews were selected for participation. All students who participated in the interviews were asked to return a signed permission slip prior to the administration of the interview.

**Pre-instruments.** The geometry courses began on August 31, 2000 and ended on January 19, 2001. Both groups were administered the Entering Geometry Student Test on the first day of classes. The researcher introduced the experimental group to *Cabri* on the TI-92 on the first day of class after students had completed the Entering Geometry Student
Test. Both groups were administered the Generalization Pretests (Geometry Challenge—Form A or Geometry Challenge—Form B) at the end of the first week of the course.

**Instruction.** As the research site has two class sets of TI-92 calculators, the students in the experimental group were able to use their own TI-92 each time that Cabri was used in class. The researcher used Cabri for exploration and discovery for every topic possible in the course. Naturally, dynamic geometry software is not applicable for certain topics in traditional geometry textbooks, such as triangle congruence proofs. Therefore, the software was not used for every section of the textbook, but wherever applicable, it was used to develop understanding of geometric ideas. Typically, students in the experimental group were given a geometric object to construct using Cabri and then asked to perform certain constructions on the object, or to take certain measurements, and in some cases, both. Students in the experimental group were then routinely asked to make conjectures about relationships that they observed. Students in the control group used the same textbook, but the teacher used traditional methods of instruction in the course. Students in the control group did not use geometry software in any form, but were permitted to use calculators (scientific or graphing) whenever they wanted to use them. The vignettes presented in this chapter provide examples of the type of instruction both groups received.

Observations by the researcher served as the basis for data regarding lessons of the experimental and control groups. Lesson plan data served as the source of data for the experimental group, while observations of the control group by the researcher served as the source of data for the control group.

**Post-instruments.** Both groups were administered the Generalization Posttests during the last week of the course. As described earlier, students who took Form A at the beginning of the course as a pretest took Form B at the end of the course as a posttest, and vice versa.
Interviews took place from January 31, 2001 to March 14, 2001. The interviews varied in length from approximately 40 minutes to approximately two and one half hours. Most interviews were approximately one hour in length. The interviews were audio-taped by the researcher. All interview candidates had a copy of the interview questions, and were permitted to write on the interview packet if they so desired. The researcher attempted to put each student at ease at the beginning of the interview by thanking them for being willing to participate and by stating that they would be asked them some questions about geometry. The students were told that some of the content would be familiar to them, but that there were some questions that involved content that they had not studied in geometry. The students were asked to try their best to answer each question. The students knew that the results of the interviews could not possibly impact their course grade, as the geometry courses had ended by the time the interviews had begun. Students were permitted to use calculators during the interviews.
CHAPTER 4

DATA ANALYSIS

This chapter presents an analysis of data collected during the research study. The internal consistency reliability of each instrument used in the study is reported. Students' scores on the Entering Geometry Student Test, the Generalization pre- and posttests, as well as results of the interviews are reported. Individual student performances on the interviews are reported in several tables with additional information such as course grades and time spent on the interviews.

RELIABILITY OF INSTRUMENTS

Each of the instruments used in the study, the Entering Geometry Student Test, Geometry Challenge—Form A (as a pretest), Geometry Challenge—Form B (as a pretest), Geometry Challenge—Form A (as a posttest), and Geometry Challenge—Form B (as a posttest), and the interview tasks were examined for internal consistency. Thirty students participated in task-based interviews after the conclusion of the course. The reliability results (Cronbach's Alpha) of all of the instruments in the study are presented in Table 4.1. As all of the values of Cronbach's Alpha are higher than .70, with most being greater than .80, it can be stated that the reliability of the instruments used in the study is of an acceptable value.
Table 4.1. Internal Consistency Reliability of Instruments.

<table>
<thead>
<tr>
<th>INSTRUMENT</th>
<th>ITEMS</th>
<th>STUDENTS</th>
<th>RELIABILITY*</th>
</tr>
</thead>
<tbody>
<tr>
<td>ENTERING GEOMETRY TEST</td>
<td>20</td>
<td>90</td>
<td>.72</td>
</tr>
<tr>
<td>GEOMETRY CHALLENGE — FORM A (PRETEST)</td>
<td>18</td>
<td>42</td>
<td>.81</td>
</tr>
<tr>
<td>GEOMETRY CHALLENGE — FORM B (PRETEST)</td>
<td>18</td>
<td>40</td>
<td>.75</td>
</tr>
<tr>
<td>GEOMETRY CHALLENGE — FORM A (POSTTEST)</td>
<td>18</td>
<td>47</td>
<td>.87</td>
</tr>
<tr>
<td>GEOMETRY CHALLENGE — FORM B (POSTTEST)</td>
<td>18</td>
<td>39</td>
<td>.84</td>
</tr>
<tr>
<td>INTERVIEW TASKS</td>
<td>13</td>
<td>30</td>
<td>.85</td>
</tr>
</tbody>
</table>

*Cronbach’s Alpha

**ENTERING GEOMETRY STUDENT TEST**

The Entering Geometry Student Test (EGST) was given to both the experimental group and the control group on the first day of classes during the fall semester. The purpose of administering this test to both groups was to determine if the groups were similar in their knowledge of geometry at the beginning of the course. The experimental group consisted of 67 students while the control group consisted of 26 students. Three students in the experimental group did not take the EGST due to the fact that they enrolled in the class shortly after the school year began. The EGST was scored on a 100-point scale, with each correct answer worth five points. Results of the EGST are presented in Table 4.2.
Table 4.2. Results of Entering Geometry Student Test (EGST).

<table>
<thead>
<tr>
<th>STUDENTS</th>
<th>MEAN</th>
<th>STD. DEV.</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXPERIMENTAL</td>
<td>64</td>
<td>57.50</td>
</tr>
<tr>
<td>CONTROL</td>
<td>26</td>
<td>51.15</td>
</tr>
</tbody>
</table>

A t-test on the means determined that the difference between the performance on the groups on the EGST was not significant, $t(88) = 1.55, p = .13$.

GENERALIZATION PRETESTS

The Generalization Tests (Geometry Challenge—Form A and Geometry Challenge—Form B) were given to both the experimental group and the control group as pretests early in the course. Approximately half of the students in each group took Form A and the remaining students took Form B. Although there were not significant differences between the groups on the generalization pretests, there was a tendency for the control group to perform better on Form A and the experimental group to perform better on Form B. On Form A, the control group had a tendency to score better than the experimental group ($r = -.16, p = .32$). On Form B, the experimental group had a tendency to score better than the control group ($r = .15, p = .32$). Table 4.3 reports the results of Generalization Pretest Form A and Table 4.4 reports the results of Generalization Pretest Form B.
Table 4.3. Results of Generalization Pretest (Form A).

<table>
<thead>
<tr>
<th>STUDENTS</th>
<th>MEAN</th>
<th>STD. DEV.</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXPERIMENTAL</td>
<td>32</td>
<td>15.31</td>
</tr>
<tr>
<td>CONTROL</td>
<td>10</td>
<td>19.20</td>
</tr>
</tbody>
</table>

Table 4.4. Results of Generalization Pretest (Form B).

<table>
<thead>
<tr>
<th>STUDENTS</th>
<th>MEAN</th>
<th>STD. DEV.</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXPERIMENTAL</td>
<td>32</td>
<td>15.03</td>
</tr>
<tr>
<td>CONTROL</td>
<td>13</td>
<td>10.69</td>
</tr>
</tbody>
</table>

GENERALIZATION POSTTESTS

The posttest scores of seven students (five from the experimental group and two from the control group) were deleted from analysis of posttest data. The researcher made this decision due to the fact that these students' posttest scores were dramatically lower (less than one-third) of their pretest scores. The cause for such a dramatic decrease from pretest to posttest can most likely be attributed to the fact that these students did not give the posttest as much effort and attention as the pretest. Although all students knew that performance on the generalization tests would not affect their course grade, these seven students may have let that fact influence their effort on the posttest. The posttest was given very near the end of the course, so these seven students may not have seen any reason to give it a great deal of effort if it was not going to affect their course grade. Table 4.5
reports the results of Generalization Posttest Form A and Table 4.6 reports the results of Generalization Posttest Form B. Discussion regarding the relatively low means (a score of 50 points was possible) on the generalization posttests is given in Chapter 5.

Table 4.5. Results of Generalization Posttest (Form A).

<table>
<thead>
<tr>
<th>STUDENTS</th>
<th>MEAN</th>
<th>STD. DEV.</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXPERIMENTAL</td>
<td>27</td>
<td>17.44</td>
</tr>
<tr>
<td>CONTROL</td>
<td>10</td>
<td>16.70</td>
</tr>
</tbody>
</table>

Table 4.6. Results of Generalization Posttest (Form B).

<table>
<thead>
<tr>
<th>STUDENTS</th>
<th>MEAN</th>
<th>STD. DEV.</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXPERIMENTAL</td>
<td>28</td>
<td>18.75</td>
</tr>
<tr>
<td>CONTROL</td>
<td>9</td>
<td>16.22</td>
</tr>
</tbody>
</table>

Analysis of covariance (ANCOVA) was used to account for initial differences between the groups when analyzing the posttest results. As there were two forms of the generalization posttests, a separate ANCOVA analysis was conducted for each form of the test. The EGST score and the generalization pretest score were used as covariates.
ANCOVA Results for Generalization Posttest (Form A). The test for homogeneity of regression \( (p = .09) \) resulted in nonsignificant differences between the experimental and control groups. The results of the ANCOVA analysis are presented in Table 4.7.

Table 4.7. Results of ANCOVA for Generalization Posttest (Form A).

<table>
<thead>
<tr>
<th>SOURCE OF VARIATION</th>
<th>SS</th>
<th>DF</th>
<th>MS</th>
<th>F</th>
<th>SIG OF F</th>
</tr>
</thead>
<tbody>
<tr>
<td>WITHIN CELLS</td>
<td>1170.36</td>
<td>27</td>
<td>43.35</td>
<td></td>
<td></td>
</tr>
<tr>
<td>REGRESSION</td>
<td>1097.30</td>
<td>2</td>
<td>548.65</td>
<td>12.66</td>
<td>.00</td>
</tr>
<tr>
<td>GROUP</td>
<td>45.64</td>
<td>1</td>
<td>45.64</td>
<td>1.05</td>
<td>.31</td>
</tr>
</tbody>
</table>

As a significant portion of the variance in posttest scores is accounted for by the two covariates \( (p = .00) \), it makes sense to use them as covariates. Results of the ANCOVA analysis \( (p = .31) \) show that there were no significant differences between the experimental and control groups on Form A of the generalization posttest.
ANCOVA Results for generalization posttest (Form B). The test for homogeneity of regression ($p = .77$) resulted in nonsignificant differences between the experimental and control groups. The results of the ANCOVA analysis are presented in Table 4.8.

Table 4.8. Results of ANCOVA for Generalization Posttest (Form B).

<table>
<thead>
<tr>
<th>SOURCE OF VARIATION</th>
<th>SS</th>
<th>DF</th>
<th>MS</th>
<th>F</th>
<th>SIG OF F</th>
</tr>
</thead>
<tbody>
<tr>
<td>WITHIN CELLS</td>
<td>1961.52</td>
<td>29</td>
<td>67.64</td>
<td></td>
<td></td>
</tr>
<tr>
<td>REGRESSION</td>
<td>838.37</td>
<td>2</td>
<td>419.19</td>
<td>6.20</td>
<td>.01</td>
</tr>
<tr>
<td>GROUP</td>
<td>105.91</td>
<td>1</td>
<td>105.91</td>
<td>1.57</td>
<td>.22</td>
</tr>
</tbody>
</table>

As a significant portion of the variance in posttest scores is accounted for by the two covariates ($p = .01$), it makes sense to use them as covariates. Results of the ANCOVA analysis ($p = .22$) show that there were no significant differences between the experimental and control groups on Form B of the generalization posttest.
INTERVIEW DATA

Baseline data. Fifteen students from each group were interviewed by the researcher on six geometry tasks. Baseline data regarding the two groups is presented in Table 4.9.

Table 4.9. Baseline Data on Interview Subjects.

<table>
<thead>
<tr>
<th></th>
<th>COURSE GRADE</th>
<th>EGST SCORE</th>
<th>GENERALIZATION PRETEST SCORE</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXPERIMENTAL</td>
<td>2.53</td>
<td>61.92 (n = 13)</td>
<td>18.29 (n = 14)</td>
</tr>
<tr>
<td>CONTROL</td>
<td>2.13</td>
<td>55.33 (n = 15)</td>
<td>16.64 (n = 14)</td>
</tr>
</tbody>
</table>

Two students from the experimental group did not take the EGST due to the fact that they enrolled in the course shortly after the school year began. One student from the experimental group did not take the generalization pretest for the same reason. One student from the control group did not take the generalization pretest due to absence from school. Due to the small number of students in the interview groups, a non-parametric test (the Mann-Whitney Test) was performed on course grades, scores on the EGST, and scores on the generalization pretest to test for differences between the groups. On course grades, the Mann-Whitney Test did not yield significant differences, $U = 89.0$, $p = .32$. On the EGST, the Mann-Whitney Test did not yield significant differences, $U = 77.5$, $p = .35$. With regards to the generalization pretest scores, the Mann-Whitney Test again yielded no significant differences between the two groups, $U = 94.5$, $p = .87$. Even though it has already been established that the entire experimental and control groups were similar, these results allow the researcher to make a claim that the interview subgroups from the
experimental and control groups were also similar. All thirty students in the interview subgroups participated in the interviews, so all data was included in the analyses in the upcoming sections.

**Scoring.** The interviews were scored by the researcher by means of sorting student responses into high, medium, and low response categories. A rubric describing essential elements of each level of response as well as sample responses is given for each interview task. Summary data for each group is organized in a table. A chi-square test of independence (Siegel, 1956; Ferguson, 1959; Fraenkel & Wallen, 1990; Hopkins, Glass, & Hopkins, 1987) was performed on the summary data for each task to determine if there was a significant relationship ($p < 0.05$) between group membership and performance on the interview tasks. Siegel (1956) discusses the requirements for the use of the chi-square test. Siegel uses Cochran’s (1954) criteria: fewer than 20 percent of the cells should have an expected frequency of less than 5, and no cells should have an expected frequency of less than 1. The current study violates the first criterion in all cases and the second one in three cases. The researcher followed the recommendations of Roscoe and Briars (1971) which state that the average expected frequency should be equal to $n/k$ where $n$ is the sample size (in this case 30), and $k$ is the number of cells (in this case 6). Following this rule, the average expected frequency for this analysis is five, which is satisfied in the current study.

**Inter-rater reliability.** A colleague of the researcher (a recent graduate of the mathematics education doctoral program at the same university) assisted in documenting inter-rater reliability. Creswell (1994) discusses the need for another investigator to audit key decisions made by the researcher and validate that they were good decisions. The responses of six students (three from the experimental group and three from the control group) were examined by the colleague of the researcher. The six students were randomly selected from the groups. The group membership of each student was unknown to the
colleague. As each interview consisted of thirteen components (one each for Tasks 1 through Task 5, and eight for Task 6), a total of 78 student responses were examined. The researcher and his colleague agreed on 72 out of the 78 responses, for an inter-rater reliability rate of 92.3%. After discussion of the six responses on which there was disagreement, all six differences were resolved. Five of the six researcher’s ratings remained the same and one was changed.

**TASK BY TASK ANALYSIS OF INTERVIEW DATA**

In the next section, each task is presented, including a rubric identifying sample responses for each category, and the results of the chi-square analysis. Following the individual tasks, a summary table of all of the chi-square results is presented.
**TASK 1**

You are working on your Geometry assignment with a friend. The topic is equilateral triangles. Your friend presents the following examples to you regarding a point, E, anywhere in the interior of the triangle, and the perpendicular segments constructed from E to each side of the triangle. Your friend states: "I think I have come up with something interesting regarding point E." Can you study the examples and try to find out what your friend has discovered?

**EXAMPLE 1**

```
K
|   |
|___|
Y   1.6 cm
|     |
|     |
|___|
E  1.1 cm
|     |
|     |
|___|
M   O   I
```

**EXAMPLE 2**

```
K
|   |
|___|
Y   3.9 cm
|     |
|     |
|___|
E  3.2 cm
|     |
|     |
|___|
M   O   I
```

**EXAMPLE 3**

```
K
|   |
|___|
Y   1.5 cm
|     |
|     |
|___|
E  1.7 cm
|     |
|     |
|___|
M   O   I
```

**EXAMPLE 4**

```
K
|   |
|___|
Y   2.1 cm
|     |
|     |
|___|
E  2.0 cm
|     |
|     |
|___|
M   O   I
```

Study the evidence presented by your friend in Examples 1, 2, and 3. What has your friend discovered? Find the missing value in Example 4.
RUBRIC FOR TASK 1

HIGH The student found the missing value of 1.8 cm. in Example 4. The student may or may not have made other observations.

Sample response(s):

• The student finds the correct value of 1.8 cm. almost immediately without looking at any other relationships.
• The student experiments and finds results such as $YE = (1/2) ET$ in Example 1, but sees that this relationship is not true in the other examples. The student then finds the correct value of 1.8 cm. in Example 4.

MEDIUM The student did not find the missing value of 1.8 cm. The student made one or more correct observations while attempting to find the missing value.

Sample response(s):

• The student experiments and finds results such as $YE = (1/2) ET$ in Example 1 or $3 * OE = ET$ in Example 2, but fails to find the missing value in Example 4.

LOW The student did not find the missing value of 1.8 cm. The student may have made limited observations, but the observations were based on reasoning that led to an incorrect value for Example 4.

Sample response(s):

• The missing value is 2.2 because it looks like there is a pattern of how the lengths are increasing in Example 4.
• It looks like $EO$ on Examples 1 and 2 added together is about the same length as $EO$ on Example 4.
• I have no clue what I'm doing here. I can't find anything in common.
The overall results of the interviews for Task 1 are presented in Table 4.10. The actual values for each cell are given in bold, while the expected values for each cell for the chi-square computation is given in parentheses.

Table 4.10. Interview Results for Task 1.

<table>
<thead>
<tr>
<th></th>
<th>HIGH</th>
<th>MEDIUM</th>
<th>LOW</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXPERIMENTAL</td>
<td>10</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>CONTROL</td>
<td>6</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

\[ \chi^2 (2, N = 30) = 3.66, p > .05 \]

A chi-square (\(\chi^2\)) test was performed on the categorical data to test for a relationship between the variables. The value of chi-square does not show a significant relationship between group membership and performance on Task 1.
**TASK 2**

You are working on your Geometry assignment with a friend. The topic you are studying is the various angles that are formed when two segments intersect inside a circle. Two results that you discovered in class are as follows:

**THEOREM 1:** In a circle, the measure of the angle formed by intersecting chords is one-half the sum of the measures of the intercepted arcs.

For the given illustration, the measure of angle \( \angle DEB \) is equal to one-half the sum of the measures of arcs \( AC \) and \( DB \). An example of Theorem 1 is as follows:

**EXAMPLE 1**

How could you interpret **THEOREM 1** in terms of **EXAMPLE 1**?

**THEOREM 2:** In a circle, the measure of an inscribed angle is one-half the measure of the intercepted arc.

For the given illustration, the measure of angle \( \angle PQR \) is one-half the measure of arc \( PR \). An example of Theorem 2 is as follows:

**EXAMPLE 2**

Your friend makes the following observation: Theorems 1 and 2 seem somehow to be related, but I cannot seem to make the connection. Do you think your friend is correct? If so, what is the connection between Theorem 1 and Theorem 2?
RUBRIC FOR TASK 2

HIGH  The student discussed movement of point E to the circle and/or adding the two smaller arcs together. The student stated or implied that in Theorem 2 there was an arc with a measure of zero degrees. One or the other claim was necessary to receive a high rating.

Sample response(s):

* If arc AC was added on to arc DB, you would have an arc of 105.6°. Half of this arc measures 52.8°, which is the measure of the vertical angles. The student has moved point E to the circle, with the combined arcs as the intercepted arc. The inscribed angle measures one-half of the intercepted arc.
* If we move point E to the circle, the measure of the inscribed angle would be half of 68.1°.
* Theorem 2 is like Theorem 1 because one of the arcs is zero, since it's a point.

MEDIUM  The student's response contains some elements of a high response, but it is not clear enough to be put into that category. The response may contain some incorrect and/or irrelevant statements.

Sample response(s):

* In Theorem 2, you have a big arc and you cut out the process of combining the two arcs.
* In Theorem 1, you add the two arcs up and divide by two. In Theorem 2, you already have the sum. You don't have two different numbers, you have one solid number.

LOW  The only connection that the student is able to make is that in both theorems you divide by two to get the result. Some students stated that they did not see a connection between the two theorems. Some students simply restated the theorems.

Sample response(s):

* In both theorems you divide by two to get the desired angle measurement.
* For Theorem 1, you have to add two arcs up to be able to divide it in half. For Theorem 2, you just divide it in half. Theorem 2 has less steps.
The student's response to the first question (How can you interpret THEOREM 1 in terms of EXAMPLE 1?) was not used to classify the response into a response category. It was what the student said later, about connecting the theorems, that was used to classify their response. The overall results of the interviews for Task 2 are presented in Table 4.11.

Table 4.11. Interview Results for Task 2.

<table>
<thead>
<tr>
<th></th>
<th>HIGH</th>
<th>MEDIUM</th>
<th>LOW</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXPERIMENTAL</td>
<td>6 (4.5)</td>
<td>5 (4)</td>
<td>4 (6.5)</td>
</tr>
<tr>
<td>CONTROL</td>
<td>3 (4.5)</td>
<td>3 (4)</td>
<td>9 (6.5)</td>
</tr>
</tbody>
</table>

$\chi^2 (2, N = 30) = 3.42, p > .05$

The value of chi-square does not show a significant relationship between group membership and performance on Task 2.
TASK 3

You are working on your Geometry assignment with a friend. The topic is circles. Your friend presents the following examples to you regarding a point, P, in the interior of the circle and the segment that passes through point P and intersects the circle at points T and S. Your friend states: “I think I have come up with something interesting regarding point P.” Can you study the examples and try to find out what your friend has discovered?

EXAMPLE 1

EXAMPLE 2

EXAMPLE 3

EXAMPLE 4

Study the evidence presented by your friend in Examples 1, 2, and 3. What has your friend discovered? Find the missing value in Example 4.
RUBRIC FOR TASK 3

HIGH
The student found the missing value of approximately 4.7 cm. to 4.9 cm. in Example 4. The student may or may not have made other observations.

Sample response(s):

*The student tried the ratio of the shorter segment to the longer segment for each one and sees that this is not the same. Then she multiplies PT by PS for Examples 1, 2, and 3 and finds that the result is always in the neighborhood of 4.8 cm.
*When I multiplied the two parts together, the answer was always about 4.8.
*By multiplying the numbers together and dividing the answer by two, I found that the result is always about 2.4.

MEDIUM
The student did not find the missing value of approximately 4.7 cm. to 4.9 cm. in Example 4. The student experimented moderately to extensively while attempting to find the missing value.

Sample response(s):

*At first, I thought it was going to be like the first one, where they added up to the same number, but it's not that. The student tried many different computations, but did not arrive at the correct result. Some additional examples of what students tried include checking to see if the shorter segment (PT) was proportionally related to the longer segment (PS) or subtracting the two lengths to see if there was a common answer.

LOW
The student did not find the missing value of approximately 4.7 cm. to 4.9 cm. in Example 4. The student experimented minimally while attempting to find the missing value.

Sample response(s):

*I think PS is 2/3 of TS in Example 1. This is the only idea that this student explored before stating that she could not find anything.
*PS is different in each picture, and I don't think there is any way to find it.
*I added up the numbers to see if it was the same in each example, but it wasn't.
The overall results of the interviews for Task 3 are presented in Table 4.12.

**Table 4.12. Interview Results for Task 3.**

<table>
<thead>
<tr>
<th></th>
<th>HIGH</th>
<th>MEDIUM</th>
<th>LOW</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXPERIMENTAL</td>
<td>3 (1.5)</td>
<td>6 (4)</td>
<td>6 (9.5)</td>
</tr>
<tr>
<td>CONTROL</td>
<td>0 (1.5)</td>
<td>2 (4)</td>
<td>13 (9.5)</td>
</tr>
</tbody>
</table>

$\chi^2 (2, N = 30) = 7.58, p < .05$

The value of chi-square shows a significant relationship between group membership and performance on Task 3.
TASK 4

You are working on your Geometry assignment with a friend. The topic you are studying is midsegments of trapezoids and triangles. Two results that you discovered in class are as follows:

**THEOREM 1:** In a trapezoid, the length of the midsegment is one-half the sum of the lengths of the bases.

For the given illustration, the length of segment EF is one-half of the sum of the lengths of segments AB and DC. An example of Theorem 1 is as follows:

**EXAMPLE 1**

![Diagram of a trapezoid with midsegment](image)

How could you interpret **THEOREM 1** in terms of **EXAMPLE 1**?

**THEOREM 2:** In a triangle, the length of the midsegment is one-half of the length of the side of the triangle to which it is parallel.

For the given illustration, the length of segment ST is one-half of the length of segment QR. An example of Theorem 2 is as follows:

**EXAMPLE 2**

![Diagram of a triangle with midsegment](image)

Your friend makes the following observation: Theorems 1 and 2 seem somehow to be related, but I cannot seem to make the connection. Do you think your friend is correct? If so, what is the connection between Theorem 1 and Theorem 2?
RUBRIC FOR TASK 4

HIGH
The student relates the theorems in some meaningful way. Typical responses include transforming one figure to become the other or relating the computation method by a stated or implied zero length for the top base of the trapezoid.

Sample response(s):
• Theorems 1 and 2 are related because if you stretch the triangle by creating a segment on the top, you turn it into a trapezoid. You could call the top x. You could use \((x + QR)/2\) to get the midsegment. Also, in Theorem 1, you could compress AB into one point. Then you would have a triangle, and the midsegment EF would be one-half of AC.
• The triangle theorem is the same as the trapezoid theorem because you are still adding two numbers together and dividing the result by two. The other number happens to be zero.
• Again like last time (Task 2), it is almost the same except that in a triangle it is a point, therefore the whole measurement is QR, which has to be divided to get the midsegment. In Theorem 1, you have two lengths. As B gets closer to A (when A and B coincide), DC will have expanded to be twice as long as EF.

MEDIUM
The student’s response contains some elements of a high response, but it was not explicit enough to be considered in that category. Some students noticed that by drawing the midsegment in a triangle, you create a trapezoid in the bottom half.

Sample response(s):
• I see a trapezoid in the bottom of the triangle. If you put in a midsegment, you can figure out the length of it using Theorem 1.
• In Theorem 1, you could extend sides DA and CB to make a triangle. Then you could find the midpoints of the sides and you would be able to compute the midsegment.
• For the first theorem, you are taking the two bases and adding them together to get a total. In the second theorem, you have the total already, it is just the one base.
The only connection that the student is able to make is that in both theorems you divide by two to get the result. Some students stated that they did not see a connection between the two theorems. Some students simply restated the theorems.

Sample response(s):

• I don't think they are connected, because you don't have to do anything in the second theorem. You just take half. This is the only connection, you use half in both of them.
• The connection is that ST is parallel to QR and half as long as QR. In Theorem 1, EF is parallel to AB and DC and is one-half of the combined lengths.
• I don't know if you would call it a connection, but in Theorem 1, you add up the bases and then divide by two to get the midsegment, and in Theorem 2, you just divide the base by two and you have the midsegment. I wouldn't really call that a connection, but that's just how you get them.

The student’s response to the first question (How can you interpret THEOREM 1 in terms of EXAMPLE 1?) was not used to classify the response into a response category. It was what the student said later, about connecting the theorems, that was used to classify their response. The overall results of the interviews for Task 4 are presented in Table 4.13.

Table 4.13. Interview Results for Task 4.

<table>
<thead>
<tr>
<th>Group</th>
<th>HIGH</th>
<th>MEDIUM</th>
<th>LOW</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXPERIMENTAL</td>
<td>7 (5)</td>
<td>6 (4)</td>
<td>2 (6)</td>
</tr>
<tr>
<td>CONTROL</td>
<td>3 (5)</td>
<td>2 (4)</td>
<td>10 (6)</td>
</tr>
</tbody>
</table>

χ² (2, N = 30) = 8.93, p < .02

The value of chi-square shows a significant relationship between group membership and performance on Task 4.
TASK 5

You are working on your Geometry assignment with a friend. The topic is cyclic quadrilaterals. Your friend presents the following examples to you regarding cyclic quadrilaterals. Your friend states: "I think I have come up with something interesting regarding cyclic quadrilaterals." Can you study the examples and try to find out what your friend has discovered?

EXAMPLE 1

EXAMPLE 2

EXAMPLE 3

EXAMPLE 4

Study the evidence presented by your friend in Examples 1, 2, 3, and 4. What relationships do you see?
RUBRIC FOR TASK 5

HIGH
The student finds that opposite angles of a cyclic quadrilateral are supplementary or a related idea or two other true observations.

Sample response(s):

• The opposite angles are supplementary on all four of the figures.
• The difference between angles Q and U in Example 4 is 7.7°. Also, the difference between angles D and A in Example 4 is 7.7°. I ask if there is something like this going on in the other examples. She explores and finds this to be true in every example. This is due to the opposite angles being supplementary, which she did not state.
• In every example, the longest of the four segments points to the smallest angle. The smallest segment goes with the biggest angle measure.

MEDIUM
Finds one true observation.

Sample response(s):

• The sum of all four angles of the quadrilateral is 360°.
• Point U has moved in each case to make angle U smaller.
• All of the quadrilaterals look like kites; there are two longer sides and two shorter sides.
• In Examples 1 to 4, DM decreases and angle D increases.
• The smallest angle is across from the largest angle in each example.

LOW
Does not make any true observations.

Sample response(s):

• I think possibly that in Example 3, QM is an angle bisector of angle Q. I tell her that we don't really have any evidence given that would allow us to conclude that. She asks if we did this in class. I tell her that we did not.
• I can't find anything that is in common on all four drawings.
• The smaller angles seem to be almost the same and the larger angles seem to be almost the same.
• I don't really have any observations.
• I'm starting to wonder who this friend is and why we are still friends!
The overall results of the interviews for Task 5 are presented in Table 4.14.

### Table 4.14. Interview Results for Task 5.

<table>
<thead>
<tr>
<th></th>
<th>HIGH</th>
<th>MEDIUM</th>
<th>LOW</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXPERIMENTAL</td>
<td>11 (7)</td>
<td>3 (5.5)</td>
<td>1 (2.5)</td>
</tr>
<tr>
<td>CONTROL</td>
<td>3 (7)</td>
<td>8 (5.5)</td>
<td>4 (2.5)</td>
</tr>
</tbody>
</table>

$\chi^2 (2, N = 30) = 8.64, p < .02$

The value of chi-square shows a significant relationship between group membership and performance on Task 5.
**TASK 6**

The sixth task is comprised of eight components. Accordingly, data is reported separately for the eight components of Task 6 first, and then overall results for Task 6 are given. The focus of Task 6 was to explore student understanding of the area interpretation of the Pythagorean Theorem. The task began with asking students to interpret a diagram of a right triangle with squares constructed on each side, followed by asking students to consider if the area argument still worked if the squares were cut in half. Then, the students were asked to consider if any rectangles could be constructed on the sides of the right triangle and still have the area argument hold. The drawing that was shown to the students purposely had a half-square rectangle constructed on one of the legs, a one-fourth square rectangle on the other leg, and a one-third square rectangle constructed on the hypotenuse. Students were then asked if they thought any other figures could be constructed on the sides of the right triangle and still have the area argument hold. Next, students were asked to consider isosceles triangles constructed on the sides. The following question asked students to consider if any triangles could be constructed on the sides and have the relationship hold. The final parts of the task were to ask students if they thought that trapezoids could be constructed on each side and have the relationship hold, and finally if there was anything else they could think of that would work that we had not discussed.
You are working on your Geometry assignment with a friend. The topic you are studying is the Pythagorean Theorem.

**THE PYTHAGOREAN THEOREM:** In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs.

How could you interpret the Pythagorean Theorem in terms of the above diagram?
RUBRIC FOR TASK 6 (Part I)

HIGH  The students correctly explains the theorem and correctly gives the meaning of the numbers 36, 64, and 100 without assistance.

Sample response(s):

• The theorem states that $6^2 + 8^2 = 10^2$. The 64 represents the area of this square (points to square DACE). The theorem means that this area (area of square CBGF) plus the area of this square (area of square DACE) is equal to the area of this square (square AIHB).

MEDIUM  The student correctly explains the theorem, but needs help coming up with the idea that the theorem is about area. The sample responses are responses to the question “What do you think the 36, 64, and 100 represent?”

Sample response(s):

• Some students first stated that these numbers represented perimeter, but then realized they could not represent perimeter. The students then correctly identified these values as the areas of the squares.

LOW  The student needs help in recalling the theorem and in understanding the area argument.

Sample response(s):

• I can’t really remember what the theorem says.
The results of the interviews for Task 6 (Part I) are presented in Table 4.15.

Table 4.15. Interview Results for Task 6 (Part I).

<table>
<thead>
<tr>
<th></th>
<th>HIGH</th>
<th>MEDIUM</th>
<th>LOW</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXPERIMENTAL</td>
<td>6 (4.5)</td>
<td>8 (7)</td>
<td>1 (3.5)</td>
</tr>
<tr>
<td>CONTROL</td>
<td>3 (4.5)</td>
<td>6 (7)</td>
<td>6 (3.5)</td>
</tr>
</tbody>
</table>

χ² (2, N = 30) = 4.86, p > .05

The value of chi-square does not show a significant relationship between group membership and performance on Task 6 (Part I).
**TASK 6 (Part II)**

Your friend poses the following question to you: What if the figures constructed on the sides were something other than squares? What if we cut each of these squares in half so that we have a rectangle on each side? What, if anything, could you say about the relationship?
RUBRIC FOR TASK 6 (Part II)

HIGH  The students correctly explains the effect of cutting the squares in half. There are no inconsistencies in the student's response.

Sample response(s):

• The areas are 18, 32, and 50, and 18 + 32 = 50.
• The sum of the areas of the two smaller rectangles would still give you the area of the larger rectangle. Computes and confirms areas.
• By cutting the squares in half, you also cut the areas in half; 18 + 32 = 50.

MEDIUM  The student has some inconsistencies or irrelevant ideas in the response, but eventually confirms the area argument.

Sample response(s):

• If you take half of 6, 8, and 10, you have 3, 4, and 5, and $3^2 + 4^2 = 5^2$.
• If you cut the squares in half, you wouldn't necessarily cut the areas in half. Then he figures the areas and finds that they do work.

LOW  The student does not think the area argument will work.

Sample response(s):

• It wouldn't be the same, because the areas would change.
The results of the interviews for Task 6 (Part II) are presented in Table 4.16.

Table 4.16. Interview Results for Task 6 (Part II).

<table>
<thead>
<tr>
<th></th>
<th>HIGH</th>
<th>MEDIUM</th>
<th>LOW</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXPERIMENTAL</td>
<td>12 (10)</td>
<td>3 (4.5)</td>
<td>0 (0.5)</td>
</tr>
<tr>
<td>CONTROL</td>
<td>8 (10)</td>
<td>6 (4.5)</td>
<td>1 (0.5)</td>
</tr>
</tbody>
</table>

\[ \chi^2 (2, N = 30) = 2.80, p > .05 \]

The value of chi-square does not show a significant relationship between group membership and performance on Task 6 (Part II).
Could I draw any rectangles on the sides and still have the relationship hold?
RUBRIC FOR TASK 6 (Part III)

**HIGH**
The student explicitly states that the squares must be cut proportionally or that the rectangles must be similar.

*Sample response(s):*

*You can't put any rectangles on the sides and have the relationship hold. They have to be proportional or else the area of the smaller rectangles won't add up to the area of the rectangle off of the hypotenuse.  
*The rectangles must be similar in order for the area argument to work.*

**MEDIUM**
The student confirms that the area argument does not work for this particular drawing, but fails to state that it can work for other rectangles than the half-square rectangles.

*Sample response(s):*

*No, you can't put any rectangles on the sides. On this drawing, this one (rectangle CBPQ) might be 1/4, this one (rectangle ASTB) might be 1/3, and this one (rectangle JACK) might be 1/2.  
*Any rectangles will not work because the side lengths are just random amounts and the others were specific. We took half of the squares.*

**LOW**
The student thinks the area argument will work for any rectangles.

*Sample response(s):*

*Yes, it will work for any rectangles, because that is what the theorem is stating.  
*Yes, you could draw any rectangles. As long as the Pythagorean Theorem holds true, then it wouldn't matter what size rectangles you draw.*
The results of the interviews for Task 6 (Part III) are presented in Table 4.17.

Table 4.17. Interview Results for Task 6 (Part III).

<table>
<thead>
<tr>
<th></th>
<th>HIGH</th>
<th>MEDIUM</th>
<th>LOW</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXPERIMENTAL</td>
<td>12 (10)</td>
<td>3 (4.5)</td>
<td>0 (0.5)</td>
</tr>
<tr>
<td>CONTROL</td>
<td>8 (10)</td>
<td>6 (4.5)</td>
<td>1 (0.5)</td>
</tr>
</tbody>
</table>

\[ \chi^2(2, N = 30) = 10.00, p < .01 \]

The value of chi-square shows a significant relationship between group membership and performance on Task 6 (Part III).
TASK 6 (Part IV)

We have seen that the Pythagorean Theorem holds for squares and some kinds of rectangles. Are there any other figures we could put on the sides so that the relationship would still hold?
RUBRIC FOR TASK 6 (Part IV)

HIGH

The student suggests a figure that will work and there are no inconsistencies in their response. The student may suggest that any similar figures will work.

Sample response(s):

• I think that any figures could go on the sides as long as they were proportional.
• Equilateral triangles should work. Proceeds to compute the areas and finds out that it does work.
• I think we could put triangles on each side, ones where the height is equal to the base.

MEDIUM

The student thinks that there may be some other figures that will work, but does not suggest how or there are some inconsistencies in the student's response.

Sample response(s):

• I think it is possible, but you'd have to spend a lot of time figuring it out.
• The student draws circles attached to each vertex. After being asked how the theorem still applies, she decides to change how they are drawn. Now she has them drawn using the sides of the triangles as diameters of the circles.

LOW

The student states that no other figures will work or suggests something that does not work.

Sample response(s):

• I don't think anything else would work.
• I think that maybe a rhombus will work. He draws a rhombus on each side. Then he adds the heights to the figures (5, 7, and 6, respectively, to go with the sides of 6, 8, and 10). This does not work.
• I think maybe a rhombus will work. The sketches appear to be more like non-rhombus parallelograms. I give her the area formula and ask her to assign heights to each figure. She picks 8, 10, and 12 for the heights. We compute the areas and see that the area argument does not hold. I asked her if she thinks it is possible, based on this information. She says she thinks it won't work.
The results of the interviews for Task 6 (Part IV) are presented in Table 4.18.

Table 4.18. Interview Results for Task 6 (Part IV).

<table>
<thead>
<tr>
<th></th>
<th>HIGH</th>
<th>MEDIUM</th>
<th>LOW</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXPERIMENTAL</td>
<td>9 (6.5)</td>
<td>4 (2.5)</td>
<td>2 (6)</td>
</tr>
<tr>
<td>CONTROL</td>
<td>4 (6.5)</td>
<td>1 (2.5)</td>
<td>10 (6)</td>
</tr>
</tbody>
</table>

\[ \chi^2 (2, N = 30) = 9.06, p < .02 \]

The value of chi-square shows a significant relationship between group membership and performance on Task 6 (Part IV).
TASK 6 (Part V)

Consider the following diagram. Suppose we drew an isosceles right triangle on each side. What can you say about the relationship?
RUBRIC FOR TASK 6 (Part V)

**HIGH**
The student computes the areas and state that the relationship holds for isosceles right triangles.

*Sample response(s):*

• These areas are half of the original squares. He proceeds to identify the correct areas and states that $18 + 32 = 50$, so the area argument still holds.

**MEDIUM**
The student has inconsistencies in their response.

*Sample response(s):*

• The triangles would be similar since they share numbers. After figuring out the length of the hypotenuse on each side (with decimals), he concludes that the triangles are not similar since the numbers are not similar.

**LOW**
The students states that the area argument does not work for isosceles right triangles.

*Sample response(s):*

• None.
The results of the interviews for Task 6 (Part V) are presented in Table 4.19.

Table 4.19. Interview Results for Task 6 (Part V).

<table>
<thead>
<tr>
<th></th>
<th>HIGH</th>
<th>MEDIUM</th>
<th>LOW</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXPERIMENTAL</td>
<td>15 (14)</td>
<td>0 (1)</td>
<td>0 (0)</td>
</tr>
<tr>
<td>CONTROL</td>
<td>13 (14)</td>
<td>2 (1)</td>
<td>0 (0)</td>
</tr>
</tbody>
</table>

$\chi^2 (1, N = 30) = 2.14$, $p > .05$

The value of chi-square does not show a significant relationship between group membership and performance on Task 6 (Part V). Since there were zeros in the third column, the data was treated as a 2 X 2 table for the chi-square analysis.
TASK 6 (Part VI)

Could I draw any triangles on the sides and still have the relationship hold?
<table>
<thead>
<tr>
<th>RUBRIC FOR TASK 6 (Part VI)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>HIGH</strong></td>
</tr>
<tr>
<td><strong>Sample response(s):</strong></td>
</tr>
<tr>
<td>• <em>You can’t draw any triangles, they have to be proportional.</em></td>
</tr>
<tr>
<td>• <em>Just any triangles would not work, because they may not be similar.</em></td>
</tr>
<tr>
<td><strong>MEDIUM</strong></td>
</tr>
<tr>
<td><strong>Sample response(s):</strong></td>
</tr>
<tr>
<td>• <em>Scalene right triangles would work, with legs of 3, 4, and 5, respectively.</em></td>
</tr>
<tr>
<td>• <em>He draws triangles with heights of 4, 6, and 8. This gives areas of 12, 24, and 40, and the relationship doesn’t hold. He thinks it will only work if the base and height are the same.</em></td>
</tr>
<tr>
<td>• <em>I think if we put equilateral triangles on each side it will work, but I don’t think any other type of triangles will work.</em></td>
</tr>
<tr>
<td><strong>LOW</strong></td>
</tr>
<tr>
<td><strong>Sample response(s):</strong></td>
</tr>
<tr>
<td>• <em>Tries putting equilateral triangles on each side, but then concludes that this will not work. Then puts a scalene right triangle on each side, but concludes that this will not work either.</em></td>
</tr>
<tr>
<td>• <em>Wants to draw equiangular triangles on each side and thinks that this may work. I help her to compute the areas and she sees that it does work. She then concludes that you could draw any triangles on the sides and the theorem will still hold.</em></td>
</tr>
<tr>
<td>• <em>I don’t think any other triangles would work.</em></td>
</tr>
<tr>
<td>• <em>I think it would only work for the specific triangles that we have looked at already.</em></td>
</tr>
</tbody>
</table>
The results of the interviews for Task 6 (Part VI) are presented in Table 4.20.

### Table 4.20. Interview Results for Task 6 (Part VI).

<table>
<thead>
<tr>
<th></th>
<th>HIGH</th>
<th>MEDIUM</th>
<th>LOW</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXPERIMENTAL</td>
<td>3 (2)</td>
<td>12 (10)</td>
<td>0 (3)</td>
</tr>
<tr>
<td>CONTROL</td>
<td>1 (2)</td>
<td>8 (10)</td>
<td>6 (3)</td>
</tr>
</tbody>
</table>

\[ \chi^2 (2, N = 30) = 7.80, p < .05 \]

The value of chi-square shows a significant relationship between group membership and performance on Task 6 (Part VI).
TASK 6 (Part VII)

Consider the following diagram. How could we construct trapezoids on each side so that the relationship will still hold? What can you say about the relationship?
RUBRIC FOR TASK 6 (Part VII)

HIGH  The student states that it can be done and gives appropriate values to confirm the area argument or states that it can be done as long as the trapezoids are similar.

Sample response(s):

- If the sides were all 3 on the smallest trapezoid, all 4 on the medium trapezoid, and all 5 on the largest trapezoid, this would work.
- As long as the trapezoids are similar, it would work.
- We can put trapezoids on as long as they are proportional.
- She puts proportional values on the sides of the trapezoid. On the small one, she puts legs of 4.5 and a shorter base of 3. On the medium one, she puts legs of 6 and a shorter base of 4. On the large one, she puts legs of 7.5 and a shorter base of 5. She is correct, these values will make the trapezoids similar to each other.

MEDIUM  The student thinks it can be done, but is not sure how.

Sample response(s):

- I don't think it can be done with trapezoids. Well, maybe it can. The area would be a certain percentage of the squares. I think that if a square and a triangle worked, then it would work with trapezoids.
- I think it will work. He gives incorrect values, but thinks it is still possible, but cannot think of anything that would work.

LOW  The student thinks it cannot be done.

Sample response(s):

- I don’t think it will hold. It’s just a feeling, but I don’t think it will hold.
- I don’t think trapezoids would work.
- I’m not sure if it’s possible to put trapezoids on each side. No, I don’t think it’s possible.
- She gives lengths of 4, 6, and 8 to the shorter bases and lengths of 3, 4, and 5 to the heights. The corresponding areas are 15, 28, and 45. She doesn’t think it would work for trapezoids.
The results of the interviews for Task 6 (Part VII) are presented in Table 4.21.

Table 4.21. Interview Results for Task 6 (Part VII).

<table>
<thead>
<tr>
<th></th>
<th>HIGH</th>
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<th>LOW</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXPERIMENTAL</td>
<td>10 (7)</td>
<td>4 (3.5)</td>
<td>1 (4.5)</td>
</tr>
<tr>
<td>CONTROL</td>
<td>4 (7)</td>
<td>3 (3.5)</td>
<td>8 (4.5)</td>
</tr>
</tbody>
</table>

$\chi^2 (2, N = 30) = 8.16, p < .02$

The value of chi-square shows a significant relationship between group membership and performance on Task 6 (Part VII).
**TASK 6 (Part VIII)**

Now we’ve looked at squares, some rectangles, some triangles, and even trapezoids. Are there any other shapes we could put on each side so that the relationship will still hold?
RUBRIC FOR TASK 6 (Part VIII)

HIGH  The student states that there are other shapes and gives an example.

Sample response(s):

• I think that a rhombus would work since it is a tilted square. He draws rhombi on each side with sides of 6, 8, and 10, and gives angle measurements of 52, and 128 for each rhombus.
• I think semicircles would work on each side. He showed that the computation worked out for the areas of the semicircles.
• You could put anything on the sides as long as they are proportional to each other.
• You could put any shapes on the outside, as long as they are similar.
• I think parallelograms would be ok. She draws in parallelograms and I ask her to mark them any way she likes. She marks all of the angles to be congruent to each other and marks the short sides of the parallelograms to be equal to 3, 4, and 5, respectively.
• Figures other than regular figures would work out, as long as they are similar. It would be harder to figure out, but they would work.

MEDIUM  The student thinks that there are other shapes, but does not give an example or has some inconsistencies in their response.

Sample response(s):

• I think that maybe we could put a rhombus on each side. Perhaps an arc.

LOW  The student does not think it would work for any other shapes.

Sample response(s):

• I don't think a diamond shape will work. I can't think of anything else that would work.
• I don't think anything else will work.
• Maybe a kite. No, that wouldn't work. I don't think anything else will work.
The results of the interviews for Task 6 (Part VIII) are presented in Table 4.22.

Table 4.22. Interview Results for Task 6 (Part VIII).

<table>
<thead>
<tr>
<th></th>
<th>HIGH</th>
<th>MEDIUM</th>
<th>LOW</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXPERIMENTAL</td>
<td>9 (6)</td>
<td>4 (3)</td>
<td>2 (6)</td>
</tr>
<tr>
<td>CONTROL</td>
<td>3 (6)</td>
<td>2 (3)</td>
<td>10 (6)</td>
</tr>
</tbody>
</table>

χ² (2, N = 30) = 9.00, p < .02

The value of chi-square shows a significant relationship between group membership and performance on Task 6 (Part VIII).
**TASK 6 (OVERALL RESULTS)**

Based on the categories for the eight parts of Task 6, students were classified into High, Medium, and Low overall performances for Task 6. Nearly all students performed well on the isosceles right triangle component of Task 6 (Part V), making it possible to disregard this information when considering overall performance on Task 6. This approach left seven parts of Task 6. The researcher determined which level of response was most characteristic of each student's performance on Task 6 by examining the results of the other seven parts. For example, a student who was in the HIGH response category on all seven of the remaining components of Task 6 was scored an overall HIGH on Task 6. A student who was in the HIGH response category for four of the remaining seven components of Task 6 and in the MEDIUM response category for the other three components was also scored an overall HIGH on Task 6. The results of students' overall performance on Task 6 are presented in Table 4.23.

**Table 4.23. Overall Results for Task 6.**

<table>
<thead>
<tr>
<th></th>
<th>HIGH</th>
<th>MEDIUM</th>
<th>LOW</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXPERIMENTAL</td>
<td>8 (5)</td>
<td>7 (7.5)</td>
<td>0 (2.5)</td>
</tr>
<tr>
<td>CONTROL</td>
<td>2 (5)</td>
<td>8 (7.5)</td>
<td>5 (2.5)</td>
</tr>
</tbody>
</table>

\[ \chi^2 (2, N = 30) = 8.67, p < .02 \]

The value of chi-square shows a significant relationship between group membership and overall performance on Task 6.
OVERALL TASK RESULTS

The overall performance of the experimental and control groups on the interview tasks are summarized in Table 4.24. The reader may refer to Table 4.26 (Experimental Group) and Table 4.27 (Control Group) for detailed summaries of how each student performed on each task.

Table 4.24. Overall Results for Interview Tasks.

<table>
<thead>
<tr>
<th></th>
<th>HIGH</th>
<th>MEDIUM</th>
<th>LOW</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXPERIMENTAL</td>
<td>6 (4)</td>
<td>8 (6.5)</td>
<td>1 (4.5)</td>
</tr>
<tr>
<td>CONTROL</td>
<td>2 (4)</td>
<td>5 (6.5)</td>
<td>8 (4.5)</td>
</tr>
</tbody>
</table>

$\chi^2 (2, N = 30) = 8.14, p < .02$

The value of chi-square shows a significant relationship between group membership and overall performance on the Interview Tasks.
**SUMMARY OF CHI-SQUARE STATISTICS**

The values of chi-square that have been reported for each individual task, the separate components of Task 6, overall performances on Task 6, and overall performances on the interview tasks are summarized in Table 4.25.

Table 4.25. Summary of Chi-square Statistics.

<table>
<thead>
<tr>
<th>TASK</th>
<th>$\chi^2$-VALUE</th>
<th>$p$-VALUE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.66</td>
<td>.16</td>
</tr>
<tr>
<td>2</td>
<td>3.42</td>
<td>.18</td>
</tr>
<tr>
<td>3</td>
<td>7.58</td>
<td>.023*</td>
</tr>
<tr>
<td>4</td>
<td>8.93</td>
<td>.011**</td>
</tr>
<tr>
<td>5</td>
<td>8.64</td>
<td>.013**</td>
</tr>
<tr>
<td>6I</td>
<td>4.86</td>
<td>.09</td>
</tr>
<tr>
<td>6II</td>
<td>2.80</td>
<td>.25</td>
</tr>
<tr>
<td>6III</td>
<td>10.00</td>
<td>.007***</td>
</tr>
<tr>
<td>6IV</td>
<td>9.06</td>
<td>.011**</td>
</tr>
<tr>
<td>6V</td>
<td>2.14</td>
<td>.14</td>
</tr>
<tr>
<td>6VI</td>
<td>7.80</td>
<td>.0202*</td>
</tr>
<tr>
<td>6VII</td>
<td>8.16</td>
<td>.017**</td>
</tr>
<tr>
<td>6VIII</td>
<td>9.00</td>
<td>.011**</td>
</tr>
<tr>
<td>6 (OVERALL)</td>
<td>8.67</td>
<td>.013**</td>
</tr>
<tr>
<td>OVERALL TASKS</td>
<td>8.14</td>
<td>.017**</td>
</tr>
</tbody>
</table>

*p < .05. **p < .02. ***p < .01.

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INDIVIDUAL STUDENT PERFORMANCES

Individual student performances on each of Tasks 1 to 5, as well as overall performance on Task 6 are summarized in Tables 4.26 and 4.27. The first character in the student identification indicates group membership (E or C). The second character indicates the student's relative position in the group (1 to 15). The third character (a letter) distinguishes between individuals tied in certain ranks. All students received a third character for consistency. Individuals were ranked for comparison within the group (explained below). There were several ties in ranking, and these students' records are grouped together in the table.

Students' individual performances were examined in order to determine an overall classification of HIGH, MEDIUM, or LOW for the interview tasks as a whole. For example, the first two students in the Experimental Group (see Table 4.26) each had five HIGH classifications and one MEDIUM classification. These students were classified as HIGH overall on the interview tasks, and their records are shown together. The next three students in the Experimental Group (see Table 4.26) each had four HIGH classifications and two MEDIUM classifications. These students were also classified as HIGH overall on the interview tasks, and their records are shown together. The data for the remaining students in each group were analyzed in a similar manner to create Tables 4.26 and 4.27.
Table 4.26. Individual Performances on Tasks (Experimental Group).

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>Task 1</th>
<th>Task 2</th>
<th>Task 3</th>
<th>Task 4</th>
<th>Task 5</th>
<th>Task 6</th>
<th>OVERALL</th>
</tr>
</thead>
<tbody>
<tr>
<td>E1A</td>
<td>H</td>
<td>H</td>
<td>M</td>
<td>H</td>
<td>H</td>
<td>H</td>
<td>HIGH</td>
</tr>
<tr>
<td>E1B</td>
<td>M</td>
<td>H</td>
<td>H</td>
<td>H</td>
<td>H</td>
<td>H</td>
<td>HIGH</td>
</tr>
<tr>
<td>E3A</td>
<td>H</td>
<td>H</td>
<td>M</td>
<td>M</td>
<td>H</td>
<td>H</td>
<td>HIGH</td>
</tr>
<tr>
<td>E3B</td>
<td>H</td>
<td>M</td>
<td>H</td>
<td>M</td>
<td>H</td>
<td>H</td>
<td>HIGH</td>
</tr>
<tr>
<td>E3C</td>
<td>M</td>
<td>H</td>
<td>M</td>
<td>H</td>
<td>H</td>
<td>H</td>
<td>HIGH</td>
</tr>
<tr>
<td>E6A</td>
<td>H</td>
<td>L</td>
<td>L</td>
<td>H</td>
<td>H</td>
<td>H</td>
<td>HIGH</td>
</tr>
<tr>
<td>E7A</td>
<td>H</td>
<td>M</td>
<td>H</td>
<td>M</td>
<td>H</td>
<td>M</td>
<td>MEDIUM</td>
</tr>
<tr>
<td>E7B</td>
<td>H</td>
<td>M</td>
<td>M</td>
<td>H</td>
<td>H</td>
<td>M</td>
<td>MEDIUM</td>
</tr>
<tr>
<td>E9A</td>
<td>H</td>
<td>H</td>
<td>L</td>
<td>M</td>
<td>H</td>
<td>M</td>
<td>MEDIUM</td>
</tr>
<tr>
<td>E10A</td>
<td>H</td>
<td>H</td>
<td>L</td>
<td>L</td>
<td>H</td>
<td>M</td>
<td>MEDIUM</td>
</tr>
<tr>
<td>E10B</td>
<td>H</td>
<td>L</td>
<td>L</td>
<td>H</td>
<td>H</td>
<td>M</td>
<td>MEDIUM</td>
</tr>
<tr>
<td>E12A</td>
<td>M</td>
<td>M</td>
<td>M</td>
<td>H</td>
<td>M</td>
<td>H</td>
<td>MEDIUM</td>
</tr>
<tr>
<td>E13A</td>
<td>H</td>
<td>L</td>
<td>M</td>
<td>M</td>
<td>M</td>
<td>M</td>
<td>MEDIUM</td>
</tr>
<tr>
<td>E14A</td>
<td>M</td>
<td>M</td>
<td>L</td>
<td>M</td>
<td>L</td>
<td>M</td>
<td>MEDIUM</td>
</tr>
<tr>
<td>E15A</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>M</td>
<td>H</td>
<td>LOW</td>
</tr>
</tbody>
</table>
### Table 4.27. Individual Performances on Tasks (Control Group).

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>Task 1</th>
<th>Task 2</th>
<th>Task 3</th>
<th>Task 4</th>
<th>Task 5</th>
<th>Task 6</th>
<th>OVERALL</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1A</td>
<td>H</td>
<td>H</td>
<td>M</td>
<td>H</td>
<td>L</td>
<td>H</td>
<td>HIGH</td>
</tr>
<tr>
<td>C2A</td>
<td>H</td>
<td>H</td>
<td>L</td>
<td>L</td>
<td>H</td>
<td>H</td>
<td>HIGH</td>
</tr>
<tr>
<td>C3A</td>
<td>H</td>
<td>L</td>
<td>M</td>
<td>L</td>
<td>H</td>
<td>M</td>
<td>MEDIUM</td>
</tr>
<tr>
<td>C3B</td>
<td>H</td>
<td>L</td>
<td>L</td>
<td>M</td>
<td>H</td>
<td>M</td>
<td>MEDIUM</td>
</tr>
<tr>
<td>C5A</td>
<td>H</td>
<td>H</td>
<td>L</td>
<td>M</td>
<td>M</td>
<td>M</td>
<td>MEDIUM</td>
</tr>
<tr>
<td>C6A</td>
<td>M</td>
<td>M</td>
<td>L</td>
<td>H</td>
<td>M</td>
<td>L</td>
<td>MEDIUM</td>
</tr>
<tr>
<td>C7A</td>
<td>M</td>
<td>M</td>
<td>L</td>
<td>L</td>
<td>M</td>
<td>M</td>
<td>MEDIUM</td>
</tr>
<tr>
<td>C8A</td>
<td>H</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>M</td>
<td>M</td>
<td>LOW</td>
</tr>
<tr>
<td>C9A</td>
<td>M</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>M</td>
<td>L</td>
<td>LOW</td>
</tr>
<tr>
<td>C9B</td>
<td>M</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>M</td>
<td>L</td>
<td>LOW</td>
</tr>
<tr>
<td>C9C</td>
<td>L</td>
<td>M</td>
<td>L</td>
<td>L</td>
<td>M</td>
<td>L</td>
<td>LOW</td>
</tr>
<tr>
<td>C9D</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>M</td>
<td>M</td>
<td>M</td>
<td>LOW</td>
</tr>
<tr>
<td>C9E</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>H</td>
<td>L</td>
<td>L</td>
<td>LOW</td>
</tr>
<tr>
<td>C14A</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>M</td>
<td>L</td>
<td>LOW</td>
</tr>
<tr>
<td>C14B</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>M</td>
<td>LOW</td>
</tr>
</tbody>
</table>
Tables 4.28 and 4.29 classify the students with respect to performance on the interview tasks and grade earned in the course.

**Table 4.28. Individual Overall Student Performances on Interview Tasks and Final Course Grades (Experimental Group).**

<table>
<thead>
<tr>
<th>STUDENT</th>
</tr>
</thead>
<tbody>
<tr>
<td>E1A</td>
</tr>
<tr>
<td>E1B</td>
</tr>
<tr>
<td>E3A</td>
</tr>
<tr>
<td>E3B</td>
</tr>
<tr>
<td>E3C</td>
</tr>
<tr>
<td>E6A</td>
</tr>
<tr>
<td>E7A</td>
</tr>
<tr>
<td>E7B</td>
</tr>
<tr>
<td>E9A</td>
</tr>
<tr>
<td>E10A</td>
</tr>
<tr>
<td>E10B</td>
</tr>
<tr>
<td>E12A</td>
</tr>
<tr>
<td>E13A</td>
</tr>
<tr>
<td>E14A</td>
</tr>
<tr>
<td>E15A</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>OVERALL TASK PERFORMANCE</th>
<th>COURSE GRADE</th>
</tr>
</thead>
<tbody>
<tr>
<td>HIGH</td>
<td>A</td>
</tr>
<tr>
<td>HIGH</td>
<td>C</td>
</tr>
<tr>
<td>HIGH</td>
<td>A</td>
</tr>
<tr>
<td>HIGH</td>
<td>B</td>
</tr>
<tr>
<td>HIGH</td>
<td>A</td>
</tr>
<tr>
<td>MEDIUM</td>
<td>C</td>
</tr>
<tr>
<td>MEDIUM</td>
<td>A</td>
</tr>
<tr>
<td>MEDIUM</td>
<td>F</td>
</tr>
<tr>
<td>MEDIUM</td>
<td>C</td>
</tr>
<tr>
<td>MEDIUM</td>
<td>D</td>
</tr>
<tr>
<td>MEDIUM</td>
<td>B</td>
</tr>
<tr>
<td>MEDIUM</td>
<td>C</td>
</tr>
<tr>
<td>MEDIUM</td>
<td>B</td>
</tr>
<tr>
<td>LOW</td>
<td>F</td>
</tr>
</tbody>
</table>
Table 4.29. Individual Overall Student Performances on Interview Tasks and Final Course Grades (Control Group).

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>OVERALL TASK PERFORMANCE</th>
<th>COURSE GRADE</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1A</td>
<td>HIGH</td>
<td>A</td>
</tr>
<tr>
<td>C2A</td>
<td>HIGH</td>
<td>F</td>
</tr>
<tr>
<td>C3A</td>
<td>MEDIUM</td>
<td>C</td>
</tr>
<tr>
<td>C3B</td>
<td>MEDIUM</td>
<td>B</td>
</tr>
<tr>
<td>C5A</td>
<td>MEDIUM</td>
<td>B</td>
</tr>
<tr>
<td>C6A</td>
<td>MEDIUM</td>
<td>B</td>
</tr>
<tr>
<td>C7A</td>
<td>MEDIUM</td>
<td>D</td>
</tr>
<tr>
<td>C8A</td>
<td>LOW</td>
<td>C</td>
</tr>
<tr>
<td>C9A</td>
<td>LOW</td>
<td>C</td>
</tr>
<tr>
<td>C9B</td>
<td>LOW</td>
<td>C</td>
</tr>
<tr>
<td>C9C</td>
<td>LOW</td>
<td>A</td>
</tr>
<tr>
<td>C9D</td>
<td>LOW</td>
<td>C</td>
</tr>
<tr>
<td>C9E</td>
<td>LOW</td>
<td>D</td>
</tr>
<tr>
<td>C14A</td>
<td>LOW</td>
<td>C</td>
</tr>
<tr>
<td>C14B</td>
<td>LOW</td>
<td>D</td>
</tr>
</tbody>
</table>
Tables 4.30 and 4.31 show how much time each student spent with the researcher for the interviews. Figure 4.1 displays the time information from Tables 4.30 and 4.31 in a visual manner.

Table 4.30. Time Spent on Interviews (Experimental Group).

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>OVERALL TASK PERFORMANCE</th>
<th>TIME SPENT (MINUTES)</th>
</tr>
</thead>
<tbody>
<tr>
<td>E1A</td>
<td>HIGH</td>
<td>65</td>
</tr>
<tr>
<td>E1B</td>
<td>HIGH</td>
<td>65</td>
</tr>
<tr>
<td>E3A</td>
<td>HIGH</td>
<td>90</td>
</tr>
<tr>
<td>E3B</td>
<td>HIGH</td>
<td>70</td>
</tr>
<tr>
<td>E3C</td>
<td>HIGH</td>
<td>65</td>
</tr>
<tr>
<td>E6A</td>
<td>HIGH</td>
<td>55</td>
</tr>
<tr>
<td>E7A</td>
<td>MEDIUM</td>
<td>65</td>
</tr>
<tr>
<td>E7B</td>
<td>MEDIUM</td>
<td>165</td>
</tr>
<tr>
<td>E9A</td>
<td>MEDIUM</td>
<td>60</td>
</tr>
<tr>
<td>E10A</td>
<td>MEDIUM</td>
<td>120</td>
</tr>
<tr>
<td>E10B</td>
<td>MEDIUM</td>
<td>55</td>
</tr>
<tr>
<td>E12A</td>
<td>MEDIUM</td>
<td>45</td>
</tr>
<tr>
<td>E13A</td>
<td>MEDIUM</td>
<td>90</td>
</tr>
<tr>
<td>E14A</td>
<td>MEDIUM</td>
<td>70</td>
</tr>
<tr>
<td>E15A</td>
<td>LOW</td>
<td>45</td>
</tr>
</tbody>
</table>

190
Table 4.31. Time Spent on Interviews (Control Group).

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>OVERALL TASK PERFORMANCE</th>
<th>TIME SPENT (MINUTES)</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1A</td>
<td>HIGH</td>
<td>50</td>
</tr>
<tr>
<td>C2A</td>
<td>HIGH</td>
<td>45</td>
</tr>
<tr>
<td>C3A</td>
<td>MEDIUM</td>
<td>40</td>
</tr>
<tr>
<td>C3B</td>
<td>MEDIUM</td>
<td>55</td>
</tr>
<tr>
<td>C5A</td>
<td>MEDIUM</td>
<td>40</td>
</tr>
<tr>
<td>C6A</td>
<td>MEDIUM</td>
<td>60</td>
</tr>
<tr>
<td>C7A</td>
<td>MEDIUM</td>
<td>60</td>
</tr>
<tr>
<td>C8A</td>
<td>LOW</td>
<td>45</td>
</tr>
<tr>
<td>C9A</td>
<td>LOW</td>
<td>50</td>
</tr>
<tr>
<td>C9B</td>
<td>LOW</td>
<td>45</td>
</tr>
<tr>
<td>C9C</td>
<td>LOW</td>
<td>55</td>
</tr>
<tr>
<td>C9D</td>
<td>LOW</td>
<td>40</td>
</tr>
<tr>
<td>C9E</td>
<td>LOW</td>
<td>40</td>
</tr>
<tr>
<td>C14A</td>
<td>LOW</td>
<td>45</td>
</tr>
<tr>
<td>C14B</td>
<td>LOW</td>
<td>65</td>
</tr>
</tbody>
</table>
Figure 4.1. Time spent on interviews.
CHAPTER 5

DISCUSSION OF RESULTS, RECOMMENDATIONS, AND CONCLUSIONS

This dissertation investigated students' abilities to make generalizations in geometry through the use of a generalization test and task-based interviews. Two groups of geometry students were compared in the study: an experimental group that made extensive use of *Cabri Geometry II* dynamic geometry software (on the TI-92) for the purpose of exploring concepts in geometry and a control group that did not use any geometry software during the course. This chapter discusses the findings of the study, comparisons with previous research, recommendations for research, recommendations for teaching, implications for theory, limitations of the study, and conclusions.

**RESEARCH QUESTIONS**

One overall research question and two subquestions guided the study. Each of these questions shall be restated and discussed in this section. Prior to discussing the findings of the study, baseline data regarding the students in the study shall be presented.

**Baseline data.** Three classes of students participated in this study. Two classes were taught by the researcher and received the experimental treatment. One class was taught by another mathematics teacher at the same school and served as a control group.
Based on pretest information (the Entering Geometry Student Test and the generalization pretest), it can be stated that the two groups (the experimental group and the control group) are similar. No significant differences were found between the experimental group and the control group on pretest data. The focus of the study was to take two similar groups and try to determine if regular use of dynamic geometry software seemed to enhance students’ abilities to make generalizations in geometry.

 **RESEARCH QUESTION:** Does experience with *Cabri* enhance students’ abilities to make generalizations in geometry? Two primary sources of data (the generalization posttest data and results of the interview tasks) and another source of data (observations made by the researcher during the period of data collection with the experimental group) will help to answer this question. While more detail will be given to the discussion of the generalization posttest and the interview tasks when reporting on the research subquestions later in this chapter, some comments will be made here to help answer the research question. No significant differences were found between the groups on the generalization posttest. If this were the only source of data available to inform the research question, the answer would have to be that the experience with *Cabri* did not seem to enhance students’ abilities to make generalizations in geometry. Based on data obtained from the interview tasks, however, a different answer seems appropriate. Results of the interview tasks show a relationship between group membership and performance on the interview tasks in ten out of fifteen categories reported (see Table 4.25). These results favor the experimental group. Considering Tasks 1 to 5 and overall performance on Task 6 as six categories, results favor the experimental group statistically in four of the six categories. Additionally, results from Tasks 1 and 2 favor the experimental group in their raw form (see Tables 4.10 and 4.11), but the results were not statistically significant. Therefore, it can be argued that results of the interview tasks overall do favor the experimental group.
Should one or the other of these results be given greater consideration in answering the research question? In the opinion of the researcher, the results of the interview tasks weigh more heavily in answering the research question. Even though the interview tasks involved only a subset of all of the students in the study, the nature of the questions on each instrument leads the researcher to make this conclusion. The type of generalizations required on the generalizations pre- and posttest were algebraic in nature. Beyond the time students actually spent taking the generalizations tests and some attention to these types of questions in Chapter 2 of the textbook, little attention was given to these types of generalizing questions during the course by either the researcher or the control group teacher. Sometimes, these types of generalizations occurred naturally in the course of study, but not often. The example that comes to mind is the Polygon Interior Angle Sum Theorem. Additionally, in this example, there is overlap on whether the generalization is algebraic or geometric. The generalization of \([180(n - 2)]^\circ\) for the sum of the interior angles of any convex polygon is algebraic in nature, but the origin of the data is geometric.

The researcher feels, therefore, that the interview tasks should weigh more heavily in answering the research question. With this in mind, it can be stated that the experience with Cabri does seem to enhance students abilities to make generalizations in geometry. Additional corroboration for this claim comes from observations that the researcher made during the course of data collection with the experimental group. With Cabri at their disposal, students were prone to experiment and make generalizations. One situation that merits discussion is when students were constructing the perpendicular bisectors of the sides of a triangle. In this situation, it was clear that dynamic geometry software offered much more to investigating the topic than any traditional method could hope to offer. Once students noted that the perpendicular bisectors were concurrent at the circumcenter, it was natural for them to grab a vertex and manipulate the triangle. In a matter of minutes, several observations were made regarding the circumcenter. Students noted that for acute
triangles the circumcenter was located in the interior of the triangle, for obtuse triangles, in
the exterior, and for right triangles, on the hypotenuse. One student pointed out that the
circumcenter was precisely at the midpoint of the hypotenuse. Figure 5.1 shows the case
of the acute and the obtuse triangle, while Figure 5.2 shows the case of the right triangle.

Figure 5.1. Circumcenters of acute and obtuse triangles.

Figure 5.2. Circumcenter of a right triangle.
After all four points of concurrency of triangles were discussed separately, the experimental group explored what might happen if one were to construct all four points on the same triangle. With the ability to hide parts of a sketch (a built-in feature of Cabri), it was easy for students to focus on the relationship between the four points themselves, instead of having to see all of the construction lines (the perpendicular bisectors of the sides, the angle bisectors, the altitudes, and the medians) on the same sketch. The result of this exploration is shown in Figure 5.3, complete with the Euler line (the line that the orthocenter, centroid, and circumcenter all lie on). Students can use the Collinear command in Cabri to check if points are collinear. Figure 5.3 also contains the information that there is a relationship among the lengths of the three points on the Euler line.

**Figure 5.3. Euler line of a triangle.**

Distance from Orthocenter to Centroid = 4.25 cm
Distance from Centroid to Circumcenter = 2.12 cm
Ratio of the two distances = 2.0
A Swiss mathematician, Leonhard Euler, proved in 1765 that the orthocenter, the centroid, and the circumcenter are all collinear. Euler also proved that the distance from the orthocenter to the centroid is twice the distance from the centroid to the circumcenter (Larson, Boswell, & Stiff, 1995, p. 229).

Another extension of this topic are the following questions: When, if ever, are all four of the points collinear? When, if ever, do two or more of the points coincide? These questions came up in the experimental group as a result of a student bringing the issues to the class, not the researcher. While examining the situation of all four points of concurrency on the same sketch, a student was manipulating the triangle and noticed that when the triangle was isosceles, all four points were collinear. Additionally, students brought up the fact that when the triangle was equilateral, all four of the points coincided. These results are shown in Figure 5.4 and Figure 5.5.
Figure 5.4. Euler line of an isosceles triangle.

Figure 5.5. All four points coincide in an equilateral triangle.
When the class was discussing the relationship between the lengths of the two segments on either side of the centroid of a triangle, one student commented that two of the three "shorter pieces" were congruent. I asked her what type of a triangle she had, and she responded that it was acute. This student brought her calculator to the front of the class and displayed her triangle for the class. Another student noted that the triangle appeared to be isosceles. The student measured the appropriate side lengths and concluded that the triangle was indeed isosceles (see Figure 5.6). I asked the class if anyone could extend this idea. Another student responded that if the triangle was equilateral, all three of the "shorter pieces" would be congruent and all three of the "longer pieces" would be congruent. The student at the overhead projector modified her triangle to where it appeared to be equilateral, made the appropriate measurements, and verified the result (see Figure 5.7).

![Figure 5.6. Medians of an isosceles triangle.](image)

AG = 3.31 cm  CG = 3.31 cm  
GE = 1.65 cm  GD = 1.65 cm
Rich explorations such as these can happen at any moment when students have tools like *Cabri* at their disposal.

**RESEARCH SUBQUESTION ONE:** Were there significant differences between the groups in scores on the generalization posttest? As stated earlier, the generalization tests contained items for which the generalizations were algebraic in nature. These tests served a useful purpose at the beginning of the study to help determine if there were any differences between the groups in their abilities to generalize. Results showed nonsignificant differences between the groups on the pretest data. Results of posttest data also showed nonsignificant differences between the groups.

Scores on the generalization pretests and posttests were low overall. The mean scores on all administrations of the test (as a pretest and as a posttest) were less than 20
(out of possible 50 points). The scores were relatively low on the pretest and did not show much improvement over the course of the study. This can be attributed, as mentioned earlier, to a lack of particular focus on these types of questions during the geometry course. Some problems like these occur in the text, but they are not a primary emphasis. Another reason the scores were low can be attributed to the method used for scoring the tests. Students could earn a total of 19 points (out of 50) for extending patterns, which is a much easier task than arriving at a generalization for the \(n\)th term of a sequence. Accordingly, \(n\)th term generalizations made up 31 points out of the 50 point total. This scoring was the same on each form of the test. Most students in both groups demonstrated some degree of success on extending patterns and collecting data, but only a few students from either group demonstrated a high degree of success on the generalizing questions.

One particular issue that merits discussion is regarding Form B, Question 5. The question is about the volume of several cubes whose sides start out with the pattern 1, 2, 4. The instructions state for the students to think carefully about the length of a side of the cube in Stage 4. The researcher created the question with doubling of side lengths in mind. Therefore, the next two cubes would have side lengths of 8 and 16, and corresponding volumes of 512 and 4096. One student in the experimental group interpreted the change in side lengths a different, yet valid, way. This student considered 7 and 11 to be the next two cases, by thinking of the side lengths changing by adding one, two, three, and then four to the length of the previous side length. The researcher did not consider this until the papers were scored. This student received full credit for this interpretation. The reader may refer to Appendix B for the full test, or to Chapter 3 for an item by item discussion of how the generalization tests were scored.

The problems on the generalization tests are excellent vehicles for teaching generalization, and could be the focus of future work by the researcher. To emphasize these problems any more than occurred with typical coverage of the text, however, would
have biased the results of the study. There have been occasions in the researcher's prior experience where a unit of study was taught that emphasized these types of generalization problems. On these occasions, students showed remarkable improvement in their ability to generalize from the beginning of the unit to the end of the unit.

**RESEARCH SUBQUESTION TWO: Was there a significant relationship between group membership and performance on the interview tasks?** In the opinion of the researcher, this question should be answered affirmatively. The summary of the chi-square statistics reported in Chapter 4 (see Table 4.25) show a relationship between group membership and performance on the interview tasks in ten of fifteen categories of analysis. The raw data favors the experimental group in all fifteen categories. These are important results which address the generalizing ability of students immersed in a *Cabri* environment. The task-by-task descriptions in this chapter further illustrate differences between the two groups. One of the most notable differences is the greater willingness of the experimental group to be persistent in looking for generalizations. The experimental group spent more time and tested a greater variety of conjectures when looking for generalizations during the interview tasks. Further discussion on time data is provided later in this chapter. Differences between the groups on each task shall now be discussed.

**Task 1.** Even though the chi-square value did not show a significant relationship between group membership and performance on the task, more students (66.67% vs. 40%) from the experimental group scored in the high category on this task. Students had to arrive at the correct value for the missing length in the fourth figure to be scored as a high for this task. Some students in both groups found the result very quickly, while others who found the correct value spent a fair amount of time experimenting with various computations (noting proportional relationships in some of the examples and looking for commonalities between the first three examples). Even though time spent on each
particular task was not recorded, it is clear from the interview summaries prepared by the researcher that more students in the experimental group spent considerable time examining the data.

**Task 2.** Again, the chi-square value cannot lead to the conclusion of a relationship between group membership and task performance. However, more students from the experimental group (40%) than the control group (20%) were classified in the high category for this task. Most students correctly interpreted Theorem 1, but there was a slightly higher tendency in the experimental group to do so. Four students in the control group needed help interpreting the theorem, while only one student in the experimental group needed help. Some students made a connection between the two theorems that was hoped for by the researcher. These students discussed the movement of point E either to some place on the circle, or to coincide with point C. Three students in the experimental group and two students from the control group made this connection. The reason that the researcher hoped for this connection is that Theorem 2 can be thought of as a special case of Theorem 1. Both theorems are about the angle(s) formed by the intersection of two chords. In the second theorem, the chords happen to intersect on the circle. In this situation, the measure of the "second arc" is zero, which two students (one from each group) specifically stated.

**Task 3.** The value of chi-square shows a significant relationship between group membership and performance on Task 3. This task proved to be harder than Task 1, where the goal was also to find a missing value. Only three students found the correct value in Example 4 (all from the experimental group). Multiplicative relationships seem harder for students to discover than additive relationships. Many students in both groups added up the lengths of the two parts of segment TS to see if there was a common sum. The notion of a common sum was the only idea that many students in the control group considered before giving up on the task. Once they found out that the sum was not the same in
Examples 1, 2, and 3, they did not try anything else. Differences between the groups were most apparent in this task on the willingness of the students in the experimental group to keep trying even though they were not having any success in finding the missing value. Two students in the experimental group commented that they did not recall this from class. They were reminded that not everything they would encounter during the interview was a topic specifically taught in class.

Notable among the interview summaries were the responses of one student in the experimental group. She compared this task to Task 1 by stating that Task 1 was easy, but it took “forever to find the result.” In fact, she did not find the correct result in this task, but she did spend a considerable amount of time on the task. This student stated that “It seems hard until you learn it, and then you think, I should have seen that!” One possible confounding issue in this task is the rounding of the product computation. For appearance and simplicity purposes, the lengths of segments TP and SP were rounded to two decimal places. This may have made it more difficult for students to be able to generalize the desired result. In Example 1, TP x SP = 4.8392. In Example 2, TP x SP = 4.8112. In Example 3, TP x SP = 4.8326. The students from the experimental group who were classified as high on this task did find that the product of the two small segments was always approximately 4.8. The fact that there was disagreement after one decimal place could have caused confusion. No students from the control group attempted to find the product of the two segments. The researcher paid careful attention to the types of computations that all students attempted during the interviews.

**Task 4.** Results of the statistical analysis show a significant relationship between group membership and performance on Task 4. This task, like Task 2, asked students to make connections between two theorems. For many students, the only connection given was that in both theorems you have to divide by two to get the result. All control group students and all but one student in the experimental group correctly interpreted Theorem 1
without any assistance. Like Task 2, this task offered an opportunity for students to transform a figure. Theorem 2 can be thought of as a special case of Theorem 1 with one of the sides having a length of zero. A few students in each group discussed how the shorter base of the trapezoid could have a zero length, or be compressed to one point. Two students in each group arrived at a different type of generalization: they stated that Task 4 was like Task 2, which, of course, it is. The fact that these two tasks are similar to each other was intentional. The researcher was interested to see if any students would actually make this particular observation.

Task 5. The value of chi-square shows a significant relationship between group membership and task performance for Task 5. This task was different than any of the others in that there was no missing value to find as in Task 1 and Task 3, nor any theorems to relate, as in Task 2 and Task 4. The task gave four examples of a cyclic quadrilateral with various angle and segment measurements given and asked students to make an observation as to what a friend had discovered. Students were encouraged to make as many observations as they could from the given data. Eleven students from the experimental group were classified in the high response category, while only three students from the control group were in that category. Many students noted that the angles of quadrilateral QUAD added up to 360°, which is true of any quadrilateral. Six students in the experimental group noted that the opposite angles of quadrilateral QUAD are supplementary, which is true of all cyclic quadrilaterals. No students in the control group made this observation. Neither group was taught anything about cyclic quadrilaterals during the geometry course. Two students from the experimental group made an interesting observation related to the fact that the opposite angles of a cyclic quadrilateral are supplementary. They noted that the difference between any two consecutive angles is the same as the difference between the other two angles. To assist the reader in understanding
what these two students discovered, Example 1 from Task 5 is shown here as Figure 5.8. Figure 5.9 explains the thinking of the two students.

![Figure 5.8. Example 1 from Task 5.](image)

The discovery figured one way...

\[ \angle U - \angle Q = 5.9^\circ \]
\[ \angle A - \angle D = 5.9^\circ \]

or a different way...

\[ \angle Q - \angle D = 29.3^\circ \]
\[ \angle U - \angle A = 29.3^\circ \]

![Figure 5.9. A discovery made during Task 5.](image)
This discovery was not one that the researcher had considered. When the first student figured this out, I had to convince myself of the truth of this generalization. Figure 5.10 demonstrates the reasons why this observation is true for all cyclic quadrilaterals.

\[ \text{NOTE: IF WE ASSUME X TO BE THE LARGEST ANGLE, THEN } 180 - X \text{ WILL BE SMALLER THAN } 180 - Y \]

\[
\begin{align*}
\text{THE DIFFERENCE OF TWO CONSECUTIVE ANGLES} &= X - Y \\
\text{THE DIFFERENCE OF THE OTHER TWO ANGLES} &= (180 - Y) - (180 - X) \\
&= 180 - Y - 180 + X \\
&= X - Y
\end{align*}
\]

Figure 5.10. Reasoning behind students' discovery made during Task 5.
Two students in the experimental group and one student from the control group made a different observation. Each of the three students expressed the idea in a slightly different way. A student from the experimental group expressed the relationship between an angle and a small segment that was “across from it.” In Example 1, this student noted that segment UM, the shortest of the four interior segments originating from point M, is “across from” angle D, the smallest of the four angles. A second student in the experimental group stated that the longest segment (in Example 1 this is segment DM) “points to” the smallest angle (angle D). A student from the control group stated that the shortest length (segment UM) is associated with the largest angle (angle U). Even though this discussion has centered around Example 1, each of these students verified that the discovery was true in each of the four examples. This task is related to the discovery in Task 3, as we have chords of a circle and a point on the chord. Even though three students made the appropriate generalization in Task 3, none of these students observed that the same relationship was present in Task 5. Several students tried adding the segment lengths together (more students from the experimental group than the control group did this), but found only that the sum of the four small segments was about approximately 12. It was not a common number to each example. Some students computed the lengths of the diagonals in each example. These students found that in Example 2, the diagonals were very close to each other in length (QA = 6.34 and DU = 6.42).

**Task 6.** Overall results for this task show a significant relationship between group membership and task performance. One of the notable things about this task is that so many students did not seem to have a grasp of the Pythagorean Theorem as an area argument. In the experimental group, seven students correctly interpreted the theorem with no help, four needed only a little help, and four needed to be told what the theorem represented. In the control group, four students correctly interpreted the theorem with no help, and eleven needed to be told what the theorem represented. The entire experimental
group and all but three students in the control group realized that we cannot put just any rectangles on the sides and have the area argument hold. Students found several ways to express their ideas: the rectangles must be similar; the rectangles must be proportional to each other; the squares must all be “cut the same way” (like 1/2, 1/3, or 1/4). Some students gave specific counterexamples. When asked if other figures might work, several students (five from the experimental group and two from the control group) suggested an equilateral triangle. On the last question of Task 6, three students from the experimental group and one student from the control group specifically stated that any figures would work as long as they were similar to each other. A majority of the students in the experimental group gave suggestions for other figures that could be constructed on the sides of the triangle and still have the area argument work. A majority of students in the control group felt that there were no other figures that would work (other than what had already been discussed in the task).

A brief discussion of the time spent on the interview tasks will now be given. Tables 4.30 and 4.31 report the length the interviews. Figure 4.1 shows these data in a visual manner. The experimental group spent more time overall in the interviews. Part of this might be attributed to the fact that they had the researcher as their geometry teacher, while the control group did not know the researcher, except as another teacher at the school. Part of the time difference, however, might be attributed to the fact that the experimental group spent a great deal of time during the course conjecturing while using Cabri. Some of this inquisitiveness may have carried over to the interview situation when they were asked to make generalizations without having the software in front of them. Students in the experimental group were routinely given tasks in class in which some level of generalization was the desired outcome. While not a daily occurrence, students in the experimental group were regularly asked to look at geometric situations using Cabri, take measurements, and to try to make some sense of what they saw on their calculator screen.
One final issue that merits discussion regarding the interview task data are the differences in grade levels of the students in the interview subgroups. The experimental subgroup consisted of ten freshmen and five sophomores. The control group consisted of four freshmen, ten sophomores, and one junior. Even though data were presented in Chapter 3 showing that there were nonsignificant differences in the make-up of the two subgroups, some readers may feel that the groups were somewhat different based on their grade-level distribution. In the opinion of the researcher, this difference did not have any major impact on the results of the study. All students were offered an opportunity to participate in the interviews, knowing that not all who volunteered would be chosen. As it turns out, not enough students volunteered for the interviews, and some students had to be solicited for participation. From a humanitarian standpoint, the researcher did not wish to turn away students from his classes (the experimental group) who volunteered to participate in the interview sessions. All students enrolled in Geometry had previously earned credit in Algebra I. No effort was made to use students' Algebra I course grades as a consideration as to whether the interview subgroups were similar. The researcher feels that adequate controls were in place (course grades, scores on the Entering Geometry Student Test, and scores on the generalization pretest) to argue that the interview subgroups were similar.

COMPARISONS WITH PREVIOUS RESEARCH

Previous research using some type geometry software can be traced back to the 1980s. Programs such as Logo, The Geometric Supposer, The Geometer's Sketchpad, Geometry Inventor, and Cabri have been used to examine teaching and learning issues in geometry classrooms. The study by Yerushalmy (1986) focused on induction and generalization in conjunction with use of The Geometric Supposer. The results of that study indicated that the computer-intensive environment positively impacted the
generalization ability of the students in the study. Results of the study by Yerushalmy, Chazan, and Gordon (1987) indicate a favorable impact on students' abilities to develop generalizations when The Geometric Supposer was used. The results of the present study also indicate a favorable impact on students' generalization abilities.

Roberts and Stephens (1999) reported on results of a study which compared the effect of differing levels of software implementation on students' geometry achievement. The study used Geometry Inventor, which is comparable to The Geometric Supposer. Few significant differences were reported, leading the researchers to claim that using computer software may not be beneficial when teaching certain topics in geometry. These findings support the claim by the researcher that dynamic geometry software may not be the best method for teaching certain topics in geometry.

Robinson (1994) concluded that strategies learned by students in a technological environment (The Geometer's Sketchpad) are transferable to paper-and-pencil situations. While the present study did not directly investigate the idea of transfer, it seems as though this is an underlying idea in the study. Students used dynamic geometry software on a regular basis during the course. During the interview tasks, however, students were not permitted to use the software. This was done in order to ensure an equal opportunity for both groups of students. Students in the experimental group would have been at an advantage during the interviews if they had access to Cabri. As the results of data analysis favored students in the experimental group, it seems that students were indeed able to transfer their conjecturing ability to a paper-and-pencil environment.

Some of the research studies in the literature review (Smyser, 1994; Bell, 1998; Round, 1998) have called for a more intense exposure to the software used in the study. By using Cabri to explore geometric ideas on a regular basis, this study has capitalized on those recommendations.
The present study compared a software group to a non-software group and found results that generally favored the software group. In the literature review in Chapter 2, several studies were discussed that compared a software group to a non-software group in some aspect of geometry learning (Yerushalmy, 1986; Yerushalmy, Chazan, & Gordon, 1987; Smyser, 1994; Dixon, 1995; Lester, 1996; Melczarek, 1996; Roberts & Stephens, 1999). All of these studies reported some positive results (some were nonsignificant, yet positive) favoring the software group. Further research needs to be conducted which considers questions posed by these researchers to more fully determine the impact of geometry software on student learning in geometry.

Several studies have examined the generalization abilities of the subjects in the study. Many of these studies involved finding the $n$th term of a sequence (Stacey, 1989; Hitt, 1994; Orton & Orton, 1994; Taplin & Robertson, 1995; Dossey, Swafford, Langrall, & Kersaint, 1997; García-Cruz & Martinón, 1998), as did the present study. Some of these studies involved adults rather than school-aged children. As reported by these studies and confirmed by the present study, generalizing to the $n$th term of a sequence is a challenging task for people of all ages. Further studies may wish to consider various treatments that may potentially have a positive impact on subjects' generalization abilities. Subjects in many of these studies could extend a sequence, but had difficulty generalizing to the $n$th term, a result found in the present study. Several of these studies asked subjects to find terms such as the 20th, 50th, and 100th terms prior to asking subjects to find the $n$th term. In the present study, students were asked to find the $n$th term after extending the sequence by a few terms. Future work by this researcher may consider these issues as they are of particular interest to the researcher, and generalization is an important aspect of school mathematics.
RECOMMENDATIONS FOR RESEARCH

As an education research community, we are just beginning to see the power of using dynamic geometry software in schools. While several studies have now been completed that have had some level of dynamic software as a component, there is much work to be done as we continue to learn about students' learning using this type of technology. It is difficult to recommend specific studies that should be done when the field is so wide open with what can be done. Conjecturing and generalizing are important processes in school mathematics, and especially important in geometry.

Further studies can be conducted to explore the role of dynamic geometry software with respect to conjecturing and generalizing. The present study could be replicated in other schools to add generalizability to the study. If the current study is replicated, this researcher recommends that the teacher in the study and the researcher be two different people. Circumstances in the present situation prevented this from occurring. If the current study is replicated, an additional consideration might be to give the Entering Geometry Student Test again at the end of the course to test for differences in the two groups on that instrument as a posttest. Replications of the study may wish to consider written data instead of interview data for the tasks merely as an alternative method of data collection. Replications may wish to consider replacing Task 3 with another task, as it proved quite difficult for the majority of the students in the study. The researcher recommends utilization of a secondary method of data collection during class time with each group. A separate observer may have been able to share some insight on differences between the groups that the researcher was unable to notice or document. Further studies can also be conducted with regards to different notions of mathematical learning (problem solving, for example) using dynamic geometry software. As stated earlier, the field for research is wide open, and very promising.
RECOMMENDATIONS FOR TEACHING

The results of the study indicate promise for the use of dynamic geometry software in geometry classrooms. I strongly recommend using dynamic geometry software in the teaching and learning of geometric ideas. As the current study focused on dynamic software, I consciously did not use the traditional tools of compasses and straightedges for constructions in geometry. All constructions were done using Cabri. While there is no doubt in my mind as to the power of dynamic geometry software to positively impact student learning, I feel that there is still a place for compass and straightedge constructions in geometry instruction. I feel that using both types of construction tools can only add to students' learning experiences in geometry. In fact, experience with compass and straightedge constructions may help students to be able to construct figures when using dynamic geometry software programs.

Traditional tools may have an advantage in teaching certain topics. In the current study, for example, dynamic geometry software was not used to teach students about the triangle congruence patterns (Side-Side-Side, Side-Angle-Side, Angle-Side-Angle, and Angle-Angle-Side). On other occasions prior to the study, I have used traditional tools to help students learn about the triangle congruence patterns. An example of this type of teaching might be to give students a paper with three segments so that they can construct a triangle. Students can be asked to choose a different segment to start the construction than the person next to them, for example. When the two students have finished their constructions, their triangles can be compared to see that these three segments do indeed form a unique triangle.

This study focused on whether dynamic geometry software can positively impact students' abilities to make generalizations in geometry. Another factor that must be considered is the disposition of the geometry teacher toward generalizing. The disposition of the teacher toward generalizing is a major factor in whether or not students increase their
level of generalization ability during a particular course. Teachers who routinely ask students to make conjectures create a different classroom environment than teachers who do not ask their students to make conjectures. Students in the experimental group were routinely asked to make observations and conjectures. The fact that these students were asked to make observations on a regular basis was certainly enabled by the use of Cabri. It is possible, of course, to ask students to make observations and conjectures without the use of dynamic geometry software. Observations of the control group teacher showed that students were asked to make conjectures in that classroom as well.

The idea of inductive versus deductive reasoning is a key issue in terms of generalization in mathematics. When students make conjectures from data they observe, inductive reasoning is used. When students prove conjectures using reasoning involving postulates, theorems, and definitions, deductive reasoning is used. There were many times during explorations with the experimental group that I highlighted the difference between the two reasoning processes. Students in the experimental group were often asked to prove results discovered using Cabri. Both inductive and deductive reasoning are important in the teaching and learning of geometry and neither should be ignored for the sake of the other.

This researcher strongly recommends (as has the National Council of Teachers of Mathematics and other professional organizations) the use of dynamic geometry software in the teaching and learning of geometry. Dynamic geometry software brings a new dimension to teaching geometry. The power of visualization is in each student's hands when they use dynamic geometry software to explore ideas in geometry. Students enjoy working with dynamic geometry software, and as the present study indicates, it positively impacts generalization ability, a key goal of school mathematics.
IMPLICATIONS FOR THEORY

This dissertation drew upon the Mathematics Learning Theory of Dienes, adapting it to a secondary situation with dynamic geometry software as a vehicle for helping students to develop generalization skills. Dienes’ theory is summarized into four principles: the Dynamic Principle, the Constructivity Principle, the Mathematical Variability Principle, and the Perceptual Variability (or Multiple Embodiment) Principle. This researcher believes that the essence of Dienes’ Mathematics Learning Theory is captured in Cabri. In Chapter 2, details are presented as to how Cabri embodies each of Dienes’ principles. In Chapter 3, lesson vignettes from the experimental group further explicate how Dienes’ principles are apparent when students use Cabri to explore concepts in geometry. Earlier in this chapter, some important classroom experiences detailing how Cabri enabled students in the experimental group to make generalizations are described. Results of this study indicate a favorable impact on generalization ability when students use dynamic geometry software such as Cabri.

Dynamic geometry software has the potential to impact students’ abilities to generalize on another level. As discussed in Chapter 2, software such as Cabri can be used to investigate Dienes’ ideas of abstraction as class formation and generalization as class extension. An example using Cabri that clarifies Dienes’ ideas on this topic is given in Chapter 2. While investigating this particular aspect of Dienes’ theory was beyond the scope of this dissertation study, it remains an ardent interest of the researcher, and could be the subject of some future work of the researcher. It seems clear that dynamic geometry software is a very promising tool for exploring Dienes’ ideas of class extension.
LIMITATIONS OF THE STUDY

A major limitation of the study is that the researcher was also a participant in the study as the teacher of the experimental group. Although I (the researcher) feel that appropriate controls were used to minimize bias, I also understand that there are individuals who may feel the results are somewhat biased due to the researcher and the teacher of the experimental group being the same person.

Another limitation is that the experimental and control groups were taught by different individuals. Every teacher has an individual style of teaching which cannot help but impact the way students learn. It is possible that differences found in the study can partially be attributed to differences in the teaching styles of the researcher and the control group teacher. Lampert (1988) emphasized the importance of the role of the teacher in a technological learning environment. As discussed in Chapter 2, Lampert feels that the power of geometry software lies not in what the software itself can do, “but in what it enables teachers to do if they are both able and disposed to use it in the way it was intended” (1988, p. 2).

Even though the planned content studied by both groups was the same, it is possible that very slight differences existed in the actual content covered. Again this is due to the fact that the two groups had different teachers. Both teachers used the same textbook and the same course of study. There are always slight differences, however, in what instructors plan to teach and what actually is taught. Any potential differences in this area were minimized by constant communication between the researcher and the control group teacher. Along these same lines, even minor issues such as particular homework problems assigned could bring about very slight differences in the content covered by both groups. Questions brought up by students, such as the circumcenter situation discussed earlier in this chapter, can cause differences in content coverage. Realistically, however, even if
both groups had been taught by the same individual, it would be virtually impossible to guarantee that there were absolutely no differences in content and delivery of instruction.

The sample size limits the generalizability of results. While it would have been advisable to include more students in the interviews, practical considerations prevented this from happening. Considering the time available in which to conduct the interviews, it was impossible to make the interview subgroups any larger than they were.

Another limitation is that it was difficult for the researcher to conduct the interviews in a reasonable time frame after the course ended. Interviews were mostly conducted during the researcher’s planning period and some interviews were not completed until March. This may have had an impact on the participants’ recall of geometric knowledge, as the geometry course ended in January.

CONCLUSIONS

This research study addressed one of the key issues of school mathematics today: the ability of students to make generalizations. Several studies have utilized geometry software programs to add to the research literature in geometry and technology. This study has added one more piece to the puzzle of studying students’ learning of geometry in a technological era. Using a theoretical framework developed by Dienes and adapted to the use of dynamic geometry software, the generalization ability of high school geometry students was the focus of the study. Even though students have been making generalizations in geometry for many years without the use of dynamic geometry software, this new tool provides a wealth of opportunities to take students further than they have gone before. Recommendations of professional organizations such as NCTM have stressed the use of dynamic geometry in the teaching and learning of geometry. The results of this study indicate that the use of dynamic geometry software seems to enhance students’
abilities to make generalizations in geometry. The future of geometry learning has taken a
new turn with the introduction of geometry software programs. In particular, dynamic
software programs such as Cabri seem to positively impact student learning in the area of
generalizations, a critical aspect of mathematics learning. For now, we must continue to
study teaching and learning geometry with exciting software such as Cabri. In the future,
we can only dream of how new technologies will impact geometry learning. Euclid would
be surprised to see the technological tools of today. We will likely be surprised to see the
technological advancements of tomorrow.
LIST OF REFERENCES


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APPENDIX A

ENTERING GEOMETRY STUDENT TEST
ENTERING GEOMETRY STUDENT TEST

Directions

Do not open this test booklet until you are told to do so.

This entering geometry student test contains 20 questions. It is not expected that you know everything on this test.

There is a test number in the top right hand corner of this page. Write this number in the corresponding place on your scan sheet.

You must use an Number 2 pencil to mark your answers on the scan sheet.

When you are told to begin:

1. Read each question carefully.

2. Decide upon the answer you think is correct. There is only one correct answer to each question. Mark the correct answer on the scan sheet.

3. You may write on the test booklet.

4. If you want to change an answer, completely erase the first answer.

5. If you need another pencil, raise your hand.

6. You will have 25 minutes for this test.

Do not begin until you are told to do so.
ENTERING GEOMETRY STUDENT TEST

1. Perpendicular lines
   A. intersect to form four right angles
   B. intersect to form two acute and two obtuse angles
   C. do not intersect at all
   D. intersect to form four acute angles
   E. none of the above

2. The area of a rectangle with length 3 inches and width 12 inches is
   A. 18 square inches
   B. 72 square inches
   C. 36 square inches
   D. 15 square inches
   E. 30 square inches

3. If two figures are similar but not congruent, then they
   A. have congruent bases and congruent altitudes
   B. have the same height
   C. both have horizontal bases
   D. have a different shape but the same size
   E. have a different size but the same shape

4. The measure of an obtuse angle is
   A. 90°
   B. between 45° and 90°
   C. less than 90°
   D. between 90° and 180°
   E. more than 180°
5. In the figure below, A, B, and D lie on a straight line. The measure of angle ABC is

A. 120°
B. 60°
C. 80°
D. 240°
E. need more information

6. Parallel lines are lines

A. in the same plane which never meet
B. which never lie in the same plane and never meet
C. which always form angles of 90° when they meet
D. which have the same length
E. none of the above

7. In the figure below, O is the center of the circle. Segment OA is called a

A. radius of the circle
B. diameter of the circle
C. chord of the circle
D. segment of the circle
E. sector of the circle
8. In the figure below, Angles 1 and 2 are called
A. opposite angles
B. parallel angles
C. alternate interior angles
D. alternate exterior angles
E. corresponding angles

9. The measure of a right angle is
A. less than 90°
B. between 90° and 180°
C. 45°
D. 90°
E. 180°

10. In the figure below, lines \( m \) and \( n \) are parallel. The measure of angle \( x \) is
A. 65°
B. 130°
C. 30°
D. 40°
E. 50°
11. An equilateral triangle has
A. all three sides the same length
B. one obtuse angle
C. two angles having the same measure and the third a different measure
D. all three sides of different lengths
E. all three angles of different measures

12. In the figure below, you are given that ABCD is a parallelogram. Which of the following statements is true?
A. ABCD is equiangular
B. Triangle ABD is congruent to triangle CDB
C. The perimeter of ABCD is four times the length of segment AB
D. Segment AC is the same length as segment BD
E. All of the above are true

13. The area of the triangle shown in the figure below is
A. 36 square cm.
B. 54 square cm.
C. 72 square cm.
D. 108 square cm.
E. 1620 square cm.
14. In the figure below, ABCD is a parallelogram. The measure of angle C is

A. 40°
B. 130°
C. 140°
D. 50°
E. need more information

![Parallelogram with angle C labeled]

15. In the figure below, the perimeter of parallelogram ABCD is

A. 25 cm.
B. 42 cm.
C. 21 cm.
D. 60 cm.
E. 90 cm.

![Parallelogram with dimensions labeled]
16. In the figure below, triangle ABC is similar to triangle DEF. The measure of segment AB is

A. 10 in.  
B. 11 in.  
C. 12 in.  
D. 13 in.  
E. 15 in.

\[ \triangle ABC \sim \triangle DEF \]

\[ AB = 14 \text{ in.} \]

\[ AC = 7 \text{ in.} \]

\[ EF = 5 \text{ in.} \]

\[ FD = 7 \text{ in.} \]

17. The plane figure produced by drawing all points exactly 6 inches from a given point is a

A. circle with a diameter of 6 inches  
B. square with a side of 6 inches  
C. sphere with a diameter of 6 inches  
D. cylinder 6 inches high and 6 inches wide  
E. circle with a radius of 6 inches

18. In the figure below, the area of the square is

A. 10 sq. in.  
B. 40 sq. in.  
C. 40 inches  
D. 100 sq. in.  
E. 100 inches
19. In the figure below, angles 1 and 2 are
   A. interior
   B. vertical
   C. supplementary
   D. complementary
   E. scalene

20. In the figure below, angle C is a right angle. The length of side AB is
   A. 8 cm.
   B. 14 cm.
   C. 10 cm.
   D. 12 cm.
   E. 18 cm.
APPENDIX B

GENERALIZATION TESTS
1. You are given the following information regarding a sequence of numbers:

<table>
<thead>
<tr>
<th>TERM IN SEQUENCE (T)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>VALUE OF TERM (V)</td>
<td>8</td>
<td>11</td>
<td>14</td>
<td>17</td>
<td>20</td>
</tr>
</tbody>
</table>

Give the value of the 6th, 7th, and nth terms in the sequence by filling in the table below.

<table>
<thead>
<tr>
<th>T</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>V</td>
<td>8</td>
<td>11</td>
<td>14</td>
<td>17</td>
<td>20</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

SPACE FOR FIGURING
2. If several people are in a room, how many handshakes are possible if each person shakes hands with everyone else once?

Collect data in the following table to help you analyze the situation.

<table>
<thead>
<tr>
<th>PEOPLE</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>HANDSHAKES</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Generalize the situation to predict how many handshakes there will be if there are \( n \) people in the room.

ANSWER: ____________________________

SPACE FOR FIGURING
3. Find a formula for the sum of the first $n$ odd positive integers.

Collect data in the following table to help you analyze the situation.

<table>
<thead>
<tr>
<th>NUMBER OF ODD INTEGERS</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>SUM</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>____</td>
<td>____</td>
</tr>
<tr>
<td>HOW TO FIGURE</td>
<td>1</td>
<td>1 + 3</td>
<td>1 + 3 + 5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Generalize the situation to find a formula for the sum of the first $n$ odd positive integers.

ANSWER: ____________________________

SPACE FOR FIGURING
4. Consider the following set of figures, each comprised of a right triangle sitting on top of a square.

![Stage 1][Stage 2][Stage 3]

Stage 1  Stage 2  Stage 3

a) Collect data in the following table regarding the area of each figure.

THE AREA OF A SQUARE IS COMPUTED BY MULTIPLYING ITS LENGTH BY ITS WIDTH.

THE AREA FORMULA FOR A TRIANGLE IS

\[ A = \frac{1}{2} b h \]

<table>
<thead>
<tr>
<th>STAGE</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>AREA</td>
<td>5</td>
<td>12</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) Generalize the situation to predict the area of the figure in the \( n \)th stage.

**ANSWER:**

**SPACE FOR FIGURING**

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5. Consider the following set of rectangles:

![Rectangles diagram]

<table>
<thead>
<tr>
<th>Stage 1</th>
<th>Stage 2</th>
<th>Stage 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>6</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>8</td>
</tr>
</tbody>
</table>

a) Complete the following table regarding the area of each rectangle.

THE AREA OF A RECTANGLE IS COMPUTED BY MULTIPLYING ITS LENGTH BY ITS WIDTH.

<table>
<thead>
<tr>
<th>STAGE</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>AREA</td>
<td>6</td>
<td>24</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) Generalize the situation to predict the area of the rectangle in the $n$th stage.

ANSWER: __________________________

SPACE FOR FIGURING
1. You are given the following information regarding a sequence of numbers:

TERM IN SEQUENCE (T)  | 1 | 2 | 3 | 4 | 5  
VALUE OF TERM (V)     | 0 | 5 | 12| 21| 32

Give the value of the 6th, 7th, and nth terms in the sequence by filling in the table below.

<table>
<thead>
<tr>
<th>T</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>V</td>
<td>0</td>
<td>5</td>
<td>12</td>
<td>21</td>
<td>32</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

SPACE FOR FIGURING
2. How many segments are necessary to connect \( n \) points on a plane if each point is connected to all the others?

The following figure will help you to visualize the situation.

Collect data in the following table to help you analyze the situation.

<table>
<thead>
<tr>
<th>POINTS</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>SEGMENTS</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Generalize the situation to predict how many segments there will be if there are \( n \) points.

ANSWER: _____________________________

SPACE FOR FIGURING
3. Find a formula for the sum of the first $n$ even positive integers.

Collect data in the following table to help you analyze the situation.

<table>
<thead>
<tr>
<th>NUMBER OF EVEN INTEGERS</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>SUM</td>
<td>2</td>
<td>6</td>
<td>12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>HOW TO FIGURE</td>
<td>2</td>
<td>2 + 4</td>
<td>2 + 4 + 6</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Generalize the situation to find a formula for the sum of the first $n$ even positive integers.

ANSWER: ____________________________

SPACE FOR FIGURING
4. Consider the following set of figures.

![Stage 1](4 5 6 5)

![Stage 2](1 2 3 4)

![Stage 3](Stage 3)

a) Collect data in the following table regarding the perimeter of each figure. The length of each small segment is 1. The numbers on the figures in Stages 1 and 2 are to help you see how the numbers in the table below were obtained.

THE PERIMETER OF A FIGURE IS DEFINED TO BE THE SUM OF THE LENGTHS OF ITS SIDES.

<table>
<thead>
<tr>
<th>STAGE</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>PERIMETER</td>
<td>4</td>
<td>8</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) Generalize the situation to predict the perimeter of the figure in the \(n\)th stage.

ANSWER: ____________________________

SPACE FOR FIGURING
5. Consider the following set of cubes:

Stage 1  Stage 2  Stage 3

Be sure to notice how the length of the sides of the cubes are changing. If the
lengths changed from 1 to 2 to 4, think carefully about the length of a side of a
cube in stage 4.

a) Complete the following table regarding the volume of each cube.

THE VOLUME OF A CUBE IS COMPUTED BY
THE FORMULA \( V = L \times W \times H \).

<table>
<thead>
<tr>
<th>STAGE</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>VOLUME</td>
<td>1</td>
<td>8</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) Generalize the situation to predict the volume of the cube in the \( n \)th
stage.

ANSWER: ________________________________

SPACE FOR FIGURING
APPENDIX C

INTERVIEW TASKS
TASK 1

You are working on your Geometry assignment with a friend. The topic is equilateral triangles. Your friend presents the following examples to you regarding a point, E, anywhere in the interior of the triangle, and the perpendicular segments constructed from E to each side of the triangle. Your friend states: "I think I have come up with something interesting regarding point E." Can you study the examples and try to find out what your friend has discovered?

EXAMPLE 1

EXAMPLE 2
Study the evidence presented by your friend in Examples 1, 2, and 3. What has your friend discovered? Find the missing value in Example 4.
**TASK 2**

You are working on your Geometry assignment with a friend. The topic you are studying is the various angles that are formed when two segments intersect inside a circle. Two results that you discovered in class are as follows:

**THEOREM 1:** In a circle, the measure of the angle formed by intersecting chords is one-half the sum of the measures of the intercepted arcs.

For the given illustration, the measure of angle DEB is equal to one-half the sum of the measures of arcs AC and DB. An example of Theorem 1 is as follows:

**EXAMPLE 1**

How could you interpret THEOREM 1 in terms of EXAMPLE 1?
THEOREM 2: In a circle, the measure of an inscribed angle is one-half the measure of the intercepted arc.

For the given illustration, the measure of angle PQR is one-half the measure of arc PR. An example of Theorem 2 is as follows:

EXAMPLE 2

Your friend makes the following observation: Theorems 1 and 2 seem somehow to be related, but I cannot seem to make the connection.

Do you think your friend is correct? If so, what is the connection between Theorem 1 and Theorem 2?
TASK 3

You are working on your Geometry assignment with a friend. The topic is circles. Your friend presents the following examples to you regarding a point, $P$, in the interior of the circle and the segment that passes through point $P$ and intersects the circle at points $T$ and $S$. Your friend states: “I think I have come up with something interesting regarding point $P$.” Can you study the examples and try to find out what your friend has discovered?

EXAMPLE 1

![Diagram of Example 1](image)

EXAMPLE 2

![Diagram of Example 2](image)
EXAMPLE 3

Study the evidence presented by your friend in Examples 1, 2, and 3. What has your friend discovered? Find the missing value in Example 4.

EXAMPLE 4

Study the evidence presented by your friend in Examples 1, 2, and 3. What has your friend discovered? Find the missing value in Example 4.
You are working on your Geometry assignment with a friend. The topic you are studying is midsegments of trapezoids and triangles. Two results that you discovered in class are as follows:

**THEOREM 1:** In a trapezoid, the length of the midsegment is one-half the sum of the lengths of the bases.

For the given illustration, the length of segment EF is one-half of the sum of the lengths of segments AB and DC. An example of Theorem 1 is as follows:

**EXAMPLE 1**

How could you interpret **THEOREM 1** in terms of **EXAMPLE 1**?
THEOREM 2: In a triangle, the length of the midsegment is one-half of the length of the side of the triangle to which it is parallel.

For the given illustration, the length of segment ST is one-half of the length of segment QR. An example of Theorem 2 is as follows:

EXAMPLE 2

Your friend makes the following observation: Theorems 1 and 2 seem somehow to be related, but I cannot seem to make the connection.

Do you think your friend is correct? If so, what is the connection between Theorem 1 and Theorem 2?

__________________________________________________________________

__________________________________________________________________

__________________________________________________________________

__________________________________________________________________

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TASK 5

You are working on your Geometry assignment with a friend. The topic is cyclic quadrilaterals. Your friend presents the following examples to you regarding cyclic quadrilaterals. Your friend states: "I think I have come up with something interesting regarding cyclic quadrilaterals." Can you study the examples and try to find out what your friend has discovered?

EXAMPLE 1

![Diagram of Example 1]

EXAMPLE 2

![Diagram of Example 2]
EXAMPLE 3

EXAMPLE 4

Study the evidence presented by your friend in Examples 1, 2, 3, and 4. What relationships do you see?
You are working on your Geometry assignment with a friend. The topic you are studying is the Pythagorean Theorem.

**THE PYTHAGOREAN THEOREM:** In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs.

How could you interpret the Pythagorean Theorem in terms of the above diagram?
Your friend poses the following question to you: What if the figures constructed on the sides were something other than squares? What if we cut each of these squares in half so that we have a rectangle on each side? What, if anything, could you say about the relationship?
Could I draw any rectangles on the sides and still have the relationship hold?
We have seen that the Pythagorean Theorem holds for squares and some kinds of rectangles. Are there any other figures we could put on the sides so that the relationship would still hold?
Consider the following diagram. Suppose we drew an isosceles right triangle on each side. What can you say about the relationship?
Could I draw *any* triangles on the sides and still have the relationship hold?
Consider the following diagram. How could we construct trapezoids on each side so that the relationship will still hold? What can you say about the relationship?
Now we've looked at squares, some rectangles, some triangles, and even trapezoids. Are there any other shapes we could put on each side so that the relationship will still hold?